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On Moser-Adams' sharp inequalities about exponential integrability for higher order weak and fractional derivatives

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Contents

Notations

We will make use of the following notations in the whole issue:

- $\mathbb{N}$ denotes the set of positive integer numbers;
- $\mathbb{N}_0$ denotes the set of non-negative integer numbers;
- $\mathbb{N}_{\geq 2}$ is the set of positive integer numbers except for 1, namely $\mathbb{N}_{\geq 2} := \mathbb{N} \backslash \{1\}$;
- for $n \in \mathbb{N}$, $\mathbb{N}^n$ is the set of the tuples of the type $(a_1, a_2, \ldots, a_n)$, where $a_i \in \mathbb{N}$, $\forall\, 1 \leq i \leq n$;
- $\mathbb{Z}$ denotes the set of integer numbers;
- $\mathbb{R}$ denotes the set of real numbers;
- $\mathbb{C}$ denotes the set of complex numbers;
- for $n \in \mathbb{N}$, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space;
- $(e_1, e_2, \ldots, e_n)$ denotes the standard orthonormal basis of $\mathbb{R}^n$;
- $x \in \mathbb{R}^n$ means that $x := (x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{R}$, $\forall\, 1 \leq i \leq n$;
- for $x, y \in \mathbb{R}^n$, the operation $x \cdot y$ denotes the standard scalar product, namely
$$x \cdot y := \sum_{i=1}^{n} x_i y_i;$$
- for $x \in \mathbb{R}^n$, $|x|$ denotes its norm, which is defined by
$$|x| := \sqrt{x \cdot x} = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}};$$
- we will denote by Ω an open and measurable subset of $\mathbb{R}^n$ whose measure is finite;
- given $\Omega \subseteq \mathbb{R}^n$, we will denote its closure by $\overline{\Omega}$, its boundary by $\partial\Omega$ and its (n-dimensional Lebesgue) measure by $|\Omega|$;

- given an arbitrary subset $\mathcal{D} \subseteq \mathbb{R}^n$ and $k \in \mathbb{N} \cup \{\infty\}$, $C^k(\mathcal{D})$ denotes the space of functions which are k times continuously differentiable in $\mathcal{D}$;
- given $\Omega \subseteq \mathbb{R}^n$, $k \in \mathbb{N}_0$ and $\alpha > 0$, $C^{k,\alpha}\big(\overline{\Omega}\big)$ denotes the space of functions which are k times continuously differentiable in $\overline{\Omega}$ and whose k-th partial derivatives are α-Hölder continuous (with the convention that $C^{0,\alpha}\big(\overline{\Omega}\big)$ is the set of α-Hölder continuous functions), endowed with the norm $||\cdot||_{C^{k,\alpha}(\overline{\Omega})}$;
- given an arbitrary subset $\mathcal{D} \subseteq \mathbb{R}^n$, $C_0^\infty(\mathcal{D})$ denotes the space of functions belonging to $C^\infty(\mathcal{D})$ which are compactly supported on $\mathcal{D}$;
- given an arbitrary subset $\mathcal{D} \subseteq \mathbb{R}^n$ and $1 \leq p \leq \infty$, $L^p(\mathcal{D})$ denotes the usual Lebesgue space endowed with the norm $||\cdot||_{L^p(\mathcal{D})}$;
- given an arbitrary subset $\mathcal{D} \subseteq \mathbb{R}^n$ and $1 \leq p \leq \infty$, $L^p_{\text{loc}}(\mathcal{D})$ denotes the space of functions lying in $L^p(\mathcal{D}')$, for every $\mathcal{D}'$ compact subset of $\mathcal{D}$;
- given an arbitrary subset $\mathcal{D} \subseteq \mathbb{R}^n$, $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, $W^{m,p}(\mathcal{D})$ denotes the usual Sobolev space of order m of functions in $L^p(\mathcal{D})$ all of whose (distribution) derivatives up to order m are also in $L^p(\mathcal{D})$, endowed with the norm $||\cdot||_{W^{m,p}(\mathcal{D})}$;
- given an arbitrary subset $\mathcal{D} \subseteq \mathbb{R}^n$, $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, $W_0^{m,p}(\mathcal{D})$ denotes the closure of $C_0^\infty(\mathcal{D})$ in $W^{m,p}(\mathcal{D})$;
- $O(n)$ denotes the n-dimensional orthogonal group, namely the group of distance-preserving transformations of $\mathbb{R}^n$;
- given a matrix $M \in O(n)$, $||M||$ denotes its matrix norm;
- given two spaces $\mathcal{A}$ and $\mathcal{B}$, the writing $\mathcal{A} \hookrightarrow \mathcal{B}$ denotes the continuous embedding of $\mathcal{A}$ into $\mathcal{B}$;
- given a function u, ∇u denotes its gradient, namely

$$\nabla u(x) := \left(\frac{\partial}{\partial x_1}\, u(x), \ldots, \frac{\partial}{\partial x_n}\, u(x)\right);$$

- given a function u, Δu denotes its Laplacian, namely

$$\Delta u(x) := \operatorname{div}\big(\nabla u(x)\big) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\, u(x);$$

- given a function u and $\beta \in \mathbb{N}^n$, the multi-index notation is

$$D^\beta u(x) := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}}\, u(x);$$

- given a function u, its support is denoted by $\text{supp}\{u\}$;
- given a function u, u_+ denotes its positive part and u_- its negative part, namely $u_+(x) := \max\{0, u(x)\}$ and $u_-(x) := -\min\{0, u(x)\}$;
- given a function u defined on a measurable set $\mathcal{D}$ such that $|\mathcal{D}| < +\infty$, we define

$$\fint_{\mathcal{D}} u(x)dx := \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} u(x)dx;$$

- given two functions u and v, $u(x) \equiv v(x)$ means that $u(x) = v(x)$ for every x, while $u(x) \not\equiv v(x)$ means that there exists x_0 such that $u(x_0) \neq v(x_0)$;
- given $x_0 \in \mathbb{R}^n$ and two functions u and v, $u(x) \sim v(x)$ near x_0 means that

$$\lim_{x \to x_0} \frac{u(x)}{v(x)} = 1;$$

- given a sequence $\{u_k\}_{k \in \mathbb{N}}$, the writing $u_k(x) \longrightarrow u(x)$ means that u_k converges to u as $k \longmapsto +\infty$;
- given a sequence $\{u_k\}_{k \in \mathbb{N}}$, the writing $u_k(x) \rightharpoonup u(x)$ means that u_k converges weakly to u as $k \longmapsto +\infty$;
- the acronym a.e. denotes the "almost everywhere" validity of a given property $\mathcal{P}$;
- given $n \in \mathbb{N}_0$, $n!$ denotes its factorial;
- given $n \in \mathbb{N}_0$, $n!!$ denotes its semifactorial;
- $\Gamma(x)$ denotes the usual gamma function of x;
- ω_{n-1} denotes the area of the surface of the unit ball in $\mathbb{R}^n$, which is equal to

$$\omega_{n-1} := \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)};$$

- given $r > 0$ and $x_0 \in \mathbb{R}^n$, $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ denotes the ball of radius r centered in x_0;
- for $x > 0$, $\lfloor x \rfloor$ denotes the integer part of x;
- $\text{dist}(\cdot, \cdot)$ denotes the distance of a point from a set or between two sets;
- for $x_0 \in \mathbb{R}^n$, $\delta_{x_0}(x)$ denotes the Dirac delta distribution centered in x_0.

Introduction

The aim of the present issue is to demonstrate some important results concerning the Sobolev spaces which generalize the classical Sobolev inequalities for a subset $\Omega \subseteq \mathbb{R}^n$ in the n-dimensional space. We will always consider (unless explicitly stated otherwise) bounded and measurable domains $\Omega \subseteq \mathbb{R}^n$, for an arbitrary $n \in \mathbb{N}_{\geq 2}$, and functions $u : \Omega \longrightarrow \mathbb{R}$, because we shall deal with results in which the measure of Ω appears explicitly. Therefore, the boundedness and the measurability of Ω are two reasonable assumptions (roughly speaking, one could omit the boundedness of the latter as long as its measure is still finite).
We begin by reminding some results about the continuous immersions in Sobolev spaces, which will be our starting point; subsequently, we will be able to find a (sharp, in some sense) estimate concerning an exponential estimate.
It is well-known that the embedding theorems due to Gagliardo, Nirenberg, Sobolev and Morrey among others, can be summarized as follows.

Theorem. *Let* $1 \leq p < \infty$, $n \in \mathbb{N}_{\geq 2}$ *and* $u \in W^{1,p}(\Omega)$. *Then:*

(i) for $1 \leq p < n$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall\, p \leq q \leq p_n^*$, *where* $p_n^* := \frac{np}{n-p} > p$;

(ii) for $p = n$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall\, n \leq q < \infty$;

(iii) for $n < p < \infty$, $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right)$, *where* $\alpha := 1 - \frac{n}{p} \in (0,1)$.

In particular, there is a stronger estimate concerning the point *(i)* of the preceding theorem.

Corollary. *Under the same hypothesis of the previous theorem, if* $1 \leq p < n$, *then*

$$||u||_{L^{p_n^*}(\Omega)} \leq c||\nabla u||_{L^p(\Omega)}$$

holds for a certain constant $c = c(n, p, \Omega)$ *depending on* n, p *and* Ω *only.*

The generalization of these results for functions possessing higher order derivatives are widely known, too, and are contained in the following statement (which is also present, for instance, in [9]).

Theorem. *Let* $1 \leq p < \infty$, $n \in \mathbb{N}_{\geq 2}$, $m \in \mathbb{N}$ *such that* $m < n$ *and* $u \in W^{1,p}(\Omega)$. *Then:*

(i) for $1 \leq p < \frac{n}{m}$, $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall\, p \leq q \leq p^*_{m,n}$, *where* $p^*_{m,n} := \frac{np}{n-mp} > p$;

(ii) for $p = \frac{n}{m}$, $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall\, \frac{n}{m} \leq q < \infty$;

(iii) for $\frac{n}{m} < p < \infty$, $W^{m,p}(\Omega) \hookrightarrow C^{k,\alpha}\left(\overline{\Omega}\right)$, *where* $k := m - \left\lfloor \frac{n}{p} \right\rfloor - 1$ *and, taken an arbitrary number* $\delta \in (0,1)$, $\alpha \in (0,1)$ *is defined by*

$$\alpha := \begin{cases} \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \notin \mathbb{N} \\ \delta & \text{if } \frac{n}{p} \in \mathbb{N} \end{cases}.$$

To be precise, it does not suffice to consider a domain Ω with the two properties given before to satisfy these statements: one must also have the extension property (which is implied, for instance, by assuming that its frontier is Lipschitz continuous). Another interesting outcome is that these immersions, in addition to being continuous, are even compact if Ω is bounded.
We shall also remind Trudinger's work in [25], where he was able to bound the integral

$$\int_\Omega e^{\alpha |u(x)|^p} dx$$

by a constant $c = c(n, q, \Omega)$ depending on n, q and Ω only (hence, independent of u), where:

- u is an arbitrary function lying in $W^{1,q}_0(\Omega)$ such that $||u||_{L^q(\Omega)} \leq 1$;
- p is the conjugate exponent of q defined by $p := \frac{q}{q-1}$;
- α is a certain positive number.

His proof is based on the power series expansion of the exponential function and some Sobolev estimates.
We start from here: in the first chapter, we introduce some notions and helpful results concerning the unidimensional decreasing rearrangement of a function and, afterwards, we will deal with Trudinger's integral estimate considering the limit case in which $p = n$, which is represented by the point *(ii)* of the theorem previously stated for the space $W^{1,p}(\Omega)$. However, we will make use of another strategy following the techniques utilized in [18], resorting to the Schwarz symmetrization and Pólya-Szegö's theorem to the scope of simplify its proof: this reasoning will allow us to demonstrate the estimate in a more direct way and, at the same time, will give us the best exponent α. In fact, it turns out that there exists a positive number α_n such that the above statement holds for $\alpha \leq \alpha_n$ and is false for $\alpha > \alpha_n$. What is interesting is that the estimate is still

valid if $\alpha = \alpha_n$, namely if α is the critical value. Proving the theorem for $\alpha < \alpha_n$ and confuting it for $\alpha > \alpha_n$ will be quite easy, while the discussion for the limit case in which $\alpha = \alpha_n$ will be more complex.
After that, in the second chapter, we will generalize this result for functions in $W^{m,p}(\Omega)$ and for $p = \frac{n}{m}$, which means that, once more, we are in the limit case given by *(ii)* of the theorem regarding the space $W^{m,p}(\Omega)$. In order to do that, we shall remind the notion of the convolution of two functions because it will play a major role in generalizing Moser's result. This will be done by following O'Neil's work in [21]. Then, resorting mainly to Adams' work (see [1]), we will achieve a similar outcome respect to the preceding one: an analogous integral estimate about an exponentiation will hold for all coefficients $\alpha \leq \alpha_{m,n}$, where this $\alpha_{m,n}$ depends, this time, also on m. Once more, this estimate will not be valid whenever $\alpha > \alpha_{m,n}$. This new outcome generalizes the previous one in the sense that, for $m = 1$ fixed, it is exactly Moser's result.
Finally, in the third and last chapter, we will replace the positive integer m with a real number $s > 0$ and make the estimate still true: in order to do that, we will first have to introduce the meaning of a non-integer number of derivatives (which will be done through the fractional Laplacian operator $(-\Delta)^s$). All this will be done using the discussion present in [17] and, surprisingly, some results already acquired in the first two chapters which will be still valid, even if dealing with fractional derivatives.

Chapter 1

1.1. Rearrangement and Schwarz symmetrization

We want here to introduce the fundamentals required for the comprehension of Moser's theorem, which will be enunciated and demonstrated at the end of the chapter. We need mainly two notions: the Schwarz symmetrization (and some of its properties) and Pólya-Szegö's theorem. The first one will be achieved in this paragraph, while the aforementioned theorem will be postponed. The Schwarz symmetrization is a particular type of rearrangement of functions: given a function u, we create an associated function having "better" properties than u. In particular, we want this new function to be radially decreasing and, if integrated on its domain of definition, to give the same value of u. After having proved this properties and achieved some preliminary results, we will be able to pass to Pólya-Szegö's theorem. These first two paragraphs are taken from [14].
Let us begin by giving the following definitions.

Definition 1. *Let u be a function. Then:*

*(i) for $t \in \mathbb{R}$, the **level set** $\{u > t\}$ is defined as*

$$\{u > t\} := \{x \in \Omega : u(x) > t\};$$

*(ii) the **distribution function** of u is the function $\mu_u : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$\mu_u(t) := |\{u > t\}|;$$

*(iii) the **unidimensional decreasing rearrangement** of u is defined as the function $u^{\#} : [0, |\Omega|] \longrightarrow \mathbb{R}$ such that*

$$u^{\#}(s) := \begin{cases} \operatorname{ess\,sup}\{u\} & \text{if } s = 0 \\ \inf_{t \in \mathbb{R}} \{\mu_u(t) < s\} & \text{if } s \in (0, |\Omega|] \end{cases}.$$

Remark 1. *Some initial observations:*

(i) *clearly, the sets* $\{u < t\}$, $\{u \geq t\}$, $\{u \leq t\}$ *and* $\{u = t\}$ *can be defined in the same way as we did for* $\{u > t\}$;

(ii) *the function* $\mu_u(t)$ *is monotonically decreasing in the variable* t *and its range is* $[0, |\Omega|]$ *(in particular, by definition, we have that* $\mu_u(t) = 0$ *for* $t \geq \operatorname{ess\,sup}\{u\}$ *and* $\mu_u(t) = |\Omega|$ *for* $t \leq \operatorname{ess\,inf}\{u\}$*);*

(iii) *essentially,* $\mu_u(t)$ *and* $u^{\#}(s)$ *are the inverse functions of each other;*

(iv) *there exist other conventions to the scope of defining the distribution function (for instance, in* [11] *and* [22]*, its definition is given by* $|\{|u| > t\}|$*).*

We enunciate here the first properties of the rearrangement $u^{\#}$ of a function u.

Proposition 1. *The unidimensional decreasing rearrangement* $u^{\#}$ *of a function* u *is a non-increasing and left-continuous function.*

Proof. Let $s_1 < s_2$. If $|\{u > t\}| < s_1$, then $|\{u > t\}| < s_2$. It means that

$$\{t \in \mathbb{R} : \mu_u(t) < s_1\} \subseteq \{t \in \mathbb{R} : \mu_u(t) < s_2\}$$

and therefore, by definition, $u^{\#}(s_1) \geq u^{\#}(s_2)$. This proves that $u^{\#}$ is non-increasing. Now, let $s \in (0, |\Omega|)$. Again by definition of $u^{\#}$, for every choice of $\varepsilon > 0$, $\exists\, t \in \mathbb{R}$ such that $u^{\#}(s) \leq t < u^{\#}(s) + \varepsilon$ and $\mu_u(t) < s$. Choosing $h > 0$ such that $\mu_u(t) < s - h < s$, one has that, $\forall\, k \in (0, h]$, $\mu_u(t) < s - k < s$. Putting the pieces together, it must be that

$$u^{\#}(s) \leq u^{\#}(s - k) \leq t < u^{\#}(s) + \varepsilon.$$

We have therefore established that $u^{\#}(s)$ is left-continuous.

□

We are starting to realize the "good" properties mentioned earlier. Another one is the equimeasurability.

Definition 2. *Two functions* u *and* v *are* ***equimeasurable*** *if they have the same distribution function, namely if* $\mu_u(t) = \mu_v(t)$.

We now show that u and $u^{\#}$ are equimeasurable.

Proposition 2. *The functions* u *and* $u^{\#}$ *are equimeasurable:* $\forall\, t \in \mathbb{R}$,

$$\mu_u(t) := |\{u > t\}| = |\{x \in \Omega : u(x) > t\}| =$$

$$= |\{s \in [0, |\Omega|] : u^{\#}(s) > t\}| = |\{u^{\#} > t\}| =: \mu_{u^{\#}}(t). \tag{1}$$

Proof. If $u^{\#}(s) > t$, then $|\{u > t\}| \geq s$ by definition of $u^{\#}$. Thus,

$$\{s \in [0, |\Omega|] : u^{\#}(s) > t\} \subseteq \{s \in [0, |\Omega|] : |\{u > t\}| \geq s\}.$$

Using the fact that $u^{\#}$ is a non-increasing function, we have

$$|\{u^{\#} > t\}| = \sup_{s \in [0, |\Omega|]} \{u^{\#}(s) > t\} \leq |\{u > t\}|. \tag{2}$$

On the other hand, if $|\{u^{\#} \geq t\}| = s$, it must be that $u^{\#}(s) = t$ by the left-continuity and the non-increasing properties of $u^{\#}$. Again by definition of $u^{\#}$, it follows that $|\{u > t\}| \leq s$ and therefore

$$|\{u > t\}| \leq s = |\{u^{\#} \geq t\}|. \tag{3}$$

Letting $h > 0$ and applying (2) and (3) for $t + h$ instead of t, we get

$$|\{u^{\#} > t + h\}| \leq |\{u > t + h\}| \leq |\{u^{\#} \geq t + h\}|,$$

which leads to

$$|\{u^{\#} > t\}| \leq |\{u > t\}| \leq |\{u^{\#} > t\}|$$

as $h \longmapsto 0^+$. The last estimate proves (1) since it reads as $\mu_{u^{\#}}(t) \leq \mu_u(t) \leq \mu_{u^{\#}}(t)$.

□

Corollary 1. *In according with the preceding notations, we have*

$$|\{u > t\}| = |\{u^{\#} > t\}|,$$
$$|\{u \geq t\}| = |\{u^{\#} \geq t\}|,$$
$$|\{u < t\}| = |\{u^{\#} < t\}|,$$
$$|\{u \leq t\}| = |\{u^{\#} \leq t\}|$$

and

$$|\{u = t\}| = |\{u^{\#} = t\}|.$$

Proof. The first of the five relations has already been established with the preceding proposition: the others will follow from this one by complementation and suitable limiting arguments.

□

Proposition 2 and the following corollary explain the reason why $u^\#$ is called a rearrangement of u. This one in particular is just one example of a wide variety of such construction: others can be defined as well. For instance, one can create a non-decreasing rearrangement $u_\#(s)$ starting from the "opposite" distribution function $\mu^u(t) := |\{u < t\}|$ (see also [13] for examples of different kinds of rearrangements). Even the Schwarz symmetrization we are going to define is another type of rearrangement.

Theorem 1. *Let u be a measurable function and let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative Borel measurable function. Then*

$$\int_\Omega F\big(u(x)\big)dx = \int_0^{|\Omega|} F\big(u^\#(s)\big)ds. \tag{4}$$

Proof. For $E := [t, +\infty)$, we set $F(\xi) := \chi_E(\xi)$. Then, by (1),

$$\int_\Omega F\big(u(x)\big)dx = |\{u > t\}| = |\{u^\# > t\}| = \int_0^{|\Omega|} F\big(u^\#(s)\big)ds.$$

In the same way, the result holds for any interval by standard arguments in view of Corollary 1. Hence, it is true for any non-negative simple function F: if F_k is a non-negative simple function, for k fixed one has

$$\int_\Omega F_k\big(u(x)\big)dx = \int_0^{|\Omega|} F_k\big(u^\#(s)\big)ds. \tag{5}$$

Therefore, since any non-negative Borel function can be expressed as the limit of an increasing sequence $\{F_k\}_{k\in\mathbb{N}}$ of non-negative simple functions, passing to the limit as $k \longmapsto +\infty$ in (5) and using the monotone convergence theorem, we get (4).

□

Corollary 2. *Let $1 \le p \le \infty$ and $u \in L^p(\Omega)$. Then $u^\# \in L^p\big([0, |\Omega|]\big)$ and*

$$||u||_{L^p(\Omega)} = ||u^\#||_{L^p([0,|\Omega|])}. \tag{6}$$

Proof. If $p = \infty$, the result easily comes from the definition of $u^\#$; if $1 \le p < \infty$, the result follows from the previous theorem once we set $F(t) := |t|^p$.

□

We continue to enunciate some results concerning the rearrangement $u^\#$.

Proposition 3. *The mapping $u \longmapsto u^\#$ is non-decreasing. In other words, if $u \le v$, then $u^\# \le v^\#$.*

Proof. The thesis follows from the definition of the rearrangement $u^\#$ noting that, since $\{u > t\} \subseteq \{v > t\}$ by hypothesis, then

$$\{t \in \mathbb{R} : |\{v > t\}| < s\} \subseteq \{t \in \mathbb{R} : |\{u > t\}| < s\}.$$

□

Proposition 4. *For $f, g \in L^1(\Omega)$, we have*

$$||f^\# - g^\#||_{L^1([0,|\Omega|])} \leq ||f - g||_{L^1(\Omega)}. \tag{7}$$

Proof. We set $h(x) := \max\{f(x), g(x)\}$. Then, since $f \leq h$ and $g \leq h$, we have that $f^\# \leq h^\#$ and $g^\# \leq h^\#$ by the previous proposition. Now,

$$|f^\#(s) - g^\#(s)| \leq |f^\#(s) - h^\#(s)| + |h^\#(s) - g^\#(s)| = 2h^\#(s) - f^\#(s) - g^\#(s).$$

Thus, using (4) for $F(t) = |t|$ (since $2h^\#(s) - f^\#(s) - g^\#(s)$ is a non-negative function), we get

$$||f^\# - g^\#||_{L^1([0,|\Omega|])} = \int_0^{|\Omega|} |f^\#(s) - g^\#(s)| ds \leq \int_0^{|\Omega|} \big[2h^\#(s) - f^\#(s) - g^\#(s)\big] ds =$$

$$= \int_\Omega \big[2h(x) - f(x) - g(x)\big] dx = \int_\Omega |f(x) - g(x)| dx = ||f - g||_{L^1(\Omega)},$$

which is the thesis.

□

Theorem 2. *Let $1 \leq p < \infty$. The mapping $u \longmapsto u^\#$ is continuous from $L^p(\Omega)$ into $L^p\big([0, |\Omega|]\big)$; in other words, given a sequence $\{u_k\}_{k\in\mathbb{N}}$ such that $u_k(x) \longrightarrow u(x)$ in $L^p(\Omega)$, then $u_k^\#(s) \longrightarrow u^\#(s)$ in $L^p\big([0, |\Omega|]\big)$.*

Proof. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence converging to u in $L^p(\Omega)$. If $p = 1$, from (7) we get

$$||u_k^\# - u^\#||_{L^1([0,|\Omega|])} \leq ||u_k - u||_{L^1(\Omega)} \longrightarrow 0.$$

For $1 < p < \infty$, by the boundedness of Ω it follows that $u_k(x) \longrightarrow u(x)$ in $L^1(\Omega)$, which means that also $u_k^\#(s) \longrightarrow u^\#(s)$ in $L^1\big([0, |\Omega|]\big)$. Hence, given a subsequence $\{u_{k_h}^\#\}_{h\in\mathbb{N}}$, we have that $u_{k_h}^\#(s) \longrightarrow u^\#(s)$ a.e. and, therefore,

$$||u_{k_h}^\#||_{L^p([0,|\Omega|])} = ||u_{k_h}||_{L^p(\Omega)} \longrightarrow ||u||_{L^p(\Omega)} = ||u^\#||_{L^p([0,|\Omega|])}$$

by (6). Further, $u_{k_h}^\#(s) \longrightarrow u^\#(s)$ in $L^p\big([0, |\Omega|]\big)$ as well due to the independence of the limit from the subsequence. So it must be that $u_k^{\#}(s) \longrightarrow u^{\#}(s)$ in $L^p\big([0,|\Omega|]\big)$, situation which represents the desired result.

$\square$

Remark 2. *The last result is also true for $p=\infty$; however, we will not make use of it and so it has been omitted.*

We are finally ready to define the Schwarz symmetrization of a function u.

Definition 3. *Let Ω^* be the open ball centered at the origin whose measure is the same as Ω and let u be a function. Then, its **Schwarz symmetrization** is the function $u^* : \Omega^* \longrightarrow \mathbb{R}$ defined by*

$$u^*(x) := u^{\#}\Big(\frac{\omega_{n-1}}{n}\,|x|^n\Big).$$

Remark 3. *If R denotes the radius of Ω^*, then*

$$\int_{\Omega^*} u^*(x)dx = \int_{\Omega^*} u^{\#}\Big(\frac{\omega_{n-1}}{n}\,|x|^n\Big)dx = \int_0^R u^{\#}\Big(\frac{\omega_{n-1}}{n}\,r^n\Big)\omega_{n-1}r^{n-1}dr =$$

$$= \int_0^{|\Omega^*|} u^{\#}(s)ds = \int_0^{|\Omega|} u^{\#}(s)ds.$$

This means that all the results obtained for $u^{\#}$ are still valid for u^*. In particular, we have that:

(i) u^* is radially symmetric and decreasing;

(ii) u, $u^{\#}$ and u^* are all equimeasurable;

(iii) if $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-negative Borel measurable function, then

$$\int_\Omega F\big(u(x)\big)dx = \int_0^{|\Omega|} F\big(u^{\#}(s)\big)ds = \int_{\Omega^*} F\big(u^*(x)\big)dx \tag{8}$$

and, consequently,

$$||u||_{L^p(\Omega)} = ||u^{\#}||_{L^p([0,|\Omega|])} = ||u^*||_{L^p(\Omega^*)}; \tag{9}$$

(iv) for $1 \le p \le \infty$, the mapping $u \longmapsto u^*$ is continuous from $L^p(\Omega)$ into $L^p(\Omega^*)$, which means that, as before, if we are given a sequence $\{u_k\}_{k\in\mathbb{N}}$ such that $u_k(x) \longrightarrow u(x)$ in $L^p(\Omega)$, then also $u_k^*(x) \longrightarrow u^*(x)$ in $L^p(\Omega^*)$ (using mainly (8) and (9)).

These properties of u^* will be essential in the discussion of future results.

We now introduce some notions and results that will gradually take us in position to understand Pólya-Szegö's theorem. See also [4] and [19] for this part.

Definition 4. *Let $E \subseteq \Omega$ a measurable set. The* ***De Giorgi perimeter*** *of E with respect to Ω, denoted $P_\Omega(E)$, is defined as the total variation of the characteristic function χ_E of E:*

$$P_\Omega(E) := \sup_{\substack{\phi(x)\in(C_0^\infty(\Omega))^n \\ \phi(x)\not\equiv 0}} \left\{ \frac{\nabla\chi_E(x)\cdot\phi(x)}{||\phi||} \right\},$$

where $\phi(x) = \big(\phi_1(x), \dots, \phi_n(x)\big)$ and

$$||\phi|| := \sqrt{\max_{x\in\overline{\Omega}} \left\{ \sum_{i=1}^{n} |\phi_i(x)|^2 \right\}}.$$

Remark 4. *If E is a sufficiently smooth domain, we are considering $\nabla\chi_E$ as a singular measure supported on ∂E, due to χ_E being differentiable a.e. (and, obviously, $\nabla\chi_E$ being null a.e.). In other words, $P_\Omega(E)$ is the surface area of the part of ∂E where a normal can be unambiguously defined.*

Remark 5. *One has*

$$\int_\Omega \nabla\chi_E(x)\cdot\phi(x)dx = -\int_\Omega \chi_E(x)\,\mathrm{div}\big(\phi(x)\big)dx = -\int_E \mathrm{div}\big(\phi(x)\big)dx$$

after integrating by parts. Therefore,

$$P_\Omega(E) = \sup_{\substack{\phi(x)\in(C_0^\infty(\Omega))^n \\ \phi(x)\not\equiv 0}} \left\{ \frac{\int_E \mathrm{div}\big(\phi(x)\big)dx}{||\phi||} \right\}. \tag{10}$$

Proposition 5. *Let $E \subseteq \Omega \subseteq \Omega'$ be three measurable domains. Then:*

(i) if Ω is smooth, we have $P_\Omega(E) = P_\Omega(\Omega\backslash E)$;

(ii) $P_\Omega(E) \leq P_{\Omega'}(E)$.

Proof. Let $\phi(x) \in \big(C_0^\infty(\Omega)\big)^n$ such that $\phi(x) \not\equiv 0$. One has that

$$\int_{\Omega\backslash E} \mathrm{div}\big(\phi(x)\big)dx = \int_\Omega \mathrm{div}\big(\phi(x)\big)dx - \int_E \mathrm{div}\big(\phi(x)\big)dx =$$

$$= \int_{\partial\Omega} \phi(x)\cdot\nu dx - \int_E \mathrm{div}\big(\phi(x)\big)dx = -\int_E \mathrm{div}\big(\phi(x)\big)dx \tag{11}$$

by the divergence theorem, and so *(i)* follows by (10).
Instead, *(ii)* easily comes by the definition itself of the De Giorgi perimeter.

□

Remark 6. *We can thus remind the classical isoperimetric inequality in* $\mathbb{R}^n$ *for* $n \geq 2$*: given a bounded, measurable and sufficiently smooth domain* $\Omega \subseteq \mathbb{R}^n$ *and chosen a suitable* $(n-1)$*-dimensional surface measure of* $\partial\Omega$*, denoted by* $|\partial\Omega|_{n-1}$*, then*

$$|\partial\Omega|_{n-1} \geq n^{1-\frac{1}{n}} \omega_{n-1}^{\frac{1}{n}} |\Omega|^{1-\frac{1}{n}}. \tag{12}$$

Furthermore, equality is attained if and only if Ω *is a sphere. For us, the measure for the boundary of* Ω *will be the De Giorgi perimeter (here is the reason why we have introduced it). The relation* (12) *will be used primarily in the following way:*

$$\int_{\{u=t\}} d\sigma \geq \int_{\{u^*=t\}} d\sigma.$$

This is due to the fact that the classical isoperimetric inequality establishes that the n-dimensional sphere is the manifold which minimize the surface area among all the ones of fixed volume.

Lemma 1. *Let* u *be a function and let* $t \in \mathbb{R}$*. Define* $E_t := \{x \in \Omega : u(x) > t\}$ *and* $F_t := \{x \in \Omega : u(x) \leq t\} = \Omega \backslash E_t$*. Define also* $b : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ *such that*

$$b(t,x) := \begin{cases} -\chi_{F_t}(x) & \text{if } t \in (-\infty, 0) \\ \chi_{E_t}(x) & \text{if } t \in [0, +\infty) \end{cases}.$$

Then

$$u(x) = \int_{-\infty}^{+\infty} b(t,x)dt. \tag{13}$$

Proof. If $u(x) \geq 0$, we have

$$\int_{-\infty}^{+\infty} b(t,x)dt = \int_0^{u(x)} dt = u(x);$$

if $u(x) < 0$, we have

$$\int_{-\infty}^{+\infty} b(t,x)dt = -\int_{u(x)}^{0} dt = u(x).$$

□

We shall now enunciate two results that are special cases of the famous co-area formula (see also [6] and [7] for this), one of the most important fact of geometric measure theory, which we need for the proof of Pólya-Szegö's theorem. The following is the first one of them (the second one will be stated in the next section).

Theorem 3 (**Fleming-Rishel**). *If $u \in W^{1,1}(\Omega)$, then*

$$\int_\Omega |\nabla u(x)| dx = \int_{-\infty}^{+\infty} P_\Omega(\{u > t\}) dt \tag{14}$$

holds.

Proof. Let $\phi \in \big(C_0^\infty(\Omega)\big)^n$. By (13) and Fubini's theorem, we have

$$\int_\Omega u(x) \operatorname{div}\big(\phi(x)\big) dx = \int_\Omega \left(\int_{-\infty}^{+\infty} b(t,x) dt \right) \operatorname{div}\big(\phi(x)\big) dx =$$

$$= \int_{-\infty}^{+\infty} \left(\int_\Omega b(t,x) \operatorname{div}\big(\phi(x)\big) dx \right) dt =$$

$$= - \int_{-\infty}^{0} \left(\int_{F_t} \operatorname{div}\big(\phi(x)\big) dx \right) dt + \int_0^{+\infty} \left(\int_{E_t} \operatorname{div}\big(\phi(x)\big) dx \right) dt$$

using the definition of $b(t,x)$. Now, since $\phi \in \big(C_0^\infty(\Omega)\big)^n$, by (11) one has

$$\int_{F_t} \operatorname{div}\big(\phi(x)\big) dx = \int_{\Omega \setminus E_t} \operatorname{div}\big(\phi(x)\big) dx = - \int_{E_t} \operatorname{div}\big(\phi(x)\big) dx.$$

Thus,

$$\int_\Omega u(x) \operatorname{div}\big(\phi(x)\big) dx = - \int_{-\infty}^{0} \left(\int_{F_t} \operatorname{div}\big(\phi(x)\big) dx \right) dt + \int_0^{+\infty} \left(\int_{E_t} \operatorname{div}\big(\phi(x)\big) dx \right) dt =$$

$$= \int_{-\infty}^{+\infty} \left(\int_{E_t} \operatorname{div}\big(\phi(x)\big) dx \right) dt$$

and the thesis will follow if we take the supremum over $\phi(x) \not\equiv 0$.

□

We enunciate here the last result of this section, which is a simple consequence of Fleming-Rishel theorem. Nevertheless, it will be fundamental for future results.

Corollary 3. *If $u \in W^{1,1}(\Omega)$ is such that $u(x) \geq t_0$ a.e., then*

$$\int_\Omega |\nabla u(x)| dx = \int_{t_0}^{+\infty} P_\Omega(\{u > t\}) dt \tag{15}$$

holds.

Proof. For all $x \in \Omega$, if $t_0 < 0$, then

$$b(t,x) = \begin{cases} 0 & \text{if } t \in (-\infty, t_0) \\ -\chi_{F_t}(x) & \text{if } t \in [t_0, 0) \\ \chi_{E_t}(x) & \text{if } t \in [0, +\infty) \end{cases} ;$$

if $t_0 \geq 0$, then

$$b(t,x) = \begin{cases} 0 & \text{if } t \in (-\infty, 0) \\ 1 & \text{if } t \in [0, t_0) \\ \chi_{E_t}(x) & \text{if } t \in [t_0, +\infty) \end{cases} .$$

Therefore, applying (13), we get

$$u(x) = \int_{t_0}^{+\infty} b(t,x)dt \tag{16}$$

if $t_0 < 0$; instead, if $t_0 \geq 0$, we get

$$u(x) = t_0 + \int_{t_0}^{+\infty} b(t,x)dt. \tag{17}$$

The thesis will now follow as in Theorem 3 using (16) and (17) instead of (13).

□

Remark 7. *If u is integrable and if the integral appearing on the right side of* (14) *is finite, then, for $\phi \in \left(C_0^\infty(\Omega)\right)^n$, there exists a constant $c \geq 0$ such that*

$$\left| \int_\Omega u(x)\, \mathrm{div}\big(\phi(x)\big) dx \right| \leq c||\phi||.$$

This means that $u \in W^{1,1}(\Omega)$ and then, applying the previous result, (14) *holds. The same is true for* (15)*, obviously.*

1.2. Pólya-Szegö's theorem

In this second paragraph, we will gradually pass to the statement and the proof of Pólya-Szegö's theorem. We begin by two technical results.

Lemma 2. *Let f, g be two functions with g integrable over Ω and let f be such that $-\infty < a \leq f(x) \leq b \leq +\infty$. Then*

$$\int_\Omega f(x)g(x)dx = a\int_\Omega g(x)dx + \int_a^b \left(\int_{\{f>t\}} g(x)dx\right)dt. \tag{18}$$

Proof. Assume $a \geq 0$ (the other case can be similarly treated). Considering the set $E_t := \{x \in \Omega : f(x) > t\}$, by (13) we get

$$f(x) = \int_{-\infty}^{+\infty} b(t,x)dt = \int_0^b \chi_{E_t}(x)dt.$$

Hence, by Fubini's theorem,

$$\int_\Omega f(x)g(x)dx = \int_\Omega \left(\int_0^b \chi_{E_t}(x)dt\right)g(x)dx = \int_0^b \left(\int_\Omega g(x)\chi_{E_t}(x)dx\right)dt =$$

$$= \int_0^a \left(\int_\Omega g(x)dx\right)dt + \int_a^b \left(\int_{E_t} g(x)dx\right)dt = a\int_\Omega g(x)dx + \int_a^b \left(\int_{\{f>t\}} g(x)dx\right)dt.$$

□

Lemma 3. *Let $1 \leq p \leq \infty$ and q its conjugate exponent. Suppose $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. If we set*

$$F(t) := \int_{\{f>t\}} g(x)\big(f(x) - t\big)dx,$$

then

$$F'(t) = -\int_{\{f>t\}} g(x)dx.$$

Proof. Let $t \in \mathbb{R}$ and define the cut-off $h(x) := \big(f(x) - t\big)_+ + t$. Since $h(x) \geq t$ by definition, by (18) we have that

$$\int_\Omega g(x)h(x)dx = t\int_\Omega g(x)dx + \int_t^{+\infty} \left(\int_{\{h>\tau\}} g(x)dx\right)d\tau.$$

However, by the definition itself of the cut-off, we also have that

$$\int_\Omega g(x)h(x)dx = t\int_\Omega g(x)dx + \int_\Omega g(x)\big(f(x) - t\big)_+ dx =$$

$$= t \int_\Omega g(x)dx + \int_{\{f>t\}} g(x)\big(f(x) - t\big)dx.$$

Comparing these two relations, one has that

$$F(t) := \int_{\{f>t\}} g(x)\big(f(x) - t\big)dx = \int_t^{+\infty} \left(\int_{\{h>\tau\}} g(x)dx \right) d\tau. \tag{19}$$

Now, we show that $\{h > \tau\} = \{f > \tau\}$ for $\tau \geq t$:

- if $f(x) > \tau$, then obviously $h(x) > \tau$ as well;
- if $h(x) > \tau$, then $h(x) > t \Longrightarrow \big(f(x) - t\big)_+ > 0 \Longrightarrow f(x) > t \Longrightarrow \big(f(x) - t\big)_+ =$ $= f(x) - t \Longrightarrow h(x) \equiv f(x) \Longrightarrow f(x) > \tau$.

Therefore, (19) becomes

$$F(t) := \int_{\{f>t\}} g(x)\big(f(x) - t\big)dx = \int_t^{+\infty} \left(\int_{\{f>\tau\}} g(x)dx \right) d\tau.$$

Finally, differentiating both sides with respect to t, we get

$$F'(t) = \frac{d}{dt}\left(\int_t^{+\infty} \left(\int_{\{f>\tau\}} g(x)dx \right) d\tau \right) = - \int_{\{f>t\}} g(x)dx,$$

which is the thesis of the lemma.

□

The following theorem is the second result which, as said in the previous section, is a special case of the co-area formula.

Theorem 4. *Let* $1 \leq p < \infty$ *and let* $u \in C_0^\infty(\Omega)$ *such that* $u(x) \geq 0$. *Then*

$$\int_\Omega |\nabla u(x)|^p dx = \int_0^M \left(\int_{\{u=t\}} |\nabla u(x)|^{p-1} d\sigma \right) dt, \tag{20}$$

where $M := \max_{x \in \overline{\Omega}} \{u(x)\}$.

Proof. We divide the proof into three steps.

Step 1: an initial consideration. By hypothesis, u is a smooth function. Therefore, by Sard's theorem, $|\nabla u(x)| \neq 0$ for a.e. t on the level set $\{u = t\}$. Thus, $\{u = t\}$ can be taken as a $(n-1)$-dimensional surface such that $\{u = t\} = \partial\{u > t\}$. Also, thanks to Remark 3, $|\{u^* = t\}| = |\{u = t\}| = 0$.

Step 2: the case $2 \leq p < \infty$. For $2 \leq p < \infty$, define the p-Laplacian of f as

$$f(x) := - \operatorname{div}\big(|\nabla u(x)|^{p-2}\, \nabla u(x)\big).$$

If $u \in W_0^{1,p}(\Omega)$, then

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) dx = \int_\Omega f(x) v(x) dx. \tag{21}$$

Let $t > 0$. Setting $v(x) := \big(u(x) - t\big)_+ \in W_0^{1,p}(\Omega)$ in (21), we get

$$\int_{\{u>t\}} |\nabla u(x)|^p dx = \int_{\{u>t\}} f(x)\big(u(x) - t\big) dx.$$

Hence, differentiating with respect to t and using Lemma 3, one has

$$-\frac{d}{dt}\bigg(\int_{\{u>t\}} |\nabla u(x)|^p dx\bigg) = -\frac{d}{dt}\bigg(\int_{\{u>t\}} f(x)\big(u(x) - t\big) dx\bigg) = \int_{\{u>t\}} f(x) dx.$$

Integrating on $[0, M]$ (the range of u), we have that

$$\int_\Omega |\nabla u(x)|^p dx = \int_0^M \bigg(\int_{\{u>t\}} f(x) dx\bigg) dt. \tag{22}$$

Now, for a.e. $t \in [0, M]$, the consideration made in Step 1 holds and so, for such a t, by the definition of f and by the divergence theorem, we get

$$\int_{\{u>t\}} f(x) dx = -\int_{\{u=t\}} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nu \, d\sigma = \int_{\{u=t\}} |\nabla u(x)|^{p-1} d\sigma, \tag{23}$$

where we used the fact that, on the level set $\{u = t\}$, the tangential derivatives of u vanish and, since $u > t$ inside this surface, $-\nabla u(x) \cdot \nu = |\nabla u(x)|$. Putting (23) into (22), we have

$$\int_\Omega |\nabla u(x)|^p dx = \int_0^M \bigg(\int_{\{u=t\}} |\nabla u(x)|^{p-1} d\sigma\bigg) dt,$$

which is exactly (20).

<u>Step 3: the case $1 \le p < 2$.</u> For $1 \le p < 2$, we use an approximation technique in order to get something similar to the previous case. Introducing $\varepsilon > 0$, we define

$$f_\varepsilon(x) := -\operatorname{div}\Big[\big(|\nabla u(x)|^2 + \varepsilon\big)^{\frac{p-2}{2}} \nabla u(x)\Big].$$

Hence, $\forall\, v \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega \big(|\nabla u(x)|^2 + \varepsilon\big)^{\frac{p-2}{2}} \nabla u(x) \cdot \nabla v(x) dx = \int_\Omega f_\varepsilon(x) v(x) dx.$$

As before, for $v(x) := \big(u(x) - t\big)_+ \in W_0^{1,p}(\Omega)$, where $t > 0$, if we operate in the exact same way as in the previous step, we will get

$$\int_\Omega \left(|\nabla u(x)|^2+\varepsilon\right)^{\frac{p-2}{2}}|\nabla u(x)|^2dx = \int_0^M \left(\int_{\{u=t\}}\left(|\nabla u(x)|^2+\varepsilon\right)^{\frac{p-2}{2}}|\nabla u(x)|d\sigma\right)dt. \quad (24)$$

We now pass to the limit for $\varepsilon \longmapsto 0^+$ on both sides of the last relation. On the left side, the integrand converges pointwise to $|\nabla u(x)|^p$; reminding that $1 \le p < 2$, we have

$$\left(|\nabla u(x)|^2+\varepsilon\right)^{\frac{p-2}{2}}|\nabla u(x)|^2 = \left(\frac{|\nabla u(x)|^2}{|\nabla u(x)|^2+\varepsilon}\right)^{\frac{2-p}{2}}|\nabla u(x)|^p \le |\nabla u(x)|^p,$$

which is an integrable quantity over Ω. Hence, by the dominated convergence theorem,

$$\lim_{\varepsilon\to 0^+}\int_\Omega \left(|\nabla u(x)|^2+\varepsilon\right)^{\frac{p-2}{2}}|\nabla u(x)|^2dx = \int_\Omega |\nabla u(x)|^p dx. \quad (25)$$

Instead, on the right side of (24), we observe that

$$\left(|\nabla u(x)|^2+\varepsilon\right)^{\frac{p-2}{2}}|\nabla u(x)| \le |\nabla u(x)|^{p-1},$$

which is, again, integrable (this time, on the set $\{u = t\}$). Another application of the dominated convergence theorem yields to

$$\lim_{\varepsilon\to 0^+}\int_{\{u=t\}}\left(|\nabla u(x)|^2+\varepsilon\right)^{\frac{p-2}{2}}|\nabla u(x)|d\sigma = \int_{\{u=t\}}|\nabla u(x)|^{p-1}d\sigma.$$

Further,

$$\int_{\{u=t\}}\left(|\nabla u(x)|^2+\varepsilon\right)^{\frac{p-2}{2}}|\nabla u(x)|d\sigma \le \int_{\{u=t\}}|\nabla u(x)|^{p-1}d\sigma$$

and

$$\int_0^M\left(\int_{\{u=t\}}|\nabla u(x)|^{p-1}d\sigma\right)dt < +\infty,$$

since $u \in C_0^\infty(\Omega)$. Thus, applying again the dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0^+}\int_0^M \left(\int_{\{u=t\}}\left(|\nabla u(x)|^2+\varepsilon\right)^{\frac{p-2}{2}}|\nabla u(x)|d\sigma\right)dt =$$

$$= \int_0^M\left(\int_{\{u=t\}}|\nabla u(x)|^{p-1}d\sigma\right)dt. \quad (26)$$

This means that, due to (25) and (26), the relation (24) gives us (20) because now we have that

$$\int_\Omega |\nabla u(x)|^p dx = \int_0^M\left(\int_{\{u=t\}}|\nabla u(x)|^{p-1}d\sigma\right)dt.$$

□

Remark 8. *The same happens for* u^* *under the same hypothesis of Theorem 4, since* $M := \max_{x\in\overline{\Omega}}\{u(x)\} = \max_{x\in\overline{\Omega^*}}\{u^*(x)\}$*:*

$$\int_{\Omega^*} |\nabla u^*(x)|^p dx = \int_0^M \left(\int_{\{u^*=t\}} |\nabla u^*(x)|^{p-1} d\sigma \right) dt. \tag{27}$$

We enunciate here a last result before discussing Pólya-Szegö's theorem because it exemplifies its proof.

Theorem 5. *Let* $u \in C_0^\infty(\Omega)$ *such that* $u(x) \geq 0$*. Then, for a.e.* t *in the range of* u*,*

$$-\mu_u'(t) = \int_{\{u=t\}} \frac{d\sigma}{|\nabla u(x)|} = \int_{\{u^*=t\}} \frac{d\sigma}{|\nabla u^*(x)|} \tag{28}$$

holds.

Proof. We split the proof into four steps.

Step 1: an initial consideration. As before, for a.e. t, the same properties of Step 1 of the proof of the preceding theorem hold.

Step 2: introduction of an auxiliary function. For $\varepsilon > 0$, we define

$$f(x) := -\operatorname{div}\left(\frac{\nabla u(x)}{|\nabla u(x)|^2 + \varepsilon} \right).$$

Multiplying by $v(x) := \big(u(x) - t\big)_+$ and integrating by parts (using the fact that, since $u \in C_0^\infty(\Omega)$, the boundary terms are null and that $\nabla u(x) = \nabla v(x)$ if $u(x) > t$), we get

$$\int_{\{u>t\}} \frac{|\nabla u(x)|^2}{|\nabla u(x)|^2 + \varepsilon}\, dx = \int_{\{u>t\}} f(x)\big(u(x) - t\big) dx.$$

Differentiating now with respect to t and by Lemma 3, we have

$$-\frac{d}{dt}\left(\int_{\{u>t\}} \frac{|\nabla u(x)|^2}{|\nabla u(x)|^2 + \varepsilon}\, dx \right) = -\frac{d}{dt}\left(\int_{\{u>t\}} f(x)\big(u(x) - t\big) dx \right) =$$

$$= \int_{\{u>t\}} f(x) dx, \tag{29}$$

relation which will help us in the following step.

Step 3: the thesis for u. Let t be such that $|\nabla u(x)| \neq 0$ on the set $\{u = t\}$. Since $|\nabla u(x)| \not\equiv 0$ for a.e. t, if $h > 0$ is small enough, the same is true regarding the set

$\{t-h \leq u \leq t+h\}$. Then, integrating (29) from $t-h$ to t, one has

$$\int_{\{t-h<u\leq t\}} \frac{|\nabla u(x)|^2}{|\nabla u(x)|^2+\varepsilon}\,dx = \int_{\{u>t-h\}} \frac{|\nabla u(x)|^2}{|\nabla u(x)|^2+\varepsilon}\,dx - \int_{\{u>t\}} \frac{|\nabla u(x)|^2}{|\nabla u(x)|^2+\varepsilon}\,dx =$$

$$= \int_{t-h}^{t} -\frac{d}{dt}\left(\int_{\{u>\tau\}} \frac{|\nabla u(x)|^2}{|\nabla u(x)|^2+\varepsilon}\,dx\right)d\tau = \int_{t-h}^{t}\left(\int_{\{u>\tau\}} f(x)dx\right)d\tau =$$

$$= \int_{t-h}^{t}\left(\int_{\{u=\tau\}} -\frac{\nabla u(x)}{|\nabla u(x)|^2+\varepsilon}\cdot \nu\, d\sigma\right)d\tau = \int_{t-h}^{t}\left(\int_{\{u=\tau\}} \frac{|\nabla u(x)|}{|\nabla u(x)|^2+\varepsilon}\,d\sigma\right)d\tau,$$

using the definition of f, the divergence theorem and the sign of u (which is such that $\nabla u = -\nu|\nabla u|$ since $\nabla u \perp \{u=t\}$). Now, due to the dominated convergence theorem, we can pass to the limit for $\varepsilon \longmapsto 0^+$ in order to obtain

$$\mu_u(t-h)-\mu_u(t) = \int_{\{t-h<u\leq t\}} dx = \lim_{\varepsilon\to 0^+}\int_{\{t-h<u\leq t\}} \frac{|\nabla u(x)|^2}{|\nabla u(x)|^2+\varepsilon}\,dx =$$

$$= \lim_{\varepsilon\to 0^+}\int_{t-h}^{t}\left(\int_{\{u=\tau\}} \frac{|\nabla u(x)|}{|\nabla u(x)|^2+\varepsilon}\,d\sigma\right)d\tau = \int_{t-h}^{t}\left(\int_{\{u=\tau\}} \frac{d\sigma}{|\nabla u(x)|}\right)d\tau.$$

Further, dividing by h and taking the limit for $h \longmapsto 0^+$, we establish (by Lebegue's differentiation theorem) that

$$-\mu_u'(t) = \int_{\{u=t\}} \frac{d\sigma}{|\nabla u(x)|},$$

which is the first relation in (28).

Step 4: the thesis for u^*. Let $r = r(t)$ be the radius of the ball given by the level set $\{u^* > t\}$. If this is the case, then $\mu_{u^*}(t) = \mu_u(t) = \frac{\omega_{n-1}}{n}\, r(t)^n$, which means that $\mu_{u^*}'(t) = \omega_{n-1} r(t)^{n-1} r'(t)$. Since $\mu_{u^*}(t)$ and $r(t)$ are both monotonically decreasing functions, they are differentiable a.e. on their domains of definition. Hence, writing $u^*(x)$ as $u^*(|x|)$ by abuse of notation and, therefore, considering it as a function of a single variable (being able to do that in view of the previously studied properties of the Schwarz symmetrization, in particular the radial symmetry and its domain of definition which is a ball), we have that $u^*\big(r(t)\big) = t$ a.e. due to the properties of the rearrangement. In conclusion, by implicit differentiation,

$$\mu_u'(t) = \mu_{u^*}'(t) = |\{u^*=t\}|_{n-1}\,\frac{1}{(u^*)'\big(r(t)\big)} = -\frac{|\{u^*=t\}|_{n-1}}{\big|\nabla u^*|_{\{u^*=t\}}\big|} = -\int_{\{u^*=t\}} \frac{d\sigma}{|\nabla u^*(x)|},$$

which is the second relation in (28).

□

We are finally able to enunciate and prove Pólya-Szegö's theorem, one of the most important result concerning the Schwarz symmetrization.

Theorem 6 (Pólya-Szegö). *Let $1 \leq p < \infty$ and let $u \in W_0^{1,p}(\Omega)$ be such that $u(x) \geq 0$. Then $u^* \in W_0^{1,p}(\Omega)$ and we have that*

$$\int_{\Omega^*} |\nabla u^*(x)|^p dx \leq \int_\Omega |\nabla u(x)|^p dx. \tag{30}$$

Proof. We split the proof noting that, if (30) holds, then $u^* \in W_0^{1,p}(\Omega)$ automatically.

Step 1: the case $p = 1$. For $p = 1$, we show that $P_\Omega(\{u > t\}) = P_{\mathbb{R}^N}(\{u > t\})$ for any $t > 0$: in fact, being $u \in W_0^{1,1}(\Omega)$ such that $u(x) \geq 0$ a.e. over Ω by hypothesis, we can extend it by setting $\tilde{u} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\tilde{u}(x) := u(x)\chi_\Omega(x)$ in order to get $\{x \in \mathbb{R}^n : \tilde{u}(x) > t\} = \{x \in \Omega : u(x) > t\}$. Thus, by (15),

$$\int_0^{+\infty} P_\Omega(\{u > t\})dt = \int_\Omega |\nabla u(x)|dx = \int_{\mathbb{R}^N} |\nabla \tilde{u}(x)|dx = \int_0^{+\infty} P_{\mathbb{R}^N}(\{\tilde{u} > t\})dt =$$

$$= \int_0^{+\infty} P_{\mathbb{R}^N}(\{u > t\})dt,$$

from which we have $P_\Omega(\{u > t\}) = P_{\mathbb{R}^N}(\{u > t\})$, since $P_\Omega(\{u > t\}) \leq P_{\mathbb{R}^N}(\{u > t\})$ in general in view of the point *(ii)* of Proposition 5. After doing that, using the classical isoperimetric inequality, for every $t > 0$ one has that $P_{\mathbb{R}^N}(\{u > t\}) \geq P_{\mathbb{R}^N}(\{u^* > t\})$ for the same consideration made in Remark 6 (because the set $\{u^* > t\}$ is a sphere). Therefore, using again (15) and having in mind the discussion made in Remark 7, we achieve the estimate

$$\int_\Omega |\nabla u(x)|dx = \int_0^{+\infty} P_\Omega(\{u > t\})dt = \int_0^{+\infty} P_{\mathbb{R}^N}(\{u > t\})dt \geq$$

$$\geq \int_0^{+\infty} P_{\mathbb{R}^N}(\{u^* > t\})dt \geq \int_0^{+\infty} P_{\Omega^*}(\{u^* > t\})dt = \int_{\Omega^*} |\nabla u^*(x)|dx,$$

which is the desired relation.

