EXTENSION OF LINEAR CONTINUOUS FUNCTIONALS

WE SAW THE (EASY) CASE OF $E \subset X$ DENSE

$\boxed{\text{DEF}}$ A FUNCTIONAL $p: X \to \mathbb{R}$ DEFINED ON A NORMED SPACE IS

HOMOGENEOUS IF $p(\lambda x) = \lambda p(x) \quad \forall x \in X, \forall \lambda \geq 0$

SUBADDITIVE IF $p(x+y) \leq p(x) + p(y) \quad \forall x, y \in X$

$\boxed{\text{EXAMPLE}}$ ANY (SEMI) NORM IS HOMOGENEOUS AND SUBADDITIVE

HAHN - BANACH THEOREM

LET X BE A NORMED SPACE, $E \subset X$ A LINEAR SUBSPACE, $p: X \to \mathbb{R}$ HOMOGENEOUS AND SUBADDITIVE AND $L: E \to \mathbb{R}$ LINEAR SUCH THAT $Lx \leq p(x)$ $\forall x \in E$

THEN, $\exists \tilde{L}: X \to \mathbb{R}$ SUCH THAT $\tilde{L}x = Lx \quad \forall x \in E$

AND $\tilde{L}x \leq p(x) \quad \forall x \in X$.

$\boxed{\text{REMARK}}$ 1) $\tilde{L}$ MAY NOT BE UNIQUE $X = \mathbb{R}^2 \quad E = \{x_2 = 0\}$

$p(x) = \|x\|_1 = |x_1| + |x_2| \qquad L: E \to \mathbb{R}$, $x \to x_1$. ANY EXTENSION $\tilde{L}: X \to \mathbb{R}$ WILL

BE OF THE KIND $\tilde{L}x = x_1 + c x_2$ FOR SOME $c \in \mathbb{R}$. WE HAVE

$\tilde{L} \leq p$ ON X IF $-1 \leq c \leq 1$

2) ON HILBERT SPACES, WE EXTEND $L: E \to \mathbb{R}$ TO $\tilde{L}: \overline{E} \to \mathbb{R}$ BY DENSITY

THEN, IF $P: X \to \overline{E}$ IS THE PROJECTION, WE SET $\tilde{\tilde{L}}x = \tilde{L}(Px)$

IN PARTICULAR, $\tilde{\tilde{L}}$ IS UNIQUE (AND WE DO NOT NEED HAHN-BANACH)

PROOF | WE USE ZORN LEMMA'S. (IF $(P, \preceq)$ IS NON-EMPTY, PARTIALLY ORDERED AND ANY $Q \subset P$ TOTALLY ORDERED HAS AN UPPER BOUND, THEN P HAS A MAXIMAL ELEMENT)

$P = \{(F, M) : F \subset X$ LINEAR SUBSPACE, $E \subset F$; $M: F \to \mathbb{R}$ LINEAR SUCH THAT $M|_E = L$ AND $M \leq p$ ON $F\}$

$(F, M) \preceq (F', M') \overset{\text{DEF}}{\iff} F \subset F', \; M'|_F = M$ IS A PARTIAL ORDER RELATION

$P \neq \emptyset$ BECAUSE $(E, L) \in P$

LET US TAKE $Q \subset P$ TOTALLY ORDERED. $F_0 := \bigcup F$. M_0 DEFINED

$P \neq \emptyset$ BECAUSE $(E, L) \in P$

LET US TAKE $Q \subset P$ TOTALLY ORDERED. $F_0 := \bigcup_{F \in Q} F$, M_0 DEFINED AS $M_0|_{F_\alpha} = M_\alpha$ FOR ANY $(F_\alpha, M_\alpha) \in Q$.

M_α IS WELL-DEFINED BECAUSE Q IS TOTALLY ORDERED

$(F_0, M_0) \in P$ BECAUSE M_0 EXTENDS L AND $M_0 \leq P$ ON F_0 BECAUSE ON ANY F_α WE HAVE $M_0|_{F_\alpha} = M_\alpha \leq P$. BY CONSTRUCTION, $(F_\alpha, M_\alpha) \lesssim (F_0, M_0) \Rightarrow$ (F_0, M_0) IS AN UPPER BOUND.

BY ZORN'S LEMMA, $\exists\ (\tilde{E}, \tilde{L})$ MAXIMAL ELEMENT. WE NEED TO SHOW $\tilde{E} = X$. ASSUME $\underbrace{\text{THAT } \exists x_0 \in X \setminus \tilde{E}}_{\text{BY CONTRADICTION}}$, WE WILL SHOW $\tilde{L}$ CAN BE EXTENDED TO SPAN $\{\tilde{E}, x_0\}$, CONTRADICTING MAXIMALITY.

