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Prescribing Gaussian and geodesic curvatures on a compact surface with boundary

Laurea Magistrale in Matematica

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Contents

1	Introduction	4
2	Obstructions to Existence	7
2.1	A Topological Obstruction	8
2.2	A Pohozaev type identity and a Kazdan-Warner identity	10
3	Variational Study of the Energy Functional	13
3.1	Reduction to Constant $\tilde{K}$ and $\tilde{h} = 0$	13
3.2	Existence results by minimization	15
3.3	Min-max structure	18
4	Blow-up analysis	25
4.1	Limit problems	25
4.1.1	Solutions of the problem in the plane	25
4.1.2	Solutions of the problem in the half-plane and their Morse index	25
4.1.3	Solution of the problem in the disk	32
4.2	The case of negative Gaussian curvature	34
4.2.1	Localization of singular points and preliminary estimates . . .	35
4.2.2	Blow-up with Unbounded Mass	40
4.2.3	Blow-up with Bounded Mass or Morse Index	45
4.3	The case of the disk	55
4.3.1	Single-Point Blow-up and Asymptotic Profile	58
4.3.2	Estimate of the remainder term	68
4.3.3	The criticality condition at the blow-up point	74
5	Constructing Solutions via Ljapunov-Schmidt Reduction	83
5.1	The inverse problem on the disk	83
5.1.1	Ansatz and error estimates	86
5.1.2	Invertibility of the Linearized Operator	94
5.1.3	The nonlinear projected problem	99

5.1.4	Estimates on the projections	102
5.1.5	The finite dimensional reduction	115
5.2	Construction of solution on a generic domain	118
5.2.1	Ansatz and Error Estimates	118
5.2.2	Linear Theory	129
5.2.3	The Non-Linear Problem	161
5.2.4	Finite-Dimensional Reduction	166
5.2.5	The Existence Theorem	171
A	Useful Tools and Computations	176
A.1	Test functions	176
A.2	An Harnack type inequality	180
A.3	Integrals and Taylor Expansions for the disk case	182

Abstract

We study the problem of prescribing Gaussian and geodesic curvature on a surface with boundary, which leads to a nonlinear elliptic PDE with a nonlinear Neumann boundary condition. We first investigate obstructions to existence, develop a variational framework, and analyze the blow-up behavior of solutions. Finally, in the case of a bounded domain in $\mathbb{R}^2$, we construct new blowing-up solutions via a Lyapunov–Schmidt reduction.

Chapter 1

Introduction

Let Σ be a compact surface with boundary $\partial\Sigma$, endowed with a Riemannian metric $\tilde{g}$ with Gaussian curvature $\tilde{K}$ and geodesic curvature $\tilde{h}$ along the boundary. Given functions $K : \Sigma \rightarrow \mathbb{R}$ and $h : \partial\Sigma \rightarrow \mathbb{R}$, we look for a conformal metric

$$g = e^u \tilde{g}$$

whose Gaussian curvature is K and whose geodesic curvature along $\partial\Sigma$ is h . The conformal factor u satisfies

$$\begin{cases} -\Delta u + 2\tilde{K}(x) = 2K(x)e^u, & \text{in } \Sigma, \\ \frac{\partial u}{\partial n} + 2\tilde{h}(x) = 2h(x)e^{u/2}, & \text{on } \partial\Sigma, \end{cases} \quad (1.1)$$

where n denotes the outward unit normal.

Two particular cases will be considered throughout the thesis. The first is the unit disk $\mathbb{D}$ equipped with the Euclidean metric. Since in this case $\tilde{K} \equiv 0$ and $\tilde{h} \equiv 1$, (1.1) becomes

$$\begin{cases} -\Delta u = 2K(x)e^u, & \text{in } \mathbb{D}, \\ \frac{\partial u}{\partial n} + 2 = 2h(x)e^{u/2}, & \text{on } \partial\mathbb{D}. \end{cases} \quad (1.2)$$

The second case is obtained by choosing the background metric so that $\tilde{K}$ is constant and $\partial\Sigma$ is geodesic, namely $\tilde{h} \equiv 0$. As shown in Proposition 3.1, this normalization can always be achieved, and the problem takes the form

$$\begin{cases} -\Delta u + 2\tilde{K} = 2K(x)e^u, & \text{in } \Sigma, \\ \frac{\partial u}{\partial n} = 2h(x)e^{u/2}, & \text{on } \partial\Sigma. \end{cases} \quad (1.3)$$

Equation (1.3) is the setting of the variational analysis in Chapter 3, while the flat problem (1.2) is the model considered in the blow-up analysis of Chapter 4 and in the construction of solutions in Chapter 5.

A first obstruction to solvability is obtained by integrating (1.1) and using the Gauss–Bonnet Theorem for both g and $\tilde{g}$:

$$\int_{\Sigma} \tilde{K} + \int_{\partial\Sigma} \tilde{h} = 2\pi\chi(\Sigma) = \int_{\Sigma} K e^u + \int_{\partial\Sigma} h e^{u/2}.$$

Thus the topology of Σ imposes a compatibility condition on the prescribed curvatures. Chapter 2 develops this point of view further, proving a topological obstruction and deriving a Pohozaev-type identity, which in the disk case yields a Kazdan–Warner identity. Together with the variational theory of Chapter 3 and the blow-up analysis of Chapter 4, with the exception of Sections 2.2 and 4.3, where we followed [20], these results follow closely the approach of López Soriano, Malchiodi and Ruiz [29].

Chapter 3 is devoted to the variational structure of (1.3). The associated energy functional is studied first in the coercive regime, where existence follows by direct minimization. When coercivity fails, a min-max construction is developed by combining the monotonicity argument of Struwe, in the form introduced by Jeanjean, with a deformation argument of Fang and Ghoussoub.

The failure of compactness of the corresponding solutions is studied in Chapter 4. We first analyze the limiting problems in the plane, the half-plane and the disk, including the relation between the possible blow-up profiles and their Morse index. We then consider the case of negative Gaussian curvature on a general compact surface and describe the possible concentration phenomena under suitable boundedness assumptions. In the case of the disk, Section 4.3 gives a more precise description of the blow-up: under the assumptions of Theorem 4.21, a blowing-up sequence concentrates at a single point $p \in \partial\mathbb{D}$, admits a sharp asymptotic profile, and the concentration point satisfies the criticality condition for

$$\varphi(p) := h(p) + \sqrt{h(p)^2 + K(p)}.$$

This part follows the analysis of Jevnikar, López Soriano, Medina and Ruiz [20].

Chapter 5 addresses the corresponding existence problem for blowing-up solutions. Section 5.1 considers the disk and, under suitable non-degeneracy assumptions at a critical point of φ , constructs a family of solutions concentrating at that point by means of a Lyapunov–Schmidt reduction, following the construction of Battaglia, Medina and Pistoia [4].

In Section 5.2 we consider an analogous construction on a smooth bounded domain $\Omega \subset \mathbb{R}^2$, allowing for m distinct concentration points on $\partial\Omega$. This is obtained by adapting the construction of Dávila, del Pino and Musso in [10], but in our case the boundary condition contains a nonlocal term depending on the global interior and boundary mass of the solution, due to the compatibility of the problem. The finite-dimensional reduction developed in this section leads to a system of equations for the locations of the concentration points and provides the existence of the resulting concentrating solutions.

Chapter 2

Obstructions to Existence

We begin by deriving necessary conditions to the existence of solutions, including some obstructions for the existence of solutions to problem (1.1), which in some sense complement the existence results given in Theorems 3.4, 3.5, and 3.14. Moreover, we establish a Pohozaev-type identity, from which we derive a Kazdan–Warner identity in the case of the disk, this Pohozaev-type identity will also be used in subsequent computations.

The first tool to derive necessary conditions is the Gauss–Bonnet theorem, applied both to the metric g and to the metric $\tilde{g}$ (after integrating and summing the equations), which yields

$$\int_{\Sigma} K + \oint_{\partial\Sigma} h = 2\pi\chi(\Sigma) = \int_{\Sigma} K e^u + \oint_{\partial\Sigma} h e^{u/2}. \quad (2.1)$$

For example, the first equality implies that a disk with Gaussian curvature -1 and constant geodesic curvature h exists only when $h > 1$. In [21], the problem of prescribing constant curvatures K , h_1 , and h_2 on an annulus is addressed. In this case, all solutions are known explicitly; in particular, the following result is proved (see [21], [Corollary 1, Section 3]):

Theorem 2.1. *Let Σ be an annulus with $K \equiv -1$ and constant geodesic curvatures h_1 and h_2 on its two boundary components. Then:*

- i) either $h_1 + h_2 > 0$ and both $h_i < 1$;*
- ii) or $h_1 + h_2 < 0$ and some $h_i > 1$;*
- iii) or $h_1 = 1$, $h_2 = -1$, or vice versa.*

As can be seen, Theorem 3.5 corresponds to case (i), whereas Theorem 3.14 is related to case (ii).

2.1 A Topological Obstruction

The assumptions of Theorems 3.4, 3.5, and 3.14 are also natural in view of the following result:

Theorem 2.2. *Let Σ be a compact surface with boundary $\partial\Sigma$, and denote by h its geodesic curvature. Assume that*

$$h(x) > \sqrt{|K^-(p)|} \quad \text{for all } x \in \partial\Sigma, \quad p \in \Sigma.$$

Then Σ is homeomorphic to a disc.

Proof. Assume by contradiction that Σ is not a topological disc, and consider the following two cases:

Case 1: $\partial\Sigma$ is not connected. Let Λ_0 be a connected component of $\partial\Sigma$, and let γ be a curve minimizing the distance between Λ_0 and $\partial\Sigma \setminus \Lambda_0$. Then the length of γ solves the problem

$$L = \inf \{ \text{Length}(\gamma) : \gamma : [0, 1] \rightarrow \Sigma, \gamma(0) \in \Lambda_0, \gamma(1) \in \partial\Sigma \setminus \Lambda_0 \}.$$

We may assume that γ is parametrized by arc-length, and set $p = \gamma(0)$, $q = \gamma(1)$. The curve γ exists since the above infimum is attained, by the Ascoli–Arzelà theorem.

Take a neighborhood $O \subset \Sigma$ of $\gamma([0, L])$, and define $\Lambda_t \subset O$ to be the parallel curve to Λ_0 at distance $t \in [0, L]$, namely

$$\Lambda_t = \{x \in O : d(x, \Lambda_0) = t\},$$

where d denotes the geodesic distance in Σ . Observe that Λ_t is the image of

$$\gamma_t(s) = \exp_{\phi(s)}(tn(s)),$$

where ϕ is a local parametrization of Λ_0 around p , and $n(s)$ is the outward unit normal to Λ_0 .

We now show that γ_t is a regular curve for each $t \in [0, L]$, provided O is sufficiently small. Suppose by contradiction that there exists $\bar{t}$ such that $\dot{\gamma}_{\bar{t}}(0) = 0$. Let λ_0 be the first eigenvalue of the operator $J = \partial_t^2 + K$ with Neumann boundary conditions. Since γ minimizes the distance between two connected

components of $\partial\Sigma$, we have $\lambda_0 > 0$. Let ϕ_0 be the corresponding (positive) eigenfunction.

Define $\psi(t) = g(\dot{\gamma}_t(0), v(t))$, where v is the parallel unit normal field along γ with $v(0) = \phi'(s_0)$. Since $\gamma_t(s) = \exp_{\phi(s)}(tn(s))$, the field $t \mapsto \dot{\gamma}_t(0)$ is a Jacobi field along γ , so ψ solves $J(\psi) = \psi'' + K\psi = 0$ with $\psi(0) = 1$. Its derivative at $t = 0$ is not zero: by the standard formula for the variation field of a normal exponential map along a curve, $\psi'(0) = -\langle \nabla_{\phi'(s_0)} n, \phi'(s_0) \rangle = -h(p)$, since Λ_0 has geodesic curvature $h(p)$ at p with respect to n . By assumption, $\psi(\bar{t}) = 0$.

This means that $\bar{t}$ is a focal point of Λ_0 along γ . But γ realizes the distance between Λ_0 and $\partial\Sigma \setminus \Lambda_0$, meeting each orthogonally; by the classical fact that a geodesic realizing the distance between two submanifolds contains no focal point of either one in the open interval $(0, L)$ (see e.g. [33], Appendix A), this is a contradiction.

Case 2: $\partial\Sigma$ is connected.

Let $\tilde{\Sigma}$ be the compact surface obtained by identifying $\partial\Sigma$ to a single point. If Σ is not a topological disc, then $\tilde{\Sigma}$ is not homeomorphic to a sphere, hence it is not simply connected. Consider the following minimization problem:

$$\inf \{ \text{Length}(\gamma) : \gamma : [0, L] \rightarrow \Sigma, \{ \gamma(0), \gamma(L) \} \subset \partial\Sigma, [\tilde{\gamma}] \text{ non-trivial} \},$$

where $\tilde{\gamma} : [0, L] \rightarrow \tilde{\Sigma}$ denotes the curve obtained from γ by applying the identification, and $[\tilde{\gamma}]$ its homotopy class.

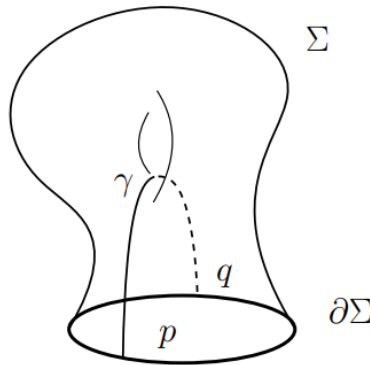


Figure 2.1: A curve minimizing the distance in Σ with non-trivial homotopy class in $\tilde{\Sigma}$.

Clearly, this infimum is attained at some γ (as in Figure 2.1, below), which is a local minimizer of the distance. We can now repeat the arguments of Case 1 and again reach a contradiction.

□

Remark 2.3. Let us point out that in case of a disk we can have $h(x) > \sqrt{|K^-(p)|}$ for all $x \in \partial\Sigma$, $p \in \Sigma$. For instance, in the Poincaré disk there are balls with constant geodesic curvature $h > 1$.

2.2 A Pohozaev type identity and a Kazdan-Warner identity

The following lemma gives a Pohozaev type identity, depending on an arbitrary field F .

Lemma 2.4. *Let u be a solution of*

$$-\Delta u + 2\tilde{K} = 2K(x)e^u, \quad \text{in } \Sigma.$$

Then, given any vector field $F : \Sigma \rightarrow T\Sigma$, there holds:

$$\begin{aligned} & \oint_{\partial\Sigma} [4K(x)e^u(F \cdot \nu) + 2(\nabla u \cdot \nu)(\nabla u \cdot F) - |\nabla u|^2 F \cdot \nu] \\ &= \int_{\Sigma} [4\tilde{K}\nabla u \cdot F + 4e^u(\nabla K \cdot F + K \nabla \cdot F) + 2DF(\nabla u, \nabla u) - \nabla \cdot F |\nabla u|^2]. \end{aligned}$$

Proof. We start from the following basic identity:

$$2\Delta u (\nabla u \cdot F) = \nabla \cdot \left(2(\nabla u \cdot F)\nabla u - |\nabla u|^2 F \right) - 2DF(\nabla u, \nabla u) + |\nabla u|^2 \nabla \cdot F, \quad (2.2)$$

where $DF(\nabla u, \nabla u) = \sum_{i,j} \partial_i F_j \partial_i u \partial_j u$.

Multiplying the equation $\Delta u + 2Ke^u = 2\tilde{K}$ by $2\nabla u \cdot F$ and integrating over Σ , we get

$$\int_{\Sigma} 2\Delta u (\nabla u \cdot F) = \int_{\Sigma} 4\tilde{K}(\nabla u \cdot F) - \int_{\Sigma} 4Ke^u(\nabla u \cdot F).$$

Using the divergence theorem in (2.2):

$$\int_{\Sigma} 2\Delta u (\nabla u \cdot F) = \int_{\partial\Sigma} \left[2(\nabla u \cdot F)(\nabla u \cdot \nu) - |\nabla u|^2 (F \cdot \nu) \right] + \int_{\Sigma} \left[-2DF(\nabla u, \nabla u) + |\nabla u|^2 \nabla \cdot F \right].$$

For the nonlinear term, again by the divergence theorem:

$$4 \int_{\Sigma} Ke^u(\nabla u \cdot F) = 4 \left[\int_{\partial\Sigma} Ke^u(F \cdot \nu) - \int_{\Sigma} e^u(\nabla K \cdot F + K \nabla \cdot F) \right].$$

Combining all these expressions, we obtain the final identity. □

From this we can obtain the following Kazdan-Warner identity.

Proposition 2.5. *Let u is a solution of (1.2). Then*

$$\int_{\mathbb{D}} e^u \nabla K \cdot F = 4 \int_{\partial \mathbb{D}} h_\tau e^{u/2} y,$$

where $F(x, y) := (1 - x^2 + y^2, -2xy)$, or in complex notation, $F(z) = 1 - z^2$.

Proof. Consider

$$f_\lambda(z) := \frac{\lambda + z}{1 + \lambda z}$$

with $\lambda \in (-1, 1)$. We define $v_\lambda(z) = u(f_\lambda(z)) + 2 \log(|f'_\lambda(z)|)$. By direct computation, using the notation $z = x + iy$, we can see that

$$\left. \frac{d}{d\lambda} v_\lambda(z) \right|_{\lambda=0} = \nabla u \cdot F - 4x.$$

Observe that on $\partial \mathbb{D}$, F is a tangential vector field, and $F \cdot \tau = -2y$, where $\tau(x, y) = (-y, x)$. Moreover, since F is holomorphic, the Cauchy-Riemann conditions yield

$$DF(\nabla u, \nabla u) - \nabla \cdot F \frac{|\nabla u|^2}{2} = 0.$$

Then, inserting in the Pohozaev-type identity of the previous Lemma, we obtain

$$\int_{\partial \mathbb{D}} (2he^{u/2} - 2)(-2y)u_\tau = \int_{\mathbb{D}} 2e^u (\nabla K \cdot F - 4xK). \quad (2.3)$$

We now multiply (1.2) by $4x$ and integrate to obtain

$$\begin{aligned} 8 \int_{\mathbb{D}} K e^u x &= 4 \int_{\mathbb{D}} u_x - 4 \int_{\partial \mathbb{D}} x \nabla u \cdot \nu = 4 \int_{\mathbb{D}} u_x - 4 \int_{\partial \mathbb{D}} x (2he^{u/2} - 2) \\ &= 4 \int_{\mathbb{D}} u_x + 4 \int_{\partial \mathbb{D}} y (2h_\tau e^{u/2} + h e^{u/2} u_\tau). \end{aligned} \quad (2.4)$$

Observe also that, integrating by parts,

$$\int_{\partial \mathbb{D}} y u_\tau = - \int_{\partial \mathbb{D}} x u = - \int_{\mathbb{D}} u_x. \quad (2.5)$$

Putting together (2.3), (2.4) and (2.5) we conclude. $\square$

Remark 2.6. By symmetry, exchanging the roles of the variables x and y and reversing the orientation of the tangential vector τ , one obtains an analogous identity. More precisely, let

$$\tilde{F}(x, y) := (1 - y^2 + x^2, -2xy),$$

which corresponds in complex notation to $\tilde{F}(z) = i(1 + z^2)$. Then, if u is a solution of (1.2), it holds

$$\int_{\mathbb{D}} e^u \nabla K \cdot \tilde{F} = -4 \int_{\partial \mathbb{D}} h_\tau e^{u/2} x.$$

Remark 2.7. Let us observe that the Kazdan-Warner identity given in Proposition 2.5 can be written as

$$\int_{\mathbb{D}} e^u (1 + x^2 + y^2)^2 \nabla K \cdot \nabla T = -8 \int_{\partial \mathbb{D}} h_\tau T_\tau e^{u/2},$$

where $T(x, y) = \frac{x}{1+x^2+y^2}$. As a consequence, if $\nabla K \cdot \nabla T$, $h_\tau T_\tau$ are both positive (or negative), then (1.2) does not admit any solution.

Chapter 3

Variational Study of the Energy Functional

3.1 Reduction to Constant $\tilde{K}$ and $\tilde{h} = 0$

The first result of the section let us show that we can always prescribe zero geodesic curvature and constant Gaussian curvature.

Proposition 3.1. *Equation 1.1 is always solvable if $h = 0$ and $K = \text{sgn}\chi(\Sigma)$. The solution is unique if $K = -1$, unique up to a constant if $K = 0$, and unique up to a Möbius transformations if $K = 1$.*

Proof. • Case $\chi(\Sigma) < 0$: We need to study the energy functional

$$\begin{aligned} J(u) &= \int_{\Sigma} \left(\frac{1}{2} |\nabla u|^2 + 2\tilde{K}u + 2e^u \right) + \oint_{\partial\Sigma} 2\tilde{h}u \\ &= \int_{\Sigma} \left(\frac{1}{2} |\nabla u|^2 + 2\tilde{K} \left(u - \oint_{\Sigma} u \right) + e^{\oint_{\Sigma} u} 2e^{u - \oint_{\Sigma} u} \right) \\ &\quad + \oint_{\partial\Sigma} 2\tilde{h}(u - \oint_{\Sigma} u) + 2 \oint_{\Sigma} u \left(\int_{\Sigma} \tilde{K} + \int_{\partial\Sigma} \tilde{h} \right). \end{aligned}$$

Let $v = u - \oint_{\Sigma} u$. By the continuous embedding $H^1(\Sigma) \hookrightarrow L^2(\partial\Sigma)$, Hölder's inequality and Poincaré's inequality, we have

$$\begin{aligned} \left| \int_{\Sigma} 2\tilde{K}v \right| &\leq 2 \|\tilde{K}\|_{L^2(\Sigma)} \|v\|_{L^2(\Sigma)} \leq C \|\nabla v\|_{L^2(\Sigma)}, \\ \left| \oint_{\partial\Sigma} 2\tilde{h}v \right| &\leq 2 \|\tilde{h}\|_{L^2(\partial\Sigma)} \|v\|_{L^2(\partial\Sigma)} \leq C \|v\|_{H^1(\Sigma)} \leq C \|\nabla v\|_{L^2(\Sigma)}. \end{aligned}$$

Moreover, by Young's inequality, we have

$$C \|\nabla v\|_{L^2(\Sigma)} \leq \frac{1}{2} \left(\frac{1}{2} \|\nabla v\|_{L^2(\Sigma)}^2 + 2C^2 \right). \quad (3.1)$$

Furthermore, by Jensen's inequality we know that

$$\oint_{\Sigma} e^v \geq 1 \quad (3.2)$$

With these estimates in hand and recalling that by the Gauss-Bonnet theorem

$$\int_{\Sigma} \tilde{K} + \int_{\partial\Sigma} \tilde{h} = 2\pi\chi(\Sigma),$$

we have the coercivity of the functional:

$$J(u) \geq \frac{1}{4} \|\nabla u\|_{L^2}^2 - C + 4\pi\chi(\Sigma) \oint_{\Sigma} u + 2|\Sigma|e \oint_{\Sigma} u \rightarrow +\infty$$

when $\|u\|_{H^1(\Sigma)} \rightarrow +\infty$.

We now prove that J is weakly lower semicontinuous: Let $u_n \rightharpoonup u$ in $H^1(\Sigma)$, then $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Sigma)$ and $u_n \rightharpoonup u$ in $L^2(\Sigma)$, so for the lower semicontinuity of the norm and since the terms with $\tilde{K}$ and $\tilde{h}$ are in $L^2(\Sigma)^*$ we have

$$\begin{aligned} \|\nabla u\|_{L^2(\Sigma)} &\leq \liminf_{n \rightarrow \infty} \|\nabla u_n\|_{L^2(\Sigma)} \\ \int_{\Sigma} 2\tilde{K}u &= \lim_{n \rightarrow \infty} \int_{\Sigma} 2\tilde{K}u_n \\ \oint_{\partial\Sigma} 2\tilde{h}u &= \lim_{n \rightarrow \infty} \oint_{\partial\Sigma} 2\tilde{h}u_n \end{aligned}$$

moreover, up to subsequences, $u_n \rightarrow u$ a.e. and so by Fatou's lemma

$$\int_{\Sigma} e^u \leq \liminf_{n \rightarrow \infty} \int_{\Sigma} e^{u_n}.$$

Thus

$$J(u) \leq \liminf_{n \rightarrow \infty} J(u_n)$$

Since the functional J is weakly lower semicontinuous and coercive it attains its minimum. Uniqueness follows from the strict convexity of the functional J .

- Case $\chi(\Sigma) = 0$: We are looking for a solution to

$$\begin{cases} -\Delta u + 2\tilde{K} = 0, & \text{in } \Sigma, \\ \frac{\partial u}{\partial n} + 2\tilde{h} = 0, & \text{on } \partial\Sigma, \end{cases}$$

which is equivalent to finding $u \in H^1(\Sigma)$ such that

$$\int_{\Sigma} \nabla u \cdot \nabla \phi \, dV = \int_{\Sigma} 2\tilde{K}\phi \, dV + \int_{\partial\Sigma} 2\tilde{h}\phi \, dS \quad \forall \phi \in H^1(\Sigma).$$

We consider the Hilbert space

$$\mathcal{H}^1(\Sigma) := \left\{ u \in H^1(\Sigma) \mid \int_{\Sigma} u \, dV = 0 \right\}.$$

On this space, the bilinear form $\int_{\Sigma} \nabla u \cdot \nabla \phi \, dV$ defines an inner product equivalent to the standard H^1 -inner product. Moreover, the right-hand side defines a continuous linear functional on $\mathcal{H}^1(\Sigma)$. Hence, by the Lax–Milgram theorem, there exists a unique solution $u \in \mathcal{H}^1(\Sigma)$ (so unique up to additive constant in $H^1(\Sigma)$).

- Case $\chi(\Sigma) > 0$: The only case is $\chi(\Sigma) = 1$, then Σ is topologically a disk. By the Uniformization Theorem, Σ is conformally equivalent to the hemisphere so the statement follows. □

Thanks to the above result, we can assume that our initial metric has constant Gaussian curvature and that $\partial\Sigma$ is geodesic. Hence our problem reduces to the study of the functional

$$I(u) = \int_{\Sigma} \left(\frac{1}{2} |\nabla u|^2 + 2\tilde{K}u - 2K(x)e^u \right) - 4 \oint_{\partial\Sigma} h(x)e^{u/2}. \quad (3.3)$$

3.2 Existence results by minimization

We start by defining the quantity that appear in the existence theorems:

$$\mathfrak{D}(x) := \frac{h(x)}{\sqrt{|K(x)|}}. \quad (3.4)$$

The following estimate will be crucial in the variational study of I .

Lemma 3.2. *Define $\bar{\mathfrak{D}} = \max\{\mathfrak{D}^+(x) : x \in \partial\Sigma\}$. For any $\varepsilon > 0$ there exists $C > 0$ such that:*

$$4 \oint_{\partial\Sigma} h e^{u/2} \leq (\bar{\mathfrak{D}} + \varepsilon) \left(\int_{\Sigma} \frac{1}{2} |\nabla u|^2 + 2|K|e^u \right) + C.$$

Proof. Take a finite partition of unity $\{\phi_j\}_{j=1}^M$ of $\partial\Sigma$ such that for all j one has $\text{diam}(\text{supp } \phi_j) < \delta$ for some fixed $\delta > 0$. We denote by Σ_j the support of ϕ_j . Take

a smooth vector field N in Σ such that $N(x) = \nu(x)$ on the boundary, $|N(x)| \leq 1$. Then, we use the divergence theorem to obtain:

$$\begin{aligned} \oint_{\partial\Sigma} \phi_j e^{u/2} &= \oint_{\partial\Sigma} \phi_j e^{u/2} N(x) \cdot \nu(x) \\ &= \int_{\Sigma} e^{u/2} \left[\nabla \phi_j \cdot N + \phi_j \operatorname{div} N + \frac{1}{2} \phi_j \nabla u \cdot N \right] \\ &\leq C \int_{\Sigma} e^{u/2} + \frac{1}{2} \int_{\Sigma} e^{u/2} |\nabla u| \phi_j. \end{aligned} \quad (3.5)$$

Let us set $\bar{h}_j = \sup_{x \in \Sigma_j} \{h^+(x)\}$. If the diameter of Σ_j is small enough, one has that $\bar{h}_j \leq (\bar{\mathfrak{D}} + \varepsilon) \sqrt{|K(x)|}$ for any $x \in \Sigma_j$ (continuity of $\mathfrak{D}^+$). Then

$$\begin{aligned} 4 \oint_{\partial\Sigma} h e^{u/2} &= 4 \sum_{j=1}^M \oint_{\partial\Sigma} \phi_j h e^{u/2} \leq 4 \sum_{j=1}^M \bar{h}_j \oint_{\partial\Sigma} \phi_j e^{u/2} \\ &\leq C \int_{\Sigma} e^{u/2} + (\bar{\mathfrak{D}} + \varepsilon) \sum_{j=1}^M 2 \int_{\Sigma} \sqrt{|K|} \phi_j e^{u/2} |\nabla u| \\ &\leq C \int_{\Sigma} e^{u/2} + (\bar{\mathfrak{D}} + \varepsilon) \left(\sum_{j=1}^M 2 \int_{\Sigma} |K| \phi_j e^u + \frac{1}{2} \int_{\Sigma} \phi_j |\nabla u|^2 \right) \\ &= C \int_{\Sigma} e^{u/2} + (\bar{\mathfrak{D}} + \varepsilon) \left(2 \int_{\Sigma} |K| e^u + \frac{1}{2} \int_{\Sigma} |\nabla u|^2 \right). \end{aligned}$$

We conclude by observing that $\int_{\Sigma} e^{u/2} \leq \varepsilon \int_{\Sigma} e^u + C$ and renaming ε appropriately. $\square$

Remark 3.3. Under the hypothesis of the Moser–Trudinger inequality on Σ , summing over all j in (3.5) and applying Young’s inequality as in the previous proof, we obtain that the family

$$\{e^{u|\partial\Sigma} : u \in H^1(\Sigma), \|u\|_{H^1(\Sigma)} \leq C\}$$

is uniformly integrable on $\partial\Sigma$.

The first existence result regard the case $\tilde{K} < 0$:

Theorem 3.4. *Assume that $\tilde{K} < 0$. Let K, h be continuous functions such that $K < 0$ and $\mathfrak{D}(p) < 1$ for all $p \in \partial\Sigma$. Then I attains its infimum and hence problem 1.1 admits a solution. If moreover $h \leq 0$, then the solution is unique.*

Proof. Recalling the notation from the previous Lemma we have $\bar{\mathfrak{D}} < 1$, so we can choose ε with $\bar{\mathfrak{D}} + \varepsilon < 1 - \varepsilon$. By inserting the inequality of the Lemma 3.2 into the

definition of I and using Jensen inequality, we obtain coercivity:

$$\begin{aligned} I(u) &\geq \int_{\Sigma} \left(\frac{\varepsilon}{2} |\nabla u|^2 + 2\varepsilon |K| e^u + 2\tilde{K}u \right) - C \\ &\geq \int_{\Sigma} \frac{\varepsilon}{2} |\nabla u|^2 + 2\varepsilon |\Sigma| \min_{\Sigma} |K| e^{\int_{\Sigma} u} + 2|\Sigma| \tilde{K} \int_{\Sigma} u - C \xrightarrow{\|u\|_{H^1} \rightarrow \infty} +\infty. \end{aligned}$$

As shown in Proposition 3.1, the terms in the integral over Σ are weakly lower semicontinuous. We now prove that the boundary contribution enjoys the same property: Let $u_n \rightharpoonup u$ in $H^1(\Sigma)$ then, up to subsequences, $u_n \rightarrow u$ a.e in $\partial\Sigma$. and by the Remark 3.3 e^{u_n} is uniformly integrable, hence by Vitali's convergence theorem

$$\oint_{\partial\Sigma} e^{u_n} \longrightarrow \oint_{\partial\Sigma} e^u$$

Therefore, I attains its minimum. If $h \leq 0$, the functional is strictly convex, and thus the solution is unique. $\square$

We now consider the case $\tilde{K} = 0$.

Theorem 3.5. *Assume that $\tilde{K} = 0$. Let K, h be continuous functions such that $K < 0$ and*

1. $\mathfrak{D}(p) < 1$ for all $p \in \partial\Sigma$;
2. $\oint_{\partial\Sigma} h(x) > 0$.

Then I attains its infimum and hence problem (1.1) admits a solution.

Proof. By Lemma 3.2 we obtain

$$I(u) \geq \int_{\Sigma} \left(\frac{\varepsilon}{2} |\nabla u|^2 + 2\varepsilon |K| e^u \right) - C. \quad (3.6)$$

This implies I is bounded from below, but we do not have coercivity in this case. First, let us show that $\inf I < 0$. Indeed, take $u = -c$, with $c > 0$. Then:

$$I(-c) = e^{-\frac{c}{2}} \left(e^{-\frac{c}{2}} \int_{\Sigma} 2|K| - 4 \oint_{\partial\Sigma} h \right)$$

which is negative for large c , since $\oint_{\partial\Sigma} h > 0$.

Take now a minimizing sequence u_n for I . By (3.6), $\int_{\Sigma} |\nabla u_n|^2$ is bounded. We conclude if we show that $\oint_{\Sigma} u_n$ is bounded. Up to a subsequence, we can assume that

$$u_n - \oint_{\Sigma} u_n \rightharpoonup u_0 \text{ in } H^1(\Sigma).$$

Then, by Moser-Trudinger inequality,

$$e^{u_n - f_\Sigma u_n} \rightarrow e^{u_0} \text{ in } L^1(\Sigma), \quad e^{(u_n - f_\Sigma u_n)/2} \rightarrow e^{u_0/2} \text{ in } L^1(\partial\Sigma).$$

If now $f_\Sigma u_n \rightarrow +\infty$, then

$$\int_\Sigma |K| e^{u_n} = e^{f_\Sigma u_n} \int_\Sigma |K| e^{u_n - f_\Sigma u_n} \rightarrow +\infty,$$

contradicting (3.6).