Step 2: the case $1 < p < \infty$ for $u \in C_0^\infty(\Omega)$. Setting

$$M := \max_{x \in \overline{\Omega}}\{u(x)\} = \max_{x \in \overline{\Omega^*}}\{u^*(x)\},$$

it suffices to show that, if $1 < p < \infty$ and $u \in C_0^\infty(\Omega)$, then

$$\int_{\{u^*=t\}} |\nabla u^*(x)|^{p-1} d\sigma \leq \int_{\{u=t\}} |\nabla u(x)|^{p-1} d\sigma$$

for a.e. $t \in [0, M]$ in view of (20) and (27), using the monotonicity of the integral. The function u is smooth: so, by Sard's theorem and the same considerations done earlier, $|\nabla u|$ does not vanish on the set $\{u = t\}$ for a.e. $t \in [0, M]$. Introducing now a measure ν on $\{u = t\}$ by $d\nu := \frac{d\sigma}{|\nabla u|}$, Hölder's inequality leads to

$$\int_{\{u=t\}} |\nabla u(x)| d\nu \leq \Big(\int_{\{u=t\}} |\nabla u(x)|^p d\nu \Big)^{\frac{1}{p}} \Big(\int_{\{u=t\}} d\nu \Big)^{\frac{p-1}{p}}.$$

Thus, by the definition of the measure ν and using again the classical isoperimetric inequality, we have

$$\int_{\{u=t\}} |\nabla u(x)|^{p-1} d\sigma = \int_{\{u=t\}} |\nabla u(x)|^p d\nu \geq \frac{\big(\int_{\{u=t\}} |\nabla u(x)| d\nu \big)^p}{\big(\int_{\{u=t\}} d\nu \big)^{p-1}} = \frac{\big(\int_{\{u=t\}} d\sigma \big)^p}{\big(\int_{\{u=t\}} d\nu \big)^{p-1}} \geq$$

$$\geq \frac{\big(\int_{\{u^*=t\}} d\sigma \big)^p}{\big(\int_{\{u=t\}} d\nu \big)^{p-1}} = \frac{|\{u^* = t\}|_{n-1}^p}{\big(- \mu_u'(t) \big)^{p-1}} = |\{u^* = t\}|_{n-1}^p \Bigg(\frac{\big| \nabla u^* |_{\{u^*=t\}} \big|}{|\{u^* = t\}|_{n-1}} \Bigg)^{p-1} =$$

$$= |\{u^* = t\}|_{n-1} \big| \nabla u^* |_{\{u^*=t\}} \big|^{p-1} = \int_{\{u^*=t\}} |\nabla u^*(x)|^{p-1} d\sigma,$$

where we make also use of (28).

Step 3: the case $1 < p < \infty$ for $u \in W_0^{1,p}(\Omega)$. The thesis for the case $1 < p < \infty$ and $u \in W_0^{1,p}(\Omega)$ will follow by density from the previous step: in fact, we know there exists a sequence of functions $\{u_k\}_{k \in \mathbb{N}}$ such that $u_k \in C_0^\infty(\Omega)$, $u_k(x) \geq 0$ and, lastly, $u_k(x) \longrightarrow u(x)$ in $W_0^{1,p}(\Omega)$, $\forall\, k \in \mathbb{N}$. Besides, by the previous step, we have that

$$\int_{\Omega^*} |\nabla u_k^*(x)|^p dx \leq \int_\Omega |\nabla u_k(x)|^p dx.$$

Hence, the sequence $\{u_k^*\}_{k \in \mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega^*)$: this means (since $1 < p < +\infty$) that there exists a weakly convergent subsequence which, by Rellich's compactness theorem (see [9]), also converges strongly in $L^p(\Omega^*)$. Nevertheless, we already know (see the point *(iv)* of Remark 3) that $u_k^*(x) \longrightarrow u^*(x)$ in $L^p(\Omega^*)$: therefore, we deduce that $u^* \in W_0^{1,p}(\Omega^*)$ and $u_k^*(x) \rightharpoonup u^*(x)$ in that space. In conclusion, by the weak lower semi-continuity of the norm, we get

$$\int_{\Omega^*} |\nabla u^*(x)|^p dx \leq \liminf_{k \to +\infty} \int_{\Omega^*} |\nabla u_k^*(x)|^p dx \leq \liminf_{k \to +\infty} \int_\Omega |\nabla u_k(x)|^p dx = \int_\Omega |\nabla u(x)|^p dx.$$

□

1.3. Moser's theorem on exponential integrability

The main result of this section is the following theorem. After stating it, we will deduce an outcome of it.

Theorem 7 (**Moser**). *Let $n \in \mathbb{N}_{\geq 2}$ and $u \in W_0^{1,n}(\Omega)$. Assume that*

$$||\nabla u||_{L^n(\Omega)} := \left(\int_\Omega |\nabla u(x)|^n dx \right)^{\frac{1}{n}} \leq 1.$$

Then, $\forall\, \alpha \in [0, \alpha_n]$, there exists a constant $c = c(n)$ depending on n only such that

$$\fint_\Omega e^{\alpha |u(x)|^p} dx \leq c, \tag{31}$$

where $p := \frac{n}{n-1}$ is the conjugate exponent of n and $\alpha_n := n\omega_{n-1}^{\frac{1}{n-1}}$.

An immediate consequence of this theorem is the following result, whose proof will be postponed. It establishes the sharpness of α_n, namely that (31) is no more valid if we choose α strictly greater than α_n.

Corollary 4. *The integral on the left side of* (31) *is actually finite for every choice of $\alpha \geq 0$ but, if $\alpha > \alpha_n$, it can become arbitrarily large by an appropriate choice of u in the sense that, for $\alpha > \alpha_n$, we have*

$$\sup_{\substack{u \in W_0^{1,n}(\Omega) \\ ||\nabla u||_{L^n(\Omega)} \leq 1}} \left\{ \fint_\Omega e^{\alpha |u(x)|^p} dx \right\} = +\infty.$$

In other words, if $\alpha > \alpha_n$, then the constant $c = c(n, u)$ is forced to depend on the function u taken into consideration as well.

In order to prove these results, we use the same methods of [18], leading back to the symmetrization of a function and its decreasing rearrangement, in particular the Schwarz symmetrization. This technique will help us greatly, transforming the general n-dimensional problem we have to deal with into a unidimensional one. The case $\alpha > \alpha_n$ is quite simple as well, and one can easily construct counter-examples. However, the limit case $\alpha = \alpha_n$ is not completely trivial and it requires a different strategy. We point out again the remarkable outcome that the result still holds even for the critical value α_n itself, which can be considered a watershed. Like the Sobolev inequalities mentioned in the introduction, Moser's theorem is helpful if dealing with non-linear partial differential equations as well as in Berger's study of conformal deformation of surfaces (see [2]). In particular, Sobolev's theorems allow to deal with equations of the type $-\Delta u = |u|^{p-1} u$, while Moser's with the ones of the type $-\Delta u = e^u$.

Proof of Theorem 7. We divide the proof of Moser's theorem in different steps in order to lighten the discourse and to avoid losing the thread of the discussion.

Step 1: rewriting the problem. Without any loss of generality, we are able to consider the function u non-negative because, if it is not the case, we are allowed to replace u by $|u|$ since this assumption does not increase the value of the integral of $|\nabla u|^n$: in fact, if $u \in W_0^{1,n}(\Omega)$, then $|u| \in W_0^{1,n}(\Omega)$ and

$$\int_\Omega |\nabla u(x)|^n dx = \int_\Omega \big|\nabla(|u(x)|)\big|^n dx.$$

This is helpful because, in such a case, we have $u^* \in W_0^{1,n}(\Omega)$. Clearly, another (banal) observation is that assuming $||\nabla u||_{L^n(\Omega)} \le 1$ is the same as assuming $||\nabla u||^n_{L^n(\Omega)} \le 1$. As previously said, we now pass to the Schwarz symmetrization u^* of u, reminding that

$$\int_{\Omega^*} |\nabla u^*(x)|^n dx \le \int_\Omega |\nabla u(x)|^n dx$$

due to (30) and

$$\int_{\Omega^*} e^{\alpha|u^*(x)|^p} dx = \int_\Omega e^{\alpha|u(x)|^p} dx$$

in view of (8). Thus the convenience of all this reasoning: in this way, we have reduced the dimension from n to 1. Now, depending u^* on $|x|$ and denoting with R the radius of Ω^*, for convenience we introduce a new variable t by

$$t := n \log\left(\frac{R}{|x|}\right) \Longleftrightarrow |x| = Re^{-\frac{t}{n}}. \tag{32}$$

Furthermore, if

$$w(t) := \alpha_n^{\frac{1}{p}} u^*\Big(Re^{-\frac{t}{n}}\Big),$$

then $w(t)$ is monotone increasing by definition, since $u^*(|x|)$ is monotone decreasing and for (32). So, we have that

$$|w'(t)|^n = \left|n^{\frac{n-1}{n}} \omega_{n-1}^{\frac{1}{n}}\Big(-\frac{R}{n}\Big)e^{-\frac{t}{n}}\, \nabla u^*\Big(Re^{-\frac{t}{n}}\Big)\right|^n = \frac{\omega_{n-1}}{n}\, R^n e^{-t}\Big|\nabla u^*\Big(Re^{-\frac{t}{n}}\Big)\Big|^n,$$

from which it follows that

$$\int_{\Omega^*} \big|\nabla u^*(|x|)\big|^n dx = \int_0^R \omega_{n-1} r^{n-1} |\nabla u^*(r)|^n dr =$$

$$= \int_0^{+\infty} \frac{\omega_{n-1}}{n}\, R^n e^{-t}\Big|\nabla u^*\Big(Re^{-\frac{t}{n}}\Big)\Big|^n dt = \int_0^{+\infty} |w'(t)|^n dt. \tag{33}$$

At this point, reminding that the n-dimensional volume $|B_1(0)|$ of the unit ball in $\mathbb{R}^n$

equals to $n\omega_{n-1}$, we get

$$\fint_{\Omega^*} e^{\alpha u^*(x)^p} dx = \int_0^R \frac{n}{R^n} r^{n-1} e^{\alpha u^*(r)^p} dr = \int_0^{+\infty} e^{\alpha u^*\left(Re^{-\frac{t}{n}}\right)^p - t} dt =$$

$$= \int_0^{+\infty} e^{\beta w(t)^p - t} dt, \tag{34}$$

where β is defined by $\beta := \frac{\alpha}{\alpha_n}$.

Consequently, thanks to (33) and (34), the theorem is equivalent to show that there exists a constant $c_1 = c_1(n)$ depending on n only such that

$$\int_0^{+\infty} e^{\beta w(t)^p - t} dt \leq c_1 \tag{35}$$

holds for every $\beta \in [0, 1]$, under the hypothesis that $n \geq 2$ is the conjugate exponent of p and $w(t) \in C^1\big([0, +\infty]\big)$ a.e. satisfies:

(i) $w(0) = 0$;

(ii) $w'(t) \geq 0$, for a.e. $t \geq 0$;

(iii) $\int_0^{+\infty} w'(t)^n dt \leq 1$.

We now split the case $\beta \in [0, 1)$ from the limit one in which $\beta = 1$.

Step 2: the case $\beta \in [0, 1)$. We shall begin considering the case $\beta \in [0, 1)$, whose proof is immediate since we get

$$w(t) = \int_0^t w'(\tau)d\tau \leq \left(\int_0^t d\tau\right)^{\frac{1}{p}} \left(\int_0^t w'(\tau)^n d\tau\right)^{\frac{1}{n}} = t^{\frac{1}{p}} \left(\int_0^t w'(\tau)^n d\tau\right)^{\frac{1}{n}} \leq t^{\frac{1}{p}} \tag{36}$$

from Hölder's inequality, and so

$$\int_0^{+\infty} e^{\beta w(t)^p - t} dt \leq \int_0^{+\infty} e^{\beta t - t} dt = \int_0^{+\infty} e^{(\beta - 1)t} dt = \frac{1}{\beta - 1}\Big[e^{(\beta-1)t}\Big]_0^{+\infty} = \frac{1}{1 - \beta}.$$

Then, in this case, the theorem is proven setting $c_1 = c_1(n) := \frac{1}{1-\beta}$.

Step 3: introduction of a useful rescalement. What remains to show is the limit case where $\beta = 1$. Due to the density of simple function in $L^n(\Omega)$, we may assume $w'(t)$ to be piecewise constant and compactly supported, namely $w(t)$ to be piecewise affine and constant for large t. In particular, we have

$$\frac{w(t)}{t^{\frac{1}{p}}} \longrightarrow 0$$

for both $t \longmapsto 0^+$ and $t \longmapsto +\infty$: therefore, $t^{1-n} w(t)^n$ attains its maximum in $(0, +\infty)$.

So let us set $\xi > 0$, which is the point where the maximum is attained, and δ such that

$$\delta := 1 - \max_{t>0}\{t^{1-n}w(t)^n\} = 1 - \xi^{1-n}w(\xi)^n \iff 1 - \delta = \xi^{1-n}w(\xi)^n. \tag{37}$$

By definition, we have that $\delta \in [0,1]$, because

$$t^{-\frac{1}{p}}w(t) \le 1 \iff 1 \ge \left(t^{-\frac{1}{p}}w(t)\right)^n = t^{-\frac{n}{p}}w(t)^n = t^{1-n}w(t)^n$$

due to (36) (and, obviously, because $t^{1-n}w(t)^n \ge 0$). Let us make some comments about δ: the relevant case is when δ is near zero. In fact, if $\delta = 1$, then the theorem is trivial because $w(t)$ would be the function identically null; if $\delta \in [\delta_0, 1)$, for a certain $\delta_0 \in (0,1)$ small enough, by definition we have that

$$t^{1-n}w(t)^n \le 1 - \delta \iff \frac{w(t)}{t^{\frac{1}{p}}} \le (1-\delta)^{\frac{1}{n}},$$

which means

$$\int_0^{+\infty} e^{w(t)^p - t}dt \le \int_0^{+\infty} e^{(1-\delta)^{\frac{1}{n-1}}t - t}dt \le \int_0^{+\infty} e^{\left[(1-\delta_0)^{\frac{1}{n-1}} - 1\right]t}dt = \frac{1}{1 - (1-\delta_0)^{\frac{1}{n-1}}}.$$

It follows that we can have in mind that $\delta \in [0, \delta_0)$, where δ_0 is in close proximity of zero. We now scale the variables by introducing a new one and setting, for convenience,

$$s := \frac{t}{\xi} \implies w(t) = \xi^{1-\frac{1}{n}}y(s) = \xi^{\frac{1}{p}}y(s) \tag{38}$$

or, equivalently,

$$y(s) = \xi^{-\frac{1}{p}}w(\xi s).$$

Doing this, we have rescaled the problem setting in 1 the watershed point that separates the two different ways the function behaves. The three conditions have, this way, become:

(i) $y(0) = \xi^{-\frac{1}{p}}w(0) = 0$;

(ii) $y'(s) = \xi^{1-\frac{1}{p}}w'(\xi s) \ge 0$, for a.e. $s \ge 0$;

(iii) $\int_0^{+\infty} y'(s)^n ds = \int_0^{+\infty}\left(\xi^{1-\frac{1}{p}}w'(\xi s)\right)^n ds = \int_0^{+\infty} \xi w'(\xi s)^n ds = \int_0^{+\infty} w'(t)^n dt \le 1.$

Clearly, in the last condition we changed variables by setting $t := \xi s$.
Besides, we have

$$s^{1-n}y(s)^n = \left(\frac{t}{\xi}\right)^{1-n}\xi^{-\frac{n}{p}}w(t)^n = t^{1-n}w(t)^n \le 1 - \delta = \xi^{1-n}w(\xi)^n =$$

$$= \left(\xi^{-\frac{1}{p}} w(t)\Big|_{s=1}\right)^n = y_1^n, \tag{39}$$

where $y_1 := y(1)$.

Step 4: introduction of three useful estimates. We want now to show that, if δ is chosen small enough (as discussed earlier), the function $y(s)$ cannot be much bigger than $\eta(s) := \min\{1, s\}$. However, before proceeding, we will prove three inequalities useful for later: the first two are

$$(n-1)y_1^{n-2} \int_0^1 \left(y'(s) - y_1\right)^2 ds + \int_1^{+\infty} y'(s)^n ds \leq \delta \tag{40}$$

and

$$y(s) \leq z(s), \tag{41}$$

with

$$z(s) := \begin{cases} s + \min\left\{(2\delta)^{\frac{1}{n}} s^{\frac{1}{p}}, c_3\big(\delta(1-s)\big)^{\frac{1}{2}}\right\} & \text{if } s \in [0,1] \\ 1 + \delta^{\frac{1}{n}}(s-1)^{\frac{1}{p}} & \text{if } s \in (1, +\infty) \end{cases},$$

where $c_3 = c_3(n)$ is a positive constant depending on n only. We specify that (41) is valid for $\delta \in \left[0, \frac{1}{2}\right)$. The last one is the following: defining the function

$$\varphi(s) := \begin{cases} s & \text{if } s \in \left[0, \frac{1}{2}\right) \\ |s-1| & \text{if } s \in \left[\frac{1}{2}, +\infty\right) \end{cases},$$

there exist positive constants $\delta_0 < \frac{1}{2}$, $c_4 = c_4(n)$ and $c_5 = c_5(n)$ depending on n only such that, $\forall\, \delta \in (0, \delta_0)$ and $\forall\, s \in [0, +\infty) \setminus I$, where $I := \{s \in [0, +\infty) \,:\, |s-1| < c_4\delta\}$,

$$z(s)^p - s \leq -c_5^{-1}\varphi(s) \tag{42}$$

holds. In order to achieve them, we use the following elementary estimates: given $n \geq 2$ and a, b such that $a + b \geq 0$, we have

$$a^n + na^{n-1}b + (n-1)a^{n-2}b^2 \leq (a+b)^n; \tag{43}$$

if, moreover, $a, b \geq 0$, then

$$a^n + na^{n-1}b + b^n \leq (a+b)^n \leq a^n + c_2(a^{n-1}b + b^n), \tag{44}$$

where $c_2 = c_2(n)$ is a positive constant depending on n only (the upper estimate holds even for $n \geq 1$, but we are focused on $n \geq 2$).

Step 5: proof of the first estimate. Setting $a = y_1 \geq 0$ and $b = y'(s) - y_1$ in (43) (being allowed since $a+b=y'(s)\geq 0$), we get

$$y_1^n + ny_1^{n-1}\big(y'(s)-y_1\big) + (n-1)y_1^{n-2}\big(y'(s)-y_1\big)^2 \leq y'(s)^n.$$

If we integrate now over $s\in[0,1]$, we obtain

$$\int_0^1 y'(s)^n ds \geq \int_0^1 \Big[y_1^n + ny_1^{n-1}\big(y'(s)-y_1\big) + (n-1)y_1^{n-2}\big(y'(s)-y_1\big)^2\Big] ds =$$

$$= y_1^n \int_0^1 ds + ny_1^{n-1}\int_0^1 y'(s)ds - ny_1^n \int_0^1 ds + (n-1)y_1^{n-2}\int_0^1 \big(y'(s)-y_1\big)^2 ds =$$

$$= y_1^n + ny_1^{n-1}\big(y_1 - y(0)\big) - ny_1^n + (n-1)y_1^{n-2}\int_0^1 \big(y'(s)-y_1\big)^2 ds =$$

$$= y_1^n + (n-1)y_1^{n-2}\int_0^1 \big(y'(s)-y_1\big)^2 ds.$$

Using the conditions on $y(s)$ and (39), we are able to write

$$\int_0^1 y'(s)^n ds = \int_0^{+\infty} y'(s)^n ds - \int_1^{+\infty} y'(s)^n ds \leq 1 - \int_1^{+\infty} y'(s)^n ds =$$

$$= \delta + y_1^n - \int_1^{+\infty} y'(s)^n ds$$

and, putting the pieces together, we get

$$\delta + y_1^n \geq \int_0^1 y'(s)^n ds + \int_1^{+\infty} y'(s)^n ds \geq$$

$$\geq y_1^n + (n-1)y_1^{n-2}\int_0^1 \big(y'(s)-y_1\big)^2 ds + \int_1^{+\infty} y'(s)^n ds \Longleftrightarrow$$

$$\Longleftrightarrow (n-1)y_1^{n-2}\int_0^1 \big(y'(s)-y_1\big)^2 ds + \int_1^{+\infty} y'(s)^n ds \leq \delta, \tag{45}$$

which is exactly (40).

Step 6: proof of the second estimate. From this, it must be that both summands in (40) are less or equal to δ thanks to the hypothesis on $y(s)$. This means that, applying Hölder's inequality, we find

$$y(s) = \int_0^1 y'(r)dr + \int_1^s y'(r)dr \leq y_1 - y(0) + \left(\int_1^s dr\right)^{\frac{1}{p}} \left(\int_1^s y'(r)^n dr\right)^{\frac{1}{n}} \leq$$

$$\leq y_1 + (s-1)^{\frac{1}{p}} \left(\int_1^{+\infty} y'(r)^n dr\right)^{\frac{1}{n}} \leq y_1 + \delta^{\frac{1}{n}}(s-1)^{\frac{1}{p}}$$

if $s > 1$. This proves (41) in the case $s > 1$. Similarly, if $s \in [0,1]$, Cauchy-Schwarz inequality, (39) and (45) yield to

$$y(s) = \int_0^s \big(y_1 + y'(r) - y_1\big)dr = y_1 s + \int_0^s \big(y'(r) - y_1\big)dr =$$

$$= (1-\delta)^{\frac{1}{n}} s - \int_0^1 \big(y'(r) - y_1\big)dr + \int_0^s \big(y'(r) - y_1\big)dr \leq s - \int_s^1 \big(y'(r) - y_1\big)dr \leq$$

$$\leq s + \left(\int_s^1 \big(y'(r) - y_1\big)^2 dr\right)^{\frac{1}{2}} (1-s)^{\frac{1}{2}} \leq s + \left(\int_0^1 \big(y'(r) - y_1\big)^2 dr\right)^{\frac{1}{2}} (1-s)^{\frac{1}{2}} \leq$$

$$\leq s + \left[\frac{\delta}{(n-1)y_1^{n-2}}\right]^{\frac{1}{2}} (1-s)^{\frac{1}{2}} = s + \left[\frac{\delta}{(n-1)(1-\delta)^{n-2}}\right]^{\frac{1}{2}} (1-s)^{\frac{1}{2}} \leq s + c_3\big(\delta(1-s)\big)^{\frac{1}{2}},$$

for a positive constant $c_3 = c_3(n)$.
To complete the proof of (41), it suffices to show that

$$y(s) \leq s + (2\delta)^{\frac{1}{n}} s^{\frac{1}{p}} \tag{46}$$

if $s \in [0,1]$. We fix $\sigma \in (0,1)$ and maximize $y(\sigma)$ for all functions $y(s)$ satisfying:

(i) $y(0) = 0$;

(ii) $y(1) = y_1$;

(iii) $\int_0^1 |y'(s)|^n ds \leq 1$.

This way, the maximum $y^*(s)$ is attained for an "extremal" function: this means that $y^*(s)$ is a segment connecting $(0,0)$ with $\big(\sigma, y^*(\sigma)\big)$ and another one passing from $\big(\sigma, y^*(\sigma)\big)$ to $(1,1)$. Thus,

$$y^*(s) = \begin{cases} \frac{y^*(\sigma)}{\sigma}\, s & \text{if } s \in [0,\sigma] \\ y^*(\sigma) + \frac{y_1 - y^*(\sigma)}{1-\sigma}\,(s-\sigma) & \text{if } s \in (\sigma,1] \end{cases},$$

and so

$$|(y^*)'(s)|^n = \begin{cases} \left(\frac{y^*(\sigma)}{\sigma}\right)^n & \text{if } s \in [0,\sigma] \\ \left|\frac{y_1 - y^*(\sigma)}{1-\sigma}\right|^n & \text{if } s \in (\sigma,1] \end{cases}.$$

The integral condition on $y^*(\sigma)$ has therefore became

$$\left(\frac{y^*(\sigma)}{\sigma}\right)^n \sigma + \left|\frac{y_1 - y^*(\sigma)}{1-\sigma}\right|^n (1-\sigma) \leq 1. \tag{47}$$

Besides, by definition of $y^*(s)$, we have $\frac{y^*(\sigma)}{\sigma} \geq y_1$: so we can affirm that $\exists\, \rho \geq 0$ such

that $y^*(\sigma) = y_1(\sigma+\rho)$. Essentially, this is due to the maximization of $y(\sigma)$: in fact, the function $\tilde{y}(\sigma) := y_1\sigma$ satisfies the three condition and so $y^*(\sigma) \geq \tilde{y}(\sigma) = y_1\sigma$.
We are now able to rewrite (47) as

$$1 \geq \left(\frac{y_1(\sigma+\rho)}{\sigma}\right)^n \sigma + \left|\frac{y_1 - y_1(\sigma+\rho)}{1-\sigma}\right|^n (1-\sigma) =$$

$$= y_1^n\left[\left(1+\frac{\rho}{\sigma}\right)^n \sigma + \left|1 - \frac{\rho}{1-\sigma}\right|^n (1-\sigma)\right] \Longleftrightarrow$$

$$\Longleftrightarrow \left(1+\frac{\rho}{\sigma}\right)^n \sigma + \left|1-\frac{\rho}{1-\sigma}\right|^n (1-\sigma) \leq \frac{1}{y_1^n} = \frac{1}{1-\delta}.$$

Using (47) and another elementary estimate given, this time, by $|1-x|^n \geq 1-nx$, valid $\forall\, x \in \mathbb{R}$ and $\forall\, n \geq 1$, we obtain

$$\frac{1}{1-\delta} \geq \left(1+\frac{\rho}{\sigma}\right)^n \sigma + \left|1-\frac{\rho}{1-\sigma}\right|^n (1-\sigma) \geq$$

$$\geq \left[1 + n\,\frac{\rho}{\sigma} + \left(\frac{\rho}{\sigma}\right)^n\right]\sigma + \left(1 - n\,\frac{\rho}{1-\sigma}\right)(1-\sigma) =$$

$$= \sigma + n\rho + \frac{\rho^n}{\sigma^{n-1}} + 1 - \sigma - n\,\frac{\rho}{1-\sigma} + n\sigma\,\frac{\rho}{1-\sigma} = \frac{\rho^n}{\sigma^{n-1}} + 1 \Longleftrightarrow$$

$$\Longleftrightarrow \frac{\rho^n}{\sigma^{n-1}} \leq \frac{1}{1-\delta} - 1 = \frac{\delta}{1-\delta} < 2\delta,$$

since we assumed $\delta \in \left[0, \frac{1}{2}\right)$. Hence, as $y_1 = (1-\delta)^{\frac{1}{n}} \leq 1$, we get

$$y(\sigma) \leq y^*(\sigma) = y_1(\sigma+\rho) \leq \sigma + \rho \leq \sigma + \left(2\delta\sigma^{\frac{1}{n-1}}\right)^{\frac{1}{n}} = \sigma + (2\delta)^{\frac{1}{n}}\sigma^p.$$

Here, σ was an arbitrary element chosen in $(0,1)$: this means that (46), and so (41), has been proven (if $s \in \{0,1\}$, the estimate (46) is trivial).

Step 7: proof of the third estimate. For the last estimate, we appeal to the formula (41) just established: using (44) and the definition of $z(s)$, in $s \in \left(0, \frac{1}{2}\right)$ we have

$$z(s)^p \leq \left(s + (2\delta)^{\frac{1}{n}} s^{\frac{1}{p}}\right)^p \leq s^p + c_2\left(s^{p-1}(2\delta)^{\frac{1}{n}} s^{\frac{1}{p}} + \left[(2\delta)^{\frac{1}{n}} s^{\frac{1}{p}}\right]^p\right) =$$

$$= s^p + c_2(2\delta)^{\frac{1}{n}} s^{p-1+\frac{1}{p}} + c_2(2\delta)^{\frac{p}{n}} s.$$

Noting that $p + \frac{1}{p} > 2$, $2\delta \in (0,1)$ and $s \in \left(0, \frac{1}{2}\right)$, one has

$$z(s)^p - s \leq s^p + c_2(2\delta)^{\frac{1}{n}} s^{p-1+\frac{1}{p}} + c_2(2\delta)^{\frac{p}{n}} s - s < s^p + c_2(2\delta)^{\frac{p}{n}} s + c_2(2\delta)^{\frac{p}{n}} s - s =$$

$$= s\Big(s^{p-1} + 2c_2(2\delta)^{\frac{p}{n}} - 1\Big) < s\Big(s^{p-1} + 2c_2(2\delta)^{\frac{1}{n}} - 1\Big) \le s\left[\left(\frac{1}{2}\right)^{\frac{p-1}{2}} - 1\right]$$

if $\delta \in (0, \delta_0)$, where δ_0 has been chosen small enough. This verifies (42) in the interval $\big[0, \frac{1}{2}\big)$ as the coefficient of s is a negative number (if $s = 0$, the proof is trivial). We now consider $s \in \big[\frac{1}{2}, 1\big)$ and set $\sigma := 1 - s \in \big(0, \frac{1}{2}\big]$ in order to apply the definition of $z(s)$, (44) and the previous case, which take us to

$$z(s)^p - s \le \Big(s + c_3\big(\delta(1-s)\big)^{\frac{1}{2}}\Big)^p - s = \Big(s + c_3(\delta\sigma)^{\frac{1}{2}}\Big)^p - s \le$$

$$\le s^p + c_2\Big(s^{p-1}c_3(\delta\sigma)^{\frac{1}{2}} + \Big[c_3(\delta\sigma)^{\frac{1}{2}}\Big]^p\Big) - s = s^p + c_2c_3s^{p-1}(\delta\sigma)^{\frac{1}{2}} + c_2c_3^p(\delta\sigma)^{\frac{p}{2}} - s <$$

$$< s(s^{p-1} - 1) + c_2c_3(\delta\sigma)^{\frac{1}{2}} + c_2c_3^p(\delta\sigma)^{\frac{1}{2}} \le s\big((1-\sigma)^{p-1} - 1\big) + c_6(\delta\sigma)^{\frac{1}{2}},$$

for a certain positive constant $c_6 = c_6(n)$ depending on n only. Using the fact that $p \in (1, 2]$ by definition and $s \in \big[\frac{1}{2}, 1\big)$, the previous relation becomes

$$z(s)^p - s \le s\big((1-\sigma)^{p-1} - 1\big) + c_6(\delta\sigma)^{\frac{1}{2}} \le s\big(-(p-1)\sigma\big) + c_6(\delta\sigma)^{\frac{1}{2}} \le$$

$$\le -\frac{1}{2}(p-1)\sigma + c_6(\delta\sigma)^{\frac{1}{2}},$$

where we used that $(1-\sigma)^{p-1} - 1 \le -(p-1)\sigma$ due to the elementary estimate introduced in the previous step, which was $|1 - x|^n \ge 1 - nx$, $\forall\, x \in \mathbb{R}$ and $\forall\, n \ge 1$. In fact, for $x := \frac{\sigma}{n-1}$, we have

$$\left(1 - \frac{\sigma}{n-1}\right)^{n-1} = (1-x)^{n-1} \ge 1 - (n-1)x = 1 - \sigma \iff$$

$$\iff \big(1 - (p-1)\sigma\big)^{\frac{1}{p-1}} \ge (1-\sigma) \iff 1 - (p-1)\sigma \ge (1-\sigma)^{p-1} \iff$$

$$\iff 1 - (1-\sigma)^{p-1} \ge (p-1)\sigma \iff (1-\sigma)^{p-1} - 1 \le -(p-1)\sigma.$$

Considering

$$\sigma > \sigma_0 := \left(\frac{4c_6}{p-1}\right)^2 \delta \iff \delta < \left(\frac{p-1}{4c_6}\right)^2 \sigma$$

(which explain why this third extimate is valid outside of what we have defined as I), one has

$$c_6(\delta\sigma)^{\frac{1}{2}} < c_6\left[\sigma\left(\frac{p-1}{4c_6}\right)^2 \sigma\right]^{\frac{1}{2}} = c_6\sigma\,\frac{p-1}{4c_6} = \frac{p-1}{4}\,\sigma \implies z(s)^p - s \le$$

$$\le -\frac{1}{2}(p-1)\sigma + c_6(\delta\sigma)^{\frac{1}{2}} < -\frac{1}{2}(p-1)\sigma + \frac{p-1}{4}\,\sigma = -\frac{p-1}{4}\,\sigma = -\frac{p-1}{4}(1-s),$$

which is the desired estimate for $s \in \left[\frac{1}{2}, 1\right)$. For the last case, we take $s \in (1, +\infty)$ and, again by the definition of $z(s)$, for (41) and for (42), we establish

$$z(s)^p - s = \left(1 + \delta^{\frac{1}{n}}(s-1)^{\frac{1}{p}}\right)^p - s \le 1 + c_2\left[\delta^{\frac{1}{n}}(s-1)^{\frac{1}{p}} + \left(\delta^{\frac{1}{n}}(s-1)^{\frac{1}{p}}\right)^p\right] - s =$$

$$= 1 + c_2\left[\delta^{\frac{1}{n}}(s-1)^{\frac{1}{p}} + \delta^{\frac{p}{n}}(s-1)\right] - s = 1 + c_2\delta^{\frac{1}{n}}(s-1)^{\frac{1}{p}} + c_2\delta^{\frac{p}{n}}(s-1) - s =$$

$$= (s-1)\Big(-1 + c_2\delta^{\frac{1}{n}}(s-1)^{\frac{1}{p}-1} + c_2\delta^{\frac{p}{n}}\Big) = (s-1)\left[-1 + c_2\left(\frac{\delta}{s-1}\right)^{\frac{1}{n}} + c_2\delta^{\frac{p}{n}}\right] \le$$

$$\le (s-1)\left[-\frac{1}{2} + c_2\left(\frac{\delta}{s-1}\right)^{\frac{1}{n}}\right],$$

for an appropriate choice of δ_0. Taking $s - 1 > (4c_2)^n\delta$ (which means we are again in $[0, +\infty)\backslash I$), we get

$$z(s)^p - s \le (s-1)\left[-\frac{1}{2} + c_2\left(\frac{\delta}{s-1}\right)^{\frac{1}{n}}\right] < (s-1)\left[-\frac{1}{2} + c_2\left(\frac{\delta}{(4c_2)^n\delta}\right)^{\frac{1}{n}}\right] =$$

$$= (s-1)\left(-\frac{1}{2} + \frac{1}{4}\right) = -\frac{1}{4}(s-1), \tag{48}$$

that represent the estimate for $s \in [1, +\infty)$ (once again, the limit case $s = 1$ is banal) if we define $c_4 = c_4(n) := \left(\frac{4c_6}{p-1}\right)^2 + (4c_2)^n$ and $c_5 = c_5(n) := \left(1 - 2^{\frac{1-p}{2}}\right)^{-1} + \frac{4}{p-1}$.
A little clarification: the choice of c_4 is quite obvious, while the one of c_5 may seem a little ambiguous: we have not forgotten the coefficient $-\frac{1}{4}$ appearing in (48). The point is that, if c_5 exceeds $\frac{4}{p-1} = 4(n-1)$, then it automatically exceeds 4.

Step 8: conclusion. Now, after finally prove (40), (41) and (42), we make use of these estimates to derive the boundedness of the integral in (35) for $\beta = 1$. By (38), (41) and (42), we have

$$w^p(t) - t = \left(\xi^{\frac{1}{p}}y(s)\right)^p - \xi s = \xi y(s)^p - \xi s = \xi\big(y(s)^p - s\big) \le \xi\big(z(s)^p - s\big) \le$$

$$\le -c_5^{-1}\xi\varphi(s) = -c_5^{-1}\xi\varphi\left(\frac{t}{\xi}\right) \tag{49}$$

excluding the interval $I := \{s \in [0, +\infty) : |s-1| < c_4\delta\} = \{t \in [0, +\infty) : |t-\xi| < c_4\xi\delta\}$. In I, using (37), one has

$$\delta := 1 - \max_{t>0}\{t^{1-n}w(t)^n\} \le 1 - t^{1-n}w(t)^n \Longleftrightarrow w(t)^n \le \big(1-\delta\big)t^{n-1} \Longleftrightarrow$$

$$\Longleftrightarrow w(t)^p \le \big[(1-\delta)t^{n-1}\big]^{\frac{p}{n}} = \big(1-\delta\big)^{p-1}t \tag{50}$$

and, taking once more advantage of an elementary estimate given, this time, by the relation $1-(1-\delta)^{p-1} \geq (p-1)\delta$ for $p \in (1,2]$, it follows that

$$\int_I e^{w(t)^p - t} dt \leq \int_I e^{(1-\delta)^{p-1}t - t} dt = \int_I e^{t[(1-\delta)^{p-1}-1]} dt \leq \int_I e^{-(p-1)\delta t} dt \leq$$

$$\leq 2c_4 \delta\xi \max_{t\in I} \{ e^{-(p-1)\delta t} \}.$$

Choosing $\delta_0 < (2c_4)^{-1}$, we have $\delta < \delta_0 < (2c_4)^{-1}$ and then $t > \xi - c_4\xi\delta = \xi(1-c_4\delta) > \frac{\xi}{2}$; therefore, we have that

$$\int_I e^{w(t)^p - t} dt \leq 2c_4\delta\xi \max_{t\in I}\{ e^{-(p-1)\delta t}\} \leq 2c_4 \delta\xi e^{-(p-1)\delta\frac{\xi}{2}} =$$

$$= 2c_4 \frac{2}{p-1} \frac{(p-1)\delta\xi}{2} e^{-(p-1)\delta\frac{\xi}{2}} \leq 4c_4 \frac{1}{(p-1)e},$$

since $xe^{-x} \leq e^{-1}$, $\forall\, x \in \mathbb{R}$. Instead, for $t \in (0,+\infty)\setminus I$, (49) leads to

$$\int_{(0,+\infty)\setminus I} e^{w(t)^p - t} dt \leq \int_{(0,+\infty)\setminus I} e^{c_5^{-1}\xi\varphi\left(\frac{t}{\xi}\right)} dt =$$

$$= \int_0^{\frac{\xi}{2}} e^{-c_5^{-1}t} dt + \int_{\frac{\xi}{2}}^{\xi} e^{-c_5^{-1}\xi\left(1-\frac{t}{\xi}\right)} ds + \int_{\xi}^{+\infty} e^{-c_5^{-1}\xi\left(\frac{t}{\xi}-1\right)} ds =$$

$$= \frac{1}{c_5^{-1}}\left(- \left[e^{-c_5^{-1}t}\right]_0^{\frac{\xi}{2}} + e^{-c_5^{-1}\xi}\left[e^{c_5^{-1}t}\right]_{\frac{\xi}{2}}^{\xi} - e^{c_5^{-1}\xi}\left[e^{-c_5^{-1}t}\right]_{\xi}^{+\infty}\right) = \frac{1}{c_5^{-1}}\left(3 - 2e^{-c_5^{-1}\frac{\xi}{2}}\right) < 3c_5.$$

Hence,

$$\int_0^{+\infty} e^{w(t)^p - t} dt = \int_I e^{w(t)^p - t} dt + \int_{(0,+\infty)\setminus I} e^{w(t)^p - t} dt \leq 4c_4 \frac{1}{(p-1)e} + 3c_5 =: c_7,$$

for $0 < \delta < \delta_0$.

Finally, if $\delta \in [\delta_0, 1]$, we make again use of (50) and the following elementary estimate to get

$$\int_0^{+\infty} e^{w(t)^p - t} dt \leq \int_0^{+\infty} e^{(1-\delta)^{p-1}t - t} dt \leq \int_0^{+\infty} e^{-(p-1)\delta t} dt =$$

$$= -\frac{1}{\delta(p-1)}\left[e^{-(p-1)\delta t}\right]_0^{+\infty} = \frac{1}{\delta(p-1)} \leq \frac{1}{\delta_0(p-1)} =: c_8.$$

Clearly, both $c_7 = c_7(n)$ and $c_8 = c_8(n)$ depend only on n.

Thus, the theorem is proven for $\beta = 1$ setting $c_1 = c_1(n) := c_7 + c_8$.

□

We next prove Corollary 4, which will be basically an application of Moser's theorem.

Proof of Corollary 4. Tracing back to the proof of Moser's theorem, we consider a function $w(t) \in C^1\big([0,+\infty]\big)$ a.e. satisfying the three usual conditions, namely:

(i) $w(0) = 0$;

(ii) $w'(t) \geq 0$, for a.e. $t \geq 0$;

(iii) $\int_0^{+\infty} w'(t)^n dt \leq 1$.

We remind here that the above conditions are equivalent to the hypothesis of Moser's theorem. We now show that the integral in (35) exists finite for any positive β. In fact, $\forall\, \varepsilon > 0$, $\exists\, T = T(\varepsilon) > 0$ depending on ε only such that

$$\int_T^{+\infty} w'(t)^n dt < \varepsilon,$$

from which we establish, given any $t \geq T$ and using Hölder's inequality, that

$$w(t) - w(T) = \int_T^t w'(\tau)d\tau \leq \left(\int_T^t d\tau\right)^{\frac{1}{p}} \left(\int_T^t w'(\tau)^n d\tau\right)^{\frac{1}{n}} \leq$$

$$\leq (t-T)^{\frac{1}{p}} \left(\int_T^{+\infty} w'(\tau)^n d\tau\right)^{\frac{1}{n}} < (t-T)^{\frac{1}{p}}\, \varepsilon^{\frac{1}{n}} \iff w(t) < w(T) + \varepsilon^{\frac{1}{n}} (t-T)^{\frac{1}{p}}.$$

Hence,

$$\lim_{t\to+\infty} \frac{w(t)}{t^{\frac{1}{p}}} < \lim_{t\to+\infty} \frac{w(T) + \varepsilon^{\frac{1}{n}}(t-T)^{\frac{1}{p}}}{t^{\frac{1}{p}}} = \varepsilon^{\frac{1}{n}} \lim_{t\to+\infty} \left(\frac{t-T}{t}\right)^{\frac{1}{p}} = \varepsilon^{\frac{1}{n}}, \tag{51}$$

which means that the term appearing into the first limit of (51) is bounded and so it can lay arbitrarily near 0 if ε is chosen small enough. Thus, $\beta w(t)^p < \frac{t}{2}$ for sufficiently large t, and this estimate makes the integral in (35) finite for every choice of β. However, if $\beta > 1$, this integral can become large at will because, if we take a number $\xi > 0$, the function $\eta(\tau) := \min\{1, \tau\}$ and set

$$\tilde{w}(t) := \xi^{\frac{1}{p}} \eta\left(\frac{t}{\xi}\right) = \begin{cases} \xi^{-\frac{1}{n}}\, t & \text{if } t \in [0,\xi) \\ \xi^{1-\frac{1}{n}} & \text{if } t \in [\xi, +\infty) \end{cases},$$

then $\tilde{w}(t) \in C^1\big([0,+\infty)\setminus\{\xi\}\big)$ and we have that:

(i) $\tilde{w}(0) = 0$;

(ii) $\tilde{w}'(t) \geq 0$, $\forall\, t \in [0,+\infty)\setminus\{\xi\}$;

(iii) $\int_0^{+\infty} \tilde{w}'(t)^n dt = \int_0^{\xi} \big(\xi^{-\frac{1}{n}}\big)^n dt = \xi^{-1} \int_0^{\xi} dt = \xi^{-1}\, \xi = 1$.

Nevertheless,

$$\int_0^{+\infty} e^{\beta\tilde{w}(t)^p - t} dt \geq \int_\xi^{+\infty} e^{\beta\tilde{w}(t)^p - t} dt = \int_\xi^{+\infty} e^{\beta\left(\xi^{1-\frac{1}{n}}\right)^p - t} dt = \int_\xi^{+\infty} e^{\beta\xi - t} dt =$$

$$= e^{\beta\xi}\Big[-e^{-t}\Big]_\xi^{+\infty} = e^{(\beta-1)\xi} \longrightarrow +\infty$$

as $\xi \longmapsto +\infty$, being $\beta > 1$.

$\square$

Chapter 2

2.1. O'Neil's estimates on convolution and rearrangement

In this first section of the second chapter of the issue, we have to familiarize with some results concerning the convolution of two functions in order to properly achieve a result due to Adams, which is the generalization of Moser's theorem previously studied for higher order derivatives. We will follow O'Neil's original work, namely [21].
We begin by recalling the notion of convolution and, afterwards, we will use again the rearrangement technique to get some properties required for Adams' theorem. Note that, in this chapter, we will deal with functions whose domains of definition are the whole $\mathbb{R}^n$, which are integrable over $\mathbb{R}^n$ and such that their real images are non-negative.

Definition 5. *Given two functions f and g, a **convolution operator** of f and g is a bilinear operator, denoted $\mathcal{T}(f,g)$, whose $L^p(\mathbb{R}^n)$-norm is not greater than the product of the single $L^p(\mathbb{R}^n)$-norms of the functions f and g for $p=1$ and $p=\infty$. In other words, for all functions f_1, f_2, g_1, g_2 and $\forall\, \alpha, \beta \in \mathbb{R}$, $\mathcal{T}(f,g)$ is an operator such that:*

(i) $\mathcal{T}\big(\alpha f_1(x) + \beta f_2(x), g(x)\big) = \alpha \mathcal{T}\big(f_1(x), g(x)\big) + \beta \mathcal{T}\big(f_2(x), g(x)\big)$;

(ii) $\mathcal{T}\big(f(x), \alpha g_1(x) + \beta g_2(x)\big) = \alpha \mathcal{T}\big(f(x), g_1(x)\big) + \beta \mathcal{T}\big(f(x), g_2(x)\big)$;

(iii) $||\mathcal{T}(f,g)||_{L^1(\mathbb{R}^n)} \leq ||f||_{L^1(\mathbb{R}^n)} ||g||_{L^1(\mathbb{R}^n)}$ *if* $f \in L^1(\mathbb{R}^n)$ *and* $g \in L^1(\mathbb{R}^n)$;

(iv) $||\mathcal{T}(f,g)||_{L^\infty(\mathbb{R}^n)} \leq ||f||_{L^1(\mathbb{R}^n)} ||g||_{L^\infty(\mathbb{R}^n)}$ *if* $f \in L^1(\mathbb{R}^n)$ *and* $g \in L^\infty(\mathbb{R}^n)$;

(v) $||\mathcal{T}(f,g)||_{L^\infty(\mathbb{R}^n)} \leq ||f||_{L^\infty(\mathbb{R}^n)} ||g||_{L^1(\mathbb{R}^n)}$ *if* $f \in L^\infty(\mathbb{R}^n)$ *and* $g \in L^1(\mathbb{R}^n)$.

Remark 9. *In order to lighten the notation, given a subset $\Omega \subseteq \mathbb{R}^n$, from now on we will denote the $L^p(\Omega)$-norm of a function f as $||f||_p$, forgetting to write the set on which the norm is taken. Whenever an ambiguity may occur, we will clarify the set on which the norm is calculated; however, we reiterate here that we will work on the whole $\mathbb{R}^n$ when dealing with convolution operators.*

Definition 5 encloses a large class of operators $\mathcal{T}$; however, for us the prototype will be the classical convolution of two functions.

Definition 6. *Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be two functions. The* ***convolution*** *of f and g is the function $f * g : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that*

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Remark 10. *Using the definition of the convolution of two functions just defined and given three functions f, g and h, one can easily prove that the aforementioned operation possesses the following properties:*

(i) commutativity, because $(f * g)(x) = (g * f)(x)$;

(ii) linearity, since $\big(f * (\alpha g + \beta h)\big)(x) = \alpha(f * g)(x) + \beta(f * h)(x)$ and, similarly, $\big((\alpha f + \beta g) * h\big)(x) = \alpha(f * h)(x) + \beta(g * h)(x)$, $\forall\, \alpha, \beta \in \mathbb{R}$;

(iii) associativity, namely $\big((f * g) * h\big)(x) = \big(f * (g * h)\big)(x)$, which allows us to write $(f * g * h)(x)$ without ambiguity.

Remark 11. *The conditions (iii), (iv) and (v) appearing in Definition 5, if viewed through Definition 6, can be summarized in what is called Young's inequality for the convolution. This result states that, given any $1 \leq p \leq \infty$ and taken two functions f and g such that $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, the function $y \longmapsto f(x-y)g(y)$ is integrable in the variable y over $\mathbb{R}^n$ for a.e. $x \in \mathbb{R}^n$ and $f * g$ belongs to $L^p(\mathbb{R}^n)$. Moreover, the estimate*

$$||f * g||_p \leq ||f||_1 ||g||_p \tag{52}$$

holds. Therefore, due to what has been just said, we can conclude that the convolution is actually a convolution operator indeed: the bilinearity is given by Remark 10 (the second point, in particular), while the remaining properties by Young's inequality for the convolution. In fact, due to its commutativity, the roles of f and g can be interchanged: this means that the estimate (52) *reads, in particular, as*

$$||f * g||_1 \leq ||f||_1 ||g||_1 \tag{53}$$

for $p = 1$, as

$$||f * g||_\infty \leq ||f||_1 ||g||_\infty \tag{54}$$

for $p = \infty$ and as

$$||f * g||_\infty \leq ||f||_\infty ||g||_1 \tag{55}$$

once more for $p = \infty$ (after using the commutativity of the convolution). Thus, the inequalities (53), (54) *and* (55) *are the same conditions appearing in Definition 5 for the $L^p(\mathbb{R}^n)$-norm of a convolution operator $\mathcal{T}(f, g)$, where $p = 1$ and $p = \infty$.*

We now need a new notion that has to do with the unidimensional decreasing rearrangement $u^\#$ of a function u.

Definition 7. *Given a function u, the* ***integral average of the unidimensional decreasing rearrangement*** *$u^\#$ is the function $u^{\#\#} : (0, +\infty) \longrightarrow \mathbb{R}$ such that*

$$u^{\#\#}(t) := \frac{1}{t} \int_0^t u^\#(s) ds. \tag{56}$$

Remark 12. *The function defined by (56), due to the properties of the unidimensional decreasing rearrangement already studied, is monotonically non-increasing. Besides, although we have discussed the unidimensional decreasing rearrangement for functions defined on a bounded and measurable domain $\Omega \subseteq \mathbb{R}^n$, we can translate all the notions and results even for $\Omega = \mathbb{R}^n$ since we are assuming that the functions we are dealing with are integrable over $\mathbb{R}^n$. Therefore, in such a case, due to (6) (with $p = 1$ and using the fact that here the functions are non-negative), its unidimensional decreasing rearrangement will be integrable over $\mathbb{R}$, too. This means that, from now on, we will be able to integrate a function over $[\xi, +\infty)$, for a certain $\xi \in \mathbb{R}$ (not only over $[\xi, |\Omega|]$ as we have done until this moment).*

After having introduced all these quantities, we are finally ready to enunciate what is the fundamental result of this section, which will be essential for the proof of the future Adams' theorem.

Theorem 8 (O'Neil). *Let $\mathcal{T}$ be a convolution operator and set $h(x) := \mathcal{T}\big(f(x), g(x)\big)$. Then, for any $t > 0$,*

$$h^{\#\#}(t) \leq t f^{\#\#}(t) g^{\#\#}(t) + \int_t^{+\infty} f^\#(s) g^\#(s) ds. \tag{57}$$

Remark 13. *Theorem 8 is not valid only for non-negative functions f and g: if they change sign, then (57) becomes*

$$h^{\#\#}(t) \leq t \big|f^{\#\#}(t)\big| \big|g^{\#\#}(t)\big| + \int_t^{+\infty} \big|f^\#(s)\big| \big|g^\#(s)\big| ds. \tag{58}$$

The same is true for the following results needed for its proof: in fact, modifying them appropriately, one can establish the more general formula given by (58). However, we are interested in (57) and so we will omit the complete discussion.

As just stated, we need first some intermediate results concerning the convolution operators and the integral average of the unidimensional decreasing rearrangement.

Lemma 4. *If a function f can be decomposed into two other functions, namely if $f(x) = f_1(x) + f_2(x)$, then*

$$f^{\#\#}(t) \leq f_1^{\#\#}(t) + f_2^{\#\#}(t). \tag{59}$$

Proof. The validity of this inequality is essentially given by the nature itself of the unidimensional decreasing rearrangement: first of all, given an arbitrary measurable set E and using (6) for $p = 1$, we have

$$\int_0^{|E|} f^{\#}(s)ds = \int_0^{|E|} \big(f_1^{\#}(s) + f_2^{\#}(s)\big)ds = \int_E \big(f_1(x) + f_2(x)\big)dx =$$

$$= \int_E f_1(x)dx + \int_E f_2(x)dx = \int_0^{|E|} f_1^{\#}(s)ds + \int_0^{|E|} f_2^{\#}(s)ds = \int_0^{|E|} \big(f_1^{\#}(s) + f_2^{\#}(s)\big)ds.$$

Then, being $u^{\#}$ a radially non-increasing function, we can write

$$\int_0^t f^{\#}(s)ds = \sup_{|E|=t}\left\{ \int_0^{|E|} f^{\#}(s)ds \right\}$$

for a fixed $t > 0$. Hence,

$$tf^{\#\#}(t) = \int_0^t f^{\#}(s)ds = \sup_{|E|=t}\left\{ \int_0^{|E|} f^{\#}(s)ds \right\} = \sup_{|E|=t}\left\{ \int_0^{|E|} \big(f_1^{\#}(s) + f_2^{\#}(s)\big)ds \right\} \leq$$

$$\leq \sup_{|E|=t}\left\{ \int_0^{|E|} f_1^{\#}(s)ds \right\} + \sup_{|E|=t}\left\{ \int_0^{|E|} f_2^{\#}(s)ds \right\} = \int_0^t f_1^{\#}(s)ds + \int_0^t f_2^{\#}(s)ds =$$

$$= tf_1^{\#\#}(t) + tf_2^{\#\#}(t),$$

which is the relation (59) if one divides for t.

□

We can characterize what we defined in (56) in view of the following proposition, which merges the notion of the unidimensional decreasing rearrangement $u^{\#}$ with its integral average $u^{\#\#}$.

Proposition 6. *For a function f, we have that*

$$tf^{\#\#}(t) = tf^{\#}(t) + \int_{f^{\#}(t)}^{+\infty} \mu_f(\tau)d\tau. \tag{60}$$

Proof. First of all, by equimeasurability, one has

$$\mu_f(\tau) = \{x \in \Omega : f(x) > \tau\} = |\{s \in [0, +\infty) : f^{\#}(s) > \tau\}| = \mu_{f^{\#}}(\tau). \tag{61}$$

Now, taking an element $s_0 \in [0, |\Omega|]$, if we set

$$A := \{(s, \tau) \in [0, +\infty) \times \mathbb{R} : \tau > f^{\#}(s_0),\ f^{\#}(s) > \tau\}$$

and

$$B := \{(s, \tau) \in [0, +\infty) \times \mathbb{R} : 0 < s < s_0,\ f^{\#}(s_0) < \tau < f^{\#}(s)\},$$

then, by their definitions, it follows immediately that $A = B$. Thus,

$$\int_{f^{\#}(s_0)}^{+\infty} \mu_{f^{\#}}(\tau) d\tau = \int_{f^{\#}(s_0)}^{+\infty} |\{f^{\#}(s) > \tau\}| d\tau = \int_0^{s_0} \left(\int_{f^{\#}(s_0)}^{f^{\#}(s)} d\tau \right) ds. \tag{62}$$

However,

$$\int_0^{s_0} \left(\int_{f^{\#}(s_0)}^{f^{\#}(s)} d\tau \right) ds = \int_0^{s_0} \big(f^{\#}(s) - f^{\#}(s_0)\big) ds = \int_0^{s_0} f^{\#}(s) ds - s_0 f^{\#}(s_0) =$$

$$= s_0 f^{\#\#}(s_0) - s_0 f^{\#}(s_0). \tag{63}$$

Finally, taking into account (61) and joining the relations (62) and (63), one has the thesis.

□

We now need a last auxiliary lemma in order to prove Theorem 8.

