Università degli Studi Roma Tre - Ingegneria delle Tecnologie per il Mare

Esercizi di Analisi Matematica I

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Argomento: Limiti di funzioni

0. Dando per buoni i limiti notevoli

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\ln(1+x)}{x}=1,$$

dimostrare i seguenti limiti:

(a) $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$;

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{1}{1+\cos x}\frac{1-\cos^2 x}{x^2}=\lim_{x\to 0}\frac{1}{1+\cos x}\lim_{x\to 0}\frac{\sin^2 x}{x^2}=\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2=\frac{1}{2}.$$

(b) $\lim_{x\to 0}\frac{\tan x}{x}=1$;

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{1}{\cos x}\frac{\sin x}{x}=\lim_{x\to 0}\frac{1}{\cos x}\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\sin x}{x}=1.$$

(c) $\lim_{x\to 0}\frac{\arctan x}{x}=1$;
Scrivendo $y=\arctan x$ avremo $y\underset{\to 0}{\to}0$ e $x=\tan y$, dunque

$$\lim_{x\to 0}\frac{\arctan x}{x}=\lim_{y\to 0}\frac{y}{\tan y}=\frac{1}{\lim_{y\to 0}\frac{\tan y}{y}}=1.$$

(d) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$;

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\lim_{x\to 0}e^{\ln\left((1+x)^{\frac{1}{x}}\right)}=\lim_{x\to 0}e^{\frac{\ln(1+x)}{x}}=e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}}=e.$$

(e) $\lim_{x\to 0}\frac{e^x-1}{x}=1$;
Scrivendo $y=e^x-1$ avremo $y\underset{\to 0}{\to}0$ e $x=\ln(1+y)$, dunque

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{y\to 0}\frac{y}{\ln(1+y)}=\frac{1}{\lim_{y\to 0}\frac{\ln(1+y)}{y}}=1.$$

(f) $\lim_{x\to 0}\frac{(1+x)^a-1}{x}=a$, $\forall a\in\mathbb{R}$;

$$\begin{aligned}\lim_{x\to 0}\frac{(1+x)^a-1}{x} &= \lim_{x\to 0}\frac{e^{\ln((1+x)^a)}-1}{x}\\ &= \lim_{x\to 0}\frac{e^{a\ln(1+x)}-1}{a\ln(1+x)}\frac{a\ln(1+x)}{x}\end{aligned}$$

$$
\begin{aligned}
&= a \lim_{x\to 0} \frac{e^{a\ln(1+x)}-1}{a\ln(1+x)} \lim_{x\to 0} \frac{\ln(1+x)}{x} \\
&= a \lim_{x\to 0} \frac{e^{a\ln(1+x)}-1}{a\ln(1+x)}.
\end{aligned}
$$

Per calcolare l'ultimo limite, scriviamo $y = a\ln(1+x)$ e avremo $y \underset{x\to 0}{\to} 0$ e

$$
\lim_{x\to 0} \frac{(1+x)^a - 1}{x} = a \lim_{y\to 0} \frac{e^y - 1}{y} = a.
$$

Calcolare, se esistono, i seguenti limiti:

1. $\displaystyle\lim_{x\to 1} \frac{e^x - e}{\ln x}$;

$$
\lim_{x\to 1} \frac{e^x - e}{\ln x} = e \lim_{x\to 1} \frac{e^{x-1}-1}{\ln x} = e \lim_{y\to 0} \frac{e^y - 1}{\ln(1+y)} = e \lim_{y\to 0} \frac{\frac{e^y-1}{y}}{\frac{\ln(1+y)}{y}} = e \frac{\lim_{y\to 0} \frac{e^y-1}{y}}{\lim_{y\to 0} \frac{\ln(1+y)}{y}} = e.
$$

2. $\displaystyle\lim_{x\to 0} \frac{e^{x^2} - \sqrt[3]{\cos x}}{x^2}$;

$$
\begin{aligned}
\lim_{x\to 0} \frac{e^{x^2} - \sqrt[3]{\cos x}}{x^2} &= \lim_{x\to 0} \frac{e^{x^2}-1}{x^2} + \lim_{x\to 0} \frac{1 - \sqrt[3]{\cos x}}{x^2} \\
&= 1 + \lim_{x\to 0} \frac{1-(\cos x)^{\frac{1}{3}}}{1-\cos x} \frac{1-\cos x}{x^2} \\
&= 1 + \lim_{x\to 0} \frac{(1+\cos x - 1)^{\frac{1}{3}} - 1}{\cos x - 1} \lim_{x\to 0} \frac{1-\cos x}{x^2} \\
&= 1 + \frac{1}{2} \lim_{y\to 0} \frac{(1+y)^{\frac{1}{3}} - 1}{y} \\
&= 1 + \frac{1}{2}\frac{1}{3} \\
&= \frac{7}{6}.
\end{aligned}
$$

