Sapienza - Università di Roma - Corso di Laurea in Ingegneria Elettrotecnica

Soluzioni degli esercizi di Analisi Matematica I

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Lezione 9 del 1 Dicembre 2016
Argomento: Integrali

1. Calcolare i seguenti integrali indefiniti:

   (a) $\int (x+1)e^{x^2+2x}\mathrm{d}x$

   $$\int (x+1)e^{x^2+2x}\mathrm{d}x = \frac{1}{2}\int (2x+2)e^{x^2+2x}\mathrm{d}x = \frac{1}{2}\int \frac{d}{dx}e^{x^2+2x}\mathrm{d}x = \frac{1}{2}e^{x^2+2x} + C.$$

   (b) $\int \frac{(\log x)^5}{x}$

   Con la sostituzione $y = \log x$ si ottiene

   $$\int \frac{(\log x)^5}{x} = \int y^5 \mathrm{d}y = \frac{y^6}{6} + C = \frac{(\log x)^6}{6} + C.$$

   (c) $\int x^5 \log x \mathrm{d}x$

   Integrando per parti si ottiene

   $$\begin{aligned}\int x^5 \log x \mathrm{d}x &= \int \left(\frac{d}{dx}\frac{x^6}{6}\right)\log x \mathrm{d}x \\ &= \frac{x^6}{6}\log x - \int \frac{x^6}{6}\frac{1}{x} + C \\ &= \frac{x^6}{6}\log x - \frac{1}{6}\int \frac{x^5}{6} + C \\ &= \frac{x^6}{6}\log x - \frac{x^6}{36} + C.\end{aligned}$$

   (d) $\int x^3 \sin\left(x^2\right)\mathrm{d}x$

   Con la sostituzione $y = \sin\left(x^2\right)$ e poi integrando per parti si ottiene

   $$\begin{aligned}\int x^3 \sin\left(x^2\right)\mathrm{d}x &= \frac{1}{2}\int y \sin y \mathrm{d}y \\ &= \frac{1}{2}\left(-y\cos y + \int \cos y \mathrm{d}y\right) + C \\ &= \frac{1}{2}\left(-y\cos y + \sin y\right) + C \\ &= -\frac{x^2}{2}\cos\left(x^2\right) + \frac{\sin\left(x^2\right)}{2} + C.\end{aligned}$$

(e) $\displaystyle\int \frac{\mathrm{d}x}{x^4-1}$

Provando a scrivere

$$\frac{1}{x^4-1} = \frac{Ax+B}{x^2+1}+\frac{C}{x-1}+\frac{D}{x+1} = \frac{(A+C+D)x^3+(B+C-D)x^2+(-A+C+D)x+(-B+C-D)}{x^4-1}$$

si ottiene $A=0, B=-\frac{1}{2}, C=\frac{1}{4}, D=-\frac{1}{4}$, dunque

$$\frac{1}{x^4-1} = \int\left(-\frac{1}{2}\frac{1}{x^2+1}+\frac{1}{4}\frac{1}{x+1}-\frac{1}{4}\frac{1}{x-1}\right) = -\frac{1}{2}\arctan x+\frac{1}{4}\log|x+1|-\frac{1}{4}\log|x-1|+C$$

(f) $\displaystyle\int \frac{x^3}{x^2-4}\mathrm{d}x$

$$\int \frac{x^3}{x^2-4} = \int\left(x+2\frac{2x}{x^2-4}\right) = \frac{x^2}{2}+\log\left|x^2-4\right|+C.$$

(g) $\displaystyle\int \frac{\mathrm{d}x}{1+\sin(x)}$

Con la sostituzione $t=\tan\frac{x}{2}$ si ottiene

$$\int \frac{\mathrm{d}x}{1+\sin(x)} = \int \frac{1}{1+\frac{2t}{1+t^2}}\frac{2}{1+t^2}\mathrm{d}t = 2\int\frac{\mathrm{d}t}{(1+t)^2} = -\frac{2}{1+t}+C = -\frac{2}{1+\tan\frac{x}{2}}+C.$$

2. Calcolare i seguenti integrali definiti:

(a) $\displaystyle\int_0^{\pi^3} \frac{\sin\sqrt[3]{x}}{x^{\frac{2}{3}}}\mathrm{d}x.$

Con la sostituzione $y=\sqrt[3]{x}$ si ottiene

$$\int_0^{\pi^3} \frac{\sin\sqrt[3]{x}}{x^{\frac{2}{3}}}\mathrm{d}x = \int_0^{\pi^3} \frac{3\sin\sqrt[3]{x}}{3x^{\frac{2}{3}}}\mathrm{d}x = \int_0^{\pi} 3\sin y\mathrm{d}y = [-3\cos(y)]_0^{\pi} = -3\cos\pi+3\cos 0 = 6.$$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} (\cos x)^5\mathrm{d}x$