So, we now consider the remaining case $f_\Sigma u_n \rightarrow -\infty$. In such a case,

$$\oint_{\partial\Sigma} h e^{u_n/2} = e^{\frac{f_\Sigma u_n}{2}} \oint_{\partial\Sigma} h e^{(u_n - f_\Sigma u_n)/2} \rightarrow 0.$$

By the definition of I we conclude that

$$\liminf_{n \rightarrow +\infty} I(u_n) \geq 0,$$

which is a contradiction. □

3.3 Min-max structure

In this section we prove the Theorem 3.14, assuming the blow-up analysis that will be carried out in the next chapter.

Lemma 3.6. *If $K < 0$ and $\mathfrak{D}(p) > 1$ for some $p \in \partial\Sigma$, then there exists a sequence u_n such that $I(u_n) \rightarrow -\infty$ and $\oint_{\partial\Sigma} e^{u_n/2} \rightarrow +\infty$.*

Proof. The proof is given in section 1 of the Appendix A. □

Lemma 3.7. *If $\tilde{K} = 0$ and $\oint_{\partial\Sigma} h < 0$, then there exists $\varepsilon > 0$, $\delta > 0$ such that*

$$I(u) > \varepsilon \quad \forall u \in H^1(\Sigma) \quad \text{with} \quad \oint_{\partial\Sigma} e^{u/2} = \delta.$$

Proof. First we claim that for any $\delta > 0$, the infimum:

$$\alpha_\delta = \inf \left\{ I(u) : u \in H^1(\Sigma) \text{ with } \oint_{\partial\Sigma} e^{u/2} = \delta \right\}$$

is attained. Recall the definition of I given in (3.3); this implies that $\alpha_\delta \neq -\infty$. Moreover, for a minimizing sequence u_n the integral $\int_\Sigma |\nabla u_n|^2$ must be bounded, otherwise $I(u_n) \rightarrow +\infty$. Up to a subsequence,

$$u_n - \oint_\Sigma u_n \rightharpoonup u_0 \text{ in } H^1(\Sigma),$$

which implies that

$$e^{u_n - f_\Sigma u_n} \rightarrow e^{u_0} \text{ in } L^1(\Sigma), \quad e^{(u_n - f_\Sigma u_n)/2} \rightarrow e^{u_0/2} \text{ in } L^1(\partial\Sigma).$$

Moreover,

$$0 < \delta = \oint_{\partial\Sigma} e^{u/2} = e^{\frac{f_\Sigma u_n}{2}} \oint_{\partial\Sigma} h e^{(u_n - f_\Sigma u_n)/2},$$

which implies that $f_\Sigma u_n$ is bounded. Hence u_n is bounded in $H^1(\Sigma)$ and the conclusion of the claim follows from Banach-Alaoglu Theorem and the lower semicontinuity of the functional.

By contradiction, assume that there exists $u_n \in H^1(\Sigma)$ such that:

$$\liminf_n I(u_n) \leq 0, \quad \oint_{\partial\Sigma} e^{u_n/2} \rightarrow 0.$$

By the definition of I it readily follows that $\int_\Sigma |\nabla u_n|^2 \rightarrow 0$, hence $u_n - f_\Sigma u_n \rightarrow 0$ in $H^1(\Sigma)$. Therefore,

$$I(u_n) = \int_\Sigma \frac{1}{2} |\nabla u_n|^2 + 2|K|e^{u_n} - 4 \int_{\partial\Sigma} h e^{u_n/2} \geq -4e^{f_\Sigma u_n/2} \oint_{\partial\Sigma} h e^{(u_n - f_\Sigma u_n)/2},$$

but $\oint_{\partial\Sigma} h e^{(u_n - f_\Sigma u_n)/2} \rightarrow \int h < 0$, hence $I(u_n) > 0$ for sufficiently large n . $\square$

Proposition 3.8. *Assume that $\tilde{K} = 0$, $K < 0$, $\oint_{\partial\Sigma} h < 0$ and $\mathfrak{D}(p) > 1$ for some $p \in \partial\Sigma$. Then there exist $u_0, u_1 \in H^1(\Sigma)$ such that:*

$$c = \inf_\gamma \max_t \{I(\gamma(t)) : t \in [0, 1], \gamma \in \Gamma\} > \max\{I(u_0), I(u_1)\} > 0,$$

where $\Gamma = \{\gamma : [0, 1] \rightarrow H^1(\Sigma) \text{ continuous} : \gamma(0) = u_0, \gamma(1) = u_1\}$.

Proof. Fix $\varepsilon > 0$, $\delta > 0$ as given by Lemma 3.7. Take a constant $c > 0$ so large that

$$I(-c) = 2 \int_\Sigma |K|e^{-c} - 4 \oint_{\partial\Sigma} h e^{-c/2} \in \left(0, \frac{\varepsilon}{2}\right),$$

and moreover $\oint_{\partial\Sigma} e^{-c/2} < \delta$. We define $u_0 = -c$.

By Lemma 3.6, there exists also u_1 with $\oint_{\partial\Sigma} e^{u_1/2} > \delta$ satisfying $I(u_1) < 0$.

Observe that for any $\gamma \in \Gamma$ there exists $t \in (0, 1)$ such that $\oint_{\partial\Sigma} e^{\gamma(t)/2} = \delta$. As a consequence, $c \geq \varepsilon > \max\{I(u_0), I(u_1)\} > 0$. $\square$

The main difficulty in proving Theorem 3.14 is that we do not know whether the Palais–Smale condition holds for I . The strategy is to construct a sequence

of solutions u_n to a perturbed version of (1.1), namely (4.22) (where $\tilde{K}_n$ and $\tilde{h}_n$ are as in Theorem 4.7), whose Morse index does not exceed 1. The main tool is a monotonicity argument, originally due to Struwe (see [31]), but used in the form given by Jeanjean (see [19]), combined with a deformation argument from [13].

For ε close to zero, we consider the following family of functionals

$$I_\varepsilon(u) = I(u) + \varepsilon J(u),$$

where I is as in (3.3), and

$$J(u) = \int_{\Sigma} (|\nabla u|^2 + e^u - u) \, d\mu.$$

Notice that, since

$$e^u - u \rightarrow +\infty \quad \text{as } u \rightarrow \pm\infty,$$

$J(u)$ is coercive on $H^1(\Sigma)$.

For a small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$ we can reason as in the proof of Proposition 3.8, to find two elements $u_0, u_1 \in H^1(\Sigma)$ for which

$$c_\varepsilon := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\varepsilon(\gamma(t)) > \max\{I_\varepsilon(u_0), I_\varepsilon(u_1)\} + \delta > \delta, \quad (3.7)$$

where $\delta > 0$ is a fixed positive number and where Γ , again, stands for the class of admissible curves

$$\Gamma = \left\{ \gamma \in C([0, 1]; H^1(\Sigma)) : \gamma(0) = u_0, \gamma(1) = u_1 \right\}.$$

Under these assumptions, the function $\varepsilon \mapsto c_\varepsilon$ is monotone non-decreasing and therefore c_ε is a.e. differentiable in ε . Let ε an arbitrary fixed value where c'_ε exists and $\{\varepsilon_n\}$ so that $\varepsilon_n \searrow \varepsilon$.

Lemma 3.9. *There exists a sequence of paths $\{\gamma_n\} \subset \Gamma$ and $M = M(c'_\varepsilon) > 0$, such that*

1. $\|\gamma_n(t)\| \leq M$, if $\gamma_n(t)$ satisfies

$$I_\varepsilon(\gamma_n(t)) \geq c_\varepsilon - (\varepsilon_n - \varepsilon)$$

2. $\max_{t \in [0,1]} I_\varepsilon(\gamma_n(t)) \leq c_\varepsilon + (2 + c'_\varepsilon)(\varepsilon_n - \varepsilon)$.

Proof. 1. Let $\{\gamma_n\}$ be a sequence such that

$$\max_{t \in [0,1]} I_{\varepsilon_n}(\gamma_n(t)) \leq c_{\varepsilon_n} + (\varepsilon_n - \varepsilon).$$

Since c'_ε exists, there is $N \in \mathbb{N}$ such that $\forall n \geq N$

$$-c'_\varepsilon - 1 \leq \frac{c_{\varepsilon_n} - c_\varepsilon}{\varepsilon_n - \varepsilon} \leq c'_\varepsilon + 1. \quad (3.8)$$

Hence $\forall n \geq N$

$$J(\gamma_n(t)) = \frac{I_{\varepsilon_n}(\gamma_n(t)) - I_\varepsilon(\gamma_n(t))}{\varepsilon_n - \varepsilon} \leq \frac{c_{\varepsilon_n} - c_\varepsilon}{\varepsilon_n - \varepsilon} + 2 \leq c'_\varepsilon + 3.$$

Therefore, boundedness follows from the coercivity of J .

2. From 3.8 we have

$$c_{\varepsilon_n} \leq c_\varepsilon + (\varepsilon_n - \varepsilon)(c'_\varepsilon + 1),$$

thus $\forall t \in [0, 1]$ we obtain

$$I_\varepsilon(\gamma_n(t)) \leq I_{\varepsilon_n}(\gamma_n(t)) \leq c_{\varepsilon_n} + (\varepsilon_n - \varepsilon) \leq c_\varepsilon + (2 + c'_\varepsilon)(\varepsilon_n - \varepsilon).$$

□

Now, for $\alpha > 0$, we define

$$F_\alpha = \{u \in H^1(\Sigma) : \|u\| \leq M + 1, |I_\varepsilon - c_\varepsilon| \leq \alpha\}$$

where the constant $M > 0$ is given by the previous lemma.

Proposition 3.10. *For all $\alpha > 0$,*

$$\inf\{\|I'_\varepsilon(u)\| : u \in F_\alpha\} = 0.$$

Proof. Suppose by contradiction that $\inf_{u \in F_\alpha} \|I'_\varepsilon(u)\| > 0$. Let u_0, u_1 and c_ε be as above, and take

$$0 < \alpha < \min \left\{ A, \frac{1}{2} (c_\varepsilon - \max\{I_\varepsilon(u_0), I_\varepsilon(u_1)\}) \right\}.$$

By the classical Deformation Lemma, there exist $\delta \in (0, \alpha)$ and a map $\eta : H^1(\Sigma) \rightarrow H^1(\Sigma)$ such that:

$$(i) \quad \eta(u) = u \text{ if } |I_\varepsilon(u) - c_\varepsilon| \geq \alpha.$$

(ii) $I_\varepsilon(\eta(u)) \leq I_\varepsilon(u)$ for all $u \in H^1(\Sigma)$.

(iii) $I_\varepsilon(\eta(u)) \leq c_\varepsilon - \delta$ for all $u \in H^1(\Sigma)$ with $\|u\| \leq M$ and $I_\varepsilon(u) \leq c_\varepsilon + \delta$.

Let $\{\gamma_n\}$ be given by the previous lemma, and choose $m \in \mathbb{N}$ sufficiently large so that $(2 + c'_\varepsilon)(\varepsilon_m - \varepsilon) < \delta$. Observe that (i) implies $\eta(\gamma_m) \in \Gamma$.

If $I_\varepsilon(\gamma_m(t)) \leq c_\varepsilon - (\varepsilon_m - \varepsilon)$, then by (ii)

$$I_\varepsilon(\eta(\gamma_m(t))) \leq c_\varepsilon - (\varepsilon_m - \varepsilon).$$

Otherwise, if $I_\varepsilon(\gamma_m(t)) > c_\varepsilon - (\varepsilon_m - \varepsilon)$, then by the first point of the previous lemma we have $\|\gamma_m\| \leq M$, and by the second point and our choice of m ,

$$I_\varepsilon(\gamma_m(t)) \leq c_\varepsilon + \delta.$$

Thus, by (iii) we obtain

$$I_\varepsilon(\eta(\gamma_m(t))) \leq c_\varepsilon - (\varepsilon_m - \varepsilon).$$

Therefore the estimate

$$I_\varepsilon(\eta(\gamma_m(t))) \leq c_\varepsilon - (\varepsilon_m - \varepsilon)$$

holds in any case and for all $t \in [0, 1]$. Taking the maximum over t in this inequality contradicts the definition of c_ε . $\square$

Theorem 3.11. I_ε has a critical point in F_α .

Proof. The previous proposition and the definition of F_α ensures that exists a bounded Palais-Smale sequence $\{u_n\}$ at level c_ε , so we have just to prove that this sequence is compact. By boundedness, up to subsequences we can suppose $u_n \rightharpoonup u$ in $H^1(\Sigma)$, $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Sigma)$ and

$$e^{u_n} \rightarrow e^u \text{ in } L^p(\Sigma) \quad e^{\frac{u_n}{2}} \rightarrow e^{\frac{u}{2}} \text{ in } L^p(\partial\Sigma).$$

We can also suppose, up to subsequences, that $u_n \rightarrow u$ in L^2 and almost everywhere. The first derivative of I_ε is

$$\langle I'_\varepsilon(v), \varphi \rangle = \int_\Sigma \left(2 \left(\frac{1}{2} + \varepsilon \right) \nabla v \cdot \nabla \varphi - (2K - \varepsilon) e^v \varphi - \varepsilon \varphi \right) - 2 \int_{\partial\Sigma} h e^{v/2} \varphi,$$

Thus

$$(1 + 2\varepsilon) \int_{\Sigma} \nabla u_n \cdot \nabla \varphi = (\langle I'_\varepsilon(u_n), \varphi \rangle) + \int_{\Sigma} ((2K - \varepsilon)e^{u_n} \varphi + \varepsilon \varphi) + 2 \oint_{\partial \Sigma} h e^{\frac{u_n}{2}} \varphi$$

By the above convergences and the fact that u_n is a Palais-Smale sequence the right term converge in $H^{-1}(\Sigma)$. Hence, $\|\nabla u_n\|_{L^2}$ converges to $\|\nabla u\|_{L^2}$ and then u_n converges to u strongly in H^1 .

□

Now we state an abstract result of Fang and Ghoussoub (Corollary 1 in [13]):

Theorem 3.12. *Let I be a real-valued C^2 -functional on a Hilbert space H and let $u_0, u_1 \in H$ be such that*

$$\max\{I(u_0), I(u_1)\} < c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)).$$

where $\Gamma = \{\gamma : [0, 1] \rightarrow H^1(\Sigma) \text{ continuous} : \gamma(0) = u_0, \gamma(1) = u_1\}$. Suppose that I' and I'' are Holder continuous in a neighborhood of $\{I = c\}$. Then there exists a Palais-Smale sequence $\{x_n\}_n$ in H such that:

For each n , if $\langle I''(x_n)u, u \rangle < -\frac{1}{n}\|u\|^2$ for all u in a subspace E of H , then $\dim(E) \leq 1$.

The sequence in the Theorem is obtained by starting from curves $\gamma(t)$ whose maximal value $\sup_{t \in [0,1]} I(\gamma(t))$ is close to c . One then applies a deformation that acts only at those points where $I(\gamma(t))$ is sufficiently large. This deformation alters the path $\gamma(t)$ only in a very small neighborhood of its highest energy region, as ensured by estimate (21) in the proof of Theorem 1.bis in [13]. Unfortunately, in this case, the strips $\{c_\varepsilon - \rho \leq I_\varepsilon \leq c_\varepsilon + \rho\}$ are unbounded, hence the above Holder continuity property is not guaranteed. However, by the just done monotonicity argument, we can use the deformation only inside the set F_α , where the Holder continuity property holds thanks to boundness of the set. Thus, we obtain a bounded Palais-Smale sequence for I_ε at level c_ε also satisfying the property of the Theorem 3.12.

We now gather the conclusions reached so far into the following proposition:

Proposition 3.13. *Suppose that we are under the assumptions of Theorem 3.14. Then there exists a sequence $\varepsilon_n \searrow 0$ and solutions u_n to the problem*

$$\begin{cases} -\Delta u_n + 2\tilde{K}_n(x) = 2K_n(x)e^{u_n}, & \text{in } \Sigma, \\ \frac{\partial u_n}{\partial n} + 2\tilde{h}_n(x) = 2h_n(x)e^{u_n/2}, & \text{on } \partial \Sigma, \end{cases} \quad (3.9)$$

with

$$\tilde{K}_n = \frac{-\varepsilon_n/2}{1+2\varepsilon_n}; \quad K_n(x) = \frac{K(x) - \varepsilon_n/2}{1+2\varepsilon_n}; \quad h_n(x) = \frac{h(x)}{1+2\varepsilon_n}; \quad \tilde{h}_n = 0.$$

Moreover $\text{ind}(u_n) \leq 1$.

We now are able to prove Theorem 3.14, assuming the validity of Theorem 4.7, which will be proved in the next section.

Theorem 3.14. *Assume that $\tilde{K} = 0$. Let K, h be C^1 functions such that $K < 0$ and*

1. $\mathfrak{D}(p) > 1$ for some $p \in \partial\Sigma$;
2. $\oint_{\partial\Sigma} h(x) < 0$;
3. $\mathfrak{D}_\tau(p) \neq 0$ for any $p \in \partial\Sigma$ with $\mathfrak{D}(p) = 1$.

Then problem (1.1) admits a solution.

Proof. Consider the sequence u_n given by the previous proposition with $\varepsilon_n \searrow 0$. Observe that $\chi_n = \int_{\Sigma} \tilde{K}_n + \oint_{\partial\Sigma} \tilde{h}_n \leq 0$. By Theorem 4.7 (4) and our assumptions on the function $\mathfrak{D}$, u_n must be uniformly bounded from above. Thanks to upper boundedness of u_n , we have that the right-hand sides of 3.9 are uniformly bounded. By elliptic regularity theory, $u_n - \oint_{\Sigma} u_n$ has a subsequence that converges in $C^1(\Sigma)$ to a function w . Let's assume by contradiction that $\inf_{\Sigma} u_n \rightarrow -\infty$. We have

$$\inf_{\Sigma} u_n = \inf_{\Sigma} \left(u_n - \oint_{\Sigma} u_n \right) + \oint_{\Sigma} u_n$$

and so $\oint_{\Sigma} u_n \rightarrow -\infty$. However this would imply

$$e^{u_n} = e^{u_n - \oint_{\Sigma} u_n} e^{\oint_{\Sigma} u_n} \rightarrow 0,$$

uniformly on Σ . As a consequence, w is harmonic in Σ and satisfies homogeneous Neumann boundary conditions. Since $\oint_{\Sigma} w = 0$, we must have $w \equiv 0$ and therefore $I_{\varepsilon_n}(u_n) \rightarrow 0$, which is a contradiction with (3.7). $\square$

Chapter 4

Blow-up analysis

4.1 Limit problems

4.1.1 Solutions of the problem in the plane

The classification of finite-mass solutions to Liouville-type problems was first obtained in [8] for the whole space $\mathbb{R}^2$ and is by now well-known. More precisely, the unique solutions of the problem

$$-\Delta v = 2K_0 e^v \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^v < +\infty, \quad (4.1)$$

exist only if $K_0 > 0$ and are given by

$$v(x) = 2 \log \left\{ \frac{2\lambda}{K_0 \lambda^2 + |x - x_0|^2} \right\}, \quad x_0 \in \mathbb{R}^2, \quad \lambda > 0. \quad (4.2)$$

All such solutions satisfy the quantization property

$$K_0 \int_{\mathbb{R}^2} e^v = 4\pi. \quad (4.3)$$

4.1.2 Solutions of the problem in the half-plane and their Morse index

We now turn to the case of the half-plane, where we will perform also a Morse index computation.

$$\begin{cases} -\Delta v = 2K_0 e^v, & \text{in } \mathbb{R}_+^2, \\ \frac{\partial v}{\partial n} = 2h_0 e^{v/2}, & \text{on } \partial\mathbb{R}_+^2, \end{cases} \quad (4.4)$$

where K_0 and h_0 are constants, and $\mathbb{R}_+^2 = \{(s, t) \in \mathbb{R}^2 : t \geq 0\}$. The classification of solutions to this problem has been established in [15, 37], and can be summarized as follows:

Theorem 4.1. *Consider the problem (4.4) in $\mathbb{R}_+^2$ with constants K_0 and h_0 , and define*

$$\mathfrak{D}_0 := \frac{h_0}{\sqrt{|K_0|}}.$$

Then:

1. *If $\mathfrak{D}_0 < 1$ and $K_0 < 0$, problem (4.4) admits no solutions.*
2. *If $\mathfrak{D}_0 = 1$ and $K_0 < 0$, the only solutions are of the form*

$$v_\lambda(s, t) = 2 \log \left(\frac{\lambda}{1 + \lambda t} \right) - \log |K_0|, \quad \lambda > 0. \quad (4.5)$$

3. *If $\mathfrak{D}_0 > 1$ or $K_0 \geq 0$, there exist solutions of finite mass, which mean*

$$\text{either } \int_{\mathbb{R}_+^2} e^v < +\infty \quad \text{or} \quad \oint_{\partial \mathbb{R}_+^2} e^{v/2} < +\infty. \quad (4.6)$$

given explicitly by

$$v_\lambda(s, t) = 2 \log \left(\frac{2\lambda}{K_0 \lambda^2 + (s - s_0)^2 + (t + \lambda h_0)^2} \right), \quad s_0 \in \mathbb{R}, \quad \lambda > 0. \quad (4.7)$$

In this case, the solutions satisfy the quantization formulas

$$\int_{\mathbb{R}_+^2} K_0 e^{v_\lambda} = 2\pi - \beta, \quad \oint_{\partial \mathbb{R}_+^2} h_0 e^{v_\lambda/2} = \beta, \quad (4.8)$$

where

$$\beta = 2\pi \frac{h_0}{\sqrt{h_0^2 + K_0}}. \quad (4.9)$$

4. *More generally, if $\mathfrak{D}_0 > 1$ and $K_0 < 0$, the solution can also be represented as*

$$v(z) = 2 \log \left(\frac{2|g'(z)|}{1 - |g(z)|^2} \right) - \log |K_0|,$$

where g is a locally univalent holomorphic map from $\mathbb{R}_+^2$ onto a disk of geodesic curvature $\mathfrak{D}_0$ in the Poincaré disk $\mathbb{H}^2$. In particular, g is a Möbius transformation if and only if the solution has finite mass, and then it reduces to (4.7).

Next, we compute the Morse index of these solutions in the case $K_0 < 0$. By performing the change of variable $v \mapsto v + \log |K_0|$, we can assume $K_0 = -1$ and rewrite the problem as

$$\begin{cases} -\Delta v = -2e^v, & \text{in } \mathbb{R}_+^2, \\ \frac{\partial v}{\partial n} = 2\mathfrak{D}_0 e^{v/2}, & \text{on } \partial\mathbb{R}_+^2. \end{cases} \quad (4.10)$$

We focus on the quadratic form

$$Q(\psi) = \int_{\mathbb{R}_+^2} |\nabla \psi|^2 dV_g + 2 \int_{\mathbb{R}_+^2} e^v \psi^2 dV_g - \mathfrak{D}_0 \oint_{\partial\mathbb{R}_+^2} e^{v/2} \psi^2 dy_g, \quad (4.11)$$

for test functions $\psi \in C_0^\infty(\mathbb{R}_+^2)$ (compactly supported, not necessarily vanishing on the boundary). The Morse index of a solution v of (4.10) is defined as

$$\text{ind}(v) = \sup \left\{ \dim(E) : E \subset C_0^\infty(\mathbb{R}_+^2), Q(\psi) < 0 \ \forall \psi \in E \right\}.$$

If the above set is unbounded, we set $\text{ind}(v) = +\infty$.

Remark 4.2. Suppose we have a strictly positive solution ψ of the linearized problem

$$\begin{cases} -\Delta \psi = f'(v)\psi, \\ \frac{\partial \psi}{\partial n} = g'(v)\psi. \end{cases}$$

Then the solution v is stable, meaning that the quadratic form

$$Q(\phi) = \int_{\Sigma} |\nabla \phi|^2 - \int_{\Sigma} f'(v)\phi^2 - \oint_{\partial\Sigma} g'(v)\phi^2$$

is nonnegative.

To see this, one simply multiplies the linearized equation by $\frac{\phi^2}{\psi}$, integrates over Σ , and integrates by parts:

$$\begin{aligned} \int_{\Sigma} f'(v)\phi^2 dx &= \int_{\Sigma} (-\Delta \psi) \frac{\phi^2}{\psi} dx \\ &= \int_{\Sigma} \nabla \psi \cdot \nabla \left(\frac{\phi^2}{\psi} \right) dx - \oint_{\partial\Sigma} \frac{\partial \psi}{\partial n} \frac{\phi^2}{\psi} dS \\ &= 2 \int_{\Sigma} \frac{\phi \nabla \psi \cdot \nabla \phi}{\psi} dx - \int_{\Sigma} \frac{\phi^2 |\nabla \psi|^2}{\psi^2} dx - \oint_{\partial\Sigma} g'(v)\phi^2 dS \\ &\leq \int_{\Sigma} |\nabla \phi|^2 dx - \oint_{\partial\Sigma} g'(v)\phi^2 dS, \end{aligned}$$

where the last step follows from Young's inequality. This establishes the non-

negativity of $Q(\phi)$.

Theorem 4.3. *Let v be a solution of problem (4.10), with $K_0 < 0$. Then:*

a) *If $\mathfrak{D}_0 = 1$, then $\text{ind}(v) = 0$, i.e., v is stable.*

b) *If $\mathfrak{D}_0 > 1$, then:*

- *If (4.6) holds, $\text{ind}(v) = 1$.*
- *Otherwise, $\text{ind}(v) = +\infty$.*

Proof. In case a), the solution v is given by (4.5). Consider the linearized problem around v_1 :

$$\begin{cases} -\Delta\psi + 2\frac{1}{(1+t)^2}\psi = 0, & \text{in } \mathbb{R}_+^2, \\ \frac{\partial\psi}{\partial n} = \psi, & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

An explicit positive solution is $\psi(s, t) = \psi(t) = \frac{1}{1+t}$. By the previous remark, this implies that v is stable.

In case b), let g be given by the classification above. Composing with a symmetry of $\mathbb{H}^2$, we may assume, without loss of generality, that $g(\mathbb{R}_+^2)$ is contained in a disk $D_R \subset \mathbb{H}^2$ centered at 0, with Euclidean radius

$$R = \mathfrak{D}_0 - \sqrt{\mathfrak{D}_0^2 - 1} < 1.$$

In this setting, the Gaussian and geodesic curvatures of D_R in $\mathbb{H}^2$ are transported to $\mathbb{R}_+^2$ via g .

In what follows we set $\rho(s) = \frac{4}{(1-s^2)^2}$. To study stability we pass to the disk D_R and consider the quadratic form

$$Q_R(\psi) = \int_{D_R} |\nabla\psi|^2 dz + 2 \int_{D_R} \psi^2 \rho(|z|) dz - \mathfrak{D}_0 \int_{\partial D_R} \psi^2 \sqrt{\rho(R)} d\sigma, \quad \psi \in C^\infty(\overline{D_R}),$$

where $d\sigma$ denotes the surface measure on ∂D_R . We claim that Q_R has Morse index 1.

To see this, consider nontrivial ψ solving the boundary value problem

$$\begin{cases} -\Delta\psi + 2\rho(|z|)\psi = 0, & \text{in } D_R, \\ \frac{\partial\psi}{\partial n} - \mathfrak{D}_0\sqrt{\rho(R)}\psi = \lambda\psi, & \text{on } \partial D_R. \end{cases} \quad (4.12)$$

Note that the only solution of (4.12) vanishing identically on ∂D_R is $\psi \equiv 0$. For any nontrivial solution of (4.12) an integration by parts gives the identity

$$Q_R(\psi) = \lambda \int_{\partial D_R} \psi^2 d\sigma,$$

so the sign of $Q_R(\psi)$ agrees with the sign of λ . The first eigenvalue admits the variational characterization (Rayleigh quotient)

$$\lambda_0 = \inf_{\psi|_{\partial D_R} \neq 0} \frac{Q_R(\psi)}{\int_{\partial D_R} \psi^2 d\sigma},$$

and higher eigenvalues are obtained by minimization on subspaces orthogonal (with respect to the $L^2(\partial D_R)$ inner product) to the previous eigenfunctions.

One easily checks that the two functions

$$\frac{x_1}{1 - |x|^2}, \quad \frac{x_2}{1 - |x|^2}$$

satisfy (4.12) with $\lambda = 0$. These elements of the kernel correspond to the invariance of the problem under translations. Using the Fourier expansion in polar coordinates

$$\psi(r, \theta) = \sum_{m=0}^{\infty} \left(u_m(r) \cos(m\theta) + v_m(r) \sin(m\theta) \right),$$

the two functions above correspond precisely to the modes $u_1(r) \cos \theta$ and $v_1(r) \sin \theta$. Note also that different Fourier modes are orthogonal for the boundary integrals appearing in Q_R .

For the radial sector ($m = 0$) there is an explicit bounded solution arising from the dilation invariance:

$$\gamma(r) = \frac{1 + r^2}{1 - r^2}.$$

The radial equation reduces to

$$\psi_0''(r) + \frac{1}{r} \psi_0'(r) - \frac{8}{(1 - r^2)^2} \psi_0(r) = 0.$$

Writing a general radial solution as $\psi_0(r) = c(r)\gamma(r)$ one finds a second, linearly independent solution of the form

$$\psi_{\text{sing}}(r) = \gamma(r) \log r + \frac{2}{1 - r^2},$$

which is singular at $r = 0$ and therefore does not belong to our admissible space (it is

not bounded in D_R). Hence the space of bounded radial solutions is one-dimensional and is spanned by γ .

The function γ is bounded on D_R for $R \in (0, 1)$ and satisfies the boundary condition

$$\frac{\partial \gamma}{\partial n} = \frac{1}{\mathfrak{D}_0} \sqrt{\rho(R)} \gamma \quad \text{on } \partial D_R.$$

Comparing this with (4.12) we see that γ is an eigenfunction of that problem with eigenvalue

$$\lambda_0 = \frac{1}{\mathfrak{D}_0} - \mathfrak{D}_0 < 0.$$

(thanks to the fact that $\mathfrak{D}_0 > 1$).

Finally, eigenvalues increase with the Fourier mode m . Indeed, for fixed radial profile $u(r)$ and integers $m \geq m' \geq 0$,

$$\frac{Q_R(u(r) \cos(m\theta))}{\int_{\partial D_R} (u(r) \cos(m\theta))^2 d\sigma} = \frac{\int_0^R \left(u'(r)^2 + \frac{m^2}{r^2} u(r)^2 + 2\rho(r) u(r)^2 \right) r dr - \mathfrak{D}_0 \sqrt{\rho(R)} R u(R)^2}{\pi R u(R)^2},$$

and since $m^2 \geq (m')^2$ the quotient for m is not smaller than that for m' . The same conclusion holds for the sine modes. Therefore the eigenvalues satisfy

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots,$$

and, using the explicit information above, we obtain $\lambda_0 < 0$, $\lambda_1 = 0$, and $\lambda_m \geq 0$ for $m \geq 2$. This proves the claim that Q_R has Morse index 1.

Define now $\psi = \gamma \circ g$. By construction, ψ satisfies

$$\begin{cases} -\Delta \psi + 2e^v \psi = 0, & \text{in } \mathbb{R}_+^2, \\ \frac{\partial \psi}{\partial n} = \frac{1}{\mathfrak{D}_0} e^{v/2} \psi, & \text{on } \partial \mathbb{R}_+^2. \end{cases} \quad (4.13)$$

We first deal with the case of finite mass, namely when (4.6) holds. Since the Möbius map g is invertible, we can transfer the second variation from the disk D_R to the half-plane $\mathbb{R}_+^2$, which immediately gives

$$Q(\psi) < 0.$$

Observe that ψ is uniformly bounded in $\mathbb{R}_+^2$; in addition, since $e^v \in L^1(\mathbb{R}_+^2)$, $e^{v/2} \in L^1(\partial \mathbb{R}_+^2)$, and ψ has a finite limit at infinity, we may approximate ψ by a compactly supported function $\tilde{\psi}$ such that still $Q(\tilde{\psi}) < 0$. Hence $\text{ind}(v) \geq 1$.

Assume by contradiction that the index is strictly larger than one. Then we

could find a two-dimensional subspace E_2 , spanned by smooth compactly supported functions, such that the quadratic form Q is negative definite on E_2 . Define the corresponding two-dimensional space on the disk D_R by

$$\widehat{E}_2 := \{ \phi \circ g^{-1} : \phi \in E_2 \}.$$

For every $\widehat{\phi} \in \widehat{E}_2$ we would have $Q_R(\widehat{\phi}) < 0$, contradicting the fact that the Morse index of Q_R is exactly 1. Thus, if (4.6) holds, we conclude that $\text{ind}(v) = 1$.

Assume now that $\oint_{\partial\mathbb{R}_+^2} e^{v/2} = +\infty$. We show that the solution v is unstable outside every compact set, which in particular implies that its Morse index is infinite.

Multiply (4.13) by $\phi^2\psi$ and integrate, where ϕ is a suitable cut-off function:

$$0 = \int_{\mathbb{R}_+^2} (-\Delta\psi + 2e^v\psi) \phi^2\psi = \int_{\mathbb{R}_+^2} \nabla\psi \cdot \nabla(\phi^2\psi) + 2e^v\psi^2\phi^2 - \oint_{\partial\mathbb{R}_+^2} \frac{1}{\mathfrak{D}_\circ} e^{v/2}\psi^2\phi^2.$$

Using the identity

$$|\nabla(\phi\psi)|^2 = \nabla\psi \cdot \nabla(\phi^2\psi) + \psi^2|\nabla\phi|^2,$$

we obtain

$$\int_{\mathbb{R}_+^2} \psi^2|\nabla\phi|^2 = \int_{\mathbb{R}_+^2} |\nabla(\phi\psi)|^2 + 2e^v(\phi\psi)^2 - \oint_{\partial\mathbb{R}_+^2} \frac{1}{\mathfrak{D}_\circ} e^{v/2}(\phi\psi)^2. \quad (4.14)$$

Note that the right-hand side resembles the expression for $Q(\phi\psi)$ in (4.11), except that the boundary coefficient is $\frac{1}{\mathfrak{D}_\circ}$ instead of $\mathfrak{D}_\circ$.