Lemma 5. *Let $\mathcal{T}$ be a convolution operator and set $h(x) := \mathcal{T}\big(f(x), g(x)\big)$, where f is a function vanishing outside of a set E whose measure is $|E| = s$ and such that $f(x) \le \alpha$, for a certain $\alpha > 0$. Then, for $t > 0$, we have that:*

(i) $h^{\#\#}(t) \le \alpha s g^{\#\#}(s)$;

(ii) $h^{\#\#}(t) \le \alpha s g^{\#\#}(t)$.

Proof. Let $\lambda > 0$. We can rewrite the function g as $g(x) = g_1(x) + g_2(x)$, where

$$g_1(x) := \begin{cases} g(x) & \text{if } g(x) \in [0, \lambda] \\ \lambda & \text{if } g(x) \in (\lambda, +\infty) \end{cases}$$

and

$$g_2(x) := \begin{cases} 0 & \text{if } g(x) \in [0, \lambda] \\ g(x) - \lambda & \text{if } g(x) \in (\lambda, +\infty) \end{cases}.$$

After doing this, applying the definition of h, we get

$$h(x) = \mathcal{T}\big(f(x), g(x)\big) = \mathcal{T}\big(f(x), g_1(x) + g_2(x)\big) = \underbrace{\mathcal{T}\big(f(x), g_1(x)\big)}_{=:\, h_1(x)} + \underbrace{\mathcal{T}\big(f(x), g_2(x)\big)}_{=:\, h_2(x)}.$$

We now have that

$$||g_1||_\infty \leq \lambda \tag{64}$$

trivially by its definition, while

$$||g_2||_1 = \int_\lambda^{+\infty} \mu_g(\tau) d\tau. \tag{65}$$

As just said, the relation (64) follows directly from the definition of g_1; instead, in order to prove (65), we write

$$\int_\Omega g_2(x)dx = \int_0^{+\infty} \mu_{g_2}(\tau)d\tau = \int_0^{+\infty} |\{g_2(x) > \tau\}|d\tau = \int_0^{+\infty} |\{g(x) - \lambda > \tau\}|d\tau =$$

$$= \int_0^{+\infty} \{g(x) > \tau + \lambda\}d\tau = \int_\lambda^{+\infty} |\{g(x) > \xi\}|d\xi = \int_\lambda^{+\infty} \mu_g(\xi)d\xi. \tag{66}$$

So, (66) yields to

$$||g_2||_1 = \int_\Omega g_2(x)dx = \int_\lambda^{+\infty} \mu_g(\xi)d\xi,$$

which is exactly (65). Using the hypothesis on f and the definition of the convolution operators, let us make some estimates: we have

$$||h_1||_\infty \leq ||f||_1 ||g_1||_\infty \leq \alpha s \lambda, \tag{67}$$

$$||h_2||_1 \leq ||f||_1 ||g_2||_1 \leq \alpha s \int_\lambda^{+\infty} \mu_g(\tau)d\tau \tag{68}$$

and

$$||h_2||_\infty \leq ||f||_\infty ||g_2||_1 \leq \alpha \int_\lambda^{+\infty} \mu_g(\tau)d\tau. \tag{69}$$

Then, fixing $\lambda = g^\#(s)$ and using (6), (56), (60), (67), (69) and Minkowski's inequality, we obtain

$$h^{\#\#}(t) := \frac{1}{t}\int_0^t h^\#(s)ds \leq ||h||_\infty \leq ||h_1||_\infty + ||h_2||_\infty \leq \alpha s g^\#(s) + \alpha \int_{g^\#(s)}^{+\infty} \mu_g(\tau)d\tau =$$

$$= \alpha\left(s g^\#(s) + \int_{g^\#(s)}^{+\infty} \mu_g(\tau)d\tau\right) = \alpha s g^{\#\#}(s),$$

which is the point *(i)* of the thesis. Further, fixing instead $\lambda = g^\#(t)$ and using (6), (56), (58), (60), (67) and (68), we have

$$th^{\#\#}(t) = \int_0^t h^{\#}(s)ds \leq \int_0^t h_1^{\#}(s)ds + \int_0^t h_2^{\#}(s)ds \leq$$

$$\leq t||h_1||_\infty + \int_0^{+\infty} h_2^{\#}(s)ds = t||h_1||_\infty + ||h_2||_1 \leq$$

$$\leq t\alpha s g^{\#}(t) + \alpha s \int_{g^{\#}(t)}^{+\infty} \mu_g(\tau)d\tau = \alpha s\left(tg^{\#}(t) + \int_{g^{\#}(t)}^{+\infty} \mu_g(\tau)d\tau \right) = \alpha s t g^{\#\#}(t)$$

which, this time, is the point *(ii)* of the thesis once we divide by t (being able to do that since $t > 0$ by hypothesis).

□

We are finally in position to prove Theorem 8, which has been enunciated earlier.

Proof of Theorem 8. We divide the proof into three steps.

<u>Step 1: initial setting.</u> After fixing a number $t > 0$, we consider a non-negative and monotonically increasing sequence $\{y_k\}_{k\in\mathbb{Z}}$ going from 0 to $+\infty$ and passing through $f^{\#}(t)$, namely $y_k \leq y_{k+1}$ for every $k \in \mathbb{Z}$, $y_0 = f^{\#}(t)$, $y_k \longrightarrow 0$ as $k \longmapsto -\infty$ and $y_k \longrightarrow +\infty$ as $k \longmapsto +\infty$. We are, consequently, able to rewrite $f(x)$ as

$$f(x) = \sum_{k\in\mathbb{Z}} f_k(x),$$

where

$$f_k(x) := \begin{cases} 0 & \text{if } f(x) \in [0, y_{k-1}] \\ f(x) - y_{k-1} & \text{if } f(x) \in (y_{k-1}, y_k] \\ y_k - y_{k-1} & \text{if } f(x) \in (y_k, +\infty) \end{cases} . \tag{70}$$

In fact, if $f(x) = 0$, then $f_k(x) = 0$ by definition for every $k \in \mathbb{Z}$. Otherwise, for x fixed, there exists $N \in \mathbb{N}$ such that $f(x) \in [y_{k-1}, y_k]$, which means that

$$\sum_{k\in\mathbb{Z}} f_k(x) = \sum_{k=-\infty}^{N-1} f_k(x) + f_N(x) + \sum_{k=N+1}^{+\infty} f_k(x) = \sum_{k=-\infty}^{N-1} (y_k - y_{k-1}) + f(x) - y_{k-1} \tag{71}$$

using the definition of $f_k(x)$ in the various cases. Since the remaining sum in the preceding relation is a telescoping series, then (71) becomes

$$\sum_{k\in\mathbb{Z}} f_k(x) = \sum_{k=-\infty}^{N-1} (y_k - y_{k-1}) + f(x) - y_{k-1} = -\lim_{k\to-\infty} y_k + y_{N-1} + f(x) - y_{N-1} = f(x).$$

Moreover, $f_k(x)$ vanishes outside the set $E_k := \{x \in \Omega : f(x) > y_{k-1}\}$, whose measure is given by $\mu_f(y_{k-1})$. Using (70), an easy check shows also that $f_k(x) \leq y_k - y_{k-1}$.

Step 2: utilizing Lemma 5. These considerations just made allow us to use Lemma 5, since we are under its hypothesis: before doing that, we rewrite h as

$$h(x) = \mathcal{T}\big(f(x), g(x)\big) = \mathcal{T}\bigg(\sum_{k\in\mathbb{Z}} f_k(x), g(x)\bigg) =$$

$$= \underbrace{\mathcal{T}\bigg(\sum_{k=-\infty}^{0} f_k(x), g(x)\bigg)}_{=:\, h_1(x)} + \underbrace{\mathcal{T}\bigg(\sum_{k=1}^{+\infty} f_k(x), g(x)\bigg)}_{=:\, h_2(x)},$$

having used the linearity of the operator $\mathcal{T}(f,g)$. As done previously, due to (58) we have that

$$h^{\#\#}(t) \leq h_1^{\#\#}(t) + h_2^{\#\#}(t).$$

Now, in order to get (57), we evaluate $h_1^{\#\#}(t)$ and $h_2^{\#\#}(t)$ using Lemma 5: however, that result used a convolution operator for two functions, while here we have a countable number of functions. It is easily seen that we can generalize Lemma 5 for a finite number of functions: from that, fixing $M \in \mathbb{N}$ and using the monotone convergence theorem to get

$$\sum_{k=1}^{M} f_k(x) \longrightarrow \sum_{k=1}^{+\infty} f_k(x)$$

as $M \longmapsto +\infty$ in $L^1(\mathbb{R}^n)$, we can affirm that

$$\mathcal{T}\bigg(\sum_{k=1}^{M} f_k(x), g(x)\bigg) \longrightarrow \mathcal{T}\bigg(\sum_{k=1}^{+\infty} f_k(x), g(x)\bigg)$$

as $M \longmapsto +\infty$ in $L^1(\mathbb{R}^n)$, since

$$\bigg|\bigg|\mathcal{T}\bigg(\sum_{k=M+1}^{+\infty} f_k, g\bigg)\bigg|\bigg|_1 \leq ||g||_1 \bigg|\bigg|\sum_{k=M+1}^{+\infty} f_k\bigg|\bigg|_1 \longrightarrow 0$$

as $M \longmapsto +\infty$, where we used the property *(iii)* of Definition 5.

Next, setting $s := \mu_f(y_{k-1})$ and $\alpha := y_k - y_{k-1} > 0$, the first part of the thesis of the aforementioned result for h_1 reads as

$$h_1^{\#\#}(t) \leq \sum_{k=-\infty}^{0} (y_k - y_{k-1})\mu_f(y_{k-1}) g^{\#\#}\big(\mu_f(y_{k-1})\big).$$

Choosing an appropriate sequence $\{y_k\}_{k\in\mathbb{Z}}$ in (70), by the approximation of Riemann sums we are able to make the sum on the right side of the above relation approach the integral

$$\int_0^{f^\#(t)} \mu_f(\tau) g^{\#\#}\big(\mu_f(\tau)\big) d\tau. \tag{72}$$

Hence, evaluating (72) by making the substitution $\tau = f^\#(r)$ and integrating by parts, we get

$$h_1^{\#\#}(t) \le \int_0^{f^\#(t)} \mu_f(\tau) g^{\#\#}\big(\mu_f(\tau)\big) d\tau =$$

$$= \int_{+\infty}^{t} \mu_f\big(f^\#(r)\big) g^{\#\#}\Big(\mu_f\big(f^\#(r)\big)\Big) \big(f^\#(r)\big)' dr =$$

$$= -\int_t^{+\infty} r g^{\#\#}(r) \big(f^\#(r)\big)' dr = \Big[- r g^{\#\#}(r) f^\#(r) \Big]_t^{+\infty} + \int_t^{+\infty} f^\#(r) g^\#(r) dr \le$$

$$\le t g^{\#\#}(t) f^\#(t) + \int_t^{+\infty} f^\#(r) g^\#(r) dr, \tag{73}$$

where, in the various steps, we have used that:

- μ_u and $u^\#$ are the inverse functions of each other, as noted in Remark 1;
- if $0 = \tau = f^\#(r)$, then $r = \mu_f\big(f^\#(r)\big) = \mu_f(0) = +\infty$;
- $\frac{d}{dr}\big(r g^{\#\#}(r)\big) = \frac{d}{dr}\big(\int_0^r g^\#(\xi) d\xi\big) = g^\#(r)$.

Instead, using the second estimate of Lemma 5 with the same s and α as before, we have that

$$h_2^{\#\#}(t) \le \sum_{k=1}^{+\infty} (y_k - y_{k-1}) \mu_f(y_{k-1}) g^{\#\#}(t) = g^{\#\#}(t) \sum_{k=1}^{+\infty} (y_k - y_{k-1}) \mu_f(y_{k-1}).$$

Again, the series on the right can become (with a proper choice of the sequence) arbitrarily close to

$$\int_{f^\#(t)}^{+\infty} \mu_f(\tau) d\tau,$$

which means that

$$h_2^{\#\#}(t) \le g^{\#\#}(t) \int_{f^\#(t)}^{+\infty} \mu_f(\tau) d\tau. \tag{74}$$

Having finally an estimate on $h_1^{\#\#}$ and one on $h_2^{\#\#}$, we easily reach the thesis.

Step 3: conclusion. Joining (73) and (74) and using also (58) and (60), we are finally done because, in this way, we obtain

$$h^{\#\#}(t) \le h_1^{\#\#}(t) + h_2^{\#\#}(t) \le$$

$$\le t g^{\#\#}(t) f^\#(t) + \int_t^{+\infty} f^\#(r) g^\#(r) dr + g^{\#\#}(t) \int_{f^\#(t)}^{+\infty} \mu_f(\tau) d\tau =$$

$$= \left[t f^{\#}(t) + \int_{f^{\#}(t)}^{+\infty} \mu_f(\tau) d\tau \right] g^{\#\#}(t) + \int_t^{+\infty} f^{\#}(r) g^{\#}(r) dr =$$

$$= t f^{\#\#}(t) g^{\#\#}(t) + \int_t^{+\infty} f^{\#}(r) g^{\#}(r) dr,$$

which is the desired estimate.

□

2.2. Adams' generalization of Moser's theorem

In this paragraph, our main target will be generalizing Moser's theorem for functions $u \in W_0^{m,\frac{n}{m}}(\Omega)$, for a certain $m < n$. This will be done using mainly [1].
Some techniques will be similar to the ones used in the proof of the aforementioned result, but there will be some differences. In fact, at the beginning of the proof of Moser's result, we assumed (without loss of generality) that $u(x) \geq 0$: this was helpful because it allowed us to consider $u^* \in W_0^{1,p}(\Omega)$; however, this is no more true if $m \geq 2$. Indeed, $u \in W_0^{1,p}(\Omega)$ if and only if $|u| \in W_0^{1,p}(\Omega)$, which means that, if the function u changes sign, then we are allowed to replace it by $|u|$ (that is exactly what we did in the proof) but, if $m \geq 2$, it can happen that $|u| \notin W_0^{m,p}(\Omega)$ even though $u \in W_0^{m,p}(\Omega)$. Then, introducing a function $w(t)$ with some special properties, we deduced the formula (35) which was equivalent to the thesis. Instead, here we will deal with a different auxiliary function with stronger properties.
Dealing with functions $u \in W_0^{1,p}(\Omega)$, we worked with the gradient of u: here, being $m \geq 2$, we have to clarify what $\nabla^m u$ means. We will use the symbol $\nabla^m u$ to denote the m-th order gradient of u: if $m = 1$, then $\nabla : \mathbb{R} \longrightarrow \mathbb{R}^n$ is such that

$$u(x) \longmapsto \nabla u(x) := \left(\frac{\partial}{\partial x_1} u(x), \dots, \frac{\partial}{\partial x_n} u(x) \right);$$

if $m = 2$, then $\nabla^2 : \mathbb{R}^n \longrightarrow \mathbb{R}$ is such that

$$\nabla u(x) \longmapsto \Delta u(x) := \operatorname{div} \big(\nabla u(x) \big) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u(x);$$

if $m = 3$, then $\nabla^3 : \mathbb{R} \longrightarrow \mathbb{R}^n$ is such that

$$\Delta u(x) \longmapsto \nabla \big(\Delta u(x) \big) = \left(\frac{\partial}{\partial x_1} \big(\Delta u(x) \big), \dots, \frac{\partial}{\partial x_n} \big(\Delta u(x) \big) \right)$$

and so on. Therefore, we have that

$$\nabla^m u(x) = \begin{cases} \Delta^{\frac{m}{2}} u(x) & \text{if } m = 2k \\ \nabla \Big(\Delta^{\frac{m-1}{2}} u(x) \Big) & \text{if } m = 2k - 1 \end{cases},$$

$\forall\, k \in \mathbb{N}$ (where $\Delta^0 u(x) := u(x)$ if $m = 1$).
The distinction between the odd case and the even one will play a major role in the future discussions. In fact, in order to prove the forthcoming theorem, we have to appeal to some intermediate results: one of them is a representation formula for functions in $C_0^\infty(\mathbb{R}^n)$ which distinguish, precisely, the odd case for m from the even one.

Theorem 9 (**Adams**). *Let $n \in \mathbb{N}_{\geq 2}$, $m \in \mathbb{N}$ such that $m < n$ and $u \in W_0^{m,\frac{n}{m}}(\Omega)$. Assume that*

$$||\nabla^m u||_{L^q(\Omega)} := \left(\int_\Omega |\nabla^m u(x)|^q dx \right)^{\frac{1}{q}} \leq 1,$$

where we set $q := \frac{n}{m} \in (1, n]$. Then, $\forall\, \alpha \in [0, \alpha_{m,n}]$, there exists a constant $c = c(m, n)$ depending on m and n only such that

$$\fint_\Omega e^{\alpha |u(x)|^p} dx \leq c, \tag{75}$$

where $p := \frac{q}{q-1} = \frac{n}{n-m}$ is the conjugate exponent of q and

$$\alpha_{m,n} := \begin{cases} \frac{n}{\omega_{n-1}} \left[\frac{2^m \pi^{\frac{n}{2}} \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right]^p & \text{if } m = 2k \\ \frac{n}{\omega_{n-1}} \left[\frac{2^m \pi^{\frac{n}{2}} \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \right]^p & \text{if } m = 2k-1 \end{cases},$$

$\forall\, k \in \mathbb{N}$.

Remark 14. *Adams' $\alpha_{1,n}$ agrees with Moser's α_n: in fact,*

$$\alpha_{1,n} := \frac{n}{\omega_{n-1}} \left[\frac{2\pi^{\frac{n}{2}} \Gamma(1)}{\Gamma\left(\frac{n}{2}\right)} \right]^{\frac{n}{n-1}} = \frac{n}{\omega_{n-1}} \left[\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \right]^{\frac{n}{n-1}} = \frac{n}{\omega_{n-1}} \left(\omega_{n-1}\right)^{\frac{n}{n-1}} = n\omega_{n-1}^{\frac{1}{n-1}} =: \alpha_n.$$

This is not a coincidence, since Theorem 9 is a generalization of Theorem 7 in the sense that, if we fixed $m = 1$, then Adams' theorem is exactly Moser's theorem. Besides, one can easily check another interesting relation, which is $\alpha_{m,2m} = 2^{2m}\pi^m\Gamma(m+1)$, valid for both even and odd m.

There is a consequence of Adams's theorem which is, in simple terms, Corollary 4 revisited for Theorem 9. It states that, even this time, the element $\alpha_{m,n}$ is sharp.

Proposition 7. *If $\alpha > \alpha_{m,n}$, the estimate (75) is no more true in the sense that*

$$\sup_{\substack{u \in W_0^{m,q}(\Omega) \\ ||\nabla^m u||_{L^q(\Omega)} \leq 1}} \left\{ \fint_\Omega e^{\alpha |u(x)|^p} dx \right\} = +\infty.$$

In other words, under the same hypothesis of Theorem 9 (except the one on α), if there exists a constant c for which

$$\fint_\Omega e^{\alpha |u(x)|^p} dx \leq c$$

holds for $\alpha > \alpha_{m,n}$, then $c = c(m, n, u)$ is forced to depend also on the function u taken into consideration as well.

To achieve these results, we need some intermediate notions: one of them is the following theorem.

Theorem 10. *Let $1 < q < \infty$ and $f \in L^q(\mathbb{R}^n)$ such that* $\text{supp}\{f\} \subseteq \Omega$. *If $f \neq 0$ a.e., then there exists a constant $c = c(q)$ depending on q only such that*

$$\fint_{\Omega} e^{\frac{n}{\omega_{n-1}}\left|\frac{(I_\beta * f)(x)}{||f||_q}\right|^p} dx \leq c, \tag{76}$$

where $p := \frac{q}{q-1}$, $\beta := \frac{n}{q}$ and $I_\beta(x) := |x|^{\beta-n}$.

Remark 15. *The previous theorem is even valid for a function f which is null a.e. if formulated differently. In fact, instead of dividing the exponent of (76) by $||f||_q^p$, we can make the constant c become dependent also on f, so that the result incorporates the trivial case in which $f = 0$ a.e.; however, we will not deal with such a case and, therefore, it is preferable to have a constant c depending on q only for our purpose.*

The quantity

$$\mathcal{I}_\beta f(x) := \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}}\, dy,$$

where

$$\gamma(\beta) := \frac{2^\beta \pi^{\frac{n}{2}} \Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{n-\beta}{2}\right)}$$

and $\beta \in (0, n)$, is called the Riesz potential of order β of f (see also [24]). Apart from the constant $\gamma(\beta)$, it is the function $I_\beta * f$ appearing in the estimate (76). It will play a major role in proving these results. Besides, notice that the constant $\gamma(\beta)$ is very similar to the definition of $\alpha_{m,n}$ in the even case (although here we are allowed to have any real number $\beta \in (0, n)$, while $\alpha_{m,n}$ includes only a positive integer number m strictly less than n). Note also that, if f is as in the hypothesis of the preceding theorem, then $I_\beta * f$ is well defined since $n - \beta < n$.
Theorem 10 is enough to deduce Theorem 9; however, we are going to enunciate a consequence of it which will be useful in the next chapter.

Proposition 8. *The constant $\frac{n}{\omega_{n-1}}$ is sharp in the sense that, if $\eta > \frac{n}{\omega_{n-1}}$, then (76) is no more valid. In other words, assuming the same hypothesis of Theorem 10, if c is a constant such that the relation*

$$\fint_{\Omega} e^{\eta\left|\frac{(I_\beta * f)(x)}{||f||_q}\right|^p} dx \leq c$$

holds for $\eta > \frac{n}{\omega_{n-1}}$, then $c = c(q, f)$ is forced to depend also on the function f.

The proof of Theorem 10 (which will be helpful in view of Theorem 9) will be based in turn on O'Neil's theorem and the following lemma.

Lemma 6. *Let* $1 < q < \infty$ *and* $p := \frac{q}{q-1}$. *Consider a non-negative Borel measurable function*

$$a : (-\infty, +\infty) \times [0, +\infty) \longrightarrow \mathbb{R}$$

such that

$$a(s,t) \leq 1 \tag{77}$$

for $s \in (0,t)$ *and*

$$b := \sup_{t>0} \left\{ \left(\int_{-\infty}^{0} a(s,t)^p ds + \int_{t}^{+\infty} a(s,t)^p ds \right)^{\frac{1}{p}} \right\} < +\infty. \tag{78}$$

If $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ *is a non-negative function such that*

$$\int_{-\infty}^{+\infty} \phi(s)^q ds \leq 1, \tag{79}$$

then there exists a constant $c = c(q,b)$ *depending on* q *and* b *only such that*

$$\int_{0}^{+\infty} e^{-F(t)} dt \leq c, \tag{80}$$

where

$$F(t) := t - \left(\int_{-\infty}^{+\infty} a(s,t)\phi(s) ds \right)^{p}. \tag{81}$$

Note that the function ϕ, if multiplied by $\chi_{\{s<t\}}$ (which here has the role of a), corresponds to w' utilized in the proof of Moser's theorem: however, this time we must require these more general properties.
Lemma 6 is the key to prove Theorem 10: we begin by it and, afterwards, we will pass to Theorem 10 and Proposition 8, after which we will be able to prove the most important result of this entire chapter, namely Adams' theorem, with the help of this following and last intermediary statement.

Lemma 7. *Let* $u \in C_0^\infty(\mathbb{R}^n)$. *Then:*

(i) if m *is an odd positive integer, we have that*

$$u(x) = (-1)^{\frac{m-1}{2}} \left(\frac{n}{\omega_{n-1}\alpha_{m,n}} \right)^{\frac{1}{p}} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-m+1}} \cdot \nabla^m u(y) dy;$$

(ii) if m *is an even positive integer, we have that*

$$u(x) = (-1)^{\frac{m}{2}} \left(\frac{n}{\omega_{n-1}\alpha_{m,n}} \right)^{\frac{1}{p}} \int_{\mathbb{R}^n} \frac{\nabla^m u(y)}{|x-y|^{n-m}} dy.$$

Remark 16. *Exactly as when dealing with the Riesz potential, the integrals appearing in the previous formulas have both a singularity (when y approaches x). However, even this time, they are well defined since $n-m<n$ for every $m\in\mathbb{N}$ in the even case, while*

$$\frac{x-y}{|x-y|^{n-m+1}}\sim\frac{1}{|x-y|^{n-m}}$$

near $y=x$ in the odd one and, therefore, the condition we get is the same, namely $n-m<n$.

We can finally begin to prove these results.

Proof of Lemma 6. We split the proof into three steps.

Step 1: introduction of two auxiliary conditions. We can rewrite (80) as

$$\int_0^{+\infty}e^{-F(t)}dt=\int_0^{+\infty}\big|\big\{\tau\geq 0:e^{-F(\tau)}>t\big\}\big|dt=\int_0^{+\infty}|\{\tau\geq 0:F(\tau)<-\log(t)\}|dt=$$

$$=\int_{-\infty}^{+\infty}|\{\tau\geq 0:F(\tau)<\lambda\}|e^{-\lambda}d\lambda=\int_{-\infty}^{+\infty}|E_\lambda|e^{-\lambda}d\lambda, \tag{82}$$

where we used the change of variables $\lambda:=-\log(t)$ and set $E_\lambda:=\{\tau\geq 0:F(\tau)<\lambda\}$. In order to bound the integral on the right side of (82), we will show that:

(i) $\exists\, c=c(q,b)$ such that $F(t)\geq -c$, $\forall\, t\geq 0$;

(ii) $\exists\, A=A(q,b)$, $B=B(q,b)$ constants such that $|E_\lambda|\leq A|\lambda|+B$.

It is clear that it suffices to show that, if *(i)* and *(ii)* hold, then we have the thesis. In fact, *(i)* suggests that $\exists\,\lambda_0\in\mathbb{R}$ such that $|E_\lambda|=0$ for $\lambda<\lambda_0$, while *(ii)* tells us that there is integrability at $+\infty$. Together, they give us

$$\int_0^{+\infty}e^{-F(t)}dt=\int_{-\infty}^{+\infty}|E_\lambda|e^{-\lambda}d\lambda=\int_{\lambda_0}^{+\infty}|E_\lambda|e^{-\lambda}d\lambda\leq\int_{\lambda_0}^{+\infty}\big(A|\lambda|+B\big)e^{-\lambda}d\lambda=$$

$$=A\int_{\lambda_0}^{+\infty}|\lambda|e^{-\lambda}d\lambda+B\int_{\lambda_0}^{+\infty}e^{-\lambda}d\lambda=A\bigg(-\int_{\lambda_0}^{0}\lambda e^{-\lambda}d\lambda+\int_0^{+\infty}\lambda e^{-\lambda}d\lambda\bigg)-B\Big[e^{-\lambda}\Big]_{\lambda_0}^{+\infty}=$$

$$=A\bigg(\Big[\lambda e^{-\lambda}\Big]_{\lambda_0}^{0}-\int_{\lambda_0}^{0}e^{-\lambda}d\lambda-\Big[\lambda e^{-\lambda}\Big]_0^{+\infty}+\int_0^{+\infty}e^{-\lambda}d\lambda\bigg)+Be^{-\lambda_0}=$$

$$=A\bigg(-\lambda_0e^{-\lambda_0}+\Big[e^{-\lambda}\Big]_{\lambda_0}^{0}-\Big[e^{-\lambda}\Big]_0^{+\infty}\bigg)+Be^{-\lambda_0}=A\big(-\lambda_0e^{-\lambda_0}-e^{-\lambda_0}+2\big)+Be^{-\lambda_0}=$$

$$=2A-\big[A(\lambda_0+1)-B\big]e^{-\lambda_0}=:c<+\infty,$$

where $c=c(q,b)$ depends on q and b only. Notice that, here, we are considering the "worst" case in which $\lambda_0\leq 0$ (if $\lambda_0>0$, the calculations are very similar and lead to the same result).

Step 2: proof of *(i)*. We start by the first claim, which is quite immediate: noting that p and q are the conjugate exponents of each other, Hölder's inequality yields

$$\left(\int_{-\infty}^{+\infty} a(s,t)\phi(s)ds\right)^p \leq \left(\int_{-\infty}^{+\infty} a(s,t)^p ds\right)\left(\int_{-\infty}^{+\infty} \phi(s)^q ds\right)^{\frac{p}{q}} \leq \int_{-\infty}^{+\infty} a(s,t)^p ds =$$

$$= \int_{-\infty}^{0} a(s,t)^p ds + \int_{0}^{t} a(s,t)^p ds + \int_{t}^{+\infty} a(s,t)^p ds \leq b^p + t = c + t \iff$$

$$\iff F(t) := t - \left(\int_{-\infty}^{+\infty} a(s,t)\phi(s)ds\right)^p \geq -c,$$

which is exactly *(i)* for $c = c(q,b) := b^p = b^{\frac{q}{q-1}}$.

Step 3: proof of *(ii)*. For the point *(ii)*, let $t \in E_\lambda$ (otherwise, if E_λ is empty, there is nothing that needs to be shown). Using (77), (78), (79), (81) and Hölder's inequality one more time, we get

$$t - \lambda < t - F(t) = \left(\int_{-\infty}^{+\infty} a(s,t)\phi(s)ds\right)^p =$$

$$= \left(\int_{-\infty}^{t} a(s,t)\phi(s)ds + \int_{t}^{+\infty} a(s,t)\phi(s)ds\right)^p \leq$$

$$\leq \left[\left(\int_{-\infty}^{t} a(s,t)^p ds\right)^{\frac{1}{p}}\left(\int_{-\infty}^{t} \phi(s)^q ds\right)^{\frac{1}{q}} + \left(\int_{t}^{+\infty} a(s,t)^p ds\right)^{\frac{1}{p}}\left(\int_{t}^{+\infty} \phi(s)^q ds\right)^{\frac{1}{q}}\right]^p \leq$$

$$\leq \left[\left(\int_{-\infty}^{0} a(s,t)^p ds + \int_{0}^{t} a(s,t)^p ds\right)^{\frac{1}{p}}\left(\int_{-\infty}^{+\infty} \phi(s)^q ds - \int_{t}^{+\infty} \phi(s) ds^q\right)^{\frac{1}{q}} + bL(t)\right]^p \leq$$

$$\leq \left[\left(b^p + t\right)^{\frac{1}{p}}\left(1 - L(t)^q\right)^{\frac{1}{q}} + bL(t)\right]^p, \tag{83}$$

having set

$$L(t) := \left(\int_{t}^{+\infty} \phi(s)^q ds\right)^{\frac{1}{q}}.$$

Now, we make use of two elementary estimates: the first one is given by

$$(\gamma + \delta)^r \leq \gamma^r + 2^{r-1} r\left(\gamma^{r-1}\delta + \delta^r\right), \tag{84}$$

valid for every non-negative real numbers γ and δ and $\forall\, r \geq 1$, while the second one is

$$(1-x)^l \leq 1 - lx, \tag{85}$$

true $\forall\, x \in [0,1]$ and $\forall\, l \leq 1$. Thus, using that $L(t) \leq 1$ by hypothesis and applying

(84) (for $\gamma := \big(b^p+t\big)^{\frac{1}{p}}\big(1-L(t)^q\big)^{\frac{1}{q}}$, $\delta := bL(t)$ and $r := p$) and (85) (for $x := L(t)^q$ and $l := \frac{1}{q}$) to (83), one has

$$t-\lambda < \Big[\big(b^p+t\big)^{\frac{1}{p}}\big(1-L(t)^q\big)^{\frac{1}{q}} + bL(t)\Big]^p \leq$$

$$\leq \big(b^p+t\big)\big(1-L(t)^q\big)^{\frac{1}{q-1}} + 2^{\frac{1}{q-1}}\,\frac{q}{q-1}\Big[bL(t)\big(b^p+t\big)^{\frac{1}{q}}\big(1-L(t)^q\big)^{\frac{1}{q(q-1)}} + b^pL(t)^p\Big] \leq$$

$$\leq \big(b^p+t\big)\bigg(1-\frac{1}{q-1}\,L(t)^q\bigg) + 2^{\frac{1}{q-1}}\,\frac{q}{q-1}\Big[bL(t)\big(b^p+t\big)^{\frac{1}{q}} + b^p\Big] =$$

$$= b^p+t-\frac{1}{q-1}\big(b^p+t\big)L(t)^q + c_1\big(b^p+t\big)^{\frac{1}{q}}L(t) + c_2 = b^p+t-\frac{1}{q-1}\,\sigma^q + c_1\sigma + c_2,$$

where $c_1 = c_1(q,b) := 2^{\frac{1}{q-1}}\,\frac{q}{q-1}\,b$, $c_2 = c_2(q,b) := 2^{\frac{1}{q-1}}\,\frac{q}{q-1}\,b^p$ and $\sigma := \big(b^p+t\big)^{\frac{1}{q}}L(t)$. We notice that, in order to use (85), we had to assume that $q \geq 2$: if it is not the case, namely if $1 < q < 2$, then it suffices to use the estimate $(1-x)^l \leq 1-x$ valid for $x \in [0,1]$ and for $l \geq 1$ instead of (85) to get a similar conclusion (the only difference is that the coefficient of σ^p will be 1). Therefore, subtracting t from both sides, reversing the relation and using Young's inequality on $c_1\sigma$, we get

$$\lambda > \frac{1}{q-1}\,\sigma^q - c_1\sigma - c_2 - b^p \geq \frac{1}{q-1}\,\sigma^q - \frac{1}{q}\,\sigma^q - \frac{1}{p}\,c_1^p - c_2 - b^p \Longleftrightarrow$$

$$\Longleftrightarrow \bigg(\frac{1}{q-1}-\frac{1}{q}\bigg)\sigma^q < \frac{1}{p}\,c_1^p + c_2 + b^p + \lambda \leq \frac{1}{p}\,c_1^p + c_2 + b^p + |\lambda| \Longleftrightarrow$$

$$\Longleftrightarrow \sigma < q^{\frac{1}{q}}(q-1)^{\frac{1}{q}}\bigg(\frac{1}{p}\,c_1^p + c_2 + b^p + |\lambda|\bigg)^{\frac{1}{q}} = \big(c_3 + c_4|\lambda|\big)^{\frac{1}{q}} \leq 2^{\frac{1}{q}}\bigg(c_3^{\frac{1}{q}} + c_4^{\frac{1}{q}}|\lambda|^{\frac{1}{q}}\bigg) =$$

$$= c_5 + c_6|\lambda|^{\frac{1}{q}}, \tag{86}$$

where we resorted to the inequality $(a+b)^d \leq 2^d\big(a^d+b^d\big)$, true whenever $a,b,d \geq 0$, and where we also set the four constants $c_3 = c_3(q,b) := \frac{q(q-1)}{p}\,c_1^p + c_2q(q-1) + q(q-1)b^p$, $c_4 = c_4(q) := q(q-1)$, $c_5 = c_5(q,b) := (2c_3)^{\frac{1}{q}}$ and $c_6 = c_6(q) := (2c_4)^{\frac{1}{q}}$.
We reiterate that (86) holds for $q \geq 2$: for the remaining cases, the discussion is almost the same (instead of multiplying for $q(q-1)$, one has to multiply for $\frac{q}{q-1}$ to get slightly different constants but same end). Now, (86) also tells us that λ is bounded from above because this is true for $\frac{1}{q-1}\,\sigma^q - \frac{1}{q}\,\sigma^q$ and λ is greater that this value. Further, using the definitions of σ and $L(t)$, we get

$$\big(b^p+t\big)^{\frac{1}{q}}\bigg(\int_t^{+\infty}\phi(s)^q ds\bigg)^{\frac{1}{q}} = \big(b^p+t\big)^{\frac{1}{q}}L(t) = \sigma < c_5 + c_6|\lambda|^{\frac{1}{q}}. \tag{87}$$

Now, let $R>0$ and suppose that $E_\lambda \cap [R,+\infty)$ is not empty. Take then two elements t_1, $t_2 \in E_\lambda \cap [R,+\infty)$ such that $t_1 < t_2$. Using again (77), (78), (79), (81), Hölder's inequality and the definition of $L(t)$, we get

$$t_2 - \lambda < t_2 - F(t_2) = \Big(\int_{-\infty}^{+\infty} a(s,t_2)\phi(s)ds\Big)^p \leq$$

$$\leq \Big(\int_{-\infty}^{t_1} a(s,t_2)\phi(s)ds + \int_{t_1}^{t_2} a(s,t_2)\phi(s)ds + \int_{t_1}^{+\infty} a(s,t_2)\phi(s)ds\Big)^p \leq$$

$$\leq \bigg(\Big(\int_{-\infty}^{t_1} a(s,t_2)^p ds\Big)^{\frac{1}{p}}\Big(\int_{-\infty}^{t_1} \phi(s)^q ds\Big)^{\frac{1}{q}} + \Big(\int_{t_1}^{t_2} a(s,t_2)^p ds\Big)^{\frac{1}{p}}\Big(\int_{t_1}^{t_2} \phi(s)^q ds\Big)^{\frac{1}{q}} +$$

$$+\Big(\int_{t_1}^{+\infty} a(s,t_2)^p ds\Big)^{\frac{1}{p}}\Big(\int_{t_1}^{+\infty} \phi(s)^q ds\Big)^{\frac{1}{q}}\bigg)^p \leq$$

$$\leq \bigg(\Big(\int_{-\infty}^{0} a(s,t_2)^p ds + \int_{0}^{t_1} a(s,t_2)^p ds\Big)^{\frac{1}{p}} + (t_2-t_1)^{\frac{1}{p}}\Big(\int_{t_1}^{+\infty} \phi(s)^q ds\Big)^{\frac{1}{q}} + bL(t_1)\bigg)^p \leq$$

$$\leq \Big(\big(b^p + t_1\big)^{\frac{1}{p}} + TL(t_1) + bL(t_1)\Big)^p = \Big(\big(b^p + t_1\big)^{\frac{1}{p}} + (T+b)L(t_1)\Big)^p,$$

where we set $T := (t_2 - t_1)^{\frac{1}{p}}$. We make another use of (84), now with $\gamma := \big(b^p + t_1\big)^{\frac{1}{p}}$ and $\delta := (T+b)L(t_1)$, and, due to (79), (87) and the previous elementary estimate, we obtain

$$t_2 - \lambda < \Big(\big(b^p + t_1\big)^{\frac{1}{p}} + (T+b)L(t_1)\Big)^p \leq$$

$$\leq b^p + t_1 + 2^{p-1}p\Big(\big(b^p+t_1\big)^{\frac{p-1}{p}}(T+b)L(t_1) + (T+b)^p L(t_1)^p\Big) =$$

$$= b^p + t_1 + 2^{\frac{1}{q-1}}\frac{q}{q-1}\Big(\big(b^p+t_1\big)^{\frac{1}{q}}L(t_1)(T+b) + (T+b)^p L(t_1)^p\Big) \leq$$

$$\leq b^p + t_1 + 2^{\frac{1}{q-1}}\frac{q}{q-1}\Big(\Big(c_5 + c_6|\lambda|^{\frac{1}{q}}\Big)(T+b) + (T+b)^p L(t_1)\Big) \leq$$

$$\leq b^p + t_1 + 2^{\frac{1}{q-1}}\frac{q}{q-1}\Big(\Big(c_5 + c_6|\lambda|^{\frac{1}{q}}\Big)(T+b) + 2^p\big(T^p + b^p\big)L(t_1)\Big) \leq$$

$$\leq t_1 + c_7 + c_8(T+b)\Big(c_5 + c_6|\lambda|^{\frac{1}{q}}\Big) + c_9 L(t_1)T^p,$$

where we set the constants $c_7 = c_7(q,b) := b^p\Big(1 + 2^{\frac{q+1}{q-1}}\frac{q}{q-1}\Big)$, $c_8 = c_8(q) := 2^{\frac{1}{q-1}}\frac{q}{q-1}$ and $c_9 = c_9(q) := 2^{\frac{q+1}{q-1}}\frac{q}{q-1}$. Therefore,

$$T^p = t_2 - t_1 <$$

$$< \lambda + c_7 + c_8(T+b)\Big(c_5 + c_6|\lambda|^{\frac{1}{q}}\Big) + c_9 L(t_1)T^p \leq$$

$$\le |\lambda| + c_7 + c_8(T+b)\Big(c_5 + c_6|\lambda|^{\frac{1}{q}}\Big) + c_9 L(t_1)T^p =$$

$$= |\lambda| + c_7 + c_5c_8T + c_6c_8T|\lambda|^{\frac{1}{q}} + c_5b + c_6b|\lambda|^{\frac{1}{q}} + c_9L(t_1)T^p =$$

$$= |\lambda| + c_{10} + c_{11}T + c_{12}T|\lambda|^{\frac{1}{q}} + c_{13}|\lambda|^{\frac{1}{q}} + c_9L(t_1)T^p, \tag{88}$$

with, now, $c_{10} = c_{10}(q,b) := c_5b + c_7$, $c_{11} = c_{11}(q,b) := c_5c_8$, $c_{12} = c_{12}(q) := c_6c_8$ and $c_{13} = c_{13}(q,b) := c_6b$. Now, we introduce two constants $\varepsilon = \varepsilon(q) > 0$ and $c_{14} = c_{14}(q)$: the first one will be fixed later, while we choose the second one so that $|\lambda|^{\frac{1}{q}} \le |\lambda| + c_{14}$. In this way, making use of Young's inequality, (88) becomes

$$T^p < |\lambda| + c_{10} + c_{11}T + c_{12}T|\lambda|^{\frac{1}{q}} + c_{13}|\lambda|^{\frac{1}{q}} + c_9L(t_1)T^p =$$

$$= |\lambda| + c_{10} + \frac{c_{11}}{\varepsilon}\,\varepsilon T + c_{12}\varepsilon T\,\frac{|\lambda|^{\frac{1}{q}}}{\varepsilon} + c_{13}|\lambda|^{\frac{1}{q}} + c_9L(t_1)T^p \le$$

$$\le |\lambda| + c_{10} + \frac{c_{11}^q}{q\varepsilon^q} + \frac{\varepsilon^pT^p}{p} + c_{12}\left(\frac{\varepsilon^pT^p}{p} + \frac{|\lambda|}{q\varepsilon^q}\right) + c_{13}\big(|\lambda| + c_{14}\big) + c_9L(t_1)T^p =$$

$$= |\lambda| + c_{10} + \frac{c_{11}^q}{q\varepsilon^q} + \frac{q-1}{q}\,\varepsilon^pT^p + c_{12}\,\frac{q-1}{q}\,\varepsilon^pT^p + \frac{c_{12}}{q\varepsilon^q}\,|\lambda| + c_{13}|\lambda| + c_{13}c_{14} + c_9L(t_1)T^p =$$

$$= \left[1 + \frac{c_{12}}{q\varepsilon^q} + c_{13}\right]|\lambda| + c_{15} + \left[(1+c_{12})\,\frac{q-1}{q}\,\varepsilon^p + c_9L(t_1)\right]T^p = c_{16}|\lambda| + c_{15} + c_{17}T^p,$$

defining, this time, $c_{15} = c_{15}(q,b) := c_{10} + \frac{c_{11}^q}{q\varepsilon^q} + c_{13}c_{14}$, $c_{16} = c_{16}(q,b) := 1 + \frac{c_{12}}{q\varepsilon^q} + c_{13}$ and $c_{17} = c_{17}(q,b) := \big[(1+c_{12})\,\frac{q-1}{q}\,\varepsilon^p + c_9L(t_1)\big]$. Consequently, we have that

$$(1 - c_{17})T^p < c_{16}|\lambda| + c_{15}:$$

fixing ε in an appropriate way, for example $\varepsilon := \big(\frac{1}{4}\,\frac{q}{q-1}\,\frac{1}{1+c_{12}}\big)^{\frac{1}{p}}$, and choosing R large enough so that $L(t_1)$ is small at will (let say $L(t_1) \le \frac{1}{4c_9}$), we are allowed to suppose that c_{17} is not greater than $\frac{1}{2}$, which means that we have achieved

$$\frac{1}{2}\,T^p \le (1 - c_{17})T^p < c_{16}|\lambda| + c_{15}$$

and so, due to the arbitrariness of t_1 and t_2,

$$|E_\lambda \cap [R, +\infty)| < 2c_{16}|\lambda| + 2c_{15}.$$

Finally, we also have that

$$|E_\lambda \cap [0, R)| \le |[0, R)| = R,$$

which means that, in conclusion, we have achieved

$$|E_\lambda| = |E_\lambda \cap [0,R)| + |E_\lambda \cap [R,+\infty)| < R + 2c_{16}|\lambda| + 2c_{15} = A|\lambda| + B$$

once we set $A = A(q,b) := 2c_{16}$ and $B = B(q,b) := 2c_{15} + R$. Therefore, we have (even more than) the point *(ii)* and, consequently, the thesis of the lemma.

□

We are now in position to prove Theorem 10.

Proof of Theorem 10. We are going to use mainly two results already acquired to get the thesis, namely O'Neil's theorem and the preceding lemma.
First of all, we consider f to be non-negative without loss of generality: in fact, if this is not the case, then

$$\fint_\Omega e^{\frac{n}{\omega_{n-1}}\left|\frac{(I_\beta * f)(x)}{||f||_q}\right|^p} dx \leq \fint_\Omega e^{\frac{n}{\omega_{n-1}}\left[\frac{(I_\beta * |f|)(x)}{||\,|f|\,||_q}\right]^p} dx,$$

since

$$|(I_\beta * f)(x)| = \left|\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}}\,dy\right| \leq \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\beta}}\,dy = (I_\beta * |f|)(x)$$

and, obviously, $||f||_q = ||\,|f|\,||_q$.
We now proceed step by step.

<u>Step 1: utilizing Theorem 8.</u> For convenience, we set $u(x) := (I_\beta * f)(x)$. In according to Definition 1, we have

$$\mu_{I_\beta}(t) := |\{I_\beta > t\}| = \left|\{|x|^{\beta-n} > t\}\right| = \left|\left\{|x| < t^{-\frac{1}{n-\beta}}\right\}\right| = \frac{\omega_{n-1}}{n}\,t^{-\frac{n}{n-\beta}}$$

and therefore, a.e.,

$$I_\beta^\#(s) := \inf_{t\in\mathbb{R}}\{\mu_{I_\beta}(t) < s\} = \inf_{t\in\mathbb{R}}\left\{\frac{\omega_{n-1}}{n}\,t^{-\frac{n}{n-\beta}} < s\right\} = \inf_{t\in\mathbb{R}}\left\{t > \left(\frac{\omega_{n-1}}{ns}\right)^{\frac{n-\beta}{n}}\right\} =$$

$$= \left(\frac{\omega_{n-1}}{ns}\right)^{\frac{n-\beta}{n}} = \left(\frac{\omega_{n-1}}{ns}\right)^{\frac{1}{p}}.$$

Hence, using the definition of p, one has that

$$I_\beta^{\#\#}(t) := \frac{1}{t}\int_0^t I_\beta^\#(s)ds = \frac{1}{t}\int_0^t \left(\frac{\omega_{n-1}}{ns}\right)^{\frac{1}{p}} ds = \frac{1}{t}\left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{p}}\int_0^t s^{-\frac{1}{p}}ds =$$

$$= \frac{1}{t}\left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{p}}\frac{p}{p-1}\,t^{1-\frac{1}{p}} = \frac{q}{t^{\frac{1}{p}}}\left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{p}} = q\left(\frac{\omega_{n-1}}{nt}\right)^{\frac{1}{p}} = qI_\beta^\#(t).$$

Now, O'Neil's theorem and the properties of the rearrangement already studied lead us to

$$u^{\#}(t) \leq u^{\#\#}(t) \leq tf^{\#\#}(t)I_{\beta}^{\#\#}(t) + \int_t^{+\infty} f^{\#}(s)I_{\beta}^{\#}(s)ds =$$

$$= t\Big(\frac{1}{t}\int_0^t f^{\#}(s)ds\Big)qI_{\beta}^{\#}(t) + \int_t^{+\infty} f^{\#}(s)\Big(\frac{\omega_{n-1}}{ns}\Big)^{\frac{1}{p}}ds =$$

$$= q\Big(\frac{\omega_{n-1}}{nt}\Big)^{\frac{1}{p}}\int_0^t f^{\#}(s)ds + \Big(\frac{\omega_{n-1}}{n}\Big)^{\frac{1}{p}}\int_t^{+\infty} s^{-\frac{1}{p}}f^{\#}(s)ds =$$

$$= \Big(\frac{\omega_{n-1}}{n}\Big)^{\frac{1}{p}}\Big[qt^{-\frac{1}{p}}\int_0^t f^{\#}(s)ds + \int_t^{|\Omega|} s^{-\frac{1}{p}}f^{\#}(s)ds\Big], \tag{89}$$

where we also used that f is supported on Ω.

Step 2: utilizing Lemma 6. Next, we change variables by setting

$$\phi(s) := \begin{cases} 0 & \text{if } s \in (-\infty, 0) \\ ||f||_q^{-1}\,|\Omega|^{\frac{1}{q}}\,e^{-\frac{s}{q}}f^{\#}\big(|\Omega|e^{-s}\big) & \text{if } s \in [0, +\infty) \end{cases}$$

in order to apply Lemma 6: we define

$$a(s,t) := \begin{cases} 0 & \text{if } s \in (-\infty, 0) \\ 1 & \text{if } s \in [0, t) \\ qe^{\frac{t-s}{p}} & \text{if } s \in [t, +\infty) \end{cases}$$

and notice that a is a non-negative Borel measurable function constantly equal to 1 if $s \in (0,t)$ (so that the hypothesis (77) of Lemma 6 is satisfied). Further, we have that

$$b := \sup_{t>0}\Big\{\Big(\int_{-\infty}^0 a(s,t)^p ds + \int_t^{+\infty} a(s,t)^p ds\Big)^{\frac{1}{p}}\Big\} < +\infty,$$

which means that also (78) is valid since, $\forall\, t \geq 0$,

$$\Big(\int_{-\infty}^0 a(s,t)^p ds + \int_t^{+\infty} a(s,t)^p ds\Big)^{\frac{1}{p}} = \Big(\int_t^{+\infty} q^p e^{t-s}ds\Big)^{\frac{1}{p}} = qe^{\frac{t}{p}}\Big(\Big[-e^{-s}\Big]_t^{+\infty}\Big)^{\frac{1}{p}} =$$

$$= qe^{\frac{t}{p}}e^{-\frac{t}{p}} = q.$$

Lastly, the hypothesis (79) is satisfied, too, because

$$\int_{-\infty}^{+\infty} \phi(s)^q ds = ||f||_q^{-q}\int_0^{+\infty} |\Omega|e^{-s}f^{\#}\big(|\Omega|e^{-s}\big)^q ds = -||f||_q^{-q}\int_{|\Omega|}^0 f^{\#}(\tau)^q d\tau =$$

$$= ||f||_q^{-q} \int_0^{|\Omega|} f^{\#}(\tau)^q d\tau = ||f||_q^{-q} \int_\Omega f(x)^q dx = 1,$$

where we defined the variable $\tau := |\Omega| e^{-s}$ and used the properties of the rearrangement (Corollary 2). So, given $F(t)$ as in (81), Lemma 6 guarantees us that there exists a constant $c = c(q,b) = c(q)$ (since $b = q$ in this case) depending on q only such that

$$\int_0^{+\infty} e^{-F(t)} dt \le c.$$

We now rewrite $F(t)$ as

$$F(t) := t - \left(\int_{-\infty}^{+\infty} a(s,t)\phi(s) ds \right)^p = t - \left(\int_0^t \phi(s) ds + \int_t^{+\infty} q e^{\frac{t-s}{p}} \phi(s) ds \right)^p =$$

$$= t - \left(\int_0^t ||f||_q^{-1} |\Omega|^{\frac{1}{q}} e^{-\frac{s}{q}} f^{\#}\big(|\Omega| e^{-s}\big) ds + \int_t^{+\infty} q e^{\frac{t-s}{p}} ||f||_q^{-1} |\Omega|^{\frac{1}{q}} e^{-\frac{s}{q}} f^{\#}\big(|\Omega| e^{-s}\big) ds \right)^p =$$

$$= t - ||f||_q^{-p} \left(\int_0^t \big(|\Omega| e^{-s}\big)^{\frac{1}{q}} f^{\#}\big(|\Omega| e^{-s}\big) ds + q e^{\frac{t}{p}} \int_t^{+\infty} e^{-\frac{s}{p}} \big(|\Omega| e^{-s}\big)^{\frac{1}{q}} f^{\#}\big(|\Omega| e^{-s}\big) ds \right)^p =$$

$$= t - ||f||_q^{-p} \left(- \int_{|\Omega|}^{|\Omega| e^{-t}} \tau^{\frac{1}{q}} f^{\#}(\tau) \frac{d\tau}{\tau} - q e^{\frac{t}{p}} \int_{|\Omega| e^{-t}}^0 \left(\frac{\tau}{|\Omega|} \right)^{\frac{1}{p}} \tau^{\frac{1}{q}} f^{\#}(\tau) \frac{d\tau}{\tau} \right)^p =$$

$$= t - ||f||_q^{-p} \left(\int_{|\Omega| e^{-t}}^{|\Omega|} \tau^{-\frac{1}{p}} f^{\#}(\tau) d\tau + q \big(|\Omega| e^{-t}\big)^{-\frac{1}{p}} \int_0^{|\Omega| e^{-t}} f^{\#}(\tau) d\tau \right)^p,$$

having used again the variable τ defined as before. It follows that

$$\int_0^{+\infty} e^{-F(t)} dt =$$

$$= \int_0^{+\infty} \exp\left[||f||_q^{-p} \left(\int_{|\Omega| e^{-t}}^{|\Omega|} \tau^{-\frac{1}{p}} f^{\#}(\tau) d\tau + q \big(|\Omega| e^{-t}\big)^{-\frac{1}{p}} \int_0^{|\Omega| e^{-t}} f^{\#}(\tau) d\tau \right)^p - t \right] dt =$$

$$= - \int_{|\Omega|}^0 \exp\left[||f||_q^{-p} \left(\int_\xi^{|\Omega|} \tau^{-\frac{1}{p}} f^{\#}(\tau) d\tau + q \xi^{-\frac{1}{p}} \int_0^\xi f^{\#}(\tau) d\tau \right)^p + \log\left(\frac{\xi}{|\Omega|} \right) \right] \frac{d\xi}{\xi} =$$

$$= \int_0^{|\Omega|} \exp\left[||f||_q^{-p} \left(q \xi^{-\frac{1}{p}} \int_0^\xi f^{\#}(\tau) d\tau + \int_\xi^{|\Omega|} \tau^{-\frac{1}{p}} f^{\#}(\tau) d\tau \right)^p \right] \frac{\xi}{|\Omega|} \frac{d\xi}{\xi} =$$

$$= \frac{1}{|\Omega|} \int_0^{|\Omega|} \exp\left[||f||_q^{-p} \left(q \xi^{-\frac{1}{p}} \int_0^\xi f^{\#}(\tau) d\tau + \int_\xi^{|\Omega|} \tau^{-\frac{1}{p}} f^{\#}(\tau) d\tau \right)^p \right] d\xi, \qquad (90)$$

where we introduced the variable $\xi := |\Omega| e^{-t}$. Finally, by Theorem 1, the definition of

u, (89) and (90), we get

$$\fint_\Omega e^{\frac{n}{\omega_{n-1}}\left[\frac{(I_\beta * f)(x)}{||f||_q}\right]^p} dx = \frac{1}{|\Omega|}\int_0^{|\Omega|} e^{\frac{n}{\omega_{n-1}}\left[\frac{u^\#(t)}{||f||_q}\right]^p} dt \leq$$

$$\leq \frac{1}{|\Omega|}\int_0^{|\Omega|} \exp\left[\frac{n}{\omega_{n-1}}\,||f||_q^{-p}\,\frac{\omega_{n-1}}{n}\left(qt^{-\frac{1}{p}}\int_0^t f^\#(s)ds + \int_t^{|\Omega|} s^{-\frac{1}{p}} f^\#(s)ds\right)^p\right]dt =$$

$$= \frac{1}{|\Omega|}\int_0^{|\Omega|} \exp\left[||f||_q^{-p}\left(qt^{-\frac{1}{p}}\int_0^t f^\#(s)ds + \int_t^{|\Omega|} s^{-\frac{1}{p}} f^\#(s)ds\right)^p\right]dt =$$

$$= \int_0^{+\infty} e^{-F(t)}dt \leq c.$$

□

We now show the validity of Proposition 8, which is strictly related to Theorem 10 (being a sort of completion of it) but will be useful for a result we will enunciate in the following chapter.