Con la sostituzione $y=\sin x$ si ottiene

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} (\cos x)^5\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \left(1-(\sin x)^2\right)^2\cos x\mathrm{d}x\\
&= \int_0^1 \left(1-y^2\right)^2\mathrm{d}y\\
&= \int_0^1 \left(1-2y^2+y^4\right)\mathrm{d}y\\
&= \left[y-\frac{2}{3}y^3+\frac{y^5}{5}\right]_0^1\\
&= 1-\frac{2}{3}+\frac{1}{5}\\
&= \frac{8}{15}.
\end{aligned}$$

(c) $\displaystyle\int_{-1}^{1} |x|e^x \mathrm{d}x$

Dividendo l'integrale tra $[-1,0]$ e $[0,1]$ e integrando entrambi per parti si ottiene

$$
\begin{aligned}
\int_{-1}^{1} |x|e^x \mathrm{d}x &= \int_{-1}^{0} -xe^x \mathrm{d}x + \int_{0}^{1} xe^x \mathrm{d}x \\
&= [-xe^x]_{-1}^{0} + \int_{-1}^{0} e^x \mathrm{d}x + [xe^x]_0^1 - \int_0^1 e^x \mathrm{d}x \\
&= -\frac{1}{e} + [e^x]_{-1}^{0} + e - [e^x]_0^1 \\
&= -\frac{1}{e} + 1 - \frac{1}{e} + e - e + 1 \\
&= 2 - \frac{2}{e}
\end{aligned}
$$

(d) $\displaystyle\int_1^e (\log x)^2 \mathrm{d}x$

Integrando per parti e ricordando che l'integrale indefinito del logaritmo è $\displaystyle\int \log x \mathrm{d}x = x\log x - x + C$ si ottiene

$$
\begin{aligned}
\int_1^e (\log x)^2 \mathrm{d}x &= [(x\log x - x)\log x]_1^e - \int_1^e (x\log(x) - x)\frac{1}{x}\mathrm{d}x \\
&= [x(\log x)^2 - x\log x]_1^e - \int_1^e (\log x - 1)\mathrm{d}x \\
&= -\int_1^e (\log x - 1)\mathrm{d}x \\
&= -[x\log x - 2x]_1^e \\
&= e - 2.
\end{aligned}
$$

(e) $\displaystyle\int_0^1 \frac{\sqrt{x}}{1+x}\mathrm{d}x$

Con la sostituzione $y = \sqrt{x}$ si ottiene

$$
\begin{aligned}
\int_0^1 \frac{\sqrt{x}}{1+x}\mathrm{d}x &= \int_0^1 \frac{2x}{1+x}\frac{1}{2\sqrt{x}}\mathrm{d}x \\
&= \int_0^1 \frac{2y^2}{1+y^2}\mathrm{d}y \\
&= 2\int_0^1 \left(1 - \frac{1}{1+y^2}\right)\mathrm{d}y \\
&= 2[y - \arctan(y)]_0^1 \\
&= 2(1 - \arctan 1) \\
&= 2 - \frac{\pi}{2}.
\end{aligned}
$$

(f) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\mathrm{d}x}{\cos x}$

Con la sostituzione $y = \cos(x)$ si ottiene

$$
\begin{aligned}
\int_0^{\frac{\pi}{6}} \frac{\mathrm{d}x}{\cos x} &= \int_0^{\frac{\pi}{6}} \frac{\cos x}{(\cos x)^2}\mathrm{d}x \\
&= \int_0^{\frac{\pi}{6}} \frac{\cos x}{1 - (\sin x)^2}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\frac{1}{2}} \frac{\mathrm{d}y}{1-y^2} \\
&= \int_0^{\frac{1}{2}} \left(\frac{1}{2}\frac{1}{y+1} - \frac{1}{2}\frac{1}{y-1}\right) \mathrm{d}y \\
&= \frac{1}{2}[\log|y+1| - \log|y-1|]_0^{\frac{1}{2}} \\
&= \frac{1}{2}\left(\log\frac{3}{2} - \log\frac{1}{2}\right) \\
&= \frac{\log 3}{2}.
\end{aligned}
$$

(g) $\displaystyle\int_0^{\log 2} \frac{\mathrm{d}x}{e^x + e^{-x} - 1}$.
Con la sostituzione $y = e^x$ si ottiene

$$
\begin{aligned}
\int_0^{\log 2} \frac{\mathrm{d}x}{e^x + e^{-x} - 1} &= \int_0^{\log 2} \frac{e^x}{e^{2x} - e^x + 1}\mathrm{d}x \\
&= \int_1^2 \frac{\mathrm{d}y}{y^2 - y + 1} \\
&= \int_1^2 \frac{\mathrm{d}y}{\left(y-\frac{1}{2}\right)^2 + \frac{3}{4}} \\
&= \frac{2}{\sqrt{3}} \int_1^2 \frac{\frac{2}{\sqrt{3}}}{\left(\frac{2y-1}{\sqrt{3}}\right)^2 + 1}\mathrm{d}y \\
&= \frac{2}{\sqrt{3}}\left[\arctan\frac{2y-1}{\sqrt{3}}\right]_1^2 \\
&= \frac{2}{\sqrt{3}}\left(\arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}}\right) \\
&= \frac{\pi}{3\sqrt{3}}.
\end{aligned}
$$