WE DEFINE $\hat{L}(\underline{x + t x_0}) = \underline{\tilde{L}x} + \underline{c}\,t \quad \forall x \in \tilde{E},\ t \in \mathbb{R}$ FOR SOME $c \in \mathbb{R}$

IF WE CAN TAKE c SUCH THAT $\hat{L}(x + t x_0) \leq P(x + t x_0)$ WE CAN EXTEND $\tilde{L}$ AND WE GET A CONTRADICTION.

$\hat{L}(x + t x_0) \leq P(x + t x_0)$, DIVIDE BY t AND USE HOMOGENEITY

$$\tilde{L}\left(\frac{x}{t}\right) + c = \hat{L}\left(\frac{x}{t} + x_0\right) \leq P\left(\frac{x}{t} + x_0\right) \quad \text{IF } t > 0$$

$$\tilde{L}\left(\frac{x}{t}\right) + c \geq -P\left(-\frac{x}{t} - x_0\right) \quad \text{IF } t < 0$$

$\Rightarrow c$ MUST SATISFY

$$(\mathrm{I})\ c \leq P\left(\frac{x}{t} + x_0\right) - \tilde{L}\left(\frac{x}{t}\right) \quad \forall x \in \tilde{E}$$

$$(\mathrm{II})\ c \geq -P\left(-\frac{x}{t} - x_0\right) - \tilde{L}\left(\frac{x}{t}\right) \quad \forall t > 0$$

SINCE $\tilde{E}$ IS A LINEAR SUBSPACE, I CALL $\frac{x}{t} = y$ IN (I)

$\Rightarrow c$ MUST SATISFY $\qquad \frac{x}{t} = z$ IN (II)

$$a = \underline{P(-z - x_0) - \tilde{L}z} \leq c \leq \underline{P(y + x_0) - \tilde{L}y} = b \quad \forall y, z \in \tilde{E}$$

I CAN TAKE c BETWEEN a AND b IF AND ONLY IF $a \leq b$

$$a - b = P(-z - x_0) - \tilde{L}z - P(y + x_0) + \tilde{L}y$$

$$= \tilde{L}(y - z) - \left(P(y + x_0) - P(-z - x_0)\right)$$

$$= \tilde{L}(y-z) - \left(P(y+x_0) - P(-z-x_0)\right)$$

$$\leq \tilde{P}(y-z) - \left(P(y+x_0) - P(-z-x_0)\right)$$

≤ 0 BECAUSE P IS SUBADDITIVE

$\Rightarrow$ WE CAN TAKE c SUCH THAT $a \leq c \leq b$ AND WE GET A CONTRADICTION.

COROLLARY LET $E \subset X$ BE A LINEAR SUBSPACE OF A NORMED SPACE X AND $L \in E^*$. THEN $\exists\, \tilde{L} \in X^*$ WHICH EXTENDS L $\left(\tilde{L}|_E = L\right)$ AND $\|\tilde{L}\|_{X^*} = \|L\|_{E^*}$

PROOF APPLY HAHN-BANACH THEOREM WITH $P(x) = \|L\|_{E^*}\|x\|$

FROM HAHN-BANACH WE HAVE $\tilde{L}x \leq P(x) = \|L\| \cdot \|x\| \rightarrow \|\tilde{L}\| \leq \|L\|$

BUT SINCE $\tilde{L}|_E = L$ WE HAVE $\|\tilde{L}\| = \|L\|$

LET US START SEEING APPLICATIONS OF HAHN-BANACH THEOREM, IN PARTICULAR TO DUAL SPACES

$\boxed{\text{LEMMA}}$ ① FOR ANY ELEMENT $x_0 \in X$, $x_0 \neq 0$ OF A NORMED SPACE $\exists\, L_{x_0} \in X^*$ SUCH THAT $\|L_{x_0}\|_{X^*} = 1$ AND $L_{x_0} x_0 = \|x_0\|$ (AS LARGE AS POSSIBLE)

② MORE GENERALLY, GIVEN $E \subset X$ LINEAR SUBSPACE AND $x_0 \notin E$ $\exists\, L_{E,x_0} \in X^*$ SUCH THAT $\|L_{E,x_0}\|_{X^*} = 1$ AND $L_{E,x_0}|_E \equiv 0$

$L_{E,x_0} x_0 = d(x_0, E)$

PROOF

① APPLY HAHN-BANACH WITH $E = \operatorname{span}\{x_0\}$ $\quad L : E \to \mathbb{R}$, $\quad tx_0 \to t\|x_0\|$ $\qquad P(x) = \|x\|$