Fix $M_0 > 0$ and let M be large. Choose $\phi = \phi_M$ a nonnegative cut-off function such that

$$\phi = 0 \text{ in } B_0(M_0), \quad \phi = 1 \text{ in } A(0; 2M_0, M), \quad \phi = 0 \text{ outside } B_0(2M),$$

and

$$\int_{\mathbb{R}^2} |\nabla\phi_M|^2 \leq C, \quad \text{with } C \text{ independent of } M.$$

From (4.14) we obtain

$$\int_{\mathbb{R}_+^2} \psi^2|\nabla\phi|^2 \leq C.$$

Since $\psi \geq 1$, we have

$$\oint_{\partial\mathbb{R}_+^2} e^{v/2}(\phi\psi)^2 \geq \oint_{\partial\mathbb{R}_+^2 \cap A(0; 2M_0, M)} e^{v/2}.$$

Because $\oint_{\partial\mathbb{R}_+^2} e^{v/2} = +\infty$ and M_0 is fixed, the last quantity diverges as $M \rightarrow \infty$. Thus, for M sufficiently large, we obtain $Q(\phi\psi) < 0$. Since M_0 is arbitrary, this proves instability outside any compact set, as claimed. $\square$

4.1.3 Solution of the problem in the disk

We focus on the analysis of the following limit problem:

$$\begin{cases} -\Delta v = 2K_0 e^v & \text{in } \mathbb{D}, \\ \frac{\partial v}{\partial \nu} + 2 = 2h_0 e^{v/2} & \text{on } \partial\mathbb{D}, \end{cases} \quad (4.15)$$

where K_0 and h_0 are real constants.

We introduce the group of conformal automorphisms of the unit disk, defined as

$$\mathcal{G} := \left\{ e^{i\theta} f_a : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}; \theta \in [0, 2\pi), f_a(z) = \frac{a+z}{1+\bar{a}z}, a \in \mathbb{D} \right\}. \quad (4.16)$$

It is well known that problem (4.15) is invariant under conformal transformations. More precisely, if v is a solution of (4.15) and $f \in \mathcal{G}$, then the function

$$v_f(z) := v(f(z)) + 2 \log |f'(z)|$$

is again a solution of (4.15). Using the Möbius map $\phi(z) = \frac{z-i}{z+i}$, we can pass from the upper half-plane to the disk, so we have the following classification result:

Lemma 4.4. *Problem (4.15) admits a solution if and only if*

$$\text{either } K_0 > 0 \quad \text{or} \quad K_0 \leq 0 \text{ and } h_0 > \sqrt{-K_0}. \quad (4.17)$$

Whenever this condition is satisfied, every solution is given by

$$v_a(z) := 2 \log \left\{ \frac{2\phi_0(1-|a|^2)}{\phi_0^2|1-\bar{a}z|^2 + K_0|z-a|^2} \right\}, \quad (4.18)$$

where $\phi_0 := h_0 + \sqrt{h_0^2 + K_0}$ and $a \in \mathbb{D}$.

Remark 4.5. In particular, there exists a unique solution to (4.15) satisfying the normalization conditions

$$\int_{\mathbb{D}} x e^{v(z)} = 0, \quad \int_{\mathbb{D}} y e^{v(z)} = 0.$$

This solution corresponds to the profile in (4.18) with $a = 0$, namely

$$v_0(z) = 2 \log \left\{ \frac{2\phi_0}{\phi_0^2 + K_0|z|^2} \right\}. \quad (4.19)$$

Lemma 4.6. *Consider the linearized problem*

$$\begin{cases} -\Delta\psi = 2K_0e^{v_0}\psi & \text{in } \mathbb{D}, \\ \frac{\partial\psi}{\partial\nu} = 2h_0e^{v_0/2}\psi & \text{on } \partial\mathbb{D}, \end{cases} \quad (4.20)$$

where v_0 is given by (4.19). Then the space of solutions to (4.20) is two-dimensional and is spanned by the functions

$$\psi_1(z) := \frac{x}{\phi_0^2 + K_0|z|^2}, \quad \psi_2(z) := \frac{y}{\phi_0^2 + K_0|z|^2},$$

where $\phi_0 := h_0 + \sqrt{h_0^2 + K_0}$. In particular, any solution ψ of (4.20) satisfying the orthogonality conditions

$$\int_{\mathbb{D}} x e^{v_0(z)} \psi(z) = 0, \quad \int_{\mathbb{D}} y e^{v_0(z)} \psi(z) = 0,$$

must be identically zero.

Proof. We present the proof only in the case $K_0 = 1$, since the remaining cases can be treated analogously. As shown in the previous lemma, the surface $(\mathbb{D}, e^{v_0}dz)$ is isometric to a spherical cap of $\mathbb{S}^2$, namely

$$U = \left\{ (x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 \geq \frac{h_0}{\sqrt{1 + h_0^2}} \right\}.$$

Through this conformal identification, problem (4.20) is transformed into

$$\begin{cases} -\Delta\psi = 2\psi & \text{in } U, \\ \frac{\partial\psi}{\partial\nu} = 2h_0\psi & \text{on } \partial U. \end{cases}$$

It is well known that the coordinate functions x and y generate the space of solutions of this problem. Pulling them back through the conformal map yields precisely the functions ψ_1 and ψ_2 . $\square$

4.2 The case of negative Gaussian curvature

Let us define the singular set:

$$S = \{p \in \Sigma : \exists x_n \rightarrow p \text{ such that } u_n(x_n) \rightarrow +\infty\}, \quad (4.21)$$

The blow-up statement that we aim to prove in this section is

Theorem 4.7. *Let u_n be a blowing-up sequence (namely, $\sup_\Sigma u_n \rightarrow +\infty$) of solutions to*

$$\begin{cases} -\Delta u_n + 2\tilde{K}_n(x) = 2K_n(x)e^{u_n}, & \text{in } \Sigma, \\ \frac{\partial u_n}{\partial n} + 2\tilde{h}_n(x) = 2h_n(x)e^{u_n/2}, & \text{on } \partial\Sigma, \end{cases} \quad (4.22)$$

with $\tilde{K}_n \rightarrow \tilde{K}$, $K_n \rightarrow K$, $\tilde{h}_n \rightarrow \tilde{h}$ and $h_n \rightarrow h$ in the C^1 sense, with $K < 0$. Define the singular set S as in (4.21), and

$$\chi_n = \int_\Sigma \tilde{K}_n + \oint_{\partial\Sigma} \tilde{h}_n. \quad (4.23)$$

Then the following assertions hold true:

1. $S \subset \{p \in \partial\Sigma : \mathfrak{D}(p) \geq 1\}$.
2. If $\int_\Sigma e^{u_n}$ is bounded, then there exists $m \in \mathbb{N}$ such that

$$S = \{p_1, \dots, p_m\} \subset \{p \in \partial\Sigma : \mathfrak{D}(p) > 1\}.$$

In this case $|K_n|e^{u_n} \rightharpoonup \sum_{i=1}^m (\beta_i - 2\pi)\delta_{p_i}$, $h_n e^{u_n/2} \rightharpoonup \sum_{i=1}^m \beta_i \delta_{p_i}$ for some suitable $\beta_i > 0$. In particular, $\chi_n \rightarrow 2\pi m$.

3. If $\int_\Sigma e^{u_n}$ is unbounded, there exists a unit positive measure σ on Σ such that:

$$\begin{aligned} a) \quad & \frac{|K_n|e^{u_n}}{\int_\Sigma |K_n|e^{u_n}} \rightharpoonup \sigma, \quad \frac{h_n e^{u_n/2}}{\oint_{\partial\Sigma} h_n e^{u_n/2}} \rightharpoonup \sigma|_{\partial\Sigma}; \\ b) \quad & \text{supp } \sigma \subset \{p \in \partial\Sigma : \mathfrak{D}(p) \geq 1, \mathfrak{D}_\tau(p) = 0\}. \end{aligned}$$

4. If there exists $m \in \mathbb{N}$ such that $\text{ind}(u_n) \leq m$ for all n , then $S = S_0 \cup S_1$, where:

$$S_0 \subset \{p \in \partial\Sigma : \mathfrak{D}(p) = 1, \mathfrak{D}_\tau(p) = 0\},$$

$$S_1 = \{p_1, \dots, p_k\} \subset \{p \in \partial\Sigma : \mathfrak{D}(p) > 1\}, \quad k \leq m.$$

If moreover $\chi_n \leq 0$, then S_1 is empty.

We emphasize that this result holds for general solutions of the PDE problem, independently on any connection to the geometric problem of prescribing curvatures.

The proof of Theorem 4.7 is divided into several steps, including preliminary properties of the singular set, the case of unbounded mass, and the case of bounded mass or bounded Morse index. These are presented in the following subsections.

4.2.1 Localization of singular points and preliminary estimates

In this section we focus on some general properties of blowing-up sequences of solutions to (4.22) in the case where Σ not a disk. In particular we will derive a proof of the first assertion in Theorem 4.7.

The spirit of the proof is simple: if $p \in S$, then one can rescale around p to obtain a solution on the half-plane, and this is possible only if $\mathfrak{D}(p) \geq 1$. The non-trivial point here is to be able to rescale and pass to a limit problem even if p is not isolated in S . This is possible by choosing carefully a sequence $x_n \in \Sigma$ (not necessarily local maxima) such that $x_n \rightarrow p$ and $u_n(x_n) \rightarrow +\infty$. Let us point out that this is a technical novelty even for the classical problem considered in [5, 25]: one can pass to the limit around a singular point without knowing if the singular set is finite.

For this purpose we shall use Ekeland's variational principle, which we recall below (see Chapter 5 in [32] for a proof):

Theorem 4.8. *Let (X, d) be a complete metric space and consider a function $\varphi : X \rightarrow (-\infty, +\infty]$ that is lower semi-continuous, bounded from below and not identical to $+\infty$. Let $\varepsilon > 0$ and $\lambda > 0$ be given and let $x \in X$ be such that $\varphi(x) \leq \inf_X \varphi + \varepsilon$. Then there exists $x_\varepsilon \in X$ such that*

1. $\varphi(x_\varepsilon) \leq \varphi(x)$,
2. $d(x_\varepsilon, x) \leq \lambda$,
3. $\varphi(x_\varepsilon) < \varphi(z) + \varepsilon \frac{1}{\lambda} d(x_\varepsilon, z)$ for every $z \neq x_\varepsilon$.

Proposition 4.9. *The singular set S defined in (4.21) satisfies*

$$S \subset \{p \in \partial\Sigma : \mathfrak{D}(p) \geq 1\}.$$

Proof. Let p be a point in S . By conformal invariance, we can pass from a neighborhood of p to a domain $B \subset \mathbb{R}^2$, where B denotes $B_0(r) \subset \mathbb{R}^2$, if $p \in \text{int}(\Sigma)$, or $B_0^+(r)$, if $p \in \partial\Sigma$. We can choose this conformal map so that p is mapped to 0.

Let us take a sequence y_n in B such that $y_n \rightarrow 0$ and $u_n(y_n) \rightarrow +\infty$, and define

$$\varepsilon_n = e^{-\frac{u_n(y_n)}{2}}.$$

We apply Theorem 4.8 taking $\varphi = e^{-\frac{u_n}{2}}$ and $\lambda = \sqrt{\varepsilon_n}$: then there exists a sequence $x_n \in B$ such that

- a) $u_n(y_n) \leq u_n(x_n)$,
- b) $d(x_n, y_n) \leq \sqrt{\varepsilon_n}$,
- c) $e^{-\frac{u_n(x_n)}{2}} < e^{-\frac{u_n(z)}{2}} + \sqrt{\varepsilon_n}d(x_n, z)$ for every $z \neq x_n$.

As a consequence of a) and b) above, $x_n \rightarrow 0$ and $u_n(x_n) \rightarrow +\infty$. The idea is that the new sequence x_n is convenient for rescaling and passing to a limit problem.

Now, define $\delta_n = e^{-\frac{u_n(x_n)}{2}}$, $B_n = B_{x_n}(\frac{r}{2}) \cap B$ and

$$v_n(x) = u_n(\delta_n x + x_n) + 2 \log \delta_n, \quad (4.24)$$

which is defined in $\tilde{B}_n = \frac{1}{\delta_n} B_n$. Clearly, $v_n(0) = 0$.

We claim that, given any $R > 0$, $\varepsilon > 0$,

$$v_n(x) \leq \varepsilon \quad \forall x \in \tilde{B}_n, \quad |x| < R, \quad (4.25)$$

for sufficiently large n . Indeed, using c), we have that if $|z - x_n| < R\delta_n$, then

$$(1 - \sqrt{\varepsilon_n}R)e^{-\frac{u_n(x_n)}{2}} < e^{-\frac{u_n(z)}{2}}.$$

From this we get:

$$u_n(z) < u_n(x_n) - 2 \log(1 - \sqrt{\varepsilon_n}R), \quad \text{if } |z - x_n| < R\delta_n.$$

And this prove the claim.

In what follows we distinguish two cases:

Case 1 $p \in \partial\Sigma$ and, up to a subsequence:

$$d(x_n, \Gamma_0^+(r)) = O(\delta_n) \quad \text{as } n \rightarrow +\infty.$$

Passing to a subsequence we can assume that $\frac{d(x_n, \Gamma_0^+(r))}{\delta_n} \rightarrow t_0 \geq 0$. Then, the

function v_n solves

$$\begin{cases} -\Delta v_n + 2\delta_n^2 \tilde{K}_n(\delta_n x + x_n) = 2K_n(\delta_n x + x_n)e^{v_n}, & \text{in } \tilde{B}_n, \\ \frac{\partial v_n}{\partial n} + 2\delta_n \tilde{h}_n(\delta_n x + x_n) = 2h_n(\delta_n x + x_n)e^{v_n/2}, & \text{on } \tilde{\Gamma}_n, \end{cases} \quad (4.26)$$

where $\tilde{\Gamma}_n$ is the straight portion of $\partial \tilde{B}_n$.

Taking into account (4.25), by the Harnack inequality (A.2) proved in the Appendix A (applied at $v_n - \varepsilon$), v_n is uniformly bounded in $L_{loc}^\infty(\mathbb{R} \times (-t_0, +\infty))$. Therefore, up to subsequence,

$$v_n \rightarrow v \quad \text{in } C_{loc}^2(\mathbb{R} \times (-t_0, +\infty)), \quad (4.27)$$

which is a solution of the equation

$$\begin{cases} -\Delta v = 2K(0)e^v, & \text{in } \mathbb{R} \times (-t_0, +\infty), \\ \frac{\partial v}{\partial n} = 2h(0)e^{v/2}, & \text{if } t = -t_0. \end{cases} \quad (4.28)$$

If $K(0) < 0$ and $h(0) > 0$, the latter problem admits solutions only if $\mathfrak{D}(0) \geq 1$, as we have seen in the previous section.

Case 2

$$\frac{d(x_n, \Gamma_0^+(r))}{\delta_n} \rightarrow +\infty, \quad \text{as } n \rightarrow +\infty.$$

In this situation the rescaled domains $\tilde{B}_n$ invade all of $\mathbb{R}^2$. Hence, reasoning as before, up to subsequence we have

$$v_n \rightarrow v \quad \text{uniformly in } C_{loc}^2(\mathbb{R}^2),$$

which is a solution of the equation

$$-\Delta v = 2K(0)e^v, \quad \text{in } \mathbb{R}^2. \quad (4.29)$$

If $K(0) < 0$, is it well known, via Liouville's formula, that problem (4.29) does not admit any solution. □

Remark 4.10. In the above proof, in Case 1, we arrive to a solution of (4.28). Moreover $v(0) = 0$ and, by (4.25), $v(x) \leq 0$ for all $x \in \mathbb{R} \times [-t_0, +\infty]$. By the maximum principle this implies that actually $t_0 = 0$. That is, one obtains the more precise conclusion $d(x_n, \Gamma_0^+(r)) = o(\delta_n)$.

We now state and prove a lemma for later use.

Lemma 4.11. *There holds:*

- a) *For any $q \in (1, 2)$, $\int_{\Sigma} |\nabla u_n^-|^q < O(1)$.*
- b) *If, moreover, $\chi_n \leq 0$, then $\int_{\Sigma} |\nabla u_n^-|^2 < O(1)$.*
- c) *For any $K \subset \Sigma$ compact set with $K \cap S = \emptyset$, we have that:*

$$\sup_K u_n - \inf_K u_n = O(1), \quad \int_K |\nabla u_n|^2 = O(1).$$

Bounds like those in c) frequently appear in Liouville-type problems and are typically derived via a Green's function representation. In our situation, however, this approach cannot be applied unless the total mass is finite. The proof, that follows, rely on estimates for u_n^- combined with local regularity techniques.

The control of u_n^- is usually proved with the Kato inequality (for example in [11]). The difficulty here is that the boundary condition satisfied by u_n^- is not immediately clear, so we use, as in [29], a smooth approximation of the map $u \mapsto u^-$.

Proof. Define $w : \mathbb{R} \rightarrow \mathbb{R}$ a C^2 function such that:

$$\begin{cases} w(u) = u & u \leq 0, \\ w(u) = 1 & u \geq 1, \\ w'(u) \geq 0 & u \in \mathbb{R}, \\ w''(u) \leq 0 & u \in \mathbb{R}. \end{cases}$$

Then the function $w_n = w(u_n)$ satisfies

$$\begin{cases} -\Delta w_n = -w''(u_n)|\nabla u_n|^2 - 2w'(u_n)(\tilde{K}_n + |K_n|e^{u_n}), & x \in \Sigma, \\ \frac{\partial w_n}{\partial \nu} + 2w'(u_n)\tilde{h}_n = 2w'(u_n)h_n e^{u_n/2}, & x \in \partial\Sigma. \end{cases} \quad (4.30)$$

Just by integrating we find that

$$0 \leq \int_{\Sigma} -w''(u_n)|\nabla u_n|^2 \leq O(1).$$

As a consequence, the right-hand side of the first equation in (4.30) is bounded in L^1 , whereas the boundary data are bounded in L^∞ . By elliptic regularity estimates, $\int_{\Sigma} |\nabla w_n|^q < O(1)$ for any $q \in (1, 2)$. This implies a).

We now prove *b*): We multiply (4.22) by u_n^- and integrate, to get

$$\int_{\Sigma} |\nabla u_n^-|^2 + 2u_n^-(\tilde{K}_n + |K_n|e^{u_n}) = 2 \oint_{\partial\Sigma} u_n^-(h_n e^{u_n/2} - \tilde{h}_n).$$

Clearly the functions $u_n^- e^{u_n}$, $u_n^- e^{u_n/2}$ are bounded in L^∞ . Hence,

$$\int_{\Sigma} |\nabla u_n^-|^2 = -2 \left[\int_{\Sigma} \tilde{K}_n u_n^- + \oint_{\partial\Sigma} \tilde{h}_n u_n^- \right] + O(1).$$

We are able to estimate the right hand side by the assumption on χ_n . We split $u_n^- = (u_n^- - \oint_{\Sigma} u_n^-) + \oint_{\Sigma} u_n^-$, and, making use of *a*), the Poincaré inequality and the Trace inequality, we obtain:

$$\left| \int_{\Sigma} \tilde{K}_n (u_n^- - \oint_{\Sigma} u_n^-) \right| \leq O(1), \quad \left| \oint_{\partial\Sigma} \tilde{h}_n (u_n^- - \oint_{\Sigma} u_n^-) \right| \leq O(1).$$

Moreover,

$$-2 \left[\int_{\Sigma} \tilde{K}_n \oint_{\Sigma} u_n^- + \oint_{\partial\Sigma} \tilde{h}_n \oint_{\Sigma} u_n^- \right] = -2\chi_n \oint_{\Sigma} u_n^- \leq 0,$$

yielding the assertion.

Finally, let us show *c*). We can assume, without loss of generality, that $K = \overline{B}_p(r)$ where $B_p(4r) \cap S = \emptyset$ with $p \in \Sigma$ and:

1. either $B_p(4r) \subset \text{int}(\Sigma)$,
2. or $p \in \partial\Sigma$.

Via a conformal map we can pass to a problem in $B_0(4r) \subset \mathbb{R}^2$ or $B_0^+(4r) \subset \mathbb{R}_+^2$.

Observe that in any case $u_n \leq C$ in $B_p(2r)$ for some $C > 0$. Recall that as in *a*), we have that the function $v_n = \min\{u_n, C\}$ satisfies $\int_{\Sigma} |\nabla v_n|^q = O(1)$ for all $q \in [1, 2)$. As a consequence, thanks to the Poincaré inequality and to the fact that $v_n = u_n$ in $B_p(2r)$, the function $\tilde{u}_n = u_n - \oint_{B_p(2r)} u_n$ is bounded in $L^{q'}(B_p(2r))$ for any $q' > 1$.

In case (1) we have that $\tilde{u}_n$ solves:

$$-\Delta \tilde{u}_n = f_n, \quad \text{in } B_0(2r),$$

with f_n bounded in L^∞ . We conclude then by local regularity estimates (see for instance [16, Theorem 8.17 and Theorem 8.32]).

In case 2 we are led to the problem:

$$\begin{cases} -\Delta \tilde{u}_n = f_n, & \text{in } B_0^+(2r), \\ \frac{\partial \tilde{u}_n}{\partial n} = g_n, & \text{on } \Gamma_0^+(2r), \end{cases}$$

with f_n, g_n bounded in L^∞ . We conclude by local regularity estimates for the Neumann problem (see for instance [27, Theorem 5.36 and Lemma 5.51]).

□

4.2.2 Blow-up with Unbounded Mass

Example 4.12. Consider the problem:

$$\begin{cases} -\Delta u = -2e^u, & \text{in } A(0; r, 1), \\ \frac{\partial u}{\partial n} + 2 = 2h_1 e^{u/2}, & \text{on } |x| = 1, \\ \frac{\partial u}{\partial n} - 2/r = 2h_2 e^{u/2}, & \text{on } |x| = r, \end{cases} \quad (4.31)$$

where h_1, h_2 are two real constants. All explicit solutions of this problem have been classified by [21]; here we exhibit some of them observing that they form families of blowing-up solutions with infinite mass.

As a first example (see [21, Equation (3) and Lemma 2]), the functions

$$u_\lambda(x) = \log \left(\frac{4}{|x|^2(\lambda + 2 \log |x|)^2} \right), \quad \text{for any } \lambda \notin [0, -2 \log r],$$

are solutions of (4.31); if $\lambda < 0$ then $h_1 = 1$ and $h_2 = -1$, whereas if $\lambda > -2 \log r$, $h_1 = -1$ and $h_2 = 1$. Observe that if λ tends to 0 or $-2 \log r$, then the functions u blow up at a whole component of the boundary, which is indeed the one with curvature equal to 1. In particular, the blowing-up set S is infinite here and the mass is infinite since

$$\int_{A(0; r, 1)} e^{u_\lambda} \sim 4\pi \left(\frac{1}{\lambda + 2 \log r} - \frac{1}{\lambda} \right)$$

Let us give a second example. Given any $h_1 > 1$, for any $\gamma \in \mathbb{N}$, there exists an explicit solution in the form (we use complex variable notation):

$$u_\gamma(z) = 2 \log \left(\frac{\gamma |z|^{\gamma-1}}{h_1 + \operatorname{Re}(z^\gamma)} \right), \quad (4.32)$$

where $h_2 = -h_1 r^{-\gamma}$.

Observe that the solutions in (4.32) form a blowing-up sequence as $\gamma \rightarrow +\infty$, keeping $h_1 > 1$ fixed. Indeed,

$$u_\gamma(z) \rightarrow -\infty \quad \text{if } |z| < 1, \quad u_\gamma(z) \rightarrow +\infty \quad \text{if } |z| = 1.$$

Hence, in this case $S = \{|z| = 1\}$ and $h_1 > 1$, the solutions concentrate entirely on the unit circle as $\gamma \rightarrow +\infty$.

Let us compute the asymptotic profile of these solutions. Fix a point of blow-up $z = -1$, and define the rescaling:

$$v_\gamma(z) = u_\gamma\left(1 - \frac{z}{\gamma}\right) - 2\log \gamma.$$

Hence v can be written in the form:

$$v_\gamma(z) = 2\log\left(\frac{|1 - \frac{z}{\gamma}|^{\gamma-1}}{h_1 + \operatorname{Re}((1 - \frac{z}{\gamma})^\gamma)}\right).$$

Clearly,

$$\lim_{\gamma \rightarrow +\infty} \left(1 - \frac{z}{\gamma}\right)^\gamma = e^{-z},$$

so v_γ converges, at least point-wise, to the function:

$$v(z) = 2\log\left(\frac{|e^{-z}|}{h_1 + \operatorname{Re}(e^{-z})}\right) = 2\log\left(\frac{e^{-t}}{h_1 + e^{-t}\cos s}\right),$$

which is defined in the half-plane $\{(s, t) \in \mathbb{R}^2 : t \geq 0\}$. This is indeed a solution to the limit problem in the half-space with $K = -1$, $h_1 > 1$, infinite mass and infinite Morse index, according to our results.

This example shows that infinite Morse index blowing up solutions exist, even in this geometric context. Moreover, the blow-up limits can be entire solutions in the half-plane with infinite Morse index. Let us point out, though, that this example is not under the conditions of Theorem 4.7 since $h_2 = -h_1 r^{-\gamma}$ diverges negatively. Whether the infinite Morse index assumption is necessary or not in Theorem 4.7 (4) remains as an interesting open problem.

We now investigate the blow-up phenomenon in the regime of diverging mass. As we shall show, under the assumption $\chi_n \leq 0$ this is the only admissible blow-up behaviour, revealing a genuinely new scenario for this class of geometric problems.

We introduce the quantity

$$\rho_n := \oint_{\partial\Sigma} h_n e^{u_n/2} = \int_{\Sigma} |K_n| e^{u_n} + O(1) \longrightarrow +\infty,$$

and, up to subsequences, obtain the weak convergences

$$\frac{h_n e^{u_n/2}}{\rho_n} \rightharpoonup \sigma, \quad \frac{|K_n| e^{u_n}}{\rho_n} \rightharpoonup \xi,$$

where σ and ξ are nonnegative Radon probability measures supported on $\partial\Sigma$ and Σ respectively.

The next statement corresponds to case (3a) of Theorem 4.7.

Proposition 4.13. *The limiting measure ξ is supported on the boundary, and coincides there with σ . More precisely,*

$$\xi|_{\text{int}(\Sigma)} = 0, \quad \xi|_{\partial\Sigma} = \sigma.$$

Proof. Let $\phi \in C^2(\Sigma)$ and multiply equation (4.22) by ϕ . Using Green's identity we obtain

$$2 \oint_{\partial\Sigma} (h_n e^{u_n/2} - \tilde{h}_n) \phi - 2 \int_{\Sigma} (\hat{K}_n \phi + |K_n| e^{u_n} \phi) = \int_{\Sigma} u_n \Delta \phi + \oint_{\partial\Sigma} u_n \frac{\partial \phi}{\partial \nu}.$$

We estimate the right-hand side by separating $u_n = u_n^+ + u_n^-$. For the positive part,

$$\left| \int_{\Sigma} u_n^+ \Delta \phi + \oint_{\partial\Sigma} u_n^+ \frac{\partial \phi}{\partial \nu} \right| \leq C \left(\int_{\Sigma} u_n^+ + \oint_{\partial\Sigma} u_n^+ \right) = o(\rho_n),$$

For the negative part we write

$$\int_{\Sigma} u_n^- \Delta \phi + \oint_{\partial\Sigma} u_n^- \frac{\partial \phi}{\partial \nu} = \int_{\Sigma} \left(u_n^- - \oint_{\Sigma} u_n^- \right) \Delta \phi + \oint_{\partial\Sigma} \left(u_n^- - \oint_{\Sigma} u_n^- \right) \frac{\partial \phi}{\partial \nu}.$$

By Lemma 4.11(a) we have, for any $q \in (1, 2)$,

$$\int_{\Sigma} |u_n^- - \oint_{\Sigma} u_n^-| + \oint_{\partial\Sigma} |u_n^- - \oint_{\Sigma} u_n^-| \leq C \int_{\Sigma} |\nabla u_n^-|^q,$$

and the last quantity is uniformly bounded. Thus the right-hand side is $o(\rho_n)$.

Using these facts, we pass to the limit in the first identity of the proof and we obtain

$$\oint_{\partial\Sigma} \phi d\sigma = \int_{\Sigma} \phi d\xi.$$

The arbitrariness of ϕ yields the claim. $\square$

By Proposition 4.9 we already know that

$$\text{supp } \sigma \subset S \subset \{p \in \partial\Sigma : \mathfrak{D}(p) \geq 1\}.$$

The goal of this subsection is to refine this information and show that

$$\text{supp } \sigma \subset \{p \in \partial\Sigma : \mathfrak{D}_\tau(p) = 0\},$$

The argument is delicate because the support of σ may be infinite and because we do not control the asymptotic Dirichlet energy. The strategy is to apply the Pohozaev-type identity from Lemma 2.4 to holomorphic vector fields, for which the Cauchy–Riemann equations force the energy contributions to vanish. A conformal change of coordinates allows us to work in an annulus where analytic extensions can be performed.

Proposition 4.14. *We have*

$$\text{supp } \sigma \subset \{p \in \partial\Sigma : \mathfrak{D}_\tau(p) = 0\}.$$

Proof. Let Λ_0 be a boundary component with $\sigma(\Lambda_0) > 0$, and let U be a smooth neighborhood. By the Uniformization Theorem for annuli (see [14, IV.7]) there exists a conformal chart sending U onto an annulus $A(0; r, 1)$, $r < 1$, mapping Λ_0 to the outer boundary $\{|x| = 1\}$.

In these coordinates the equation becomes

$$\begin{cases} -\Delta_g u_n + 2\tilde{K}_n = 2K_n e^{u_n} & \text{in } A(0; r, 1), \\ \partial_\nu u_n + 2\tilde{h}_n = 2h_n e^{u_n/2} & \text{on } \Lambda_0, \end{cases}$$

where $g = e^v g_0$ is conformal to the Euclidean metric. Setting $\hat{u}_n = u_n + v$ one obtains the Euclidean system

$$\begin{cases} -\Delta \hat{u}_n + 2\hat{K}_n = 2K_n e^{\hat{u}_n} & \text{in } A(0; r, 1), \\ \partial_\nu \hat{u}_n + 2\hat{h}_n = 2h_n e^{\hat{u}_n/2} & \text{on } \Lambda_0. \end{cases} \quad (4.33)$$

Let f be a real analytic function on Λ_0 , and define $F(p) = f(p)\tau(p)$, where τ is the unit tangent vector. Analytic continuation extends F holomorphically to $A(0; r, 1)$ (possibly after shrinking r). Choose a cut-off function ϕ supported near Λ_0 and set $\tilde{F} = F\phi$. Applying Lemma 2.4 to $\tilde{F}$ yields

$$\oint_{\Lambda_0} 4f(h_n e^{\hat{u}_n/2} - \hat{h}_n)(\hat{u}_n)_\tau = \int_{A(0; r_0, 1)} \left[4\hat{K}_n \nabla \hat{u}_n \cdot F + 4e^{\hat{u}_n} (\nabla K_n \cdot F + K_n \nabla \cdot F) \right] + O(1).$$

Since F is holomorphic, the Cauchy–Riemann equations imply

$$2DF(\nabla \hat{u}_n, \nabla \hat{u}_n) - (\nabla \cdot F) |\nabla \hat{u}_n|^2 = 0.$$

We now estimate each term. The contributions involving $\nabla \hat{u}_n^-$ are bounded by Lemma 4.11, while those involving $\hat{u}_n^+$ are $o(\rho_n)$ by integration by parts:

$$\begin{aligned} \left| \int_{A(0;r_0,1)} 4\hat{K}_n \nabla \hat{u}_n^+ \cdot F \right| &\leq \left| \oint_{\Lambda_0} 4\hat{K}_n \hat{u}_n^+ F \cdot \nu - \int_{A(0;r_0,1)} \hat{u}_n^+ \nabla \cdot (\hat{K}_n F) \right| + O(1) \\ &\leq C \left(\oint_{\Lambda_0} \hat{u}_n^+ + \int_{A(0;r_0,1)} \hat{u}_n^+ \right) + O(1) = o(\rho_n). \end{aligned}$$

A similar argument shows

$$\oint_{\Lambda_0} f \hat{h}_n(\hat{u}_n)_\tau = o(\rho_n).$$

Integrating by parts on Λ_0 gives

$$4 \oint_{\Lambda_0} h_n f e^{\hat{u}_n/2} (\nabla \hat{u}_n \cdot F) = -8 \oint_{\Lambda_0} (h_n f)_\tau e^{\hat{u}_n/2}.$$

Combining all terms we obtain

$$-8 \oint_{\Lambda_0} ((h_n)_\tau f + h_n f_\tau) e^{\hat{u}_n/2} = \int_{A(0;r_0,1)} (4\nabla K_n \cdot F + 4K_n \nabla \cdot F) e^{\hat{u}_n} + o(\rho_n).$$

Since on Λ_0 we have $\nabla \cdot F = 2f_\tau$, using Proposition 4.13 and dividing by ρ_n yields in the limit

$$-8 \oint_{\Lambda_0} \left(\frac{h_\tau}{h} f + f_\tau \right) d\sigma = - \oint_{\Lambda_0} \left(4f \frac{K_\tau}{K} + 8f_\tau \right) d\sigma.$$

The f_τ terms cancel, and we obtain

$$\oint_{\Lambda_0} \left(2\frac{h_\tau}{h} - \frac{K_\tau}{K} \right) f d\sigma = 0 \quad \text{for all analytic } f.$$

Set

$$\mu := \left(2\frac{h_\tau}{h} - \frac{K_\tau}{K} \right) \sigma.$$

Since analytic functions are dense in $C^0(\Lambda_0)$, the above identity forces $\mu \equiv 0$.

