SOBOLEV EMBEDDING THEOREM

IF $u \in W^{1,p}((a,b))$ THEN $u \in L^\infty((a,b)) \cap C^{0,1-\frac{1}{p}}((a,b))$ $(1 \le p \le \infty)$

IF $a,b \in \mathbb{R}$ THEN $W^{1,p}((a,b)) \hookrightarrow C^{0,1-\frac{1}{p}}((a,b))$ CONTINUOUS "CRITICAL" EMBEDDING

IF $p > 1$ AND $\alpha < 1 - \frac{1}{p}$ THEN $W^{1,p}((a,b)) \hookrightarrow C^{0,\alpha}((a,b))$ COMPACT EMBEDDING "SUB-CRITICAL"

RECALL: $C^{0,\alpha}((a,b)) := \left\{ u \in C((a,b)) ; \|u\|_{C^{0,\alpha}((a,b))} := \left(\|u\|_{L^\infty((a,b))} + \sup_{\substack{x,y \in (a,b) \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \right) \right\}$

ASCOLI-ARZELÀ THEOREM

$a,b \in \mathbb{R}$ (BOUNDED INTERVALS)

LET $\{u_n\}$ BE A BOUNDED SEQUENCE ON $C([a,b])$. IF $\{u_n\}$ IS EQUI-CONTINUOUS, THAT IS

$$\forall \varepsilon > 0 \ \exists \underbrace{\delta > 0}_{\text{NOT DEPENDING ON } n} \text{ SUCH THAT } |x-y| < \delta \Rightarrow |u_n(x) - u_n(y)| < \varepsilon \quad \forall n \in \mathbb{N}$$

THEN $\{u_n\}$ HAS A CONVERGING SUB-SEQUENCE.

PROOF OF SOBOLEV EMBEDDING THEOREM

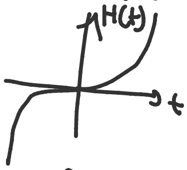
$a,b \in \mathbb{R}$

TAKE $u \in W^{1,p}$, SINCE $u \in L^p$, $\exists x_0$ SUCH THAT $|u(x_0)| \le \frac{\|u\|_{L^p}}{(b-a)^{\frac{1}{p}}}$

IF (a,b) IS UNBOUNDED, I CAN TAKE ANY SMALL $\delta > 0$ AND x_0 SUCH THAT $|u(x_0)| \le \delta \|u\|_{L^p}$

WE CONSIDER $H(t) = |t|^{p-1} t$

$H'(t) = p|t|^{p-1}$



IF $p > 1$ THEN $H(t) \in C^1(\mathbb{R})$

$\Rightarrow H(u) \in W^{1,p}_{loc}((a,b))$ $\Rightarrow$ I CAN APPLY THE FUNDAMENTAL THEOREM OF CALCULUS TO $H(u)$.

$$\underbrace{H(u(x))}_{|u(x)|^{p-1} u(x)} = H(u(x_0)) + \int_{x_0}^{x} H'(u(y))\, u'(y) = |u(x_0)|^{p-1} u(x_0) + \int_{x_0}^{x} p\, |u(y)|^{p-1} u'(y)\, dy$$

$\Downarrow$ ABS. VALUE

$$(*) \ |u(x)|^p \le |u(x_0)|^p + \int_{x_0}^{x} p\, |u|^{p-1} |u'| \underset{\text{HÖLDER}}{\le} \delta^p \|u\|^p_{L^p((a,b))} + p\, \|u\|^{p-1}_{L^p} \|u'\|_{L^p} < +\infty$$

$\forall x \in (a,b)$

$\Rightarrow \|u\|_{L^\infty} < +\infty$

$$\text{HÖLDER CONTINUITY: } (**) \ |u(x) - u(y)| = \left| \int_x^y u' \right| \le \int_x^y |u'| \cdot 1 \underset{\text{HÖLDER}}{\le} \left(\int_x^y |u'|^p \right)^{\frac{1}{p}} |x-y|^{1-\frac{1}{p}}$$

$$\le \|u'\| \ |x-y|^{1-\frac{1}{p}}$$

HÖLDER CONTINUITY: $|u(x) - u(y)| \le \int_x^y |u'| \le \left(\int_x^y |u'|^p\right)^{\frac{1}{p}} |x-y|^{1-\frac{1}{p}}$

$\le \|u'\|_{L^p} |x-y|^{1-\frac{1}{p}}$

FROM (*) AND (**) WE GET $\|u\|_{C^{0,1-\frac{1}{p}}} \le C\left(\|u\|_{L^p} + \|u'\|_{L^p}\right)$

THEREFORE THE EMBEDDING $W^{1,p}((a,b)) \hookrightarrow C^{0,1-\frac{1}{p}}((a,b))$ IS CONTINUOUS

COMPACT EMBEDDING IN $C^0((a,b))$: TAKE $\{u_n\}$ BOUNDED IN $W^{1,p}$.
BECAUSE OF THE CONTINUOUS EMBEDDING, $\{u_n\}$ IS BOUNDED IN C^0.
MOREOVER, AS IN (**), IF $|x-y| < \delta$ THEN
$|u_n(x) - u_n(y)| \le \|u_n'\|_{L^p} |x-y|^{1-\frac{1}{p}} \le C\delta^{\frac{1}{p}} =: \varepsilon \Rightarrow \{u_n\}$ IS EQUI-CONTINUOUS, SO BY ASCOLI-ARZELÀ THEOREM, $u_n \to u$ IN $C^0((a,b))$ UP TO SUBSEQ.