Proof of Proposition 8. Our aim is to show that it is not possible to take any number $\eta > \frac{n}{\omega_{n-1}}$ and still maintain the constant c independent of u. We can assume that Ω contains $B := B_1(0)$, the unit ball of $\mathbb{R}^n$ (we are always allowed to consider this kind of situation after an appropriate change of coordinates involving the dilatation and the translation). So, we assume (76) to be true with η replacing $\frac{n}{\omega_{n-1}}$ and show that, in such a case, we must have $\eta \leq \frac{n}{\omega_{n-1}}$.
We divide the discussion.

<u>Step 1: introduction of the Riesz conductor capacity $\mathcal{R}_{\beta,q}(E,B)$.</u> Let $f(x) \geq 0$ be as in the hypothesis of Theorem 10 and such that $(I_\beta * f)(x) \geq 1$ whenever $x \in \overline{B_r(0)}$, for a certain radius $r \in (0,1)$. If (76) holds with η instead of $\frac{n}{\omega_{n-1}}$, then

$$c \geq \fint_B e^{\eta\left|\frac{(I_\beta * f)(x)}{||f||_q}\right|^p} dx = \fint_B e^{\eta\left[\frac{(I_\beta * f)(x)}{||f||_q}\right]^p} dx \geq \frac{1}{|B|}\int_{\overline{B_r(0)}} e^{\eta\left[\frac{(I_\beta * f)(x)}{||f||_q}\right]^p} dx \geq$$

$$\geq \frac{1}{|B|}\int_{B_r(0)} e^{\frac{\eta}{||f||_q^p}} dx = \frac{|B_r(0)|}{|B|}\, e^{\frac{\eta}{||f||_q^p}} \iff \frac{\eta}{||f||_q^p} \leq \log\left(\frac{|B|}{|B_r(0)|}\,c\right) \iff$$

$$\iff \eta \leq \log\left(\frac{|B|}{|B_r(0)|}\,c\right)||f||_q^p.$$

Hence, if we define

$$\mathcal{R}_{\beta,q}(E,B) := \inf_{f\in\mathcal{A}(E,B)}\left\{||f||_q^q\right\}, \tag{91}$$

where E is a compact subset of B and

$$\mathcal{A}(E,B) := \big\{ f \in L^q(\mathbb{R}^n) : f(x) \geq 0,\ \mathrm{supp}\{f\} \subseteq B \text{ and } (I_\beta * f)(x) \geq 1 \text{ on } E \big\},$$

then

$$\eta \leq \log\left(\frac{|B|}{|B_r(0)|}\,c\right)\mathcal{R}_{\beta,q}\Big(\overline{B_r(0)},B\Big)^{\frac{1}{q-1}} = \log\left(\frac{\omega_{n-1}}{n}\,\frac{n}{\omega_{n-1}}\,\frac{c}{r^n}\right)\mathcal{R}_{\beta,q}\Big(\overline{B_r(0)},B\Big)^{\frac{1}{q-1}} =$$

$$= \left[\log(c) + n\log\left(\frac{1}{r}\right)\right]\mathcal{R}_{\beta,q}\Big(\overline{B_r(0)},B\Big)^{\frac{1}{q-1}} =$$

$$= n\log\left(\frac{1}{r}\right)\mathcal{R}_{\beta,q}\Big(\overline{B_r(0)},B\Big)^{\frac{1}{q-1}}\left[1 + \frac{\log(c)}{n\log\left(\frac{1}{r}\right)}\right].$$

The quantity in (91) is called the Riesz conductor capacity of the compact subset E. Now, if $r \longmapsto 0^+$, the constant c does not appear anymore because

$$\frac{\log(c)}{n\log\left(\frac{1}{r}\right)} \longrightarrow 0.$$

Therefore, we get

$$\eta \leq n \liminf_{r\to 0^+} \log\left(\frac{1}{r}\right)\mathcal{R}_{\beta,q}\Big(\overline{B_r(0)},B\Big)^{\frac{1}{q-1}}. \tag{92}$$

We now have to focus on the Riesz conductor capacity $\mathcal{R}_{\beta,q}\Big(\overline{B_r(0)},B\Big)$.

<u>Step 2: a useful relation.</u> Here, we make some calculations which will be useful later. Given $x \in \mathbb{R}^n$ and $\rho \in (|x|,+\infty)$, we have

$$\frac{1}{\omega_{n-1}}\int_{|y|\leq\rho}\frac{dy}{|x-y|^{n-\beta}|y|^\beta} = \frac{1}{\omega_{n-1}}\int_{\{|\sigma|=1\}}\left(\int_0^\rho |x-\sigma s|^{\beta-n}s^{-\beta}s^{n-1}ds\right)d\sigma =$$

$$= -\frac{1}{\omega_{n-1}}\int_{\{|\sigma|=1\}}\left(\int_{+\infty}^{\frac{|x|}{\rho}}\left|x-\sigma\,\frac{|x|}{t}\right|^{\beta-n}\left(\frac{|x|}{t}\right)^{n-1-\beta}|x|\,\frac{dt}{t^2}\right)d\sigma =$$

$$= \frac{1}{\omega_{n-1}}\int_{\{|\sigma|=1\}}\left(\int_{\frac{|x|}{\rho}}^{+\infty}\left|x-\sigma\,\frac{|x|}{t}\right|^{\beta-n}\left(\frac{|x|}{t}\right)^{n-\beta}\frac{dt}{t}\right)d\sigma =$$

$$= \frac{1}{\omega_{n-1}}\int_{\{|\sigma|=1\}}\left(\int_{\frac{|x|}{\rho}}^{+\infty}\left(\frac{|x|}{t}\right)^{\beta-n}\left|t\,\frac{x}{|x|}-\sigma\right|^{\beta-n}\left(\frac{|x|}{t}\right)^{n-\beta}\frac{dt}{t}\right)d\sigma =$$

$$= \frac{1}{\omega_{n-1}}\int_{\{|\sigma|=1\}}\left(\int_{\frac{|x|}{\rho}}^{+\infty}\left|t\,\frac{x}{|x|}-\sigma\right|^{\beta-n}\frac{dt}{t}\right)d\sigma = \int_{\frac{|x|}{\rho}}^{+\infty}\frac{u_\beta(t)}{t}\,dt, \tag{93}$$

where we defined

$$u_\beta(t) := \frac{1}{\omega_{n-1}} \int_{\{|\sigma|=1\}} \left| t\,\frac{x}{|x|} - \sigma \right|^{\beta-n} d\sigma$$

for $t \geq 0$, as done also in [8]. Notice that, in (93), we changed coordinates twice (by setting $s := \frac{y}{\sigma}$ in the first passage and $t := \frac{|x|}{s}$ in the second one) and defined the function u_β after an application of Fubini's theorem. Furthermore, u_β turns out to be, apart from the usual constant appearing in front of the integral, the Riesz potential of order β of the unit mass on the unit sphere in $\mathbb{R}^n$ evaluated at a point of distance t from the origin.

We observe some properties of u_β: firstly, it does not depend on x since we are allowed to replace the term $\frac{x}{|x|}$ by any unit vector. In fact, by definition, if we introduce a matrix $M \in O(n)$ and set $\tilde{x} := Mx$, we get

$$\int_{\{|\sigma|=1\}} \left| t\,\frac{\tilde{x}}{|\tilde{x}|} - \sigma \right|^{\beta-n} d\sigma = \int_{\{|\sigma|=1\}} \left| t\,\frac{Mx}{|Mx|} - \sigma \right|^{\beta-n} d\sigma = \int_{\{|\sigma|=1\}} \left| t\,\frac{Mx}{|x|} - \sigma \right|^{\beta-n} d\sigma =$$

$$= \int_{\{|\theta|=1\}} \left| t\,\frac{Mx}{|x|} - M\theta \right|^{\beta-n} d\theta = \int_{\{|\theta|=1\}} \left| M\left(t\,\frac{x}{|x|} - \theta \right) \right|^{\beta-n} d\theta = \int_{\{|\theta|=1\}} \left| t\,\frac{x}{|x|} - \theta \right|^{\beta-n} d\theta,$$

where we set $\theta := \frac{\sigma}{M}$ and used the fact that $||M|| = 1$ (which implies that $|Mx| = |x|$ and $|\theta| = |\sigma|$). Moreover, there is a definition problem only for $t = 1$ (in such a case, the integrand would be null because $\frac{x}{|x|}$ and σ are both unit vectors). Furthermore, it is easily seen that:

- $u_\beta(0) = 1$ because

$$u_\beta(0) = \frac{1}{\omega_{n-1}} \int_{\{|\sigma|=1\}} |-\sigma|^{\beta-n} d\sigma = \frac{1}{\omega_{n-1}} \int_{\{|\sigma|=1\}} d\sigma = 1;$$

- for $t \neq 1$, we have

$$u'_\beta(t) = \frac{1}{\omega_{n-1}} \int_{\{|\sigma|=1\}} \frac{\beta-n}{2} \left| t\,\frac{x}{|x|} - \sigma \right|^{\beta-n-2} 2\left(t\,\frac{x}{|x|} - \sigma \right) \cdot \frac{x}{|x|}\, d\sigma =$$

$$= \frac{\beta-n}{\omega_{n-1}} \int_{\{|\sigma|=1\}} \left| t\,\frac{x}{|x|} - \sigma \right|^{\beta-n-2} \left(t\,\frac{x}{|x|} - \sigma \right) \cdot \frac{x}{|x|}\, d\sigma,$$

which means that u_β is differentiable for every $t \neq 1$ and that

$$u'_\beta(0) = \frac{\beta-n}{\omega_{n-1}} \int_{\{|\sigma|=1\}} |-\sigma|^{\beta-n-2} (-\sigma) \cdot \frac{x}{|x|}\, d\sigma = \frac{\beta-n}{\omega_{n-1}} \int_{\{|\sigma|=1\}} (-\sigma) \cdot \frac{x}{|x|}\, d\sigma = 0$$

by symmetry (because, for every unit vector ξ, there is its corresponding antipodal one that makes the above scalar product the opposite as compared to ξ);

- u_β is integrable in a neighborhood of $t=1$ because, $\forall\, \varepsilon \in (0,1)$, Fubini's theorem tells us that

$$\int_{1-\varepsilon}^{1+\varepsilon} u_\beta(t)dt = \frac{1}{\omega_{n-1}} \int_{\{|\sigma|=1\}} \left(\int_{1-\varepsilon}^{1+\varepsilon} \left| t\,\frac{x}{|x|} - \sigma \right|^{\beta-n} dt \right) d\sigma,$$

  where

$$\int_{1-\varepsilon}^{1+\varepsilon} \left| t\,\frac{x}{|x|} - \sigma \right|^{\beta-n} dt \sim \left| \frac{x}{|x|} - \sigma \right|^{\beta-n+1}$$

  near $t=1$, which makes u_β indeed integrable near this point since, as $\beta \in (0,n)$, then $\beta - n + 1 > -n+1$.

Hence, if

$$v_\beta(t) := \begin{cases} \frac{u_\beta(t)-1}{t} & \text{if } t \in (0,1) \\ \frac{u_\beta(t)}{t} & \text{if } t \in (1,+\infty) \end{cases},$$

then v_β is bounded near $t=0$ because, as $t \longmapsto 0^+$, the previous calculations imply that $v_\beta(t) \longrightarrow 0$ (after an application of L'Hôpital's rule). Moreover, we also have that v_β is integrable over $(0,+\infty)$: in fact, near $t=0$, it is due to its boundedness; in a neighborhood of $t=1$, it is due to the integrability of u_β (which implies that of v_β by definition); in proximity of $+\infty$, it is due to the fact that

$$\int_1^{+\infty} v_\beta(t)dt = \int_1^{+\infty} \frac{u_\beta(t)}{t}\,dt = \frac{1}{\omega_{n-1}} \int_{\{|\sigma|=1\}} \left(\int_1^{+\infty} \left| t\,\frac{x}{|x|} - \sigma \right|^{\beta-n} \frac{dt}{t} \right) d\sigma$$

after applying Fubini's theorem, with (using that here t is positive)

$$\int_1^{+\infty} \left| t\,\frac{x}{|x|} - \sigma \right|^{\beta-n} \frac{dt}{t} = \int_1^{+\infty} |t|^{\beta-n} \left| \frac{x}{|x|} - \frac{\sigma}{t} \right|^{\beta-n} \frac{dt}{t} = \int_1^{+\infty} t^{\beta-n-1} \left| \frac{x}{|x|} - \frac{\sigma}{t} \right|^{\beta-n} dt \leq$$

$$\leq \int_1^{+\infty} t^{\beta-n-1} \left(\left| \frac{x}{|x|} \right| + \left| \frac{\sigma}{t} \right| \right)^{\beta-n} dt = \int_1^{+\infty} t^{\beta-n-1} \left(1 + \frac{1}{t} \right)^{\beta-n} dt < \int_1^{+\infty} t^{\beta-n-1} dt,$$

where we also used that $\left(1+\frac{1}{t}\right)^{\beta-n} < 1$, from which we have the integrability because $\beta - n - 1 < -1$, since $\beta \in (0,n)$. Note also that, by its definition, u_β is everywhere non-negative and, therefore, v_β is a non-negative function, again by definition, on the interval $(1,+\infty)$. Thus, being bounded near $t=0$, well defined on $(0,1)$ and integrable on a neighborhood of $t=1$, even $|v_\beta|$ is integrable on $(0,+\infty)$.
Further, (93) becomes

$$\frac{1}{\omega_{n-1}} \int_{|y|\leq\rho} \frac{dy}{|x-y|^{n-\beta}|y|^\beta} = \int_{\frac{|x|}{\rho}}^{+\infty} \frac{u_\beta(t)}{t}\,dt = \int_{\frac{|x|}{\rho}}^{1} \frac{u_\beta(t)}{t}\,dt + \int_1^{+\infty} \frac{u_\beta(t)}{t}\,dt =$$

$$= \int_{\frac{|x|}{\rho}}^{1} \frac{u_\beta(t)-1}{t}\,dt + \int_{\frac{|x|}{\rho}}^{1} \frac{dt}{t} + \int_{1}^{+\infty} \frac{u_\beta(t)}{t}\,dt = \int_{\frac{|x|}{\rho}}^{1} v_\beta(t)\,dt + \int_{\frac{|x|}{\rho}}^{1} \frac{dt}{t} + \int_{1}^{+\infty} v_\beta(t)dt =$$

$$= \Big[\log(t)\Big]_{\frac{|x|}{\rho}}^{1} + \int_{\frac{|x|}{\rho}}^{+\infty} v_\beta(t)dt = \log\left(\frac{\rho}{|x|}\right) + \int_{\frac{|x|}{\rho}}^{+\infty} v_\beta(t)dt. \tag{94}$$

The above formula will come in handy immediately.

Step 3: estimation of η through an auxiliary function. To complete our estimate for η in (92), we define the radial function

$$g(x) := \begin{cases} 0 & \text{if } |x| \in [0,r] \cup (1,+\infty) \\ \left[\omega_{n-1}\log\left(\frac{1}{r}\right)|x|^\beta\right]^{-1} & \text{if } |x| \in (r,1] \end{cases}$$

and note that, by definition, it is a non-negative function belonging to $L^q(\mathbb{R}^n)$ such that $\operatorname{supp}\{g\} \subseteq B$, since $r \in (0,1)$. Further, for $|x| < r$,

$$(I_\beta * g)(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\beta}}\,dy = \int_{r<|y|\le 1} \frac{dy}{|x-y|^{n-\beta}\omega_{n-1}\log\left(\frac{1}{r}\right)|y|^\beta} =$$

$$= \frac{1}{\log\left(\frac{1}{r}\right)}\left(\frac{1}{\omega_{n-1}}\int_{|y|\le 1} \frac{dy}{|x-y|^{n-\beta}|y|^\beta} - \frac{1}{\omega_{n-1}}\int_{|y|\le r} \frac{dy}{|x-y|^{n-\beta}|y|^\beta}\right) =$$

$$= \frac{1}{\log\left(\frac{1}{r}\right)}\left[\log\left(\frac{1}{|x|}\right) + \int_{|x|}^{+\infty} v_\beta(t)dt - \log\left(\frac{r}{|x|}\right) - \int_{\frac{|x|}{r}}^{+\infty} v_\beta(t)dt\right] =$$

$$= \frac{1}{\log\left(\frac{1}{r}\right)}\left[\log\left(\frac{1}{r}\right) + \int_{|x|}^{\frac{|x|}{r}} v_\beta(t)dt\right] = 1 + \frac{1}{\log\left(\frac{1}{r}\right)}\int_{|x|}^{\frac{|x|}{r}} v_\beta(t)dt, \tag{95}$$

where we used (94) twice (with $\rho := 1$ the first time and with $\rho := r$ the second one). Now, given an arbitrary $\varepsilon \in (0,1)$, there exists a number r_0 small enough such that, for every $r \in (0, r_0]$, we have

$$\left|\frac{1}{\log\left(\frac{1}{r}\right)}\int_{|x|}^{\frac{|x|}{r}} v_\beta(t)dt\right| \le \frac{1}{\log\left(\frac{1}{r_0}\right)}\int_0^{+\infty} |v_\beta(t)|dt < \varepsilon,$$

where in the last passage we used the integrability of the function $|v_\beta|$. Utilizing the above relation in (95), we obtain

$$\big|(I_\beta * g)(x)\big| = \left|1 + \frac{1}{\log\left(\frac{1}{r}\right)}\int_{|x|}^{\frac{|x|}{r}} v_\beta(t)dt\right| \ge 1 - \left|\frac{1}{\log\left(\frac{1}{r}\right)}\int_{|x|}^{\frac{|x|}{r}} v_\beta(t)dt\right| > 1-\varepsilon.$$

So, for small enough r, defining the function $\tilde{g}(x) := \frac{g(x)}{1-\varepsilon}$, using the non-negativity of the function g (which implies the one of $I_\beta * g$) and recalling the linearity of the convolution (see Remark 10), we get

$$(I_\beta * \tilde{g})(x) = \left[I_\beta * \left(\frac{g}{1-\varepsilon}\right)\right](x) = \frac{(I_\beta * g)(x)}{1-\varepsilon} = \frac{\big|(I_\beta * g)(x)\big|}{1-\varepsilon} > 1,$$

which means that $\tilde{g} \in \mathcal{A}\Big(\overline{B_r(0)}, B\Big)$. Consequently,

$$\mathcal{R}_{\beta,q}\Big(\overline{B_r(0), B}\Big) \leq ||\tilde{g}||_q^q = \int_{\mathbb{R}^n} \left|\frac{g(x)}{1-\varepsilon}\right|^q dx =$$

$$= \left[\frac{1}{(1-\varepsilon)\,\omega_{n-1}\log\left(\frac{1}{r}\right)}\right]^q \int_{r<|x|\leq 1} |x|^{-\beta q} dx =$$

$$= \left[\frac{1}{(1-\varepsilon)\,\omega_{n-1}\log\left(\frac{1}{r}\right)}\right]^q \int_r^1 \omega_{n-1}\xi^{n-1}\xi^{-\beta q} d\xi =$$

$$= \omega_{n-1}\left[\frac{1}{(1-\varepsilon)\,\omega_{n-1}\log\left(\frac{1}{r}\right)}\right]^q \int_r^1 \xi^{n-\beta q-1} d\xi = \omega_{n-1}^{1-q}\,(1-\varepsilon)^{-q}\log\left(\frac{1}{r}\right)^{-q}\int_r^1 \frac{d\xi}{\xi} =$$

$$= \omega_{n-1}^{1-q}\,(1-\varepsilon)^{-q}\log\left(\frac{1}{r}\right)^{-q}\Big[\log(\xi)\Big]_r^1 = -\omega_{n-1}^{1-q}\,(1-\varepsilon)^{-q}\log\left(\frac{1}{r}\right)^{-q}\log(r) =$$

$$= \omega_{n-1}^{1-q}\,(1-\varepsilon)^{-q}\log\left(\frac{1}{r}\right)^{1-q},$$

using the definition of β. Hence, recalling also the one of p, we have

$$\mathcal{R}_{\beta,q}\Big(\overline{B_r(0)}, B\Big)^{\frac{1}{q-1}} \leq \left[\omega_{n-1}^{1-q}\,(1-\varepsilon)^{-q}\log\left(\frac{1}{r}\right)^{1-q}\right]^{\frac{1}{q-1}} = \frac{(1-\varepsilon)^{-p}}{\omega_{n-1}\log\left(\frac{1}{r}\right)}. \tag{96}$$

Finally, using (96) inside (92), one has

$$\eta \leq n \liminf_{r\to 0^+} \log\left(\frac{1}{r}\right)\mathcal{R}_{\beta,q}\Big(\overline{B_r(0)}, B\Big)^{\frac{1}{q-1}} \leq \frac{n}{\omega_{n-1}}\,(1-\varepsilon)^{-p}\liminf_{r\to 0^+}\frac{\log\left(\frac{1}{r}\right)}{\log\left(\frac{1}{r}\right)} =$$

$$= \frac{n}{\omega_{n-1}}\,(1-\varepsilon)^{-p} \longrightarrow \frac{n}{\omega_{n-1}}$$

as $\varepsilon \longmapsto 0^+$.

□

We still have Lemma 7 left to be proven in order to deal with Adams' theorem. It is the last result we prove in this section.

Proof of Lemma 7. The strategy is that of showing the formula for the odd case by induction and, once we have it, inferring the one for m even.
We split this process into four steps.

Step 1: introduction of the auxiliary function φ_k. We begin by a helpful equality which will be used during the various computations: using the definition of $\alpha_{m,n}$, it is easlily seen that

$$(-1)^{\frac{m-1}{2}}\left(\frac{n}{\omega_{n-1}\alpha_{m,n}}\right)^{\frac{1}{p}} = (-1)^{\frac{m-1}{2}}\left[\frac{n}{\omega_{n-1}}\,\frac{\omega_{n-1}}{n}\left(\frac{\Gamma(\frac{n-m+1}{2})}{2^m\pi^{\frac{n}{2}}\Gamma(\frac{m+1}{2})}\right)^p\right]^{\frac{1}{p}} =$$

$$= (-1)^{\frac{m-1}{2}}\frac{\Gamma(\frac{n-m+1}{2})}{2^m\pi^{\frac{n}{2}}\Gamma(\frac{m+1}{2})} \tag{97}$$

for odd m and

$$(-1)^{\frac{m}{2}}\left(\frac{n}{\omega_{n-1}\alpha_{m,n}}\right)^{\frac{1}{p}} = (-1)^{\frac{m}{2}}\left[\frac{n}{\omega_{n-1}}\,\frac{\omega_{n-1}}{n}\left(\frac{\Gamma(\frac{n-m}{2})}{2^m\pi^{\frac{n}{2}}\Gamma(\frac{m}{2})}\right)^p\right]^{\frac{1}{p}} =$$

$$= (-1)^{\frac{m}{2}}\frac{\Gamma(\frac{n-m}{2})}{2^m\pi^{\frac{n}{2}}\Gamma(\frac{m}{2})} \tag{98}$$

for even m. Next, we resort to an auxiliary function: for $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$ both fixed, we define, for every $y \neq x$, the function

$$\varphi_k(y) := \frac{1}{k|x-y|^k} \tag{99}$$

without writing the dependence of φ_k on x in order to lighten the notation. Thus, we have that

$$\frac{\partial}{\partial y_i}\varphi_k(y) = \frac{\partial}{\partial y_i}\left(\frac{1}{k|x-y|^k}\right) = -\frac{1}{k}\,\frac{-2k(x_i-y_i)}{2|x-y|^{k+2}} = \frac{x_i-y_i}{|x-y|^{k+2}},$$

$\forall\, 1 \leq i \leq n$. Consequently,

$$\nabla\varphi_k(y) = \frac{x-y}{|x-y|^{k+2}}. \tag{100}$$

Furthermore,

$$\Delta\varphi_k(y) = \sum_{i=1}^{n}\frac{\partial^2}{\partial y_i^2}\varphi_k(y) = \sum_{i=1}^{n}\frac{\partial}{\partial y_i}\left(\frac{x_i-y_i}{|x-y|^{k+2}}\right) =$$

$$= \sum_{i=1}^{n}\frac{-|x-y|^{k+2}+(x_i-y_i)(k+2)(x_i-y_i)|x-y|^k}{|x-y|^{2(k+2)}} =$$

$$= \sum_{i=1}^{n}\frac{(k+2)(x_i-y_i)^2-|x-y|^2}{|x-y|^{k+4}} = \frac{(k+2)}{|x-y|^{k+4}}\sum_{i=1}^{n}(x_i-y_i)^2 - \frac{n|x-y|^2}{|x-y|^{k+4}} =$$

$$= \frac{(k+2)|x-y|^2}{|x-y|^{k+4}} - \frac{n}{|x-y|^{k+2}} = \frac{(k+2)}{|x-y|^{k+2}} - \frac{n}{|x-y|^{k+2}} = \frac{k+2-n}{|x-y|^{k+2}}. \tag{101}$$

In particular, φ_{n-2} turns out to be a harmonic function (although it is not noteworthy for our aim). The relation (100) and (101) will come in handy afterwards.

Step 2: proof of the base cases. Now, if $m = 1$, we have that a function $u \in C_0^\infty(\mathbb{R})$, by the fundamental theorem of calculus, can be written as

$$u(x) = \int_{-\infty}^{x} u'(t)dt = \int_{+\infty}^{0} -u'(x-s)ds = \int_{0}^{+\infty} u'(x-s)ds$$

once we use the change of variables $s := x - t$. The analogous n-dimensional formula is

$$u(x) = \int_{0}^{+\infty} \nabla u(x - \xi t) \cdot \xi dt,$$

where ξ is any unit vector. The above relation, integrated over the unit sphere in $\mathbb{R}^n$, becomes

$$\omega_{n-1} u(x) = \int_{\{|\xi|=1\}} u(x) d\sigma = \int_{\{|\xi|=1\}} \left(\int_{0}^{+\infty} \nabla u(x - \xi t) \cdot \xi dt \right) d\xi,$$

which gives us

$$u(x) = \frac{1}{\omega_{n-1}} \int_{\{|\xi|=1\}} \left(\int_{0}^{+\infty} \nabla u(x - \xi t) \cdot \xi dt \right) d\xi =$$

$$= \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{1}{|y|^{n-1}} \nabla u(x-y) \cdot \frac{y}{|y|} dy = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \nabla u(x-y) \cdot \frac{y}{|y|^n} dy =$$

$$= \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \nabla u(z) \cdot \frac{x-z}{|x-z|^n} dz$$

passing first to Cartesian coordinates and then setting $z := x - y$. This is *(i)* for $m = 1$ because, in such a case, (97) reads as

$$\left(\frac{n}{\omega_{n-1} \alpha_{1,n}} \right)^{\frac{1}{p}} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} \Gamma(1)} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} = \frac{1}{\omega_{n-1}}.$$

We continue by showing the base case for even m: for $m = 2$, we have to utilize the generalized integration by parts (or, more precisely, Green's first identity). Being u supported on a bounded and compact domain by hypothesis, we are allowed to integrate on the whole space (since $u(x) \equiv 0$ on $\mathbb{R}^n \setminus \mathrm{supp}\{u\}$) instead of only on $\mathrm{supp}\{u\}$ and to omit the integration over the frontier of $\mathrm{supp}\{u\}$ (since its normal derivative is everywhere null on that set). This means we will have

$$\int_{\mathbb{R}^n} \nabla \varphi_k(y) \cdot \nabla u(y) dy = - \int_{\mathbb{R}^n} \varphi_k(y) \Delta u(y) dy. \tag{102}$$

The latter relation will be used to transfer the gradient from φ_k to $\nabla^m u$, utilizing the definition itself and the properties of the operator ∇^m. Note that the integral of φ_k over $\mathbb{R}^n$ is not always well defined: in fact, if $k \geq n$, then it is infinite. Clearly, the same happens with $\nabla\varphi_k$ and $\Delta\varphi_k$ (this time, by (100) and (101), it is infinite whenever $k \geq n-1$ and $k \geq n-2$ respectively). However, as stated in Remark 16, we are dealing with integrable functions even though they have a point of singularity: indeed, we are going to use (102) with the right k, namely with the exponent k which makes all the forthcoming quantities integrable.

So, by the formula already acquired for $m = 1$, (99), (100) and (102), one has

$$u(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} \cdot \nabla u(y) dy = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \nabla\varphi_{n-2}(y) \cdot \nabla u(y) dy =$$

$$= -\frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \varphi_{n-2}(y) \Delta u(y) dy = -\frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\Delta u(y)}{(n-2)|x-y|^{n-2}} \, dy =$$

$$= -\frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\nabla^2 u(y)}{|x-y|^{n-2}} \, dy,$$

which is *(ii)* for $m = 2$ since, this time, (98) implies that

$$-\left(\frac{n}{\omega_{n-1}\alpha_{2,n}}\right)^{\frac{1}{p}} = -\frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}\Gamma(1)} = -\frac{2\Gamma(\frac{n}{2})}{4\pi^{\frac{n}{2}}(n-2)} = -\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}(n-2)} = -\frac{1}{(n-2)\omega_{n-1}}.$$

Notice that we also make use of the recursive formula for the Gamma function, namely $\Gamma(z+1) = z\Gamma(z)$, $\forall\, z \in \mathbb{C}$.

Step 3: proof of *(i)*. Now, assume that m is odd and that Lemma 7 is valid for $m-2$, which means that

$$u(x) = (-1)^{\frac{m-3}{2}} \left(\frac{n}{\omega_{n-1}\alpha_{m-2,n}}\right)^{\frac{1}{p}} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-m+3}} \cdot \nabla^{m-2} u(y) dy =$$

$$= (-1)^{\frac{m-1}{2}-1} \frac{\Gamma(\frac{n-m+1}{2}+1)}{2^{m-2}\pi^{\frac{n}{2}}\Gamma(\frac{m-1}{2})} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-m+3}} \cdot \nabla^{m-2} u(y) dy.$$

Thus, by (99), (100), (101) and (102), we get

$$u(x) = (-1)^{\frac{m-1}{2}-1} \frac{\Gamma(\frac{n-m+1}{2}+1)}{2^{m-2}\pi^{\frac{n}{2}}\Gamma(\frac{m-1}{2})} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-m+3}} \cdot \nabla^{m-2} u(y) dy =$$

$$= (-1)^{\frac{m-1}{2}-1} \frac{\Gamma(\frac{n-m+1}{2}+1)}{2^{m-2}\pi^{\frac{n}{2}}\Gamma(\frac{m-1}{2})} \int_{\mathbb{R}^n} \nabla\varphi_{n-m+1}(y) \cdot \nabla\Big(\Delta^{\frac{m-1}{2}-1} u(y)\Big) dy =$$

$$= (-1)^{\frac{m-1}{2}} \frac{\Gamma(\frac{n-m+1}{2}+1)}{2^{m-2}\pi^{\frac{n}{2}}\Gamma(\frac{m-1}{2})} \int_{\mathbb{R}^n} \varphi_{n-m+1}(y)\Delta\Big(\Delta^{\frac{m-1}{2}-1}u(y)\Big)dy =$$

$$= (-1)^{\frac{m-1}{2}} \frac{\Gamma(\frac{n-m+1}{2}+1)}{2^{m-2}\pi^{\frac{n}{2}}\Gamma(\frac{m-1}{2})} \int_{\mathbb{R}^n} \frac{\Delta^{\frac{m-1}{2}}u(y)}{(n-m+1)|x-y|^{n-m+1}}\,dy =$$

$$= (-1)^{\frac{m-1}{2}} \frac{2\Gamma(\frac{n-m+1}{2}+1)}{2^{m-1}\pi^{\frac{n}{2}}(n-m+1)\Gamma(\frac{m-1}{2})} \int_{\mathbb{R}^n} \frac{\Delta^{\frac{m-1}{2}}u(y)}{|x-y|^{n-m+1}}\,dy =$$

$$= (-1)^{\frac{m-1}{2}} \frac{\Gamma(\frac{n-m+1}{2})}{2^{m-1}\pi^{\frac{n}{2}}\Gamma(\frac{m-1}{2})} \int_{\mathbb{R}^n} \frac{\Delta\varphi_{n-m-1}(y)}{1-m}\,\Delta^{\frac{m-1}{2}}u(y)dy =$$

$$= (-1)^{\frac{m-1}{2}+1} \frac{2\Gamma(\frac{n-m+1}{2})}{2^{m}\pi^{\frac{n}{2}}(m-1)\Gamma(\frac{m-1}{2})} \int_{\mathbb{R}^n} \Delta\varphi_{n-m-1}(y)\Delta^{\frac{m-1}{2}}u(y)dy =$$

$$= (-1)^{\frac{m-1}{2}} \frac{\Gamma(\frac{n-m+1}{2})}{2^{m}\pi^{\frac{n}{2}}\Gamma(\frac{m+1}{2})} \int_{\mathbb{R}^n} \nabla\varphi_{n-m-1}(y)\cdot\nabla\Big(\Delta^{\frac{m-1}{2}}u(y)\Big)dy =$$

$$= (-1)^{\frac{m-1}{2}} \frac{\Gamma(\frac{n-m+1}{2})}{2^{m}\pi^{\frac{n}{2}}\Gamma(\frac{m+1}{2})} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-m+1}}\cdot\nabla^m u(y)dy,$$

where we used twice the property of the Gamma function previously mentioned. The above relation, recalling (97), is exactly the point *(i)* of the thesis.

Step 4: proof of *(ii)*. Finally, if now m is even, the formula for $m-1$ tells us that

$$u(x) = (-1)^{\frac{m}{2}-1}\left(\frac{n}{\omega_{n-1}\alpha_{m-1,n}}\right)^{\frac{1}{p}} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-m+2}}\cdot\nabla^{m-1}u(y)dy =$$

$$= (-1)^{\frac{m}{2}-1} \frac{\Gamma(\frac{n-m}{2}+1)}{2^{m-1}\pi^{\frac{n}{2}}\Gamma(\frac{m}{2})} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-m+2}}\cdot\nabla^{m-1}u(y)dy.$$

Therefore, with a quite identical computation than before, due to (99), (100), (101) and (102), we achieve

$$u(x) = (-1)^{\frac{m}{2}-1} \frac{\Gamma(\frac{n-m}{2}+1)}{2^{m-1}\pi^{\frac{n}{2}}\Gamma(\frac{m}{2})} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-m+2}}\cdot\nabla^{m-1}u(y)dy =$$

$$= (-1)^{\frac{m}{2}-1} \frac{\Gamma(\frac{n-m}{2}+1)}{2^{m-1}\pi^{\frac{n}{2}}\Gamma(\frac{m}{2})} \int_{\mathbb{R}^n} \nabla\varphi_{n-m}(y)\cdot\nabla\Big(\Delta^{\frac{m}{2}-1}u(y)\Big)dy =$$

$$= (-1)^{\frac{m}{2}} \frac{\Gamma(\frac{n-m}{2}+1)}{2^{m-1}\pi^{\frac{n}{2}}\Gamma(\frac{m}{2})} \int_{\mathbb{R}^n} \varphi_{n-m}(y)\Delta\Big(\Delta^{\frac{m}{2}-1}u(y)\Big)dy =$$

$$= (-1)^{\frac{m}{2}} \frac{\Gamma(\frac{n-m}{2}+1)}{2^{m-1}\pi^{\frac{n}{2}}\Gamma(\frac{m}{2})} \int_{\mathbb{R}^n} \frac{\Delta^{\frac{m}{2}}u(y)}{(n-m)|x-y|^{n-m}}\,dy =$$

$$= (-1)^{\frac{m}{2}} \frac{2\Gamma(\frac{n-m}{2}+1)}{2^m \pi^{\frac{n}{2}}(n-m)\Gamma(\frac{m}{2})} \int_{\mathbb{R}^n} \frac{\nabla^m u(y)}{|x-y|^{n-m}}\, dy =$$

$$= (-1)^{\frac{m}{2}} \frac{\Gamma(\frac{n-m}{2})}{2^m \pi^{\frac{n}{2}}\Gamma(\frac{m}{2})} \int_{\mathbb{R}^n} \frac{\nabla^m u(y)}{|x-y|^{n-m}}\, dy,$$

which is the point *(ii)* of the thesis once we recall (98).

□

2.3. Proof of Adams' theorem and sharpness of the constant

We begin by proving Theorem 9, being in possession of the necessary prerequisites. Its proof will be quite immediate, because the hard work has been already done in the previous paragraph.

Proof of Theorem 9. Our aim is that of applying Theorem 10 after writing a function $u \in W_0^{m,q}(\Omega)$ (due to the density of $C_0^\infty(\mathbb{R}^n)$ into the latter) in terms of its m-th order gradient by applying Lemma 7.
As usual, we divide the discussion to make it more compact.

<u>Step 1: an initial consideration.</u> First of all, we have to notice the trivial conclusion for the function null a.e., in which case we can choose the constant $c = c(m,n)$ to be just 1. Moreover, we observe that $||\nabla^m u||_q = 0 \iff u = 0$ a.e. due to the Poincaré inequality, which affirms that there exists a constant $c_0 = c_0(q,\Omega)$ depending on q and Ω only such that

$$\int_\Omega |u(x)|^q dx \leq c_0 \int_\Omega |\nabla u(x)|^q dx,$$

$\forall\, u \in W_0^{1,q}(\Omega)$. So, if $u \in W_0^{m,q}(\Omega)$, applying the inequality m times takes us to

$$\int_\Omega |u(x)|^q dx \leq \tilde{c}_0 \int_\Omega |\nabla^m u(x)|^q dx,$$

for a certain constant $\tilde{c}_0 = \tilde{c}_0(m,q,\Omega)$ depending on m, q and Ω only, which assures us that $||\nabla^m u||_q = 0 \Longrightarrow u = 0$ a.e. (clearly, the reverse is obvious). This means that, if u is not null a.e. and is as in the hypothesis, then $||\nabla^m u||_q \in (0,1]$.

<u>Step 2: proof of the odd case.</u> Now, for odd m and $u \in W_0^{m,q}(\Omega)$ not null a.e., we appeal to Lemma 7 which tells us that

$$u(x) = (-1)^{\frac{m-1}{2}} \left(\frac{n}{\omega_{n-1}\alpha_{m,n}}\right)^{\frac{1}{p}} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-m+1}} \cdot \nabla^m u(y) dy,$$

and so

$$|u(x)|^p = \left| (-1)^{\frac{m-1}{2}} \left(\frac{n}{\omega_{n-1}\alpha_{m,n}}\right)^{\frac{1}{p}} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-m+1}} \cdot \nabla^m u(y) dy \right|^p \leq$$

$$\leq \left[\left(\frac{n}{\omega_{n-1}\alpha_{m,n}}\right)^{\frac{1}{p}} \int_{\mathbb{R}^n} \frac{|\nabla^m u(y)|}{|x-y|^{n-m}} dy \right]^p = \frac{n}{\omega_{n-1}\alpha_{m,n}} \left| \int_{\mathbb{R}^n} \frac{|\nabla^m u(y)|}{|x-y|^{n-m}} dy \right|^p =$$

$$= \frac{n}{\omega_{n-1}\alpha_{m,n}} \left| \big(I_m * |\nabla^m u|\big)(x) \right|^p.$$

Hence,

$$\alpha_{m,n}|u(x)|^p \leq \frac{n}{\omega_{n-1}} \left|\left(I_m * |\nabla^m u|\right)(x)\right|^p$$

which means that, by the monotonicity of the exponential and $\forall\, \alpha \in [0, \alpha_{m,n}]$, we have

$$\fint_\Omega e^{\alpha|u(x)|^p} dx \leq \fint_\Omega e^{\alpha_{m,n}|u(x)|^p} dx \leq \fint_\Omega \exp\left(\frac{n}{\omega_{n-1}} \left|\left(I_m * |\nabla^m u|\right)(x)\right|^p\right) dx \leq$$

$$\leq \fint_\Omega \exp\left(\frac{n}{\omega_{n-1}} \left|\frac{(I_m * |\nabla^m u|)(x)}{||\nabla^m u||_q}\right|^p\right) dx,$$

using the banal relation $||\nabla^m u|| = ||\,|\nabla^m u|\,||$. Then, being $1 < q < \infty$ and $m = \frac{n}{q}$, Theorem 10 assures us that there exists a constant $c = c(q) = c(m, n)$ depending on m and n only such that

$$\fint_\Omega e^{\alpha|u(x)|^p} dx \leq \fint_\Omega \exp\left(\frac{n}{\omega_{n-1}} \left|\frac{(I_m * |\nabla^m u|)(x)}{||\nabla^m u||_q}\right|^p\right) dx \leq c,$$

which gives us the thesis for the odd case.

Step 3: proof of the even case. Similarly, if m is even, Lemma 7 reads as

$$u(x) = (-1)^{\frac{m}{2}} \left(\frac{n}{\omega_{n-1}\alpha_{m,n}}\right)^{\frac{1}{p}} \int_{\mathbb{R}^n} \frac{\nabla^m u(y)}{|x-y|^{n-m}}\, dy,$$

which means that

$$|u(x)|^p = \left|(-1)^{\frac{m}{2}} \left(\frac{n}{\omega_{n-1}\alpha_{m,n}}\right)^{\frac{1}{p}} \int_{\mathbb{R}^n} \frac{\nabla^m u(y)}{|x-y|^{n-m}}\, dy\right|^p =$$

$$= \left[\left(\frac{n}{\omega_{n-1}\alpha_{m,n}}\right)^{\frac{1}{p}} \left|\int_{\mathbb{R}^n} \frac{\nabla^m u(y)}{|x-y|^{n-m}}\, dy\right|\right]^p = \frac{n}{\omega_{n-1}\alpha_{m,n}} \left|\int_{\mathbb{R}^n} \frac{\nabla^m u(y)}{|x-y|^{n-m}}\, dy\right|^p =$$

$$= \frac{n}{\omega_{n-1}\alpha_{m,n}} \left|\left(I_m * (\nabla^m u)\right)(x)\right|^p.$$

So, we have that

$$\alpha_{m,n}|u(x)|^p = \frac{n}{\omega_{n-1}} \left|\left(I_m * (\nabla^m u)\right)(x)\right|^p.$$

Therefore, for every $\alpha \in [0, \alpha_{m,n}]$ and c as before, Theorem 10 brings us to

$$\fint_\Omega e^{\alpha|u(x)|^p} dx \leq \fint_\Omega e^{\alpha_{m,n}|u(x)|^p} dx = \fint_\Omega \exp\left(\frac{n}{\omega_{n-1}} \left|\left(I_m * (\nabla^m u)\right)(x)\right|^p\right) dx \leq$$

$$\leq \fint_\Omega \exp\left(\frac{n}{\omega_{n-1}} \left|\frac{(I_m * (\nabla^m u))(x)}{||\nabla^m u||_q}\right|^p\right) dx \leq c.$$

□

What remains to show is that the number $\alpha_{m,n}$ is the greatest value for which Theorem 9 holds, namely Proposition 7. However, contrary to the corollary of Moser's theorem, this result is not a simple application of Adams' theorem and needs a whole discussion. The technique we are going to use is the same of Proposition 8 (indeed, the initial considerations and the first step will be almost identical to the ones of the aforementioned result). However, some computations will differ and, basically, the discussion will be longer.

Proof of Proposition 7. The proof of Adams' theorem showed that, if $\alpha \le \alpha_{m,n}$, then (75) holds. We want here to establish that $\alpha_{m,n}$ is the greatest value for which this estimate is true, namely that the constant $c = c(m,n)$ depends also on the function $u \in W_0^{m,\frac{n}{m}}(\Omega)$ taken into consideration if $\alpha > \alpha_{m,n}$.
Let then α be such that the estimate (75) is valid with $c = c(m,n)$ depending on m and n only and assume that Ω contains $B := B_1(0)$, the unit ball of $\mathbb{R}^n$ (this is not a loss of generality as noted in the proof of Proposition 8). We assume also that $m \ge 2$, because the case for $m = 1$ is exactly Theorem 7 which has been already proved. The proof will be complete if we show that, under such hypothesis, α will be $\alpha_{m,n}$ at most. Once more, the proof will contain several steps.

Step 1: introduction of the conductor capacity $\mathcal{C}_{m,q}(E,B)$. We have

$$⨍_B e^{\alpha|u(x)|^p} dx \le c,$$

$\forall\, u \in W_0^{m,\frac{n}{m}}(B)$ such that $||\nabla^m u||_q^q \le 1$. In particular, by the density of $C_0^\infty(\Omega)$ into $W_0^{m,\frac{n}{m}}(\Omega)$, the inequality will be true even for a function $u \in C_0^\infty(B)$ which identically equals 1 on a compact subset E of B and such that $||\nabla^m u||_q$ is as small (but non-null) as possible. However, as noted in the previous proof, one has that $||\nabla^m u||_q = 0 \iff u(x) \equiv 0$ a.e. (and this is not our case since $u(x) \equiv 1$ on E and u is null on ∂B).
In other words, given a compact subset E of B, if we define the set

$$\mathcal{D}(E,B) := \big\{ u \in C_0^\infty(B) : u(x) \equiv 1 \text{ on } E \big\},$$

choose a function $u \in \mathcal{D}(E,B)$ and consider $v(x) := \frac{u(x)}{||\nabla^m u||_q}$, then we have $||\nabla^m v||_q^q = 1$. Hence, being satisfied the hypothesis of Theorem 9, we must have

$$c \ge ⨍_B e^{\alpha|v(x)|^p} dx = ⨍_B \exp\left[\alpha \frac{|u(x)|^p}{||\nabla^m u||_q^p}\right] dx \ge \frac{1}{|B|}\int_E \exp\left[\alpha \frac{|u(x)|^p}{||\nabla^m u||_q^p}\right] dx =$$

$$= \frac{1}{|B|}\int_E \exp\left[\frac{\alpha}{||\nabla^m u||_q^p}\right]dx = \frac{|E|}{|B|}\,e^{\frac{\alpha}{||\nabla^m u||_q^p}} \Longleftrightarrow \frac{\alpha}{||\nabla^m u||_q^p} \leq \log\left(\frac{|B|}{|E|}\,c\right) \Longleftrightarrow$$

$$\Longleftrightarrow \alpha \leq \log\left(\frac{|B|}{|E|}\,c\right)||\nabla^m u||_q^p.$$

If we introduce the quantity

$$\mathcal{C}_{m,q}(E,B) := \inf_{u\in\mathcal{D}(E,B)}\big\{||\nabla^m u||_q^q\big\} \tag{103}$$

(which is, as the one in (91), a type of conductor capacity), then

$$\alpha \leq \log\left(\frac{|B|}{|E|}\,c\right)\mathcal{C}_{m,q}(E,B)^{\frac{m}{n-m}}.$$

Now, if $E := \overline{B_r(0)}$ is the closed ball of $\mathbb{R}^n$ centered at the origin with radius $r \in (0,1)$, one has that

$$\alpha \leq \log\left(\frac{|B|}{|B_r(0)|}\,c\right)\mathcal{C}_{m,q}\Big(\overline{B_r(0)},B\Big)^{\frac{m}{n-m}} = \log\left(\frac{\omega_{n-1}}{n}\,\frac{n}{\omega_{n-1}}\,\frac{c}{r^n}\right)\mathcal{C}_{m,q}\Big(\overline{B_r(0)},B\Big)^{\frac{m}{n-m}} =$$

$$= \left[\log(c) + n\log\left(\frac{1}{r}\right)\right]\mathcal{C}_{m,q}\Big(\overline{B_r(0)},B\Big)^{\frac{m}{n-m}} =$$

$$= n\log\left(\frac{1}{r}\right)\mathcal{C}_{m,q}\Big(\overline{B_r(0)},B\Big)^{\frac{m}{n-m}}\left[1 + \frac{\log(c)}{n\log\left(\frac{1}{r}\right)}\right],$$

being clearly the measure of $\overline{B_r(0)}$ the same as $B_r(0)$. If $r \longmapsto 0^+$, since c is a constant, the fraction involving it goes to 0: we therefore get

$$\alpha \leq n\liminf_{r\to 0^+}\log\left(\frac{1}{r}\right)\mathcal{C}_{m,q}\Big(\overline{B_r(0)},B\Big)^{\frac{m}{n-m}}. \tag{104}$$

We have then to work on $\mathcal{C}_{m,q}\Big(\overline{B_r(0)},B\Big)^{\frac{m}{n-m}}$ appearing in (104).

<u>Step 2: introduction of some auxiliary functions.</u> We shall appeal to some auxiliary functions. Let $\phi \in C^\infty\big([0,1]\big)$ such that:

- $\phi^{(j)}(0) = 0$, $\forall\, 0 \leq j \leq m-1$, where $\phi^{(0)}(t) := \phi(t)$;
- $\phi(1) = 1 = \phi'(1)$;
- $\phi^{(k)}(1) = 0$, $\forall\, 2 \leq k \leq m-1$ (unless $m = 2$, in which case this last condition is not present).

Let then $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $h : (0, +\infty) \longrightarrow \mathbb{R}$ such that

$$h(t) := \begin{cases} \varepsilon\phi\left(\frac{t}{\varepsilon}\right) & \text{if } t \in (0, \varepsilon] \\ t & \text{if } t \in (\varepsilon, 1-\varepsilon] \\ 1 - \varepsilon\phi\left(\frac{1-t}{\varepsilon}\right) & \text{if } t \in (1-\varepsilon, 1] \\ 1 & \text{if } t \in (1, +\infty) \end{cases}.$$

Here, the function h is similar to the auxiliary function w utilized in the proof of Moser's theorem, since they share similar (nearly equal) properties: in fact, if ε is small enough, the function h is almost the identity in the interval $(0, 1)$ and identically 1 in $(1, +\infty)$. We now rescale the variables by considering

$$\psi(x) := h\left(\frac{\log(|x|)}{\log(r)}\right) = \begin{cases} \varepsilon\phi\left(\frac{\log(|x|)}{\varepsilon \log(r)}\right) & \text{if } |x| \in [r^{\varepsilon}, 1) \\ \frac{\log(|x|)}{\log(r)} & \text{if } |x| \in [r^{1-\varepsilon}, r^{\varepsilon}) \\ 1 - \varepsilon\phi\left(\frac{\log(r)-\log(|x|)}{\varepsilon \log(r)}\right) & \text{if } |x| \in [r, r^{1-\varepsilon}) \\ 1 & \text{if } |x| \in (0, r) \end{cases}$$

and notice that $\psi(x) \in W_0^{m,q}(B)$, since h (and therefore ψ) depends only on the function $\phi \in C^\infty\big([0, 1]\big)$. Besides, if $|x| \leq r$, then $\psi(x) = 1$ by definition. Therefore, in such a case, we will get

$$\int_{B_r(0)} |\nabla^m \psi(x)|^q dx = 0, \tag{105}$$

relation which will come in handy later.
We have now to evaluate $\nabla^m \psi(x)$.

Step 3: computation of $\nabla^m \psi(x)$. Next, we evaluate $\nabla^m \psi(x)$ in general: we will make use of the definition of the m-th order gradient of a function given at the beginning of this paragraph. Our aim is to get the formulas

$$\nabla^m \psi(x) = \Delta^{\frac{m}{2}} \psi(x) = s(m)\beta(m, n) \frac{1}{|x|^m \log(r)} h'\left(\frac{\log(|x|)}{\log(r)}\right) + o\left(\frac{1}{|x|^m \log(r)}\right) \tag{106}$$

for m even and, $\forall\, 1 \leq i \leq n$,

$$\frac{\partial}{\partial x_i}\left(\nabla^{m-1}\psi(x)\right) = \frac{\partial}{\partial x_i}\left(\Delta^{\frac{m-1}{2}}\psi(x)\right) =$$

$$= s(m)\beta(m, n) \frac{x_i}{|x|^{m+1} \log(r)} h'\left(\frac{\log(|x|)}{\log(r)}\right) + o\left(\frac{x_i}{|x|^{m+1} \log(r)}\right) \tag{107}$$

for m odd, where

$$s(m) := \begin{cases} (-1)^{\frac{m}{2}-1} & \text{if } m = 2k \\ (-1)^{\frac{m-1}{2}} & \text{if } m = 2k-1 \end{cases}$$

and

$$\beta(m,n) := \begin{cases} 2^{\frac{m}{2}-1}\Gamma\big(\frac{m}{2}\big)\,\frac{(n-2)!!}{(n-m-2)!!} & \text{if } m = 2k \\ 2^{\frac{m-1}{2}}\Gamma\big(\frac{m+1}{2}\big)\,\frac{(n-2)!!}{(n-m-1)!!} & \text{if } m = 2k-1 \end{cases},$$

$\forall\, k \in \mathbb{N}$. First of all, we show the first case (when $m = 2$). Even though we are not interested in the case where $m = 1$, we have to start from here. So, if $m = 1$, then

$$\frac{\partial}{\partial x_i}\,\psi(x) = \frac{2x_i|x|^{-1}}{2|x|\log(r)}\,h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) = \frac{x_i}{|x|^2\log(r)}\,h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg),$$

$\forall\, 1 \le i \le n$. Therefore, if $m = 2$, we obtain

$$\nabla^2\psi(x) = \Delta\psi(x) = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\,\psi(x) = \sum_{i=1}^{n}\bigg[\frac{|x|^2 - 2x_i^2}{|x|^4\log(r)}\,h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{x_i^2}{|x|^4\log(r)}\bigg)\bigg] =$$

$$= \sum_{i=1}^{n}\bigg(\frac{1}{|x|^2\log(r)} - \frac{2x_i^2}{|x|^4\log(r)}\bigg)h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^2\log(r)}\bigg) =$$

$$= \bigg(\frac{n}{|x|^2\log(r)} - \frac{2|x|^2}{|x|^4\log(r)}\bigg)h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^2\log(r)}\bigg) =$$

$$= \frac{n-2}{|x|^2\log(r)}\,h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^2\log(r)}\bigg).$$

What we got is exactly (106) for $m = 2$. Now, by induction on the even numbers, we will obtain the formula (106) and, once we have it, we will deduce also (107). So let us assume that m is even and that (106) is true for $m-2$: hence, by definition of $\nabla^m\psi(x)$,

$$\nabla^{m-2}\psi(x) = \Delta^{\frac{m}{2}-1}\psi(x) =$$

$$= s(m-2)\beta(m-2,n)\,\frac{1}{|x|^{m-2}\log(r)}\,h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^{m-2}\log(r)}\bigg).$$

Thus,

$$\nabla^m\psi(x) = \Delta^{\frac{m}{2}}\psi(x) = \Delta\Big(\Delta^{\frac{m}{2}-1}\psi(x)\Big) =$$

$$= \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\bigg[s(m-2)\beta(m-2,n)\,\frac{1}{|x|^{m-2}\log(r)}\,h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^{m-2}\log(r)}\bigg)\bigg] =$$

$$= \frac{s(m-2)\beta(m-2,n)}{\log(r)}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\bigg[\frac{1}{|x|^{m-2}}\,h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^{m-2}}\bigg)\bigg] =$$

$$= \frac{s(m-2)\beta(m-2,n)}{\log(r)} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\frac{2\big(1-\frac{m}{2}\big)x_i}{|x|^m}\, h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{x_i}{|x|^m \log(r)}\bigg) \right] =$$

$$= -(m-2)\, \frac{s(m-2)\beta(m-2,n)}{\log(r)} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\frac{x_i}{|x|^m}\, h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{x_i}{|x|^m}\bigg) \right] =$$

$$= -(m-2)\, \frac{s(m-2)\beta(m-2,n)}{\log(r)} \sum_{i=1}^{n} \left[\frac{|x|^m - m x_i^2 |x|^{m-2}}{|x|^{2m}}\, h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + \right.$$

$$\left. + o\bigg(\frac{|x|^m + x_i^2|x|^{m-2}}{|x|^{2m}\log(r)}\bigg) \right] =$$

$$= -(m-2)\, \frac{s(m-2)\beta(m-2,n)}{\log(r)} \sum_{i=1}^{n} \left[\bigg(\frac{1}{|x|^m} - \frac{m x_i^2}{|x|^{m+2}}\bigg) h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + \right.$$

$$\left. + o\bigg(\frac{1}{|x|^m} + \frac{x_i^2}{|x|^{m+2}\log(r)}\bigg) \right] =$$

$$= -(m-2)(n-m)\, \frac{s(m-2)\beta(m-2,n)}{|x|^m \log(r)}\, h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^m \log(r)}\bigg) =$$

$$= -(m-2)(n-m)\, \frac{(-1)^{\frac{m}{2}-2}\, 2^{\frac{m}{2}-2}\, \Gamma\big(\frac{m}{2}-1\big)(n-2)!!}{(n-m)!!\, |x|^m \log(r)}\, h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^m \log(r)}\bigg) =$$

$$= (-1)^{\frac{m}{2}-1}\, 2^{\frac{m}{2}-1}\, \frac{\big(\frac{m}{2}-1\big)\Gamma\big(\frac{m}{2}-1\big)(n-2)!!}{(n-m-2)!!\, |x|^m \log(r)}\, h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^m \log(r)}\bigg) =$$

$$= (-1)^{\frac{m}{2}-1}\, 2^{\frac{m}{2}-1}\, \frac{\Gamma\big(\frac{m}{2}\big)(n-2)!!}{(n-m-2)!!\, |x|^m \log(r)}\, h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^m \log(r)}\bigg) =$$

$$= s(m)\beta(m,n)\, \frac{1}{|x|^m \log(r)}\, h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^m \log(r)}\bigg),$$

where we also used the properties of the semifactorial and the Gamma function. This proves the validity of the formula (106). Now, being in possession of it, we get also (107) because, if this time m is odd, then

$$\frac{\partial}{\partial x_i}\Big(\nabla^{m-1}\psi(x)\Big) = \frac{\partial}{\partial x_i}\Big(\Delta^{\frac{m-1}{2}}\psi(x)\Big) =$$

$$= \frac{\partial}{\partial x_i}\left[s(m-1)\beta(m-1,n)\, \frac{1}{|x|^{m-1}\log(r)}\, h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^{m-1}\log(r)}\bigg) \right] =$$

$$= \frac{s(m-1)\beta(m-1,n)}{\log(r)}\, \frac{\partial}{\partial x_i} \left[\frac{1}{|x|^{m-1}}\, h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{1}{|x|^{m-1}}\bigg) \right] =$$

$$= \frac{s(m-1)\beta(m-1,n)}{\log(r)} \left[\frac{2(1-m)x_i}{2|x|^{m+1}}\, h'\bigg(\frac{\log(|x|)}{\log(r)}\bigg) + o\bigg(\frac{x_i}{|x|^{m+1}\log(r)}\bigg) \right] =$$

$$= -(m-1)\,\frac{s(m-1)\beta(m-1,n)}{\log(r)}\,\frac{x_i}{|x|^{m+1}}\,h'\!\left(\frac{\log(|x|)}{\log(r)}\right) + o\!\left(\frac{x_i}{|x|^{m+1}\log(r)}\right) =$$

$$= -(m-1)\,\frac{(-1)^{\frac{m-1}{2}-1}\,2^{\frac{m-1}{2}-1}\,\Gamma\big(\frac{m-1}{2}\big)\,(n-2)!!\,x_i}{(n-m-1)!!\,|x|^{m+1}\log(r)}\,h'\!\left(\frac{\log(|x|)}{\log(r)}\right) + o\!\left(\frac{x_i}{|x|^{m+1}\log(r)}\right) =$$

$$= (-1)^{\frac{m-1}{2}}\,2^{\frac{m-1}{2}}\,\frac{\big(\frac{m-1}{2}\big)\Gamma\big(\frac{m-1}{2}\big)\,(n-2)!!\,x_i}{(n-m-1)!!\,|x|^{m+1}\log(r)}\,h'\!\left(\frac{\log(|x|)}{\log(r)}\right) + o\!\left(\frac{x_i}{|x|^{m+1}\log(r)}\right) =$$

$$= (-1)^{\frac{m-1}{2}}\,2^{\frac{m-1}{2}}\,\frac{\Gamma\big(\frac{m+1}{2}\big)\,(n-2)!!\,x_i}{(n-m-1)!!\,|x|^{m+1}\log(r)}\,h'\!\left(\frac{\log(|x|)}{\log(r)}\right) + o\!\left(\frac{x_i}{|x|^{m+1}\log(r)}\right) =$$

$$= s(m)\beta(m,n)\,\frac{x_i}{|x|^{m+1}\log(r)}\,h'\!\left(\frac{\log(|x|)}{\log(r)}\right) + o\!\left(\frac{x_i}{|x|^{m+1}\log(r)}\right),$$

$\forall\, 1 \le i \le n$. Hence, (107) is proven, too. It follows that, if m is odd,

$$\nabla^m\psi(x) = s(m)\beta(m,n)\,\frac{x}{|x|^{m+1}\log(r)}\,h'\!\left(\frac{\log(|x|)}{\log(r)}\right) + o\!\left(\frac{x}{|x|^{m+1}\log(r)}\right)$$

and consequently, for both even and odd m,

$$|\nabla^m\psi(x)| = \left|s(m)\beta(m,n)\,\frac{1}{|x|^m\log(r)}\,h'\!\left(\frac{\log|x|)}{\log(r)}\right) + o\!\left(\frac{1}{|x|^m\log(r)}\right)\right|, \tag{108}$$

relation which we are going to use immediately.