Finally,

$$2\frac{h_\tau}{h} - \frac{K_\tau}{K} = 2\frac{\mathfrak{D}_\tau}{\mathfrak{D}},$$

which implies $\mathfrak{D}_\tau(p) = 0$ for all $p \in \text{supp } \sigma$. □

4.2.3 Blow-up with Bounded Mass or Morse Index

In this section, we focus on the case where the sequence u_n either has bounded mass or bounded Morse index. We leverage this property to provide a more detailed description of the blow-up phenomena, thereby proving statements (2) and (4) in Theorem 4.7. The crucial point is that if $\mathfrak{D}(p) > 1$, then the only limit profiles with bounded mass or bounded Morse index are the bubbles of the form (4.7).

We begin with the following lemma:

Lemma 4.15. *Assume that $\int_{\Sigma} e^{u_n}$ is uniformly bounded in n . Then,*

$$S \subset \{p \in \partial\Sigma : \mathfrak{D}(p) > 1\}.$$

Proof. Let $p \in S$ and consider the function v_n defined in (4.24). Using Fatou's lemma, the classification of entire solutions on the upper half-plane, and Remark 4.10, we deduce that for some $r > 0$:

$$C \geq \lim_{n \rightarrow +\infty} \int_{B_p^+(r)} |K_n| e^{u_n} = \lim_{n \rightarrow +\infty} \int_{\tilde{B}_n} |K_n| e^{v_n} \geq \int_{\mathbb{R}_+^2} |K(p_0)| e^v,$$

where $\tilde{B}_n = B_{\frac{x_n - p}{\delta_n}}^+(\frac{r}{\delta_n})$ and v is a solution of the limit problem (4.28). Observe that if $\mathfrak{D}(p) = 1$, then v takes the form (4.5), violating the volume bound. Hence, $\mathfrak{D}(p) > 1$. $\square$

The main outcome of this section is the following proposition, which completes the proof of Theorem 4.7 and describes the asymptotic behavior of blowing-up sequences near singular points.

Proposition 4.16. *Let u_n be a blowing-up sequence (i.e., $\sup_{\Sigma} u_n \rightarrow +\infty$) of solutions to (4.22) under the assumptions of Theorem 4.7, and assume further that*

$$\text{either } \int_{\Sigma} e^{u_n} \text{ or } \text{ind}(u_n) \text{ is bounded.} \quad (4.34)$$

Then $S = S_0 \cup S_1$ with

$$S_0 \subset \{p \in \partial\Sigma : \mathfrak{D}(p) = 1, \mathfrak{D}_{\tau}(p) = 0\}, \quad S_1 = \{p_1, \dots, p_m\} \subset \{p \in \partial\Sigma : \mathfrak{D}(p) > 1\}.$$

Moreover, the following asymptotic profiles hold:

- a) *If $p \in S_0$, there exist sequences $x_n \rightarrow p$, $\lambda_n \rightarrow 0$, $R_n \rightarrow +\infty$ such that $d(x_n, \partial\Sigma) = o(\lambda_n)$ and*

$$u_n(y) = v_{\lambda_n}(0, t) + o(1), \quad y \in B_{x_n}(R_n \delta_n), \quad (4.35)$$

where v_λ is the one-dimensional solution in (4.5) and $t = d(y, \partial\Sigma)$.

b) If $p \in S_1$, there exist sequences $(s_n, t_n) \rightarrow p$ and $\lambda_n \rightarrow 0$ such that

$$u_n(s, t) = 2 \log \left(\frac{2\lambda_n}{(s - s_n)^2 + (t + \mathfrak{D}(p)\lambda_n)^2 - \lambda_n^2} \right) + O(1), \quad (4.36)$$

in $B_p^+(r) \subset \mathbb{R}^2$, with $d(p, p_j) > 2r$ for all $j = 1, \dots, m$, where (s, t) are isothermal coordinates centered at p . Moreover,

$$\nabla u_n(x) = -4 \frac{x - p}{|x - p|^2} + o \left(\frac{1}{|x - p|^2} \right), \quad x \in B_p^+(r) \setminus B_p^+(\lambda_n \log(1/\lambda_n)). \quad (4.37)$$

c) Finally, if $\chi_n \leq 0$, then S_1 is empty.

This proposition is a consequence of the following lemmas.

Lemma 4.17. *Under assumption (4.34), the set S_1 is finite. Moreover, for any $p \in S_1$, the blow-up profile v constructed in the proof of Proposition 4.9 has finite volume.*

Proof. Recall the profile v is defined as the limit

$$v_n(x) = u_n(\delta_n x + x_n) + 2 \log \delta_n, \quad \delta_n = e^{-u_n(x_n)/2}.$$

If v did not have finite mass, Fatou's lemma would imply $\int_\Sigma e^{u_n} \rightarrow \infty$. Moreover, according to Theorem 4.3 b), we can select compactly supported functions (with mutually disjoint supports) $\psi_1, \dots, \psi_{m+1}$ such that

$$a_i := \int_{\mathbb{R}_+^2} [|\nabla \psi_i|^2 + 2e^v \psi_i^2] - \int_{\partial \mathbb{R}_+^2} \frac{1}{\mathfrak{D}(p)} e^{v/2} \psi_i^2 < 0$$

for each $i = 1, \dots, m+1$.

Next, define the rescaled functions

$$\psi_{i,n} := \psi_i \left(\frac{x - x_n}{\delta_n} \right).$$

By changing variables and using the expression

$$Q_n(\psi) = \int_\Sigma [|\nabla \psi|^2 + 2|K_n(x)|e^{u_n} \psi^2] - \oint_{\partial \Sigma} h_n e^{u_n/2} \psi^2, \quad (4.38)$$

it is straightforward to check that

$$Q_n(\psi_{i,n}) = a_i + o_n(1) < 0, \quad i = 1, \dots, m+1.$$

Since the supports of the $\psi_{i,n}$ are also disjoint, we conclude that $\text{ind}(u_n) \geq m + 1$ for sufficiently large n , leading to a contradiction with (4.34).

We now observe that at each point in S_1 , one can construct a blow-up profile v satisfying (4.6), with Morse index equal to 1. Furthermore, by (4.9),

$$\liminf_{n \rightarrow +\infty} \int_{\Sigma} |K_n| e^{u_n} \geq \int_{\mathbb{R}_+^2} |K(p_0)| e^v = 2\pi \left(\frac{h(p_0)}{\sqrt{h^2(p_0) + K(p_0)}} - 1 \right) > \delta,$$

for some suitable $\delta > 0$ (independent of the choice of p_0). Since each blow-up point in S_1 contributes at least δ to the integral, the total number of blow-up points must be finite in order to satisfy the uniform bound imposed by condition (4.34). Therefore, the set S of blow-up points is finite. □

Lemma 4.18. *Let $p \in S_1$. Then there exist constants $r, C > 0$ such that*

$$\int_{B_p(r) \cap \Sigma} e^{u_n} \leq C, \quad \oint_{B_p(r) \cap \partial \Sigma} e^{u_n/2} \leq C, \quad u_n \rightarrow -\infty \text{ on } \partial B_p(r) \cap \Sigma.$$

Proof. First, using a conformal local chart, we reduce to the Euclidean case. Next we divide the proof into several steps:

Step 1 Blowing a first bubble: From Lemma 4.17, we already know that the subset S_1 of the blow-up set S is finite. Hence, by subharmonicity, for sufficiently large n (up to a subsequence), the functions u_n attain local maxima x_n on $\Gamma_0^+(r)$, with $x_n \rightarrow p$.

We then rescale the solutions u_n as

$$v_n(x) := u_n(x_n + \delta_n x) + 2 \log \delta_n, \quad \text{with } e^{-2\delta_n} = u_n(x_n).$$

Reasoning as in the proof of Proposition 4.9, we can find a sequence $R_n \rightarrow +\infty$ (growing slowly) such that

$$\|v_n - v\|_{C^2(B_0^+(2R_n))} \rightarrow 0,$$

where v is a solution of (4.28).

By the classification result 4.1, the limit v must have the form

$$v(s, t) = 2 \log \left(\frac{2\lambda}{-\lambda^2 + (s - s_0)^2 + (t + t_0)^2} \right),$$

for some $\lambda > 0$, $s_0 \in \mathbb{R}$, and t_0 determined by $t_0 = \mathfrak{D}(p)\lambda$. Notice that, at infinity, v satisfies

$$v(s, t) \sim -4 \log |(s, t) - (s_0, -t_0)|. \quad (4.39)$$

We also recall that this solution has Morse index one, according to Theorem 4.3.

Step 2 Blowing (possibly) other bubbles: Let r be as before and let $R_n \rightarrow +\infty$ as in the previous step. We then consider the maximization problem

$$\sup_{R_n \delta_n \leq |x - x_n| \leq r/2} (u_n(x) + 2 \log |x - x_n|).$$

If this supremum diverges, then, following the argument in the proof of Lemma 4 in [25], one could again rescale u_n near a maximum point of the function $u_n + 2 \log |x - x_n|$ to obtain a new limiting profile.

Since all limiting profiles are of the same form as the function v described above, and each has Morse index one and a fixed mass, condition (4.34) ensures that this procedure can produce only finitely many bubbles.

Consequently, there exist an integer k , with $1 \leq k \leq m$, sequences of boundary points $x_n^i \in B_0^+(r)$ for $i = 1, \dots, k$, sequences $(\delta_n^i)_n$ with $\delta_n^i \rightarrow 0$, and sequences $(R_n^i)_n$ with $R_n^i \rightarrow \infty$, as well as a fixed constant C , such that the following properties hold:

(i) The rescaled functions

$$v_n^i(x) := u_n(x_n^i + \delta_n^i x) + 2 \log \delta_n^i$$

converge in $C^2(B_0^+(R_n^i))$ to the profile v ;

(ii) The balls are disjoint:

$$B_{x_n^i}^+(4R_n^i \delta_n^i) \cap B_{x_n^j}^+(4R_n^j \delta_n^j) = \emptyset \quad \text{for } i \neq j;$$

(iii) Outside the rescaled balls, the function is uniformly bounded:

$$\sup_{B_0^+(r/2) \setminus \bigcup_{i=1}^k B_{x_n^i}^+(R_n^i \delta_n^i)} (u_n(x) + 2 \log \min_i |x - x_n^i|) \leq C.$$

Step 3 Harnack inequality: For $s_n^i \in [2R_n^i \delta_n^i, \frac{1}{8} \min_{j \neq i} |x_n^i - x_n^j|]$ with $s_n^i \leq r/8$, we consider the rescaled function

$$w_n^i(x) := u_n(x_n^i + s_n^i x) + 2 \log s_n^i, \quad x \in B_0^+(4) \setminus B_0^+(1/2).$$

Thanks to the uniform bound on u_n given in **(iii)**, the functions w_n^i are uniformly bounded from above, and they satisfy a differential equation with uniformly bounded coefficients in $B_0^+(4) \setminus B_0^+(1/2)$.

Arguing as in (4.27), we deduce a Harnack inequality for $e^{w_n^i}$ in $B_0^+(2) \setminus B_0^+(1)$, which implies that w_n^i has uniformly bounded oscillation in this region.

Next, define the (semi-circular) average

$$\bar{u}_n^i(s) := \frac{1}{|\partial^+ B_{x_n^i}(s)|} \oint_{\partial^+ B_{x_n^i}(s)} u_n.$$

The Harnack inequality then gives a constant $C > 0$, independent of n and s , such that

$$u_n(x) \leq \bar{u}_n^i(s) + C, \quad x \in B_{x_n^i}^+(2s) \setminus B_{x_n^i}^+(s/2),$$

for all $s \in [2R_n^i \delta_n^i, \frac{1}{8} \min_{j \neq i} |x_n^i - x_n^j|]$.

Step 4 Local radial decay: Let us first assume that there exists $x_n^j \neq x_n^i$ as in Step 2 such that $|x_n^i - x_n^j| \rightarrow 0$. We aim to show that, after an appropriate rescaling, the functions u_n retain the same blow-up profile as in (4.39) for $|x - x_n^i| \leq O(\min_{j \neq i} |x_n^i - x_n^j|)$. In particular, this implies that the accumulation of volume is finite up to that scale.

Let ν_s denote the outer unit normal to $\partial^+ B_{x_n^i}(s)$. Then we have

$$|\partial^+ B_{x_n^i}(s)| \frac{d}{ds} \bar{u}_n^i(s) = \oint_{\partial^+ B_{x_n^i}(s)} \frac{\partial u_n}{\partial \nu_s}. \quad (4.40)$$

Fix a small positive number δ . We claim that

$$\bar{u}_n^i(s) \leq \bar{u}_n^i(R_n^i \delta_n^i) - (4 - \delta) (\log s - \log(R_n^i \delta_n^i)) \quad (4.41)$$

for all $s \in [2R_n^i \delta_n^i, \frac{1}{8} \min_{j \neq i} |x_n^i - x_n^j|]$.

Assume by contradiction that (4.41) is false, and let t_n^i be the infimum of the radii s such that

$$\bar{u}_n^i(s) > \bar{u}_n^i(R_n^i \delta_n^i) - (4 - \delta) (\log s - \log(R_n^i \delta_n^i)).$$

By definition of t_n^i , we then also have

$$t_n^i \frac{d}{ds} \Big|_{s=t_n^i} \bar{u}_n(s) \geq \delta - 4.$$

Using (4.40) and the fact that $|\partial^+ B_{x_n^i}(s)| = \pi s$, we obtain

$$\oint_{\partial^+ B_{x_n^i}(t_n^i)} \frac{\partial u_n}{\partial \nu_{t_n^i}} \geq (\delta - 4)\pi \left(1 + O((t_n^i)^2)\right).$$

On the other hand, by the convergence in Step 1 and the asymptotic profile (4.39), we have

$$\oint_{\partial^+ B_{x_n^i}(R_n^i \delta_n^i)} \frac{\partial u_n}{\partial \nu_{R_n^i \delta_n^i}} \rightarrow -4\pi \quad \text{as } n \rightarrow +\infty.$$

Integrating (4.22) and using the above two formulas, together with $t_n^i \rightarrow 0$, we find

$$\begin{aligned} & 2 \int_{B_{x_n^i}^+(t_n^i) \setminus B_{x_n^i}^+(R_n^i \delta_n^i)} (K_n e^{u_n} - \tilde{K}_n u_n) + 2 \oint_{\Gamma_{x_n^i}^+(t_n^i) \setminus \Gamma_{x_n^i}^+(R_n^i \delta_n^i)} (h_n e^{u_n/2} - \tilde{h}_n u_n) \\ & \leq -\delta\pi + o_n(1). \end{aligned}$$

However, by the definition of t_n^i and the Harnack inequality, in the same regions we have

$$e^{u_n(x)} \leq C(\delta_n^i)^{-2}(R_n^i)^{-4} \left(\frac{R_n^i \delta_n^i}{|x|} \right)^{4-\delta}, \quad |x| \in [R_n^i \delta_n^i, t_n^i], \quad (4.42)$$

and, moreover, by the asymptotics of u_n ,

$$|u_n(x) - u_n \left(\frac{\delta_n^i R_n^i}{|x|} x \right)| \leq C \log \frac{|x|}{\delta_n^i R_n^i}, \quad |x| \in [R_n^i \delta_n^i, t_n^i].$$

Hence, where u_n is positive we can bound it by its exponential, and where it is negative we can estimate it by the logarithmic bound above. Together, these estimates show that the integrals in (4.2.3) converge to zero, giving a contradiction.

Finally, if there is no $x_n^j \neq x_n^i$ (i.e., only one bubble exists), we choose the upper bound for t_n^i in the formula after (4.41) to be a small fixed number $\tilde{r} > 0$. Then all the above arguments remain valid, except that the right-hand

side of (4.2.3) becomes $-\delta\pi + O(\tilde{r}^2) + o_n(1)$. Choosing $\tilde{r}$ small relative to δ yields the same contradiction.

Step 5 Global radial decay: This step can be omitted when only a single bubble appears in the previous stage. If instead multiple bubbles arise, for simplicity we assume that there are exactly two of them, namely x_n^1 and x_n^2 . The general situation is analogous, and can be handled by suitably grouping the points into clusters.

From the previous step, fixing any small $\delta > 0$, we obtain inequality (4.41); moreover, we also have

$$\bar{u}_n^2(s) \leq \bar{u}_n^2(R_n^2 \delta_n^2) - (4 - \delta) \left(\log s - \log(R_n^2 \delta_n^2) \right),$$

for every

$$s \in \left[2R_n^2 \delta_n^2, \frac{1}{8} |x_n^1 - x_n^2| \right].$$

By the Harnack inequality obtained in Step 3, the difference between the radial averages satisfies

$$\left| \bar{u}_n^2\left(\frac{1}{8}|x_n^1 - x_n^2|\right) - \bar{u}_n^1\left(\frac{1}{8}|x_n^1 - x_n^2|\right) \right| \leq C,$$

uniformly in n . Furthermore, using also (4.42), we have

$$|u_n(x) - \bar{u}_n^1\left(\frac{1}{8}|x_n^1 - x_n^2|\right)| \leq C \quad \text{for every } x \in \Omega_n,$$

and

$$\int_{\Omega_n} e^{u_n} \rightarrow 0, \quad \oint_{\Xi_n} e^{u_n/2} \rightarrow 0.$$

Here we denoted

$$\Omega_n = B_{x_n^1}^+(8|x_n^1 - x_n^2|) \setminus \left(B_{x_n^1}^+\left(\frac{1}{8}|x_n^1 - x_n^2|\right) \cup B_{x_n^2}^+\left(\frac{1}{8}|x_n^1 - x_n^2|\right) \right),$$

$$\Xi_n = \Gamma_{x_n^1}^+(8|x_n^1 - x_n^2|) \setminus \left(\Gamma_{x_n^1}^+\left(\frac{1}{8}|x_n^1 - x_n^2|\right) \cup \Gamma_{x_n^2}^+\left(\frac{1}{8}|x_n^1 - x_n^2|\right) \right).$$

Integrating (4.22) and arguing as in the previous step, we deduce that

$$\oint_{\partial^+ B_{x_n^1}(8|x_n^1 - x_n^2|)} \frac{\partial u_n}{\partial \nu_n} \longrightarrow -8\pi \quad \text{as } n \rightarrow +\infty,$$

because the total flux of ∇u_n through $\partial\Omega_n$ tends to zero. Repeating the

argument of Step 4 then yields

$$\bar{u}_n^1(s) \leq \bar{u}_n^1(8|x_n^1 - x_n^2|) - 2(4 - \delta) \left(\log s - \log(8|x_n^1 - x_n^2|) \right) + C,$$

for all

$$s \in [8|x_n^1 - x_n^2|, \tilde{r}],$$

where $C > 0$ is a fixed constant and $\tilde{r} > 0$ is small but independent of n . These estimates also imply that on the region Ω_n , both $\int_{\Omega_n} e^{u_n}$ and $\oint_{\Xi_n} e^{u_n/2}$ tend to zero.

Step 6 Conclusion: Combining the conclusions of Step 4 and Step 5, we obtain that the bulk integral of e^{u_n} (and similarly the boundary integral of $e^{u_n/2}$) remains uniformly bounded in some fixed neighborhood of p . This proves the first claim of the proposition. It also follows directly that u_n tends to $-\infty$ on $\partial^+ B_p(r)$. $\square$

Lemma 4.19. *Under the assumptions of Proposition 4.16, the asymptotic profiles described in (4.35) and (4.36) hold. Moreover, if $\chi_n \leq 0$, then the set S_1 is empty.*

Proof. If $p \in S_0$, the conclusion follows directly from Case 1 in the proof of Proposition 4.9. Recall that when $\mathfrak{D}(p) = 1$, equation (4.28) admits only the explicit solution in (4.5) (see Section 4). In this regard we also refer to Remark 4.10.

We now focus on the case $p \in S_1$, where the main issue is excluding the formation of a blow-up cluster. Using a conformal chart, the problem may be transferred to a half-ball $B_0^+(r) \subset \mathbb{R}^2$ where p corresponds to the origin. In this setting one has the distributional convergence

$$(|K_n|e^{u_n})|_{B_p(r) \cap \Sigma} \rightharpoonup \tilde{m}(2\pi - \beta)\delta_p, \quad h_n e^{u_n/2}|_{\partial B_p(r) \cap \Sigma} \rightharpoonup \tilde{m}\beta\delta_p, \quad (4.43)$$

for some $\tilde{m} \in \mathbb{N}$, where

$$\beta = 2\pi \left(\frac{h(p)}{\sqrt{h^2(p) + K(p)}} \right). \quad (4.44)$$

To identify $\tilde{m}$, we employ a Pohozaev-type identity. Choose $r > 0$ small enough so that $B_0^+(r) \cap S = \{0\}$ and apply Lemma 2.4 to $B_0^+(r)$ with $F(x) = x$. This yields

$$\begin{aligned} \oint_{\partial B_0^+(r)} [4K_n e^{u_n}(x \cdot \nu) + 2(\nabla u_n \cdot x)(\nabla u_n \cdot \nu) - |\nabla u_n|^2 x \cdot \nu] \\ = \int_{B_0^+(r)} [4\tilde{K}_n \nabla u_n \cdot x + 8K_n e^{u_n} + (\nabla K_n \cdot x)e^{u_n}]. \end{aligned}$$

Splitting the boundary into $\Gamma_0^+(r)$ and $\partial^+ B_0(r)$ gives

$$\oint_{\Gamma_0^+(r)} \left[4h_n e^{u_n/2} (\nabla u_n \cdot x) - 4\tilde{h}_n (\nabla u_n \cdot x) \right] + \quad (4.45)$$

$$r \oint_{\partial^+ B_0(r)} \left[4K_n e^{u_n} + 2 \left(\frac{\nabla u_n \cdot x}{r} \right)^2 - |\nabla u_n|^2 \right] \quad (4.46)$$

$$= \int_{B_0^+(r)} \left[4\tilde{K}_n \nabla u_n \cdot x + 8K_n e^{u_n} + (\nabla K_n \cdot x) e^{u_n} \right]. \quad (4.47)$$

Since $|\nabla K_n| \leq C$ in $B_0^+(r)$, we obtain

$$\int_{B_0^+(r)} (\nabla K_n \cdot x) e^{u_n} \leq r \int_{B_0^+(r)} |\nabla K_n| e^{u_n} = O(r). \quad (4.48)$$

Integrating by parts on $\Gamma_0^+(r)$ yields

$$\oint_{\Gamma_0^+(r)} 4h_n e^{u_n/2} (\nabla u_n \cdot x) = \left[8h_n e^{u_n/2} x_1 \right]_{-r}^r - 8 \oint_{\Gamma_0^+(r)} \nabla h_n \cdot x e^{u_n/2} - 8 \oint_{\Gamma_0^+(r)} h_n e^{u_n/2}.$$

Using $|\nabla h_n| \leq C$ we deduce

$$\oint_{\Gamma_0^+(r)} \nabla h_n \cdot x e^{u_n/2} = O(r). \quad (4.49)$$

By Lemma 5.6,

$$\left[h_n e^{u_n/2} x_1 \right]_{-r}^r \rightarrow 0, \quad \int_{\partial^+ B_0(r)} K_n e^{u_n} \rightarrow 0. \quad (4.50)$$

Next, it is needed to estimate the gradient terms. In order to do it, we introduce the function $v_n = u_n - \inf_{\partial^+ B_0(r)} u_n - w_n$, where v_n, w_n satisfy

$$\begin{cases} -\Delta v_n + 2\tilde{K}_n = 2K_n e^{u_n}, & \text{in } B_0^+(r), \\ \frac{\partial v_n}{\partial n} + 2\tilde{h}_n = 2h_n e^{u_n/2}, & \text{on } \Gamma_0^+(r), \\ v_n = 0, & \text{on } \partial^+ B_0(r), \end{cases}$$

$$\begin{cases} \Delta w_n = 0, & \text{in } B_0^+(r), \\ \frac{\partial w_n}{\partial n} = 0, & \text{on } \Gamma_0^+(r), \\ w_n = u_n - \inf_{\partial^+ B_0(r)} u_n, & \text{on } \partial^+ B_0(r). \end{cases}$$

Recall the Green's representation formula for v_n

$$\begin{aligned}
v_n(x) &= \frac{-1}{\pi} \int_{B_0^+(r)} \log|x-y| (2K_n(y)e^{u_n(y)} - 2\tilde{K}_n(y)) dy \\
&\quad - \frac{1}{\pi} \oint_{\Gamma_0^+(r)} \log|x-y| (2h_n(y)e^{u_n(y)/2} - 2\tilde{h}_n(y)) dy + R_n(x), \quad (4.51)
\end{aligned}$$

where R_n is uniformly bounded.

Let us define the function

$$\varphi_n = u_n - \inf_{B_0^+(r)} u_n \geq 0.$$

By Lemma 4.11, c), we know that $\varphi_n \leq C$ on $\partial^+ B_0(r)$, so using the Green's representation formula for φ_n we obtain that

$$\begin{aligned}
\varphi_n(x) &= \frac{-1}{\pi} \int_{B_0^+(r)} \log|x-y| (2K_n(y)e^{u_n(y)} - 2\tilde{K}_n(y)) dy \\
&\quad - \frac{1}{\pi} \oint_{\Gamma_0^+(r)} \log|x-y| (2h_n(y)e^{u_n(y)/2} - 2\tilde{h}_n(y)) dy + O(1).
\end{aligned}$$

Therefore,

$$\varphi_n \rightarrow \varphi = -4\tilde{m} \log|x| + \phi \quad \text{uniformly in } C^2(B_0^+(r) \setminus \{0\}),$$

where ϕ is a regular function on $B_0^+(r)$. As a consequence, we obtain that

$$\nabla u_n = \nabla \varphi_n \rightarrow -4\tilde{m} \frac{x}{|x|^2} + \nabla \phi. \quad (4.52)$$

Now, by the boundedness of $\tilde{K}_n$, $\tilde{h}_n$ and (4.52) one gets

$$\int_{B_0^+(r)} \tilde{K}_n(\nabla u_n \cdot x) = O(r), \quad \int_{\Gamma_0^+(r)} \tilde{h}_n(\nabla u_n \cdot x) = O(r). \quad (4.53)$$

By (4.81), (4.82), (4.83), (4.53) and (4.52), we can pass to the limit in (4.45) as $r \rightarrow 0$ to conclude that $\tilde{m} = 1$.

We now turn to the pointwise estimates in (4.36). The argument parallels the proof of Theorem 1.6 in [2], combining Green's representation, a Pohozaev identity, and (4.27) together with Lemma 4.11 (c). Considering the limit equation (4.28), for the scaled function v_n introduced in (4.24) one obtains

$$v_n(s, t) = 2 \log \left(\frac{2\lambda_0}{s^2 + (t + t_0)^2 - \lambda_0^2} \right) + O(1), \quad \text{in } B_0^+ \left(\frac{r}{\delta_n} \right),$$

where $t_0 = \mathfrak{D}(0)\lambda_0$ and $\lambda_0 = \frac{2}{\mathfrak{D}^2(0)-1}$ (because $v(0) = 0$). Using Remark 4.10 and that $x_n \in \Gamma_0^+(r)$, this expansion translates into (4.36) for u_n , with $\lambda_n = \lambda_0\delta_n$. The gradient estimate (4.37) follows similarly.

Finally, combining (4.36) and (4.37) we get

$$\int_{B_0^+(r)} |\nabla u_n^-|^2 \geq 16 \int_{B_0^+(r) \setminus B_0^+(\delta_n \log \delta_n)} \frac{1}{|x|^2} + o\left(\frac{1}{|x|^2}\right) = -4 \log(\delta_n \log \delta_n) + o(\log(\delta_n \log \delta_n)).$$

Thus

$$\int_{B_0^+(r)} |\nabla u_n^-|^2 \rightarrow +\infty, \quad (4.54)$$

which contradicts Lemma 4.11 (b). Hence the proof is complete. $\square$

Lemma 4.20. *For every $p \in S_0$, one has $\mathfrak{D}_\tau(p) = 0$.*

Proof. Assume by contradiction that $\mathfrak{D}_\tau(p) \neq 0$ for some $p \in S_0$. Then p is isolated in S_0 , and since S_1 is finite by the previous lemma, it is also isolated in S .

Using a conformal chart we reduce the problem to a half-ball $B_p^+(r) \subset \mathbb{R}^2$ where p is the only singular point. At points in S_0 the blow-up profiles are given by (4.5), which implies that the amount of local volume diverges to $+\infty$. Consequently $\rho_n(u_n) \rightarrow +\infty$.

We now select $F(s, t) = (1, 0)$ in Lemma 2.4, cutting it off near p . Repeating the argument of Proposition 4.14, but replacing the *global* mass ρ_n with the *localized* quantity $\int_{B_p^+(r)} e^{\hat{u}_n}$ (for a fixed small r), and using Lemma 4.11 (c), we again conclude that $\mathfrak{D}_\tau(p) = 0$, a contradiction. $\square$

4.3 The case of the disk

The main result of the blow-up analysis on the disk, which will be used in Chapter 5 to build solutions, is the following

Theorem 4.21. *Let u_n be a sequence of solutions of the problem*

$$\begin{cases} -\Delta u_n = 2K_n e^{u_n} & \text{in } \mathbb{D}, \\ \frac{\partial u_n}{\partial \nu} + 2 = 2h_n e^{u_n/2} & \text{on } \partial\mathbb{D}, \end{cases} \quad (4.55)$$

where $K_n \rightarrow K$ in $C^2(\overline{\mathbb{D}})$ and $h_n \rightarrow h$ in $C^2(\partial\mathbb{D})$ as $n \rightarrow +\infty$. Assume moreover that u_n is blowing-up, namely, $\sup\{u_n\} \rightarrow +\infty$, with bounded global mass,

$$\int_{\mathbb{D}} e^{u_n} + \int_{\partial\mathbb{D}} e^{u_n/2} \leq C. \quad (4.56)$$

Then there exists a unique point $p \in \partial\mathbb{D}$ such that

i) If $K(p) \leq 0$ then $h(p) > \sqrt{-K(p)}$.

ii) There exists $a_n \rightarrow p$ so that

$$u_n(z) = u_{a_n}(z) + \psi_n(z),$$

$$u_{a_n}(z) := 2 \log \left\{ \frac{2\hat{\phi}_n(1 - |a_n|^2)}{\hat{\phi}_n^2 |1 - \bar{a}_n z|^2 + \hat{k}_n |z - a_n|^2} \right\},$$

with

$$\hat{\phi}_n := \phi_n \left(\frac{a_n}{|a_n|} \right), \quad \hat{k}_n := K_n \left(\frac{a_n}{|a_n|} \right),$$

where

$$\phi_n(z) := h_n(z) + \sqrt{h_n(z)^2 + K_n(z)}. \quad (4.57)$$

Moreover, the error term ψ_n satisfies:

$$\|\psi_n\|_{C^{0,\alpha}(\mathbb{D})} \rightarrow 0 \quad \text{for any } \alpha \in (0, 1/2).$$

iii) Let H denote the harmonic extension of h , that is,

$$\begin{cases} \Delta H = 0 & \text{in } \mathbb{D}, \\ H = h & \text{on } \partial\mathbb{D}. \end{cases}$$

Define:

$$\varphi(z) := H(z) + \sqrt{H(z)^2 + K(z)}, \quad (4.58)$$

which, by i), is positive and well defined at least in a neighborhood of p . Then:

$$\nabla \varphi(p) = 0. \quad (4.59)$$

We note that condition (4.59) is equivalent to

$$h_\tau(p) = -\frac{K_\tau(p)}{2\varphi(p)}, \quad (-\Delta)^{1/2}h(p) = -\frac{K_\nu(p)}{2\varphi(p)}.$$

Here τ denotes the tangent vector to $\partial\mathbb{D}$ at the point p . As a consequence, we obtain a compactness criterion: if no such point p exists, then the sequence u_n is uniformly bounded from above and hence precompact. This criterion depends on local quantities involving K and h , as well as on the half-Laplacian of h evaluated at the blow-up point.

Remark 4.22. There are examples in which the finite mass hypothesis (4.56) holds. If $h > 0$ and $K > 0$, it follows from the Gauss–Bonnet theorem; moreover, there are also cases in which either h or K is zero, as pointed out in the Appendix of [20], Proposition A.2, using the isoperimetric inequality and Toponogov’s length bound.

