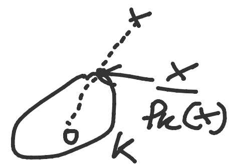
GEOMETRIC FORMS OF HAHN-BANACH THEOREM

$\boxed{\text{DEF}}$ LET X BE A NORMED SPACE AN $K \subset X$ <u>CONVEX</u> SUCH THAT $B_\delta(0) \subset K$. WE DEFINE THE <u>MINKOWSKI FUNCTIONAL</u> ASSOCIATED TO K AS

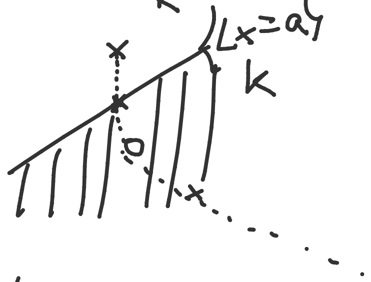
$$P_K(x) := \inf \left\{ r > 0 : \underline{\frac{x}{r} \in K} \right\}$$



$\boxed{\text{EXAMPLES}}$ ① $K = \overline{B_R(0)}$ CLOSED BALL $\quad P_K(x) = \frac{\|x\|}{R}$

② $K = \{ x \in X ;\ Lx \le a \}$ $L \in X^*$ $a > 0$

$$P_K(x) = \begin{cases} \frac{Lx}{a} & \text{IF } Lx > 0 \\ 0 & \text{IF } Lx \le 0 \end{cases}$$



$\boxed{\text{LEMMA}}$ LET X BE A NORMED SPACE AND $K \subset X$ CONVEX SUCH THAT $B_\delta(0) \subset K$ FOR SOME $\delta > 0$. THEN:

① P_K IS HOMOGENEOUS AND SUBADDITIVE

② $\underline{P_K(x) \le C\|x\|}$ FOR SOME $C > 0$ AND IN PARTICULAR P_K IS CONTINUOUS

③ IF $K \subset K'$ THEN $P_{K'}(x) \le P_K(x)$ $\forall x \in X$

④ $\underline{P_{\mathring{K}} = P_K = P_{\overline{K}}}$ AND $P_K(x) \le 1 \iff x \in \overline{K}$

$P_K(x) < 1 \iff x \in \mathring{K}$

⑤ IF K IS <u>BALANCED</u> ($x \in K \iff -x \in K$) THEN P_K IS A <u>SEMINORM</u>

⑥ IF K IS BALANCED AND <u>BOUNDED</u> THEN P_K IS A <u>NORM</u> EQUIVALENT TO $\|\cdot\|$

<u>DIM</u> ① $P_K(x) \ne +\infty$ $\forall x \in X$: $x = 0 \Rightarrow P_K(0) = 0$ BY DEF

$x \ne 0 \Rightarrow \frac{\delta}{2\|x\|} x \in B_\delta(0) \subset K \Rightarrow P_K(x) \le \frac{2\|x\|}{\delta} < +\infty$

HOMOGENEITY:

$P_K(\lambda x) := \inf \left\{ r > 0 : \frac{x}{\frac{r}{\lambda}} \in K \right\} \underset{s = \frac{r}{\lambda}}{=} \inf \left\{ \lambda s > 0 : \frac{x}{s} \in K \right\} = \lambda P_K(x)$
($\lambda > 0$)

SUBADDITIVITY:

GIVEN $x, y \in X$, $\forall \varepsilon > 0$ $\exists r_\varepsilon, s_\varepsilon$ SUCH THAT $\frac{x}{r}, \frac{y}{s} \in K$, $\underline{P_K(x) \ge r - \varepsilon}$

$x+y$ ... r x ... s y

GIVEN $x, y \in X$, $\forall \varepsilon > 0$ $\exists\, r_\varepsilon, s_\varepsilon$ SUCH THAT $\frac{x}{r}, \frac{y}{s} \in K$, $P_K(x) \geq r - \varepsilon$, $P_K(y) \geq s - \varepsilon$

$$\frac{x+y}{r+s} = \frac{r}{r+s}\underbrace{\frac{x}{r}}_{\in K} + \frac{s}{r+s}\underbrace{\frac{y}{s}}_{\in K} \in K \text{ BY CONVEXITY} \Rightarrow P_K(x+y) \leq r+s$$

$$\Rightarrow P_K(x+y) \leq r+s \leq P_K(x) + P_K(y) + 2\varepsilon \quad \forall \varepsilon > 0 \Rightarrow P_K(x+y) \leq P_K(x) + P_K(y)$$

(2) WE SAW $P_K \leq C\|\cdot\|$ WITH $C = \frac{2}{\delta}$, MOREOVER, P_K IS SUBADDITIVE $\Rightarrow P_K(x) - P_K(y) \leq P_K(x-y) \leq C\|x-y\|$

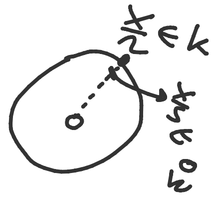
SIMILARLY, $P_K(y) - P_K(x) \leq C\|x-y\| \Rightarrow |P_K(x) - P_K(y)| \leq C\|x-y\|$

SO P_K IS LIPSCHITZ CONTINUOUS.

