AM110 - Analisi matematica 1

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Argomenti: integrali

Esercizio 1.
Calcolare l'integrale

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x \mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \sin^2 x \left(1 - \sin^2 x\right) \cos x \mathrm{d}x \\
&\overset{(y=\sin x)}{=} \int_0^1 y^2 \left(1 - y^2\right) \mathrm{d}y \\
&= \left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 \\
&= \frac{2}{15}.
\end{aligned}$$

Esercizio 2.
Calcolare l'integrale

$$\int_1^e \frac{1}{x\left(\log^2 x - 2\log x + 4\right)} \mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_1^e \frac{1}{x\left((\log^2 x - 2\log x + 4\right)} \mathrm{d}x &\overset{(y=\log x)}{=} \int_0^1 \frac{1}{y^2 - 2y + 4} \mathrm{d}y \\
&= \frac{1}{\sqrt{3}} \int_0^1 \frac{\frac{1}{\sqrt{3}}}{\left(\frac{y-1}{\sqrt{3}}\right)^2 + 1} \mathrm{d}y \\
&= \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{y-1}{\sqrt{3}}\right)\right]_0^1 \\
&= \frac{\pi}{6\sqrt{3}}.
\end{aligned}$$

Esercizio 3.
Calcolare l'integrale

$$\int_0^1 e^{\sqrt[3]{x}} \mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_0^1 e^{\sqrt[3]{x}}\mathrm{d}x &\overset{(y=\sqrt[3]{x})}{=} 3\int_0^1 y^2 e^y \mathrm{d}y \\
&= 3\left(\left[y^2e^y\right]_0^1 - 2\int_0^1 ye^y\mathrm{d}y\right) \\
&= 3\left(e - 2\left([ye^y]_0^1 - \int_0^1 e^y\mathrm{d}y\right)\right) \\
&= 3\left(e - 2\left(e - [e^y]_0^1\right)\right) \\
&= 3e - 6.
\end{aligned}$$

Esercizio 4.
Calcolare l'integrale

$$\int_0^1 \frac{x}{\sqrt{x^2+2x+6}}\mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_0^1 \frac{x}{\sqrt{x^2+2x+6}}\mathrm{d}x &\overset{(y=x+\sqrt{x^2+2x+6})}{=} \int_{\sqrt{6}}^2 \frac{\frac{6-y^2}{2(y-1)}}{y+\frac{6-y^2}{2(y-1)}}\frac{-y^2+2y-6}{2(y-1)^2}\mathrm{d}y \\
&= \int_2^{\sqrt{6}}\left(\frac{1}{2}+\frac{1}{y-1}-\frac{5}{2(y-1)^2}\right)\mathrm{d}y \\
&= \left[\frac{y}{2}+\log|y-1|+\frac{5}{2(y-1)}\right]_2^{\sqrt{6}} \\
&= \frac{\sqrt{6}}{2}+\log\left(\sqrt{6}-1\right)+\frac{5}{2\left(\sqrt{6}-1\right)}-\frac{7}{2} \\
&= 3-\sqrt{6}+\log\left(\sqrt{6}-1\right).
\end{aligned}$$

Esercizio 5.
Calcolare l'integrale

$$\int_{-1}^1 x\left(e^{x^4}+e^{2x}\right)\mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_{-1}^1 x\left(e^{x^4}+e^{2x}\right)\mathrm{d}x &= \int_{-1}^1 xe^{2x}\mathrm{d}x + \int_0^1 xe^{x^4}\mathrm{d}x + \int_{-1}^0 xe^{x^4}\mathrm{d}x \\
&= \left[x\frac{e^{2x}}{2}\right]_{-1}^1 - \int_0^1 e^{\frac{2x}{2}}\mathrm{d}x + \int_0^1 xe^{x^4}\mathrm{d}x - \int_0^{-1} ye^{y^4}\mathrm{d}y \\
&= \frac{e^2+e^{-2}}{2} - \left[\frac{e^{2x}}{4}\right]_{-1}^1 \\
&= \frac{e^2}{4}+\frac{3}{4e^2}.
\end{aligned}$$

Esercizio 6 (Assegnato per casa)**.**
Calcolare l'integrale

$$\int_{\frac{\log\pi-\log 3}{2}}^{\frac{\log\pi-\log 2}{2}} \frac{e^{2x}}{1+\cos(e^{2x})}\mathrm{d}x.$$

Soluzione:

$$
\begin{aligned}
\int_{\frac{\log\pi-\log 3}{2}}^{\frac{\log\pi-\log 2}{2}} \frac{e^{2x}}{1+\cos(e^{2x})}\mathrm{d}x &\overset{(y=e^{2x})}{=} \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1+\cos y} \\
&\overset{(z=\tan\frac{y}{2})}{=} \frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{1} \frac{1}{1+\frac{1-z^2}{1+z^2}}\frac{2}{1+z^2}\mathrm{d}z \\
&= \frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{1} \mathrm{d}z \\
&= \frac{1}{2}-\frac{\sqrt{3}}{6}.
\end{aligned}
$$

Esercizio 7 (Assegnato per casa)**.**
Calcolare l'integrale

$$
\int_0^1 x^2 \log\left(x^6+1\right)\mathrm{d}x.
$$

Soluzione:

$$
\begin{aligned}
\int_0^1 x^2\log\left(x^6+1\right)\mathrm{d}x &\overset{(y=x^3)}{=} \frac{1}{3}\int_0^1 \log\left(y^2+1\right)\mathrm{d}y \\
&= \frac{1}{3}\left(\left[y\log\left(y^2+1\right)\right]_0^1 - \int_0^1 \frac{2y^2}{y^2+1}\mathrm{d}y\right) \\
&= \frac{1}{3}\left(\log 2 - \int_0^1\left(2-\frac{2}{y^2+1}\right)\mathrm{d}y\right) \\
&= \frac{1}{3}\left(\log 2 - \left[2y-2\arctan y\right]_0^1\right) \\
&= \frac{\log 2}{3}-\frac{2}{3}+\frac{\pi}{6}.
\end{aligned}
$$