AM110 - Analisi matematica 1

Luca Battaglia

Esercitazione 10 di giovedì 5 dicembre 2024

Argomenti: integrali

Esercizio 1.
Calcolare l'integrale

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin x} \mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin x} \mathrm{d}x &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin x}{1-\cos^2 x} \mathrm{d}x \\
&\overset{(y=\cos x)}{=} \int_0^{\frac{1}{2}} \frac{1}{1-y^2} \mathrm{d}y \\
&= \int_0^{\frac{1}{2}} \left(\frac{1}{2}\frac{1}{y+1} - \frac{1}{2}\frac{1}{y-1}\right) \mathrm{d}y \\
&= \left[\frac{1}{2}\log|y+1| - \frac{1}{2}\log|y-1|\right]_0^{\frac{1}{2}} \\
&= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\log\frac{1}{2} \\
&= \frac{1}{2}\log 3.
\end{aligned}$$

Esercizio 2.
Calcolare l'integrale

$$\int_{\frac{1}{e}}^{1} \frac{1}{x\left(\log^2 x + 2\log x + 4\right)} \mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_{\frac{1}{e}}^{1} \frac{1}{x\left(\log^2 x + 2\log x + 4\right)} \mathrm{d}x &\overset{(y=\log x)}{=} \int_{-1}^{0} \frac{1}{y^2+2y+4} \mathrm{d}y \\
&= \frac{1}{\sqrt{3}} \int_{-1}^{0} \frac{\frac{1}{\sqrt{3}}}{\left(\frac{y+1}{\sqrt{3}}\right)^2 + 1} \mathrm{d}y \\
&= \frac{1}{\sqrt{3}} \left[\arctan\left(\frac{y+1}{\sqrt{3}}\right)\right]_{-1}^{0} \\
&= \frac{\pi}{6\sqrt{3}}.
\end{aligned}$$

Esercizio 3.
Calcolare l'integrale

$$\int_0^8 e^{\sqrt[3]{x}}\mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_0^8 e^{\sqrt[3]{x}}\mathrm{d}x &\overset{(y=\sqrt[3]{x})}{=} 3\int_0^2 y^2e^y\mathrm{d}y \\
&= 3\left(\left[y^2e^y\right]_0^2 - 2\int_0^2 ye^y\mathrm{d}y\right) \\
&= 3\left(4e^2 - 2\left([ye^y]_0^2 - \int_0^2 e^y\mathrm{d}y\right)\right) \\
&= 3\left(4e^2 - 2\left(2e^2 - [e^y]_0^2\right)\right) \\
&= 6e^2 - 6.
\end{aligned}$$

Esercizio 4.
Calcolare l'integrale

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{(x-x^2)^{\frac{3}{2}}}\mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{(x-x^2)^{\frac{3}{2}}}\mathrm{d}x &= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{\left(\left(\frac{1}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2\right)^{\frac{3}{2}}}\mathrm{d}x \\
&\overset{(y=\arcsin(2x-1))}{=} \int_{-\frac{\pi}{6}}^{0} \frac{1}{\left(\frac{1}{4}-\frac{\sin^2 y}{4}\right)^{\frac{3}{2}}}\frac{\cos y}{2}\mathrm{d}y \\
&= \int_{-\frac{\pi}{6}}^{0} \frac{4}{\cos^2 y}\mathrm{d}y \\
&= 4[\tan y]_{-\frac{\pi}{6}}^{0} + c \\
&= \frac{4}{\sqrt{3}}.
\end{aligned}$$

Esercizio 5 (Assegnato per casa)**.**
Calcolare l'integrale

$$\int_0^{\frac{\pi}{4}} \frac{\cos^2 x}{1+\sin x\cos x}\mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int \frac{\cos^2 x}{1+\sin x\cos x}\mathrm{d}x &\overset{(y=\tan x)}{=} \int \frac{\frac{1}{1+y^2}}{1+\frac{y}{1+y^2}}\frac{1}{y^2+1}\mathrm{d}y \\
&= \int_0^1 \left(\frac{1}{\sqrt{3}}\frac{\frac{1}{\sqrt{3}}}{\left(\frac{2y+1}{\sqrt{3}}\right)^2+1} - \frac{1}{2}\frac{2y}{1+y^2} + \frac{1}{2}\frac{1+2y}{1+y+y^2}\right)\mathrm{d}y \\
&= \left[\frac{1}{\sqrt{3}}\arctan\frac{2y+1}{\sqrt{3}} - \frac{\log\left(1+y^2\right)}{2} + \frac{\log\left(1+y+y^2\right)}{2}\right]_0^1 \\
&= \frac{\pi}{6\sqrt{3}} - \frac{\log 2}{2} + \frac{\log 3}{2}.
\end{aligned}$$

Esercizio 6 (Assegnato per casa)**.**
Calcolare l'integrale

$$\int_0^{\log 2} \sqrt{e^x - 1}\mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_0^{\log 2} \sqrt{e^x - 1}\mathrm{d}x &\overset{(y=e^x)}{=} \int_1^2 \frac{\sqrt{y-1}}{y}\mathrm{d}y \\
&\overset{(z=\sqrt{y-1})}{=} \int_0^1 \frac{z}{z^2+1} 2z\mathrm{d}z \\
&= \int_0^1 \left(2 - \frac{2}{z^2+1}\right)\mathrm{d}z \\
&= [2z - 2\arctan z]_0^1 \\
&= 2 - \frac{\pi}{2}.
\end{aligned}$$