COMPACT EMBEDDING IN $C^{0,\alpha}((a,b))$ $\forall \alpha < \gamma := 1-\frac{1}{p}$
WE SUFFICE TO SHOW THAT $\|u_n - u\|_{C^{0,\alpha}((a,b))} \to 0$

$$\sup_{x \neq y} \frac{|u_n(x) - u(x) - (u_n(y) - u(y))|}{|x-y|^{\alpha}}$$

$$= \left(\frac{|u_n(x) - u(x) - (u_n(y) - u(y))|}{|x-y|^{\gamma}}\right)^{\frac{\alpha}{\gamma}} \left(|u_n(x) - u(x) - (u_n(y) - u(y))|\right)^{1-\frac{\alpha}{\gamma}}$$

$$\le \left(\frac{|u_n(x) - u_n(y)|}{|x-y|^{\gamma}} + \frac{|u(x) - u(y)|}{|x-y|^{\gamma}}\right)^{\frac{\alpha}{\gamma}} \left(|u_n(x) - u(x)| + |u_n(y) - u(y)|\right)^{1-\frac{\alpha}{\gamma}}$$

$$\le \underbrace{\left(\|u_n\|_{C^{0,\gamma}} + \|u\|_{C^{0,\gamma}}\right)}_{\le C}^{\frac{\alpha}{\gamma}} \left(2\underbrace{\|u_n - u\|_{C^0}}_{\to 0}\right)^{1-\frac{\alpha}{\gamma}} \xrightarrow[n \to \infty]{} 0$$

$\Rightarrow u_n \to u$ IN $C^{0,\alpha}$, THAT IS THE EMBEDDING IS COMPACT.

EXAMPLES (1) $u(x) = |x|^{\alpha}$ $0 < \alpha < 1$, WE SAW THAT $u \in W^{1,p}((-1,1))$
$\Leftrightarrow p < \frac{1}{1-\alpha} \Leftrightarrow \alpha < 1 - \frac{1}{p}$ AND MOREOVER $u \in C^{0,\alpha}((-1,1))$

(2) $u(x) = |x|^{1-\frac{1}{p}}$, $u \in W^{1,p}((-\frac{1}{2}, \frac{1}{2}))$ (BECAUSE $u' \sim \frac{1}{\ldots}$ ...)

② $u(x) = \dfrac{|x|^{1-\frac{1}{p}}}{\left(\log\frac{1}{|x|}\right)^{\frac{2}{p}}}$ $\quad u \in W^{1,p}\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ $\left(\text{BECAUSE } u' \sim \dfrac{1}{|x|^{\frac{1}{p}}\left(\log\frac{1}{|x|}\right)^{\frac{2}{p}}}\right)$

AND $u \in C^{0,\alpha} \Leftrightarrow \alpha \leq 1-\frac{1}{p} \Rightarrow \alpha = 1-\frac{1}{p}$ IS SHARP.

$\boxed{\text{REMARKS}}$ ① THE EMBEDDING $C^{0,\alpha}((a,b)) \hookrightarrow L^p((a,b))$ IS CONTINUOUS $a,b\in\mathbb{R}$ $\forall p \in [1,+\infty]$. SO THE EMBEDDING $W^{1,q}((a,b)) \hookrightarrow L^p((a,b))$ IS COMPACT $\forall q>1$ (COMPACT + CONTINUOUS = COMPACT)

② IF $a=-\infty$ OR $b=+\infty$, EMBEDDINGS ARE NEVER COMPACT.

TAKE $u_0 \in C^1_0((a,b))$, $u_0 \not\equiv 0$, $u_n(x) := u(x+n)$

$\|u_n\|_{W^{1,p}} = \|u_0\|_{W^{1,p}} \leq C$, IF $u_n \to u$ IN SOME $C^{0,\alpha}$ OR L^q

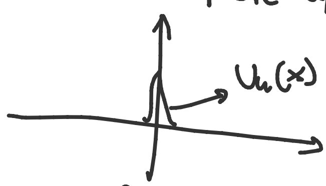
$\|u_n\|_{C^{0,\alpha}} = \|u_0\|_{C^{0,\alpha}}$

$\|u_n\|_{L^q} = \|u_0\|_{L^q}$

THEN $u \equiv 0$, BECAUSE $u\equiv 0$ IS THE POINTWISE LIMIT, BUT THIS IS IMPOSSIBLE BECAUSE $\|u_n - \underset{0}{u}\| = \|u_0\| \not\to 0$

