Università degli Studi Roma Tre - Ingegneria delle Tecnologie per il Mare

Esercizi di Analisi Matematica I

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Soluzioni degli esercizi 5 del 5 Novembre 2019
Argomento: Limiti con sviluppo in serie

Calcolare, se esistono, i seguenti limiti:

1. $\lim_{x\to 0} \dfrac{x^2}{3\sin x - \ln(1+3x)}$;
Ricordiamo gli sviluppi delle funzioni

$$\begin{aligned} \sin x &= x + O\left(x^3\right) \\ \ln(1+x) &= x - \frac{x^2}{2} + O\left(x^3\right), \end{aligned}$$

e applichiamo quest'ultimo con $3x$ al posto di x:

$$\ln(1+3x) = 3x - \frac{(3x)^2}{2} + O\left((3x)^3\right) = 3x - \frac{9}{2}x^2 + O\left(x^3\right).$$

Si ottiene dunque

$$\begin{aligned} \lim_{x\to 0} \frac{x^2}{3\sin x - \ln(1+3x)} &= \lim_{x\to 0} \frac{x^2}{3\left(x + O\left(x^3\right)\right) - \left(3x - \frac{9}{2}x^2 + O\left(x^3\right)\right)} \\ &= \lim_{x\to 0} \frac{x^2}{3x + O\left(x^3\right) - 3x + \frac{9}{2}x^2 + O\left(x^3\right)} \\ &= \lim_{x\to 0} \frac{x^2}{\frac{9}{2}x^2 + O\left(x^3\right)} \\ &= \lim_{x\to 0} \frac{1}{\frac{9}{2} + O(x)} \\ &= \frac{2}{9}. \end{aligned}$$

2. $\lim_{x\to 0} \dfrac{e^{-\frac{x}{2}} - \cos\sqrt{x}}{x^2}$;
Ricordiamo gli sviluppi delle funzioni

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + O\left(x^3\right) \\ \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + O\left(x^6\right), \end{aligned}$$

e applichiamoli rispettivamente $-\dfrac{x}{2}$ e $\sqrt{x}$:

$$\begin{aligned} e^{-\frac{x}{2}} &= 1 - \frac{x}{2} + \frac{\left(-\frac{x}{2}\right)}{2} + O\left(\left(-\frac{x}{2}\right)^3\right) \\ &= 1 - \frac{x}{2} + \frac{x^2}{8} + O\left(x^3\right) \end{aligned}$$

$$
\begin{aligned}
\cos\sqrt{x} &= 1-\frac{\left(\sqrt{x}\right)^2}{2}+\frac{\left(\sqrt{x}\right)^4}{24}+O\left(\left(\sqrt{x}\right)^6\right)\\
&= 1-\frac{x}{2}+\frac{x^2}{24}+O\left(x^3\right).
\end{aligned}
$$

Dunque si ottiene

$$
\begin{aligned}
\lim_{x\to 0}\frac{e^{-\frac{x}{2}}-\cos\sqrt{x}}{x^2} &= \lim_{x\to 0}\frac{1-\frac{x}{2}+\frac{x^2}{8}+O\left(x^3\right)-\left(1-\frac{x}{2}+\frac{x^2}{24}+O\left(x^3\right)\right)}{x^2}\\
&= \lim_{x\to 0}\frac{\frac{x^2}{8}+O\left(x^3\right)-\frac{x^2}{24}+O\left(x^3\right)}{x^2}\\
&= \lim_{x\to 0}\frac{\frac{x^2}{12}+O\left(x^3\right)}{x^2}\\
&= \lim_{x\to 0}\frac{1}{12}+O(x)\\
&= \frac{1}{12}.
\end{aligned}
$$