② APPLY HAHN-BANACH ON $\operatorname{span}\{E, x_0\}$ $\quad L(x + tx_0) = t\, d(x_0, E)$ $\quad x \in E,\ t \in \mathbb{R}$

$$\|L\|_{\operatorname{span}\{E,x_0\}^*} = \sup_{\substack{x \in E \\ t \in \mathbb{R}}} \frac{|t|\, d(x_0,E)}{\|x + tx_0\|} = \sup_{x,t} \frac{d(x_0,E)}{\left\|\frac{x}{t} + x_0\right\|} = \sup_{y \in E} \frac{d(x_0,E)}{\|x_0 - y\|} = 1$$

$\boxed{\text{EXAMPLES}}$ ① $X = L^p(\mu)$ $\quad 1 \leq p < +\infty \Rightarrow X^*$ IS ISOMETRIC TO $L^{p'}(\mu)$

EXAMPLES ① $X = L^p(\mu)$ $1 < p < +\infty \Rightarrow X^*$ IS ISOMETRIC TO $L^{p'}(\mu)$
$\frac{1}{p} + \frac{1}{p'} = 1$, ANY $L \in X^*$ IS OF THE KIND $L : f \to \int_X fg \, d\mu$
FOR SOME $g \in L^{p'}(\mu)$. GIVEN $f_0 \in L^p(\mu)$, L_{f_0} IS GIVEN BY
$$g = \frac{|f_0|^{p-2} f_0}{\|f_0\|^{p-1}}$$

② X HILBERT SPACE $\Rightarrow X^*$ IS ISOMETRIC TO X, ANY $L \in X^*$ IS
$L : x \to (x, h)$. GIVEN x_0, L_{x_0} IS GIVEN BY $h = \frac{x_0}{\|x_0\|}$

COROLLARY ① IF $X \neq \{0\}$ IS A NONTRIVIAL NORMED SPACE, THEN $X^* \neq \{0\}$
② MOREOVER, IF $\dim X = +\infty$ THEN $\dim X^* = +\infty$

PROOF ① IF $\exists x_0 \in X \setminus \{0\}$, FROM THE LEMMA WE FIND $L_{x_0} \in X^*$ WITH $\|L_{x_0}\| = 1$
$\Rightarrow L_{x_0} \neq 0 \Rightarrow X^* \neq \{0\}$
② IF $\dim X = +\infty$, $\exists \{E_n\}_{n \in \mathbb{N}}$ LINEAR SUBSPACES $E_n \subsetneq E_{n+1}$. TAKE
$x_n \in E_{n+1} \setminus E_n$ AND, BY THE LEMMA, $\exists L_n = L_{E_n, x_n}$ SUCH THAT
$L_n|_{E_n} \equiv 0$ $L_n x_n = d(x_n, E_n) \neq 0$. $L_n \notin \text{SPAN}\{L_1, \dots, L_{n-1}\}$
BECAUSE $L_n|_{E_n} \equiv 0$ WHILE NO $L \in \text{SPAN}\{L_1 \dots L_{n-1}\}$ VANISHES
ON E_n. THEREFORE, $\{L_n\}_{n \in \mathbb{N}}$ ARE LINEARLY INDEPENDENT
$\Downarrow$
$\dim X^* = +\infty$

REMARK THE COROLLARY IS FALSE IF X IS JUST A METRIC VECTOR SPACE. FOR INSTANCE $X = L^p((0,1))$ $0 < p < 1$, $d(f,g) = \int_0^1 |f-g|^p$
$d(f,g)$ IS A DISTANCE AND X IS COMPLETE, BUT $X^* = \{0\}$.
ASSUME, BY CONTRADICTION, $\exists L \in X^* \setminus \{0\} \Rightarrow \exists f \in X$ SUCH THAT $Lf \geq 1$
CONSIDER $t \to \int_0^t |f|^p = d(f\chi_{(0,t)}, 0)$ IS CONTINUOUS (BECAUSE OF DOMINATED CONVERGENCE THEOREM) $\Rightarrow \exists t_0 \in (0,1)$ SUCH THAT
$\int_0^{t_0} |f|^p = \frac{1}{2} \int_0^1 |f|^p \Rightarrow d(g,0) = d(h,0) = \frac{d(f,0)}{2}$ WHERE $g = f\chi_{(0,t_0)}$
$f = g + h$

$\int_0 |f| \quad \frac{1}{2}\int_0 |f| \Rightarrow d(g,0) = d(h,0) = \frac{d(f,0)}{2}$ WHERE $g = f\chi_{(0,t_0)}$, $h = f\chi_{(t_0,1)}$