3. $\displaystyle\lim_{x\to 0} \frac{\tan(\pi e^x)}{x}$;
   Poiché $\tan(\pi + y) = \tan y$ per ogni $y \in \mathbb{R}$, avremo

$$
\lim_{x\to 0} \frac{\tan(\pi e^x)}{x} = \lim_{x\to 0} \frac{\tan(\pi(e^x-1))}{x} = \pi \lim_{x\to 0} \frac{\tan(\pi(e^x-1))}{\pi(e^x-1)} \frac{e^x-1}{x} = \pi \lim_{y\to 0} \frac{\tan y}{y} \lim_{x\to 0} \frac{e^x-1}{x} = \pi.
$$

4. $\displaystyle\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$;

$$
\begin{aligned}
\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} &= \lim_{x\to 0} e^{\ln\left((\cos x)^{\frac{1}{x^2}}\right)} \\
&= \lim_{x\to 0} e^{\frac{\ln(\cos x)}{x^2}} \\
&= \lim_{x\to 0} e^{\frac{\ln(\cos x)}{1-\cos x} \frac{1-\cos x}{x^2}}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-\lim_{x\to 0}\frac{\ln(1+\cos x-1)}{\cos x-1}\lim_{x\to 0}\frac{1-\cos x}{x^2}}\\
&= e^{-\frac{1}{2}\lim_{y\to 0}\frac{\ln(1+y)}{y}}\\
&= e^{-\frac{1}{2}}\\
&= \frac{1}{\sqrt{e}}.
\end{aligned}
$$

5. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$;

$$
\begin{aligned}
\lim_{x\to 0}\frac{\tan x-\sin x}{x^3} &= \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}\\
&= \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\frac{1}{\cos x}-1}{x^2}\\
&= \lim_{x\to 0}\frac{1}{\cos x}\frac{1-\cos x}{x^2}\\
&= \lim_{x\to 0}\frac{1}{\cos x}\lim_{x\to 0}\frac{1-\cos x}{x^2}\\
&= \frac{1}{2}.
\end{aligned}
$$

6. $\lim\limits_{x\to +\infty} x\ln\dfrac{x+1}{x-1}$;

$$
\begin{aligned}
\lim_{x\to +\infty} x\ln\frac{x+1}{x-1} &= \lim_{x\to +\infty} x\ln\left(1+\frac{2}{x-1}\right)\\
&= \lim_{x\to +\infty}\left(2\frac{x-1}{2}\ln\left(1+\frac{2}{x-1}\right)+\ln\left(1+\frac{2}{x-1}\right)\right)\\
&= 2\lim_{x\to +\infty}\frac{x-1}{2}\ln\left(1+\frac{2}{x-1}\right)+\lim_{x\to +\infty}\ln\left(1+\frac{2}{x-1}\right)\\
&= 2\lim_{x\to +\infty}\frac{x-1}{2}\ln\left(1+\frac{2}{x-1}\right)\\
&= 2\lim_{y\to 0}\frac{\ln(1+y)}{y}\\
&= 2.
\end{aligned}
$$

7. $\lim\limits_{x\to +\infty}\dfrac{\ln\left(x^2+x+1\right)-2\ln x}{\sin\frac{1}{x^2}+\arctan\frac{1}{x}}$;

$$
\begin{aligned}
\lim_{x\to +\infty}\frac{\ln\left(x^2+x+1\right)-2\ln x}{\sin\frac{1}{x^2}+\arctan\frac{1}{x}} &= \lim_{x\to +\infty}\frac{\ln\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{\sin\frac{1}{x^2}+\arctan\frac{1}{x}}\\
&= \lim_{y\to 0}\frac{\ln\left(1+y+y^2\right)}{\sin\left(y^2\right)+\arctan y}\\
&= \lim_{y\to 0}\frac{\ln\left(1+y+y^2\right)}{y+y^2}\lim_{y\to 0}\frac{y+y^2}{\sin\left(y^2\right)+\arctan y}\\
&= \lim_{y\to 0}\frac{y+y^2}{y^2\frac{\sin(y^2)}{y^2}+y\frac{\arctan y}{y}}\\
&= \lim_{y\to 0}\frac{1+y}{y\frac{\sin(y^2)}{y^2}+\frac{\arctan y}{y}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\lim_{y\to 0}(1+y)}{\lim_{y\to 0} y \lim_{y\to 0} \frac{\sin(y^2)}{y^2} + \lim_{y\to 0} \frac{\arctan y}{y}} \\
&= 1.
\end{aligned}
$$