Step 4: estimation of α through the function ψ. By the definition of q and reminding (105), the relation (108) takes us to

$$||\nabla^m\psi||_q^q = \int_B |\nabla^m\psi(x)|^q dx =$$

$$= \int_B \left|s(m)\beta(m,n)\,\frac{1}{|x|^m\log(r)}\,h'\!\left(\frac{\log(|x|)}{\log(r)}\right) + o\!\left(\frac{1}{|x|^m\log(r)}\right)\right|^{\frac{n}{m}} dx =$$

$$= \frac{\beta(m,n)^{\frac{n}{m}}}{\log\big(\frac{1}{r}\big)^{\frac{n}{m}}}\int_B \left|\frac{s(m)}{|x|^m}\,h'\!\left(\frac{\log(|x|)}{\log(r)}\right) + o\!\left(\frac{1}{|x|^m\log(r)}\right)\right|^{\frac{n}{m}} dx =$$

$$= \frac{\beta(m,n)^{\frac{n}{m}}}{\log\big(\frac{1}{r}\big)^{\frac{n}{m}}}\int_r^1 \omega_{n-1}\xi^{n-1}\left|\frac{s(m)}{\xi^m}\,h'\!\left(\frac{\log(\xi)}{\log(r)}\right) + o\!\left(\frac{1}{\xi^m\log(r)}\right)\right|^{\frac{n}{m}} d\xi =$$

$$= \frac{\omega_{n-1}\beta(m,n)^{\frac{n}{m}}}{\log\big(\frac{1}{r}\big)^{\frac{n}{m}}}\int_r^1 \left|s(m)\,h'\!\left(\frac{\log(\xi)}{\log(r)}\right) + o\!\left(\frac{1}{\log(r)}\right)\right|^{\frac{n}{m}} \frac{d\xi}{\xi} =$$

$$= \frac{\omega_{n-1}\beta(m,n)^{\frac{n}{m}}}{\log\big(\frac{1}{r}\big)^{\frac{n}{m}-1}}\,\underbrace{\frac{1}{\log\big(\frac{1}{r}\big)}\int_r^1 \left|s(m)\,h'\!\left(\frac{\log(\xi)}{\log(r)}\right) + o\!\left(\frac{1}{\log(r)}\right)\right|^{\frac{n}{m}} \frac{d\xi}{\xi}}_{=:\vartheta(r,\varepsilon,m,n)}. \tag{109}$$

Using the definition of h, we note that $||h'||_\infty = ||\phi'||_\infty$ in $[r, r^{1-\varepsilon}) \cup [r^\varepsilon, 1)$; therefore, we evaluate the function $\vartheta(r,\varepsilon,m,n)$ obtaining

$$\vartheta(r,\varepsilon,m,n) := \frac{1}{\log\left(\frac{1}{r}\right)} \int_r^1 \left| s(m)\, h'\left(\frac{\log(\xi)}{\log(r)}\right) + o\left(\frac{1}{\log(r)}\right)\right|^{\frac{n}{m}} \frac{d\xi}{\xi} \leq$$

$$\leq \frac{1}{\log\left(\frac{1}{r}\right)} \int_r^1 \left[\left|h'\left(\frac{\log(\xi)}{\log(r)}\right)\right| + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} \frac{d\xi}{\xi} =$$

$$= \frac{1}{\log\left(\frac{1}{r}\right)} \left(\int_r^{r^{1-\varepsilon}} \left[\left|h'\left(\frac{\log(\xi)}{\log(r)}\right)\right| + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} \frac{d\xi}{\xi} + \right.$$

$$+ \int_{r^{1-\varepsilon}}^{r^\varepsilon} \left[\left|h'\left(\frac{\log(\xi)}{\log(r)}\right)\right| + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} \frac{d\xi}{\xi} +$$

$$\left. + \int_{r^\varepsilon}^1 \left[\left|h'\left(\frac{\log(\xi)}{\log(r)}\right)\right| + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} \frac{d\xi}{\xi} \right) \leq$$

$$\leq \frac{1}{\log\left(\frac{1}{r}\right)} \left[||\phi'||_\infty + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} \left(\int_r^{r^{1-\varepsilon}} \frac{d\xi}{\xi} + \int_{r^\varepsilon}^1 \frac{d\xi}{\xi} \right) +$$

$$+ \frac{1}{\log\left(\frac{1}{r}\right)} \int_{r^{1-\varepsilon}}^{r^\varepsilon} \left[1 + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} \frac{d\xi}{\xi} =$$

$$= \frac{1}{\log\left(\frac{1}{r}\right)} \left[||\phi'||_\infty + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} \Big(\log\left(r^{1-\varepsilon}\right) - \log(r) - \log\left(r^\varepsilon\right) \Big) +$$

$$+ \frac{1}{\log\left(\frac{1}{r}\right)} \left[1 + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} \int_{r^{1-\varepsilon}}^{r^\varepsilon} \frac{d\xi}{\xi} =$$

$$= \frac{1}{\log\left(\frac{1}{r}\right)} \left[||\phi'||_\infty + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} \big(-2\varepsilon \log(r) \big) +$$

$$+ \frac{1}{\log\left(\frac{1}{r}\right)} \left[1 + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} \Big(\log\left(r^\varepsilon\right) - \log\left(r^{1-\varepsilon}\right) \Big) =$$

$$= 2\varepsilon \left[||\phi'||_\infty + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} + \frac{1}{\log\left(\frac{1}{r}\right)} \left[1 + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} (2\varepsilon - 1)\log(r) =$$

$$= 2\varepsilon \left[||\phi'||_\infty + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}} + (1 - 2\varepsilon)\left[1 + o\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right]^{\frac{n}{m}}. \tag{110}$$

The relation (110) tells us that the (non-negative) function $\vartheta(r,\varepsilon,m,n)$ is less or equal to a quantity which goes to 1 as $r,\, \varepsilon \longmapsto 0^+$, which means that $\vartheta(r,\varepsilon,m,n) \leq 1$ in such a case. Furthermore, by (103) and the calculations made in (109), we get

$$\liminf_{r\to 0^+} \log\left(\frac{1}{r}\right)\mathcal{C}_{m,q}\Big(\overline{B_r(0)}, B\Big)^{\frac{m}{n-m}} \le \liminf_{r\to 0^+} \log\left(\frac{1}{r}\right)\big(||\nabla^m\psi||_q^q\big)^{\frac{m}{n-m}} =$$

$$= \liminf_{r\to 0^+} \log\left(\frac{1}{r}\right)\left(\frac{\omega_{n-1}\,\beta(m,n)^{\frac{n}{m}}}{\log\left(\frac{1}{r}\right)^{\frac{n}{m}-1}}\,\vartheta(r,\varepsilon,m,n)\right)^{\frac{m}{n-m}} =$$

$$= \omega_{n-1}^{\frac{m}{n-m}}\beta(m,n)^{\frac{n}{n-m}} \liminf_{r\to 0^+} \frac{\log\left(\frac{1}{r}\right)}{\log\left(\frac{1}{r}\right)}\,\vartheta(r,\varepsilon,m,n)^{\frac{m}{n-m}} =$$

$$= \omega_{n-1}^{\frac{m}{n-m}}\beta(m,n)^{\frac{n}{n-m}} \liminf_{r\to 0^+} \vartheta(r,\varepsilon,m,n)^{\frac{m}{n-m}} \le \omega_{n-1}^{\frac{m}{n-m}}\beta(m,n)^{\frac{n}{n-m}}$$

as $\varepsilon \longmapsto 0^+$. Therefore, (104) becomes

$$\alpha \le n \liminf_{r\to 0^+} \log\left(\frac{1}{r}\right)\mathcal{C}_{m,q}\Big(\overline{B_r(0)}, B\Big)^{\frac{m}{n-m}} \le n\omega_{n-1}^{\frac{m}{n-m}}\beta(m,n)^{\frac{n}{n-m}} =$$

$$= \frac{n}{\omega_{n-1}}\big(\omega_{n-1}\beta(m,n)\big)^{\frac{n}{n-m}}. \tag{111}$$

We now need two final relations (the first for the odd case and the other for the even one) and we are done.

Step 5: two last helpful relations. Finally, in order to complete the proof, we just need that the relations

$$\omega_{n-1}\beta(m,n) = \frac{2^m\pi^{\frac{n}{2}}\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \tag{112}$$

and

$$\omega_{n-1}\beta(m,n) = \frac{2^m\pi^{\frac{n}{2}}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \tag{113}$$

hold for even and odd m respectively. We first remind that, if n is odd, we have

$$\Gamma\left(\frac{n}{2}\right) = \frac{(n-2)!!}{2^{\frac{n-1}{2}}}\sqrt{\pi}. \tag{114}$$

The formula (114) tells us that we need to further divide the cases:

- if n and m are both even, then

$$\omega_{n-1}\beta(m,n) = 2\pi^{\frac{n}{2}}\,2^{\frac{m}{2}-1}\,\frac{\Gamma\big(\frac{m}{2}\big)}{\Gamma\big(\frac{n}{2}\big)}\,\frac{(n-2)!!}{(n-m-2)!!} = \pi^{\frac{n}{2}}\,2^{\frac{m}{2}}\,\frac{\Gamma\big(\frac{m}{2}\big)}{\Gamma\big(\frac{n}{2}\big)}\,\frac{(n-2)!!}{(n-m-2)!!} =$$

$$= \pi^{\frac{n}{2}}\,2^{\frac{m}{2}}\,\frac{\Gamma\big(\frac{m}{2}\big)}{\big(\frac{n}{2}-1\big)!}\,\frac{(n-2)!!}{(n-m-2)!!} = \pi^{\frac{n}{2}}\,2^{\frac{m}{2}}\,\frac{2^{\frac{n}{2}-1}\Gamma\big(\frac{m}{2}\big)}{(n-2)!!}\,\frac{(n-2)!!}{(n-m-2)!!} =$$

$$= \pi^{\frac{n}{2}} 2^m \frac{2^{\frac{n-m}{2}-1}\Gamma\left(\frac{m}{2}\right)}{(n-m-2)!!} = \frac{2^m \pi^{\frac{n}{2}} \Gamma\left(\frac{m}{2}\right)}{\left(\frac{n-m}{2}-1\right)!} = \frac{2^m \pi^{\frac{n}{2}} \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)},$$

using the properties of the semifactorial and the Gamma function as before, and so we get exactly (112);

- if n is even and m is odd, similarly to the previous case, we get

$$\omega_{n-1}\beta(m,n) = 2\pi^{\frac{n}{2}} 2^{\frac{m-1}{2}} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{(n-2)!!}{(n-m-1)!!} =$$

$$= \pi^{\frac{n}{2}} 2^{\frac{m+1}{2}} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{(n-2)!!}{(n-m-1)!!} = 2\pi^{\frac{n}{2}} 2^{\frac{m+1}{2}} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\left(\frac{n}{2}-1\right)!} \frac{(n-2)!!}{(n-m-1)!!} =$$

$$= \pi^{\frac{n}{2}} 2^{\frac{m+1}{2}} \frac{2^{\frac{n}{2}-1}\Gamma\left(\frac{m+1}{2}\right)}{(n-2)!!} \frac{(n-2)!!}{(n-m-1)!!} = \pi^{\frac{n}{2}} 2^m \frac{2^{\frac{n-m+1}{2}-1}\Gamma\left(\frac{m+1}{2}\right)}{(n-m-1)!!} =$$

$$= \frac{2^m \pi^{\frac{n}{2}} \Gamma\left(\frac{m+1}{2}\right)}{\left(\frac{n-m+1}{2}-1\right)!} = \frac{2^m \pi^{\frac{n}{2}} \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)},$$

which, this time, is (113);

- if n is odd and m is even, we have to resort twice to (114), which takes us to

$$\omega_{n-1}\beta(m,n) = 2\pi^{\frac{n}{2}} 2^{\frac{m}{2}-1} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{(n-2)!!}{(n-m-2)!!} = \pi^{\frac{n}{2}} 2^{\frac{m}{2}} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{(n-2)!!}{(n-m-2)!!} =$$

$$= \pi^{\frac{n}{2}} 2^{\frac{m}{2}} \frac{2^{\frac{n-1}{2}}\Gamma\left(\frac{m}{2}\right)}{(n-2)!!\sqrt{\pi}} \frac{(n-2)!!}{(n-m-2)!!} = \pi^{\frac{n}{2}} 2^m \frac{2^{\frac{n-m-1}{2}}\Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\,(n-m-2)!!} =$$

$$= \frac{2^m \pi^{\frac{n}{2}} \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)},$$

which is (112) once more;

- if n and m are both odd, in the same way as before we obtain

$$\omega_{n-1}\beta(m,n) = 2\pi^{\frac{n}{2}} 2^{\frac{m-1}{2}} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{(n-2)!!}{(n-m-1)!!} =$$

$$= \pi^{\frac{n}{2}} 2^{\frac{m+1}{2}} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{(n-2)!!}{(n-m-1)!!} = \pi^{\frac{n}{2}} 2^{\frac{m+1}{2}} \frac{2^{\frac{n-1}{2}}\Gamma\left(\frac{m+1}{2}\right)}{(n-2)!!\sqrt{\pi}} \frac{(n-2)!!}{(n-m-1)!!} =$$

$$= \pi^{\frac{n}{2}} 2^m \frac{2^{\frac{n-m}{2}}\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi}\,(n-m-1)!!} = \frac{2^m \pi^{\frac{n}{2}} \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)}$$

and, therefore, (113) has been reached even in this case.

These computations allow us to affirm that the relations present at the beginning of this step are valid.

Step 6: conclusion. Finally, using the definition of $\alpha_{m,n}$ and p, the relations (112) and (113) tells us that

$$\frac{n}{\omega_{n-1}}\left(\omega_{n-1}\beta(m,n)\right)^{\frac{n}{n-m}} = \alpha_{m,n} := \begin{cases} \frac{n}{\omega_{n-1}}\left[\frac{2^m\pi^{\frac{n}{2}}\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^p & \text{if } m = 2k \\ \frac{n}{\omega_{n-1}}\left[\frac{2^m\pi^{\frac{n}{2}}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\right]^p & \text{if } m = 2k-1 \end{cases}, \tag{115}$$

$\forall\, k \in \mathbb{N}$. Hence, the proof is complete since now, due to (115), (111) reads as

$$\alpha \leq \frac{n}{\omega_{n-1}}\left(\omega_{n-1}\beta(m,n)\right)^{\frac{n}{n-m}} = \alpha_{m,n}.$$

□

Remark 17. *As stated in Remark 14, Adams' $\alpha_{1,n}$ agrees with Moser's α_n. Moreover, although we assumed that $m \geq 2$, the computations just made work for Proposition 7 even if $m = 1$ (which is basically Corollary 4) as long as we have some general changes. In fact, in such a case, the conditions in Step 2 are $\phi(0) = 0$ and $\phi(1) = 1 = \phi'(1)$. Then, without introducing a parameter ε, it suffices to take the identity $\phi(t) = t$ and, therefore, the function*

$$h(t) := \begin{cases} \phi(t) & \text{if } t \in (0,1] \\ 1 & \text{if } t \in (1,+\infty) \end{cases}.$$

Subsequently, from Step 3 forward, the same calculations with $m = 1$ fixed take us to

$$\alpha \leq n \liminf_{r\to 0^+} \log\left(\frac{1}{r}\right)\mathcal{C}_{1,n}\left(\overline{B_r(0)}, B\right)^{\frac{1}{n-1}} \leq n\omega_{n-1}^{\frac{1}{n-1}} \liminf_{r\to 0^+} \frac{\log\left(\frac{1}{r}\right)}{\log\left(\frac{1}{r}\right)} = n\omega_{n-1}^{\frac{1}{n-1}} =$$

$$= \alpha_{1,n} = \alpha_n.$$

Chapter 3

3.1. Unitary Fourier transform and fractional Laplacian

In this third and final part, we further generalize the results obtained in the previous chapters, following mainly [17]. We will work with spaces of functions analogous to the Sobolev spaces considered so far; however, these ones contain functions which are, in some sense, differentiable s times, where s is any positive real number.
The meaning of a non-integer number of derivatives will be explained through the fractional Laplacian, which will be defined in this paragraph. Clearly, whenever s is a positive integer, all this generalized notions we are going to give must coincide with the well-known ones.
We will make use of the Fourier transform, which will be essential in view of a useful result that merges it with the notion of the fractional Laplacian. The idea behind this reasoning is that, as the Fourier transform is a multiplier operator if dealing with derivatives, so the fractional Laplacian must be, too. In fact, for every $m \in \mathbb{N}$, one has

$$\mathscr{F}\big((-\Delta)^m u\big)(\xi) = |\xi|^{2m} \mathscr{F}u(\xi):$$

we want this relation to be true even if we have a positive real number s instead of m.

Definition 8. *We define:*

(i) the ***Schwartz space****, denoted $\mathscr{S}(\mathbb{R}^n)$, as the set of all functions of class $C^\infty(\mathbb{R}^n)$ whose derivatives decay faster than all negative powers, namely*

$$\mathscr{S}(\mathbb{R}^n) := \Big\{ f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} \big\{ |x|^N \big|D^\beta f(x)\big| \big\} < +\infty, \, \forall\, N \in \mathbb{N} \text{ and } \forall\, \beta \in \mathbb{N}^n \Big\};$$

(ii) the ***unitary Fourier transform*** *of u, whenever $u \in \mathscr{S}(\mathbb{R}^n)$, as the operator*

$$\mathscr{F}u(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)dx;$$

(iii) the ***unitary Fourier anti-transform*** *of a function* $u \in \mathscr{S}(\mathbb{R}^n)$ *as*

$$\mathscr{F}^{-1}u(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\xi\cdot x} u(\xi) d\xi.$$

Remark 18. *There exist other definitions for the Fourier transform (some of them still make the operator* $\mathscr{F}$ *unitary, others not), all differing by a multiplicative constant; for our aim, the one just given is the most convenient. Besides, the operator* $\mathscr{F}$ *is the inverse of the one given by* $\mathscr{F}^{-1}$ *and vice versa: in fact, for every* $u \in \mathscr{S}(\mathbb{R}^n)$*, one has that* $\mathscr{F}^{-1}\big(\mathscr{F}u(\xi)\big)(x) = u(x)$ *and* $\mathscr{F}\big(\mathscr{F}^{-1}u(x)\big)(\xi) = u(\xi)$*.*

Remark 19. *One defines the Fourier transform for functions in* $\mathscr{S}(\mathbb{R}^n)$ *in order to have integrability in its definition: in fact, if* $u \in \mathscr{S}(\mathbb{R}^n)$*, it automatically belongs to* $L^1(\mathbb{R}^n)$ *and, therefore,* $\mathscr{F}u(\xi)$ *is well defined for every* $\xi \in \mathbb{R}^n$*. However, since we will continue to work on a bounded domain* Ω*, we do not have to focus on the best functional space if we use the convention to extend* u *outside* Ω *in the banal way, namely introducing the function* $\tilde{u}(x) := u(x)\chi_\Omega(x)$ *(and still calling it* u *with abuse of notation). Roughly speaking, we are able to use the integral operator* $\mathscr{F}$ *whenever it is well defined, namely whenever*

$$\int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x) dx < +\infty$$

holds. Nevertheless, we still need the definition of the Schwartz space because it is fundamental to the definition of the fractional Laplacian.

Remark 20. *The Fourier transform possesses the following properties:*

i) $\mathscr{F}(\alpha u + \beta v)(\xi) = \alpha \mathscr{F}u(\xi) + \beta \mathscr{F}v(\xi)$, $\forall\, \alpha, \beta \in \mathbb{R}$ *and* $\forall\, u, v \in \mathscr{S}(\mathbb{R}^n)$*, since*

$$\mathscr{F}(\alpha u + \beta v)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} (\alpha u + \beta v)(x) dx =$$

$$= \frac{\alpha}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x) dx + \frac{\beta}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} v(x) dx = \alpha \mathscr{F}u(\xi) + \beta \mathscr{F}v(\xi);$$

(ii) defining the function $v(x) := u(x + y)$ *for a fixed* $y \in \mathbb{R}^n$*, we achieve the relation* $\mathscr{F}v(\xi) = e^{i\xi\cdot y} \mathscr{F}u(\xi)$*, since*

$$\mathscr{F}v(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} v(x) dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x + y) dx =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot (z-y)} u(z) dz = \frac{e^{i\xi\cdot y}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot z} u(z) dz = e^{i\xi\cdot y} \mathscr{F}u(\xi),$$

where we set $z := x + y$*;*

(iii) defining the function $w(x) := u(\lambda x)$ for a fixed $\lambda > 0$, we also get the identity $\mathscr{F} w(\xi) = \frac{1}{\lambda^n} \mathscr{F} u\big(\frac{\xi}{\lambda}\big)$, since

$$\mathscr{F} w(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} w(x)dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(\lambda x)dx =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot \frac{y}{\lambda}} u(y)\, \frac{dy}{\lambda^n} = \frac{1}{\lambda^n} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\frac{\xi}{\lambda}\cdot y} u(y)dy = \frac{1}{\lambda^n} \mathscr{F} u\left(\frac{\xi}{\lambda}\right),$$

where we set $y := \lambda x$;

(iv) utilizing twice the integration by parts (Green's first identity, to be precise) and defining $\vartheta_\xi(x) := e^{-i\xi\cdot x}$, we have lastly that $\mathscr{F}(-\Delta u)(\xi) = |\xi|^2 \mathscr{F} u(\xi)$, since

$$\mathscr{F}(-\Delta u)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} (-\Delta) u(x)dx = -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \vartheta_\xi(x) \Delta u(x)dx =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \nabla \vartheta_\xi(x) \cdot \nabla u(x)dx = -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \Delta \vartheta_\xi(x) u(x)dx =$$

$$= -\frac{i^2 \xi\cdot\xi}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \vartheta_\xi(x) u(x)dx = \frac{|\xi|^2}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)dx = |\xi|^2 \mathscr{F} u(\xi),$$

from which it follows (immediately by induction using the composition property of the Laplacian) the generalized formula previously stated

$$\mathscr{F}\big((-\Delta)^m u\big)(\xi) = |\xi|^{2m} \mathscr{F} u(\xi),$$

valid for every $m \in \mathbb{N}$.

Clearly, by definition, it is immediate to show that the unitary Fourier anti-transform has these properties, too (it is due to the fact that $\mathscr{F}^{-1} u(x) = \mathscr{F} u(-\xi)$, essentially).

We enunciate here two technical results concerning the Fourier transform (present, with slight differences, in [15]), which will be useful for future statements.

Proposition 9. *If $f, g \in \mathscr{S}(\mathbb{R}^n)$, then $\mathscr{F}(f * g)(\xi) = (2\pi)^{\frac{n}{2}} \mathscr{F} f(\xi) \mathscr{F} g(\xi)$.*

Proof. Since $f, g \in \mathscr{S}(\mathbb{R}^n)$, we see that $f, g \in L^1(\mathbb{R}^n)$ as noted in Remark 19. This means that the Fourier transform of $f * g$ is well defined because, due to (53), the latter belongs to $L^1(\mathbb{R}^n)$. Thus, by definition and Fubini's theorem,

$$\mathscr{F}(f * g)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \left(\int_{\mathbb{R}^n} f(x-y) g(y)dy \right) dx =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(y) \left(\int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x-y)dx \right) dy =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(y) \bigg(\int_{\mathbb{R}^n} e^{-i\xi\cdot(y+z)} f(z) dz \bigg) dy =$$

$$= (2\pi)^{\frac{n}{2}} \left[\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot y} g(y) dy \right] \left[\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot z} f(z) dz \right] = (2\pi)^{\frac{n}{2}} \mathscr{F} f(\xi) \mathscr{F} g(\xi),$$

where we set $z := x - y$.

□

Proposition 10. *Let* $\lambda > 0$ *and define the Gaussian function* $h_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ *such that* $x \longmapsto e^{-\lambda\pi|x|^2}$. *Then*

$$\mathscr{F} h_\lambda(\xi) = \frac{e^{-\frac{|\xi|^2}{4\pi\lambda}}}{(2\pi\lambda)^{\frac{n}{2}}}.$$

Proof. In view of the point *(ii)* of Remark 20, it suffices to consider the case $\lambda = 1$. Furthermore, since

$$h_1(x) := e^{-\pi|x|^2} = \prod_{i=1}^{n} e^{-\pi x_i^2},$$

we can assume that $n = 1$ without any loss of generality. Next, being $h_1 \in \mathscr{S}(\mathbb{R})$, by definition we have

$$\mathscr{F} h_1(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} e^{-\pi|x|^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\pi x^2 - i\xi x} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\pi x^2 - i\xi x + \frac{\xi^2}{4\pi} - \frac{\xi^2}{4\pi}} dx = \frac{e^{-\frac{\xi^2}{4\pi}}}{\sqrt{2\pi}} \underbrace{\int_{\mathbb{R}} e^{-\pi\left(x + i\frac{\xi}{2\pi}\right)^2} dx}_{=: f(\xi)} = \frac{1}{\sqrt{2\pi}} h_1\left(\frac{\xi}{2\pi}\right) f(\xi).$$

The function f can be differentiated an arbitrary number of times under the integral sign, which means that $f \in C^\infty(\mathbb{R})$ and

$$f'(\xi) = \frac{d}{d\xi} \left(\int_{\mathbb{R}} e^{-\pi\left(x + i\frac{\xi}{2\pi}\right)^2} dx \right) = \int_{\mathbb{R}} \frac{d}{d\xi} \left(e^{-\pi\left(x + i\frac{\xi}{2\pi}\right)^2} \right) dx =$$

$$= \left[e^{-\pi\left(x + i\frac{\xi}{2\pi}\right)^2} \right]_{-\infty}^{+\infty} = 0.$$

Hence, the function f is constant and, utilizing the elementary formula

$$\int_{\mathbb{R}} a e^{-bx^2} dx = a\sqrt{\frac{\pi}{b}},$$

valid for every $a \in \mathbb{R}$ and $b \in \mathbb{R}\setminus\{0\}$, we get

$$f(0) = \int_{\mathbb{R}} e^{-\pi x^2} dx = 1.$$

Therefore, $f(\xi) \equiv 1$ and, consequently,

$$\mathscr{F} h_1(\xi) = \frac{1}{\sqrt{2\pi}} h_1\left(\frac{\xi}{2\pi}\right) f(\xi) = \frac{1}{\sqrt{2\pi}} h_1\left(\frac{\xi}{2\pi}\right) = \frac{e^{-\frac{\xi^2}{4\pi}}}{\sqrt{2\pi}}.$$

The latter equality gives the thesis for the case λ and n equal to 1.
Finally, in the general case, using the definition of h_λ and the property of scaling of the operator $\mathscr{F}$, we obtain

$$\mathscr{F} h_\lambda(\xi) = \mathscr{F} h_1(\sqrt{\lambda}\,\xi) = \frac{1}{\lambda^{\frac{n}{2}}} \mathscr{F} h_1\left(\frac{\xi}{\sqrt{\lambda}}\right) = \frac{1}{\lambda^{\frac{n}{2}}} \frac{1}{(2\pi)^{\frac{n}{2}}} h_1\left(\frac{\xi}{2\pi\sqrt{\lambda}}\right) =$$

$$= \frac{e^{-\pi\left|\frac{\xi}{2\pi\sqrt{\lambda}}\right|^2}}{(2\pi\lambda)^{\frac{n}{2}}} = \frac{e^{-\frac{|\xi|^2}{4\pi\lambda}}}{(2\pi\lambda)^{\frac{n}{2}}},$$

which is exactly the thesis.

□

After defining the unitary Fourier transform and noticing some of its properties, we are in position to give the definition of the fractional Laplacian operator.

Definition 9. *Let $u \in \mathscr{S}(\mathbb{R}^n)$ and $s > 0$. The* ***fractional Laplacian*** *of order s of u is defined by*

$$(-\Delta)^s u(x) := \mathscr{F}^{-1}\big(|\xi|^{2s} \mathscr{F} u(\xi)\big)(x). \tag{116}$$

Note that we introduced the operator $(-\Delta)^s$ only for functions belonging to $\mathscr{S}(\mathbb{R}^n)$: we have, therefore, to generalize this notion. In order to do that, we first give a result which is, basically, an alternative definition for the fractional Laplacian when the order is less than 1 (present also in [5]).

Proposition 11. *If $\varepsilon > 0$, $s \in (0,1)$ and $u \in \mathscr{S}(\mathbb{R}^n)$, then*

$$(-\Delta)^s u(x) = C(s,n) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy, \tag{117}$$

where

$$C(s,n) := \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+2s}}\, d\xi \right)^{-1}. \tag{118}$$

Remark 21. *Some considerations:*

(i) since $s \in (0,1)$, which implies that $n+2s > n$ and $n+2s-2 < n$, the (positive) constant $C(s,n)$ is well defined for both large and small $|\xi|$ (it is immediate if $|\xi| \longmapsto +\infty$, while we have that

$$\frac{1-\cos(\xi_1)}{|\xi|^{n+2s}} \sim \frac{\xi_1^2}{2|\xi|^{n+2s}} < \frac{|\xi|^2}{|\xi|^{n+2s}} = \frac{1}{|\xi|^{n+2s-2}}$$

near $|\xi| = 0$);

(ii) the integral in (117) is finite for $|y| \longmapsto +\infty$ since $n+2s > n$;

(iii) the quantity

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

is what is commonly called P.V., the abbreviation of "in the principal value sense";

(iv) we have to introduce the principal value because the right side of (117) is not well defined in general (however, whenever $s \in \left(0, \frac{1}{2}\right)$, we have integrability because, given any $R > 0$ and using the properties of $u \in \mathscr{S}(\mathbb{R}^n)$, we obtain

$$\left| \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \right| \leq \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}}\, dy =$$

$$= \int_{B_R(x)} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}}\, dy + \int_{\mathbb{R}^n \setminus B_R(x)} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}}\, dy \leq$$

$$\leq ||\nabla u||_{L^\infty(\mathbb{R}^n)} \int_{B_R(x)} \frac{|x-y|}{|x-y|^{n+2s}}\, dy + 2||u||_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_R(x)} \frac{dy}{|x-y|^{n+2s}} \leq$$

$$\leq 2\big(||\nabla u||_{L^\infty(\mathbb{R}^n)} + ||u||_{L^\infty(\mathbb{R}^n)}\big) \left(\int_{B_R(x)} \frac{dy}{|x-y|^{n+2s-1}} + \int_{\mathbb{R}^n \setminus B_R(x)} \frac{dy}{|x-y|^{n+2s}} \right) =$$

$$= 2\omega_{n-1}\big(||\nabla u||_{L^\infty(\mathbb{R}^n)} + ||u||_{L^\infty(\mathbb{R}^n)}\big) \left(\int_0^R \frac{d\rho}{\rho^{2s}} + \int_R^{+\infty} \frac{d\rho}{\rho^{2s+1}} \right) < +\infty,$$

since $2s < 1$ and $2s+1 > 1$, where we introduced polar coordinates in the last passage);

(v) roughly speaking, even though Proposition 11 is valid for functions $u \in \mathscr{S}(\mathbb{R}^n)$, we are allowed to use the formula (117) (and, more generally, the formula (116) in Definition 9) whenever the integral on its right side is finite, regardless of whether $u \in \mathscr{S}(\mathbb{R}^n)$ or not.

We need the following technical lemma in order to prove Proposition 11 (which, as the previous one, is in [5]).

Lemma 8. *If $s \in (0,1)$ and $u \in \mathscr{S}(\mathbb{R}^n)$, then*

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy = -\frac{1}{2}\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy. \tag{119}$$

Proof. By setting $z := y - x$, one has

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy = -\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(y)-u(x)}{|x-y|^{n+2s}}\,dy =$$

$$= -\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dy. \tag{120}$$

Next, defining the variable $\zeta := -z$, we obtain

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dy = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x-\zeta)-u(x)}{|\zeta|^{n+2s}}\,dy,$$

which means that (after properly relabeling the variables)

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dy =$$

$$= \frac{1}{2}\left(\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dy + \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dy\right) =$$

$$= \frac{1}{2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x+z)+u(x-z)-2u(x)}{|z|^{n+2s}}\,dy.$$

In conclusion, by inserting the latter relation in (120), we get

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy = -\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dy =$$

$$= -\frac{1}{2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x+z)+u(x-z)-2u(x)}{|z|^{n+2s}}\,dy,$$

which is exactly (119) once we note that the integral appearing at the end of the latter relation is integrable near 0. In fact, we have that

$$\frac{u(x+z)+u(x-z)-2u(x)}{|z|^2} = \frac{1}{|z|}\left[\frac{u(x+z)-u(x)}{|z|} + \frac{u(x-z)-u(x)}{|z|}\right] =$$

$$= \frac{\nabla u(x) - \nabla u(x-z)}{|z|} = \nabla^2 u(x),$$

which implies that

$$\frac{|u(x+z)+u(x-z)-2u(x)|}{|z|^{n+2s}} \leq \frac{||\nabla^2 u||_{L^\infty(\mathbb{R}^n)}}{|z|^{n+2s-2}},$$

from which the integrability follows because, since $s \in (0,1)$, then $n+2s-2<n$.

□

We can now prove Proposition 11 which, consequently, will give us an equivalent definition for the fractional Laplacian of order $s \in (0,1)$ for function $u \in \mathscr{S}(\mathbb{R}^n)$.

Proof of Proposition 11. In view of (119), we define the linear operator

$$\mathcal{L}u(x) := -\frac{1}{2}\,C(s,n)\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy,$$

with $C(s,n)$ defined in (118). Therefore, we are searching for the function $v : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{L}u(x) = \mathscr{F}^{-1}\big(v(\xi)\mathscr{F}u(\xi)\big)(x). \tag{121}$$

In order to get (117), it suffices that

$$v(\xi) = |\xi|^{2s} \tag{122}$$

because, if this is the case, then (119) and (121) imply that

$$C(s,n)\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy =$$

$$= -\frac{1}{2}\,C(s,n)\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy =: \mathcal{L}u(x) =$$
$$= \mathscr{F}^{-1}\big(|\xi|^{2s}\mathscr{F}u(\xi)\big)(x) =: (-\Delta)^s u(x).$$

Now, since

$$\frac{|u(x+y)+u(x-y)-2u(x)|}{|y|^{n+2s}}$$

is an integrable quantity in the variable y over $\mathbb{R}^n$ (as noted in the last part of the proof of Lemma 8), by Fubini's theorem we can exchange the integral in y with the Fourier transform in x and get

$$v(\xi)\mathscr{F}u(\xi) = \mathscr{F}(\mathcal{L}u)(\xi) =$$

$$= -\frac{1}{2}\,C(s,n)\int_{\mathbb{R}^n} \frac{\mathscr{F}\big(u(x+y)+u(x-y)-2u(x)\big)(\xi)}{|y|^{n+2s}}\,dy =$$

$$= -\frac{1}{2}\,C(s,n)\int_{\mathbb{R}^n}\frac{e^{i\xi\cdot y}+e^{-i\xi\cdot y}-2}{|y|^{n+2s}}\,\mathscr{F}u(\xi)dy =$$

$$= C(s,n)\mathscr{F}u(\xi)\int_{\mathbb{R}^n}\frac{2-e^{i\xi\cdot y}-e^{-i\xi\cdot y}}{2|y|^{n+2s}}\,dy =$$

$$= C(s,n)\mathscr{F}u(\xi)\int_{\mathbb{R}^n}\frac{1-\cos(\xi\cdot y)}{|y|^{n+2s}}\,dy,$$

using the definition of $\mathcal{L}u$, the considerations done in Remark 20, (121) and that $\cos(\zeta) = \frac{e^{i\zeta}+e^{-i\zeta}}{2}$, $\forall\,\zeta\in\mathbb{C}$. Thus,

$$v(\xi) = C(s,n)\int_{\mathbb{R}^n}\frac{1-\cos(\xi\cdot y)}{|y|^{n+2s}}\,dy.$$

Therefore, in order to get (122), it suffices that the relation

$$\int_{\mathbb{R}^n}\frac{1-\cos(\xi\cdot y)}{|y|^{n+2s}}\,dy = C(s,n)^{-1}|\xi|^{2s} \tag{123}$$

holds. Next, we define the function $g:\mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$g(\xi) := \int_{\mathbb{R}^n}\frac{1-\cos(\xi\cdot y)}{|y|^{n+2s}}\,dy$$

and show that it is invariant under rotations, namely that

$$g(\xi) = g(|\xi|e_1). \tag{124}$$

Clearly, if $n=1$ it is obvious because, since $\cos(t)=\cos(-t)$, we have $g(-\xi)=g(\xi)$. For $n\geq 2$, we consider a rotation $R\in O(n)$ such that $R(|\xi|e_1)=\xi$, whose transpose is denoted by ${}^tR$, and set $z := {}^tRy$, so that we get

$$g(\xi) := \int_{\mathbb{R}^n}\frac{1-\cos(\xi\cdot y)}{|y|^{n+2s}}\,dy = \int_{\mathbb{R}^n}\frac{1-\cos\big(R(|\xi|e_1)\cdot y\big)}{|y|^{n+2s}}\,dy =$$

$$= \int_{\mathbb{R}^n}\frac{1-\cos\big(|\xi|e_1\cdot({}^tRy)\big)}{|y|^{n+2s}}\,dy = \int_{\mathbb{R}^n}\frac{1-\cos(|\xi|e_1\cdot z)}{|z|^{n+2s}}\,dz =: g(|\xi|e_1)$$

using the properties of R (basically, utilizing that $||R||=1$), which proves (124). Hence, using (118) and (124), we finally obtain

$$\int_{\mathbb{R}^n}\frac{1-\cos(\xi\cdot y)}{|y|^{n+2s}}\,dy =: g(\xi) = g(|\xi|e_1) := \int_{\mathbb{R}^n}\frac{1-\cos(|\xi|e_1\cdot y)}{|y|^{n+2s}}\,dy =$$

$$= \int_{\mathbb{R}^n}\frac{1-\cos(|\xi|y_1)}{|y|^{n+2s}}\,dy = \int_{\mathbb{R}^n}\frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}}\,|\xi|^{n+2s}\,\frac{d\zeta}{|\xi|^n} = |\xi|^{2s}\int_{\mathbb{R}^n}\frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}}\,d\zeta =$$

$$= C(s,n)^{-1}|\xi|^{2s}$$

after making the substitution $\zeta := |\xi|y$. The result is consequently proven because what we got is exactly (123).

□

Remark 22. *One can immediately notice some properties of the fractional Laplacian of order $s > 0$ concerning functions $u \in \mathscr{S}(\mathbb{R}^n)$. In fact, it turns out that:*

(i) $\forall\, s, t > 0$, *one has*

$$\big((-\Delta)^s \circ (-\Delta)^t\big)u(x) = \mathscr{F}^{-1}\big(|\xi|^{2s}\mathscr{F}\big((-\Delta)^t u(\xi)\big)\big)(x) =$$

$$= \mathscr{F}^{-1}\Big(|\xi|^{2s}\mathscr{F}\big(\mathscr{F}^{-1}\big(|\xi|^{2t}\mathscr{F}u(\xi)\big)(x)\big)(\xi)\Big)(x) =$$

$$= \mathscr{F}^{-1}\big(|\xi|^{2(s+t)}\mathscr{F}u(\xi)\big)(x) =: (-\Delta)^{s+t}u(x);$$

(ii) if $s = m \in \mathbb{N}$, *then*

$$(-\Delta)^s u(x) := \mathscr{F}^{-1}\big(|\xi|^{2s}\mathscr{F}u(\xi)\big)(x) = \mathscr{F}^{-1}\big(|\xi|^{2m}\mathscr{F}u(\xi)\big)(x) = (-\Delta)^m u(x),$$

which means that the fractional Laplacian coincides with the classical one for any positive integer order;

(iii) if $v \in \mathscr{S}(\mathbb{R}^n)$ *and* $\alpha, \beta \in \mathbb{R}$, *we have that*

$$(-\Delta)^s\big(\alpha u(x) + \beta v(x)\big) := \mathscr{F}^{-1}\big(|\xi|^{2s}\mathscr{F}(\alpha u + \beta v)(\xi)\big)(x) =$$

$$= \mathscr{F}^{-1}\big(|\xi|^{2s}(\alpha\mathscr{F}u(\xi) + \beta\mathscr{F}v(\xi))\big)(x) =$$

$$= \alpha\mathscr{F}^{-1}\big(|\xi|^{2s}\mathscr{F}u(\xi)\big)(x) + \beta\mathscr{F}^{-1}\big(|\xi|^{2s}\mathscr{F}v(\xi)\big)(x) = \alpha(-\Delta)^s u(x) + \beta(-\Delta)^s v(x).$$

All this means that the fractional Laplacian for functions $u \in \mathscr{S}(\mathbb{R}^n)$ *can be viewed as the classical one of a certain order composed with the one defined in* (116)*; in other words, given any* $s > 0$, *there exists a positive integer* k *and a number* $\sigma \in [0,1)$ *such that* $s = k + \sigma$*: consequently, due to what has been just said, we have that* $(-\Delta)^s u(x) = \big((-\Delta)^k \circ (-\Delta)^\sigma\big)u(x)$, *where* $(-\Delta)^k$ *is the classical Laplacian operator and* $(-\Delta)^\sigma$ *is the one defined in* (116) *(or, equivalently, given by* (117)*), with the usual convention that* $(-\Delta)^0 u(x) := u(x)$.

We are now in position to give a more general definition for the fractional Laplacian which extends the preceding one. To this scope, we have to resort to the notion of tempered distribution.

Definition 10. *Let* $u \in L_s(\mathbb{R}^n)$*, where* $s > 0$ *and*

$$L_s(\mathbb{R}^n) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}}\, dx < +\infty \right\}.$$

Given a function $\varphi \in \mathscr{S}(\mathbb{R}^n)$*, the* ***fractional Laplacian*** $(-\Delta)^s u$ *of order* s *of* u *is defined as a tempered distribution via the formula*

$$\big\langle (-\Delta)^s u, \varphi \big\rangle := \int_{\mathbb{R}^n} u(x)(-\Delta)^s \varphi(x) dx. \tag{125}$$

Remark 23. *Two more observations:*

(i) one can replace s *with* $\frac{s}{2}$ *everywhere inside Definition 10 and get the analogous definition of* $(-\Delta)^{\frac{s}{2}} u$ *(sometimes, this reasoning is advantageous and, later on, we will make use of it);*

(ii) the equation (125) *is a generalization of* (116) *because, similarly for the case regarding the classical Laplacian, they coincide whenever* $u \in \mathscr{S}(\mathbb{R}^n)$ *(in such a case, both* φ *and* u *can be chosen as a test function) since, using Plancherel's theorem (see again* [15]*), we get*

$$\big\langle (-\Delta)^s u, \varphi \big\rangle := \int_{\mathbb{R}^n} u(x)(-\Delta)^s \varphi(x) dx = \int_{\mathbb{R}^n} u(x) \mathscr{F}^{-1}\big(|\xi|^{2s} \mathscr{F}\varphi(\xi)\big)(x) dx =$$

$$= \int_{\mathbb{R}^n} \mathscr{F}u(\xi) \mathscr{F}\Big(\mathscr{F}^{-1}\big(|\xi|^{2s}\mathscr{F}\varphi(\xi)\big)(x)\Big)(\xi) d\xi = \int_{\mathbb{R}^n} \mathscr{F}\varphi(\xi)|\xi|^{2s}\mathscr{F}u(\xi) d\xi =$$

$$= \int_{\mathbb{R}^n} \mathscr{F}\varphi(\xi) \mathscr{F}\Big(\mathscr{F}^{-1}\big(|\xi|^{2s}\mathscr{F}u(\xi)\big)(x)\Big)(\xi) d\xi =$$

$$= \int_{\mathbb{R}^n} \varphi(x) \mathscr{F}^{-1}\big(|\xi|^{2s}\mathscr{F}u(\xi)\big)(x) dx = \int_{\mathbb{R}^n} \varphi(x)(-\Delta)^s u(x) dx =: \big\langle (-\Delta)^s \varphi, u \big\rangle.$$

Before proceeding further, we have to prove that Definition 10 is consistent, namely that the right side of (125) is finite. This is established by the following proposition, which appears in [12].

Proposition 12. *For any number* $s > 0$ *and function* $\varphi \in \mathscr{S}(\mathbb{R}^n)$*, there exists a constant* $c = c(s, n)$ *depending on* s *and* n *only such that*

$$|(-\Delta)^s \varphi(x)| \le \frac{c}{|x|^{n+2s}}.$$

Remark 24. *Proposition 12 makes indeed* (125) *consistent because we now have that*

$$\big\langle(-\Delta)^s u,\varphi\big\rangle := \int_{\mathbb{R}^n} u(x)(-\Delta)^s\varphi(x)dx \le \int_{\mathbb{R}^n} |u(x)||(-\Delta)^s\varphi(x)|dx \le$$

$$\le c\int_{\mathbb{R}^n} \frac{|u(x)|}{|x|^{n+2s}}\,dx = c\bigg(\int_{\{|x|>1\}} \frac{|u(x)|}{|x|^{n+2s}}\,dx + \int_{\{|x|\le 1\}} \frac{|u(x)|}{|x|^{n+2s}}\,dx\bigg) < +\infty,$$

with the constant $c = c(s,n)$ as before and where we used that:

- *for large $x \in \mathbb{R}^n$, one has*

$$\int_{\{|x|>1\}} \frac{|u(x)|}{|x|^{n+2s}}\,dx \sim \int_{\{|x|>1\}} \frac{|u(x)|}{1+|x|^{n+2s}}\,dx,$$

  the latter being finite by hypothesis because $u \in L_s(\mathbb{R}^n)$;

- *for $|x| \in [0,1]$,*

$$\int_{\{|x|\le 1\}} \frac{|u(x)|}{|x|^{n+2s}}\,dx = 2^{n+2s}\int_{\{|x|\le 1\}} \frac{|u(x)|}{(2|x|)^{n+2s}}\,dx \le$$

$$\le 2^{n+2s}\int_{\{|x|\le 1\}} \frac{|u(x)|}{(1+|x|)^{n+2s}}\,dx < 2^{n+2s}\int_{\{|x|\le 1\}} \frac{|u(x)|}{1+|x|^{n+2s}}\,dx$$

  holds (where we also resort to the estimate $(1+|x|)^{n+2s} > 1+|x|^{n+2s}$), which gives again the integrability to us since $u \in L_s(\mathbb{R}^n)$.

Proof of Proposition 12. We divide the proof in two steps: first, a sufficient condition to get the thesis will be introduced and, subsequently, it will be proven.

Step 1: rewriting the problem. We introduce the spaces

$$\mathscr{S}_k(\mathbb{R}^n) := \{\varphi \in \mathscr{S}(\mathbb{R}^n) : D^\alpha \mathscr{F}\varphi(0) = 0,\ \forall\,|\alpha| \le k\} =$$

$$= \left\{\varphi \in \mathscr{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} y^\alpha \varphi(y)dy = 0,\ \forall\,|\alpha| \le k\right\}$$

for $k \in \mathbb{N}_0$ and

$$\mathscr{S}_{-1}(\mathbb{R}^n) := \mathscr{S}(\mathbb{R}^n).$$

If we show that there exists a constant $c_0 = c_0(\sigma,k,n)$ depending on σ, k and n only such that, for $k \in \mathbb{N}_0 \cup \{-1\}$,

$$|(-\Delta)^\sigma\psi(x)| \le \frac{c_0}{|x|^{n+2\sigma+k+1}} \tag{126}$$

holds $\forall\,\psi \in \mathscr{S}_k(\mathbb{R}^n)$ and $\sigma \in (0,1)$, then we are done noticing first that, $\forall\,k \in \mathbb{N}_0$ and $\forall\,\psi \in \mathscr{S}_{-1}(\mathbb{R}^n)$, we have that $\Delta^k\psi \in \mathscr{S}_{2k-1}(\mathbb{R}^n)$: in fact, given α such that $|\alpha| \le 2k-1$,

we see that
$$\int_{\mathbb{R}^n} y^\alpha \Delta^k \psi(y) dy = \int_{\mathbb{R}^n} \Delta^k (y^\alpha) \psi(y) dy = 0$$
once we integrate by parts $2k$ times (where the various boundary conditions are absent due to the properties of the Schwartz space).

As we were saying, if (126) holds, then we can write $s = k + \sigma$ for two certain elements $k \in \mathbb{N}_0$ and $\sigma \in (0,1)$, take into consideration the function $\psi(x) := (-\Delta)^k \varphi(x)$ belonging to $\mathscr{S}_{2k-1}(\mathbb{R}^n)$ and get
$$|(-\Delta)^s \varphi(x)| = |(-\Delta)^{k+\sigma} \varphi(x)| = \big|\big((-\Delta)^\sigma \circ (-\Delta)^k\big)\varphi(x)\big| = |(-\Delta)^\sigma \psi(x)| \leq$$
$$\leq \frac{c_0}{|x|^{n+2\sigma+2k-1+1}} = \frac{c_0}{|x|^{n+2(k+\sigma)}} = \frac{c_0}{|x|^{n+2s}},$$
where we used the composition property of the fractional Laplacian (see Remark 22). Therefore, we are now focused on (126) which, once proven, gives the desired result to us (with $c = c(s,n) := c_0$ depending on s and n only, since $s = k + \sigma$).