MOREOVER, $1 \leq Lf \overset{f=g+h}{=} \frac{L(2g)}{2} + \frac{L(2h)}{2} \Rightarrow$ AT LEAST ONE BETWEEN $L(2g)$, $L(2h)$ IS ≥ 1. I SET $f_1 = 2g$ OR $f_1 = 2h$ SO THAT $Lf_1 \geq 1$ AND $d(f_1, 0) = \frac{d(f,0)}{2^{1-p}}$. NOW I ITERATE: I WRITE $f_1 = g_1 + h_1$ AND SET $f_2 := g_1$ OR $f_2 := h_1$ SO THAT $Lf_2 \geq 1$, $d(f_2,0) = \frac{d(f_1,0)}{2^{1-p}}$. I GET A SEQUENCE $\{f_n\}$ SATISFYING $Lf_n \geq 1$, $d(f_n,0) = \frac{d(f_{n-1},0)}{2^{1-p}} = \dots = \frac{d(f,0)}{2^{n(1-p)}} \xrightarrow[n\to\infty]{} 0$

BUT $Lf_n \not\to L0 = 0$. CONTRADICTION WITH CONTINUITY OF L.

$\boxed{\text{PROP}}$ FOR ANY NORMED SPACE X $\exists$ $\underline{X} \xrightarrow{J} X^{**}$ ISOMETRY

$x \longrightarrow \Lambda : L \to \underline{Lx}$

PROOF J IS CLEARLY LINEAR. MOREOVER, $\|J(x)\| = \sup_{\|L\|_{X^*}\leq 1} |Lx| \leq \sup_{\|L\|\leq 1} \|L\| \cdot \|x\| = \|x\|$. TO GET THE OTHER INEQUALITY ($\Rightarrow J$ IS ISOMETRY) FOR ANY $x_0 \in X$ TAKE L_{x_0} SUCH THAT $\|L_{x_0}\| = 1$, $L_{x_0} x_0 = \|x_0\|$
$\Rightarrow \|J(x_0)\| = \sup_{\|L\|\leq 1} |Lx_0| \geq |L_{x_0} x_0| = \|x_0\|$.

$\boxed{\text{DEF}}$ IF J IS A SURJECTIVE ISOMETRY, WE SAY X IS REFLEXIVE

$\boxed{\text{REMARK}}$ IN GENERAL, $\|L\|_{X^*} = \sup_{\|x\|\leq 1} |Lx|$ IS NOT A MAXIMUM.

$\|\Lambda\|_{X^{**}} = \sup_{\|L\|_{X^*}\leq 1} |\Lambda L|$ " " "

BUT $\|x\| = \sup_{\|L\|\leq 1} |Lx|$ IS A MAXIMUM, ATTAINED BY L_{x_0} (AS IN THE LEMMA)
IF $\Lambda = J(x)$ FOR SOME $x \in X$, ALSO $\|\Lambda\|_{X^{**}}$ IS A MAXIMUM.
IF $\|\Lambda\|_{X^{**}}$ IS NOT ATTAINED BY SOME $\Lambda \in X^{**}$ THEN X IS NOT

IF $\Lambda = J(x)$ FOR SOME $x \in X$, ALSO $\|\Lambda\|_{X^{**}}$ IS A MAXIMUM.
IF $\|\Lambda\|_{X^{**}}$ IS NOT ATTAINED BY SOME $\Lambda \in X^{**}$, THEN X IS NOT REFLEXIVE.

$\boxed{\text{EXAMPLES}}$ ① $L^p(\mu)$ IS REFLEXIVE FOR $1<p<+\infty$
ALSO, HILBERT SPACES ARE REFLEXIVE

② $X = L^1((0,1))$ IS NOT REFLEXIVE. IN FACT, X^* IS ISOMETRIC TO $L^\infty((0,1))$ BUT $L^\infty((0,1))^*$ IS NOT ISOMETRIC TO $L^1((0,1))$
LET US APPLY HAHN-BANACH THEOREM ON $C([0,1]) =: E$

$$L : E \to \mathbb{R}, \quad f \to f(0)$$

IS CONTINUOUS (WITH RESPECT TO $\|\cdot\|_\infty$)

SO L CAN BE EXTENDED TO $\tilde{L} \in L^\infty((0,1))^*$ SUCH THAT $\tilde{L} f = L f = f(0)$ $\forall f \in C([0,1])$. IF $\tilde{L} f = \int_0^1 fg$ FOR SOME $g \in L^1((0,1))$
WE WOULD HAVE $f(0) = \int_0^1 fg$ $\forall f \in C([0,1])$, IMPOSSIBLE.