Lemma 4.23. *Let ψ be a solution of*

$$\begin{cases} -\Delta\psi = c(z) & \text{in } \mathbb{D}, \\ \frac{\partial\psi}{\partial\nu} = d(z) & \text{on } \partial\mathbb{D}, \end{cases} \quad \text{with} \quad \int_{\mathbb{D}} \psi = 0.$$

Then, for any $q > 1$, there exists a constant $C_q > 0$ such that

$$\|\psi\|_{W^{1+1/q,q}(\mathbb{D})} \leq C_q \left(\|c\|_{L^q(\mathbb{D})} + \|d\|_{L^q(\partial\mathbb{D})} \right).$$

Lemma 4.24. *For any $q > 1$ there exists $C_q > 0$ such that the following holds. Let ψ be a solution of the nonhomogeneous linearized problem*

$$\begin{cases} -\Delta\psi = 2K_0 e^{v_0}\psi + c(z) & \text{in } \mathbb{D}, \\ \frac{\partial\psi}{\partial\nu} = 2h_0 e^{v_0/2}\psi + d(z) & \text{on } \partial\mathbb{D}, \end{cases} \quad (4.60)$$

and assume that ψ satisfies the orthogonality conditions

$$\int_{\mathbb{D}} x e^{v_0(z)} \psi(z) = 0, \quad \int_{\mathbb{D}} y e^{v_0(z)} \psi(z) = 0. \quad (4.61)$$

Then

$$\|\psi\|_{W^{1+1/q,q}(\mathbb{D})} \leq C_q \left(\|c\|_{L^q(\mathbb{D})} + \|d\|_{L^q(\partial\mathbb{D})} \right).$$

Proof. Let ψ_n be a sequence of solutions to

$$\begin{cases} -\Delta\psi_n = 2K_0 e^{v_0}\psi_n + c_n(z) & \text{in } \mathbb{D}, \\ \frac{\partial\psi_n}{\partial\nu} = 2h_0 e^{v_0/2}\psi_n + d_n(z) & \text{on } \partial\mathbb{D}. \end{cases}$$

We may assume that the functions ψ_n are sufficiently smooth; otherwise, the argument follows by density. Arguing by contradiction, suppose that there exist sequences c_n , d_n and corresponding solutions ψ_n such that

$$\|c_n\|_{L^q(\mathbb{D})} + \|d_n\|_{L^q(\partial\mathbb{D})} = 1, \quad \|\psi_n\|_{W^{1+1/q,q}(\mathbb{D})} \rightarrow +\infty.$$

Define the normalized functions

$$\tilde{\psi}_n := \|\psi_n\|_{W^{1+1/q,q}(\mathbb{D})}^{-1} \psi_n.$$

Up to a subsequence, we have $\tilde{\psi}_n \rightharpoonup \tilde{\psi}_0$ weakly in $W^{1+1/q,q}(\mathbb{D})$. By compactness,

$$2K_0 e^{v_0} \tilde{\psi}_n + \frac{c_n}{\|\psi_n\|_{W^{1+1/q,q}(\mathbb{D})}} \rightarrow 2K_0 e^{v_0} \tilde{\psi}_0 \quad \text{in } L^q(\mathbb{D}),$$

and

$$2h_0 e^{v_0/2} \tilde{\psi}_n + \frac{d_n}{\|\psi_n\|_{W^{1+1/q,q}(\mathbb{D})}} \rightarrow 2h_0 e^{v_0/2} \tilde{\psi}_0 \quad \text{in } L^q(\partial\mathbb{D}).$$

Applying Lemma 4.23, we deduce that $\tilde{\psi}_n \rightarrow \tilde{\psi}_0$ strongly in $W^{1+1/q,q}(\mathbb{D})$. In particular,

$$\|\tilde{\psi}_0\|_{W^{1+1/q,q}(\mathbb{D})} = 1.$$

Moreover, the orthogonality conditions (4.61) are preserved in the limit. Lemma 4.6 then implies $\tilde{\psi}_0 = 0$, which yields a contradiction. $\square$

4.3.1 Single-Point Blow-up and Asymptotic Profile

The purpose of this section is to provide a proof of Proposition 4.25 stated below.

Under the assumptions of Theorem 4.21, we introduce the sequence

$$u_n(\tilde{z}_n) := \max_{\mathbb{D}} u_n(z) \rightarrow +\infty. \quad (4.62)$$

We define the singular set associated with the blowing-up sequence as

$$\mathcal{S} := \{p \in \overline{\mathbb{D}} : \exists z_n \in \overline{\mathbb{D}}, z_n \rightarrow p \text{ such that } u_n(z_n) \rightarrow +\infty\}. \quad (4.63)$$

By (4.62), after possibly passing to a subsequence, we may assume that $\tilde{z}_n \rightarrow p \in \mathcal{S}$ as $n \rightarrow +\infty$.

The main result of this section is the following.

Proposition 4.25. *Under the assumptions of Theorem 4.21, $\mathcal{S} = \{p\} \subset \partial\mathbb{D}$ and, up to a subsequence,*

$$\begin{aligned} u_n &\rightarrow -\infty && \text{locally uniform in } \overline{\mathbb{D}} \setminus \{p\}, \\ h_n e^{u_n/2} &\rightharpoonup \beta \delta_p, && K_n e^{u_n} \rightharpoonup (2\pi - \beta) \delta_p, \end{aligned} \quad (4.64)$$

in the sense of measures, where

$$\beta := 2\pi \frac{h(p)}{\sqrt{h^2(p) + K(p)}}. \quad (4.65)$$

Moreover,

$$u_n(z) = 2 \log \left\{ \frac{2\phi_n(p)(1 - |\zeta_n|^2)}{\phi_n^2(p)|1 - \bar{\zeta}_n z|^2 + K_n(p)|z - \zeta_n|^2} \right\} + O(1), \quad (4.66)$$

where $z \in \bar{\mathbb{D}}$, the function ϕ_n is defined in (4.57) and $\zeta_n \in \mathbb{D}$, with $\zeta_n \rightarrow p$.

Proposition 4.25 represents a preliminary step toward the derivation of the asymptotic behavior stated in Theorem 4.21. In the next section, we shall refine the above description by improving the error term in (4.66) from $O(1)$ to $o(1)$.

The proof of Proposition 4.25 relies on the following result, which concerns a broader class of Liouville-type equations.

Proposition 4.26. *Let u_n be a sequence of solutions of the problems*

$$\begin{cases} -\Delta u_n + 2\tilde{K}_n = 2K_n e^{u_n} & \text{in } \mathbb{D}, \\ \frac{\partial u_n}{\partial \nu} + 2\tilde{h}_n = 2h_n e^{u_n/2} & \text{on } \partial\mathbb{D}, \end{cases} \quad (4.67)$$

where $K_n \rightarrow K$, $\tilde{K}_n \rightarrow \tilde{K}$ in $C^2(\bar{\mathbb{D}})$ and $h_n \rightarrow h$, $\hat{h}_n \rightarrow \hat{h}$ in $C^2(\partial\mathbb{D})$ as $n \rightarrow +\infty$. Assume moreover $\sup\{u_n\} \rightarrow +\infty$, but it has bounded global mass:

$$\int_{\mathbb{D}} e^{u_n} + \int_{\partial\mathbb{D}} e^{u_n/2} \leq C. \quad (4.68)$$

Then the singular set $\mathcal{S}$ is finite. Moreover,

- i) If $p \in \mathcal{S} \setminus \partial\mathbb{D}$, then $K(p) > 0$;
- ii) If $p \in \mathcal{S} \cap (\partial\mathbb{D})$ and $K(p) \leq 0$, then $h(p) > \sqrt{-K(p)}$;
- iii) $u_n \rightarrow -\infty$ locally uniformly in $\bar{\mathbb{D}} \setminus \mathcal{S}$;
- iv)

$$K_n e^{u_n} \rightharpoonup \sum_{p \in \mathcal{S} \setminus (\partial\mathbb{D})} 4\pi \delta_p + \sum_{p \in \mathcal{S} \cap \partial\mathbb{D}} \gamma_p \delta_p, \quad (4.69)$$

and

$$h_n e^{u_n/2} \rightharpoonup \sum_{p \in \mathcal{S} \cap \partial\mathbb{D}} \gamma'_p \delta_p, \quad (4.70)$$

where $\gamma_p + \gamma'_p = 2\pi$ and

$$\gamma_p := 2\pi \frac{h(p)}{\sqrt{h^2(p) + K(p)}}. \quad (4.71)$$

First, we introduce a minimal-mass lemma:

Lemma 4.27. *Under the assumptions of Proposition 4.26, let $p \in \mathbb{D}$ and $r > 0$ be such that $B_r(p) \subset \mathbb{D}$. If*

$$\int_{B_r(p)} K_n^+ e^{u_n} \leq \varepsilon < 2\pi, \quad (4.72)$$

then u_n^+ is uniformly bounded in $L^\infty(B_{\frac{r}{2}}(p))$.

If instead $p \in \partial\mathbb{D}$ and

$$\int_{B_r(p) \cap \overline{\mathbb{D}}} K_n^+ e^{u_n} \leq \varepsilon < \frac{\pi}{2}, \quad \int_{B_r(p) \cap \partial\mathbb{D}} h_n^+ e^{u_n/2} \leq \varepsilon < \frac{\pi}{2}, \quad (4.73)$$

then u_n^+ is uniformly bounded in $L^\infty(B_{\frac{r}{2}}(p) \cap \overline{\mathbb{D}})$.

Proof. The first part of the statement follows directly from [5], Corollary 4.

For the second part, we decompose

$$u_n = u_{1n,+} + u_{1n,-} + u_{2n,+} + u_{2n,-} + u_{3n},$$

where $u_{1n,\pm}$ and $u_{2n,\pm}$ satisfy the following problems:

$$\begin{cases} -\Delta u_{1n,\pm} = 2\tilde{K}_n^\pm e^{u_n} = \tilde{f}_n^\pm & \text{in } B_r^+(0), \\ \partial_\nu u_{1n,\pm} = 0 & \text{on } \Gamma_r(0), \\ u_{1n,\pm} = 0 & \text{on } \partial^+ B_r(0), \end{cases}$$

$$\begin{cases} \Delta u_{2n,\pm} = 0 & \text{in } B_r^+(0), \\ \partial_\nu u_{2n,\pm} = 2\tilde{h}_n^\pm e^{u_n/2} = \tilde{g}_n^\pm & \text{on } \Gamma_r(0), \\ u_{2n,\pm} = 0 & \text{on } \partial^+ B_r(0). \end{cases}$$

Clearly, $u_{1n,-}, u_{2n,-} \leq 0$. By extending $u_{1n,+}$ and $\tilde{f}_n^+$ evenly to $B_r(0)$, we can apply Corollary 4 of [5] to obtain

$$\int_{B_r^+(0)} e^{p|u_{1n,+}|} \leq C,$$

for some $p > 1$. On the other hand, according to [22, Lemma 3.2], for $0 < \delta_1 < 4\pi$ and $0 < \delta_2 < 2\pi$, we have

$$\int_{B_r^+(0)} e^{\frac{4\pi-\delta_1}{\|\tilde{g}_n^+\|_{L^1}} |u_{2n,+}|} \leq C, \quad \int_{\Gamma_r(0)} e^{\frac{2\pi-\delta_2}{\|\tilde{g}_n^+\|_{L^1}} |u_{2n,+}|} \leq C,$$

where $\|\tilde{g}_n^+\|_{L^1} = \int_{\Gamma_r(0)} \tilde{g}_n^+$.

Since $\|\tilde{g}_n^+\|_{L^1} < 2\pi$, it follows that

$$\int_{B_r^+(0)} e^{p_1|u_{2n,+}|} \leq C, \quad \int_{\Gamma_r(0)} e^{p_2|u_{2n,+}|} \leq C,$$

for some $p_1, p_2 > 1$. Consequently, $u_{1n,+}, u_{2n,+} \in L^1(B_r^+(0))$, and moreover, $u_n^+ \in L^1(B_r^+(0))$ by the bounded-mass assumption.

Observe that $u_{3n}^+ = u_n^+ + u_{1n,+} + u_{2n,+}$, so applying the mean-value property of harmonic functions, we deduce $u_{3n}^+ \in L^\infty(B_{r/2}^+(0))$.

Therefore, $\tilde{f}_n \in L^q(B_{r/2}^+(0))$ and $\tilde{g}_n \in L^q(\Gamma_{r/2}(0))$ for some $q > 1$. Finally, standard elliptic estimates imply the desired uniform bound. $\square$

As a consequence, the singular set $\mathcal{S}$ (defined in (4.63)) is necessarily finite. Moreover, we have the following weak convergences:

$$K_n e^{u_n} \rightharpoonup \sum_{p \in \mathcal{S} \cap \mathbb{D}} \alpha_p \delta_p + \sum_{p \in \mathcal{S} \cap \partial \mathbb{D}} \gamma_p \delta_p + \Psi, \quad (4.74)$$

and

$$h_n e^{u_n/2} \rightharpoonup \sum_{p \in \mathcal{S} \cap \partial \mathbb{D}} \gamma'_p \delta_p + \Psi', \quad (4.75)$$

where $\alpha_p \geq 2\pi$, $\gamma_p \geq \frac{\pi}{2}$ or $\gamma'_p \geq \frac{\pi}{2}$, and $\Psi \in L^1(\mathbb{D}) \cap L^\infty_{\text{loc}}(\overline{\mathbb{D}} \setminus \mathcal{S})$, $\Psi' \in L^1(\partial \mathbb{D}) \cap L^\infty_{\text{loc}}(\partial \mathbb{D} \setminus \mathcal{S})$.

Taking (4.68) into account, for any point $p \in \mathcal{S} \cap \mathbb{D}$ we obtain

$$\begin{aligned} 2\pi &\leq \int_{B_r(p)} K_n^+ e^{u_n} \leq \int_{B_r(p)} \left(K_n^+(p) + r \|\nabla K_n^+\|_{L^\infty(B_r(p))} \right) e^{u_n} \\ &\leq C \left(K_n^+(p) + r \|\nabla K_n^+\|_{L^\infty} \right). \end{aligned}$$

Choosing $r > 0$ sufficiently small yields $K(p) > 0$, and thus statement (i) of Proposition 4.26 follows.

Remark 4.28. We point out that, outside the set $\mathcal{S}$, e^{u_n} can be assumed to be uniformly bounded. Moreover, the proof of point (c) in Lemma 4.11 remains valid in the case of the disk, which implies that u_n has bounded oscillation away from $\mathcal{S}$. In particular, we have

$$|u_n(z_1) - u_n(z_2)| \leq C \quad \forall z_1, z_2 \in \overline{\mathbb{D}} \setminus \bigcup_{p \in \mathcal{S}} B_r(p), \quad (4.76)$$

for some fixed $r > 0$.

Consider now a point $p \in \mathcal{S} \cap \partial\mathbb{D}$ and a neighborhood around it that does not contain any other singular points. Using a conformal map, we can reduce the problem to a half-ball. More precisely, we may assume without loss of generality that $p = (1, 0)$. Then, via the Möbius transformation

$$\begin{aligned} f_1 : \mathbb{R}_+^2 &\longrightarrow \mathbb{D} \\ w &\longmapsto z = \frac{i-w}{w+i}, \end{aligned}$$

we can map a half-ball $B_1^+(0) \subset \mathbb{R}_+^2$ into the right half of the unit disk, so that p corresponds to the origin. Define

$$v_n(w) := u_n(f_1(w)) + 2 \log |f_1'(w)| = u_n(f_1(w)) + 2 \log \frac{2}{|w+i|^2}.$$

Then v_n satisfies the problem

$$\begin{cases} -\Delta v_n + 2\hat{K}_n = 2K_n e^{v_n} & \text{in } B_{r_0}^+(0), \\ \frac{\partial v_n}{\partial \nu} + 2\hat{h}_n = 2h_n e^{v_n/2} & \text{on } \Gamma_{r_0}(0) := (-r_0, r_0) \times \{0\} \end{cases} \quad (4.77)$$

Reasoning as in the proof of Lemma 5.6 we have the following Lemma:

Lemma 4.29. *There exist a finite number of sequences $\Sigma_n := \{w_1^n, \dots, w_\ell^n, \hat{w}_1^n, \dots, \hat{w}_k^n\} \rightarrow 0$ and positive sequences $\varepsilon_1^n, \dots, \varepsilon_\ell^n \rightarrow 0$, $\hat{\varepsilon}_1^n, \dots, \hat{\varepsilon}_k^n \rightarrow 0$ such that*

- a) $v_n(w_i^n) = \max\{v_n(w) : w \in B_{\varepsilon_i^n}(w_i^n) \cap \overline{\mathbb{R}_+^2}\}$ for $i = 1, \dots, \ell$,
- b) $\hat{\varepsilon}_i^n \leq \text{dist}(\hat{w}_i^n, \Gamma_{r_0}(0))$ and $v_n(\hat{w}_i^n) = \max\{v_n(w) : w \in B_{\hat{\varepsilon}_i^n}(\hat{w}_i^n)\}$ for $i = 1, \dots, k$,
- c) If $K(0) \leq 0$ then $k = 0$ and $h(0) > \sqrt{-K(0)}$,
- d) $\int_{B_r^+(0)} K_n e^{v_n} + \int_{\Gamma_r(0)} h_n e^{v_n/2} \rightarrow 2\pi\ell + 4\pi k$,
- e) $v_n \rightarrow -\infty$ uniformly on compact subsets of $B_r^+(0) \setminus \{0\}$.

Remark 4.30. In the analysis performed in the proof of Lemma 5.6 only the points w_i^n appear. This is because in that case we were considering only the case where K is strictly negative and hence no solution in the entire plane exists (see also Case 2 in the proof of proposition 4.9).

We point out that the coefficients γ, γ' appearing in the expansions (4.74) and (4.75) (we omit the dependence on p) satisfy

$$\gamma := \lim_{r \rightarrow 0} \lim_{n \rightarrow +\infty} \int_{B_r^+(0)} K_n e^{v_n}, \quad (4.78)$$

$$\gamma' := \lim_{r \rightarrow 0} \lim_{n \rightarrow +\infty} \int_{\Gamma_r(0)} h_n e^{v_n/2}. \quad (4.79)$$

We next prove the quantization of these values by means of a suitable Pohozaev-type identity together with the asymptotic estimates obtained above.

Lemma 4.31. *Let γ and γ' be as in (4.78) and (4.79). Then*

$$\gamma + \gamma' = 2\pi. \quad (4.80)$$

Proof. Apply a Pohozaev-type identity in $B_r^+(0)$ for $0 < r < r_0$, with r_0 as in (4.77). More precisely, using Lemma 2.4 with $F = w$, we obtain

$$\begin{aligned} \int_{\partial B_r^+(0)} \left[2K_n e^{v_n} (w \cdot \nu) + (2h_n e^{v_n/2} - 2\hat{h}_n)(\nabla v_n \cdot w) - \frac{|\nabla v_n|^2}{2} (w \cdot \nu) \right] dw \\ = \int_{B_r^+(0)} \left[2e^{v_n} (\nabla K_n \cdot w + 2K_n) + 2\hat{K}_n (\nabla v_n \cdot w) \right] dw. \end{aligned}$$

Splitting $\partial B_r^+ = \partial^+ B_r \cup \Gamma_r$ yields

$$\begin{aligned} \int_{\partial B_r^+(0)} \left[2K_n e^{v_n} (w \cdot \nu) + (2h_n e^{v_n/2} - 2\hat{h}_n)(\nabla v_n \cdot w) - \frac{|\nabla v_n|^2}{2} (w \cdot \nu) \right] dw \\ = \int_{\partial^+ B_r(0)} \left[2r K_n e^{v_n} + (2h_n e^{v_n/2} - 2\hat{h}_n)(\nabla v_n \cdot w) - \frac{|\nabla v_n|^2}{2} r \right] dw \\ + \int_{\Gamma_r(0)} (2h_n e^{v_n/2} - 2\hat{h}_n) \partial_1 v_n w_1 dw, \end{aligned}$$

where we wrote $w = w_1 + iw_2$. Since $|\nabla K_n| \leq C$ uniformly in $B_r^+(0)$, we have

$$\int_{B_r^+(0)} (\nabla K_n \cdot w) e^{v_n} dw \leq r \int_{B_r^+(0)} |\nabla K_n| e^{v_n} dw = O(r). \quad (4.81)$$

Integrating by parts on the boundary term gives

$$\begin{aligned} \int_{\Gamma_r(0)} 2h_n e^{v_n/2} (\nabla v_n \cdot w) dw = 4 \left[h_n e^{v_n/2} w_1 \right]_{w=(-r,0)}^{(r,0)} - 4 \int_{\Gamma_r(0)} (\nabla h_n \cdot w) e^{v_n/2} dw \\ - 4 \int_{\Gamma_r(0)} h_n e^{v_n/2} dw. \end{aligned}$$

Again, since $|\nabla h_n| \leq C$, we obtain

$$\int_{\Gamma_r(0)} (\nabla h_n \cdot w) e^{v_n/2} dw = O(r). \quad (4.82)$$

Moreover, by Lemma 4.29(e) we deduce

$$\left[h_n e^{v_n/2} w_1 \right]_{w=(-r,0)}^{(r,0)} \rightarrow 0, \quad \int_{\partial^+ B_r(0)} K_n e^{v_n} r dw \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (4.83)$$

Using the Green representation formula we have

$$\begin{aligned}\nabla v_n(w) &= \frac{1}{\pi} \int_{B_r^+(0)} \frac{w-z}{|w-z|^2} 2K_n(z) e^{v_n(z)} dz \\ &\quad + \frac{1}{\pi} \int_{\Gamma_r(0)} \frac{w-z}{|w-z|^2} 2h_n(z) e^{v_n(z)/2} dz + O(r),\end{aligned}$$

and, using (4.74) and (4.75),

$$\nabla v_n(w) \rightarrow \nabla \mathcal{G} = \frac{2}{\pi}(\gamma + \gamma') \frac{w}{|w|^2} + O(r) \quad \text{outside the origin.}$$

Hence

$$\begin{aligned}\int_{\partial^+ B_r(0)} \nabla v_n \cdot w \, dw &\rightarrow \int_{\partial^+ B_r(0)} \nabla \mathcal{G} \cdot w \, dw = O(r), \\ \int_{B_r(0)} \nabla v_n \cdot w \, dw &\rightarrow \int_{B_r(0)} \nabla \mathcal{G} \cdot w \, dw = O(r),\end{aligned}$$

and

$$\int_{\partial^+ B_r(0)} \frac{|\nabla v_n|^2}{2} r \, dw \rightarrow \int_{\partial^+ B_r(0)} \frac{|\nabla \mathcal{G}|^2}{2} r \, dw = 2(\gamma + \gamma')^2 \frac{1}{\pi} + O(r).$$

Plugging these limits into the Pohozaev identity above yields

$$2(\gamma + \gamma')^2 \frac{1}{\pi} = 4(\gamma + \gamma') + O(r),$$

and letting $r \rightarrow 0$ we obtain the claim. $\square$

Observe that the previous lemma, together with Lemma 4.29(d), implies that $\ell = 1$ and $k = 0$. Moreover, by (4.65) we have

$$\gamma = 2\pi \frac{h(0)}{h^2(0) + K(0)}.$$

This completes the proof of Proposition 4.26.

We now return to the proof of 4.25. In this particular setting we first prove that the singular set consists of a single point lying on $\partial \mathbb{D}$.

Lemma 4.32. *The blow-up set is $S = \{p\} \subset \partial \mathbb{D}$.*

Proof. From (2.1) we have

$$\int_{\mathbb{D}} K_n e^{u_n} + \int_{\partial \mathbb{D}} h_n e^{u_n/2} = 2\pi.$$

Moreover, Proposition 4.26 shows that

$$\int_{\mathbb{D}} K_n e^{u_n} + \int_{\partial \mathbb{D}} h_n e^{u_n/2} \rightarrow 4\pi N + 2\pi M,$$

where $N = |\mathcal{S} \cap \mathbb{D}|$ and $M = |\mathcal{S} \cap \partial\mathbb{D}|$. Therefore the only possibility is $N = 0$ and $M = 1$. $\square$

Next, we derive global pointwise estimates for the bubbling sequence near the origin.

Lemma 4.33. *If $K(p) \leq 0$, then $h(p) > \sqrt{-K(p)}$. Moreover, the solutions v_n of (4.77) satisfy*

$$v_n(w_1, w_2) = 2 \log \left\{ \frac{2\lambda_n}{K_n(0)\lambda_n^2 + (w_1 - w_{1,n})^2 + (w_2 - w_{2,n} + h_n(0)\lambda_n)^2} \right\} + O(1), \quad (4.84)$$

in $B_{r_0}^+(0)$, where $w_n = (w_{1,n}, w_{2,n}) \rightarrow 0$ is so that

$$v_n(w_n) = \max_{\overline{B_{r_0}^+(0)}} v_n(w) \rightarrow +\infty$$

and

$$\lambda_n := \frac{2\delta_n}{K_n(0) + h_n^2(0)} \quad \text{with} \quad \delta_n := e^{-\frac{v_n(w_n)}{2}}. \quad (4.85)$$

Proof. Define the rescaled function

$$\tilde{v}_n(w) := v_n(w_n + \delta_n w) + 2 \log \delta_n,$$

that satisfies the problem

$$\begin{cases} -\Delta \tilde{v}_n = 2K_n(w_n + \delta_n w) e^{\tilde{v}_n} & \text{in } B_{r_0/\delta_n}^+ \left(-\frac{w_n}{\delta_n} \right), \\ \frac{\partial \tilde{v}_n}{\partial \nu} = 2h_n(w_n + \delta_n w) e^{\frac{\tilde{v}_n}{2}} & \text{on } \Gamma_{r_0/\delta_n} \left(-\frac{w_n}{\delta_n} \right), \end{cases} \quad (4.86)$$

where

$$\begin{aligned} B_{r_0/\delta_n}^+ \left(-\frac{w_n}{\delta_n} \right) &:= \left\{ (w_1, w_2) \in B_{r_0/\delta_n} \left(-\frac{w_n}{\delta_n} \right) : w_2 \geq -\frac{w_n}{\delta_n} \right\}, \\ \Gamma_{r_0/\delta_n} \left(-\frac{w_n}{\delta_n} \right) &:= (-r_0/\delta_n, r_0/\delta_n) \times \left\{ -\frac{w_n}{\delta_n} \right\}. \end{aligned}$$

By simplicity we assume $-\frac{w_n}{\delta_n} \rightarrow q = 0$. Thus, by the Harnack inequality (see Appendix A):

$$\tilde{v}_n \text{ is uniformly bounded in } L_{loc}^\infty(\mathbb{R}_+^2), \quad (4.87)$$

and

$$\tilde{v}_n \rightarrow v_0 \quad \text{in } C_{loc}^2(\mathbb{R}_+^2), \quad (4.88)$$

where v_0 is the entire solution of the problem

$$\begin{cases} -\Delta v_0 = 2K(0)e^{v_0} & \text{in } \mathbb{R}_+^2, \\ \frac{\partial v_0}{\partial \nu} = 2h(0)e^{\frac{v_0}{2}} & \text{on } \partial\mathbb{R}_+^2, \\ v_0(0) = 0, \end{cases} \quad (4.89)$$

with

$$\int_{\mathbb{R}_+^2} e^{v_0} + \int_{\partial\mathbb{R}_+^2} e^{v_0/2} < C.$$

According to (4.1), if $K(0) \leq 0$, then $h(0) > \sqrt{-K(0)}$. Moreover, v_0 takes the form:

$$v_0(w_1, w_2) = 2 \log \left\{ \frac{2\lambda_0}{K(0)\lambda_0^2 + w_1^2 + (w_2 + \lambda_0 h(0))^2} \right\}, \quad (4.90)$$

where $\lambda_0 := \frac{2}{K(0)+h(0)^2} > 0$. In addition, we have that

$$|w_n| \leq C\delta_n. \quad (4.91)$$

We are now concerned with the global $O(1)$ estimate. By the analysis in Lemma (5.6)(see in particular Step 4), we conclude that for every ε there exists $R_\varepsilon > 1$ such that

$$\tilde{v}_n(w) \leq (4 - \varepsilon) \log |w| + C_\varepsilon, \quad \text{for } |w| \geq 2R_\varepsilon. \quad (4.92)$$

Taking into account (4.76) and (4.88), by Green representation formula one has that

$$\begin{aligned} \tilde{v}_n(w) + \frac{M_n}{\pi} \log |w| &= \frac{1}{\pi} \int_{B_{r_0/\delta_n}^+(-\frac{w_n}{\delta_n})} \log \left(\frac{|w||z|}{|w-z|} \right) 2K_n(w_n + \delta_n z) e^{\tilde{v}_n(z)} dz \\ &\quad + \frac{1}{\pi} \int_{\Gamma_{r_0/\delta_n}(-\frac{w_n}{\delta_n})} \log \left(\frac{|w||z|}{|w-z|} \right) 2h_n(w_n + \delta_n z) e^{\frac{\tilde{v}_n(z)}{2}} dz + \Psi_n, \end{aligned}$$

where Ψ_n is uniformly bounded in $\overline{B_{r_0/\delta_n}^+(-\frac{w_n}{\delta_n})}$ and M_n is defined as

$$\begin{aligned} M_n &:= \int_{B_{r_0}^+(0)} 2K_n e^{v_n} + \int_{\Gamma_{r_0}(0)} 2h_n e^{\frac{v_n}{2}} \\ &= \int_{B_{r_0/\delta_n}^+(-\frac{w_n}{\delta_n})} 2K_n(w_n + \delta_n w) e^{\tilde{v}_n} + \int_{\Gamma_{r_0/\delta_n}(-\frac{w_n}{\delta_n})} 2h_n(w_n + \delta_n w) e^{\frac{\tilde{v}_n}{2}}. \end{aligned}$$

We can estimate the expression above using the asymptotic behavior given by (4.92)

in order to conclude that

$$\left| \tilde{v}_n(w) + \frac{M_n}{\pi} \log |w| \right| = O(1) \quad \text{for } 3 \leq |w| \leq \frac{r_0}{\delta_n}. \quad (4.93)$$

See Lemma 4.2.4. in [34] for further details on the computations of the logarithmic terms.

Finally, using a Pohozaev type identity in $B_{-\delta_n \log \delta_n}^+(0)$ one can obtain that

$$|M_n - 4\pi| = O(|\log \delta_n|^{-1}). \quad (4.94)$$

We refer the reader to Lemma 4.2.5. in [34] for more details.

Combining (4.87), (4.88), (4.93) and (4.94), we conclude

$$\left| \tilde{v}_n(w_1, w_2) - 2 \log \left\{ \frac{2\lambda_0}{K(0)\lambda_0^2 + w_1^2 + (w_2 + h(0)\lambda_0)^2} \right\} \right| \leq C, \quad \text{in } \overline{B}_{r_0/\delta_n}^+(0).$$

Using the definition of $\tilde{v}_n$ and w_n , scaling back to v_n we obtain (4.84). $\square$

We conclude the proof of Proposition 4.25 by establishing the estimate (4.66). To this aim, we transform the solution of (4.77) into a solution defined on the set

$$\mathcal{D} := \left\{ z = x + iy \in \overline{\mathbb{D}} : x > \frac{1 - r_0}{1 + r_0} \right\}.$$

More precisely, we have

$$u_n(z) = v_n(f_1^{-1}(z)) + 2 \log \frac{2}{|z + 1|^2},$$

where f_1^{-1} is the inverse of the Möbius transformation f_1 , namely

$$\begin{aligned} f_1^{-1} : \overline{\mathbb{D}} &\longrightarrow \mathbb{R}_+^2 \\ z &\longmapsto w = i \frac{1-z}{z+1}. \end{aligned}$$

In particular, this gives

$$u_n(z) = 2 \log \left\{ \frac{4\lambda_n}{\left[K_n(0)\lambda_n^2 + \left(\frac{2y}{|z+1|^2} - w_{1,n} \right)^2 + \left(\frac{1-|z|^2}{|z+1|^2} - w_{2,n} + h_n(0)\lambda_n \right)^2 \right] |z + 1|^2} \right\}.$$

To complete the proof, we introduce the sequence

$$\zeta_n := \left(1 - \lambda_n \frac{\phi_n^2(p) + K_n(p)}{\phi_n(p)}\right) p.$$

Using (4.85), we observe that $\zeta_n \rightarrow p$ as $n \rightarrow +\infty$ and $|\zeta_n| < 1$. Moreover, (4.91) holds by the definition of ζ_n .

We then consider the function

$$\tilde{u}_n(z) := 2 \log \left\{ \frac{2\phi_n(p)(1 - |\zeta_n|^2)}{\phi_n^2(p)|1 - \bar{\zeta}_n z|^2 + K_n(p)|z - \zeta_n|^2} \right\}, \quad z \in \mathcal{D}. \quad (4.95)$$

It can be shown that

$$|\tilde{u}_n(z) - u_n(z)| = O(1),$$

that is,

$$u_n(z) = \tilde{u}_n(z) + O(1), \quad z \in \mathcal{D}.$$

Finally, we extend this estimate to the entire disk. For any $z \in \mathbb{D}$ and $z_1 \in \mathcal{D}$, by (4.76) and (4.95) we obtain

$$\begin{aligned} & \left| u_n(z) - 2 \log \left\{ \frac{2\phi_n(p)(1 - |\zeta_n|^2)}{\phi_n^2(p)|1 - \bar{\zeta}_n z|^2 + K_n(p)|z - \zeta_n|^2} \right\} \right| \\ & \leq \left| u_n(z_1) - 2 \log \left\{ \frac{2\phi_n(p)(1 - |\zeta_n|^2)}{\phi_n^2(p)|1 - \bar{\zeta}_n z|^2 + K_n(p)|z - \zeta_n|^2} \right\} \right| + |u_n(z) - u_n(z_1)| \\ & \leq C, \end{aligned}$$

which provides the desired conclusion.

4.3.2 Estimate of the remainder term

Proposition 4.34. *Assume that the hypotheses of Theorem 4.21 hold. Then there exists a sequence $a_n \in \mathbb{D}$ such that, setting*

$$v_n(z) := u_n(f_{a_n}(z)) + 2 \log \frac{1 - |a_n|^2}{|1 + \bar{a}_n z|^2}, \quad f_a(z) := \frac{a + z}{1 + \bar{a}z},$$

the following properties are satisfied:

(a) v_n solves

$$\begin{cases} -\Delta v_n = 2K_n(f_{a_n}(z))e^{v_n(z)} & \text{in } \mathbb{D}, \\ \frac{\partial v_n}{\partial \nu} + 2 = 2h_n(f_{a_n}(z))e^{v_n(z)/2} & \text{on } \partial\mathbb{D}; \end{cases} \quad (4.96)$$

(b)

$$\int_{\mathbb{D}} x e^{v_n(z)} dz = \int_{\mathbb{D}} y e^{v_n(z)} dz = 0;$$

(c) $a_n \rightarrow p$, where $p \in \partial\mathbb{D}$ is the blow-up point;

(d) the sequence v_n is uniformly bounded;

(e) v_n admits the decomposition

$$v_n(z) = 2 \log \left(\frac{2\hat{\phi}_n}{\hat{\phi}_n^2 + \hat{k}_n |z|^2} \right) + \xi_n(z),$$

where

$$\hat{\phi}_n := \phi_n \left(\frac{a_n}{|a_n|} \right), \quad \hat{k}_n := K_n \left(\frac{a_n}{|a_n|} \right),$$

ϕ_n is defined in (4.57), and

$$\|\xi_n\|_{C^{0,\alpha}(\overline{\mathbb{D}})} \leq C(1 - |a_n|)^{1-\alpha} \quad \text{for every } \alpha \in (0, 1/2).$$

Proof. Property (a) follows directly from the definition of v_n . We next prove (b). For a given continuous function u , define the map $\Gamma : \mathbb{D} \rightarrow \mathbb{R}^2$ by

$$\Gamma(a) := \int_{\mathbb{D}} z e^{v_a(z)} dz, \quad v_a(z) := u(f_a(z)) + 2 \log |f'_a(z)|.$$

We claim that there exists $a \in \mathbb{D}$ such that $\Gamma(a) = 0$. Indeed, if $a_n \rightarrow a \in \partial\mathbb{D}$, then by the change of variables $z' = f_{a_n}(z)$ and the Lebesgue dominated convergence theorem,

$$\begin{aligned} \int_{\mathbb{D}} z e^{v_{a_n}(z)} dz &= \int_{\mathbb{D}} z e^{u(f_{a_n}(z))} |f'_{a_n}(z)|^2 dz = \int_{\mathbb{D}} f_{-a_n}(z') e^{u(z')} dz' \\ &\longrightarrow -a \int_{\mathbb{D}} e^{u(z')} dz'. \end{aligned}$$

Hence Γ extends continuously to $\partial\mathbb{D}$ by setting $\Gamma(a) = -a \int_{\mathbb{D}} e^u$. It follows that the Brouwer degree of Γ is equal to 1, and the claim is proved.