Step 2: proof of (126). It suffices to prove the inequality (126) for large $x \in \mathbb{R}^n$ since $(-\Delta)^\sigma \varphi \in C^\infty(\mathbb{R}^n)$ whenever $\varphi \in \mathscr{S}(\mathbb{R}^n)$. For a fixed $x_0 \in \mathbb{R}^n \setminus \{0\}$, we split $\mathbb{R}^n$ into $B_1 := B_{\frac{|x_0|}{2}}(0)$ and $B_2 := \mathbb{R}^n \setminus B_1$. Using (119), we have that
$$|(-\Delta)^\sigma \varphi(x)| = \frac{1}{2}\, C(\sigma,n) \left| \int_{\mathbb{R}^n} \frac{\varphi(x+y) + \varphi(x-y) - 2\varphi(x)}{|y|^{n+2\sigma}}\, dy \right| \leq$$
$$\leq \frac{1}{2}\, C(\sigma,n) \big(I_1(x) + I_2(x)\big),$$
where
$$I_1(x) := \left| \int_{B_1} \frac{\varphi(x+y) + \varphi(x-y) - 2\varphi(x)}{|y|^{n+2\sigma}}\, dy \right|$$
and
$$I_2(x) := \left| \int_{B_2} \frac{\varphi(x+y) + \varphi(x-y) - 2\varphi(x)}{|y|^{n+2\sigma}}\, dy \right|.$$
With the same computations already done at the end of the proof of Lemma 8 and set $E := B_{\frac{|x_0|}{2}}(x_0)$, we have
$$|\varphi(x+y) + \varphi(x-y) - 2\varphi(x)| \leq ||\nabla^2 \varphi||_{L^\infty(E)} |y|^2,$$
which takes us to
$$I_1(x) := \left| \int_{B_1} \frac{\varphi(x+y) + \varphi(x-y) - 2\varphi(x)}{|y|^{n+2\sigma}}\, dy \right| \leq ||\nabla^2 \varphi||_{L^\infty(E)} \int_{B_1} \frac{dy}{|y|^{n+2\sigma-2}} =$$

$$= \omega_{n-1}||\nabla^2\varphi||_{L^\infty(E)} \int_0^{\frac{|x|}{2}} \frac{\rho^{n-1}}{\rho^{n+2\sigma-2}}\,d\rho = \omega_{n-1}||\nabla^2\varphi||_{L^\infty(E)} \int_0^{\frac{|x|}{2}} \frac{d\rho}{\rho^{2\sigma-1}} =$$
$$= c_1||\nabla^2\varphi||_{L^\infty(E)}|x|^{2-2\sigma}$$

once we pass to polar coordinates, for a certain constant $c_1 = c_1(\sigma, n)$ depending on σ and n only (where we used that $2\sigma - 1 < 1$ to establish the integrability of $\rho^{1-2\sigma}$). On the other hand, using that $\varphi(x-y) \geq \varphi(x+y)$ due to the hypothesis on φ and passing again to polar coordinates, we get

$$I_2(x) := \left| \int_{B_2} \frac{\varphi(x+y) + \varphi(x-y) - 2\varphi(x)}{|y|^{n+2\sigma}}\,dy \right| \leq$$
$$\leq 2|\varphi(x)| \int_{B_2} \frac{dy}{|y|^{n+2\sigma}} + 2\left| \int_{B_2} \frac{\varphi(x-y)}{|y|^{n+2\sigma}}\,dy \right| =$$
$$= 2\omega_{n-1}|\varphi(x)| \int_{\frac{|x|}{2}}^{+\infty} \frac{\rho^{n-1}}{\rho^{n+2\sigma}}\,d\rho + 2\left| \int_{B_2} \frac{\varphi(x-y)}{|y|^{n+2\sigma}}\,dy \right| =$$
$$= 2\omega_{n-1}|\varphi(x)| \int_{\frac{|x|}{2}}^{+\infty} \frac{d\rho}{\rho^{2\sigma+1}} + 2\left| \int_{B_2} \frac{\varphi(x-y)}{|y|^{n+2\sigma}}\,dy \right| =$$
$$= c_2|\varphi(x)||x|^{-2\sigma} + 2\underbrace{\left| \int_{B_2} \frac{\varphi(x-y)}{|y|^{n+2\sigma}}\,dy \right|}_{=:I_3(x)}$$

with, this time, $\rho^{-2\sigma-1}$ integrable since $2\sigma + 1 > 1$, which gives us another constant $c_2 = c_2(\sigma, n)$ as before. Next, setting $z := x - y$, we obtain

$$I_3(x) := \left| \int_{B_2} \frac{\varphi(x-y)}{|y|^{n+2\sigma}}\,dy \right| = \left| \int_{\left\{|x-z| \geq \frac{|x|}{2}\right\}} \frac{\varphi(z)}{|x-z|^{n+2\sigma}}\,dz \right| \leq$$
$$\leq \left| \int_{\left\{|x-z| \geq \frac{|x|}{2},\, |z| \geq \frac{|x|}{2}\right\}} \frac{\varphi(z)}{|x-z|^{n+2\sigma}}\,dz \right| + \left| \int_{\left\{|z| < \frac{|x|}{2}\right\}} \frac{\varphi(z)}{|x-z|^{n+2\sigma}}\,dz \right| \leq$$
$$\leq \left| \int_{\left\{|x-z| \geq \frac{|x|}{2}\right\}} \frac{\varphi(z)}{|x-z|^{n+2\sigma}}\,dz \right| + \left| \int_{B_1} \frac{\varphi(z)}{|x-z|^{n+2\sigma}}\,dz \right| \leq$$
$$\leq c_3||\varphi||_{L^\infty(B_2)}|x|^{-2\sigma} + \underbrace{\left| \int_{B_1} \frac{\varphi(z)}{|x-z|^{n+2\sigma}}\,dz \right|}_{=:I_4(x)},$$

for a certain constant $c_3 = c_3(\sigma, n)$ depending on σ and n only (where we used the integrability of the first integrand in the same way as before). Therefore, we have to bound I_4: using that

$$\int_{\mathbb{R}^n} y^\alpha \varphi(y)dy = 0$$

since $\varphi \in \mathscr{S}_k(\mathbb{R}^n)$ and defining

$$f(x) := \frac{1}{|x|^{n+2\sigma}}$$

for $x \in \mathbb{R}^n \backslash \{0\}$, we get

$$\sum_{|\alpha|\leq k} \frac{D^\alpha f(x)}{\alpha!} \int_{\mathbb{R}^n} y^\alpha \varphi(y)dy = 0.$$

Consequently,

$$\int_{B_1} \frac{\varphi(y)}{|x-y|^{n+2\sigma}}\, dy = \int_{B_1} \frac{\varphi(y)}{|x-y|^{n+2\sigma}}\, dy - \sum_{|\alpha|\leq k} \frac{D^\alpha f(x)}{\alpha!} \int_{\mathbb{R}^n} y^\alpha \varphi(y)dy =$$

$$= \int_{B_1} \frac{\varphi(y)}{|x-y|^{n+2\sigma}}\, dy - \sum_{|\alpha|\leq k} \frac{D^\alpha f(x)}{\alpha!} \int_{B_1} y^\alpha \varphi(y)dy - \sum_{|\alpha|\leq k} \frac{D^\alpha f(x)}{\alpha!} \int_{B_2} y^\alpha \varphi(y)dy =$$

$$= \int_{B_1} \varphi(y) \left[f(x-y) - \sum_{|\alpha|\leq k} y^\alpha \frac{D^\alpha f(x)}{\alpha!} \right] dy - \sum_{|\alpha|\leq k} \frac{D^\alpha f(x)}{\alpha!} \int_{B_2} y^\alpha \varphi(y)dy =$$

$$= \int_{B_1} \varphi(y) \sum_{|\beta|=k+1} y^\beta \mathcal{R}_\beta(\xi_y)dy - \sum_{|\alpha|\leq k} \frac{D^\alpha f(x)}{\alpha!} \int_{B_2} y^\alpha \varphi(y)dy,$$

where $\mathcal{R}_\beta(\xi_y)$ satisfies

$$\sum_{|\beta|=k+1} y^\beta \mathcal{R}_\beta(\xi_y) = f(x-y) - \sum_{|\alpha|\leq k} y^\alpha \frac{D^\alpha f(x)}{\alpha!}$$

for $y \in B_1$ and $\xi_y \in E$. We have written ξ_y instead of only ξ to highlight the dependence of ξ on y. Basically, this $\mathcal{R}_\beta(\xi_y)$ is the reminder of the right side of the previous equality and it is such that

$$|\mathcal{R}_\beta(\xi_y)| \leq c_4 \sup_{\substack{z\in E \\ |\alpha|=k+1}} \left\{ |D^\alpha f(z)| \right\} \leq \frac{c_5}{|x|^{n+2\sigma+k+1}},$$

for two certain constants $c_4 = c_4(\sigma, k, n)$ and $c_5 = c_5(\sigma, k, n)$ depending on σ, k and n only. Therefore,

$$I_4(x) := \left| \int_{B_1} \frac{\varphi(z)}{|x-z|^{n+2\sigma}}\, dz \right| \leq$$

$$\leq \sum_{|\beta|=k+1} \int_{B_1} |\varphi(y)||y|^{|\beta|} |\mathcal{R}_\beta(\xi_y)|dy + \sum_{|\alpha|\leq k} \frac{|D^\alpha f(x)|}{\alpha!} \int_{B_2} |y|^{|\alpha|} |\varphi(y)|dy \leq$$

$$\leq \frac{c_6}{|x|^{n+2\sigma+k+1}} \int_{\mathbb{R}^n} |\varphi(y)||y|^{k+1} dy + \frac{c_7}{|x|^{n+2\sigma+k+1}} \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |y|^{|\alpha|} |\varphi(y)| dy < +\infty,$$

where we used the hypothesis on φ and set two additional constants $c_6 = c_6(\sigma, k, n)$ and $c_7 = c_7(\sigma, k, n)$ depending on σ, k and n only. Consequently, we have succeeded to bound I_1 and I_2 (and, hence, the left side of (126) in the previous step) through a constant depending on σ, k and n only, which means that we get the desired result.

□

Remark 25. *Being the definition of the fractional Laplacian for functions $u \in L_s(\mathbb{R}^n)$ heavily based on the one for functions $\varphi \in \mathscr{S}(\mathbb{R}^n)$, it is easily seen that the properties listed in Remark 22 are still valid. In fact, given such u and φ, one has that:*

(i) $\forall\, s, t > 0$,

$$\big\langle \big((-\Delta)^s \circ (-\Delta)^t\big) u, \varphi \big\rangle := \int_{\mathbb{R}^n} u(x) \big((-\Delta)^s \circ (-\Delta)^t\big) \varphi(x) dx =$$

$$= \int_{\mathbb{R}^n} u(x) (-\Delta)^{s+t} \varphi(x) dx =: \big\langle (-\Delta)^{s+t} u, \varphi \big\rangle;$$

(ii) if $s = m \in \mathbb{N}$, then

$$\big\langle (-\Delta)^s u, \varphi \big\rangle := \int_{\mathbb{R}^n} u(x) (-\Delta)^s \varphi(x) dx = \int_{\mathbb{R}^n} u(x) (-\Delta)^m \varphi(x) dx = \big\langle (-\Delta)^m u, \varphi \big\rangle,$$

where $(-\Delta)^m$ is the classical Laplacian operator;

(iii) if $v \in \mathscr{S}(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{R}$, then

$$\big\langle (-\Delta)^s (\alpha u + \beta v), \varphi \big\rangle := \int_{\mathbb{R}^n} \big(\alpha u(x) + \beta v(x)\big) (-\Delta)^s \varphi(x) dx =$$

$$= \alpha \int_{\mathbb{R}^n} u(x) (-\Delta)^s \varphi(x) dx + \beta \int_{\mathbb{R}^n} v(x) (-\Delta)^s \varphi(x) dx =$$

$$= \alpha \big\langle (-\Delta)^s u, \varphi \big\rangle + \beta \big\langle (-\Delta)^s v, \varphi \big\rangle.$$

Remark 26. *Having in mind the previous results concerning the fractional Laplacian, one can affirm that it has some similarities with the classical one (see, for instance, Remark 25). However, there are also some differences: the (probably) most important one is that the operator $(-\Delta)^s$ is not local in the sense that, even if $u(x) \equiv 0$ on a given ball $B_r(x)$, it can happen that $(-\Delta)^s u(x) \not\equiv 0$ on $B_r(x)$ (clearly, this phenomenon does not happen with the classical Laplacian operator). The following result is in* [3] *and establishes this (let say unusual) behavior the fractional Laplacian may have.*

Proposition 13. *If $s \in (0,1)$ and $w_s : \mathbb{R} \longrightarrow \mathbb{R}$ is such that $w_s(x) := x_+^s$, then there exists a positive constant $c = c(s)$ depending on s only for which*

$$(-\Delta)^s w_s(x) = \begin{cases} -c(s)|x|^{-s} & \text{if } x \in (-\infty, 0) \\ 0 & \text{if } x \in (0, +\infty) \end{cases}$$

holds.

This proposition shows the validity of Remark 26. Note, in fact, that w_s is identically null on the negative semiaxis and strictly increasing on the positive one, while the behavior of $(-\Delta)^s w_s$ is quite the opposite because it is strictly decreasing on the negative semiaxis and identically null on the positive one. In order to prove it, we need three intermediate results.

Lemma 9. *For any $s \in (0,1)$, we have that*

$$\int_0^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\,dt + \int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\,dt = \frac{1}{s}.$$

Proof. Introducing $\varepsilon \in (0,1)$ and integrating by parts, we get

$$\int_\varepsilon^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\,dt = -\frac{1}{2s}\int_\varepsilon^1 \big[(1+t)^s + (1-t)^s - 2\big]\frac{d}{dt}\big(t^{-2s}\big)dt =$$

$$= \left[\frac{(1+t)^s + (1-t)^s - 2}{t^{2s}}\right]_\varepsilon^1 + \frac{1}{2s}\int_\varepsilon^1 \frac{s\big[(1+t)^{s-1} - (1-t)^{s-1}\big]}{t^{2s}}\,dt =$$

$$= -\frac{1}{2s}\left[2^s - 2 - \frac{(1+\varepsilon)^s + (1-\varepsilon)^s - 2}{\varepsilon^{2s}}\right] + \frac{1}{2}\int_\varepsilon^1 \frac{(1+t)^{s-1} - (1-t)^{s-1}}{t^{2s}}\,dt =$$

$$= \frac{g(s,\varepsilon) - 2^s + 2}{2s} + \frac{1}{2}\left(\int_\varepsilon^1 \frac{(1+t)^{s-1}}{t^{2s}}\,dt - \int_\varepsilon^1 \frac{(1-t)^{s-1}}{t^{2s}}\,dt\right), \tag{127}$$

where

$$g(s,\varepsilon) := \frac{(1+\varepsilon)^s + (1-\varepsilon)^s - 2}{\varepsilon^{2s}} \longrightarrow 0$$

as $\varepsilon \longmapsto 0^+$. In fact, by applying twice L'Hôpital's rule, we see that

$$\lim_{\varepsilon\to 0^+} g(s,\varepsilon) = \lim_{\varepsilon\to 0^+} \frac{(1+\varepsilon)^s + (1-\varepsilon)^s - 2}{\varepsilon^{2s}} = \lim_{\varepsilon\to 0^+} \frac{s(1+\varepsilon)^{s-1} - s(1-\varepsilon)^{s-1}}{2s\varepsilon^{2s-1}} =$$

$$= \frac{1}{2}\lim_{\varepsilon\to 0^+} \frac{(1+\varepsilon)^{s-1} - (1-\varepsilon)^{s-1}}{\varepsilon^{2s-1}} = \frac{1}{2}\lim_{\varepsilon\to 0^+} \frac{(s-1)(1+\varepsilon)^{s-2} + (s-1)(1-\varepsilon)^{s-2}}{(2s-1)\varepsilon^{2(s-1)}} =$$

$$= \frac{s-1}{2(2s-1)}\lim_{\varepsilon\to 0^+} \frac{(1+\varepsilon)^{s-2} + (1-\varepsilon)^{s-2}}{\varepsilon^{2(s-1)}} =$$

$$= \frac{s-1}{2(2s-1)} \lim_{\varepsilon\to 0^+} \left[\frac{\varepsilon^{2(1-s)}}{(1+\varepsilon)^{2-s}} + \frac{\varepsilon^{2(1-s)}}{(1-\varepsilon)^{2-s}} \right] = 0$$

for $s \in (0,1)\backslash\left\{\frac{1}{2}\right\}$ while, applying it only once, we get

$$\lim_{\varepsilon\to 0^+} g\left(\frac{1}{2},\varepsilon\right) = \lim_{\varepsilon\to 0^+} \frac{(1+\varepsilon)^{\frac{1}{2}} + (1-\varepsilon)^{\frac{1}{2}} - 2}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{(1+\varepsilon)^{-\frac{1}{2}} - (1-\varepsilon)^{-\frac{1}{2}}}{2} =$$

$$= \frac{1}{2} \lim_{\varepsilon\to 0^+} \left[(1+\varepsilon)^{-\frac{1}{2}} - (1-\varepsilon)^{-\frac{1}{2}}\right] = 0$$

for $s = \frac{1}{2}$. Moreover, setting $\tau := \frac{t}{1-t}$ (which means that $t = \frac{\tau}{1+\tau}$), we have that

$$\int_\varepsilon^1 \frac{(1-t)^{s-1}}{t^{2s}}\,dt = \int_{\frac{\varepsilon}{1-\varepsilon}}^{+\infty} \left(1 - \frac{\tau}{1+\tau}\right)^{s-1} \left(\frac{\tau}{1+\tau}\right)^{-2s} \frac{d\tau}{(1+\tau)^2} =$$

$$= \int_{\frac{\varepsilon}{1-\varepsilon}}^{+\infty} \left(\frac{1}{1+\tau}\right)^{s-1} \left(\frac{\tau}{1+\tau}\right)^{-2s} \frac{d\tau}{(1+\tau)^2} = \int_{\frac{\varepsilon}{1-\varepsilon}}^{+\infty} \frac{\tau^{-2s}}{(1+\tau)^{s-1-2s+2}}\,d\tau =$$

$$= \int_{\frac{\varepsilon}{1-\varepsilon}}^{+\infty} \frac{(1+\tau)^{s-1}}{\tau^{2s}}\,d\tau.$$

By plugging this into (127) (and relabeling the variables), we obtain

$$\int_\varepsilon^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\,dt =$$

$$= \frac{g(s,\varepsilon) - 2^s + 2}{2s} + \frac{1}{2}\left(\int_\varepsilon^1 \frac{(1+t)^{s-1}}{t^{2s}}\,dt - \int_\varepsilon^1 \frac{(1-t)^{s-1}}{t^{2s}}\,dt \right) =$$

$$= \frac{g(s,\varepsilon) - 2^s + 2}{2s} + \frac{1}{2}\left(\int_\varepsilon^1 \frac{(1+t)^{s-1}}{t^{2s}}\,dt - \int_{\frac{\varepsilon}{1-\varepsilon}}^{+\infty} \frac{(1+t)^{s-1}}{t^{2s}}\,dt \right) =$$

$$= \frac{g(s,\varepsilon) - 2^s + 2}{2s} + \frac{1}{2}\left(\int_\varepsilon^{\frac{\varepsilon}{1-\varepsilon}} \frac{(1+t)^{s-1}}{t^{2s}}\,dt - \int_1^{+\infty} \frac{(1+t)^{s-1}}{t^{2s}}\,dt \right). \tag{128}$$

Now, since

$$\frac{d}{dt}\left[\frac{(1+t)^{s-1}}{t^{2s}}\right] = \frac{(s-1)(1+t)^{s-2}t^{2s} - 2s(1+t)^{s-1}t^{2s-1}}{t^{4s}} =$$

$$= -\left[\frac{(1-s)(1+t)^{s-2}}{t^{2s}} + \frac{2s(1+t)^{s-1}}{t^{4s+1}}\right] < 0$$

for every $t > 0$ and $\forall\, s \in (0,1)$, we can write

$$\left| \int_\varepsilon^{\frac{\varepsilon}{1-\varepsilon}} \frac{(1+t)^{s-1}}{t^{2s}}\,dt \right| = \int_\varepsilon^{\frac{\varepsilon}{1-\varepsilon}} \frac{(1+t)^{s-1}}{t^{2s}}\,dt < \int_\varepsilon^{\frac{\varepsilon}{1-\varepsilon}} \frac{(1+\varepsilon)^{s-1}}{\varepsilon^{2s}}\,dt =$$

$$= \left[\frac{(1+\varepsilon)^{s-1}}{\varepsilon^{2s}}\, t\right]_{\varepsilon}^{\frac{\varepsilon}{1-\varepsilon}} = \frac{(1+\varepsilon)^{s-1}}{\varepsilon^{2s}}\left(\frac{\varepsilon}{1-\varepsilon} - \varepsilon\right) = \frac{(1+\varepsilon)^{s-1}}{\varepsilon^{2s}}\,\frac{\varepsilon^2}{1-\varepsilon} =$$

$$= \frac{\varepsilon^{2(1-s)}}{(1+\varepsilon)^{1-s}(1-\varepsilon)} \longrightarrow 0$$

as $\varepsilon \longmapsto 0^+$. Hence,

$$\lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{\frac{\varepsilon}{1-\varepsilon}} \frac{(1+t)^{s-1}}{t^{2s}}\, dt < \lim_{\varepsilon\to 0^+} \frac{\varepsilon^{2(1-s)}}{(1+\varepsilon)^{1-s}(1-\varepsilon)} = 0.$$

Consequently, by passing to the limit in (128), we get

$$\int_0^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt = \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt =$$

$$= \lim_{\varepsilon\to 0^+} \left[\frac{g(s,\varepsilon) - 2^s + 2}{2s} + \frac{1}{2}\left(\int_{\varepsilon}^{\frac{\varepsilon}{1-\varepsilon}} \frac{(1+t)^{s-1}}{t^{2s}}\, dt - \int_1^{+\infty} \frac{(1+t)^{s-1}}{t^{2s}}\, dt\right)\right] =$$

$$= \frac{2-2^s}{2s} - \frac{1}{2}\int_1^{+\infty} \frac{(1+t)^{s-1}}{t^{2s}}\, dt. \tag{129}$$

Integrating once more by parts yields to

$$\int_1^{+\infty} \frac{(1+t)^{s-1}}{t^{2s}}\, dt = \frac{1}{s}\int_1^{+\infty} t^{-2s}\frac{d}{dt}\big[(1+t)^s\big]\, dt =$$

$$= \frac{1}{s}\left(\left[\frac{(1+t)^s}{t^{2s}}\right]_1^{+\infty} + 2s\int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\, dt\right) = -\frac{2^s}{s} + 2\int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\, dt$$

which, inserted into (129), takes us to

$$\int_0^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt = \frac{2-2^s}{2s} - \frac{1}{2}\int_1^{+\infty} \frac{(1+t)^{s-1}}{t^{2s}}\, dt =$$

$$= \frac{2-2^s}{2s} - \frac{1}{2}\left[-\frac{2^s}{s} + 2\int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\, dt\right] = \frac{2-2^s}{2s} + \frac{2^s}{2s} - \int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\, dt.$$

Note that both $\frac{(1+t)^{s-1}}{t^{2s}}$ and $\frac{(1+t)^s}{t^{1+2s}}$ are integrable for $t \longmapsto +\infty$ since their asymptotic behavior is the same as t^{-s-1} which is, indeed, integrable because $-s-1 < -1$, being $s \in (0,1)$. The latter relation implies the desired result, since now we have that

$$\int_0^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt + \int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\, dt =$$

$$= \frac{2-2^s}{2s} + \frac{2^s}{2s} - \int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\, dt + \int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\, dt =$$

$$= \frac{2 - 2^s + 2^s}{2s} = \frac{2}{2s} = \frac{1}{s},$$

which is finally the thesis.

□

Corollary 5. *If w_s is as in the statement of Proposition 13, then $(-\Delta)^s w_s(1) = 0$.*

Proof. Since the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that $h(t) := (1+t)^s + (1-t)^s - 2$ is even, we have that

$$\int_{-1}^{1} \frac{(1+t)^s + (1-t)^s - 2}{|t|^{1+2s}}\, dt = 2 \int_{0}^{1} \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt. \tag{130}$$

Furthermore, setting $\tau := -t$, one has

$$\int_{-\infty}^{-1} \frac{(1-t)^s - 2}{|t|^{1+2s}}\, dt = -\int_{+\infty}^{1} \frac{(1+\tau)^s - 2}{|\tau|^{1+2s}}\, d\tau = \int_{1}^{+\infty} \frac{(1+\tau)^s - 2}{\tau^{1+2s}}\, d\tau. \tag{131}$$

Therefore, due to the definition of w_s, (130), (131) and Lemma 9, we get

$$\int_{\mathbb{R}} \frac{w_s(1+t) + w_s(1-t) - 2w_s(1)}{|t|^{1+2s}}\, dt =$$

$$= \int_{-\infty}^{-1} \frac{(1-t)^s - 2}{|t|^{1+2s}}\, dt + \int_{-1}^{1} \frac{(1+t)^s + (1-t)^s - 2}{|t|^{1+2s}}\, dt + \int_{1}^{+\infty} \frac{(1+t)^s - 2}{t^{1+2s}}\, dt =$$

$$= 2 \int_{0}^{1} \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt + 2 \int_{1}^{+\infty} \frac{(1+t)^s - 2}{t^{1+2s}}\, dt =$$

$$= 2\left[\int_{0}^{1} \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt + \int_{1}^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\, dt - 2 \int_{1}^{+\infty} \frac{dt}{t^{1+2s}} \right] =$$

$$= 2\left(\frac{1}{s} - 2 \int_{1}^{+\infty} \frac{dt}{t^{1+2s}} \right) = 2\left(\frac{1}{s} - 2\left[\frac{t^{-2s}}{-2s} \right]_{1}^{+\infty} \right) = 2\left(\frac{1}{s} - \frac{2}{2s} \right) = 2\left(\frac{1}{s} - \frac{1}{s} \right) = 0.$$

Note that here we have used both Lemma 8 and Proposition 11 in writing the fractional Laplacian because, although $w_s \notin \mathscr{S}(\mathbb{R}^n)$, we are allowed to do that since the preceding computation gives a finite number, in according to the consideration done in the point *(v)* of Remark 21. Hence,

$$(-\Delta)^s w_s(1) = -\frac{1}{2}\, C(s,1) \int_{\mathbb{R}} \frac{w_s(1+t) + w_s(1-t) - 2w_s(1)}{|t|^{1+2s}}\, dt = 0.$$

□

The last result we need to deal with Proposition 13 is the following, which is no more than a simple observation.

Lemma 10. *If w_s is as in the statement of Proposition 13, then* $(-\Delta)^s w_s(-1) < 0$.

Proof. By definition,

$$w_s(-1+t) + w_s(-1-t) - 2w_s(-1) = (-1+t)_+^s + (-1-t)_+^s \geq 0$$

and, since $(-1+t)_+^s + (-1-t)_+^s$ is not identically null (it is equal to 1 for $t = 2$, for example), it must be that

$$(-\Delta)^s w_s(-1) = -\frac{1}{2}\, C(s,1) \int_{\mathbb{R}} \frac{w_s(-1+t) + w_s(-1-t) - 2w_s(-1)}{|t|^{1+2s}}\, dt < 0.$$

□

We have now the required elements to prove Proposition 13.

Proof of Proposition 13. We split the discussion in two steps.

Step 1: a useful relation. Let $x \in \mathbb{R}\backslash\{0\}$ and denote by $\sigma \in \{-1, 1\}$ its sign. We begin by introducing a useful relation, which is

$$\int_{-\infty}^{+\infty} \frac{w_s\big(\sigma(1+t)\big) + w_s\big(\sigma(1-t)\big) - 2w_s(\sigma)}{|t|^{1+2s}}\, dt =$$

$$= \int_{-\infty}^{+\infty} \frac{w_s(\sigma+t) + w_s(\sigma-t) - 2w_s(\sigma)}{|t|^{1+2s}}\, dt. \tag{132}$$

This relation is tautological and does not need to be proven: in fact, when $x > 0$, substituting σ with 1 gives immediately the desired result while, if $x > 0$, the same happens writing -1 instead of σ.

Step 2: conclusion. Now, noticing that

$$w_s(|x|r) := (|x|r)_+^s = |x|^s r_+^s = |x|^s w_s(r),$$

$\forall\, r \in \mathbb{R}$, which implies that

$$w_s(xr) = w_s(\sigma|x|r) = |x|^s w_s(\sigma r),$$

we obtain

$$\int_{\mathbb{R}} \frac{w_s(x+t) + w_s(x-t) - 2w_s(x)}{|t|^{1+2s}}\, dt =$$

$$= \sigma x \int_{\mathbb{R}} \frac{w_s\big(x(1+\xi)\big) + w_s\big(x(1-\xi)\big) - 2w_s(x)}{|x|^{1+2s}|\xi|^{1+2s}}\, d\xi =$$

$$= |x| \int_{\mathbb{R}} \frac{|x|^s w_s\big(\sigma(1+\xi)\big) + |x|^s w_s\big(\sigma(1-\xi)\big) - 2|x|^s w_s(\sigma)}{|x|^{1+2s}|\xi|^{1+2s}}\, d\xi =$$

$$= \frac{1}{|x|^s} \int_{\mathbb{R}} \frac{w_s\big(\sigma(1+\xi)\big) + w_s\big(\sigma(1-\xi)\big) - 2w_s(\sigma)}{|\xi|^{1+2s}}\, d\xi =$$

$$= \frac{1}{|x|^s} \int_{\mathbb{R}} \frac{w_s(\sigma+\xi) + w_s(\sigma-\xi) - 2w_s(\sigma)}{|\xi|^{1+2s}}\, d\xi,$$

where we set $\xi := \frac{t}{x}$ and used (132) in the last passage. Consequently,

$$(-\Delta)^s w_s(x) = -\frac{1}{2}\, C(s,1) \int_{\mathbb{R}} \frac{w_s(x+t) + w_s(x-t) - 2w_s(x)}{|t|^{1+2s}}\, dt =$$

$$= -\frac{1}{2|x|^s}\, C(s,1) \int_{\mathbb{R}} \frac{w_s(\sigma+\xi) + w_s(\sigma-\xi) - 2w_s(\sigma)}{|\xi|^{1+2s}}\, d\xi = \frac{(-\Delta)^s w_s(\sigma)}{|x|^s}$$

or equivalently, once we divide the cases for σ,

$$(-\Delta)^s w_s(x) = \frac{1}{|x|^s}(-\Delta)^s w_s(\sigma) = \begin{cases} |x|^{-s}(-\Delta)^s w_s(-1) & \text{if } x \in (-\infty, 0) \\ |x|^{-s}(-\Delta)^s w_s(1) & \text{if } x \in (0, +\infty) \end{cases},$$

which finally gives the thesis to us due to Corollary 5 and Lemma 10, because they tell us that $(-\Delta)^s w_s(1) = 0$ and $(-\Delta)^s w_s(-1) < 0$.

□

This (apparently counterintuitive) property of the fractional Laplacian is due to the nature itself of the latter. It is not to be confused with the classical Laplacian which deals with the ordinary derivatives: this new notion generalizes it (since, as we saw before, they coincide when the order is a positive integer), which means there is no reason why the operator $(-\Delta)^s$, applied to a function f, should be identically null on a neighborhood of a point in which f is identically null. Clearly, as we said before, this phenomenon happens when $s \in \mathbb{N}$.
Hence, we have now to introduce the functional spaces we will work on to generalize Adams' theorem.

Definition 11. *For $1 \leq p \leq \infty$, we define:*

(i) the ***fractional Laplacian space*** *of order s as*

$$H^{s,p}(\mathbb{R}^n) := \left\{ u \in L^p(\mathbb{R}^n) : (-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}^n) \right\},$$

endowed with the norm

$$||u||_{H^{s,p}(\mathbb{R}^n)} := ||u||_{L^p(\mathbb{R}^n)} + \left|\left|(-\Delta)^{\frac{s}{2}} u\right|\right|_{L^p(\mathbb{R}^n)};$$

*(ii) the **bounded fractional Laplacian space** of order s as*

$$H_0^{s,p}(\Omega) := \{u \in H^{s,p}(\mathbb{R}^n) : u(x) \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega\},$$

whose norm is

$$||u||_{H_0^{s,p}(\Omega)} := ||u||_{L^p(\Omega)} + \left|\left|(-\Delta)^{\frac{s}{2}} u\right|\right|_{L^p(\Omega)}.$$

Remark 27. *Some considerations:*

(i) as noted in Remark 23, we are now considering the fractional Laplacian of u of order $\frac{s}{2}$ instead of s (it will be helpful in view of future results, although now it may seem ambiguous to define a space of order s in which the fractional Laplacian of order $\frac{s}{2}$ appears);

(ii) the norm of a function $u \in H_0^{s,p}(\Omega)$ contains the one of $(-\Delta)^{\frac{s}{2}} u$ on Ω and not on the whole $\mathbb{R}^n$ because, although they can be different even if $u(x) \equiv 0$ on $\mathbb{R}^n \setminus \Omega$ (as established with Proposition 13), they are equivalent (see [10] for this fact);

(iii) when s is an even positive integer, it is well-known that $H^{s,p}(\Omega) = W^{s,p}(\Omega)$ and $H_0^{s,p}(\Omega) = W_0^{s,p}(\Omega)$ (in particular, for $p = 2$, they turn out to be Hilbert spaces).

Our penultimate result of this section concerns the fundamental solution of the operator $(-\Delta)^{\frac{s}{2}}$, namely it identifies the function whose fractional Laplacian of order $\frac{s}{2}$ (in the sense of tempered distributions as in Definition 11) is equal to the Dirac delta distribution centered at the origin. Surprisingly, the answer is the same of the case regarding the classical Laplacian.

Theorem 11. *The fundamental solution of $(-\Delta)^{\frac{s}{2}}$ for $s \in (0, n)$ on $\mathbb{R}^n$ is the function*

$$\mathcal{F}_s(x) := \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} |x|^{s-n}. \tag{133}$$

In other words, given such a $\mathcal{F}_s$, we have that $\mathcal{F}_s \in L_{\frac{s}{2}}(\mathbb{R}^n)$ and $(-\Delta)^{\frac{s}{2}} \mathcal{F}_s(x) = \delta_0(x)$, where the latter identity is meant in the sense of distributions in according to Definition 10, namely

$$\int_{\mathbb{R}^n} \mathcal{F}_s(x)(-\Delta)^{\frac{s}{2}} \varphi(x) dx = \int_{\mathbb{R}^n} \delta_0(x)\varphi(x) dx = \varphi(0),$$

valid for every $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Moreover,

$$(-\Delta)^{\frac{s}{2}} (\mathcal{F}_s * f)(x) = f(x) \tag{134}$$

holds for, again, every $f \in \mathscr{S}(\mathbb{R}^n)$.

Proof. We divide the proof into four steps.

Step 1: an initial consideration. Firstly, we show that $\mathcal{F}_s \in L_{\frac{s}{2}}(\mathbb{R}^n)$: this is immediate because, for large $x \in \mathbb{R}^n$, we have that

$$\int_{\mathbb{R}^n} \frac{|\mathcal{F}_s(x)|}{1+|x|^{n+s}}\, dx \sim \int_{\mathbb{R}^n} \frac{|x|^{s-n}}{1+|x|^{n+s}}\, dx \sim \int_{\mathbb{R}^n} \frac{dx}{|x|^{2n}} < +\infty,$$

since $2n > n$ for every $n \in \mathbb{N}$.

Step 2: a useful relation. Next, given a function $\varphi \in \mathscr{S}(\mathbb{R}^n)$, our aim is to achieve the relation

$$\mathscr{F}^{-1}\left(\frac{2^s \pi^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)}{|\xi|^s}\, \mathscr{F}\varphi(\xi)\right)(x) = \frac{\Gamma\left(\frac{n-s}{2}\right)}{\pi^{\frac{n-s}{2}}}\, \left(\varphi * |x|^{s-n}\right)(x), \tag{135}$$

which will be useful in the next step. Note that this identity appears also in [15].
Our starting point is the elementary formula

$$\frac{2^s \pi^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)}{|\xi|^s} = \int_0^{+\infty} \lambda^{\frac{s}{2}-1} e^{-\frac{|\xi|^2}{4\pi}\lambda} d\lambda,$$

valid (at least) for every $s \in (0, n)$. It comes from an immediate computation after writing the definition of the Gamma function: indeed, being

$$\Gamma\left(\frac{s}{2}\right) := \int_0^{+\infty} \lambda^{\frac{s}{2}-1} e^{-\lambda} d\lambda,$$

we see that

$$\int_0^{+\infty} \lambda^{\frac{s}{2}-1} e^{-\frac{|\xi|^2}{4\pi}\lambda} d\lambda = \frac{4\pi}{|\xi|^2} \int_0^{+\infty} \left(\frac{4\pi}{|\xi|^2}\, t\right)^{\frac{s}{2}-1} e^{-t} dt =$$

$$= \frac{4\pi}{|\xi|^2} \left(\frac{4\pi}{|\xi|^2}\right)^{\frac{s}{2}-1} \int_0^{+\infty} t^{\frac{s}{2}-1} e^{-t} dt = \left(\frac{4\pi}{|\xi|^2}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) = \frac{2^s \pi^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)}{|\xi|^s},$$

where we set $t := \frac{|\xi|^2}{4\pi}\, \lambda$. Further, this identity becomes

$$\frac{2^s \pi^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)}{|\xi|^s} = \int_0^{+\infty} \lambda^{\frac{s}{2}-1} e^{-\frac{|\xi|^2}{4\pi}\lambda} d\lambda = (2\pi)^{\frac{n}{2}} \int_0^{+\infty} \lambda^{\frac{s-n}{2}-1} \lambda^{\frac{n}{2}}\, \frac{e^{-\frac{|\xi|^2}{4\pi}\lambda}}{(2\pi)^{\frac{n}{2}}}\, d\lambda =$$

$$= (2\pi)^{\frac{n}{2}} \int_0^{+\infty} \lambda^{\frac{s-n}{2}-1} \mathscr{F} h_{\frac{1}{\lambda}}(\xi) d\lambda \tag{136}$$

once we resort to Proposition 10 applied to the function

$$h_{\frac{1}{\lambda}}(x) := e^{-\frac{\pi|x|^2}{\lambda}},$$

being allowed to do that since $\frac{1}{\lambda} > 0 \Longleftrightarrow \lambda > 0$.

Using the definition of the unitary Fourier anti-transform, the formula (136) just stated, Fubini's theorem, Proposition 9, Definition 6 and the definition of the Gaussian function $h_{\frac{1}{\lambda}}$, we obtain

$$\mathscr{F}^{-1}\left(\frac{2^s\pi^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{|\xi|^s}\mathscr{F}\varphi(\xi)\right)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\,\frac{2^s\pi^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{|\xi|^s}\mathscr{F}\varphi(\xi)d\xi =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\left((2\pi)^{\frac{n}{2}}\int_0^{+\infty}\lambda^{\frac{s-n}{2}-1}\mathscr{F}h_{\frac{1}{\lambda}}(\xi)d\lambda\right)\mathscr{F}\varphi(\xi)d\xi =$$

$$= \int_0^{+\infty}\lambda^{\frac{s-n}{2}-1}\left(\int_{\mathbb{R}^n} e^{i\xi\cdot x}\mathscr{F}h_{\frac{1}{\lambda}}(\xi)\mathscr{F}\varphi(\xi)d\xi\right)d\lambda =$$

$$= \int_0^{+\infty}\lambda^{\frac{s-n}{2}-1}\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\mathscr{F}\left(h_{\frac{1}{\lambda}}*\varphi\right)(\xi)d\xi\right)d\lambda =$$

$$= \int_0^{+\infty}\lambda^{\frac{s-n}{2}-1}\mathscr{F}^{-1}\left(\mathscr{F}\left(h_{\frac{1}{\lambda}}*\varphi\right)(\xi)\right)(x)d\lambda = \int_0^{+\infty}\lambda^{\frac{s-n}{2}-1}\left(h_{\frac{1}{\lambda}}*\varphi\right)(x)d\lambda =$$

$$= \int_0^{+\infty}\lambda^{\frac{s-n}{2}-1}\left(\int_{\mathbb{R}^n} h_{\frac{1}{\lambda}}(x-y)\varphi(y)dy\right)d\lambda = \int_0^{+\infty}\lambda^{\frac{s-n}{2}-1}\left(\int_{\mathbb{R}^n} e^{-\frac{\pi|x-y|^2}{\lambda}}\varphi(y)dy\right)d\lambda =$$

$$= \int_{\mathbb{R}^n}\left(\int_0^{+\infty}\lambda^{\frac{s-n}{2}-1}e^{-\frac{\pi|x-y|^2}{\lambda}}d\lambda\right)\varphi(y)dy. \tag{137}$$

Now, setting $\tau := \frac{(2\pi)^2}{\lambda}$ and using again (136), we get

$$\int_0^{+\infty}\lambda^{\frac{s-n}{2}-1}e^{-\frac{\pi|x-y|^2}{\lambda}}d\lambda = -\int_{+\infty}^{0}\left[\frac{(2\pi)^2}{\tau}\right]^{\frac{s-n}{2}-1}e^{-\frac{\pi|x-y|^2}{(2\pi)^2}\tau}\,\frac{(2\pi)^2}{\tau^2}d\tau =$$

$$= \int_0^{+\infty}\frac{(2\pi)^{s-n-2}}{\tau^{\frac{s-n}{2}-1}}e^{-\frac{|x-y|^2}{4\pi}\tau}\,\frac{(2\pi)^2}{\tau^2}d\tau = (2\pi)^{s-n}\int_0^{+\infty}\tau^{\frac{n-s}{2}-1}e^{-\frac{|x-y|^2}{4\pi}\tau}d\tau =$$

$$= (2\pi)^{s-n}\,\frac{2^{n-s}\pi^{\frac{n-s}{2}}\Gamma\left(\frac{n-s}{2}\right)}{|x-y|^{n-s}} = \frac{\pi^{\frac{s-n}{2}}\Gamma\left(\frac{n-s}{2}\right)}{|x-y|^{n-s}} = \frac{\Gamma\left(\frac{n-s}{2}\right)}{\pi^{\frac{n-s}{2}}}\,|x-y|^{s-n}. \tag{138}$$

Hence, putting (138) inside (137) and using the commutativity of the convolution (see Remark 10), we see that

$$\mathscr{F}^{-1}\left(\frac{2^s\pi^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{|\xi|^s}\mathscr{F}\varphi(\xi)\right)(x) = \int_{\mathbb{R}^n}\left(\int_0^{+\infty}\lambda^{\frac{s-n}{2}-1}e^{-\frac{\pi|x-y|^2}{\lambda}}d\lambda\right)\varphi(y)dy =$$

$$= \frac{\Gamma\left(\frac{n-s}{2}\right)}{\pi^{\frac{n-s}{2}}}\int_{\mathbb{R}^n}|x-y|^{s-n}\varphi(y)dy = \frac{\Gamma\left(\frac{n-s}{2}\right)}{\pi^{\frac{n-s}{2}}}\left(\varphi*|x|^{s-n}\right)(x),$$

which is exactly the relation (135).

Step 3: proof of $(-\Delta)^{\frac{s}{2}}\mathcal{F}_s(x) = \delta_0(x)$. Utilizing the auxiliary formula proven in Step 2, we infer

$$\mathscr{F}^{-1}\left(\frac{2^s\pi^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{|\xi|^s}\mathscr{F}\varphi(\xi)\right)(x)=\frac{\Gamma\left(\frac{n-s}{2}\right)}{\pi^{\frac{n-s}{2}}}\left(\varphi*|x|^{s-n}\right)(x)\iff$$

$$\iff\frac{2^s\pi^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{|\xi|^s}\mathscr{F}\varphi(\xi)=\frac{\Gamma\left(\frac{n-s}{2}\right)}{\pi^{\frac{n-s}{2}}}\mathscr{F}\left(\varphi*|x|^{s-n}\right)(\xi)\iff$$

$$\iff\mathscr{F}\varphi(\xi)=\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s\pi^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\pi^{\frac{n-s}{2}}}|\xi|^s\mathscr{F}\left(\varphi*|x|^{s-n}\right)(\xi)=$$

$$=(2\pi)^{\frac{n}{2}}\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s\pi^{\frac{n}{2}}\Gamma\left(\frac{s}{2}\right)}|\xi|^s\mathscr{F}\varphi(\xi)\mathscr{F}\left(|x|^{s-n}\right)(\xi)=$$

$$=(2\pi)^{\frac{n}{2}}|\xi|^s\mathscr{F}\varphi(\xi)\mathscr{F}\left(\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s\pi^{\frac{n}{2}}\Gamma\left(\frac{s}{2}\right)}|x|^{s-n}\right)(\xi)=(2\pi)^{\frac{n}{2}}|\xi|^s\mathscr{F}\varphi(\xi)\mathscr{F}\mathcal{F}_s(\xi)\iff$$

$$\iff(2\pi)^{\frac{n}{2}}|\xi|^s\mathscr{F}\mathcal{F}_s(\xi)=1\iff\mathscr{F}\mathcal{F}_s(\xi)=\frac{1}{(2\pi)^{\frac{n}{2}}|\xi|^s},\tag{139}$$

where we also used the formula for the Fourier transform of the convolution established in Proposition 9 and the definition of $\mathcal{F}_s$ written in (133). Hence, we found a writing for the Fourier transform of the function $\mathcal{F}_s$, which will prove useful even in the following corollary. Further, we have that

$$\mathscr{F}\delta_0(\xi):=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-i\xi\cdot x}\delta_0(x)dx=\frac{e^{-i\xi\cdot 0}}{(2\pi)^{\frac{n}{2}}}=\frac{1}{(2\pi)^{\frac{n}{2}}}.\tag{140}$$

Note that, here, we have used the Fourier transform in relation to the Dirac delta distribution, which is a measure and not a function. Nevertheless, we are able to do that because, since the result is a number, it still makes sense (it is, in fact, a standard operation widely used). Therefore, by Plancherel's theorem, Definition 9, (139) and (140), it follows that, $\forall\,\varphi\in\mathscr{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\mathcal{F}_s(x)(-\Delta)^{\frac{s}{2}}\varphi(x)dx=\int_{\mathbb{R}^n}\mathscr{F}\mathcal{F}_s(\xi)\mathscr{F}\left((-\Delta)^{\frac{s}{2}}\varphi\right)(\xi)d\xi=$$

$$=\int_{\mathbb{R}^n}\frac{\mathscr{F}\left(\mathscr{F}^{-1}\left(|\xi|^s\mathscr{F}\varphi(\xi)\right)(x)\right)(\xi)}{(2\pi)^{\frac{n}{2}}|\xi|^s}d\xi=\int_{\mathbb{R}^n}\frac{|\xi|^s\mathscr{F}\varphi(\xi)}{(2\pi)^{\frac{n}{2}}|\xi|^s}d\xi=$$

$$=\int_{\mathbb{R}^n}\frac{\mathscr{F}\varphi(\xi)}{(2\pi)^{\frac{n}{2}}}d\xi=\int_{\mathbb{R}^n}\mathscr{F}\delta_0(\xi)\mathscr{F}\varphi(\xi)d\xi=\int_{\mathbb{R}^n}\delta_0(x)\varphi(x)dx=\varphi(0),$$

which is the desired result.

Step 4: proof of (134). Finally, we show the remaining relation given by (134): at this point, it is immediate once we use again Proposition 9 and the formula for the Fourier transform of $\mathcal{F}_s$ appearing in (139). In fact, we obtain

$$(-\Delta)^{\frac{s}{2}}(\mathcal{F}_s * f)(x) := \mathscr{F}^{-1}\big(|\xi|^s \mathscr{F}(\mathcal{F}_s * f)(\xi)\big)(x) =$$

$$= \mathscr{F}^{-1}\big((2\pi)^{\frac{n}{2}}|\xi|^s \mathscr{F}\mathcal{F}_s(\xi)\mathscr{F}f(\xi)\big)(x) = \mathscr{F}^{-1}\left((2\pi)^{\frac{n}{2}}|\xi|^s \, \frac{\mathscr{F}f(\xi)}{(2\pi)^{\frac{n}{2}}|\xi|^s}\right)(x) =$$

$$= \mathscr{F}^{-1}\big(\mathscr{F}f(\xi)\big)(x) = f(x),$$

$\forall\, f \in \mathscr{S}(\mathbb{R}^n)$.

□

We state here a consequence of the preceding theorem which will be used later (and which appears also in [15]).

Corollary 6. *For $\alpha, \beta \in (0,n)$ such that $\alpha+\beta \in (0,n)$, we have*

$$\big(|x|^{\alpha-n} * |x|^{\beta-n}\big)(x) = \frac{\pi^{\frac{n}{2}}\Gamma\big(\frac{\alpha}{2}\big)\Gamma\big(\frac{\beta}{2}\big)\Gamma\big(\frac{n-\alpha-\beta}{2}\big)}{\Gamma\big(\frac{\alpha+\beta}{2}\big)\Gamma\big(\frac{n-\alpha}{2}\big)\Gamma\big(\frac{n-\beta}{2}\big)}\,|x|^{\alpha+\beta-n}. \tag{141}$$

Proof. We start from the formula (139) appearing in the previous proof: thus, using the definition of the function $\mathcal{F}_s$ and the properties of the operator $\mathscr{F}$, we see that

$$\mathscr{F}\mathcal{F}_s(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}|\xi|^s} \Longleftrightarrow |\xi|^{-s} = (2\pi)^{\frac{n}{2}}\mathscr{F}\mathcal{F}_s(\xi) = (2\pi)^{\frac{n}{2}}\mathscr{F}\left(\frac{\Gamma\big(\frac{n-s}{2}\big)}{2^s\pi^{\frac{n}{2}}\Gamma\big(\frac{s}{2}\big)}\,|x|^{s-n}\right)(\xi) =$$

$$= (2\pi)^{\frac{n}{2}}\,\frac{\Gamma\big(\frac{n-s}{2}\big)}{2^s\pi^{\frac{n}{2}}\Gamma\big(\frac{s}{2}\big)}\,\mathscr{F}\big(|x|^{s-n}\big)(\xi) \Longleftrightarrow \mathscr{F}\big(|x|^{s-n}\big)(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\,\frac{2^s\pi^{\frac{n}{2}}\Gamma\big(\frac{s}{2}\big)}{\Gamma\big(\frac{n-s}{2}\big)}\,|\xi|^{-s}.$$

These relations hold whenever $s \in (0,n)$: therefore, by hypothesis, they still hold if we replace s with α, β or $\alpha+\beta$. Note also that we did not exemplify further the calculations for reasons of convenience.
We shall now compute $\mathscr{F}\big(|x|^{\alpha-n} * |x|^{\beta-n}\big)(\xi)$: before doing that, it is imperative to note that, although $\big(|x|^{\alpha-n} * |x|^{\beta-n}\big)(x) \notin \mathscr{S}(\mathbb{R}^n)$, we are still allowed to make such a calculation. In fact, since

$$\big(|x|^{\alpha-n} * |x|^{\beta-n}\big)(x) := \int_{\mathbb{R}^n} |x-y|^{\alpha-n}|y|^{\beta-n}dy = \int_{\mathbb{R}^n}\frac{dy}{|x-y|^{n-\alpha}|y|^{n-\beta}},$$

then:

- we have that
$$\big(|x|^{\alpha-n} * |x|^{\beta-n}\big)(x) \sim \int_{\mathbb{R}^n}\frac{dy}{|y|^{n-\beta}} < +\infty$$
near $|y| = 0$, because $n-\beta < n$;

- we have also that

$$\big(|x|^{\alpha-n} * |x|^{\beta-n}\big)(x) \sim \int_{\mathbb{R}^n} \frac{dy}{|x-y|^{n-\alpha}} < +\infty$$

near $y = x$, because $n - \alpha < n$;

- we have lastly that

$$\big(|x|^{\alpha-n} * |x|^{\beta-n}\big)(x) \sim \int_{\mathbb{R}^n} \frac{dy}{|y|^{2n-(\alpha+\beta)}} < +\infty$$

for large $y \in \mathbb{R}^n$, because $2n - (\alpha + \beta) > 2n - n = n$.

Consequently,

$$\big(|x|^{\alpha-n} * |x|^{\beta-n}\big)(x) = \int_{\mathbb{R}^n} \frac{dy}{|x-y|^{n-\alpha}|y|^{n-\beta}} < +\infty.$$

Furthermore, using also Proposition 9, it follows that

$$\mathscr{F}\big(|x|^{\alpha-n} * |x|^{\beta-n}\big)(\xi) = (2\pi)^{\frac{n}{2}}\mathscr{F}\big(|x|^{\alpha-n}\big)(\xi)\mathscr{F}\big(|x|^{\beta-n}\big)(\xi) =$$

$$= (2\pi)^{\frac{n}{2}} \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{2^{\alpha}\pi^{\frac{n}{2}}\Gamma\big(\frac{\alpha}{2}\big)}{\Gamma\big(\frac{n-\alpha}{2}\big)} |\xi|^{-\alpha} \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{2^{\beta}\pi^{\frac{n}{2}}\Gamma\big(\frac{\beta}{2}\big)}{\Gamma\big(\frac{n-\beta}{2}\big)} |\xi|^{-\beta} =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{2^{\alpha+\beta}\pi^{n}\Gamma\big(\frac{\alpha}{2}\big)\Gamma\big(\frac{\beta}{2}\big)}{\Gamma\big(\frac{n-\alpha}{2}\big)\Gamma\big(\frac{n-\beta}{2}\big)} |\xi|^{-(\alpha+\beta)} =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{2^{\alpha+\beta}\pi^{n}\Gamma\big(\frac{\alpha}{2}\big)\Gamma\big(\frac{\beta}{2}\big)}{\Gamma\big(\frac{n-\alpha}{2}\big)\Gamma\big(\frac{n-\beta}{2}\big)} (2\pi)^{\frac{n}{2}} \frac{\Gamma\big(\frac{n-\alpha-\beta}{2}\big)}{2^{\alpha+\beta}\pi^{\frac{n}{2}}\Gamma\big(\frac{\alpha+\beta}{2}\big)} \mathscr{F}\big(|x|^{\alpha+\beta-n}\big)(\xi) =$$

$$= \mathscr{F}\left(\frac{1}{(2\pi)^{\frac{n}{2}}} \frac{2^{\alpha+\beta}\pi^{n}\Gamma\big(\frac{\alpha}{2}\big)\Gamma\big(\frac{\beta}{2}\big)}{\Gamma\big(\frac{n-\alpha}{2}\big)\Gamma\big(\frac{n-\beta}{2}\big)} (2\pi)^{\frac{n}{2}} \frac{\Gamma\big(\frac{n-\alpha-\beta}{2}\big)}{2^{\alpha+\beta}\pi^{\frac{n}{2}}\Gamma\big(\frac{\alpha+\beta}{2}\big)} |x|^{\alpha+\beta-n}\right)(\xi) =$$

$$= \mathscr{F}\left(\frac{\pi^{\frac{n}{2}}\Gamma\big(\frac{\alpha}{2}\big)\Gamma\big(\frac{\beta}{2}\big)\Gamma\big(\frac{n-\alpha-\beta}{2}\big)}{\Gamma\big(\frac{\alpha+\beta}{2}\big)\Gamma\big(\frac{n-\alpha}{2}\big)\Gamma\big(\frac{n-\beta}{2}\big)} |x|^{\alpha+\beta-n}\right)(\xi),$$

which is equivalent to

$$\big(|x|^{\alpha-n} * |x|^{\beta-n}\big)(x) = \frac{\pi^{\frac{n}{2}}\Gamma\big(\frac{\alpha}{2}\big)\Gamma\big(\frac{\beta}{2}\big)\Gamma\big(\frac{n-\alpha-\beta}{2}\big)}{\Gamma\big(\frac{\alpha+\beta}{2}\big)\Gamma\big(\frac{n-\alpha}{2}\big)\Gamma\big(\frac{n-\beta}{2}\big)} |x|^{\alpha+\beta-n}.$$

□

3.2. Fractional Moser-Adams' theorem

Adams' techniques do not work if dealing with the fractional Laplacian. The critical issue is the non-local property of this operator: in the previous chapter, in fact, we deduced Theorem 9 from Theorem 10 applied to the function $f(x) := \nabla^m u(x)$ for both even and odd cases. However, in order to apply Theorem 10, it was crucial to have a function f such that $\text{supp}\{f\} \subseteq \Omega$: clearly, there is no problem for a function f defined as above but, if we take $f(x) := (-\Delta)^s u(x)$ for a certain $s \in (0, +\infty) \setminus \mathbb{N}$, the aforementioned hypothesis is no more valid in view of Remark 26.
In order to circumvent this issue, instead of introducing the Riesz potential, we will write u in terms of a Green representation formula (similarly to what has been done in Lemma 7). To this scope, we will mainly use Theorem 11 of the preceding section and a density theorem due to Netrusov (see [20] for the latter), as well as some technical results which will be enunciated afterwards. After that, we will be able to finally use Theorem 10 (and Proposition 8, which was previously stated and proven but never utilized) and get the thesis.
Another interesting outcome is that, this time, the following theorem holds even for the unidimensional case. In fact, Moser's theorem does not concern such a case because, by hypothesis, $p := \frac{n}{n-1}$; similarly, in Adams' theorem one has two positive integers n and m with the condition that $n > m$ (which is clearly not possible if we fix $n = 1$). However, Theorem 10 requires no hypothesis on the dimension and we will not directly use Theorem 7 or Theorem 9: therefore, since now $s > 0$, the condition $s < n$ can be satisfied for some s even if $n = 1$.