(3) $\frac{x}{r} \in K$ $\forall r > P_K(x)$. IF $K \subset K'$, THEN $\frac{x}{r} \in K'$ $\forall r > P_K(x)$

TAKE $\inf_{r>0}$ $P_{K'}(x) \leq r$ $\forall r > P_K(x) \Rightarrow P_{K'}(x) \leq P_K(x)$.

(4) WE OBSERVE THAT $\frac{x}{r} \in \overline{K} \Leftrightarrow \frac{x}{s} \in \mathring{K}$ $\forall s > r$



$$P_{\overline{K}}(x) = \inf\{r > 0 : \tfrac{x}{r} \in \overline{K}\} = \inf\{s > 0 : \tfrac{x}{s} \in \mathring{K}\} = P_{\mathring{K}}(x)$$

$$\mathring{K} \subset K \subset \overline{K} \overset{(3)}{\Longrightarrow} P_{\overline{K}} \leq P_K \leq P_{\mathring{K}} = P_{\overline{K}} \Rightarrow P_K = P_{\mathring{K}} = P_{\overline{K}}$$

FINALLY, $x \in \overline{K} \Leftrightarrow (1-\varepsilon)x \in K$ $\forall \varepsilon > 0 \Leftrightarrow P_K(x) \leq 1$

$x \in \mathring{K} \Leftrightarrow (1+\varepsilon)x \in K$ $\forall \varepsilon > 0$ SMALL $\Leftrightarrow P_K(x) \leq \frac{1}{1+\varepsilon} < 1$

(5) ASSUME K BALANCED

$$P_K(x) = \inf\{r > 0 : \tfrac{x}{r} \in K\} = \inf\{r > 0 : -\tfrac{x}{r} \in K\} \overset{*}{=} P_K(-x)$$

$P_K \geq 0$ BY DEFINITION, P_K IS SUBADDITIVE, MOREOVER

$$P_K(\lambda x) \overset{*}{=} P_K(|\lambda| x) \underset{\text{HOMOGENEITY}}{=} |\lambda| P(x)$$

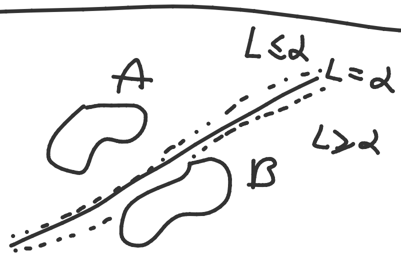
(6) IF K IS BOUNDED, $\frac{x}{r} \notin K$ IF $r < \frac{\|x\|}{\operatorname{diam} K}$

$$\Rightarrow P_K(x) \geq \frac{\|x\|}{\operatorname{diam} K} > 0 \text{ IF } x \neq 0$$

IF K IS ALSO BALANCED, p_K IS SEMINORM $\Rightarrow$ p_K IS A NORM

$p_K(x) \le C\|x\|$ $\Rightarrow$ p_K AND $\|\cdot\|$ ARE EQUIVALENT.

SEPARATION OF CONVEX SET



$\boxed{\text{DEF}}$ A SUBSET $H \subset X$ OF A NORMED SPACE IS CALLED <u>CLOSED HYPERPLANE</u> IF $\exists\, \alpha \in \mathbb{R}$, $L \in X^* \setminus \{0\}$ SUCH THAT

$$H = \{L=\alpha\} = \{x \in X : Lx = \alpha\}$$

TWO SUBSETS $A, B \subset X$ ARE <u>SEPARATED</u> BY $H = \{L=\alpha\}$ IF $Lx \le \alpha \ \forall x \in A$,
$Lx \ge \alpha \ \forall x \in B$, THAT IS $\sup_A L \le \inf_B L$

$A, B \subset X$ ARE <u>STRICTLY</u> SEPARATED BY $H = \{L=\alpha\}$ IF $\exists\, \varepsilon > 0$ SUCH THAT $Lx \le \alpha - \varepsilon \ \forall x \in A$,
$Lx \ge \alpha + \varepsilon \ \forall x \in B$, THAT IS $\sup_A L < \inf_B L$

$\boxed{\text{PROP}}$ LET $L : X \to \mathbb{R}$ BE A LINEAR FUNCTIONAL (X NORMED SPACE). THEN, L IS CONTINUOUS IF AND ONLY IF $\ker L = \{L=0\}$ IS CLOSED. MOREOVER, IF L IS <u>NOT</u> CONTINUOUS, $\ker L$ IS DENSE.

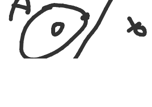
<u>PROOF</u> IF L IS CONTINUOUS, $\ker L = L^{-1}(\{0\})$ IS CLOSED BECAUSE PRE-IMAGE OF A CLOSED SET ($\hookrightarrow$ CLOSED IN $\mathbb{R}$) WITH RESPECT TO A CONTINUOUS MAP.