We now establish (c). Suppose by contradiction that, up to a subsequence, $a_n \rightarrow a_0 \in \overline{\mathbb{D}}$ with $a_0 \neq p$. Since $f_{a_n} \rightarrow f_{a_0}$ uniformly on $B_p(r)$ for small $r > 0$, and recalling from Proposition 4.25 that $K_n e^{u_n} \rightharpoonup \beta \delta_p$, we compute

$$\begin{aligned} 0 &= \int_{\mathbb{D}} z e^{v_n(z)} dz = \int_{\mathbb{D}} z e^{u_n(f_{a_n}(z))} |f'_{a_n}(z)|^2 dz \\ &= \int_{\mathbb{D}} f_{-a_n}(z) e^{u_n(z)} dz \longrightarrow c_0 \frac{-a_0 + p}{1 - \overline{a_0} p}, \end{aligned}$$

for some constant $c_0 \neq 0$, yielding a contradiction. Therefore $a_n \rightarrow p$.

Let u_n be as in Proposition 4.25 and define

$$v_n(z) := u_n(f_{a_n}(z)) + 2 \log \frac{1 - |a_n|^2}{|1 + \bar{a}_n z|^2}.$$

Using the explicit expansion of u_n , we obtain

$$v_n(z) = 2 \log \left(\frac{2\phi_n(p)(1 - |\zeta_n|^2)(1 - |a_n|^2)}{\phi_n^2(p)|1 - \bar{\zeta}_n a_n + z(\bar{a}_n - \bar{\zeta}_n)|^2 + K_n(p)|a_n - \zeta_n + z(1 - \bar{a}_n \zeta_n)|^2} \right) + O(1).$$

Introducing

$$\tilde{a}_n := \frac{a_n - \zeta_n}{1 - \bar{a}_n \zeta_n} \in \mathbb{D},$$

this expression can be rewritten as

$$v_n(z) = 2 \log \left(\frac{2\phi_n(p)(1 - |\tilde{a}_n|^2)}{\phi_n^2(p)|1 - \bar{\tilde{a}}_n z|^2 + K_n(p)|z - \tilde{a}_n|^2} \right) + O(1).$$

If $\tilde{a}_n \rightarrow \tilde{a} \in \partial\mathbb{D}$, then $v_n(\tilde{a}_n) \rightarrow +\infty$ and $e^{v_n} \rightarrow 0$ locally in $\mathbb{D} \setminus \{\tilde{a}\}$, contradicting the fact that v_n has vanishing barycenter. Hence $|\tilde{a}_n| \leq 1 - \varepsilon$ for some $\varepsilon > 0$, which proves (d).

We finally prove (e). By (d), the right-hand sides of (4.96) are uniformly bounded in L^∞ . Lemma 4.23 yields uniform bounds for v_n in $W^{1+1/q, q}(\mathbb{D})$ for all $q > 1$, and by Sobolev embeddings, v_n is bounded in $C^{0, \alpha}(\overline{\mathbb{D}})$. Up to a subsequence, $v_n \rightarrow v_0$ in $C^{0, \alpha}(\overline{\mathbb{D}})$, where v_0 solves

$$\begin{cases} -\Delta v_0 = 2K(p)e^{v_0} & \text{in } \mathbb{D}, \\ \frac{\partial v_0}{\partial \nu} + 2 = 2h(p)e^{v_0/2} & \text{on } \partial\mathbb{D}, \end{cases}$$

and satisfies $\int_{\mathbb{D}} x e^{v_0} = \int_{\mathbb{D}} y e^{v_0} = 0$. By Remark 4.5, v_0 coincides with the radial profile (4.19).

Set

$$\tilde{v}_n(z) := 2 \log \left(\frac{2\hat{\phi}_n}{\hat{\phi}_n^2 + \hat{k}_n |z|^2} \right), \quad \xi_n := v_n - \tilde{v}_n.$$

Then, $\xi_n \rightarrow 0$ in $C^{0, \alpha}(\overline{\mathbb{D}})$ for every $\alpha \in (0, 1)$. We now aim to obtain a quantitative rate for this convergence.

Notice that ξ_n solves the following problem:

$$\begin{cases} -\Delta \xi_n = 2K_n\left(f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right) e^{\tilde{v}_n} \xi_n + c_n(z) & \text{in } \mathbb{D}, \\ \frac{\partial \xi_n}{\partial \nu} = 2h_n\left(f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right) e^{\tilde{v}_n/2} \xi_n + d_n(z) & \text{on } \partial \mathbb{D}, \end{cases}$$

where

$$\begin{aligned} c_n(z) &:= 2\left[K_n(f_{a_n}(z)) - K_n\left(f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right)\right] e^{v_n} \\ &\quad + 2K_n\left(f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right) e^{\tilde{v}_n} (e^{\xi_n} - 1 - \xi_n), \\ d_n(z) &:= 2\left[h_n(f_{a_n}(z)) - h_n\left(f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right)\right] e^{v_n/2} \\ &\quad + 2h_n\left(f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right) e^{\tilde{v}_n/2} (e^{\xi_n/2} - 1 - \xi_n/2). \end{aligned}$$

By the mean value theorem, there exists a constant $C > 0$ such that

$$\left|K_n(f_{a_n}(z)) - K_n\left(f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right)\right| \leq C \left|f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right|,$$

and similarly

$$\left|h_n(f_{a_n}(z)) - h_n\left(f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right)\right| \leq C \left|f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right|.$$

Applying Lemma 4.36 below, we deduce that for every $1 < q < 2$,

$$\int_{\mathbb{D}} \left|f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right|^q \leq C(1 - |a_n|)^q, \quad \int_{\partial \mathbb{D}} \left|f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right|^q \leq C(1 - |a_n|).$$

Since $\xi_n \rightarrow 0$ uniformly, we also have

$$\|e^{\xi_n} - 1 - \xi_n\|_{L^q(\mathbb{D})} \leq C \|\xi_n\|_{L^\infty(\mathbb{D})} \|\xi_n\|_{L^q(\mathbb{D})}.$$

We can therefore apply Lemma 4.24 to obtain

$$\|\xi_n\|_{W^{1+1/q,q}(\mathbb{D})} \leq C_q(1 - |a_n|)^{1/q} + o_n(1) \left(\|\xi_n\|_{L^q(\mathbb{D})} + \|\xi_n\|_{L^q(\partial \mathbb{D})} \right). \quad (4.97)$$

On the other hand, the Sobolev trace estimate yields

$$\|\xi_n\|_{L^q(\mathbb{D})} + \|\xi_n\|_{L^q(\partial \mathbb{D})} \leq C \|\xi_n\|_{W^{1+1/q,q}(\mathbb{D})}.$$

Hence, choosing n sufficiently large, inequality (4.97) gives

$$\|\xi_n\|_{W^{1+1/q,q}(\mathbb{D})} \leq (C_q + 2)(1 - |a_n|)^{1/q}.$$

Finally, recalling the continuous embedding

$$W^{1+1/q,q}(\mathbb{D}) \hookrightarrow C^{0,\alpha}(\overline{\mathbb{D}}), \quad \alpha = 1 - \frac{1}{q},$$

we conclude the proof of Proposition 4.34. □

Remark 4.35. We are now able to prove part *i*) and *ii*) of Theorem 4.21: From the previous discussion, conclusion *i*) follows immediately. Furthermore, using the notation of Proposition 4.34, assertion *ii*) is achieved by setting

$$u_n(z) := v_n(f_{-a_n}(z)) - 2 \log \frac{1 - |a_n|^2}{|1 + \overline{a_n}z|^2}.$$

Note that

$$\psi_n(z) = \xi_n(f_{-a_n}(z)),$$

and that $|f'_{-a_n}(z)| \leq C(1 - |a_n|)$. Consequently, the mean value theorem yields the following estimate for the $C^{0,\alpha}$ norm of ψ_n :

$$\begin{aligned} |\psi_n(z_1) - \psi_n(z_2)| &= |\xi_n(f_{-a_n}(z_1)) - \xi_n(f_{-a_n}(z_2))| \\ &\leq C(1 - |a_n|)^{1-\alpha} |f_{-a_n}(z_1) - f_{-a_n}(z_2)|^\alpha \\ &\leq C(1 - |a_n|)^{1-2\alpha} |z_1 - z_2|^\alpha. \end{aligned}$$

For the sake of clarity of the exposition, we isolate the purely technical estimates of the previous proposition in the following lemma.

Lemma 4.36. *Let $1 < q < 2$ and let $\{a_n\} \subset \mathbb{D}$ be a sequence such that $|a_n| \rightarrow 1$ as $n \rightarrow +\infty$. Then there exists a constant $C > 0$, independent of n , such that*

i)

$$\int_{\mathbb{D}} \left| f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right) \right|^q \leq C(1 - |a_n|)^q,$$

ii)

$$\int_{\partial\mathbb{D}} \left| f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right) \right|^q \leq C(1 - |a_n|),$$

where $f_a(z) := \frac{a + z}{1 + \overline{a}z}$.

Proof. By the explicit expression of the Möbius transformation f_a , we compute

$$\begin{aligned} \left| f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right) \right| &= \left| \frac{a_n + z}{1 + \overline{a_n}z} - \frac{a_n}{|a_n|} \right| \\ &= \frac{1 - |a_n|}{|a_n|} \left| \frac{a_n - |a_n|z}{1 + \overline{a_n}z} \right| \leq C(1 - |a_n|) \frac{1}{|1 + \overline{a_n}z|}, \end{aligned}$$

for $|a_n|$ sufficiently close to 1.

Consequently,

$$\int_{\mathbb{D}} \left| f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right) \right|^q \leq C(1 - |a_n|)^q \int_{\mathbb{D}} \frac{1}{|1 + \overline{a_n}z|^q}. \quad (4.98)$$

The integrand is singular at the point

$$z_0 = -\frac{1}{\overline{a_n}} = -\frac{a_n}{|a_n|^2},$$

which satisfies $|z_0| > 1$ and hence does not belong to $\mathbb{D}$. Since $1 < q < 2$, the integral on the right-hand side of (4.98) is finite, and this yields assertion (i).

We now turn to the estimate on the boundary. For $z \in \partial\mathbb{D}$, we observe that

$$|1 + \overline{a_n}z| = |\overline{z} + \overline{a_n}| = |z + a_n| = \text{dist}(z, -a_n).$$

Arguing as above, we deduce

$$\begin{aligned} \int_{\partial\mathbb{D}} \left| f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right) \right|^q &\leq C(1 - |a_n|)^q \int_{\partial\mathbb{D}} \frac{1}{|1 + \overline{a_n}z|^q} \\ &= C(1 - |a_n|)^q \int_{\partial\mathbb{D}} \frac{1}{\text{dist}(z, -a_n)^q}. \end{aligned} \quad (4.99)$$

Notice that

$$\text{dist}(z, -a_n) \geq \text{dist}\left(-\frac{a_n}{|a_n|}, -a_n\right) = 1 - |a_n|.$$

We decompose the boundary as

$$\Sigma_1 := \partial\mathbb{D} \setminus B_1\left(-\frac{a_n}{|a_n|}\right), \quad \Sigma_2 := \partial\mathbb{D} \cap B_1\left(-\frac{a_n}{|a_n|}\right).$$

If $z \in \Sigma_1$, there exists a constant $c > 0$, independent of n , such that $\text{dist}(z, -a_n) \geq c$. Therefore,

$$\int_{\Sigma_1} \frac{1}{\text{dist}(z, -a_n)^q} \leq c^{-q} |\Sigma_1| \leq c^{-q} |\partial\mathbb{D}|. \quad (4.100)$$

To estimate the contribution of Σ_2 , we partition it into annular arcs:

$$L_1 := B_{(1-|a_n|)}\left(-\frac{a_n}{|a_n|}\right) \cap \Sigma_2,$$

$$L_j := \left(B_{j(1-|a_n|)}\left(-\frac{a_n}{|a_n|}\right) \setminus B_{(j-1)(1-|a_n|)}\left(-\frac{a_n}{|a_n|}\right)\right) \cap \Sigma_2, \quad j = 2, \dots, J_n,$$

where $J_n := \lceil (1 - |a_n|)^{-1} \rceil$.

We note that

$$|L_j| \simeq 2(1 - |a_n|),$$

and that

$$\text{dist}(z, -a_n) \geq 1 - |a_n|, \quad z \in L_1,$$

$$\text{dist}(z, -a_n) \geq (j - 1)(1 - |a_n|), \quad z \in L_j, \quad j \geq 2.$$

Hence,

$$\int_{L_1} \frac{1}{\text{dist}(z, -a_n)^q} \leq C(1 - |a_n|)^{1-q},$$

and for $j \geq 2$,

$$\int_{L_j} \frac{1}{\text{dist}(z, -a_n)^q} \leq \frac{C}{(j - 1)^q} (1 - |a_n|)^{1-q}.$$

Summing over j we obtain

$$\begin{aligned} \int_{\Sigma_2} \frac{1}{\text{dist}(z, -a_n)^q} &\leq C(1 - |a_n|)^{1-q} \left(1 + \sum_{j=1}^{+\infty} \frac{1}{j^q}\right) \\ &\leq C(1 - |a_n|)^{1-q}. \end{aligned} \tag{4.101}$$

Combining (4.100) and (4.101) with (4.99), we conclude the proof of (ii). $\square$

4.3.3 The criticality condition at the blow-up point

In this section we complete the proof of Theorem 4.21 by establishing point (iii). By testing equation (4.55) against a suitably chosen vector field and integrating, we derive a first necessary blow-up condition.

Proposition 4.37. *Let $p \in S$ and let φ be defined as in (4.58). Then the following identity holds:*

$$2h_\tau(p) + \frac{K_\tau(p)}{\varphi(p)} = 0, \tag{4.102}$$

or, equivalently, $\varphi_\tau(p) = 0$.

Proof. Let us introduce the vector field $F : \overline{\mathbb{D}} \rightarrow \mathbb{R}^2$ given by

$$F(x, y) = (-y, x).$$

Notice that F is tangent to $\partial\mathbb{D}$. Applying Lemma 2.4 and performing an integration by parts, we obtain

$$-2 \int_{\partial\mathbb{D}} (u_n)_\tau = 4 \int_{\partial\mathbb{D}} (h_n)_\tau e^{u_n/2} + 2 \int_{\mathbb{D}} \nabla K_n \cdot F e^{u_n}, \quad (4.103)$$

where u_n denotes the solution of (4.55) satisfying assumptions (i) and (ii) of Theorem 4.21. In deriving (4.103) we have used that $\nabla \cdot F = 0$ and that $DF(\nabla u_n, \nabla u_n) = 0$.

Arguing as in the proof of the distributional convergences (4.64), and exploiting the mass quantization stated in Proposition 4.25, we deduce that

$$(h_n)_\tau e^{u_n/2} \rightharpoonup 2\pi \frac{h_\tau(p)}{\sqrt{h^2(p) + K(p)}} \delta_p,$$

and

$$\varphi_n e^{u_n} \rightharpoonup 2\pi \frac{1}{\sqrt{h^2(p) + K(p)}} \delta_p,$$

where Φ is defined in (4.58).

Using these convergences, together with pointwise estimates on u_n , and multiplying and dividing appropriately by φ_n , we infer that the right-hand side of (4.103) converges to

$$8\pi \frac{h_\tau(p)}{\sqrt{h^2(p) + K(p)}} + 4\pi \frac{K_\tau(p)}{\varphi(p) \sqrt{h^2(p) + K(p)}}. \quad (4.104)$$

Since

$$\int_{\partial\mathbb{D}} (u_n)_\tau = 0,$$

the limit in (4.104) must vanish, which yields (4.37). $\square$

From now on, without loss of generality, we assume that $p = (1, 0)$. By composing K_n and h_n with suitable rotations, we may also suppose that $\lambda_n \in \mathbb{R}$ and $\lambda_n \rightarrow 1$. We then consider the family

$$u_{\lambda_n}(x, y) := 2 \log \left(\frac{2(1 - \lambda_n^2) \hat{\phi}_n}{\hat{\phi}_n^2(1 - \lambda_n x)^2 + \hat{\phi}_n^2(\lambda_n y)^2 + \hat{k}_n(x - \lambda_n)^2 + \hat{k}_n y^2} \right), \quad (4.105)$$

which corresponds to the blow-up profile described in Theorem 4.21. Observe that

the denominator is strictly positive as a consequence of the condition

$$\varphi(p)^2 + K(p) > 0.$$

Lemma 4.38. *Assume that $u_n = u_{\lambda_n} + \psi_n$, where u_{λ_n} is defined by (4.105) and*

$$\|\psi_n\|_{C^{0,\alpha}} \leq C(1 - \lambda_n)^{1-2\alpha}, \quad \alpha \in (0, 1/2).$$

Then the following asymptotic expansions hold:

(i)

$$\int_{\partial\mathbb{D}} (h_n)_\tau e^{\frac{u_n}{2}} y = e^{\frac{\psi_n(p)}{2}} \int_{\partial\mathbb{D}} (h_n)_\tau e^{\frac{u_{\lambda_n}}{2}} y + o_n(1)(1 - \lambda_n).$$

(ii)

$$\begin{aligned} & \int_{\mathbb{D}} \left((K_n)_y xy - (K_n)_x (1 - x^2 + y^2) \right) e^{u_n} \\ &= e^{\psi_n(p)} \int_{\mathbb{D}} \left((K_n)_y xy - (K_n)_x (1 - x^2 + y^2) \right) e^{u_{\lambda_n}} + o_n(1)(1 - \lambda_n). \end{aligned}$$

Here $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. By definition,

$$e^{\frac{u_n}{2}} = e^{\frac{u_{\lambda_n} + \psi_n}{2}} = e^{\frac{\psi_n(p)}{2}} e^{\frac{u_{\lambda_n}}{2}} e^{\frac{\psi_n - \psi_n(p)}{2}} = e^{\frac{\psi_n(p)}{2}} e^{\frac{u_{\lambda_n}}{2}} \left(1 + O(\psi_n - \psi_n(p)) \right).$$

Therefore,

$$\begin{aligned} \int_{\partial\mathbb{D}} (h_n)_\tau e^{\frac{u_n}{2}} y &= e^{\frac{\psi_n(p)}{2}} \int_{\partial\mathbb{D}} (h_n)_\tau e^{\frac{u_{\lambda_n}}{2}} y \\ &+ e^{\frac{\psi_n(p)}{2}} \int_{\partial\mathbb{D}} (h_n)_\tau e^{\frac{u_{\lambda_n}}{2}} y O(\psi_n - \psi_n(p)) =: I. \end{aligned}$$

Using the Hölder bound on ψ_n , we estimate

$$\begin{aligned} |I| &\leq C(1 - \lambda_n)^{1-2\alpha} \int_{\partial\mathbb{D}} e^{\frac{u_{\lambda_n}}{2}} |y| |z - 1|^\alpha dz \\ &\leq C(1 - \lambda_n)^{2-2\alpha} \int_{\partial\mathbb{D}} \frac{|y| |z - 1|^\alpha}{\hat{\phi}_n^2(1 - \lambda_n x)^2 + \hat{\phi}_n^2(\lambda_n y)^2 + \hat{k}_n(x - \lambda_n)^2 + \hat{k}_n y^2} dz \\ &\leq C(1 - \lambda_n)^{2-2\alpha}, \end{aligned}$$

which proves (i).

The proof of (ii) follows along the same lines. Indeed,

$$\begin{aligned} & \int_{\mathbb{D}} \left((K_n)_y xy - (K_n)_x (1 - x^2 + y^2) \right) e^{u_n} \\ &= e^{\psi_n(p)} \int_{\mathbb{D}} \left((K_n)_y xy - (K_n)_x (1 - x^2 + y^2) \right) e^{u_{\lambda_n}} \\ &+ e^{\psi_n(p)} \int_{\mathbb{D}} \left((K_n)_y xy - (K_n)_x (1 - x^2 + y^2) \right) e^{u_{\lambda_n}} O(\psi_n - \psi_n(p)) =: II, \end{aligned}$$

and one checks that

$$\begin{aligned} |II| &\leq C(1 - \lambda_n)^{2-2\alpha} \int_{\mathbb{D}} \frac{(|x||y| + |1 - x| + |y|^2)|z - 1|^\alpha (1 + \lambda_n)}{(\hat{\phi}_n^2(1 - \lambda_n x)^2 + \hat{\phi}_n^2(\lambda_n y)^2 + \hat{k}_n(x - \lambda_n)^2 + \hat{k}_n y^2)^2} dz \\ &\leq C(1 - \lambda_n)^{2-2\alpha}, \end{aligned}$$

which completes the proof. $\square$

The following result completes the proof of Theorem 4.21.

Theorem 4.39. *Let φ be defined as in (4.58). Then the identity*

$$2(-\Delta)^{1/2} h(p) + \frac{K_x(p)}{\varphi(p)} = 0 \quad (4.106)$$

holds, or equivalently $\varphi_\nu(p) = 0$.

Proof. Let $u_n(z) = u_{\lambda_n}(z) + \psi_n(z)$ be a solution of (4.55) satisfying assumption (ii) of Theorem 4.21, where u_{λ_n} is given by (4.105) and $\lambda_n \in (-1, 1)$. By Proposition 2.5 we have

$$2 \int_{\mathbb{D}} (K_n)_y e^{u_n} xy - \int_{\mathbb{D}} (K_n)_x e^{u_n} (1 - x^2 + y^2) = -4 \int_{\partial \mathbb{D}} (h_n)_\tau e^{u_n/2} y. \quad (4.107)$$

Applying Lemma 4.38, the previous identity becomes

$$2 \int_{\mathbb{D}} (K_n)_y e^{u_{\lambda_n}} xy - \int_{\mathbb{D}} (K_n)_x e^{u_{\lambda_n}} (1 - x^2 + y^2) = -4 \int_{\partial \mathbb{D}} (h_n)_\tau e^{u_{\lambda_n}/2} y + o_n(1)(1 - \lambda_n). \quad (4.108)$$

We first analyze the boundary term on the right-hand side. Integrating by parts

and inserting the explicit expression of u_{λ_n} , we obtain

$$\begin{aligned}
-4 \int_{\partial \mathbb{D}} (h_n)_\tau e^{u_{\lambda_n}/2} y &= -4 \int_{\partial \mathbb{D}} (h_n - h_n(p))_\tau e^{u_{\lambda_n}/2} y \\
&= 4 \int_{\partial \mathbb{D}} (h_n - h_n(p)) \left(e^{u_{\lambda_n}/2} \frac{(u_{\lambda_n})_\tau}{2} y + e^{u_{\lambda_n}/2} x \right) \\
&= 8(1 - \lambda_n^2) \frac{\hat{\phi}_n}{\hat{\phi}_n^2 + \hat{k}_n} \int_{\partial \mathbb{D}} (h_n - h_n(p)) \frac{x(1 - \lambda_n)^2 + 2\lambda_n(x - 1)}{((1 - \lambda_n)^2 - 2\lambda_n(x - 1))^2}.
\end{aligned}$$

We now turn to the left-hand side of (4.108). A further integration by parts yields

$$\begin{aligned}
&2 \int_{\mathbb{D}} (K_n)_y e^{u_{\lambda_n}} xy - \int_{\mathbb{D}} (K_n)_x e^{u_{\lambda_n}} (1 - x^2 + y^2) \\
&= \int_{\mathbb{D}} (K_n - K_n(p)) \left[(e^{u_{\lambda_n}})_x (1 - x^2 + y^2) - 2(e^{u_{\lambda_n}})_y xy - 4x e^{u_{\lambda_n}} \right] \\
&= -16(1 - \lambda_n^2)^2 \hat{\phi}_n^2 (\hat{\phi}_n^2 + \hat{k}_n) \int_{\mathbb{D}} (K_n - K_n(p)) \\
&\quad \times \frac{-\lambda_n((1 - x)^2 + y^2) + (1 - \lambda_n)^2 x}{\left[\hat{\phi}_n^2 (1 - \lambda_n x)^2 + \hat{\phi}_n^2 (\lambda_n y)^2 + \hat{k}_n (x - \lambda_n)^2 + \hat{k}_n y^2 \right]^3}.
\end{aligned}$$

Setting

$$\begin{aligned}
I_n &:= \int_{\partial \mathbb{D}} (h_n - h_n(p)) \frac{x(1 - \lambda_n)^2 + 2\lambda_n(x - 1)}{((1 - \lambda_n)^2 - 2\lambda_n(x - 1))^2}, \\
II_n &:= (1 - \lambda_n) \int_{\mathbb{D}} \frac{(K_n - K_n(p)) \left(-\lambda_n((1 - x)^2 + y^2) + (1 - \lambda_n)^2 x \right)}{(\hat{\phi}_n^2 (1 - \lambda_n x)^2 + \hat{\phi}_n^2 (\lambda_n y)^2 + \hat{k}_n (x - \lambda_n)^2 + \hat{k}_n y^2)^3},
\end{aligned}$$

equation (4.108) is equivalent to

$$I_n = -2(1 + \lambda_n) \hat{\phi}_n (\hat{\phi}_n^2 + \hat{k}_n)^2 II_n + o_n(1). \quad (4.109)$$

We now pass to the limit as $n \rightarrow \infty$, corresponding to the concentration of the bubble at p .

Fix $\varepsilon > 0$ and assume $1 - \lambda_n < \varepsilon$. Writing

$$f_n(x, y) := \frac{x(1 - \lambda_n)^2 + 2\lambda_n(x - 1)}{((1 - \lambda_n)^2 - 2\lambda_n(x - 1))^2},$$

we decompose

$$\begin{aligned}
I_n &= \int_{\partial \mathbb{D} \setminus B_\varepsilon(p)} (h_n - h_n(p)) f_n + \int_{\partial \mathbb{D} \cap (B_\varepsilon(p) \setminus B_{1-\lambda_n}(p))} (h_n - h_n(p)) f_n \\
&\quad + \int_{\partial \mathbb{D} \cap B_{1-\lambda_n}(p)} (h_n - h_n(p)) f_n =: I_1 + I_2 + I_3.
\end{aligned}$$

Since the integrand is uniformly integrable on $\partial\mathbb{D} \setminus B_\varepsilon(p)$, by dominated convergence we find

$$\lim_{n \rightarrow \infty} I_1 = \int_{\partial\mathbb{D} \setminus B_\varepsilon(p)} \frac{h(p) - h}{2(1-x)}.$$

Using the symmetry of f_n with respect to y , together with uniform C^2 bounds on h_n , one checks that

$$|I_2| \leq C\varepsilon, \quad |I_3| \leq C(1 - \lambda_n),$$

hence

$$\lim_{n \rightarrow \infty} I_2 = O(\varepsilon), \quad \lim_{n \rightarrow \infty} I_3 = 0.$$

We now estimate II_n . Writing

$$g_n(x, y) := \frac{-\lambda_n((1-x)^2 + y^2) + (1-\lambda_n)^2 x}{(\hat{\phi}_n^2(1-\lambda_n x)^2 + \hat{\phi}_n^2(\lambda_n y)^2 + \hat{k}_n(x-\lambda_n)^2 + \hat{k}_n y^2)^3},$$

we decompose

$$II_n = (1-\lambda_n) \int_{\mathbb{D} \setminus B_\rho(p)} (K_n - K_n(p)) g_n + (1-\lambda_n) \int_{\mathbb{D} \cap B_\rho(p)} (K_n - K_n(p)) g_n =: II_1 + II_2.$$

We start by examining the region $\mathbb{D} \setminus B_\rho(p)$. For a fixed $0 < \rho < 1$ and for $z = (x, y)$ one has $|x-1| \geq c(\rho) > 0$ for some constant depending on ρ . Consequently,

$$|x - \lambda_n| = |x - 1 - \lambda_n + 1| \geq |x - 1| - |1 - \lambda_n| \geq c(\rho) - |1 - \lambda_n| \geq \frac{c(\rho)}{2}$$

for λ_n sufficiently close to 1. Moreover,

$$(1 - \lambda_n x)^2 \geq (x - \lambda_n)^2,$$

so, assuming $\Phi(p)^2 + K(p) > 0$, for λ_n near 1 we obtain

$$|II_1| \leq C(1 - \lambda_n) \|K\|_{L^\infty(\mathbb{D})},$$

where C is a positive constant independent of n . Hence

$$\lim_{n \rightarrow \infty} II_1 = 0.$$

We now estimate II_2 . Using the expansion

$$K_n(x, y) - K_n(p) - (K_n)_y(p)y = (K_n)_x(p)(x-1) + O(|z-1|^2),$$

and the fact that g_n is even in y , we split the integral as

$$\begin{aligned} II_2 &= (1 - \lambda_n)(K_n)_x(p) \int_{\mathbb{D} \cap B_\rho(p)} (x - 1)g_n + (1 - \lambda_n) \int_{\mathbb{D} \cap B_\rho(p)} O(|z - 1|^2)g_n \\ &=: II_{21} + II_{22}. \end{aligned}$$

After the change of variables

$$1 - x = \tilde{x}(1 - \lambda_n), \quad y = \tilde{y}(1 - \lambda_n),$$

we get

$$II_{21} = (K_n)_x(p) \int_{(\mathbb{R}_+ \times \mathbb{R}) \cap B_{\frac{\rho}{1-\lambda_n}}(0)} \frac{\tilde{x}(\tilde{x}(1 - \lambda_n) - 1 + \lambda_n(\tilde{x}^2 + \tilde{y}^2)) d\tilde{y} d\tilde{x}}{(\hat{\phi}_n^2(1 + \lambda_n\tilde{x})^2 + \hat{\phi}_n^2(\lambda_n\tilde{y})^2 + \hat{k}_n(1 - \tilde{x})^2 + \hat{k}_n\tilde{y}^2)^3}.$$

First suppose $K(p) > 0$. Since $\tilde{x} \geq 0$ and $(1 + \lambda_n\tilde{x})^2 \geq 1 + \lambda_n^2\tilde{x}^2$, for λ_n sufficiently close to 1 we have the pointwise bound

$$\begin{aligned} &\left| \frac{\tilde{x}(\tilde{x}(1 - \lambda_n) - 1 + \lambda_n(\tilde{x}^2 + \tilde{y}^2))}{(\hat{\phi}_n^2(1 + \lambda_n\tilde{x})^2 + \hat{\phi}_n^2(\lambda_n\tilde{y})^2 + \hat{k}_n(1 - \tilde{x})^2 + \hat{k}_n\tilde{y}^2)^3} \chi_{(\mathbb{R}_+ \times \mathbb{R}) \cap B_{\frac{\rho}{1-\lambda_n}}(0)} \right| \\ &\leq C \frac{\tilde{x}(2(\tilde{x}^2 + \tilde{y}^2) + 1)}{(\varphi(p)^2 + \varphi(p)^2 \frac{\tilde{x}^2}{2} + K(p)\tilde{y}^2)^3}, \end{aligned}$$

which belongs to $L^1(\mathbb{R}_+ \times \mathbb{R})$.