Theorem 12 (**fractional Moser-Adams**). *Let $n \in \mathbb{N}$, $s > 0$ such that $s < n$ and $u \in H_0^{s,\frac{n}{s}}(\Omega)$. Assume that*

$$||(-\Delta)^{\frac{s}{2}} u||_{L^q(\Omega)} := \left(\int_\Omega |(-\Delta)^{\frac{s}{2}} u(x)|^q dx \right)^{\frac{1}{q}} \leq 1,$$

where we set $q := \frac{n}{s} \in (1, +\infty)$. Then, $\forall\, \alpha \in [0, \alpha_{s,n}]$, there exists a constant $c = c(s,n)$ depending on s and n only such that

$$\fint_\Omega e^{\alpha |u(x)|^p} dx \leq c, \tag{142}$$

where $p := \frac{q}{q-1} = \frac{n}{n-s}$ is the conjugate exponent of q and

$$\alpha_{s,n} := \frac{n}{\omega_{n-1}} \left[\frac{2^s \pi^{\frac{n}{2}} \Gamma(\frac{s}{2})}{\Gamma(\frac{n-s}{2})} \right]^p.$$

Remark 28. *Obviously, due to their definitions, this $\alpha_{s,n}$ coincides with Adams' $\alpha_{m,n}$ when s is an even positive integer (that is the reason we have chosen the same notation for $\alpha_{s,n}$). In such a case, Theorem 12 is automatically true because it comes down to Adams' theorem for even m (remembering that $H_0^{2k,\frac{n}{2k}}(\Omega) = W_0^{2k,\frac{n}{2k}}(\Omega)$ for $k \in \mathbb{N}$, as noted in Remark 27).*

We enunciate now the usual result concerning the sharpness of the constant $\alpha_{s,n}$.

Corollary 7. *If $\alpha > \alpha_{s,n}$, the estimate (142) is no more true in the sense that*

$$\sup_{\substack{u \in H_0^{s,q}(\Omega) \\ ||(-\Delta)^{\frac{s}{2}} u||_{L^q(\Omega)} \le 1}} \left\{ \fint_{\Omega} e^{\alpha |u(x)|^p} dx \right\} = +\infty.$$

In other words, if there exists a constant c for which

$$\fint_{\Omega} e^{\alpha |u(x)|^p} dx \le c$$

holds for $\alpha > \alpha_{s,n}$ when the remaining hypothesis of Theorem 12 are satisfied, then $c = c(s, n, u)$ is forced to depend also on the function u as well.

In order to prove Theorem 12, we need two intermediate results which allow us to write u in terms of a Green representation formula, as stated previously.

Proposition 14. *Let $\sigma \in (0, 2]$ such that $\sigma < n$. Then, for every $x_0 \in \Omega$, there exists a function $\mathcal{G}_\sigma(x_0, y) \in L^1(\mathbb{R}^n)$ satisfying*

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} \mathcal{G}_\sigma(x_0, y) = \delta_{x_0}(y) & \text{if } y \in \Omega \\ \mathcal{G}_\sigma(x_0, y) = 0 & \text{if } y \in \mathbb{R}^n \backslash \Omega \end{cases}, \tag{143}$$

where the first equation of (143), in according to Definition 10, is meant in the sense of distributions, namely

$$\int_{\mathbb{R}^n} \mathcal{G}_\sigma(x_0, y) (-\Delta)^{\frac{\sigma}{2}} \varphi(y) dy = \int_{\mathbb{R}^n} \delta_{x_0}(y) \varphi(y) dy = \varphi(x_0),$$

$\forall \varphi \in C_0^\infty(\Omega)$. Moreover, given $\mathcal{F}_s$ as in (133), we have that

$$0 \le \mathcal{G}_\sigma(x_0, y) \le \mathcal{F}_\sigma(x_0 - y) \tag{144}$$

for a.e. $y \in \Omega$ such that $y \neq x_0$. Furthermore, if $1 \le p < \infty$ and $u \in H_0^{\sigma,p}(\Omega)$,

$$u(x) = \int_{\Omega} \mathcal{G}_\sigma(x, y) (-\Delta)^{\frac{\sigma}{2}} u(y) dy \tag{145}$$

holds for a.e. $x \in \Omega$.

Remark 29. *The Green representation formula of u is consistent, namely the right side of* (145) *is well defined due to Theorem 11 and* (144)*. Furthermore, the first equation of* (143)*, intended in the sense of distributions, is valid for functions belonging to* $C_0^\infty(\Omega)$ *and not to* $\mathscr{S}(\mathbb{R}^n)$*, another supposition we are allowed to make.*

Proposition 15. *Let* $s > 0$ *such that* $s = 2k+\sigma < n$*, for some* $k \in \mathbb{N}_0$ *and* $\sigma \in (0,2]$*. Define*

$$\mathcal{G}_s(x,y) := \int_\Omega \mathcal{G}_2(x,y_1)\left[\int_\Omega \mathcal{G}_2(y_1,y_2)\left[\cdots\left[\int_\Omega \mathcal{G}_2(y_{k-1},y_k)\mathcal{G}_\sigma(y_k,y)dy_k\right]\cdots\right]dy_2\right]dy_1,$$

where the functions $\mathcal{G}_2$ *and* $\mathcal{G}_\sigma$ *are as in Proposition 14. Then*

$$0 \leq \mathcal{G}_s(x,y) \leq (\underbrace{\mathcal{F}_2 * \mathcal{F}_2 * \cdots * \mathcal{F}_2}_{k\ times} * \mathcal{F}_\sigma)(x-y) = \mathcal{F}_s(x-y). \tag{146}$$

Moreover, if $1 \leq p < \infty$ *and* $u \in H_0^{s,p}(\Omega)$*, then*

$$u(x) = \int_\Omega \mathcal{G}_s(x,y)(-\Delta)^{\frac{s}{2}}u(y)dy \tag{147}$$

holds for a.e. $x \in \Omega$*.*

Remark 30. *More generally, the Green representation formula given by* (147) *is valid for functions u which can be approximated by a sequence* $\{u_k\}_{k\in\mathbb{N}}$ *lying in* $C_0^\infty(\Omega)$ *due to Netrusov's density theorem, which affirms that, given* $s > 0$ *and* $1 \leq p < \infty$*,* $C_0^\infty(\Omega)$ *is dense in* $H_0^{s,p}(\Omega)$*. Hence, in such a situation, one gets* $u_k(x) \longrightarrow u(x)$ *and* $(-\Delta)^{\frac{s}{2}}u_k(x) \longrightarrow (-\Delta)^{\frac{s}{2}}u(x)$ *both in* $L^1(\Omega)$*.*

We state, one after the other, four results which the previous two propositions are, in turn, based on. Doing this, we divide the discussion into several parts in order to soften the discourse. The following is present in [5].

Lemma 11. *For* $\sigma \in (0,2)$ *and* $u \in W^{\frac{\sigma}{2},2}(\mathbb{R}^n)$*, define*

$$[u]_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\sigma}}\,dx\right)dy\right)^{\frac{1}{2}}.$$

Then

$$[u]_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)} = \sqrt{2}\,C\left(\frac{\sigma}{2},n\right)^{-\frac{1}{2}}\big|\big|(-\Delta)^{\frac{\sigma}{4}}u\big|\big|_{L^2(\mathbb{R}^n)}, \tag{148}$$

where $C\left(\frac{\sigma}{2},n\right)$ *is the usual constant appearing in* (118)*. In other words, we have that*

$$H^{\frac{\sigma}{2},2}(\mathbb{R}^n) = W^{\frac{\sigma}{2},2}(\mathbb{R}^n) := \left\{u \in L^2(\mathbb{R}^n) : [u]_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)} < +\infty\right\}.$$

Lemma 12. *Define the bilinear form*

$$\mathcal{B}_\sigma(u,v) := \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+\sigma}}\, dx \right) dy \tag{149}$$

for $u, v \in H^{\frac{\sigma}{2},2}(\mathbb{R}^n)$. *Given* $\sigma \in (0,2)$, $f \in L^2(\Omega)$ *and* $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ *such that*

$$\int_\Omega \left(\int_{\mathbb{R}^n} \frac{\big(g(x)-g(y)\big)^2}{|x-y|^{n+\sigma}}\, dx \right) dy < +\infty, \tag{150}$$

there exists a unique function $\bar{u} \in H_0^{\frac{\sigma}{2},2}(\Omega)$ *such that the function* $\tilde{u}(x) := \bar{u}(x) + g(x)$ *solves, for every function* $v \in H_0^{\frac{\sigma}{2},2}(\mathbb{R}^n)$, *the problem*

$$\mathcal{B}_\sigma(u,v) = \int_{\mathbb{R}^n} f(x)v(x)dx. \tag{151}$$

Moreover, such $\tilde{u}$ *satisfies the relation* $(-\Delta)^{\frac{\sigma}{2}}\tilde{u}(x) = \frac{1}{2}C\big(\frac{\sigma}{2},n\big)f(x)$ *in* Ω *in the sense of distributions, namely the relation*

$$\int_{\mathbb{R}^n} \tilde{u}(x)(-\Delta)^{\frac{\sigma}{2}}\varphi(x)dx = \frac{1}{2}C\left(\frac{\sigma}{2},n\right)\int_{\mathbb{R}^n} f(x)\varphi(x)dx \tag{152}$$

holds for every $\varphi \in C_0^\infty(\Omega)$, *where the constant* $C\big(\frac{\sigma}{2},n\big)$ *is the one defined in* (118). *Conversely, if* $\tilde{u}$ *satisfies* (152), *then it also satisfies* (151).

Remark 31. *The right side of* (149) *is well defined thanks to Lemma 11 (and Hölder's inequality), being* $u, v \in H^{\frac{\sigma}{2},2}(\mathbb{R}^n)$.

Corollary 8. *Under the same hypothesis of Lemma 12, if we add the requests that* $f(x) \geq 0$ *on* Ω *and* $g(x) \geq 0$ *on* $\mathbb{R}^n \backslash \Omega$, *then the unique function* $\tilde{u}(x) := \bar{u}(x) + g(x)$ *solving* (151) *is such that* $\tilde{u}(x) \geq 0$ *a.e. on* $\mathbb{R}^n$.

Remark 32. *The latter result is a sort of generalization of the classical maximum principle for the fractional Laplacian.*

Lemma 13. *Let* $\mathcal{F}_\sigma$ *be the fundamental solution of* $(-\Delta)^{\frac{\sigma}{2}}$ *for* $\sigma \in (0,2)$ *on* $\mathbb{R}^n$, *with* $\sigma < n$. *Then there exist a number* $\lambda > 0$ *and a function* $\Psi_\lambda \in C^1(\mathbb{R}^n)$ *such that*

$$\Psi_\lambda(x) \leq \mathcal{F}_\sigma(x) \tag{153}$$

on $\mathbb{R}^n$,

$$\Psi_\lambda(x) = \mathcal{F}_\sigma(x) \tag{154}$$

on $\mathbb{R}^n \backslash B_\lambda(0)$ *and, again on* $\mathbb{R}^n$,

$$(-\Delta)^{\frac{\sigma}{2}}\Psi_\lambda(x) \geq 0. \tag{155}$$

Remark 33. *The previous lemma is taken from* [23] *and, basically, allows to consider a new function based on the fundamental solution $\mathcal{F}_s$ (it is, indeed, an approximation of $\mathcal{F}_s$) which is, however, better than the latter since it removes the singularity present in the origin.*

We now pass to the proof of these auxiliary statements.

Proof of Lemma 11. We divide the proof in two steps.

Step 1: rewriting the problem. The relation (148), due to Plancherel's formula (being here $p = 2$) appearing in [15] applied to the function $(-\Delta)^{\frac{\sigma}{4}}u$ and using Definition 9, is achieved if we show that

$$[u]^2_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)} = 2C\left(\frac{\sigma}{2}, n\right)^{-1} \int_{\mathbb{R}^n} |\xi|^{\sigma} |\mathscr{F}u(\xi)|^2 d\xi. \tag{156}$$

In fact, if this relation holds, then we get

$$[u]^2_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)} = 2C\left(\frac{\sigma}{2}, n\right)^{-1} \int_{\mathbb{R}^n} |\xi|^{\sigma} |\mathscr{F}u(\xi)|^2 d\xi =$$

$$= 2C\left(\frac{\sigma}{2}, n\right)^{-1} \big|\big| \, |\xi|^{\frac{\sigma}{2}} \mathscr{F}u \big|\big|^2_{L^2(\mathbb{R}^n)} = 2C\left(\frac{\sigma}{2}, n\right)^{-1} \Big|\Big| \mathscr{F}\left(\mathscr{F}^{-1}\big(|\xi|^{\frac{\sigma}{2}} \mathscr{F}u(\xi)\big)(x)\right) \Big|\Big|^2_{L^2(\mathbb{R}^n)} =$$

$$= 2C\left(\frac{\sigma}{2}, n\right)^{-1} \big|\big| \mathscr{F}^{-1}\big(|\xi|^{\frac{\sigma}{2}} \mathscr{F}u(\xi)\big) \big|\big|^2_{L^2(\mathbb{R}^n)} = 2C\left(\frac{\sigma}{2}, n\right)^{-1} \big|\big| (-\Delta)^{\frac{\sigma}{4}} u \big|\big|^2_{L^2(\mathbb{R}^n)} \iff$$

$$\iff [u]_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)} = \sqrt{2}\, C\left(\frac{\sigma}{2}, n\right)^{-\frac{1}{2}} \big|\big| (-\Delta)^{\frac{\sigma}{4}} u \big|\big|_{L^2(\mathbb{R}^n)}.$$

Therefore, it is enough if we reach (156).

Step 2: proof of (156). For a fixed $y \in \mathbb{R}^n$, we set $z := x - y$, use Fubini's theorem and resort again to Plancherel's formula in order to get

$$[u]^2_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+\sigma}} dx \right) dy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|u(y + z) - u(y)|^2}{|z|^{n+\sigma}} dz \right) dy =$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| \frac{u(y + z) - u(y)}{|z|^{\frac{n+\sigma}{2}}} \right|^2 dy \right) dz = \int_{\mathbb{R}^n} \left|\left| \frac{u(\cdot + z) - u(\cdot)}{|z|^{\frac{n+\sigma}{2}}} \right|\right|^2_{L^2(\mathbb{R}^n)} dz =$$

$$= \int_{\mathbb{R}^n} \left|\left| \mathscr{F}\left(\frac{u(\cdot + z) - u(\cdot)}{|z|^{\frac{n+\sigma}{2}}} \right) \right|\right|^2_{L^2(\mathbb{R}^n)} dz. \tag{157}$$

Subsequently, using the properties of the unitary Fourier transform already studied, the equation (123) and Fubini's theorem once more, we see that

$$\int_{\mathbb{R}^n} \left|\left| \mathscr{F}\left(\frac{u(\cdot+z)-u(\cdot)}{|z|^{\frac{n+\sigma}{2}}}\right) \right|\right|^2_{L^2(\mathbb{R}^n)} dz = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| \mathscr{F}\left(\frac{u(\xi+z)-u(\xi)}{|z|^{\frac{n+\sigma}{2}}}\right)\right|^2 d\xi \right) dz =$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot z}\mathscr{F}u(\xi) - \mathscr{F}u(\xi)|^2}{|z|^{n+\sigma}} d\xi \right) dz = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot z}-1|^2}{|z|^{n+\sigma}} |\mathscr{F}u(\xi)|^2 d\xi \right) dz =$$

$$= 2\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot z)}{|z|^{n+\sigma}} dz \right) |\mathscr{F}u(\xi)|^2 d\xi = 2C\left(\frac{\sigma}{2}, n\right)^{-1} \int_{\mathbb{R}^n} |\xi|^\sigma |\mathscr{F}u(\xi)|^2 d\xi, \quad (158)$$

where, in the fourth passage, we used that

$$|e^{i\xi\cdot z}-1|^2 = |e^{2i\xi\cdot z} - 2e^{i\xi\cdot z}+1| = \left| \frac{e^{-i\xi\cdot z}(e^{2i\xi\cdot z} - 2e^{i\xi\cdot z}+1)}{e^{-i\xi\cdot z}} \right| =$$

$$= \frac{|e^{i\xi\cdot z} - 2 + e^{-i\xi\cdot z}|}{|e^{-i\xi\cdot z}|} = |2\cos(\xi\cdot z) - 2| = 2|\cos(\xi\cdot z)-1| = 2\big(1-\cos(\xi\cdot z)\big),$$

noticing that $\cos(\zeta) = \frac{e^{i\zeta}+e^{-i\zeta}}{2}$ and $\cos(\zeta) \leq 1$, $\forall\, \zeta \in \mathbb{C}$.
In conclusion, joining (157) and (158), we obtain

$$[u]^2_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left|\left| \mathscr{F}\left(\frac{u(\cdot+z)-u(\cdot)}{|z|^{\frac{n+\sigma}{2}}}\right) \right|\right|^2_{L^2(\mathbb{R}^n)} dz = 2C\left(\frac{\sigma}{2}, n\right)^{-1} \int_{\mathbb{R}^n} |\xi|^\sigma |\mathscr{F}u(\xi)|^2 d\xi,$$

which is exactly the identity (156).

□

Proof of Lemma 12. We have to prove three statements: therefore, we divide the proof in three steps.

Step 1: proof of the first statement. In order to prove the first claim, we resort to the abstract Dirichlet principle (see [9]): we start by noticing that, given an arbitrary function $v \in W_0^{\frac{\sigma}{2},2}(\mathbb{R}^n)$, the generalized Poincaré inequality states that

$$||v||_{L^2(\Omega)} \leq c[v]_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)}$$

holds for a certain constant $c = c(s,n)$ depending on s and n only. Hence, after setting $H^{\frac{\sigma}{2}}(\Omega) := W^{\frac{\sigma}{2},2}(\mathbb{R}^n)$ and, similarly, $H_0^{\frac{\sigma}{2}}(\Omega) := W_0^{\frac{\sigma}{2},2}(\mathbb{R}^n)$ for reasons of convenience, the linear operator $\mathcal{T} : H_0^{\frac{\sigma}{2}}(\Omega) \longrightarrow \mathbb{R}$ such that

$$f \longmapsto \int_\Omega f(x)v(x)dx$$

is continuous (thanks to Hölder's inequality) for every $v \in H_0^{\frac{\sigma}{2}}(\Omega)$. Furthermore, if we now define

$$||v||_{H^{\frac{\sigma}{2}}(\Omega)}:=\mathcal{B}_\sigma(v,v)^{\frac{1}{2}}=\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{\big(v(x)-v(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+\sigma}}\,dx\right)dy\right)^{\frac{1}{2}}=$$

$$=\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^2}{|x-y|^{n+\sigma}}\,dx\right)dy\right)^{\frac{1}{2}}=:[v]_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)},$$

then that $||\cdot||_{H^{\frac{\sigma}{2}}(\Omega)}$ is an equivalent norm on the Hilbert space $H_0^{\frac{\sigma}{2}}(\Omega)$ due to (148). Besides, the linear functional $\mathcal{L}:H_0^{\frac{\sigma}{2}}(\Omega)\longrightarrow\mathbb{R}$ such that

$$v\longmapsto\int_\Omega f(x)v(x)dx-\mathcal{B}_\sigma(g,v)$$

is bounded because

$$|\mathcal{L}v(x)|=\left|\int_\Omega f(x)v(x)dx-\mathcal{B}_\sigma(g,v)\right|\le\int_\Omega|f(x)||v(x)|dx+|\mathcal{B}_\sigma(g,v)|\le$$

$$\le||f||_{L^2(\Omega)}||v||_{L^2(\Omega)}+\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|g(x)-g(y)||v(x)-v(y)|}{|x-y|^{n+\sigma}}\,dx\right)dy\le$$

$$\le||f||_{L^2(\Omega)}||v||_{H^{\frac{\sigma}{2}}(\Omega)}+\int_\Omega\left(\int_{\mathbb{R}^n}\frac{|g(x)-g(y)|}{|x-y|^{\frac{n+\sigma}{2}}}\,\frac{|v(x)-v(y)|}{|x-y|^{\frac{n+\sigma}{2}}}\,dx\right)dy\le$$

$$\le||f||_{L^2(\Omega)}||v||_{H^{\frac{\sigma}{2}}(\Omega)}+\int_\Omega\left(\int_{\mathbb{R}^n}\frac{|g(x)-g(y)|^2}{|x-y|^{n+\sigma}}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^2}{|x-y|^{n+\sigma}}\,dx\right)^{\frac{1}{2}}dy\le$$

$$\le||f||_{L^2(\Omega)}||v||_{H^{\frac{\sigma}{2}}(\Omega)}+$$

$$+\left(\int_\Omega\left(\int_{\mathbb{R}^n}\frac{|g(x)-g(y)|^2}{|x-y|^{n+\sigma}}\,dx\right)dy\right)^{\frac{1}{2}}\left(\int_\Omega\left(\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^2}{|x-y|^{n+\sigma}}\,dx\right)dy\right)^{\frac{1}{2}}=$$

$$=||f||_{L^2(\Omega)}||v||_{H^{\frac{\sigma}{2}}(\Omega)}+\left(\int_\Omega\left(\int_{\mathbb{R}^n}\frac{|g(x)-g(y)|^2}{|x-y|^{n+\sigma}}\,dx\right)dy\right)^{\frac{1}{2}}[v]_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)}=$$

$$=||f||_{L^2(\Omega)}||v||_{H^{\frac{\sigma}{2}}(\Omega)}+\left(\int_\Omega\left(\int_{\mathbb{R}^n}\frac{|g(x)-g(y)|^2}{|x-y|^{n+\sigma}}\,dx\right)dy\right)^{\frac{1}{2}}||v||_{H^{\frac{\sigma}{2}}(\Omega)}=$$

$$=\left[||f||_{L^2(\Omega)}+\left(\int_\Omega\left(\int_{\mathbb{R}^n}\frac{|g(x)-g(y)|^2}{|x-y|^{n+\sigma}}\,dx\right)dy\right)^{\frac{1}{2}}\right]||v||_{H^{\frac{\sigma}{2}}(\Omega)}\le M||v||_{H^{\frac{\sigma}{2}}(\Omega)},$$

for a certain number $M=M(f,g)$ depending on the functions f and g only, where we used Hölder's inequality, the hypothesis on f and v (especially the vanishing of the latter outside Ω) and (150). Therefore, the boundedness of $\mathcal{L}$ follows since

$$||\mathcal{L}||_{\text{op}} := \sup_{\substack{v \in H_0^{\frac{\sigma}{2}}(\Omega) \\ ||v||_{H^{\frac{\sigma}{2}}(\Omega)} \leq 1}} \left\{ |\mathcal{L}v(x)| \right\} \leq \sup_{\substack{v \in H_0^{\frac{\sigma}{2}}(\Omega) \\ ||v||_{H^{\frac{\sigma}{2}}(\Omega)} \leq 1}} \left\{ M ||v||_{H^{\frac{\sigma}{2}}(\Omega)} \right\} \leq M.$$

Consequently, the Dirichlet principle tells us that the functional

$$\mathcal{K}(v) := \frac{1}{2} ||v||^2_{H^{\frac{\sigma}{2}}(\Omega)} - \mathcal{L}v(x)$$

has a unique minimizer $\bar{u} \in H_0^{\frac{\sigma}{2}}(\Omega)$, which means that the function $\tilde{u}(x) := \bar{u}(x) + g(x)$ is the only solution of (151): in fact, we see that

$$\mathcal{K}(v) := \frac{1}{2} ||v||^2_{H^{\frac{\sigma}{2}}(\Omega)} - \mathcal{L}v(x) = \frac{1}{2} \mathcal{B}_\sigma(v,v) - \left[\int_\Omega f(x)v(x)dx - \mathcal{B}_\sigma(g,v) \right] =$$

$$= \frac{1}{2} \mathcal{B}_\sigma(v,v) + \mathcal{B}_\sigma(g,v) - \int_\Omega f(x)v(x)dx =$$

$$= \frac{1}{2} \big[\mathcal{B}_\sigma(v,v) + 2\mathcal{B}_\sigma(g,v) + \mathcal{B}_\sigma(g,g) \big] - \frac{1}{2} \mathcal{B}_\sigma(g,g) - \int_\Omega f(x)v(x)dx =$$

$$= \frac{1}{2} \mathcal{B}_\sigma(v+g, v+g) - \frac{1}{2} \mathcal{B}_\sigma(g,g) - \int_\Omega f(x)\big(v(x) + g(x)\big)dx + \int_\Omega f(x)g(x)dx =$$

$$= \frac{1}{2} \mathcal{B}_\sigma(v+g, v+g) - \int_\Omega f(x)\big(v(x) + g(x)\big)dx + c,$$

where we defined the constant

$$c = c(f,g) := \int_\Omega f(x)g(x)dx - \frac{1}{2} \mathcal{B}_\sigma(g,g)$$

depending on the functions f and g only and used the property of linearity of the bilinear form $\mathcal{B}_\sigma$, namely the identity

$$\mathcal{B}_\sigma(v+g, v+g) = \mathcal{B}_\sigma(v,v) + 2\mathcal{B}_\sigma(g,v) + \mathcal{B}_\sigma(g,g).$$

Therefore, again by the Dirichlet principle (and ignoring the constant term which does not play any role), we can affirm that $\tilde{u}(x) := \bar{u}(x) + g(x)$ is the unique minimizer of

$$\frac{1}{2} \mathcal{B}_\sigma(v,v) - \int_\Omega f(x)v(x)dx$$

and, besides, it is also the unique solution of

$$\mathcal{B}_\sigma(u,v) = \int_{\mathbb{R}^n} f(x)v(x)dx,$$

for every $v \in H_0^{\frac{\sigma}{2},2}(\mathbb{R}^n)$.

Step 2: proof of the second statement. Here, we want to show that the function $\tilde{u}$ found in the previous step satisfies also the relation (152). Since, by Definition 11, we have that $C_0^\infty(\Omega) \subseteq H_0^{\frac{\sigma}{2},2}(\Omega) \cap \mathscr{S}(\mathbb{R}^n)$, then Proposition 11 implies that

$$\int_{\mathbb{R}^n} \tilde{u}(x)(-\Delta)^{\frac{\sigma}{2}}\varphi(x)dx = \int_{\mathbb{R}^n} \tilde{u}(x)\left[C\left(\frac{\sigma}{2}, n\right) \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+\sigma}}\, dy\right] dx =$$

$$= C\left(\frac{\sigma}{2}, n\right) \int_{\mathbb{R}^n} \tilde{u}(x)\left[\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+\sigma}}\, dy\right] dx =$$

$$= \frac{1}{2}\, C\left(\frac{\sigma}{2}, n\right)\mathcal{B}(\tilde{u}, \varphi) = \frac{1}{2}\, C\left(\frac{\sigma}{2}, n\right) \int_{\mathbb{R}^n} f(x)\varphi(x)dx,$$

where, in the penultimate passage, we used the fact that, once defined the quantities

$$A := \int_{\mathbb{R}^n} \tilde{u}(x)\left[\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+\sigma}}\, dy\right] dx$$

and

$$B := \int_{\mathbb{R}^n} \tilde{u}(y)\left[\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{\varphi(y)-\varphi(x)}{|x-y|^{n+\sigma}}\, dx\right] dy,$$

then clearly $A = B = \frac{1}{2}(A+B)$, which implies that

$$\int_{\mathbb{R}^n} \tilde{u}(x)\left[\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+\sigma}}\, dy\right] dx =$$

$$= \frac{1}{2}\Bigg(\int_{\mathbb{R}^n} \tilde{u}(x)\left[\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+\sigma}}\, dy\right] dx +$$

$$+ \int_{\mathbb{R}^n} \tilde{u}(y)\left[\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{\varphi(y)-\varphi(x)}{|x-y|^{n+\sigma}}\, dx\right] dy\Bigg) =$$

$$= \frac{1}{2}\Bigg(\int_{\mathbb{R}^n} \left[\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \tilde{u}(x)\, \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+\sigma}}\, dx\right] dy +$$

$$+ \int_{\mathbb{R}^n} \left[\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \tilde{u}(y)\, \frac{\varphi(y)-\varphi(x)}{|x-y|^{n+\sigma}}\, dx\right] dy\Bigg) =$$

$$= \frac{1}{2}\Bigg(\int_{\mathbb{R}^n} \left[\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{\big(\tilde{u}(x)-\tilde{u}(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+\sigma}}\, dx\right] dy\Bigg) =$$

$$= \frac{1}{2}\Bigg(\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \frac{\big(\tilde{u}(x)-\tilde{u}(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+\sigma}}\, dx\right] dy\Bigg) = \frac{1}{2}\, \mathcal{B}(\tilde{u}, \varphi)$$

using Fubini's theorem.

Step 3: proof of the third statement. This last part is very similar to the second one: in fact, if $\tilde{u}$ satisfies (152), then

$$\frac{1}{2}C\left(\frac{\sigma}{2},n\right)\int_{\mathbb{R}^n} f(x)\varphi(x)dx = \int_{\mathbb{R}^n} \tilde{u}(x)(-\Delta)^{\frac{\sigma}{2}}\varphi(x)dx =$$

$$= \int_{\mathbb{R}^n} \tilde{u}(x)\left[C\left(\frac{\sigma}{2},n\right)\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+\sigma}}\,dy\right]dx =$$

$$= \frac{1}{2}C\left(\frac{\sigma}{2},n\right)\mathcal{B}(\tilde{u},\varphi),$$

for every function $\varphi \in C_0^\infty(\Omega)$, where in the penultimate identity we made use of the exact same computation done in the preceding step. All this clearly means that

$$\mathcal{B}(\tilde{u},\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx.$$

In conclusion, by the density of $C_0^\infty(\Omega)$ into $H_0^{\frac{\sigma}{2},2}(\Omega)$ due to Netrusov's theorem, we have that the above relation is still valid even for an arbitrary function $v \in H_0^{\frac{\sigma}{2},2}(\Omega)$ replacing φ.

□

Proof of Corollary 8. Since $\tilde{u}_-$ lies in $W^{\frac{\sigma}{2},2}(\mathbb{R}^n)$ (and the same is true regarding the function $\tilde{v}(x) := -\tilde{u}_-(x)$, clearly), in view of Definition 11 and (148) we have that $\tilde{v}$ belongs to $H_0^{\frac{\sigma}{2},2}(\Omega)$, too, where the vanishing of it outside Ω is due to the fact that the minimum between $\tilde{u}$ and 0 is the latter outside Ω by hypothesis and its definition, since $\tilde{u}(x) := \bar{u}(x) + g(x)$, with $\bar{u}(x) = 0$ and $g(x) \geq 0$ in $\mathbb{R}^n\setminus\Omega$. Therefore, from (151) and the definition of the function $\tilde{v}$, it follows that

$$0 \geq \int_{\mathbb{R}^n} f(x)\tilde{v}(x)dx = \mathcal{B}_\sigma(\tilde{u},\tilde{v}) := \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{\big(\tilde{u}(x)-\tilde{u}(y)\big)\big(\tilde{v}(x)-\tilde{v}(y)\big)}{|x-y|^{n+\sigma}}\,dx\right)dy =$$

$$= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{\big(\tilde{u}_+(x)-\tilde{u}_-(x)-\tilde{u}_+(y)+\tilde{u}_-(y)\big)\big(\tilde{v}(x)-\tilde{v}(y)\big)}{|x-y|^{n+\sigma}}\,dx\right)dy =$$

$$= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{\big(\tilde{u}_+(x)+\tilde{v}(x)-\tilde{u}_+(y)-\tilde{v}(y)\big)\big(\tilde{v}(x)-\tilde{v}(y)\big)}{|x-y|^{n+\sigma}}\,dx\right)dy =$$

$$= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{\big(\tilde{u}_+(x)-\tilde{u}_+(y)\big)\big(\tilde{v}(x)-\tilde{v}(y)\big)+\big(\tilde{v}(x)-\tilde{v}(y)\big)^2}{|x-y|^{n+\sigma}}\,dx\right)dy \geq$$

$$\geq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{\big(\tilde{v}(x)-\tilde{v}(y)\big)^2}{|x-y|^{n+\sigma}}\,dx\right)dy \geq 0,$$

where we used that $f(x)\tilde{v}(x) \leq 0$ (because $f(x) \geq 0$ and $\tilde{v}(x) \leq 0$) and that

$$\big(\tilde{u}_+(x) - \tilde{u}_+(y)\big)\big(\tilde{v}(x) - \tilde{v}(y)\big) = \tilde{u}_+(x)\tilde{v}(x) - \tilde{u}_+(x)\tilde{v}(y) - \tilde{u}_+(y)\tilde{v}(x) + \tilde{u}_+(y)\tilde{v}(y) =$$

$$= -\big(\tilde{u}_+(x)\tilde{v}(y) + \tilde{u}_+(y)\tilde{v}(x)\big) \geq 0,$$

since $\tilde{u}_+(x)\tilde{v}(x) = 0 = \tilde{u}_+(y)\tilde{v}(y)$ and $\tilde{u}_+(x)\tilde{v}(y), \tilde{u}_+(y)\tilde{v}(x) \leq 0$, $\forall\, x, y \in \mathbb{R}^n$.
Hence, it must be that

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{\big(\tilde{v}(x) - \tilde{v}(y)\big)^2}{|x-y|^{n+\sigma}}\,dx\right)dy = 0,$$

which implies that $\tilde{v}(x) \equiv 0$ a.e., and so does $\tilde{u}_-(x)$. This proves that $\tilde{u}(x) \geq 0$ a.e. on the whole $\mathbb{R}^n$.

□

Proof of Lemma 13. It is easy to create a function Ψ_λ starting from $\mathcal{F}_\sigma$ lying (at least) in $C^1(\mathbb{R}^n)$ and which satisfies (153) and (154). In fact, if we take an auxiliary function Ψ which identically equals $\mathcal{F}_\sigma$ on $\mathbb{R}^n \setminus B_1(0)$ and which is, inside $B_1(0)$, a paraboloid from below attaining its maximum at the origin that connects, in a sufficiently smooth way so that $\Psi \in C^1(\mathbb{R}^n)$, itself with $\mathcal{F}_\sigma$ over $\partial B_1(0)$, then it is enough to take an element $\lambda > 0$ and define

$$\Psi_\lambda(x) := \frac{\Psi\big(\frac{x}{\lambda}\big)}{\lambda^{n-\sigma}}.$$

Indeed, that Ψ_λ belongs to $C^1(\mathbb{R}^n)$ and satisfies the first two requests, namely the relations (153) and (154), is an immediate check that follows from the definitions of Ψ_λ and $\mathcal{F}_\sigma$, since they are equal on $\mathbb{R}^n \setminus B_\lambda(0)$ and, inside $B_\lambda(0)$, $\mathcal{F}_\sigma$ has a singularity in correspondence of the origin.
To show that even the last property of Ψ_λ is satisfied, we appeal to Proposition 11 (being allowed since $\sigma \in (0,2)$ by hypothesis) and write

$$(-\Delta)^{\frac{\sigma}{2}}\Psi_\lambda(x) = C\left(\frac{\sigma}{2}, n\right) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{\Psi_\lambda(x) - \Psi_\lambda(y)}{|x-y|^{n+\sigma}}\,dy, \tag{159}$$

where $C\big(\frac{\sigma}{2}, n\big)$ is the usual constant defined in (118). We reiterate again that, in according to the point *(v)* of Remark 21, we can make use of (117) whenever the integrand is finite (and this is the case because $n + \sigma > n$). We already know, thanks to Theorem 11, that $(-\Delta)^{\frac{s}{2}}\mathcal{F}_s(x) = \delta_0(x)$ if $s \in (0,n)$: hence, if $x_0 \in \mathbb{R}^n \setminus B_\lambda(0)$, then

$$(-\Delta)^{\frac{\sigma}{2}}\Psi_\lambda(x_0) = C\left(\frac{\sigma}{2}, n\right) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x_0)} \frac{\Psi_\lambda(x_0) - \Psi_\lambda(y)}{|x_0-y|^{n+\sigma}}\,dy >$$

$$> C\left(\frac{\sigma}{2}, n\right) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x_0)} \frac{\mathcal{F}_\sigma(x_0) - \mathcal{F}_\sigma(y)}{|x_0 - y|^{n+\sigma}} \, dy = 0,$$

where we used that $\Psi_\lambda(x_0) = \mathcal{F}_\sigma(x_0)$ and, for every $y \in \mathbb{R}^n$, that $\Psi_\lambda(y) \le \mathcal{F}_\sigma(y)$ (with $\Psi_\lambda(y) < \mathcal{F}_\sigma(y)$ over $B_\lambda(0)$, in particular) due to (154) and (153) respectively.
Instead, if $x_0 \in B_\lambda(0) \setminus \{0\}$, there exist an element $x_1 \in \mathbb{R}^n \setminus \{x_0\}$ and a number $\delta > 0$ such that the function $\tilde{\mathcal{F}}(x, \sigma, x_1, \delta) := \mathcal{F}_\sigma(x - x_1) + \delta$ touches Ψ_λ from above at the point x_0 (it follows by the definition itself of Ψ_λ). Therefore, once again, we get

$$(-\Delta)^{\frac{\sigma}{2}} \Psi_\lambda(x_0) = C\left(\frac{\sigma}{2}, n\right) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x_0)} \frac{\Psi_\lambda(x_0) - \Psi_\lambda(y)}{|x_0 - y|^{n+\sigma}} \, dy >$$

$$> C\left(\frac{\sigma}{2}, n\right) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x_0)} \frac{\tilde{\mathcal{F}}(x_0, \sigma, x_1, \delta) - \tilde{\mathcal{F}}(y, \sigma, x_1, \delta)}{|x_0 - y|^{n+\sigma}} \, dy =$$

$$= C\left(\frac{\sigma}{2}, n\right) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x_0)} \frac{\mathcal{F}_\sigma(x_0 - x_1) + \delta - \mathcal{F}_\sigma(y - x_1) - \delta}{|x_0 - y|^{n+\sigma}} \, dy =$$

$$= C\left(\frac{\sigma}{2}, n\right) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x_0)} \frac{\mathcal{F}_\sigma(x_0 - x_1) - \mathcal{F}_\sigma(y - x_1)}{|x_0 - y|^{n+\sigma}} \, dy = 0,$$

since now $\Psi_\lambda(x_0) = \tilde{\mathcal{F}}(x_0, \sigma, x_1, \delta) := \mathcal{F}_\sigma(x_0 - x_1) + \delta$ and, for an arbitrary $y \in \mathbb{R}^n$, $\Psi_\lambda(y) \le \mathcal{F}_\sigma(y - x_1) < \mathcal{F}_\sigma(y - x_1) + \delta =: \tilde{\mathcal{F}}(y, \sigma, x_1, \delta)$.
Lastly, Ψ_λ attains its maximum at the origin by definition, and so

$$(-\Delta)^{\frac{\sigma}{2}} \Psi_\lambda(0) = C\left(\frac{\sigma}{2}, n\right) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x_0)} \frac{\Psi_\lambda(0) - \Psi_\lambda(y)}{|y|^{n+\sigma}} \, dy > 0$$

because we are integrating a positive quantity (since $\Psi_\lambda(0) > \Psi_\lambda(y)$, $\forall\, y \in \mathbb{R}^n \setminus \{0\}$).
In conclusion, we managed to achieve (even more than) the relation (155), being $(-\Delta)^{\frac{\sigma}{2}} \Psi_\lambda(x) > 0$ on the whole $\mathbb{R}^n$.

□

Remark 34. *In the previous proof, we did not define numerically the function* Ψ *(and, consequently, we do not possess a rigorous writing of* Ψ_λ*) because it is not relevant: in fact, in the forthcoming proof of Proposition 14, we will make use of Lemma 12 which gives a unique function to us, let say* u*, due to the abstract Dirichlet principle, as seen in its proof. Therefore, since it allows us to determine such a function but not to construct it, there is no need to find rigorously the definition of* Ψ_λ *because we will use it along with* u*, considering indeed the function* $\Psi_\lambda - u$*.*

We are now able to prove the two propositions concerning the Green representation formula for u: the first one dealt with the case for the fractional Laplacian of order not greater than 1, while the second one generalized it allowing to consider any order.

Proof of Proposition 14. We proceed step by step.

Step 1: the case $\sigma = 2$. The limit case in which $\sigma = 2$ is well-known, because it comes down to the integer case: here, in fact, we have the classical Laplacian operator $(-\Delta)$. Hence, we already know that the relations (143), (144) and (145) hold (the last one is exactly the point *(ii)* of the thesis of Lemma 7 once we reconstruct the function $\mathcal{G}_2$). Consequently, from now on, we are focused on the case $\sigma \in (0, 2)$.

Step 2: proof of (143). Next, we take an element $x_0 \in \Omega$, set $\delta := \frac{1}{2}\,\mathrm{dist}(x, \partial\Omega) > 0$ and choose any function $g_{x_0} \in C^1(\mathbb{R}^n)$ such that $g_{x_0}(y) := \mathcal{F}_\sigma(x_0 - y)$ for $y \in \mathbb{R}^n \setminus B_\delta(x_0)$, where $\mathcal{F}_\sigma$ is the fundamental solution of $(-\Delta)^{\frac{s}{2}}$ for $s \in (0, n)$ on $\mathbb{R}^n$ defined in (133). Note that $B_\delta(x_0) \subsetneq \Omega$ by definition of δ. We claim that

$$\int_\Omega \left(\int_{\mathbb{R}^n} \frac{\big(g_{x_0}(y) - g_{x_0}(z)\big)^2}{|y - z|^{n+\sigma}}\, dy \right) dz < +\infty. \tag{160}$$

In order to prove that, we split the internal integral for a fixed $z_0 \in \Omega$, obtaining

$$\int_{\mathbb{R}^n} \frac{\big(g_{x_0}(y) - g_{x_0}(z)\big)^2}{|y - z|^{n+\sigma}}\, dy =$$

$$= \underbrace{\int_{B_1(z_0)} \frac{\big(g_{x_0}(y) - g_{x_0}(z)\big)^2}{|y - z|^{n+\sigma}}\, dy}_{=: I_1(z)} + \underbrace{\int_{\mathbb{R}^n \setminus B_1(z_0)} \frac{\big(g_{x_0}(y) - g_{x_0}(z)\big)^2}{|y - z|^{n+\sigma}}\, dy}_{=: I_2(z)}.$$

Thus, we see that there exists a constant $c_1 = c_1(\sigma, n)$ depending on σ and n only (since $n + \sigma - 2 < n$) such that

$$I_1(z) := \int_{B_1(z_0)} \frac{\big(g_{x_0}(y) - g_{x_0}(z)\big)^2}{|y - z|^{n+\sigma}}\, dy = \int_{B_1(z_0)} \frac{|g_{x_0}(y) - g_{x_0}(z)|^2}{|y - z|^{n+\sigma}}\, dy \leq$$

$$\leq \int_{B_1(z_0)} \frac{\big(||\nabla g_{x_0}||_{L^\infty(B_1(z_0))} |y - z|\big)^2}{|y - z|^{n+\sigma}}\, dy = ||\nabla g_{x_0}||^2_{L^\infty(B_1(z_0))} \int_{B_1(z_0)} \frac{dy}{|y - z|^{n+\sigma-2}} \leq$$

$$\leq c_1 ||\nabla g_{x_0}||^2_{L^\infty(B_1(z_0))}.$$

Similarly, there exists also another constant $c_2 = c_2(\sigma, n)$ depending on σ and n only (since $n + \sigma > n$) such that

$$I_2(z) := \int_{\mathbb{R}^n \setminus B_1(z_0)} \frac{\big(g_{x_0}(y) - g_{x_0}(z)\big)^2}{|y - z|^{n+\sigma}}\, dy = \int_{\mathbb{R}^n \setminus B_1(z_0)} \frac{|g_{x_0}(y) - g_{x_0}(z)|^2}{|y - z|^{n+\sigma}}\, dy \leq$$

$$\leq \int_{\mathbb{R}^n \setminus B_1(z_0)} \frac{\big(|g_{x_0}(y)| + |g_{x_0}(z)|\big)^2}{|y - z|^{n+\sigma}}\, dy \leq \int_{B_1(z_0)} \frac{\big(2||g_{x_0}||_{L^\infty(\mathbb{R}^n)}\big)^2}{|y - z|^{n+\sigma}}\, dy =$$

$$= 4||g_{x_0}||^2_{L^\infty(\mathbb{R}^n)} \int_{B_1(z_0)} \frac{dy}{|y-z|^{n+\sigma}} \leq 4c_2||g_{x_0}||^2_{L^\infty(\mathbb{R}^n)}.$$

Hence,

$$\int_\Omega \left(\int_{\mathbb{R}^n} \frac{\big(g_{x_0}(y) - g_{x_0}(z)\big)^2}{|y-z|^{n+\sigma}} dy \right) dz = \int_\Omega \big(I_1(z) + I_2(z)\big) dz \leq$$

$$\leq \int_\Omega \Big(c_1||\nabla g_{x_0}||^2_{L^\infty(B_1(z_0))} + 4c_2||g_{x_0}||^2_{L^\infty(\mathbb{R}^n)} \Big) dz =$$

$$= \Big(c_1||\nabla g_{x_0}||^2_{L^\infty(B_1(z_0))} + 4c_2||g_{x_0}||^2_{L^\infty(\mathbb{R}^n)} \Big) |\Omega| < +\infty,$$

since $g_{x_0} \in C^1(\mathbb{R}^n)$. This proves (160).
Now, thanks to the estimate just proven, the hypothesis of Lemma 12 are satisfied if we take $f(x) \equiv 0$ which, trivially, belongs to $L^2(\Omega)$: consequently, we are able to use it and affirm that there exists a unique function $\bar{\mathcal{H}}_\sigma \in H_0^{\frac{\sigma}{2},2}(\Omega)$ such that $(-\Delta)^{\frac{\sigma}{2}} \mathcal{H}_\sigma(x_0, y) = 0$ in Ω (in the sense of distributions), where $\mathcal{H}_\sigma(x_0, y) := \bar{\mathcal{H}}_\sigma(x_0, y) + g_{x_0}(y)$. Furthermore, using the definition of the space $H_0^{\frac{\sigma}{2},2}(\Omega)$, we can summarize the conditions obtained by writing

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} \mathcal{H}_\sigma(x_0, y) = 0 & \text{if } y \in \Omega \\ \mathcal{H}_\sigma(x_0, y) = g_{x_0}(y) & \text{if } y \in \mathbb{R}^n \backslash \Omega \end{cases}. \tag{161}$$

Moreover, being $f(x) \equiv 0$ on Ω and $g_{x_0}(y) > 0$ on $\mathbb{R}^n \backslash \Omega$ by definition, Corollary 8 tells us that $\mathcal{H}_\sigma(x_0, y) \geq 0$ a.e. on the whole $\mathbb{R}^n$.
Next, if we define the function

$$h(x_0, y) := -\mathcal{H}_\sigma(x_0, y) + \sup_{z \in \mathbb{R}^n \backslash \Omega} \{g_{x_0}(z)\},$$

then h still satisfies the relation $(-\Delta)^{\frac{\sigma}{2}} h(x_0, y) = 0$ because

$$(-\Delta)^{\frac{\sigma}{2}} h(x_0, y) = (-\Delta)^{\frac{\sigma}{2}} \left(-\mathcal{H}_\sigma(x_0, y) + \sup_{z \in \mathbb{R}^n \backslash \Omega} \{g_{x_0}(z)\} \right) =$$

$$= -(-\Delta)^{\frac{\sigma}{2}} \mathcal{H}_\sigma(x_0, y) + (-\Delta)^{\frac{\sigma}{2}} \left(\sup_{z \in \mathbb{R}^n \backslash \Omega} \{g_{x_0}(z)\} \right) = 0$$

by (161) and the fact that the supremum appearing above is a constant (which makes the integral representation of $(-\Delta)^{\frac{\sigma}{2}} u$ in (117) null). Consequently, if we replace the function g_{x_0} with

$$\tilde{g}_{x_0}(y) := \sup_{z \in \mathbb{R}^n \backslash \Omega} \{g_{x_0}(z)\} - g_{x_0}(y)$$

in (160), the latter is still valid using the same computations done earlier (we, indeed, used mainly the fact that $|g_{x_0}(y)| \leq ||g_{x_0}||_{L^\infty(\mathbb{R}^n)}$ for every $y \in \mathbb{R}^n$ and, here, it is still the case): therefore, by Lemma 11, h satisfies the problem (151) with $f(x) \equiv 0$ and, hence, it must be a non-negative function by Corollary 8 (since, by definition, $\tilde{g}_{x_0}(y) \geq 0$ on the whole $\mathbb{R}^n$), which means that

$$0 \leq h(x_0, y) := -\mathcal{H}_\sigma(x_0, y) + \sup_{z \in \mathbb{R}^n \setminus \Omega} \{g_{x_0}(z)\} \Longleftrightarrow \sup_{z \in \mathbb{R}^n \setminus \Omega} \{g_{x_0}(z)\} \geq \mathcal{H}_\sigma(x_0, y).$$

Thus, by definition of g_{x_0}, we get

$$\sup_{z \in \mathbb{R}^n \setminus \Omega} \{\mathcal{F}_\sigma(x_0 - z)\} \geq \mathcal{H}_\sigma(x_0, y).$$

What we obtained is that

$$0 \leq \mathcal{H}_\sigma(x_0, y) \leq \sup_{z \in \mathbb{R}^n \setminus \Omega} \{\mathcal{F}_\sigma(x_0 - z)\}, \tag{162}$$

for a.e. $y \in \Omega$ (notice that, by definition of $\mathcal{H}_\sigma(x_0, y)$, the validity of (162) is immediate if $y \in \mathbb{R}^n \setminus \Omega$, which means that it is true everywhere).
Further, if we now define $\mathcal{G}_\sigma(x_0, y) := \mathcal{F}_\sigma(x_0 - y) - \mathcal{H}_\sigma(x_0, y)$, then $\mathcal{G}_\sigma$ satisfies (143): in fact, using the linearity of the fractional Laplacian, Theorem 11 (up to a translation), (161) and the definition of g_{x_0}, we infer

$$(-\Delta)^{\frac{\sigma}{2}} \mathcal{G}_\sigma(x_0, y) = (-\Delta)^{\frac{\sigma}{2}} \big(\mathcal{F}_\sigma(x_0 - y) - \mathcal{H}_\sigma(x_0, y)\big) =$$

$$= (-\Delta)^{\frac{\sigma}{2}} \mathcal{F}_\sigma(x_0 - y) - (-\Delta)^{\frac{\sigma}{2}} \mathcal{H}_\sigma(x_0, y) = \delta_{x_0}(y)$$

for $y \in \Omega$, while

$$\mathcal{G}_\sigma(x_0, y) := \mathcal{F}_\sigma(x_0 - y) - \mathcal{H}_\sigma(x_0, y) = g_{x_0}(y) - g_{x_0}(y) = 0$$

for $y \in \mathbb{R}^n \setminus \Omega$. Note that this $\mathcal{G}_\sigma$ belongs to $L^1(\mathbb{R}^n)$: in fact, by definition, it is not greater than $\mathcal{F}_\sigma$, which is in $L^1_{\text{loc}}(\mathbb{R}^n)$ because, by Theorem 11, it lies in $L_{\frac{\sigma}{2}}(\mathbb{R}^n)$. Hence, it must be that $\mathcal{G}_\sigma \in L^1(\Omega)$ and, since in this case they are equivalent conditions due to the vanishing of $\mathcal{G}_\sigma$ outside Ω, that $\mathcal{G}_\sigma \in L^1(\mathbb{R}^n)$.

Step 3: proof of (144). To the scope of having (144), it suffices to bound $\mathcal{G}_\sigma$ from below because, in view of (162), we have that $\mathcal{H}_\sigma(x_0, y) \geq 0$ a.e., which implies that

$$\mathcal{G}_\sigma(x_0, y) := \mathcal{F}_\sigma(x_0 - y) - \mathcal{H}_\sigma(x_0, y) \leq \mathcal{F}_\sigma(x_0 - y).$$

Therefore, it remains to show that $\mathcal{G}_\sigma(x_0, y) \geq 0$ for a.e. $y \in \Omega$. Since, again by (162), $\mathcal{H}_\sigma$ is bounded, choosing an element $\varepsilon \in (0, \delta]$ sufficiently small takes us to have $\mathcal{F}_\sigma(x_0 - y) > \mathcal{H}_\sigma(x_0, y)$ for a.e. $y \in B_\varepsilon(x_0)$ because, in proximity of x_0, we reach the singularity of the fundamental solution $\mathcal{F}_\sigma$. Therefore,

$$\mathcal{G}_\sigma(x_0, y) := \mathcal{F}_\sigma(x_0 - y) - \mathcal{H}_\sigma(x_0, y) > 0$$

in $B_\varepsilon(x_0)$. Next, using Lemma 13, we modify the function $\mathcal{F}_\sigma$ in $B_\varepsilon(x_0)$ in order to obtain a new function Ψ_{x_0} lying in $C^1(\mathbb{R}^n)$ and satisfying the relations (153), (154) and (155), namely it is such that $\Psi_{x_0}(y) \leq \mathcal{F}_\sigma(x_0 - y)$ on $\mathbb{R}^n$, $\Psi_{x_0}(y) = \mathcal{F}_\sigma(x_0 - y)$ on $\mathbb{R}^n \backslash B_\varepsilon(x_0)$ and $(-\Delta)^{\frac{\sigma}{2}} \Psi_{x_0}(y) \geq 0$ on $\mathbb{R}^n$. We reiterate that the last condition can be even more strict, namely we could consider $(-\Delta)^{\frac{\sigma}{2}} \Psi_{x_0}(y)$ to be everywhere positive; nevertheless, it is enough if we have that $(-\Delta)^{\frac{\sigma}{2}} \Psi_{x_0}(y) \geq 0$ on the whole $\mathbb{R}^n$.
We have to clarify that, using an abuse of notation, we wrote Ψ_{x_0} instead of Ψ_ε (which was the notation utilized in Lemma 13) in order to highlight the dependence of the aforementioned function on x_0 and to hide the one on ε, since the latter is not relevant. Besides, notice also that we used Lemma 13 after a translation.
Now, we make another claim by affirming that $\Psi_{x_0} - \mathcal{H}_\sigma \in H_0^{\frac{\sigma}{2},2}(\Omega)$. First of all, the function $\Psi_{x_0} - g_{x_0}$ belongs to $C^1(\mathbb{R}^n)$ because $\Psi_{x_0} \in C^1(\mathbb{R}^n)$ and $g_{x_0} \in C^1(\mathbb{R}^n)$. Furthermore, it vanishes outside Ω since, over (at least) $\mathbb{R}^n \backslash \Omega$, we have that

$$\Psi_{x_0}(y) = \mathcal{F}_\sigma(x_0 - y) = g_{x_0}(y),$$

being $B_\varepsilon(x_0) \subseteq B_\delta(x_0) \subsetneq \Omega$. Moreover, a similar computation respect to the ones done to prove (160) takes us to state that

$$\int_{\mathbb{R}^n} \bigg(\int_{\mathbb{R}^n} \frac{\big[\big(\Psi_{x_0}(y) - g_{x_0}(y)\big) - \big(\Psi_{x_0}(z) - g_{x_0}(z)\big)\big]^2}{|y - z|^{n+\sigma}} \, dy \bigg) dz < +\infty.$$

Indeed, using the elementary estimate $|a + b|^p \leq 2^p(|a|^p + |b|^p)$ valid for every $a, b \in \mathbb{R}$ and $p \geq 1$, we have that

$$\int_{\mathbb{R}^n} \bigg(\int_{\mathbb{R}^n} \frac{\big[\big(\Psi_{x_0}(y) - g_{x_0}(y)\big) - \big(\Psi_{x_0}(z) - g_{x_0}(z)\big)\big]^2}{|y - z|^{n+\sigma}} \, dy \bigg) dz =$$

$$= \int_{\Omega} \bigg(\int_{\mathbb{R}^n} \frac{\big|\big(\Psi_{x_0}(y) - \Psi_{x_0}(z)\big) - \big(g_{x_0}(y) - g_{x_0}(z)\big)\big|^2}{|y - z|^{n+\sigma}} \, dy \bigg) dz \leq$$

$$\leq 4 \int_{\Omega} \bigg(\int_{\mathbb{R}^n} \frac{|\Psi_{x_0}(y) - \Psi_{x_0}(z)|^2 + |g_{x_0}(y) - g_{x_0}(z)|^2}{|y - z|^{n+\sigma}} \, dy \bigg) dz =$$

$$= 4 \bigg[\int_{\Omega} \bigg(\int_{\mathbb{R}^n} \frac{|\Psi_{x_0}(y) - \Psi_{x_0}(z)|^2}{|y - z|^{n+\sigma}} \, dy \bigg) dz + \int_{\Omega} \bigg(\int_{\mathbb{R}^n} \frac{|g_{x_0}(y) - g_{x_0}(z)|^2}{|y - z|^{n+\sigma}} \, dy \bigg) dz \bigg] \leq$$

$$\leq 4 \bigg[\Big(c_3 ||\nabla \Psi_{x_0}||^2_{L^\infty(B_1(z_0))} + 4c_4 ||\Psi_{x_0}||^2_{L^\infty(\mathbb{R}^n)} \Big) |\Omega| +$$

$$+\Big(c_1||\nabla g_{x_0}||^2_{L^\infty(B_1(z_0))}+4c_2||g_{x_0}||^2_{L^\infty(\mathbb{R}^n)}\Big)|\Omega|\Big]<+\infty,$$

where we consider the same $z_0 \in \Omega$ and compute the same calculations used in Step 2, which allow us to state that, since $\Psi_{x_0} \in C^1(\mathbb{R}^n)$,

$$\int_\Omega\bigg(\int_{\mathbb{R}^n}\frac{|\Psi_{x_0}(y)-\Psi_{x_0}(z)|^2}{|y-z|^{n+\sigma}}\,dy\bigg)dz \le \Big(c_3||\nabla\Psi_{x_0}||^2_{L^\infty(B_1(z_0))}+4c_4||\Psi_{x_0}||^2_{L^\infty(\mathbb{R}^n)}\Big)|\Omega|$$

holds, similarly as before, for two constants $c_3 = c_3(\sigma, n)$ and $c_4 = c_4(\sigma, n)$ depending on σ and n only (the constants c_1 and c_2 are, in fact, the same). Consequently, it follows that

$$[\Psi_{x_0}-g_{x_0}]_{W^{\frac{\sigma}{2},2}(\mathbb{R}^n)} := \bigg(\int_{\mathbb{R}^n}\bigg(\int_{\mathbb{R}^n}\frac{\big|\big(\Psi_{x_0}(y)-g_{x_0}(y)\big)-\big(\Psi_{x_0}(z)-g_{x_0}(z)\big)\big|^2}{|y-z|^{n+\sigma}}\,dy\bigg)dz\bigg)^{\frac{1}{2}} =$$

$$=\bigg(\int_{\mathbb{R}^n}\bigg(\int_{\mathbb{R}^n}\frac{\big[\big(\Psi_{x_0}(y)-g_{x_0}(y)\big)-\big(\Psi_{x_0}(z)-g_{x_0}(z)\big)\big]^2}{|y-z|^{n+\sigma}}\,dy\bigg)dz\bigg)^{\frac{1}{2}}$$

is finite, which implies (due to Lemma 11) that $\Psi_{x_0} - g_{x_0} \in W^{\frac{\sigma}{2},2}(\mathbb{R}^n) = H^{\frac{\sigma}{2},2}(\mathbb{R}^n)$. Hence, by the vanishing of the latter outside Ω, we have also that $\Psi_{x_0} - g_{x_0} \in H_0^{\frac{\sigma}{2},2}(\Omega)$. Therefore, since we already know that $\mathcal{H}_\sigma(x_0, y) - g_{x_0}(y) \in H_0^{\frac{\sigma}{2},2}(\Omega)$, it must be that

$$\Psi_{x_0}(y)-\mathcal{H}_\sigma(x_0,y)=\big(\Psi_{x_0}(y)-g_{x_0}(y)\big)-\big(\mathcal{H}_\sigma(x_0,y)-g_{x_0}(y)\big)$$

lies in $H_0^{\frac{\sigma}{2},2}(\Omega)$, as claimed. Thus, this statement means that $\Psi_{x_0}(y) - \mathcal{H}_\sigma(x_0, y) \equiv 0$ over $\mathbb{R}^n\backslash\Omega$.