3. $\lim_{x\to 0}\dfrac{x\cos x-\sin x}{e^x+\ln(1-x)-1}$;
Ricordiamo gli sviluppi delle funzioni

$$
\begin{aligned}
\cos x &= 1-\frac{x^2}{2}+O\left(x^4\right)\\
\sin x &= x-\frac{x^3}{6}+O\left(x^4\right)\\
e^x &= 1+x+\frac{x^2}{2}+\frac{x^3}{6}+O\left(x^4\right)\\
\ln(1-x) &= (-x)-\frac{(-x)^2}{2}+\frac{(-x)^3}{3}+O\left(x^4\right)\\
&= -x-\frac{x^2}{2}-\frac{x^3}{3}+O\left(x^4\right),
\end{aligned}
$$

da cui si ottiene

$$
\begin{aligned}
\lim_{x\to 0}\frac{x\cos x-\sin x}{e^x+\ln(1-x)-1} &= \lim_{x\to 0}\frac{x\left(1-\frac{x^2}{2}+O\left(x^4\right)\right)-\left(x-\frac{x^3}{6}+O\left(x^4\right)\right)}{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+O\left(x^4\right)\right)+\left(-x-\frac{x^2}{2}-\frac{x^3}{3}+O\left(x^4\right)\right)-1}\\
&= \lim_{x\to 0}\frac{x-\frac{x^3}{2}+O\left(x^5\right)-x+\frac{x^3}{6}+O\left(x^4\right)}{\frac{x^3}{6}+O\left(x^4\right)-\frac{x^3}{3}+O\left(x^4\right)}\\
&= \lim_{x\to 0}\frac{-\frac{x^3}{3}+O\left(x^4\right)}{-\frac{x^3}{6}+O\left(x^4\right)}\\
&= \lim_{x\to 0}\frac{-\frac{1}{3}+O(x)}{-\frac{1}{6}+O(x)}\\
&= 2.
\end{aligned}
$$

4. $\lim_{x\to 0}\dfrac{1-\cos(2x)}{\ln\left(1+x+x^2\right)-\sin x}$;
Utilizzando gli sviluppi

$$
\begin{aligned}
\cos(2x) &= 1-2x^2+O\left(x^3\right)\\
\ln\left(1+x+x^2\right) &= x+x^2-\frac{\left(x+x^2\right)^2}{2}+O\left(\left(x+x^2\right)^3\right)
\end{aligned}
$$

$$
\begin{aligned}
&= x + x^2 - \frac{x^2 + O\left(x^3\right)}{2} + O\left(x^3\right)\\
&= x + \frac{x^2}{2} + O\left(x^3\right)\\
\sin x &= x + O\left(x^3\right),
\end{aligned}
$$

si ottiene

$$
\lim_{x\to 0} \frac{1-\cos(2x)}{\ln\left(1+x+x^2\right)-\sin x} = \lim_{x\to 0} \frac{1-\left(1-2x^2+O\left(x^3\right)\right)}{\left(x+\frac{x^2}{2}+O\left(x^3\right)\right)-\left(x+O\left(x^3\right)\right)} = \lim_{x\to 0} \frac{2x^2+O\left(x^3\right)}{\frac{x^2}{2}+O\left(x^3\right)} = \lim_{x\to 0} \frac{2+O(x)}{\frac{1}{2}+O(x)} = 4.
$$

5. $\lim_{x\to 0} \frac{x\ln(\cos x)}{e^{\arctan x}-e^{\sin x}}$;
Utilizzando gli sviluppi

$$
\begin{aligned}
\ln(1+x) &= x + O\left(x^2\right)\\
\cos x &= 1 - \frac{x^2}{2} + O\left(x^4\right)\\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O\left(x^4\right)\\
\arctan x &= x - \frac{x^3}{3} + O\left(x^4\right)\\
\sin x &= x - \frac{x^3}{6} + O\left(x^4\right)
\end{aligned}
$$

e componendoli tra di loro si ottiene:

$$
\begin{aligned}
\ln(\cos x) &= \cos x - 1 + O\left((\cos x - 1)^2\right)\\
&= -\frac{x^2}{2} + O\left(x^4\right) + O\left(O\left(x^2\right)^2\right)\\
&= -\frac{x^2}{2} + O\left(x^4\right)\\
e^{\arctan x} &= 1 + \arctan x + \frac{\arctan^2 x}{2} + \frac{\arctan^3 x}{6} + O\left(\arctan^4 x\right)\\
&= 1 + \left(x - \frac{x^3}{3} + O\left(x^4\right)\right) + \frac{\left(x+O\left(x^3\right)\right)^2}{2} + \frac{\left(x+O\left(x^3\right)\right)^3}{6} + O\left(O(x)^4\right)\\
&= 1 + x - \frac{x^3}{3} + O\left(x^4\right) + \frac{x^2+O\left(x^4\right)}{2} + \frac{x^3+O\left(x^4\right)}{6} + O\left(x^4\right)\\
&= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + O\left(x^4\right)\\
e^{\sin x} &= 1 + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{6} + O\left(\sin^4 x\right)\\
&= 1 + \left(x - \frac{x^3}{6} + O\left(x^4\right)\right) + \frac{\left(x+O\left(x^3\right)\right)^2}{2} + \frac{\left(x+O\left(x^3\right)\right)^3}{6} + O\left(O(x)^4\right)\\
&= 1 + x - \frac{x^3}{6} + O\left(x^4\right) + \frac{x^2+O\left(x^4\right)}{2} + \frac{x^3+O\left(x^4\right)}{6} + O\left(x^4\right)\\
&= 1 + x + \frac{x^2}{2} + O\left(x^4\right);
\end{aligned}
$$

pertanto,

$$
\lim_{x\to 0} \frac{x\ln(\cos x)}{e^{\arctan x}-e^{\sin x}} = \lim_{x\to 0} \frac{x\left(-\frac{x^2}{2}+O\left(x^4\right)\right)}{\left(1+x+\frac{x^2}{2}-\frac{x^3}{6}+O\left(x^4\right)\right)-\left(1+x+\frac{x^2}{2}+O\left(x^4\right)\right)}
$$

$$
\begin{aligned}
&= \lim_{x\to 0} \frac{-\frac{x^3}{2} + O\left(x^5\right)}{-\frac{x^3}{6} + O\left(x^4\right)} \\
&= \lim_{x\to 0} \frac{-\frac{1}{2} + O(x)}{-\frac{1}{6} + O(x)} \\
&= 3.
\end{aligned}
$$

6. $\lim\limits_{x\to 0} \left(\dfrac{1}{\sin^2 x} - \dfrac{1}{\arctan\left(x^2\right)} \right);$

Utilizzando gli sviluppi

$$
\begin{aligned}
\sin^2 x &= \left(x - \frac{x^3}{6} + O\left(x^5\right) \right)^2 \\
&= x\left(x - \frac{x^3}{6} + O\left(x^5\right) \right) - \frac{x^3}{6}\left(x + O\left(x^3\right)\right) + O\left(x^5\right) O(x) \\
&= x^2 - \frac{x^4}{3} + O\left(x^6\right) \\
\arctan\left(x^2\right) &= x^2 + O\left(\left(x^2\right)^3\right) \\
&= x^2 + O\left(x^6\right),
\end{aligned}
$$

si ottiene

$$
\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{\arctan\left(x^2\right)} \right) &= \lim_{x\to 0} \frac{\arctan\left(x^2\right) - \sin^2 x}{\sin^2 x \arctan\left(x^2\right)} \\
&= \lim_{x\to 0} \frac{x^2 + O\left(x^6\right) - \left(x^2 - \frac{x^4}{3} + O\left(x^6\right)\right)}{\left(x^2 - \frac{x^4}{3} + O\left(x^6\right)\right)\left(x^2 + O\left(x^6\right)\right)} \\
&= \lim_{x\to 0} \frac{\frac{x^4}{3} + O\left(x^6\right)}{x^2\left(x^2 + O\left(x^6\right)\right) + O\left(x^4\right) O\left(x^2\right)} \\
&= \lim_{x\to 0} \frac{\frac{x^4}{3} + O\left(x^6\right)}{x^4 + O\left(x^6\right)} \\
&= \lim_{x\to 0} \frac{\frac{1}{3} + O\left(x^2\right)}{1 + O\left(x^2\right)} \\
&= \frac{1}{3}.
\end{aligned}
$$