If instead $K(p) \leq 0$, then for λ_n close to 1 one has

$$\begin{aligned} \hat{\phi}_n^2(1 + \lambda_n\tilde{x})^2 + \hat{k}_n(1 - \tilde{x})^2 &= \hat{\phi}_n^2 + \hat{k}_n + \tilde{x}^2(\hat{\phi}_n^2\lambda_n^2 + \hat{k}_n) - 2\tilde{x}(-\lambda_n\hat{\phi}_n^2 + \hat{k}_n) \\ &\geq \hat{\phi}_n^2 + \hat{k}_n + \tilde{x}^2(\hat{\phi}_n^2\lambda_n^2 + \hat{k}_n), \end{aligned}$$

because $\tilde{x} \geq 0$ and $\Phi(p)^2 + K(p) > 0$. Therefore there exists $c_0 > 0$, independent of λ_n , such that

$$\begin{aligned} &\left| \frac{\tilde{x}(\tilde{x}(1 - \lambda_n) - 1 + \lambda_n(\tilde{x}^2 + \tilde{y}^2))}{(\hat{\phi}_n^2(1 + \lambda_n\tilde{x})^2 + \hat{\phi}_n^2(\lambda_n\tilde{y})^2 + \hat{k}_n(1 - \tilde{x})^2 + \hat{k}_n\tilde{y}^2)^3} \chi_{(\mathbb{R}_+ \times \mathbb{R}) \cap B_{\frac{\rho}{1-\lambda_n}}(0)} \right| \\ &\leq c_0 \frac{\tilde{x}(2(\tilde{x}^2 + \tilde{y}^2) + 1)}{(1 + \tilde{x}^2 + \tilde{y}^2)^3} \in L^1(\mathbb{R}_+ \times \mathbb{R}). \end{aligned}$$

By dominated convergence we can pass to the limit in the integral and obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} II_{21} &= K_x(p) \left[\int_{\mathbb{R}_+ \times \mathbb{R}} \frac{\tilde{x}((\tilde{x}^2 + \tilde{y}^2) - 1) d\tilde{y} d\tilde{x}}{(\varphi(p)^2(1 + \tilde{x})^2 + (\varphi(p)^2 + K(p))\tilde{y}^2 + K(p)(1 - \tilde{x})^2)^3} \right] \\ &= \frac{\pi}{8\varphi(p)^2(K(p) + \varphi(p)^2)^2} K_x(p). \end{aligned}$$

Next, for II_{22} we estimate

$$\begin{aligned}
|II_{22}| &\leq C(1 - \lambda_n) \int_{(\mathbb{R}_+ \times \mathbb{R}) \cap B_{\frac{\rho}{1-\lambda_n}}(0)} \frac{(\tilde{x}^2 + \tilde{y}^2) |\tilde{x}(1 - \lambda_n) - 1 + \lambda_n(\tilde{x}^2 + \tilde{y}^2)| d\tilde{y} d\tilde{x}}{|\hat{\phi}_n^2(1 + \lambda_n \tilde{x})^2 + \hat{\phi}_n^2(\lambda_n \tilde{y})^2 + \hat{k}_n(1 - \tilde{x})^2 + \hat{k}_n \tilde{y}^2|^3} \\
&\leq C(1 - \lambda_n) \int_{(\mathbb{R}_+ \times \mathbb{R}) \cap B_{\frac{\rho}{1-\lambda_n}}(0)} \frac{(\tilde{x}^2 + \tilde{y}^2)(2(\tilde{x}^2 + \tilde{y}^2) + 1) d\tilde{y} d\tilde{x}}{(1 + \tilde{x}^2 + \tilde{y}^2)^3} \\
&= O((1 - \lambda_n) \log(1 - \lambda_n)),
\end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} II_{22} = 0.$$

Taking the limit in (4.109) yields

$$\int_{\partial \mathbb{D} \setminus B_\varepsilon(p)} \frac{h(p) - h}{2(1 - x)} + O(\varepsilon) = -\frac{\pi}{2\Phi(p)} K_x(p).$$

Letting $\varepsilon \rightarrow 0$ we obtain

$$\text{p.v.} \int_{\partial \mathbb{D}} \frac{h(p) - h}{2(1 - x)} = -\frac{\pi}{2\Phi(p)} K_x(p).$$

Recalling the definition of $(-\Delta)^{1/2}$ on $\mathbb{S}^1$, we conclude:

$$2(-\Delta)^{1/2} h(p) + \frac{K_x(p)}{\varphi(p)} = 0.$$

□

Remark 4.40. As shown above, any concentration point of a blowing-up sequence must be a critical point of φ . This can be interpreted in a variational way. Consider the energy functional associated to (1.2),

$$I(u) := \int_{\mathbb{D}} \frac{1}{2} |\nabla u|^2 - 2K e^u + \int_{\partial \mathbb{D}} 2u - 4h e^{u/2}, \quad I : H^1(\mathbb{D}) \rightarrow \mathbb{R}.$$

Evaluate I on the family u_{λ_n} from (4.105) as $\lambda_n \rightarrow 1$. Due to concentration, one has

$$I(u_{\lambda_n}) = \tilde{I}(u_{\lambda_n}) + o_n(1),$$

where $\tilde{I}$ is the limiting functional obtained by freezing the curvatures at p ,

$$\tilde{I}(u) := \int_{\mathbb{D}} \frac{1}{2} |\nabla u|^2 - 2K(p) e^u + \int_{\partial \mathbb{D}} 2u - 4h(p) e^{u/2}.$$

Note that $\tilde{I}(u_{\lambda_n})$ does not depend on λ_n , so for convenience we may set $\lambda_n = 0$.

Thus

$$u_0(z) = 2 \log \left(\frac{2\varphi(p)}{\varphi(p)^2 + K(p)|z|^2} \right).$$

We compute the terms

$$\begin{aligned} \int_{\mathbb{D}} \frac{1}{2} |\nabla u_0|^2 &= 8 \frac{K(p)^2}{\varphi(p)^4} 2\pi \int_0^1 \frac{r^3 dr}{(1 + \frac{K(p)}{\varphi(p)^2} r^2)^2} = 8\pi \int_0^{\frac{K(p)}{\varphi(p)^2}} \frac{s ds}{(1+s)^2} \\ &= 8\pi \left(\log \left(1 + \frac{K(p)}{\varphi(p)^2} \right) + \frac{\varphi(p)^2}{\varphi(p)^2 + K(p)} - 1 \right), \end{aligned}$$

and

$$\int_{\partial \mathbb{D}} 2u_0 = 8\pi \log \frac{2\varphi(p)}{\varphi(p)^2 + K(p)}.$$

Using the known values of the exponential integrals (see (4.64)), we obtain

$$\begin{aligned} \tilde{I}(u_0) &= 8\pi \left(\log(2/\varphi(p)) + \frac{\varphi(p)^2}{\varphi(p)^2 + K(p)} - 1 \right) - 4\pi \left(1 + \frac{h(p)}{h^2(p) + K(p)} \right) \\ &= -8\pi + 4\pi \left(2\log(2/\varphi(p)) + \frac{2\varphi(p)^2}{\varphi(p)^2 + K(p)} - 1 - \frac{h(p)}{h^2(p) + K(p)} \right). \end{aligned}$$

By the defining relation $\frac{2\varphi(p)^2}{\varphi(p)^2 + K(p)} - 1 - \frac{h(p)}{h^2(p) + K(p)} = 0$, it follows that

$$\tilde{I}(u_0) = -8\pi(1 + \log(\varphi(p)/2)).$$

Chapter 5

Constructing Solutions via Ljapunov-Schmidt Reduction

5.1 The inverse problem on the disk

Motivated by the Theorem 4.21, one can ask the following natural question: *If $p \in \partial\mathbb{D}$ is a critical point of the function φ , does there exist a family of solutions which blow-up at p ?* In this chapter, we give a positive answer and obtain blow-up solutions via a method that accurately describes their behavior.

Up to compose with a rotation, we can suppose $p = 1$, thus the condition of Theorem 4.21 can be rewritten as

$$h(1)^2 + K(1) > 0, \quad \varphi(1) = h(1) + \sqrt{h(1)^2 + K(1)} > 0. \quad (5.1)$$

and

$$\partial_x K(1) + 2\varphi(1)(-\Delta)^{\frac{1}{2}}h(1) = 0 \quad \text{and} \quad \partial_y K(1) + 2\varphi(1)\partial_\tau h(1) = 0. \quad (5.2)$$

We impose the following *non-degeneracy* assumptions at the critical point:

$$\Delta K(1) + 4|\nabla H(1)|^2 \neq 0, \quad (5.3)$$

$$\partial_{yy} K(1) + 2\varphi(1)h''(1) \neq 0, \quad (5.4)$$

and we also assume

$$2K(1) + \varphi(1)h(1) \neq 0. \quad (5.5)$$

We next introduce suitable linear perturbations of the functions K and h leading

to blowing-up solutions. More precisely, we set

$$K_\varepsilon(z) := K(z) + \varepsilon G(z), \quad h_\varepsilon(z) := h(z) + \varepsilon I(z),$$

where $G \in C^2(\overline{\mathbb{D}^2})$ and $I \in C^2(\partial\mathbb{D}^2)$. For simplicity, we assume that the perturbations vanish at the concentration point, namely

$$G(1) = I(1) = 0, \tag{5.6}$$

which is not a restrictive condition.

In addition, we impose the generic assumption

$$\partial_1 G(1) + 2\varphi(1)(-\Delta)^{\frac{1}{2}} I(1) \neq \left(\partial_2 G(1) + 2\varphi(1)I'(1) \right) \frac{\partial_{12}K(1) + 2\varphi(1)(-\Delta)^{\frac{1}{2}} h'(1)}{\partial_{yy}K(1) + 2\varphi(1)h''(1)}. \tag{5.7}$$

Although (5.7) may appear technical, it represents a natural transversality condition on the perturbations G and I . Indeed, if we define the perturbed map

$$\tilde{\varphi}(\xi, \varepsilon) = H_\varepsilon(\xi) + \sqrt{H_\varepsilon(\xi)^2 + K_\varepsilon(\xi)},$$

where H_ε denotes the harmonic extension of h_ε , a direct computation shows that (5.7) is equivalent to requiring that the vectors $\partial_{\xi_2} \nabla_\xi \tilde{\Phi}(0, 0)$ and $\partial_\varepsilon \nabla_\xi \tilde{\Phi}(0, 0)$ are not parallel, namely

$$\det \begin{pmatrix} \partial_{\xi_1, \xi_2} \tilde{\varphi}(0, 0) & \partial_{\xi_1, \varepsilon} \tilde{\varphi}(0, 0) \\ \partial_{\xi_2, \xi_2} \tilde{\varphi}(0, 0) & \partial_{\xi_2, \varepsilon} \tilde{\varphi}(0, 0) \end{pmatrix} \neq 0.$$

This is the main result of the chapter:

Theorem 5.1. *Assume (5.1)-(5.7). Then, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ or for every $\varepsilon \in (-\varepsilon_0, 0)$ there exists a solution u_ε of*

$$\begin{cases} -\Delta u = 2K_\varepsilon(z)e^u & \text{in } \mathbb{D}, \\ \partial_\nu u + 2 = 2h_\varepsilon(z)e^{\frac{u}{2}} & \text{on } \partial\mathbb{D}, \end{cases} \tag{5.8}$$

which blows-up at 1 as $\varepsilon \rightarrow 0$.

Moreover, there exist $\delta_\varepsilon > 0$ and $\xi_\varepsilon \in \partial\mathbb{D}$ with

$$\delta_\varepsilon = O\left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}}\right) \quad \text{and} \quad \xi_\varepsilon = 1 + O(|\varepsilon|)$$

such that

$$u_\varepsilon(f_{\delta_\varepsilon, \xi_\varepsilon}(z)) + 2 \log |f'_{\delta_\varepsilon, \xi_\varepsilon}(z)| - 2 \log \left(\frac{2\varphi(1)}{\varphi(1)^2 + K(1)|z|^2} \right) = O \left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}} \right) \quad \text{in } H^1(\mathbb{D}).$$

Here $f_{\delta, \xi}$ is the conformal map

$$f = f_{\delta, \xi}(z) := \frac{z + (1 - \delta)\xi}{1 + (1 - \delta)\bar{\xi}z}, \quad \delta \in \mathbb{R}, \quad z, \xi \in \mathbb{C}. \quad (5.9)$$

From now on we identify $\mathbb{R}^2$ with $\mathbb{C}$, and thus we regard every point z, ξ as a complex number.

Let us briefly outline the proof and clarify the role of the assumptions. Observe that u solves (5.8) if and only if

$$v(z) := u(f(z)) + 2 \log |f'(z)|$$

solves

$$\begin{cases} -\Delta v = 2K_\varepsilon(f(z))e^v & \text{in } \mathbb{D}, \\ \partial_\nu v + 2 = 2h_\varepsilon(f(z))e^{\frac{v}{2}} & \text{on } \partial\mathbb{D}, \end{cases} \quad (5.10)$$

where $f = f_{\delta, \xi}$ is given by (5.9). We choose $\delta = \delta_\varepsilon \rightarrow 0$ and $\xi = \xi_\varepsilon \rightarrow 1$ (equivalently, $\xi = e^{i\eta}$ with $\eta_\varepsilon \rightarrow 0$).

To prove Theorem 5.1, we perform a Ljapunov-Schmidt reduction. Roughly speaking, we look for a solution of the form

$$v(z) = V(z) + o_\varepsilon(1),$$

where

$$V_\xi(z) = 2 \log \left(\frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)|z|^2} \right), \quad (5.11)$$

which solves

$$\begin{cases} -\Delta V = 2K(\xi)e^V & \text{in } \mathbb{D}, \\ \partial_\nu V + 2 = 2h(\xi)e^{\frac{V}{2}} & \text{on } \partial\mathbb{D}. \end{cases} \quad (5.12)$$

However, taking V as first approximation produces an error that is too large. To refine the ansatz, we introduce the correction term

$$W_\xi(z) = -\frac{2}{\pi} \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}} \log |z - w| (h(f(w)) - h(\xi)) dw, \quad (5.13)$$

which solves

$$\begin{cases} -\Delta W = 0 & \text{in } \mathbb{D}, \\ \partial_\nu W = 2(h(f(z)) - h(\xi))e^{\frac{V}{2}} - \frac{1}{\pi}e^{\frac{V}{2}} \int_{\partial\mathbb{D}} (h(f(w)) - h(\xi)) dw & \text{on } \partial\mathbb{D}. \end{cases} \quad (5.14)$$

Since V is constant on $\partial\mathbb{D}^2$ and the solution of (5.14) is defined up to additive constants, we further modify the ansatz by adding a small parameter $\tau = \tau_\xi$. We therefore look for a solution in the form

$$v(z) = V(z) + W(z) + \tau + \phi(z),$$

where ϕ is a small term obtained through a suitable linear theory.

The introduction of the parameter τ is essential, since the linear analysis is carried out in a Hilbert space that excludes constant functions; this is possible thanks to assumption (5.5).

Once ϕ is constructed, the problem reduces to determining the finite-dimensional parameters δ , ξ , and τ . Their existence follows from assumptions (5.3), (5.4), (5.6), and (5.7).

5.1.1 Ansatz and error estimates

We look for a solution of (5.10) of the form

$$\boxed{v(z) = V(z) + W(z) + \tau + \phi(z),}$$

where $V = V_\xi$ and $W = W_\xi$ are given by (5.12) and (5.14), $\tau = \tau_\xi$ is a constant, and $\phi(z) = \phi_{\xi, \delta, \tau}(z)$ is a small correction term. Our goal is to determine the three parameters $\delta = \delta_\varepsilon > 0$ with $\delta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$, $\xi = \xi_\varepsilon \in \partial\mathbb{D}$ with $\xi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 1$ (equivalently, $\xi = e^{i\eta}$ with $\eta = \eta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$), and $\tau_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$ such that ϕ satisfies

$$\begin{cases} -\Delta(V + W + \tau + \phi) = 2K_\varepsilon(f(z))e^{V+W+\tau+\phi} & \text{in } \mathbb{D}, \\ \partial_\nu(V + W + \tau + \phi) + 2 = 2h_\varepsilon(f(z))e^{\frac{V+W+\tau+\phi}{2}} & \text{on } \partial\mathbb{D}, \end{cases}$$

where $f = f_{\delta, \xi}$ denotes the conformal map defined in (5.9). The positivity of δ ensures that $(1 - \delta)\xi$ forms a sequence of points approaching ξ_0 from the interior of the disk.

Equivalently, the problem can be rewritten as

$$\begin{aligned} -\Delta\phi - 2K(\xi)e^V\phi &= 2\left(K_\varepsilon(f(z))e^{W+\tau} - K(\xi)\right)e^V \\ &\quad + 2\left(K_\varepsilon(f(z))e^{W+\tau} - K(\xi)\right)e^V\phi \\ &\quad + 2K_\varepsilon(f(z))e^{V+W+\tau}\left(e^\phi - 1 - \phi\right) \quad \text{in } \mathbb{D}, \end{aligned}$$

with boundary condition

$$\begin{aligned} \partial_\nu\phi - h(\xi)e^{\frac{V}{2}}\phi &= 2\left(h_\varepsilon(f(z))e^{\frac{W+\tau}{2}} - h(f(z))\right)e^{\frac{V}{2}} + \frac{1}{\pi}e^{\frac{V}{2}} \int_{\partial\mathbb{D}^2} (h(f(w)) - h(\xi)) dw \\ &\quad + \left(h_\varepsilon(f(z))e^{\frac{W+\tau}{2}} - h(\xi)\right)e^{\frac{V}{2}}\phi \\ &\quad + 2h_\varepsilon(f(z))e^{\frac{V+W+\tau}{2}}\left(e^{\frac{\phi}{2}} - 1 - \frac{\phi}{2}\right) \quad \text{on } \partial\mathbb{D}, \end{aligned}$$

which can be compactly expressed as

$$\begin{cases} \mathcal{L}_0^{\text{Int}}\phi = \mathcal{E}^{\text{Int}} + \mathcal{L}^{\text{Int}}\phi + \mathcal{N}^{\text{Int}}(\phi) & \text{in } \mathbb{D}, \\ \mathcal{L}_0^\partial\phi = \mathcal{E}^\partial + \mathcal{L}^\partial\phi + \mathcal{N}^\partial(\phi) & \text{on } \partial\mathbb{D}, \end{cases} \quad (5.15)$$

with

$$\begin{aligned} \mathcal{L}_0^{\text{Int}}\phi &:= -\Delta\phi - 2K(\xi)e^V\phi, \\ \mathcal{E}^{\text{Int}} &:= 2\left(K_\varepsilon(f(z))e^{W+\tau} - K(\xi)\right)e^V, \\ \mathcal{L}^{\text{Int}}\phi &:= 2\left(K_\varepsilon(f(z))e^{W+\tau} - K(\xi)\right)e^V\phi, \\ \mathcal{N}^{\text{Int}}(\phi) &:= 2K_\varepsilon(f(z))e^{V+W+\tau}(e^\phi - 1 - \phi), \end{aligned} \quad (5.16)$$

and, for $z \in \partial\mathbb{D}^2$,

$$\begin{aligned} \mathcal{L}_0^\partial\phi &:= \partial_\nu\phi - h(\xi)e^{\frac{V}{2}}\phi, \\ \mathcal{E}^\partial &:= 2\left(h_\varepsilon(f(z))e^{\frac{W+\tau}{2}} - h(f(z))\right)e^{\frac{V}{2}} + \frac{1}{\pi}e^{\frac{V}{2}} \int_{\partial\mathbb{D}} (h(f(w)) - h(\xi)) dw, \\ \mathcal{L}^\partial\phi &:= \left(h_\varepsilon(f(z))e^{\frac{W+\tau}{2}} - h(\xi)\right)e^{\frac{V}{2}}\phi, \\ \mathcal{N}^\partial(\phi) &:= 2h_\varepsilon(f(z))e^{\frac{V+W+\tau}{2}}\left(e^{\frac{\phi}{2}} - 1 - \frac{\phi}{2}\right). \end{aligned} \quad (5.17)$$

We impose the scaling

$$\boxed{\delta = O\left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}}\right), \quad \tau = O\left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}}\right), \quad \eta = O(|\varepsilon|),} \quad (5.18)$$

where $\xi = e^\eta$. These orders are justified at the final step of the finite-dimensional reduction and will be assumed throughout for notational simplicity.

Lemma 5.2. *The correction W satisfies*

$$W(z) = \frac{8\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \delta \left((-\Delta)^{1/2} h(\xi) \log |z + \xi| + h'(\xi) \arctan \frac{\langle z, \xi^\perp \rangle}{1 + \langle z, \xi \rangle} \right) + O\left(\frac{\delta^2}{\delta + |z + \xi|} \left(1 + \left|\log \frac{|z + \xi|}{\delta}\right|\right)\right).$$

In particular,

$$W(z) = O\left(\delta \left(1 + \log \frac{1}{|z + \xi|}\right)\right), \quad (5.19)$$

and for any $p \in [1, +\infty)$ and δ sufficiently small,

$$\|W\|_{L^p(\mathbb{D})} + \|W\|_{L^p(\partial\mathbb{D})} = O(\delta), \quad \|e^{|W|}\|_{L^p(\mathbb{D})} + \left\|e^{\frac{|W|}{2}}\right\|_{L^p(\partial\mathbb{D})} = O(1). \quad (5.20)$$

Proof. By Proposition A.3 we write

$$\begin{aligned} W(z) &= -\frac{2}{\pi} \int_{\partial\mathbb{D}^2} \log |z - w| (h(f(w)) - h(\xi)) \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} dw \\ &= -\frac{4\varphi(\xi)}{\pi(\varphi(\xi)^2 + K(\xi))} \left(\log |z + \xi| \underbrace{\int_{\partial\mathbb{D}^2} (h(f(w)) - h(\xi)) dw}_{=: I_1} \right. \\ &\quad \left. + \delta h'(\xi) \underbrace{\int_{\partial\mathbb{D}^2} \log \frac{|z - w|}{|z + \xi|} \Theta(w) dw}_{=: I_2} \right. \\ &\quad \left. + \underbrace{\int_{\partial\mathbb{D}^2} \log \frac{|z - w|}{|z + \xi|} \left(h(f(w)) - h(\xi) - \delta h'(\xi) \Theta(w) \right) dw}_{=: I_3} \right), \end{aligned}$$

where

$$\Theta(w) := \frac{2\langle w, \xi^\perp \rangle}{1 + (1 - \delta)^2 + 2(1 - \delta)\langle w, \xi \rangle}.$$

From Proposition A.4 we know that

$$I_1 = -2\pi\delta(-\Delta)^{\frac{1}{2}}h(\xi) + O(\delta^2);$$

hence the result follows if we prove

$$I_2 = -2\pi \arctan \frac{\langle z, \xi^\perp \rangle}{1 + \langle z, \xi \rangle} + O\left(\frac{\delta}{\delta + |z + \xi|}\right), \quad (5.21)$$

$$I_3 = O(A(z)), \quad (5.22)$$

where

$$A(z) = A_{\delta, \xi}(z) := \frac{\delta^2}{\delta + |z + \xi|} \left(1 + \left|\log \frac{|z + \xi|}{\delta}\right|\right).$$

To deal with (5.21) we use the fact that Θ is odd with respect to the map $w \mapsto \xi^2 \bar{w}$, so that

$$\int_{\partial \mathbb{D}^2} \Theta(w) dw = 0,$$

and, by Green's representation formula, I_2 is the solution to

$$\begin{cases} -\Delta I_2 = 0 & \text{in } \mathbb{D}^2, \\ \partial_\nu I_2 = -\pi \Theta & \text{on } \partial \mathbb{D}^2, \end{cases}$$

that is

$$I_2 = -\frac{2\pi}{1 - \delta} \arctan \frac{(1 - \delta) \langle z, \xi^\perp \rangle}{1 + (1 - \delta) \langle z, \xi \rangle}.$$

Using the fundamental theorem of calculus and estimates in the proof of Proposition A.3 we get

$$\begin{aligned} & -\frac{2\pi}{1 - \delta} \arctan \frac{(1 - \delta) \langle z, \xi^\perp \rangle}{1 + (1 - \delta) \langle z, \xi \rangle} - \left(-2\pi \arctan \frac{\langle z, \xi^\perp \rangle}{1 + \langle z, \xi \rangle} \right) \\ &= 2\pi \left(\arctan \frac{\langle z, \xi^\perp \rangle}{1 + \langle z, \xi \rangle} - \arctan \frac{(1 - \delta) \langle z, \xi^\perp \rangle}{1 + (1 - \delta) \langle z, \xi \rangle} \right) \\ & \quad + 2\pi \frac{\delta}{1 - \delta} \arctan \frac{(1 - \delta) \langle z, \xi^\perp \rangle}{1 + (1 - \delta) \langle z, \xi \rangle} \\ &= -2\pi \int_0^\delta \frac{\langle z, \xi^\perp \rangle}{|1 + (1 - s) \bar{\xi} z|^2} ds + O(\delta) \\ &= O\left(\int_0^\delta \frac{|z + \xi|}{(s + |z + \xi|)^2} ds + \delta\right) \\ &= O\left(\frac{\delta}{\delta + |z + \xi|}\right), \end{aligned}$$

hence (5.21) is proved.

We are left with showing (5.22); thanks to Proposition A.3, we need to show:

$$\left| \int_{\partial \mathbb{D}^2} \log \frac{|z-w|}{|z+\xi|} \left(\frac{\delta}{\delta+|w+\xi|} \right)^2 dw \right| = O(A(z)).$$

We split the integral in three parts, depending on whether z is much closer to w than to $-\xi$, much farther, or the distances are comparable.

If $|z-w| \leq \frac{|z+\xi|}{2}$, then $|w+\xi| \geq \frac{|z+\xi|}{2}$, and

$$\begin{aligned} & \left| \int_{\{|z-w| \leq \frac{|z+\xi|}{2}\}} \log \frac{|z-w|}{|z+\xi|} \left(\frac{\delta}{\delta+|w+\xi|} \right)^2 dw \right| \\ &= O \left(\left(\frac{\delta}{\delta + \frac{|z+\xi|}{2}} \right)^2 \int_{\{|z-w| \leq \frac{|z+\xi|}{2}\}} \log \frac{|z+\xi|}{|z-w|} dw \right) \\ &= O \left(\left(\frac{\delta}{\delta + |z+\xi|} \right)^2 |z+\xi| \int_{\{t < \frac{1}{2}\}} \log \frac{1}{t} dt \right) \\ &= O \left(\frac{\delta^2}{\delta + |z+\xi|} \right) \\ &= O(A(z)). \end{aligned}$$

If $\frac{|z+\xi|}{2} < |z-w| \leq 2|z+\xi|$, then

$$\left| \log \frac{|z-w|}{|z+\xi|} \right| = \left| \log \left(1 + \frac{|z-w| - |z+\xi|}{|z+\xi|} \right) \right| = O \left(\frac{|z-w| - |z+\xi|}{|z+\xi|} \right) = O \left(\frac{|w+\xi|}{|z+\xi|} \right);$$

moreover, $|w + \xi| \leq 3|z + \xi|$, therefore

$$\begin{aligned}
& \int_{\{\frac{|z+\xi|}{2} < |z-w| \leq 2|z+\xi|\}} \left| \log \frac{|z-w|}{|z+\xi|} \right| \left(\frac{\delta}{\delta + |w+\xi|} \right)^2 dw \\
&= O \left(\int_{\{|w+\xi| \leq 3|z+\xi|\}} \frac{|w+\xi|}{|z+\xi|} \left(\frac{\delta}{\delta + |w+\xi|} \right)^2 dw \right) \\
&= O \left(\int_{\{|w+\xi| \leq \min\{\delta, 3|z+\xi|\}\}} \frac{|w+\xi|}{|z+\xi|} dw \right. \\
&\quad \left. + \int_{\{\min\{\delta, 3|z+\xi|\} \leq |w+\xi| \leq 3|z+\xi|\}} \frac{\delta^2}{|z+\xi||w+\xi|} dw \right) \\
&= O \left(\frac{\min\{\delta, 3|z+\xi|\}^2}{|z+\xi|} + \frac{\delta^2}{|z+\xi|} \log^+ \frac{3|z+\xi|}{\delta} \right) \\
&= O(A(z)),
\end{aligned}$$

Finally, if $\frac{|z-w|}{|z+\xi|} > 2$, then

$$\log \frac{|z-w|}{|z+\xi|} = O \left(1 + \log \left(\frac{|z-w|}{|z+\xi|} - 1 \right) \right) = O \left(1 + \log \frac{|w+\xi|}{|z+\xi|} \right);$$

moreover, $|w + \xi| \geq |z + \xi|$, therefore

$$\begin{aligned}
& \int_{\{|z-w|>2|z+\xi|\}} \log \frac{|z-w|}{|z+\xi|} \left(\frac{\delta}{\delta + |w+\xi|} \right)^2 dw \\
&= O \left(\int_{\{|w+\xi|>|z+\xi|\}} \left(1 + \log \frac{|w+\xi|}{|z+\xi|} \right) \left(\frac{\delta}{\delta + |w+\xi|} \right)^2 dw \right) \\
&= O \left(\int_{\{|z+\xi|<|w+\xi|\leq\delta\}} \left(1 + \log \frac{|w+\xi|}{|z+\xi|} \right) dw \right. \\
&\quad \left. + \int_{\{|w+\xi|>\delta\}} \left(1 + \log \frac{|w+\xi|}{|z+\xi|} \right) \frac{\delta^2}{|w+\xi|^2} dw \right) \\
&= O \left(\int_{\{1<t\leq\frac{\delta}{|z+\xi|}\}} |z+\xi|(1+\log t) dt \right. \\
&\quad \left. + \int_{\{t>\frac{\delta}{|z+\xi|}\}} \frac{\delta^2}{|z+\xi|} \frac{1+\log t}{t^2} dt \right) \\
&= O \left(\delta \log \frac{\delta}{|z+\xi|} + \frac{\delta^2}{|z+\xi|} \right) \\
&= O(A(z)).
\end{aligned}$$

The pointwise estimate (5.19) and the L^p -estimates follow immediately. $\square$

Proposition 5.3. *Let $\mathcal{E}^{\text{Int}}$ and $\mathcal{E}^\partial$ denote the interior and boundary error terms defined in (5.16) and (5.17), respectively. Then, for $1 < p < 2$,*

$$\|\mathcal{E}\|_p := \|\mathcal{E}^{\text{Int}}\|_{L^p(\mathbb{D})} + \|\mathcal{E}^\partial\|_{L^p(\partial\mathbb{D})} = O \left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}} \right).$$

Proof. We begin by estimating the interior contribution in $\mathbb{D}^2$. We decompose

$$\begin{aligned}
\mathcal{E}^{\text{Int}} &= \underbrace{2(K(f(z)) + \varepsilon G(f(z)))(e^{W+\tau} - 1)e^V}_{=:\mathcal{E}_1^{\text{Int}}} \\
&+ \underbrace{2(K(f(z)) - K(\xi) + \varepsilon(G(f(z)) - G(\xi)))e^V}_{=:\mathcal{E}_2^{\text{Int}}} + \underbrace{2\varepsilon G(\xi)e^V}_{=:\mathcal{E}_3^{\text{Int}}}. \quad (5.23)
\end{aligned}$$

To estimate the first term we use the elementary bound

$$e^{W+\tau} - 1 = O((1 + e^{W+\tau})(W + \tau))$$

together with (5.20). This yields

$$\|\mathcal{E}_1^{\text{Int}}\|_{L^p(\mathbb{D})} = O\left(\|K + \varepsilon G\|_{L^\infty(\mathbb{D})} \|1 + e^{W+\tau}\|_{L^{2p}(\mathbb{D})} \|W + \tau\|_{L^{2p}(\mathbb{D})} \|e^V\|_{L^\infty(\mathbb{D})}\right) = O(\delta + |\tau|).$$

For the second term we apply the mean value theorem and Proposition A.3, obtaining

$$\|\mathcal{E}_2^{\text{Int}}\|_{L^p(\mathbb{D})} = O\left(\|\nabla(K + \varepsilon G)\|_{L^\infty(\mathbb{D})} \|f(z) - \xi\|_{L^p(\mathbb{D})} \|e^V\|_{L^\infty(\mathbb{D})}\right) = O(\delta).$$

For the third term we use that $G(1) = 0$, which implies $G(\xi) = O(|\eta|)$. Hence

$$\|\mathcal{E}_3^{\text{Int}}\|_{L^p(\mathbb{D})} = O(|\eta||\varepsilon|).$$

Combining the above estimates we obtain

$$\|\mathcal{E}^{\text{Int}}\|_{L^p(\mathbb{D})} = O(\delta + |\eta||\varepsilon| + |\tau|). \quad (5.24)$$

We now turn to the boundary contribution. We write

$$\mathcal{E}^\partial = \underbrace{2h(f(z))(e^{\frac{W+\tau}{2}} - 1)e^{\frac{V}{2}}}_{=:\mathcal{E}_1^\partial} + \underbrace{2\varepsilon I(f(z))e^{\frac{V+W+\tau}{2}}}_{=:\mathcal{E}_2^\partial} + \underbrace{\frac{1}{\pi}e^{\frac{V}{2}} \int_{\partial\mathbb{D}} (h(f(w)) - h(\xi)) dw}_{=:\mathcal{E}_3^\partial}. \quad (5.25)$$

The first term is estimated exactly as above using (5.20), which gives

$$\begin{aligned} \|\mathcal{E}_1^\partial\|_{L^p(\partial\mathbb{D})} &= O\left(\|h\|_{L^\infty(\partial\mathbb{D})} \|1 + e^{W+\tau}\|_{L^{2p}(\partial\mathbb{D})} \|W + \tau\|_{L^{2p}(\partial\mathbb{D})} \|e^V\|_{L^\infty(\partial\mathbb{D})}\right) \\ &= O(\delta + \tau). \end{aligned} \quad (5.26)$$

For the second boundary term we recall that, by $I(1) = 0$,

$$I(f(z)) = O(|f(z) - \xi| + |\eta|).$$

Using Proposition A.3 together with (5.20) we obtain

$$\begin{aligned} \|\mathcal{E}_2^\partial\|_{L^p(\partial\mathbb{D})} &= O\left(|\varepsilon| \|f(z) - \xi\| + |\eta| \|e^{\frac{V+W+\tau}{2}}\|_{L^{2p}(\partial\mathbb{D})}\right) \\ &= O(|\varepsilon| (\delta^{\frac{1}{2p}} + |\eta|)) \\ &= O(|\varepsilon| (\delta^{\frac{1}{4}} + |\eta|)). \end{aligned} \quad (5.27)$$

Finally, since e^V is constant on $\partial\mathbb{D}$, Proposition A.4 implies

$$\|\mathcal{E}_3^\partial\|_{L^p(\partial\mathbb{D})} = O(\delta).$$

This concludes the proof. $\square$

5.1.2 Invertibility of the Linearized Operator

We define

$$\mathfrak{C} := \left\{ \xi \in \partial\mathbb{D} : 2K(\xi) \int_{\mathbb{D}} e^{V_\xi} + h(\xi) \int_{\partial\mathbb{D}} e^{\frac{V_\xi}{2}} = 2K(\xi) \int_{\mathbb{D}} e^{V_\xi} + h(\xi) \frac{4\pi\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \neq 0 \right\}.$$

For any $\xi \in \mathfrak{C}$ we introduce the Hilbert space

$$\mathbf{H}_\xi := \left\{ \phi \in H^1(\mathbb{D}^2) : 2K(\xi) \int_{\mathbb{D}} e^{V_\xi} \phi + h(\xi) \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}} \phi = 0 \right\},$$

endowed with the scalar product

$$\langle u, v \rangle := \int_{\mathbb{D}} \nabla u \cdot \nabla v$$

and the corresponding norm

$$\|u\| := \|\nabla u\|_{L^2(\mathbb{D})} = \left(\int_{\mathbb{D}} |\nabla u|^2 \right)^{1/2}.$$

Notice that non-zero constant functions do not belong to $\mathbf{H}_\xi$.