Moreover, one has that

$$(-\Delta)^{\frac{\sigma}{2}}\big(\Psi_{x_0}(y)-\mathcal{H}_\sigma(x_0,y)\big)=(-\Delta)^{\frac{\sigma}{2}}\Psi_{x_0}(y)-(-\Delta)^{\frac{\sigma}{2}}\mathcal{H}_\sigma(x_0,y)=(-\Delta)^{\frac{\sigma}{2}}\Psi_{x_0}(y)\ge 0$$

on Ω by (161) and the properties of Ψ_{x_0}. Taking now the (positive) function

$$f(y) := 2C\Big(\frac{\sigma}{2},n\Big)^{-1}(-\Delta)^{\frac{\sigma}{2}}\big(\Psi_{x_0}(y)-\mathcal{H}_\sigma(x_0,y)\big),$$

which belongs to $L^2(\Omega)$ by Definition 11 because $\Psi_{x_0}(y) - \mathcal{H}_\sigma(x_0, y) \in H_0^{\frac{\sigma}{2},2}(\Omega)$, we have that

$$(-\Delta)^{\frac{\sigma}{2}}\big(\Psi_{x_0}(y)-\mathcal{H}_\sigma(x_0,y)\big)=\frac{1}{2}C\Big(\frac{\sigma}{2},n\Big)2C\Big(\frac{\sigma}{2},n\Big)^{-1}(-\Delta)^{\frac{\sigma}{2}}\big(\Psi_{x_0}(y)-\mathcal{H}_\sigma(x_0,y)\big)=$$

$$=\frac{1}{2}C\Big(\frac{\sigma}{2},n\Big)f(y).$$

Further, the function $\Psi_{x_0}(y)$ satisfies (150) because, on $\mathbb{R}^n \backslash B_\delta(x_0)$, it coincides with

g_{x_0} (which makes the aforementioned estimate true) and, since they both lie in $C^1(\mathbb{R}^n)$, the methods utilized to prove (160) are still valid once we replace g_{x_0} with $\Psi_{x_0}(y)$. It follows at once that, since $\Psi_{x_0}(y) - \mathcal{H}_\sigma(x_0, y)$ satisfies the above relation with such a function f, it also satisfies (151) (by Lemma 11) and that, since $\Psi_{x_0}(y)$ is non-negative, $\Psi_{x_0}(y) - \mathcal{H}_\sigma(x_0, y) \geq 0$ (by Corollary 8). Hence, we have that $\Psi_{x_0}(y) - \mathcal{H}_\sigma(x_0, y) \geq 0$ inside Ω, which implies that

$$\mathcal{G}_\sigma(x_0, y) := \mathcal{F}_\sigma(x_0 - y) - \mathcal{H}_\sigma(x_0, y) \geq \Psi_{x_0}(y) - \mathcal{H}_\sigma(x_0, y) \geq 0$$

in Ω, proving (144).

Step 4: proof of (145). To the scope of showing the last part of this result, we begin by considering a function $u \in C_0^\infty(\Omega)$. Then, using u as a test function in (143), we infer

$$u(x) = \int_{\mathbb{R}^n} \delta_x(y) u(y) dy = \int_\Omega \mathcal{G}_\sigma(x, y)(-\Delta)^{\frac{\sigma}{2}} u(y) dy =: \big\langle (-\Delta)^{\frac{\sigma}{2}} \mathcal{G}_\sigma, u \big\rangle, \tag{163}$$

reminding that $\mathcal{G}_\sigma$ vanishes outside Ω. If now $u \in H_0^{\sigma,p}(\Omega)$, let $\{u_k\}_{k\in\mathbb{N}}$ belonging to $C_0^\infty(\Omega)$ and converging to u in $H_0^{\sigma,p}(\Omega)$, which means that $u_k(x) \longrightarrow u(x)$ and $(-\Delta)^{\frac{\sigma}{2}} u_k(x) \longrightarrow (-\Delta)^{\frac{\sigma}{2}} u(x)$ both in $L^p(\mathbb{R}^n)$ (and, therefore, in $L^1(\Omega)$ due to the boundedness of Ω). Note that this is possible thanks to the density theorem of $C_0^\infty(\Omega)$ into $H_0^{\sigma,p}(\Omega)$ achieved by Netrusov. Using now (163), one has that

$$u_k(x) = \big\langle (-\Delta)^{\frac{\sigma}{2}} \mathcal{G}_\sigma, u_k \big\rangle := \int_\Omega \mathcal{G}_\sigma(x, y)(-\Delta)^{\frac{\sigma}{2}} u_k(y) dy. \tag{164}$$

Next, we show that

$$\int_\Omega \mathcal{G}_\sigma(x, y)(-\Delta)^{\frac{\sigma}{2}} u_k(y) dy \longrightarrow \int_\Omega \mathcal{G}_\sigma(x, y)(-\Delta)^{\frac{\sigma}{2}} u(y) dy \tag{165}$$

in $L^1(\Omega)$. In fact, by (144) and Fubini's theorem,

$$\int_\Omega \bigg| \int_\Omega \mathcal{G}_\sigma(x, y)(-\Delta)^{\frac{\sigma}{2}} u_k(y) dy - \int_\Omega \mathcal{G}_\sigma(x, y)(-\Delta)^{\frac{\sigma}{2}} u(y) dy \bigg| dx =$$

$$= \int_\Omega \bigg| \int_\Omega \mathcal{G}_\sigma(x, y)\big[(-\Delta)^{\frac{\sigma}{2}} u_k(y) - (-\Delta)^{\frac{\sigma}{2}} u(y)\big] dy \bigg| dx \leq$$

$$\leq \int_\Omega \bigg(\int_\Omega |\mathcal{G}_\sigma(x, y)| \big|(-\Delta)^{\frac{\sigma}{2}} u_k(y) - (-\Delta)^{\frac{\sigma}{2}} u(y)\big| dy \bigg) dx \leq$$

$$\leq \int_\Omega \bigg(\int_\Omega \mathcal{F}_\sigma(x - y) \big|(-\Delta)^{\frac{\sigma}{2}} u_k(y) - (-\Delta)^{\frac{\sigma}{2}} u(y)\big| dy \bigg) dx =$$

$$= \int_\Omega \Big(\int_\Omega \mathcal{F}_\sigma(x-y)dx \Big) \big|(-\Delta)^{\frac{\sigma}{2}} u_k(y) - (-\Delta)^{\frac{\sigma}{2}} u(y)\big| dy =$$

$$= \int_\Omega ||\mathcal{F}_\sigma(\cdot - y)||_{L^1(\Omega)} \big|(-\Delta)^{\frac{\sigma}{2}} u_k(y) - (-\Delta)^{\frac{\sigma}{2}} u(y)\big| dy \leq$$

$$\leq \sup_{y\in\Omega} \big\{ ||\mathcal{F}_\sigma(\cdot - y)||_{L^1(\Omega)} \big\} \int_\Omega \big|(-\Delta)^{\frac{\sigma}{2}} u_k(y) - (-\Delta)^{\frac{\sigma}{2}} u(y)\big| dy =$$

$$= \sup_{y\in\Omega} \big\{ ||\mathcal{F}_\sigma(\cdot - y)||_{L^1(\Omega)} \big\} \big|\big|(-\Delta)^{\frac{\sigma}{2}} u_k(y) - (-\Delta)^{\frac{\sigma}{2}} u(y)\big|\big|_{L^1(\Omega)} \longrightarrow 0$$

as $k \longmapsto +\infty$, where we also used that $||\mathcal{F}_\sigma(\cdot - y)||_{L^1(\Omega)}$ is bounded by a constant depending on σ and n only (hence, independent of y) for every $y \in \Omega$. In fact, for every $y_0 \in \Omega$ fixed, using Minkowski's inequality and the definition of $\mathcal{F}_\sigma$ takes us to

$$||\mathcal{F}_\sigma(\cdot - y_0)||_{L^1(\Omega)} \leq ||\mathcal{F}_\sigma(\cdot - y_0)||_{L^1(B_1(y_0))} + ||\mathcal{F}_\sigma(\cdot - y_0)||_{L^1(\Omega\setminus B_1(y_0))} =$$

$$= \int_{B_1(y_0)} \mathcal{F}_\sigma(x-y_0)dx + \int_{\Omega\setminus B_1(y_0)} \mathcal{F}_\sigma(x-y_0)dx =$$

$$= \frac{\Gamma\big(\frac{n-\sigma}{2}\big)}{2^\sigma \pi^{\frac{n}{2}} \Gamma\big(\frac{\sigma}{2}\big)} \Big(\int_{B_1(y_0)} |x-y_0|^{\sigma-n} dx + \int_{\Omega\setminus B_1(y_0)} |x-y_0|^{\sigma-n} dx \Big) \leq$$

$$\leq \frac{\Gamma\big(\frac{n-\sigma}{2}\big)}{2^\sigma \pi^{\frac{n}{2}} \Gamma\big(\frac{\sigma}{2}\big)} \Big(\int_{B_1(y_0)} \frac{dx}{|x-y_0|^{n-\sigma}} + \int_{\Omega\setminus B_1(y_0)} dx \Big) \leq \frac{\Gamma\big(\frac{n-\sigma}{2}\big)}{2^\sigma \pi^{\frac{n}{2}} \Gamma\big(\frac{\sigma}{2}\big)} \big(c_5 + |\Omega|\big),$$

where we used the inequality $|x-y_0|^{\sigma-n} \leq 1$ valid over $\Omega \setminus B_1(y_0)$ and introduced a constant $c_5 = c_5(\sigma, n)$ depending on σ and n only, being able to do that since $n-\sigma < n$. Thus, we got (165) which, along with (164), tells us that

$$u(x) \longleftarrow u_k(x) = \int_\Omega \mathcal{G}_\sigma(x,y)(-\Delta)^{\frac{\sigma}{2}} u_k(y)dy \longrightarrow \int_\Omega \mathcal{G}_\sigma(x,y)(-\Delta)^{\frac{\sigma}{2}} u(y)dy$$

both times in $L^1(\Omega)$. Hence, we are finally done by the uniqueness of the limit, which allows us to affirm that

$$u(x) = \int_\Omega \mathcal{G}_\sigma(x,y)(-\Delta)^{\frac{\sigma}{2}} u(y)dy.$$

□

Proof of Proposition 15. We divide the proof into two steps.

Step 1: proof of (146). Using the relation (144), we will deduce (146). In fact, that $\mathcal{G}_s(x,y) \geq 0$ is clear since $\mathcal{G}_\sigma$ is, for every $\sigma \in (0,2]$, a non-negative function, which implies also the non-negativity of $\mathcal{G}_s$ by its definition. Subsequently, using (144), we have that

$$\mathcal{G}_s(x,y) := \int_\Omega \mathcal{G}_2(x,y_1)\bigg[\int_\Omega \mathcal{G}_2(y_1,y_2)\bigg[\cdots\bigg[\int_\Omega \mathcal{G}_2(y_{k-1},y_k)\mathcal{G}_\sigma(y_k,y)dy_k\bigg]\cdots\bigg]dy_2\bigg]dy_1 \leq$$

$$\leq \int_\Omega \mathcal{F}_2(x-y_1)\bigg[\int_\Omega \mathcal{F}_2(y_1-y_2)\bigg[\cdots\bigg[\int_\Omega \mathcal{F}_2(y_{k-1}-y_k)\mathcal{F}_\sigma(y_k-y)dy_k\bigg]\cdots\bigg]dy_2\bigg]dy_1 =$$

$$= (\underbrace{\mathcal{F}_2 * \mathcal{F}_2 * \cdots * \mathcal{F}_2}_{\text{k times}} * \mathcal{F}_\sigma)(x-y),$$

where the last identity follows after a change of variables by setting $z_j := y_j - y_{j+1}$, for every $1 \leq j \leq k-1$, and $z_k := y_k - y$. In fact, doing that, the determinant of the Jacobian matrix (which is the sum of the identity with the matrix having -1 on the supradiagonal and 0 elsewhere) is 1 and we find that the above convolution becomes equal to

$$\int_\Omega \mathcal{F}_2\bigg(x-y-\sum_{j=1}^{k} z_j\bigg)\bigg[\int_\Omega \mathcal{F}_2(z_1)\bigg[\cdots\bigg[\int_\Omega \mathcal{F}_2(z_{k-1})\mathcal{F}_\sigma(z_k)dz_k\bigg]\cdots\bigg]dz_2\bigg]dz_1 =$$

$$= \int_\Omega \mathcal{F}_2(x-y_1)\bigg[\int_\Omega \mathcal{F}_2(y_1-y_2)\bigg[\cdots\bigg[\int_\Omega \mathcal{F}_2(y_{k-1}-y_k)\mathcal{F}_\sigma(y_k-y)dy_k\bigg]\cdots\bigg]dy_2\bigg]dy_1,$$

since the sum appearing in the first integral is telescoping and gives $y_1 - y$ as result. Lastly, it remains to show that

$$(\underbrace{\mathcal{F}_2 * \mathcal{F}_2 * \cdots * \mathcal{F}_2}_{\text{k times}} * \mathcal{F}_\sigma)(x-y) = \mathcal{F}_s(x-y) := \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} |x-y|^{s-n}.$$

This identity follows by the definition of the function $\mathcal{F}_s$ and Corollary 6 once we proceed by induction on $s = 2k + \sigma$. Firstly, if $s \in (0,2]$, then $k = 0$ and, therefore,

$$\mathcal{F}_\sigma(x-y) := \frac{\Gamma\left(\frac{n-\sigma}{2}\right)}{2^\sigma \pi^{\frac{n}{2}} \Gamma\left(\frac{\sigma}{2}\right)} |x-y|^{\sigma-n}.$$

Now, for every $k \in \mathbb{N}$, we show that

$$(\underbrace{\mathcal{F}_2 * \mathcal{F}_2 * \cdots * \mathcal{F}_2}_{\text{k times}})(x-y) = \mathcal{F}_{2k}(x-y) := \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k} \pi^{\frac{n}{2}} \Gamma\left(\frac{2k}{2}\right)} |x-y|^{2k-n}.$$

If $k = 1$, there is nothing that needs to be shown since, in such a situation, we have exactly the definition on $\mathcal{F}_2$. We continue with the case in which $k = 2$, where we get

$$(\mathcal{F}_2 * \mathcal{F}_2)(x-y) = \left[\left(\frac{\Gamma\left(\frac{n-2}{2}\right)}{2^2 \pi^{\frac{n}{2}}} |x|^{2-n}\right) * \left(\frac{\Gamma\left(\frac{n-2}{2}\right)}{2^2 \pi^{\frac{n}{2}}} |x|^{2-n}\right)\right](x-y) =$$

$$= \frac{\Gamma\left(\frac{n-2}{2}\right)^2}{2^4\pi^n}\left(|x|^{2-n} * |x|^{2-n}\right)(x-y) = \frac{\Gamma\left(\frac{n-2}{2}\right)^2}{2^4\pi^n}\,\frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{n-4}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}\,|x-y|^{4-n} =$$

$$= \frac{\Gamma\left(\frac{n-4}{2}\right)}{2^4\pi^{\frac{n}{2}}}\,|x-y|^{4-n} = \frac{\Gamma\left(\frac{n-4}{2}\right)}{2^4\pi^{\frac{n}{2}}\Gamma\left(\frac{4}{2}\right)}\,|x-y|^{4-n} =: \mathcal{F}_4(x-y),$$

where we used the usual properties of the Gamma function and the identity (141) with $\alpha := 2 =: \beta$, being here $\alpha+\beta = 4 < 4+\sigma = s < n$. Next, assuming the identity true for $k-1$, we obtain

$$(\underbrace{\mathcal{F}_2 * \mathcal{F}_2 * \cdots * \mathcal{F}_2}_{\text{k times}})(x-y) = (\underbrace{\mathcal{F}_2 * \mathcal{F}_2 * \cdots * \mathcal{F}_2}_{\text{k-1 times}} * \mathcal{F}_2)(x-y) = (\mathcal{F}_{2k-2} * \mathcal{F}_2)(x-y) =$$

$$= \left[\left(\frac{\Gamma\left(\frac{n+2-2k}{2}\right)}{2^{2k-2}\pi^{\frac{n}{2}}\Gamma\left(\frac{2k-2}{2}\right)}\,|x|^{2k-2-n}\right) * \left(\frac{\Gamma\left(\frac{n-2}{2}\right)}{2^2\pi^{\frac{n}{2}}}\,|x|^{2-n}\right)\right](x-y) =$$

$$= \frac{\Gamma\left(\frac{n+2-2k}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}{2^{2k}\pi^n\Gamma\left(\frac{2k-2}{2}\right)}\left(|x|^{2k-2-n} * |x|^{2-n}\right)(x-y) =$$

$$= \frac{\Gamma\left(\frac{n+2-2k}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}{2^{2k}\pi^n\Gamma\left(\frac{2k-2}{2}\right)}\,\frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{2k-2}{2}\right)\Gamma\left(\frac{n-2k}{2}\right)}{\Gamma\left(\frac{2k}{2}\right)\Gamma\left(\frac{n+2-2k}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}\,|x-y|^{2k-n} =$$

$$= \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma\left(\frac{2k}{2}\right)}\,|x-y|^{2k-n} =: \mathcal{F}_{2k}(x-y),$$

where we used again (141) with, this time, $\alpha := 2k-2$ and $\beta := 2$, being once more allowed to resort to Corollary 6 since, now, $\alpha+\beta = 2k < 2k+\sigma = s < n$.
Having this formula, we can complete the proof of (146) because we infer

$$(\underbrace{\mathcal{F}_2 * \mathcal{F}_2 * \cdots * \mathcal{F}_2}_{\text{k times}} * \mathcal{F}_\sigma)(x-y) = (\mathcal{F}_{2k} * \mathcal{F}_\sigma)(x-y) =$$

$$= \left[\left(\frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma\left(\frac{2k}{2}\right)}\,|x|^{2k-n}\right) * \left(\frac{\Gamma\left(\frac{n-\sigma}{2}\right)}{2^\sigma\pi^{\frac{n}{2}}\Gamma\left(\frac{\sigma}{2}\right)}\,|x|^{\sigma-n}\right)\right](x-y) =$$

$$= \frac{\Gamma\left(\frac{n-2k}{2}\right)\Gamma\left(\frac{n-\sigma}{2}\right)}{2^{2k+\sigma}\pi^n\Gamma\left(\frac{2k}{2}\right)\Gamma\left(\frac{\sigma}{2}\right)}\left(|x|^{2k-n} * |x|^{\sigma-n}\right)(x-y) =$$

$$= \frac{\Gamma\left(\frac{n-2k}{2}\right)\Gamma\left(\frac{n-\sigma}{2}\right)}{2^{2k+\sigma}\pi^n\Gamma\left(\frac{2k}{2}\right)\Gamma\left(\frac{\sigma}{2}\right)}\,\frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{2k}{2}\right)\Gamma\left(\frac{\sigma}{2}\right)\Gamma\left(\frac{n-2k-\sigma}{2}\right)}{\Gamma\left(\frac{2k+\sigma}{2}\right)\Gamma\left(\frac{n-2k}{2}\right)\Gamma\left(\frac{n-\sigma}{2}\right)}\,|x-y|^{2k+\sigma-n} =$$

$$= \frac{\Gamma\left(\frac{n-2k-\sigma}{2}\right)}{2^{2k+\sigma}\pi^{\frac{n}{2}}\Gamma\left(\frac{2k+\sigma}{2}\right)}\,|x-y|^{2k+\sigma-n} = \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s\pi^{\frac{n}{2}}\Gamma\left(\frac{s}{2}\right)}\,|x-y|^{s-n} =: \mathcal{F}_s(x-y),$$

where we used, for the third and last time, (141) after setting $\alpha := 2k$ and $\beta := \sigma$, being able to do that since $\alpha+\beta = 2k+\sigma = s < n$.

Step 2: proof of (147). In order to prove (147), we first consider, as before, a function $u \in C_0^\infty(\Omega)$. Thanks to the properties of the fractional Laplacian (see Remark 22 and Remark 25), we are able to write

$$(-\Delta)^{\frac{s}{2}} u(x) = \big(\underbrace{(-\Delta) \circ (-\Delta) \circ \cdots \circ (-\Delta)}_{\text{k times}} \circ (-\Delta)^{\frac{\sigma}{2}} \big) u(x). \tag{166}$$

Consequently, using $k+1$ times the identity (145) (the first k times with $\sigma := 2$), one obtains

$$\begin{aligned}
u(x) &= \int_\Omega \mathcal{G}_2(x, y_1)(-\Delta)u(y_1)dy_1 = \\
&= \int_\Omega \mathcal{G}_2(x, y_1)\left[\int_\Omega \mathcal{G}_2(y_1, y_2)(-\Delta)^2 u(y_2)dy_2 \right] dy_1 = \cdots = \\
&= \int_\Omega \mathcal{G}_2(x, y_1)\left[\cdots \left[\int_\Omega \mathcal{G}_2(y_{k-2}, y_{k-1})\left[\int_\Omega \mathcal{G}_2(y_{k-1}, y_k)(-\Delta)^k u(y_k)dy_k \right] dy_{k-1} \right] \cdots \right] dy_1 = \\
&= \int_\Omega \mathcal{G}_2(x, y_1)\left[\cdots \left[\int_\Omega \mathcal{G}_2(y_{k-1}, y_k)\left[\int_\Omega \mathcal{G}_\sigma(y_k, y)(-\Delta)^{\frac{k+\sigma}{2}} u(y)dy \right] dy_k \right] \cdots \right] dy_1 = \\
&= \int_\Omega \mathcal{G}_2(x, y_1)\left[\cdots \left[\int_\Omega \mathcal{G}_2(y_{k-1}, y_k)\left[\int_\Omega \mathcal{G}_\sigma(y_k, y)(-\Delta)^{\frac{s}{2}} u(y)dy \right] dy_k \right] \cdots \right] dy_1 = \\
&= \int_\Omega \left[\int_\Omega \mathcal{G}_2(x, y_1)\left[\cdots \left[\int_\Omega \mathcal{G}_2(y_{k-1}, y_k)\mathcal{G}_\sigma(y_k, y)dy_k \right] \cdots \right] dy_1 \right] (-\Delta)^{\frac{s}{2}} u(y)dy = \\
&= \int_\Omega \mathcal{G}_s(x, y)(-\Delta)^{\frac{s}{2}} u(y)dy,
\end{aligned}$$

where we also used (166), Fubini's theorem repeatedly and the definition of $\mathcal{G}_s$.
When u is not smooth, we can replicate the techniques utilized in the last part of the preceding proof and will get the thesis (basically, we can use again Netrusov's theorem as done before and (147) will follow by density).

□

3.3. Proof of Moser-Adams' theorem and its sharpness

In this last section of the issue, we finally prove Theorem 12 (which has now become a simple application of Proposition 15 and Theorem 10) and the usual result concerning the sharpness of the constant $\alpha_{s,n}$.

Proof of Theorem 12. We split the proof in two steps.

Step 1: an initial consideration. Given an arbitrary function $u \in H_0^{s,\frac{n}{s}}(\Omega)$, we take $f(x) := \left|(-\Delta)^{\frac{s}{2}} u(x)\right|$ as our auxiliary function. By hypothesis, we have that $f \in L^q(\Omega)$ and $||f||_{L^q(\Omega)} \leq 1$, where $q := \frac{n}{s}$.
We first notice that, similarly to what happened in the proof of Adams' theorem, we get $f(x) \equiv 0 \iff u(x) \equiv 0$ over Ω. Indeed, if $u(x) \equiv 0$, then clearly $f(x) \equiv 0$ by the definition of the Fourier transform and the properties of the fractional Laplacian. Conversely, if $f(x) \equiv 0$, then $0 \equiv |f(x)| = \left|(-\Delta)^{\frac{s}{2}} u(x)\right|$ and, using (134), we infer that

$$\left|(-\Delta)^{\frac{s}{2}} u(x)\right| = |f(x)| = \left|(-\Delta)^{\frac{s}{2}} (\mathcal{F}_s * f)(x)\right|,$$

which means that, up to a sign, u is equal to $\mathcal{F}_s * f$: this is due to the fact that, whenever we are given two functions g_1 and g_2 (belonging, for instance, to $C_0^\infty(\mathbb{R}^n)$, so that we can subsequently use the density of the latter into $H_0^{s,\frac{n}{s}}(\Omega)$ and the initial definition for the fractional Laplacian through the Fourier transform and anti-transform given in the first section of this chapter) such that

$$(-\Delta)^{\frac{s}{2}} g_1(x) \equiv (-\Delta)^{\frac{s}{2}} g_2(x),$$

then one has necessarily that

$$\mathscr{F}^{-1}\big(|\xi|^s \mathscr{F} g_1(\xi)\big)(x) =: (-\Delta)^{\frac{s}{2}} g_1(x) \equiv (-\Delta)^{\frac{s}{2}} g_2(x) := \mathscr{F}^{-1}\big(|\xi|^s \mathscr{F} g_2(\xi)\big)(x) \iff$$

$$\iff |\xi|^s \mathscr{F} g_1(\xi) \equiv |\xi|^s \mathscr{F} g_2(\xi) \iff \mathscr{F} g_1(\xi) \equiv \mathscr{F} g_2(\xi) \iff g_1(x) \equiv g_2(x)$$

using the properties of the Fourier transform. Thus, we had that $u(x) \equiv (\mathcal{F}_s * f)(x)$ which, however, is null because $f(x) \equiv 0$. This shows that $u(x) \equiv 0$ on Ω.
Therefore, if $u(x) \equiv 0$, then the thesis is obvious (it suffices to replace the constant c with 1 in (142) and the integral is bounded). Instead, if $u(x) \not\equiv 0$, then $f(x) \not\equiv 0$, too, which means that $||f||_{L^q(\Omega)} \in (0, 1]$.

Step 2: conclusion. After having dealt with the trivial case in which $u(x) \equiv 0$, we shall consider a function u which is not identically null. As done earlier, this will prove helpful because, in such a situation, we are allowed to divide by $||f||_{L^q(\Omega)}$ since

$f(x) \not\equiv 0$. Due to Proposition 15, we can bound u and get

$$|u(x)| = \left| \int_\Omega \mathcal{G}_s(x,y)(-\Delta)^{\frac{s}{2}} u(y) dy \right| = \left| \int_\Omega \mathcal{G}_s(x,y) f(y) dy \right| =$$

$$= \int_\Omega \mathcal{G}_s(x,y) f(y) dy \leq \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} \int_\Omega \frac{f(y)}{|x-y|^{n-s}} \, dy = \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} \, (I_s * f)(x), \qquad (167)$$

where $I_s(x) := |x|^{s-n}$ is exactly the same function introduced in Theorem 10. Note also that, this time, we have the complete Riesz potential, since we remind that it was defined as

$$\mathcal{I}_\beta f(x) := \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} \, dy,$$

where $\beta \in (0, n)$ and

$$\gamma(\beta) := \frac{2^\beta \pi^{\frac{n}{2}} \Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{n-\beta}{2}\right)},$$

which implies that

$$\mathcal{I}_s f(x) := \frac{1}{\gamma(s)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}} \, dy = \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} \, (I_s * f)(x).$$

Consequently, (167) implies that

$$\alpha_{s,n} |u(x)|^p \leq \alpha_{s,n} \left[\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} \, (I_s * f)(x) \right]^p =$$

$$= \frac{n}{\omega_{n-1}} \left[\frac{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)} \right]^p \left[\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} \right]^p (I_s * f)(x)^p = \frac{n}{\omega_{n-1}} \, (I_s * f)(x)^p,$$

using the definition of $\alpha_{s,n}$. In conclusion, being here $p := \frac{q}{q-1}$, $s = \frac{n}{q}$ and $I_s(x) := |x|^{s-n}$, we are allowed to apply Theorem 10 and get

$$\fint_\Omega e^{\alpha |u(x)|^p} dx \leq \fint_\Omega e^{\alpha_{s,n} |u(x)|^p} dx \leq \fint_\Omega e^{\frac{n}{\omega_{n-1}} (I_s * f)(x)^p} dx \leq \fint_\Omega e^{\frac{n}{\omega_{n-1}} \left[\frac{(I_s * f)(x)}{||f||_q} \right]^p} dx =$$

$$= \fint_\Omega e^{\frac{n}{\omega_{n-1}} \left| \frac{(I_s * f)(x)}{||f||_q} \right|^p} dx \leq c,$$

for every $\alpha \in [0, \alpha_{s,n}]$ and for a certain constant $c = c(s,n)$ depending on s and n only, which is the desired result.

□

After proving the main theorem of this chapter, it remains to show the sharpness of the constant $\alpha_{s,n}$. To this scope, we have to introduce another helpful lemma, which will be the last auxiliary result before finally proving Corollary 7.

Lemma 14. *Let* $\vartheta \in C_0^\infty\big(B_1(0)\big)$ *be a function such that* $\vartheta(x) \in [0,1]$ *on* $\mathbb{R}^n$ *and, in particular,* $\vartheta(x) \equiv 1$ *in* $B_{\frac{1}{2}}(0)$*. Given* $s > 0$, $\beta \in (0,n)$, $\rho \in \big(0, \frac{1}{8}\big]$ *and* $1 < q < \infty$, *there exists a constant* $c = c(s,n,q,\beta)$ *depending on* s, n, q *and* β *only such that*

$$\big|\big|(-\Delta)^{\frac{s}{2}}\big((1-\vartheta)(I_\beta * f)\big)\big|\big|_{L^q(\mathbb{R}^n)} \leq c\rho^{\frac{n}{p}}||f||_{L^q(B_\rho(0))} \tag{168}$$

holds for every $f \in C_0^\infty\big(B_\rho(0)\big)$, *where* $p := \frac{q}{q-1}$ *and* $I_\beta(x) := |x|^{\beta-n}$ *as always.*

Proof. Once again, we divide the discussion into two steps.

Step 1: a useful estimate. Our starting point is to prove that, given $\psi \in \mathscr{S}(\mathbb{R}^n)$ and set $B := B_{\frac{1}{8}}(0)$ for convenience, the estimate

$$\big|\big|I_\beta * \big((1-\vartheta)(-\Delta)^{\frac{s}{2}}\psi\big)\big|\big|_{L^\infty(B)} \leq c_0||\psi||_{L^p(\mathbb{R}^n)} \tag{169}$$

holds for a certain constant $c_0 = c_0(s,n,q,\beta)$ depending on s, n, q and β only, where $p := \frac{q}{q-1}$ is the conjugate exponent of q.
We set $\vartheta_1(x) := 1 - \vartheta(x)$ and choose a function $\vartheta_2 \in C_0^\infty\big(B_{\frac{1}{4}}(0)\big)$ such that $\vartheta_2(x) \geq 0$ over $\mathbb{R}^n$ and $\vartheta_2(x) \equiv 1$ in B. Hence, by definition, $\vartheta_1(x) \in [0,1]$ and $\vartheta_1(x) \equiv 0$ in $B_{\frac{1}{2}}(0)$, which means that the supports of ϑ_1 and ϑ_2 are disjoint and, in particular, one has that $\operatorname{dist}(\operatorname{supp}\{\vartheta_1\}, \operatorname{supp}\{\vartheta_2\}) \geq \frac{1}{4}$.
Next, we consider the function

$$g(x,y) := \frac{\vartheta_1(y)\vartheta_2(x)}{|x-y|^{n-\beta}}.$$

That g is smooth (in both x and y) thanks to the disjointness of $\operatorname{supp}\{\vartheta_1\}$ and $\operatorname{supp}\{\vartheta_1\}$ just established. Besides, by definition, it is non-negative for every choice of x and y. Moreover, $\forall\, x \in B$ and for every $\psi \in \mathscr{S}(\mathbb{R}^n)$, we get

$$\big(I_\beta * \big((1-\vartheta)(-\Delta)^{\frac{s}{2}}\psi\big)\big)(x) = \vartheta_2(x)\big(I_\beta * \big(\vartheta_1(-\Delta)^{\frac{s}{2}}\psi\big)\big)(x) =$$

$$= \vartheta_2(x)\int_{\mathbb{R}^n} \frac{\vartheta_1(y)(-\Delta)^{\frac{s}{2}}\psi(y)}{|x-y|^{n-\beta}}\,dy = \int_{\mathbb{R}^n} \frac{\vartheta_1(y)\vartheta_2(x)}{|x-y|^{n-\beta}}\,(-\Delta)^{\frac{s}{2}}\psi(y)dy =$$

$$= \int_{\mathbb{R}^n} g(x,y)(-\Delta)^{\frac{s}{2}}\psi(y)dy = \int_{\mathbb{R}^n} (-\Delta_y)^{\frac{s}{2}}g(x,y)\psi(y)dy = \int_{\mathbb{R}^n} h(x,y)\psi(y)dy, \tag{170}$$

where we resorted to the definition of the function g, utilized the properties of the fractional Laplacian (in particular, the one appearing in the point *(ii)* of Remark 23) and set $h(x,y) := (-\Delta_y)^{\frac{s}{2}}g(x,y)$. Note also that we had to denote by $(-\Delta_y)^{\frac{s}{2}}$ the fractional Laplacian operator because, this time, we are dealing with a function in two variables and, therefore, one has to declare in which variable it is done. Moreover, h is smooth (because, basically, g is).

We are now focused on the asymptotic behavior of the function h: it turns out that it decays, at most, like $|y|^{-n-s}$ uniformly with respect to $x \in B$ (see also [16] for this fact). Indeed, as usual, we can write $s = 2k + \sigma$ for some $k \in \mathbb{N}_0$ and $\sigma \in (0, 2)$ so that, using its properties, we are able to split the operator $(-\Delta)^{\frac{s}{2}}$ into $(-\Delta)^k \circ (-\Delta)^{\frac{\sigma}{2}}$. Subsequently, using (117), we get

$$(-\Delta_y)^{\frac{\sigma}{2}} g(x, y) = C(\sigma, n) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{g(x, y) - g(x, z)}{|y - z|^{n+\sigma}} \, dz,$$

where the constant $C(\sigma, n)$ is given by (118), as always. For $x \in B$ and for sufficiently large y, we get

$$\begin{aligned} (-\Delta_y)^{\frac{\sigma}{2}} g(x, y) &= C(\sigma, n) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{g(x, y) - g(x, z)}{|y - z|^{n+\sigma}} \, dz < \\ &< C(\sigma, n) g(x, y) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{dz}{|y - z|^{n+\sigma}}, \end{aligned} \tag{171}$$

where we used that $g(x, z) \geq 0$ and, in particular, that it is strictly greater than 0 outside $B_1(0)$. This implies that

$$|y|^{n+\sigma} (-\Delta_y)^{\frac{\sigma}{2}} g(x, y) < C(\sigma, n) g(x, y) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|y|^{n+\sigma}}{|y - z|^{n+\sigma}} \, dz \leq c_1,$$

for a certain constant $c_1 = c_1(\sigma, n, \beta)$ depending on σ, n and β only (where we used that $n + \sigma > n$ and that, for large y, the function g is not greater than 1). Hence,

$$(-\Delta_y)^{\frac{\sigma}{2}} g(x, y) < \frac{c_1}{|y|^{n+\sigma}}.$$

Thus, using (171) and differentiating repeatedly, we get

$$\begin{aligned} (-\Delta_y)^{\frac{s}{2}} g(x, y) &= \big((-\Delta_y)^k \circ (-\Delta_y)^{\frac{\sigma}{2}} \big) g(x, y) = \\ &= (-\Delta_y)^k \left[C(\sigma, n) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{g(x, y) - g(x, z)}{|y - z|^{n+\sigma}} \, dz \right] < \\ &< (-\Delta_y)^k \left[C(\sigma, n) g(x, y) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{dz}{|y - z|^{n+\sigma}} \right] = \\ &= C(\sigma, n) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} (-\Delta_y)^k \left(\frac{g(x, y)}{|y - z|^{n+\sigma}} \right) dz \leq \\ &\leq c_2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{dz}{|y - z|^{n+2k+\sigma}} = c_2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{dz}{|y - z|^{n+s}}, \end{aligned}$$

where $c_2 = c_2(s, n, \beta)$ is another constant depending on s, n and β only. Therefore, with a similar computation as before, we now have that

$$|y|^{n+s}(-\Delta_y)^{\frac{s}{2}}g(x,y) < c_2 \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{|y|^{n+s}}{|y-z|^{n+s}}\,dz \le c_3,$$

with $c_3 = c_3(s,n,\beta)$ depending, again, on s, n and β only. This means that we finally achieve

$$(-\Delta_y)^{\frac{s}{2}}g(x,y) < \frac{c_3}{|y|^{n+s}},$$

which is the desired estimate. Hence, we can affirm that h decays less than $|y|^{-n-s}$ uniformly with respect to $x \in B$. Next, by Hölder's inequality, we obtain

$$\operatorname*{ess\,sup}_{x\in B}\left\{\int_{\mathbb{R}^n} h(x,y)\psi(y)dy\right\} \le \operatorname*{ess\,sup}_{x\in B}\big\{||h(x,\cdot)||_{L^q(\mathbb{R}^n)}\big\}||\psi||_{L^p(\mathbb{R}^n)} \le$$

$$\le c_4||\psi||_{L^p(\mathbb{R}^n)}, \tag{172}$$

where $c_4 = c_4(s,n,q,\beta)$ is a constant depending on s, n, q and β only. The existence of such a constant is due to the fact that h is smooth and, decaying as $|y|^{-n-s}$, belongs to $L^q(\mathbb{R}^n)$ since $|y|^{-n-s}$ does (the reason of the latter follows because, as usual, we have that $q(n+s) > n+s > n$). Thus, joining (170) and (172), we see that

$$\big|\big|I_\beta * \big((1-\vartheta)(-\Delta)^{\frac{s}{2}}\psi\big)\big|\big|_{L^\infty(B)} := \operatorname*{ess\,sup}_{x\in B}\big\{\big(I_\beta * \big((1-\vartheta)(-\Delta)^{\frac{s}{2}}\psi\big)\big)(x)\big\} =$$

$$= \operatorname*{ess\,sup}_{x\in B}\left\{\int_{\mathbb{R}^n} h(x,y)\psi(y)dy\right\} \le c_4||\psi||_{L^p(\mathbb{R}^n)},$$

proving (169) once one sets $c_0 = c_0(s,n,q,\beta) := c_4$.

Step 2: conclusion. To the scope of proving (168), by duality (being $1 < q < \infty$) it suffices to have that, for every $\psi \in \mathscr{S}(\mathbb{R}^n)$,

$$\big|\big|I_\beta * \big((1-\vartheta)(-\Delta)^{\frac{s}{2}}\psi\big)\big|\big|_{L^p(B_\rho(0))} \le c_5\rho^{\frac{n}{p}}||\psi||_{L^p(\mathbb{R}^n)} \tag{173}$$

holds for a certain constant $c_5 = c_5(s,n,q,\beta)$ depending on s, n, q and β only. This estimates follows by (169) since, using it, we get

$$\big|\big|I_\beta * \big((1-\vartheta)(-\Delta)^{\frac{s}{2}}\psi\big)\big|\big|_{L^p(B_\rho(0))} := \left(\int_{B_\rho(0)} \big|\big(I_\beta * \big((1-\vartheta)(-\Delta)^{\frac{s}{2}}\psi\big)\big)(x)\big|^p dx\right)^{\frac{1}{p}} \le$$

$$\le \left(\big|\big|I_\beta * \big((1-\vartheta)(-\Delta)^{\frac{s}{2}}\psi\big)\big|\big|^p_{L^\infty(B)}\int_{B_\rho(0)} dx\right)^{\frac{1}{p}} =$$

$$= \big|\big|I_\beta * \big((1-\vartheta)(-\Delta)^{\frac{s}{2}}\psi\big)\big|\big|_{L^\infty(B)}\Big(\frac{\omega_{n-1}}{n}\rho^n\Big)^{\frac{1}{p}} =$$

$$= \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{p}} \rho^{\frac{n}{p}} \big|\big| I_\beta * \big((1-\vartheta)(-\Delta)^{\frac{s}{2}}\psi\big)\big|\big|_{L^\infty(B)} \leq$$

$$\leq \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{p}} c_0 \rho^{\frac{n}{p}} ||\psi||_{L^p(\mathbb{R}^n)} = c_5 \rho^{\frac{n}{p}} ||\psi||_{L^p(\mathbb{R}^n)},$$

where we set $c_5 = c_5(s,n,q,\beta) := \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{p}} c_0$. This proves the estimate (173) and, consequently, also the one in (168).

□

Remark 35. *The previous result is valid even for $s = 0$, since such a situation does not create any problems in its proof. We reiterate that, as always, $(-\Delta)^0 u$ means just u (in fact, in view of Definition 9, one would have, let say heuristically, that*

$$(-\Delta)^0 u(x) = \mathscr{F}^{-1}\big(|\xi|^0 \mathscr{F}u(\xi)\big)(x) = \mathscr{F}^{-1}\big(\mathscr{F}u(\xi)\big)(x) = u(x)$$

holds).

We lastly prove Corollary 7 and, afterwards, the discussion will be concluded.

Proof of Corollary 7. The reasoning will contain similar considerations and techniques compared to the ones utilized when proving Proposition 7 and Proposition 8. We, in fact, suppose (without loss of generality, as previously noted) that $B_1(0) \subseteq \Omega$ and consider functions belonging to $C_0^\infty\big(B_1(0)\big)$, being able to do that thanks to Netrusov's density theorem. Next, we fix $\alpha > \alpha_{s,n}$, choose a number $\rho \in \big(0, \frac{1}{8}\big]$ (which will be later fixed), take a cut-off function $\vartheta \in C_0^\infty\big(B_1(0)\big)$ such that $\vartheta(x) \in [0,1]$ on $\mathbb{R}^n$ with, in particular, $\vartheta(x) \equiv 1$ in $B_{\frac{1}{2}}(0)$ and consider an arbitrary function $f \in C_0^\infty\big(B_\rho(0)\big)$ such that $||f||_{L^q(B_\rho(0))} = 1$. Just to fix ideas, the latter request can be satisfied by choosing a positive mollifier ψ supported on $B_\rho(0)$ which, by definition, is such that

$$\int_{B_\rho(0)} \psi(x)dx = 1$$

and, then, consider the function $\tilde{\psi}(x) := \psi(x)^{\frac{1}{q}}$, so that one gets

$$||\tilde{\psi}||_{L^q(B_\rho(0))} := \left(\int_{B_\rho(0)} |\tilde{\psi}(x)|^q dx\right)^{\frac{1}{q}} = \left(\int_{B_\rho(0)} \left|\psi(x)^{\frac{1}{q}}\right|^q dx\right)^{\frac{1}{q}} =$$

$$= \left(\int_{B_\rho(0)} \left[\psi(x)^{\frac{1}{q}}\right]^q dx\right)^{\frac{1}{q}} = \left(\int_{B_\rho(0)} \psi(x)dx\right)^{\frac{1}{q}} = 1.$$

We now proceed step by step.

Step 1: introduction of two auxiliary functions. Taking an element $\varepsilon > 0$ which makes the relation

$$\tilde{\alpha} := \frac{\alpha}{(1+\varepsilon)^p} > \alpha_{s,n}$$

satisfied, we define the functions

$$v(x) := \vartheta(x)\mathcal{I}_s f(x) = \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} \vartheta(x)(I_s * f)(x)$$

and

$$u(x) := \frac{v(x)}{1+\varepsilon},$$

where $\mathcal{I}_s f$ is exactly the Riesz potential of order s of f defined in the previous chapter. Now, with the help of Theorem 11 (by (133) and (134), in particular), we get

$$(-\Delta)^{\frac{s}{2}} v(x) = (-\Delta)^{\frac{s}{2}} \left[\vartheta(x)\mathcal{I}_s f(x)\right] = (-\Delta)^{\frac{s}{2}} \left[\left(1 + \vartheta(x) - 1\right)\mathcal{I}_s f(x)\right] =$$

$$= (-\Delta)^{\frac{s}{2}} \left[\mathcal{I}_s f(x) - \left(1 - \vartheta(x)\right)\mathcal{I}_s f(x)\right] =$$

$$= (-\Delta)^{\frac{s}{2}} \left[\mathcal{I}_s f(x)\right] - (-\Delta)^{\frac{s}{2}} \left[\left(1 - \vartheta(x)\right)\mathcal{I}_s f(x)\right] =$$

$$= (-\Delta)^{\frac{s}{2}} \left[\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} \left(|x|^{s-n} * f\right)(x)\right] - (-\Delta)^{\frac{s}{2}} \left[\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} \left(1 - \vartheta(x)\right)(I_s * f)(x)\right] =$$

$$= (-\Delta)^{\frac{s}{2}} (\mathcal{F}_s * f)(x) - \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} (-\Delta)^{\frac{s}{2}} \left[\left(1 - \vartheta(x)\right)(I_s * f)(x)\right] =$$

$$= f(x) - \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} (-\Delta)^{\frac{s}{2}} \left[\left(1 - \vartheta(x)\right)(I_s * f)(x)\right], \tag{174}$$

where we also used the linearity of the fractional Laplacian and the definition of $\mathcal{I}_s f$. Thus, being satisfied all the hypothesis of Lemma 14, we are able to affirm that there exists a constant $c = c(s, n, q, \beta)$ depending on s, n, q and β only such that

$$\left|\left|(-\Delta)^{\frac{s}{2}} \left((1-\vartheta)(I_\beta * f)\right)\right|\right|_{L^q(\mathbb{R}^n)} \leq c\rho^{\frac{n}{p}} ||f||_{L^q(B_\rho(0))} = c\rho^{\frac{n}{p}}.$$

Hence,

$$\left|\left|\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} (-\Delta)^{\frac{s}{2}} \left((1-\vartheta)(I_\beta * f)\right)\right|\right|_{L^q(\mathbb{R}^n)} =$$

$$= \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} \left|\left|(-\Delta)^{\frac{s}{2}} \left((1-\vartheta)(I_\beta * f)\right)\right|\right|_{L^q(\mathbb{R}^n)} \leq \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} c\rho^{\frac{n}{p}} = c_0 \rho^{\frac{n}{p}}, \tag{175}$$

where we clearly set

$$c_0 = c_0(s, n, q, \beta) := \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} c.$$

We shall now fix the radius ρ.

Step 2: a useful estimate. Now, we fix $\rho \in \left(0, \frac{1}{8}\right]$ such that the estimates

$$\frac{1 + c_0 \rho^{\frac{n}{p}}}{1+\varepsilon} \leq 1 \tag{176}$$

holds. Since

$$\frac{1 + c_0 \rho^{\frac{n}{p}}}{1+\varepsilon} \leq 1 \Longleftrightarrow 1 + c_0 \rho^{\frac{n}{p}} \leq 1+\varepsilon \Longleftrightarrow \rho^{\frac{n}{p}} \leq \frac{\varepsilon}{c_0} \Longleftrightarrow \rho \leq \left(\frac{\varepsilon}{c_0}\right)^{\frac{p}{n}} = \varepsilon^{\frac{p}{n}} c_0^{-\frac{p}{n}},$$

this can be done by just setting $\rho := \min\left\{\frac{1}{8}, \varepsilon^{\frac{p}{n}} c_0^{-\frac{p}{n}}\right\}$. Therefore, we obtain

$$\begin{aligned}
\left|\left|(-\Delta)^{\frac{s}{2}} u\right|\right|_{L^q(\mathbb{R}^n)} &= \left|\left|(-\Delta)^{\frac{s}{2}}\left(\frac{v}{1+\varepsilon}\right)\right|\right|_{L^q(\mathbb{R}^n)} = \frac{1}{1+\varepsilon}\left|\left|(-\Delta)^{\frac{s}{2}} v\right|\right|_{L^q(\mathbb{R}^n)} = \\
&= \frac{1}{1+\varepsilon}\left|\left| f - \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} (-\Delta)^{\frac{s}{2}}\left((1-\vartheta)(I_s * f)\right)\right|\right|_{L^q(\mathbb{R}^n)} \leq \\
&\leq \frac{1}{1+\varepsilon}\left(||f||_{L^q(\mathbb{R}^n)} + \left|\left|\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} (-\Delta)^{\frac{s}{2}}\left((1-\vartheta)(I_s * f)\right)\right|\right|_{L^q(\mathbb{R}^n)}\right) \leq \\
&\leq \frac{1}{1+\varepsilon}\left(1 + c_0 \rho^{\frac{n}{p}}\right) = \frac{1 + c_0 \rho^{\frac{n}{p}}}{1+\varepsilon} \leq 1
\end{aligned} \tag{177}$$

using the definitions of the functions u and v, (174), Minkowski's inequality, (175) and (176).

Step 3: conclusion. The estimate (177) just proven tells us that the function u satisfies the hypothesis of Theorem 12. Hence, using also the definition of $\tilde{\alpha}$ and the hypothesis on ϑ, we now have that

$$\begin{aligned}
\int_{B_1(0)} e^{\alpha |u(x)|^p} dx &= \int_{B_1(0)} e^{\alpha \left|\frac{v(x)}{1+\varepsilon}\right|^p} dx = \int_{B_1(0)} e^{\frac{\alpha}{(1+\varepsilon)^p} |v(x)|^p} dx = \\
&= \int_{B_1(0)} \exp\left(\tilde{\alpha}\left|\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)} \vartheta(x)(I_s * f)(x)\right|^p\right) dx = \\
&= \int_{B_1(0)} \exp\left(\tilde{\alpha}\left[\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)}\right]^p |\vartheta(x)(I_s * f)(x)|^p\right) dx \geq \\
&\geq \int_{B_\rho(0)} \exp\left(\tilde{\alpha}\left[\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)}\right]^p |(I_s * f)(x)|^p\right) dx = \\
&= \int_{B_\rho(0)} \exp\left(\tilde{\alpha}\left[\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s \pi^{\frac{n}{2}} \Gamma\left(\frac{s}{2}\right)}\right]^p \left|\frac{(I_s * f)(x)}{||f||_q}\right|^p\right) dx,
\end{aligned} \tag{178}$$

where $||f||_q$ denotes the $L^q(\mathbb{R}^n)$-norm of f. Since $\tilde{\alpha} > \alpha_{s,n}$, one has that

$$\tilde{\alpha}\left[\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s\pi^{\frac{n}{2}}\Gamma\left(\frac{s}{2}\right)}\right]^p > \alpha_{s,n}\left[\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s\pi^{\frac{n}{2}}\Gamma\left(\frac{s}{2}\right)}\right]^p = \frac{n}{\omega_{n-1}}\left[\frac{2^s\pi^{\frac{n}{2}}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)}\right]^p\left[\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s\pi^{\frac{n}{2}}\Gamma\left(\frac{s}{2}\right)}\right]^p = \frac{n}{\omega_{n-1}}.$$

Therefore,

$$\eta := \tilde{\alpha}\left[\frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s\pi^{\frac{n}{2}}\Gamma\left(\frac{s}{2}\right)}\right]^p$$

is strictly greater than $\frac{n}{\omega_{n-1}}$: consequently, Proposition 8 assures us that it is not possible to bound

$$\fint_{B_\rho(0)} e^{\eta\left|\frac{(I_s*f)(x)}{||f||_q}\right|^p}dx$$

by a constant depending on s and n only. In conclusion, by (178), we obtain

$$\rho^{-n}\fint_{B_1(0)} e^{\alpha|u(x)|^p}dx = \frac{\rho^{-n}}{|B_1(0)|}\int_{B_1(0)} e^{\alpha|u(x)|^p}dx \geq \frac{\rho^{-n}}{|B_1(0)|}\int_{B_\rho(0)} e^{\eta\left|\frac{(I_s*f)(x)}{||f||_q}\right|^p}dx =$$

$$= \rho^{-n}\frac{|B_\rho(0)|}{|B_1(0)|}\fint_{B_\rho(0)} e^{\eta\left|\frac{(I_s*f)(x)}{||f||_q}\right|^p}dx = \rho^{-n}\,\frac{\omega_{n-1}}{n}\,\rho^n\,\frac{n}{\omega_{n-1}}\fint_{B_\rho(0)} e^{\eta\left|\frac{(I_s*f)(x)}{||f||_q}\right|^p}dx =$$

$$= \fint_{B_\rho(0)} e^{\eta\left|\frac{(I_s*f)(x)}{||f||_q}\right|^p}dx,$$

which means that neither

$$\rho^{-n}\fint_{B_1(0)} e^{\alpha|u(x)|^p}dx$$

(and, obviously,

$$\fint_{B_1(0)} e^{\alpha|u(x)|^p}dx$$

because, since ρ has been fixed, ρ^{-n} is now a constant) can be bounded by a constant depending on s and n only. This proves the sharpness of $\alpha_{s,n}$.

$\square$

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