AM110 - Analisi matematica 1

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Argomenti: limiti di successioni

Esercizio 1.
Calcolare, se esiste, il limite

$$\lim_{n\to\infty} n^3\left(\sin\left(\frac{1}{n}\right)-\tan\left(\frac{1}{n}\right)\right).$$

Soluzione:

$$\begin{aligned}
\lim_{n\to\infty} n^3\left(\sin\left(\frac{1}{n}\right)-\tan\left(\frac{1}{n}\right)\right) &= \lim_{n\to+\infty}\frac{\sin\left(\frac{1}{n}\right)-\tan\left(\frac{1}{n}\right)}{\frac{1}{n^3}}\\
&= \lim_{n\to+\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}\frac{1-\frac{1}{\cos\frac{1}{n}}}{\frac{1}{n^2}}\\
&= \lim_{n\to+\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}\left(-\frac{1}{\cos\frac{1}{n}}\right)\frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}}\\
&= -\lim_{n\to+\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}\lim_{n\to+\infty}\frac{1}{\cos\frac{1}{n}}\lim_{n\to+\infty}\frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}}\\
&= -\frac{1}{2}.
\end{aligned}$$

Esercizio 2.
Calcolare, se esiste, il limite

$$\lim_{n\to\infty} n\log\frac{n+1}{n-2}.$$

Soluzione:

$$\begin{aligned}
\lim_{n\to\infty} n\log\frac{n+1}{n-2} &= \lim_{n\to\infty} n\log\left(1+\frac{3}{n-2}\right)\\
&= \lim_{n\to\infty}\left(3\frac{n-2}{3}\log\left(1+\frac{3}{n-2}\right)+2\log\left(1+\frac{3}{n-2}\right)\right)\\
&= 3\lim_{n\to\infty}\frac{\log\left(1+\frac{3}{n-2}\right)}{\frac{3}{n-2}}+2\lim_{n\to\infty}\log\left(1+\frac{3}{n-2}\right)\\
&= 3.
\end{aligned}$$

Esercizio 3.
Calcolare, se esiste, il limite

$$\lim_{n\to\infty}\left(\cos\frac{1}{\sqrt{n-1}}\right)^n.$$

Soluzione:

$$\begin{aligned}
\lim_{n\to\infty}\left(\cos\frac{1}{\sqrt{n-1}}\right)^n &= \lim_{n\to\infty} e^{\ln\left(\left(\cos\frac{1}{\sqrt{n-1}}\right)^n\right)}\\
&= \lim_{n\to\infty} e^{\frac{\ln\left(\cos\frac{1}{\sqrt{n-1}}\right)}{\frac{1}{n}}}\\
&= \lim_{n\to\infty} e^{-\frac{\ln\left(1+\cos\frac{1}{\sqrt{n-1}}-1\right)}{\cos\frac{1}{\sqrt{n-1}}-1}\frac{1-\cos\frac{1}{\sqrt{n-1}}}{\frac{1}{n-1}}\frac{\frac{1}{n-1}}{\frac{1}{n}}}\\
&= e^{-\left(\lim_{n\to+\infty}\frac{\ln\left(1+\cos\frac{1}{\sqrt{n-1}}-1\right)}{\cos\frac{1}{\sqrt{n-1}}-1}\right)\left(\lim_{n\to+\infty}\frac{1-\cos\frac{1}{\sqrt{n-1}}}{\frac{1}{n-1}}\right)\left(\lim_{n\to+\infty}\frac{n}{n-1}\right)}\\
&= e^{-\frac{1}{2}}\\
&= \frac{1}{\sqrt{e}}.
\end{aligned}$$

Esercizio 4.
Calcolare, se esiste, il limite

$$\lim_{n\to\infty}\frac{\sin\left(\pi e^{\frac{1}{n}}\right)}{\sqrt[3]{1+\frac{1}{n}}-1}.$$

Soluzione:

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sin\left(\pi e^{\frac{1}{n}}\right)}{\sqrt[3]{1+\frac{1}{n}}-1} &= \lim_{n\to\infty}\frac{-\sin\left(\pi\left(e^{\frac{1}{n}}-1\right)\right)}{\frac{\sqrt[3]{1+\frac{1}{n}}-1}{\frac{1}{n}}}\frac{1}{n}\\
&= -\lim_{n\to+\infty}\frac{\frac{\sin\left(\pi\left(e^{\frac{1}{n}}-1\right)\right)}{\pi\left(e^{\frac{1}{n}}-1\right)}\pi\frac{e^{\frac{1}{n}}-1}{\frac{1}{n}}}{\frac{\sqrt[3]{1+\frac{1}{n}}-1}{\frac{1}{n}}}\\
&= -\pi\frac{\lim_{n\to+\infty}\frac{\sin\left(\pi\left(e^{\frac{1}{n}}-1\right)\right)}{\pi\left(e^{\frac{1}{n}}-1\right)}\lim_{n\to+\infty}\frac{e^{\frac{1}{n}}-1}{\frac{1}{n}}}{\lim_{n\to+\infty}\frac{\sqrt[3]{1+\frac{1}{n}}-1}{\frac{1}{n}}}\\
&= -3\pi.
\end{aligned}$$