③ IF $\gamma = 1-\frac{1}{p}$ THE EMBEDDING IS NOT COMPACT FOR $a,b\in\mathbb{R}$

FIX $u_0 \in C^1_0((a,b))$, $u_n(x) := \dfrac{u(nx)}{n^{1-\frac{1}{p}}}$



$\|u_n\|_{L^p} = \dfrac{\|u_0\|_{L^p}}{n}$ $\Rightarrow \{u_n\}$ IS BOUNDED IN $W^{1,p}$ BUT IT DOES NOT

$\|u_n'\|_{L^p} = \|u_0'\|_{L^p}$ CONVERGE IN $C^{0,\gamma}$: IF $u_n \to u$, THEN $u\equiv 0$

BECAUSE $u_n \to 0$ POINTWISE ALMOST EVERYWHERE.

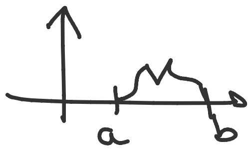
$\|u_n\|_{L^\infty} = \dfrac{\|u_0\|_{L^\infty}}{n^{1-\frac{1}{p}}} \to 0$ BUT $\sup_{x\neq y} \dfrac{|u_n(x)-u_n(y)|}{|x-y|^\gamma} = \sup_{x\neq y}\dfrac{|u(nx)-u(ny)|}{|nx-ny|^\gamma} = \sup_{x\neq y}\dfrac{|u_0(x)-u_0(y)|}{|x-y|^\gamma} \not\to 0$

$\boxed{\text{DEF}}$ WE DEFINE THE SOBOLEV SPACE OF FUNCTIONS WHICH ARE ZERO AT THE BOUNDARY $a,b\in\mathbb{R}$ (BOUNDED INTERVALS)

$W^{1,p}_0((a,b)) = \{u \in W^{1,p}((a,b)) : u(a) = u(b) = 0\}$ $\quad p \in [1, +\ldots$

$W_0^{1,p}((a,b)) = \{u \in W^{1,p}((a,b)) : u(a) = u(b) = 0\}$ $\quad p \in [1, +\infty]$

WELL-DEFINED BECAUSE u IS CONTINUOUS AT a, b.



$\boxed{\text{REMARK}}$ $W_0^{1,p}((a,b)) \triangleleft W^{1,p}((a,b))$: LINEARITY IS EASY

IF $u_n \to u$ IN $W^{1,p}$ AND $u_n(a) = u_n(b) = 0 \Rightarrow |u(a)| = |u(a) - u_n(a)| \le \|u_n - u\| \to 0$

SIMILARLY, ALSO $u(b) = 0$

THEOREM (POINCARÉ INEQUALITY)

IF $u \in W_0^{1,p}((a,b))$, $\exists C > 0$, INDEPENDENT OF u, SUCH THAT

$$\|u\|_{W^{1,p}((a,b))} \le C \|u'\|_{L^p((a,b))}$$

IN PARTICULAR $\|u'\|_{L^p}$ IS AN EQUIVALENT NORM ON $W_0^{1,p}$

PROOF

I SUFFICE TO SHOW $\|u\|_{L^p} \le C \|u'\|_{L^p}$.

$$p < +\infty \Rightarrow \|u\|_{L^p}^p = \int_a^b |u(x)|^p \underset{\substack{\text{FUND. THM. CALCULUS} \\ u(a)=0}}{=} \int_a^b \left| \int_a^x u' \right|^p \overset{\text{HÖLDER}}{\le} \int_a^b \int_a^x |u'|^p \cdot |x-a|^{p-1} \le \int_a^b |u'|^p (b-a)^p$$

$$\Rightarrow \|u\|_{L^p} \le (b-a) \|u'\|_{L^p}$$

$$p = +\infty \quad \to |u(x)| = \left| \int_a^x u' \right| \le \int_a^x |u'| \le \|u'\|_\infty (x-a) \le (b-a) \|u'\|_\infty$$

$$\Rightarrow \|u\|_\infty \le (b-a) \|u'\|_\infty$$

$\boxed{\text{REMARKS}}$ ① POINCARÉ INEQUALITY DOES NOT HOLD IN $W^{1,p}((a,b))$: IF $u \equiv C$ IS CONSTANT, $\|u'\|_{L^p} \equiv 0 \not\equiv \|u\|_{L^p}$.

② $p = 2$ $\quad W_0^{1,2}$ IS A HILBERT SPACE WITH AN EQUIVALENT SCALAR PRODUCT $(u,v) = \int_a^b u'v'$ OR, MORE IN GENERAL WE CAN CHOOSE

$$(u,v) = \int_a^b (p(x)\, u'v' + q(x)\, uv) \quad \text{WITH } p, q \in L^\infty \quad q \ge 0, \; p \ge \delta_0 > 0$$