ASSUME NOW L IS NOT CONTINUOUS, LET US SHOW $\ker L$ IS DENSE. TAKE <u>$y \in X$</u> AND BUILD $y_n \in \ker L$ SUCH THAT $y_n \to y$. SINCE L IS NOT CONTINUOUS, $\exists \{x_n\}$ SUCH THAT $\|x_n\| = 1$, $Lx_n \to +\infty$

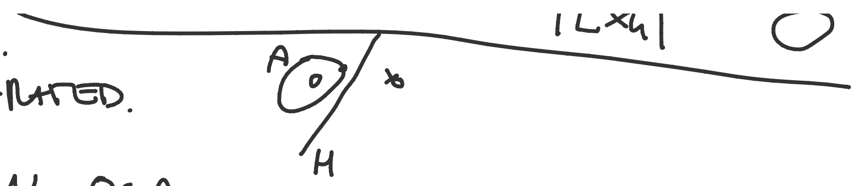
$$y_n := y - \frac{Ly}{Lx_n} x_n \Rightarrow Ly_n = Ly - \frac{Ly}{\cancel{Lx_n}} \cancel{Lx_n} = 0 \Rightarrow y_n \in \ker L$$

$$\|y_n - y\| = +\frac{|Ly|}{|Lx_n|}\|x_n\| \le \frac{C}{|Lx_n|} \longrightarrow 0$$

$\boxed{\text{LEMMA}}$ LET $A \subset X$ BE AN <u>OPEN</u>, <u>CONVEX</u> SUBSPACE OF A NORMED SPACE AND $x_0 \in X \setminus A$. THEN, $\{x_0\}$ AND A ARE SEPARATED.



NORMED SPACE AND $x_0 \in X \setminus A$.
THEN, $\{x_0\}$ AND A ARE SEPARATED.



PROOF UP TO TRANSLATION, $0 \in A$, SINCE A IS OPEN $B_\delta(0) \subset A$
THEN WE CAN APPLY HAHN-BANACH THEOREM WITH $p = p_A$.

$E = \text{span}\{x_0\}$ $\quad L: E \to \mathbb{R}$, $tx_0 \to t$. LET US VERIFY $L \le p$: (ON E)

WE OBSERVE THAT $p(x_0) \geq 1$ BECAUSE $x_0 \notin A$

$L(tx_0) = t \le t\, p(x_0) = p(tx_0)$ $\quad (t \geq 0)$

$L(tx_0) = t < 0 \le p(tx_0)$ $\quad (t < 0)$

$p \geq 0$ ALWAYS

$\Rightarrow L \le p$ AND CAN BE EXTENDED TO $\tilde{L}: X \to \mathbb{R}$

$\tilde{L} \le p \le C\|\cdot\| \Rightarrow \tilde{L}$ IS CONTINUOUS

$\Rightarrow H = \{\tilde{L} = 1\}$ IS A CLOSED HYPERPLANE.

$\tilde{L}x_0 = Lx_0 = 1$

$x \in A \Rightarrow \tilde{L}x \le p(x) < 1$ BECAUSE OF PROPERTIES OF p_A.

$\boxed{\text{THEOREM}}$ I GEOMETRIC FORM OF HAHN-BANACH THEOREM

LET $A, B \subset X$ BE DISJOINTED CONVEX, A IS OPEN
THEN, A, B ARE SEPARATED.

PROOF APPLY THE PREVIOUS LEMMA TO $K := A - B = \{a - b : a \in A, b \in B\}$, $x_0 = 0$

LET US VERIFY THE HYPOTHESES:

$0 \notin K$ BECAUSE $A \cap B = \emptyset$. K IS OPEN BECAUSE $K = \bigcup_{x \in B} A - x$ UNION OF OPEN SET $\Downarrow$ OPEN

K IS CONVEX BECAUSE A, B ARE CONVEX:

TAKE $k_1, k_2 \in K$, $t \in [0,1]$ $\quad k_1 = a_1 - b_1$, $k_2 = a_2 - b_2$ FOR SOME $a_1, a_2 \in A$, $b_1, b_2 \in B$

$$\Rightarrow (1-t)k_1 + tk_2 = \underbrace{(1-t)a_1 + ta_2}_{\in A} - \underbrace{((1-t)b_1 + tb_2)}_{\in B} \in A - B = K$$

$\Rightarrow$ I CAN APPLY THE LEMMA:

$\exists L \in X^*$ SUCH THAT $Lx \le L0 = 0$, BUT $x = a - b$ FOR $a \in A$, $b \in B$

$\exists L \in X^*$ SUCH THAT $Lx \leq L0 = 0$ $\forall x \in K$, BUT $x = a - b$ FOR $a \in A$, $b \in B$

$\Rightarrow L(a-b) \leq 0$ $\forall a \in A, b \in B$, THAT IS $La \leq Lb \Leftrightarrow A, B$ SEPARATED.