Esercizio 5.
Calcolare, se esiste, il limite

$$\lim_{n\to\infty}\frac{(n+2)!}{n!}\left(e-e^{\cos\frac{1}{n}}\right).$$

Soluzione:

$$\begin{aligned}
\lim_{n\to\infty}\frac{(n+2)!}{n!}\left(e-e^{\cos\frac{1}{n}}\right) &= \lim_{n\to\infty}(n+2)(n+1)e\left(1-e^{\cos\left(\frac{1}{n}\right)-1}\right)\\
&= e\lim_{n\to+\infty}(n+2)(n+1)\frac{1-e^{\cos\left(\frac{1}{n}\right)-1}}{\cos\left(\frac{1}{n}\right)-1}\frac{\cos\left(\frac{1}{n}\right)-1}{\frac{1}{n^2}}\frac{1}{n^2}\\
&= e\lim_{n\to+\infty}\frac{(n+2)(n+1)}{n^2}\frac{e^{\cos\left(\frac{1}{n}\right)-1}-1}{\cos\left(\frac{1}{n}\right)-1}\frac{1-\cos\left(\frac{1}{n}\right)}{\frac{1}{n^2}}\\
&= \frac{e}{2}.
\end{aligned}$$

Esercizio 6.
Calcolare, se esiste, il limite

$$\lim_{n\to\infty} \frac{\log\left(1+n^2\right)-2\log n}{\left(\arctan\frac{1}{n}\right)^2}.$$

Soluzione:

$$\lim_{n\to\infty} \frac{\log\left(1+n^2\right)-2\log n}{\left(\arctan\frac{1}{n}\right)^2} = \lim_{n\to\infty} \frac{\log\left(\frac{1}{n^2}+1\right)}{\left(\arctan\frac{1}{n}\right)^2} = \lim_{n\to\infty} \frac{\frac{\log\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}}}{\left(\frac{\arctan\frac{1}{n}}{\frac{1}{n}}\right)^2} = \frac{\lim_{n\to\infty} \frac{\log\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}}}{\lim_{n\to\infty}\left(\frac{\arctan\frac{1}{n}}{\frac{1}{n}}\right)^2} = 1.$$

Esercizio 7.
Calcolare, se esiste, il limite

$$\lim_{n\to\infty} \left(\sqrt{n+\sqrt{n}}-\sqrt{n-\sqrt{n}}\right).$$

Soluzione:

$$\begin{aligned}
\lim_{n\to\infty} \left(\sqrt{n+\sqrt{n}}-\sqrt{n-\sqrt{n}}\right) &= \lim_{n\to\infty} \frac{\left(\sqrt{n+\sqrt{n}}-\sqrt{n-\sqrt{n}}\right)\left(\sqrt{n+\sqrt{n}}+\sqrt{n-\sqrt{n}}\right)}{\sqrt{n+\sqrt{n}}+\sqrt{n-\sqrt{n}}} \\
&= \lim_{n\to+\infty} \frac{n+\sqrt{n}-(n-\sqrt{n})}{\sqrt{n+\sqrt{n}}+\sqrt{n-\sqrt{n}}} \\
&= 2\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n-\sqrt{n}}} \\
&= 2\lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}+\sqrt{1-\frac{1}{\sqrt{n}}}} \\
&= 1.
\end{aligned}$$

Esercizio 8 (Assegnato per casa)**.**
Calcolare, se esiste, il limite

$$\lim_{n\to\infty} \frac{e^n}{n^{\sqrt{n}}}.$$

Soluzione:

$$\lim_{n\to\infty} \frac{e^n}{n^{\sqrt{n}}} = \lim_{n\to\infty} \frac{e^n}{e^{\ln\left(n^{\sqrt{n}}\right)}} = \lim_{n\to\infty} \frac{e^n}{e^{\sqrt{n}\ln n}} = \lim_{n\to+\infty} e^{n-\sqrt{n}\log n} = e^{+\infty} = +\infty.$$

Esercizio 9 (Assegnato per casa)**.**
Calcolare, se esiste, il limite

$$\lim_{n\to\infty} \left(\sqrt{n^2+2n}-n\right)^n.$$

Soluzione:

$$\begin{aligned}
\lim_{n\to+\infty} \left(\sqrt{n^2+2n}-n\right)^n &= \lim_{n\to+\infty} \left(\frac{2n}{\sqrt{n^2+2n}+n}\right)^n \\
&= \lim_{n\to+\infty} \left(1-\frac{\sqrt{n^2+2n}-n}{\sqrt{n^2+2n}+n}\right)^n
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to+\infty}\left(1-\frac{2n}{\left(\sqrt{n^2+2n}+n\right)^2}\right)^n \\
&= \lim_{n\to+\infty}\left(1-\frac{2}{\left(\sqrt{n+2}+\sqrt{n}\right)^2}\right)^n \\
&= \lim_{n\to+\infty}\left(1-\frac{1}{n+1+2\sqrt{n^2+2n}}\right)^n \\
&= \left(\lim_{n\to+\infty}\left(1-\frac{1}{n+1+2\sqrt{n^2+2n}}\right)^{-\left(n+1+2\sqrt{n^2+2n}\right)}\right)^{-\lim_{n\to+\infty}\frac{n}{n+1+2\sqrt{n^2+2n}}} \\
&= e^{-\frac{1}{2}} \\
&= \frac{1}{\sqrt{e}}.
\end{aligned}
$$