③ $X = \ell_1$ IS NOT REFLEXIVE, BECAUSE $(\ell_\infty)^*$ IS NON ISOMETRIC TO ℓ_1. LET US APPLY HAHN-BANACH ON

$$c = \{x : \mathbb{N} \to \mathbb{R} \text{ SUCH THAT } \exists \lim_{k\to\infty} x(k) \in \mathbb{R}\}$$

$$L : c \to \mathbb{R}, \quad x \to \lim_{k\to\infty} x(k)$$

CAN BE EXTENDED TO $\tilde{L} \in (\ell_\infty)^*$

SUPPOSE, BY CONTRADICTION, $\tilde{L} x = \sum_{k=1}^\infty x(k) \underline{y(k)}$ $\forall x \in \ell_\infty$ FOR SOME $y \in \ell_1$

TAKE $x = e_n = (0 \ldots, 0, \underset{n}{1}, 0 \ldots 0)$ $\Rightarrow \tilde{L} e_n = y(n)$, $\tilde{L} e_n = L e_n = 0$

$\Rightarrow y(n) = 0$ $\forall n \in \mathbb{N} \Rightarrow \tilde{L} \equiv 0$

BUT IF $x = (1, \ldots, 1, \ldots) \Rightarrow \tilde{L} x = L x = 1$. WE FOUND A CONTRADICTION

④ $X = C([0,1])$ IS NOT REFLEXIVE. TAKE $L_0 : X \to \mathbb{R}$

$$L_0 : f \to \int f(\chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}) \qquad f \to \int_0^{\frac{1}{2}} f - \int_{\frac{1}{2}}^1 f$$

FROM THE LEMMA, $\exists \Lambda_{L_0} \in X^{**}$ SUCH THAT
$\|\Lambda_{L_0}\| = 1$ AND $\Lambda. L_0 = \|L_0\| = 1$

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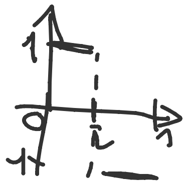
$\|\Lambda_{L_0}\| = 1$ AND $\Lambda_{L_0} L_0 = \|L_0\| = 1$

IF $\Lambda_{L_0} = J(f)$, THEN $\|f\|_\infty = 1$ AND $\int_0^{\frac{1}{2}} f - \int_{\frac{1}{2}}^{1} f = 1$



IMPOSSIBLE BECAUSE $f = \chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}$ BUT IS NON CONTINUOUS

$\boxed{\text{REMARK}}$ ℓ_1 IS NOT ISOMETRIC TO $(\ell_\infty)_*$ BUT IT IS ISOMETRIC TO $(c_0)^*$. $c_0 := \{x : \mathbb{N} \to \mathbb{R} : x(k) \xrightarrow[k\to\infty]{} 0\}$

$(c_0)_* \xrightarrow{\phi} \ell_1$ ϕ IS AN ISOMETRY

$L \longrightarrow (\underline{Le_1}, \underline{Le_2}, \dots, \underline{Le_k}, \dots)$

$\phi(L) \in \ell_1$: TAKE $x_n = (\text{SIGN}(Le_1), \dots, \text{SIGN}(Le_n), 0, \dots, 0) \in c_0$

$\|x_n\| = 1$

$\sum_{i=1}^{n} |Le_i| = Lx_n \leq \|L\| \cdot \|x_n\| \leq \|L\|$ $\forall n \in \mathbb{N} \Rightarrow \sum_{i=1}^{\infty} |Le_i| \leq \|L\|$

$\|\phi(L)\|_{\ell_1}$

$\phi(L)$ IS WELL-DEFINED, IS LINEAR (EASY)

$\phi(L)$ IS SURJECTIVE, BECAUSE $y \in \ell_1$, $y \in \phi(L)$

$L : x \to \sum_{k=1}^{\infty} x(k) y(k)$

TO GET $\|\phi(L)\| \geq \|L\|$, WE SEE THAT $\forall x \in \underline{c_0}$ WE HAVE

$$x = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} x(k) e_k}_{= (x(1), \dots, x(n), 0, \dots, 0)} \quad (\text{IN } \|\cdot\|_\infty)$$

$$Lx = \sum_{k=1}^{\infty} x(k) Le_k \leq \|x\|_\infty \sum_{k=1}^{\infty} |Le_k| = \|x\|_\infty \|\phi(L)\|_1$$

$\Rightarrow$ PASSING TO $\sup_{\|x\|_\infty \leq 1}$, $\|L\| \leq \|\phi(L)\|$.