Moreover, in view of Proposition A.9, the condition $\xi \in \mathfrak{C}$ is equivalent to

$$2K(\xi) + \varphi(\xi)h(\xi) \neq 0.$$

Since we assume (5.5), it follows that $\xi \in \mathfrak{C}$ for any ξ sufficiently close to 1.

Lemma 5.4. *1. For any compact set $\mathfrak{C}' \subseteq \mathfrak{C}$ there exists a constant $C > 0$ such that*

$$\|\phi\|_{L^2(\mathbb{D}^2)} \leq C \|\nabla \phi\|_{L^2(\mathbb{D}^2)} \quad \forall \phi \in \mathbf{H}_\xi, \xi \in \mathfrak{C}'.$$

2. The norm $\|\cdot\|$ is equivalent to the standard norm of $H^1(\mathbb{D}^2)$ and the embeddings

$$\mathbf{H}_\xi \hookrightarrow L^p(\mathbb{D}^2), \quad \mathbf{H}_\xi \hookrightarrow L^p(\partial\mathbb{D}^2)$$

are continuous and compact for any $p > 1$.

3. There exists a constant C (depending only on p and on the compact set $\mathfrak{C}'$) such that

$$\|\phi\|_{L^p(\mathbb{D}^2)} + \|\phi\|_{L^p(\partial\mathbb{D}^2)} \leq C\|\phi\| \quad \forall \phi \in \mathbf{H}_\xi, \xi \in \mathfrak{C}'.$$

Proof. We only prove (1), since (2) and (3) follow directly from it. We argue by contradiction. Suppose that there exist sequences $\xi_n \in \mathfrak{C}'$ for some $\mathfrak{C}' \subseteq \mathfrak{C}$ and $\phi_n \in H^1(\mathbb{D}^2)$ such that

$$2K(\xi_n) \int_{\mathbb{D}^2} e^{V_{\xi_n}} \phi_n + h(\xi_n) \frac{2\varphi(\xi_n)}{\varphi(\xi_n)^2 + K(\xi_n)} \int_{\partial\mathbb{D}^2} \phi_n = 0, \quad (5.28)$$

and

$$\|\phi_n\|_{L^2(\mathbb{D}^2)} = 1, \quad \|\nabla \phi_n\|_{L^2(\mathbb{D}^2)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (5.29)$$

Up to a subsequence we may assume that

$$\xi_n \rightarrow \xi \in \mathfrak{C}, \quad \phi_n \rightarrow \phi \text{ in } L^2(\mathbb{D}^2), \quad \nabla \phi_n \rightharpoonup \nabla \phi \text{ in } L^2(\mathbb{D}^2).$$

From (5.29) we deduce that $\|\phi\|_{L^2(\mathbb{D}^2)} = 1$ and $\|\nabla \phi\|_{L^2(\mathbb{D}^2)} = 0$. Hence ϕ must be constant on $\mathbb{D}^2$, namely

$$\phi \equiv \pm \frac{1}{\sqrt{\pi}}.$$

On the other hand, passing to the limit in (5.28) and using

$$2K(\xi_n)e^{V_{\xi_n}} \rightarrow 2K(\xi)e^{V_\xi} \quad \text{uniformly in } \mathbb{D}^2 \quad (5.30)$$

and

$$h(\xi_n) \frac{2\varphi(\xi_n)}{\varphi(\xi_n)^2 + K(\xi_n)} \rightarrow h(\xi) \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \quad \text{uniformly on } \partial\mathbb{D}^2, \quad (5.31)$$

we obtain

$$\pm \frac{1}{\sqrt{\pi}} \underbrace{\left(2K(\xi) \int_{\mathbb{D}^2} e^{V_\xi} + h(\xi) \frac{4\pi\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \right)}_{\neq 0 \text{ since } \xi \in \mathfrak{C}} = 0,$$

which is a contradiction. $\square$

We also remark that a Moser–Trudinger type inequality holds in $\mathbf{H}_\xi$.

Lemma 5.5. *For any $\phi \in \mathbf{H}_\xi$ and any $p > 1$ one has*

$$\left\| e^{|\phi|} \right\|_{L^p(\mathbb{D}^2)} + \left\| e^{\frac{|\phi|}{2}} \right\|_{L^p(\partial\mathbb{D}^2)} = O\left(e^{O(\|\phi\|^2)}\right).$$

Proof. Since $\phi \in \mathbf{H}_\xi$, using the definition of V_ξ in (5.11) we obtain

$$8\varphi(\xi)^2 K(\xi) \int_{\mathbb{D}^2} \frac{\phi(z)}{(\varphi(\xi)^2 + K(\xi)|z|^2)^2} dz + \frac{2h(\xi)\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \phi = 0.$$

Hence, by Proposition A.9, we can write

$$\phi = \underbrace{\phi - \frac{1}{2\pi} \int_{\partial\mathbb{D}^2} \phi}_{=: \phi_0} - \frac{2\varphi(\xi)^2 K(\xi)(\varphi(\xi)^2 + K(\xi))}{\pi(2K(\xi) + \varphi(\xi)h(\xi))} \underbrace{\int_{\mathbb{D}^2} \left(\phi(z) - \frac{1}{2\pi} \int_{\partial\mathbb{D}^2} \phi \right) \frac{dz}{(\varphi(\xi)^2 + K(\xi)|z|^2)^2}}_{=: c}.$$

By the Poincaré–Wirtinger inequality we have $|c| = O(\|\phi\|)$. Moreover, since $\int_{\partial\mathbb{D}^2} \phi_0 = 0$, we can estimate $\|e^{|\phi|}\|_{L^p(\mathbb{D}^2)}$ using a Moser–Trudinger inequality from [7], obtaining

$$\int_{\mathbb{D}^2} e^{p|\phi_0|} = O\left(e^{\frac{p^2\|\phi_0\|^2}{8\pi}} \int_{\mathbb{D}^2} e^{\frac{2\pi}{\|\phi_0\|^2} \phi_0^2}\right) = O\left(e^{O(\|\phi_0\|^2)}\right).$$

Therefore

$$\|e^{|\phi|}\|_{L^p(\mathbb{D}^2)} = O\left(e^{O(|c|)} \|e^{|\phi_0|}\|_{L^p(\mathbb{D}^2)}\right) = O\left(e^{O(\|\phi\|^2)}\right).$$

Similarly, using the boundary Moser–Trudinger inequality from [24], we obtain

$$\int_{\partial\mathbb{D}^2} e^{p\frac{|\phi|}{2}} = O\left(e^{O(\|\phi\|^2)}\right),$$

which yields

$$\left\|e^{\frac{|\phi|}{2}}\right\|_{L^p(\partial\mathbb{D}^2)} = O\left(e^{O(\|\phi\|^2)}\right).$$

□

Define the functions

$$\mathcal{Z}_1(z) := \frac{\langle z, \xi \rangle}{\varphi(\xi)^2 + K(\xi)|z|^2}, \quad \mathcal{Z}_2(z) := \frac{\langle z, \xi^\perp \rangle}{\varphi(\xi)^2 + K(\xi)|z|^2}, \quad (5.32)$$

that satisfy

$$\begin{cases} -\Delta \mathcal{Z}_i = 2K(\xi)e^V \mathcal{Z}_i & \text{in } \mathbb{D}^2 \\ \partial_\nu \mathcal{Z}_i = h(\xi)e^{\frac{V}{2}} \mathcal{Z}_i & \text{on } \partial\mathbb{D}^2, \end{cases} \quad i = 1, 2. \quad (5.33)$$

Thus, we can state the following linear invertibility result.

Theorem 5.6. *Fix $p > 1$ and let $\mathfrak{C}' \in \mathfrak{C}$. For any $\xi \in \mathfrak{C}'$ and any $c \in L^p(\mathbb{D}^2)$, $d \in L^p(\partial\mathbb{D}^2)$ satisfying*

$$\int_{\mathbb{D}^2} c + \int_{\partial\mathbb{D}^2} d = 0, \quad \int_{\mathbb{D}^2} c \mathcal{Z}_i + \int_{\partial\mathbb{D}^2} d \mathcal{Z}_i = 0, \quad i = 1, 2, \quad (5.34)$$

there exists a unique solution $\phi \in H^1(\mathbb{D}^2)$ of

$$\begin{cases} -\Delta\phi = 2K(\xi)e^{V_\xi}\phi + c & \text{in } \mathbb{D}^2, \\ \partial_\nu\phi = h(\xi)e^{\frac{V_\xi}{2}}\phi + d & \text{on } \partial\mathbb{D}^2, \\ 2K(\xi) \int_{\mathbb{D}^2} e^{V_\xi}\phi + \frac{h(\xi)2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \phi = 0, \\ 2K(\xi) \int_{\mathbb{D}^2} e^{V_\xi}\phi \mathcal{Z}_i + \frac{h(\xi)2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \phi \mathcal{Z}_i = 0, \quad i = 1, 2. \end{cases}$$

Moreover,

$$\|\phi\| \leq C_p \left(\|c\|_{L^p(\mathbb{D}^2)} + \|d\|_{L^p(\partial\mathbb{D}^2)} \right), \quad (5.35)$$

where the constant C_p depends only on p and on the compact set $\mathfrak{C}'$.

Proof. For $p > 1$ we introduce the Banach space

$$\mathbf{L} := \left\{ (f, g) \in L^p(\mathbb{D}^2) \times L^p(\partial\mathbb{D}^2) : \int_{\mathbb{D}^2} f + \int_{\partial\mathbb{D}^2} g = 0 \right\},$$

endowed with the norm

$$\|(f, g)\| := \|f\|_{L^p(\mathbb{D}^2)} + \|g\|_{L^p(\partial\mathbb{D}^2)}.$$

Let $\mathcal{J}_\xi : \mathbf{L} \rightarrow \mathbf{H}_\xi$ be the operator defined by $\mathcal{J}_\xi(f, g) = u$, where u is the unique element of $\mathbf{H}_\xi$ solving

$$\begin{cases} -\Delta u = f & \text{in } \mathbb{D}^2, \\ \partial_\nu u = g & \text{on } \partial\mathbb{D}^2. \end{cases}$$

Equivalently, u satisfies

$$\int_{\mathbb{D}^2} \nabla u \nabla \zeta = \int_{\mathbb{D}^2} f \zeta + \int_{\partial\mathbb{D}^2} g \zeta \quad \forall \zeta \in H^1(\mathbb{D}^2). \quad (5.36)$$

By Lemma 5.4 (3) we deduce the estimate

$$\|u\| = \|\mathcal{J}_\xi(f, g)\| \leq C_p \left(\|f\|_{L^p(\mathbb{D}^2)} + \|g\|_{L^p(\partial\mathbb{D}^2)} \right), \quad (5.37)$$

where C_p depends only on p and on the compact set $\mathfrak{C}'$.

Consider now the linear problem

$$\begin{cases} -\Delta\phi = 2K(\xi)e^{V_\xi}\phi + c & \text{in } \mathbb{D}^2, \\ \partial_\nu\phi = h(\xi)e^{\frac{V_\xi}{2}}\phi + d & \text{on } \partial\mathbb{D}^2. \end{cases}$$

If $\phi \in \mathbf{H}_\xi$, this equation can be rewritten as

$$\phi = \underbrace{\mathcal{J}_\xi(2K(\xi)e^{V_\xi}\phi, h(\xi)e^{\frac{V_\xi}{2}}\phi)}_{=:\mathcal{K}_\xi(\phi)} + \mathcal{J}_\xi(c, d), \quad \text{i.e.} \quad (\mathcal{I} - \mathcal{K}_\xi)\phi = \mathcal{J}_\xi(c, d). \quad (5.38)$$

From the Lemma 4.6 we know that the kernel of $\mathcal{I} - \mathcal{K}_\xi$ in $\mathbf{H}_\xi$ is two-dimensional and is generated by the functions

$$\mathcal{Z}_{\xi_1}(z) = \mathcal{Z}_1(z), \quad \mathcal{Z}_{\xi_2}(z) = \mathcal{Z}_2(z)$$

defined in (5.32).

Set

$$\mathbf{K}_\xi := \text{span}\{\mathcal{Z}_{\xi_1}, \mathcal{Z}_{\xi_2}\}.$$

Then

$$\mathbf{K}_\xi^\perp = \left\{ \phi \in \mathbf{H}_\xi : \langle \phi, \mathcal{Z}_{\xi_i} \rangle = 0, \ i = 1, 2 \right\},$$

where

$$\langle \phi, \mathcal{Z}_{\xi_i} \rangle = 2K(\xi) \int_{\mathbb{D}^2} e^{V_\xi} \phi \mathcal{Z}_{\xi_i} + \frac{h(\xi)2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \phi \mathcal{Z}_{\xi_i}.$$

Since, by Lemma 5.4(2), the operator

$$\mathcal{K}_\xi : \mathbf{H}_\xi \rightarrow \mathbf{H}_\xi$$

is compact, the Fredholm alternative implies that (5.38) admits a unique solution $\phi \in \mathbf{K}_\xi^\perp$ if and only if

$$\mathcal{J}_\xi(c, d) \in \mathbf{K}_\xi^\perp,$$

namely

$$0 = \langle \mathcal{J}_\xi(c, d), \mathcal{Z}_{\xi_i} \rangle = \int_{\mathbb{D}^2} c \mathcal{Z}_{\xi_i} + \int_{\partial\mathbb{D}^2} d \mathcal{Z}_{\xi_i}, \quad i = 1, 2,$$

which coincides with condition (5.34).

It remains to prove that estimate (5.35) is uniform with respect to $\xi \in \mathfrak{C}'$. We claim that there exists $C > 0$ such that

$$\|\mathcal{I} - \mathcal{K}_\xi\|_{\mathcal{L}(\mathbf{K}_\xi^\perp)} \geq C \quad \forall \xi \in \mathfrak{C}'. \quad (5.39)$$

Indeed, if (5.39) holds then, from $(\mathcal{I} - \mathcal{K}_\xi)\phi = \mathcal{J}_\xi(c, d)$ and using (5.37), we obtain

$$\|\phi\| \leq \|\mathcal{I} - \mathcal{K}_\xi\|^{-1} \|(\mathcal{I} - \mathcal{K}_\xi)\phi\| \leq C \left(\|c\|_{L^p(\mathbb{D}^2)} + \|d\|_{L^p(\partial\mathbb{D}^2)} \right).$$

We prove (5.39) by contradiction. Suppose that there exist sequences $\xi_n \in \mathfrak{C}'$

and $\phi_n, \psi_n \in \mathbf{K}_{\xi_n}^\perp$ such that

$$\|\phi_n\| = 1, \quad \|\psi_n\| \rightarrow 0, \quad (\mathcal{I} - \mathcal{K}_{\xi_n})\phi_n = \psi_n.$$

Up to a subsequence,

$$\xi_n \rightarrow \xi \in \mathfrak{C}, \quad \phi_n \rightarrow \phi \text{ in } L^2(\mathbb{D}^2), \quad \nabla \phi_n \rightharpoonup \nabla \phi \text{ in } L^2(\mathbb{D}^2).$$

Using (5.30)–(5.31) we deduce that $\phi \in \mathbf{K}_\xi^\perp$.

Moreover, from $(\mathcal{I} - \mathcal{K}_{\xi_n})\phi_n = \psi_n$ and (5.36), for any $\zeta \in H^1(\mathbb{D}^2)$ we have

$$\int_{\mathbb{D}^2} \nabla \phi_n \nabla \zeta - 2K(\xi_n) \int_{\mathbb{D}^2} e^{V_{\xi_n}} \phi_n \zeta - \frac{h(\xi_n)2\varphi(\xi_n)}{\varphi(\xi_n)^2 + K(\xi_n)} \int_{\partial\mathbb{D}^2} \phi_n \zeta = \int_{\mathbb{D}^2} \psi_n \zeta. \quad (5.40)$$

Passing to the limit we obtain

$$\int_{\mathbb{D}^2} \nabla \phi \nabla \zeta - 2K(\xi) \int_{\mathbb{D}^2} e^{V_\xi} \phi \zeta - \frac{h(\xi)2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \phi \zeta = 0,$$

that is,

$$(\mathcal{I} - \mathcal{K}_\xi)\phi = 0.$$

Since $\phi \in \mathbf{K}_\xi^\perp$, this implies $\phi \equiv 0$. However, choosing $\zeta = \phi_n$ in the identity (5.40) yields

$$1 - 2K(\xi) \int_{\mathbb{D}^2} e^{V_\xi} \phi^2 - \frac{h(\xi)2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \phi^2 = 0,$$

which shows that $\phi \not\equiv 0$, giving a contradiction. This concludes the proof. $\square$

5.1.3 The nonlinear projected problem

To solve (5.15), we first address the corresponding projected problem

$$\begin{cases} \mathcal{L}_0^{\text{Int}} \phi = \mathcal{E}^{\text{Int}} + \mathcal{L}^{\text{Int}} \phi + \mathcal{N}^{\text{Int}}(\phi) + c_0 + 2K(\xi)e^V(c_1\mathcal{Z}_1 + c_2\mathcal{Z}_2) & \text{in } \mathbb{D}^2, \\ \mathcal{L}_0^\partial \phi = \mathcal{E}^\partial + \mathcal{L}^\partial \phi + \mathcal{N}^\partial(\phi) + c_0 + h(\xi)e^{\frac{V}{2}}(c_1\mathcal{Z}_1 + c_2\mathcal{Z}_2) & \text{on } \partial\mathbb{D}^2, \end{cases} \quad (5.41)$$

where the operators $\mathcal{L}_0^{\text{Int}}$, $\mathcal{E}^{\text{Int}}$, $\mathcal{L}^{\text{Int}}$, $\mathcal{N}^{\text{Int}}$, $\mathcal{L}_0^\partial$, $\mathcal{E}^\partial$, $\mathcal{L}^\partial$, and $\mathcal{N}^\partial$ are those introduced in (5.15)–(5.17), while $\mathcal{Z}_1, \mathcal{Z}_2$ are defined in (5.32). The constants $c_0, c_1, c_2 \in \mathbb{R}$ arise from the orthogonality conditions.

For convenience, we introduce the notation

$$\mathcal{L}_0 := (\mathcal{L}_0^{\text{Int}}, \mathcal{L}_0^\partial), \quad \mathcal{E} := (\mathcal{E}^{\text{Int}}, \mathcal{E}^\partial), \quad \mathcal{L} := (\mathcal{L}^{\text{Int}}, \mathcal{L}^\partial), \quad \mathcal{N} := (\mathcal{N}^{\text{Int}}, \mathcal{N}^\partial).$$

The next lemmas provide the estimates needed for the fixed point argument.

Lemma 5.7. *Let $\phi \in \mathbf{H}_\xi$. Then, for every $1 < p < \frac{4}{3}$,*

$$\|\mathcal{L}\phi\|_p := \|\mathcal{L}^{\text{Int}}\phi\|_{L^p(\mathbb{D}^2)} + \|\mathcal{L}^\partial\phi\|_{L^p(\partial\mathbb{D}^2)} = O\left(\left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}}\right)^{\frac{1}{2}} \|\phi\|\right).$$

Proof. By definition, the boundary contribution can be decomposed as

$$\mathcal{L}^\partial = (h(f(z)) - h(\xi))e^{\frac{W+\tau+V}{2}} + h(\xi)\left(e^{\frac{W+\tau}{2}} - 1\right)e^{\frac{V}{2}} + \varepsilon I(f(z))e^{\frac{W+\tau+V}{2}}.$$

Proceeding exactly as in the estimates of $\mathcal{E}_1^\partial$ and $\mathcal{E}_2^\partial$ in (5.25), one obtains

$$\begin{aligned} \left\| (h(f(\cdot)) - h(\xi))e^{\frac{W(\cdot)+\tau+V}{2}} \right\|_{L^p(\partial\mathbb{D}^2)} &= O(\delta^{\frac{1}{p}}), \\ \left\| h(\xi)\left(e^{\frac{W(\cdot)+\tau}{2}} - 1\right)e^{\frac{V}{2}} \right\|_{L^p(\partial\mathbb{D}^2)} &= O(\delta + \tau), \\ \left\| \varepsilon I(f(\cdot))e^{\frac{W(\cdot)+\tau+V}{2}} \right\|_{L^p(\partial\mathbb{D}^2)} &= O(|\varepsilon|(\delta^{\frac{1}{p}} + |\eta|)). \end{aligned}$$

Since $\mathcal{L}^{\text{Int}}\phi = \mathcal{E}^{\text{Int}}\phi$, combining these bounds with (5.24) yields

$$\begin{aligned} \|\mathcal{L}\phi\|_p &= O\left(\|\mathcal{E}^{\text{Int}}\phi\|_{L^p(\mathbb{D}^2)} + \|\mathcal{L}^\partial\phi\|_{L^p(\partial\mathbb{D}^2)}\right) \\ &= O\left(\|\mathcal{E}^{\text{Int}}\|_{L^{\frac{3}{2}p}(\mathbb{D}^2)} \|\phi\|_{L^{3p}(\mathbb{D}^2)} + \|\mathcal{L}^\partial\|_{L^{\frac{3}{2}p}(\partial\mathbb{D}^2)} \|\phi\|_{L^{3p}(\partial\mathbb{D}^2)}\right) \\ &= O\left((\delta^{\frac{1}{2}} + \delta + \delta^{\frac{1}{4}}|\varepsilon| + |\eta||\varepsilon| + |\tau|)\|\phi\|\right). \end{aligned}$$

This concludes the proof. □

Lemma 5.8. *Let $\phi, \phi' \in \mathbf{H}_\xi$. For every $p > 1$,*

$$\begin{aligned} \|\mathcal{N}(\phi) - \mathcal{N}(\phi')\|_p &:= \|\mathcal{N}^{\text{Int}}(\phi) - \mathcal{N}^{\text{Int}}(\phi')\|_{L^p(\mathbb{D}^2)} + \|\mathcal{N}^\partial(\phi) - \mathcal{N}^\partial(\phi')\|_{L^p(\partial\mathbb{D}^2)} \\ &= O\left(\|\phi - \phi'\|(\|\phi\| + \|\phi'\|)e^{O(\|\phi\|^2 + \|\phi'\|^2)}\right). \end{aligned}$$

In particular, choosing $\phi' = 0$, one obtains

$$\|\mathcal{N}(\phi)\|_p = O\left(\|\phi\|^2 e^{O(\|\phi\|^2)}\right).$$

Proof. Using (5.20), together with Lemma 5.4 and Lemma 5.5, we estimate

$$\|\mathcal{N}^{\text{Int}}(\phi) - \mathcal{N}^{\text{Int}}(\phi')\|_{L^p(\mathbb{D}^2)} = \|2K_\varepsilon(f(z))e^{V+W+\tau}(e^\phi - \phi - e^{\phi'} + \phi')\|_{L^p(\mathbb{D}^2)}$$

$$\begin{aligned}
&= O\left(\left\|e^{W+\tau}|\phi - \phi'|(|\phi| + |\phi'|)(1 + e^{\phi+\phi'})\right\|_{L^p(\mathbb{D}^2)}\right) \\
&= O\left(\left\|e^{W+\tau}\right\|_{L^{4p}(\mathbb{D}^2)} \|\phi - \phi'\|_{L^{4p}(\mathbb{D}^2)} (\|\phi\|_{L^{4p}(\mathbb{D}^2)} + \|\phi'\|_{L^{4p}(\mathbb{D}^2)}) \left\|1 + e^{\phi+\phi'}\right\|_{L^{4p}(\mathbb{D}^2)}\right) \\
&= O\left(\|\phi - \phi'\|(\|\phi\| + \|\phi'\|)e^{O(\|\phi+\phi'\|^2)}\right).
\end{aligned}$$

Here we used the elementary inequality

$$e^t - t - e^s + s = O\left(|s - t|(|s| + |t|)(1 + e^{s+t})\right).$$

The boundary term $\mathcal{N}^\partial$ is treated in the same way. $\square$

Proposition 5.9. *Assume that $\delta, |\eta|, |\tau|, |\varepsilon| \leq \varepsilon_0 \ll 1$. Then there exists a unique quadruple*

$$(\phi, c_0, c_1, c_2) \in \mathbf{H}_\xi \times \mathbb{R}^3$$

solving (5.41). Moreover,

$$\|\phi\| = O\left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}}\right). \quad (5.42)$$

Proof. Consider the operator

$$\mathcal{A} := \Pi_{\mathbf{H}} \circ \mathcal{J}_\xi \circ \Pi_{\mathbf{L}} \circ \mathcal{L}_0 : \mathbf{K}_\xi^\perp \rightarrow \mathbf{K}_\xi^\perp,$$

where $\Pi_{\mathbf{H}} : \mathbf{H}_\xi \rightarrow \mathbf{K}_\xi^\perp$ denotes the orthogonal projection, $\mathcal{J}_\xi : \mathbf{L} \rightarrow \mathbf{H}_\xi$ is the operator introduced in the proof of Theorem 5.6, and $\Pi_{\mathbf{L}}$ is defined by

$$\Pi_{\mathbf{L}}(f, g) = \left(f - \frac{1}{3\pi} \left(\int_{\mathbb{D}^2} f + \int_{\partial\mathbb{D}^2} g\right), g - \frac{1}{3\pi} \left(\int_{\mathbb{D}^2} f + \int_{\partial\mathbb{D}^2} g\right)\right).$$

Equivalently, $\psi = \mathcal{A}\phi$ if and only if

$$\begin{cases} -\Delta\psi = \mathcal{L}_0^{\text{Int}}\phi + a_0 + a_1 2K(\xi)e^V \mathcal{Z}_1 + a_2 2K(\xi)e^V \mathcal{Z}_2 & \text{in } \mathbb{D}^2, \\ \partial_\nu\psi = \mathcal{L}_0^\partial\phi + a_0 + a_1 h(\xi)e^{\frac{V}{2}} \mathcal{Z}_1 + a_2 h(\xi)e^{\frac{V}{2}} \mathcal{Z}_2 & \text{on } \partial\mathbb{D}^2, \end{cases}$$

for suitable constants a_0, a_1, a_2 chosen so that the orthogonality conditions (5.34) are satisfied.

By Theorem 5.6 and Sobolev embeddings, $\mathcal{A}$ is invertible and

$$\|\mathcal{A}^{-1}\phi\| = O(\|\phi\|)$$

uniformly.

Hence solving (5.41) is equivalent to finding a fixed point of

$$\mathcal{T}_\xi : \mathbf{K}_\xi^\perp \rightarrow \mathbf{K}_\xi^\perp, \quad \mathcal{T}_\xi(\phi) = (\mathcal{A}^{-1} \circ \Pi_{\mathbf{H}} \circ \mathcal{J}_\xi \circ \Pi_{\mathbf{L}})(\mathcal{E} + \mathcal{L}\phi + \mathcal{N}(\phi)).$$

Using Proposition 5.3 together with Lemmas 5.7 and 5.8, we obtain

$$\begin{aligned} \|\mathcal{T}_\xi(\phi)\| &= O(\|\mathcal{E}\|_p + \|\mathcal{L}\phi\|_p + \|\mathcal{N}(\phi)\|_p) \\ &= O\left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}} + \left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}}\right)^{\frac{1}{2}} \|\phi\| + \|\phi\|^2 e^{O(\|\phi\|^2)}\right) \\ &= O\left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}} + \|\phi\|^2 e^{O(\|\phi\|^2)}\right), \end{aligned} \tag{5.43}$$

and similarly

$$\begin{aligned} \|\mathcal{T}_\xi(\phi) - \mathcal{T}_\xi(\phi')\| &= O(\|\mathcal{L}(\phi - \phi')\|_p + \|\mathcal{N}(\phi) - \mathcal{N}(\phi')\|_p) \\ &= O\left(\|\phi - \phi'\| \left(\left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}}\right)^{\frac{1}{2}} + (\|\phi\| + \|\phi'\|) e^{O(\|\phi\|^2 + \|\phi'\|^2)} \right)\right). \end{aligned} \tag{5.44}$$

Choosing $R > 0$ sufficiently large and $|\varepsilon|$ sufficiently small, (5.43) we have

$$\|\phi\| \leq R \frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}} \implies \|\mathcal{T}_\xi(\phi)\| \leq R \frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}}.$$

Moreover, if $\delta, |\eta|, |\tau|, |\varepsilon|$ are sufficiently small, estimate (5.44) ensures that

$$\sup_{\phi \neq \phi'} \frac{\|\mathcal{T}_\xi(\phi) - \mathcal{T}_\xi(\phi')\|}{\|\phi - \phi'\|} < 1.$$

Therefore $\mathcal{T}_\xi$ is a contraction on a suitable ball of $\mathbf{K}_\xi^\perp$, and Banach's fixed point theorem yields the existence and uniqueness of a fixed point satisfying (5.42). $\square$

5.1.4 Estimates on the projections

Let ϕ be the solution to problem (5.41) provided by Proposition 5.9. Notice that if we can show

$$c_0 = c_1 = c_2 = 0,$$

then ϕ actually solves (5.15). Therefore, it is necessary to identify the exact expressions of these constants.

Multiplying (5.41) by $\mathcal{Z}_1$ and integrating, we find that, due to the angular symmetry of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ and the radially of V ,

$$\begin{aligned} 2K(\xi) \int_{\mathbb{D}^2} e^V \mathcal{Z}_1 \mathcal{Z}_2 &= \frac{2h(\xi)\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \mathcal{Z}_1 \mathcal{Z}_2 = 2K(\xi) \int_{\mathbb{D}^2} e^V \mathcal{Z}_i \\ &= \frac{2h(\xi)\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \mathcal{Z}_i = 0, \quad i = 1, 2, \end{aligned}$$

and hence

$$\begin{aligned} c_1 2K(\xi) \int_{\mathbb{D}^2} e^V \mathcal{Z}_1^2 &= \int_{\mathbb{D}^2} \mathcal{L}_0^{\text{Int}} \phi \mathcal{Z}_1 - \int_{\mathbb{D}^2} \mathcal{E}^{\text{Int}} \mathcal{Z}_1 - \int_{\mathbb{D}^2} \mathcal{L}^{\text{Int}} \phi \mathcal{Z}_1 - \int_{\mathbb{D}^2} \mathcal{N}^{\text{Int}}(\phi) \mathcal{Z}_1, \\ c_1 \frac{2h(\xi)\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \mathcal{Z}_1^2 &= \int_{\partial\mathbb{D}^2} \mathcal{L}_0^\partial \phi \mathcal{Z}_1 - \int_{\partial\mathbb{D}^2} \mathcal{E}^\partial \mathcal{Z}_1 - \int_{\partial\mathbb{D}^2} \mathcal{L}^\partial \phi \mathcal{Z}_1 - \int_{\partial\mathbb{D}^2} \mathcal{N}^\partial(\phi) \mathcal{Z}_1. \end{aligned}$$

Integrating by parts and using (5.33), we obtain

$$\int_{\mathbb{D}^2} (\mathcal{L}_0^{\text{Int}} \phi) \mathcal{Z}_1 + \int_{\partial\mathbb{D}^2} (\mathcal{L}_0^\partial \phi) \mathcal{Z}_1 = \int_{\mathbb{D}^2} (\mathcal{L}_0^{\text{Int}} \mathcal{Z}_1) \phi + \int_{\partial\mathbb{D}^2} (\mathcal{L}_0^\partial \mathcal{Z}_1) \phi = 0,$$

and therefore

$$\begin{aligned} c_1 \left(2K(\xi) \int_{\mathbb{D}^2} e^V \mathcal{Z}_1^2 + \frac{2h(\xi)\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \mathcal{Z}_1^2 \right) &= - \int_{\mathbb{D}^2} \mathcal{E}^{\text{Int}} \mathcal{Z}_1 - \int_{\mathbb{D}^2} \mathcal{L}^{\text{Int}} \phi \mathcal{Z}_1 - \int_{\mathbb{D}^2} \mathcal{N}^{\text{Int}}(\phi) \mathcal{Z}_1 \\ &\quad - \int_{\partial\mathbb{D}^2} \mathcal{E}^\partial \mathcal{Z}_1 - \int_{\partial\mathbb{D}^2} \mathcal{L}^\partial \phi \mathcal{Z}_1 - \int_{\partial\mathbb{D}^2} \mathcal{N}^\partial(\phi) \mathcal{Z}_1. \end{aligned}$$

A similar formula holds for $\mathcal{Z}_2$. Moreover, since $\phi \in \mathbf{H}_\xi$, we have

$$\int_{\mathbb{D}^2} \mathcal{L}_0^{\text{Int}} \phi + \int_{\partial\mathbb{D}^2} \mathcal{L}_0^\partial \phi = 0,$$

and integrating (5.41) gives

$$3\pi c_0 = - \int_{\mathbb{D}^2} \mathcal{E}^{\text{Int}} - \int_{\mathbb{D}^2} \mathcal{L}^{\text{Int}} \phi - \int_{\mathbb{D}^2} \mathcal{N}^{\text{Int}}(\phi) - \int_{\partial\mathbb{D}^2} \mathcal{E}^\partial - \int_{\partial\mathbb{D}^2} \mathcal{L}^\partial \phi - \int_{\partial\mathbb{D}^2} \mathcal{N}^\partial(\phi).$$

Recalling (5.18), we now estimate all the integral terms involved.

Proposition 5.10.

$$\begin{aligned} \int_{\mathbb{D}^2} \mathcal{E}^{\text{Int}} \mathcal{Z}_1 &= \frac{4\pi}{(\varphi(\xi)^2 + K(\xi))^2} \delta \left(\partial_x K(1) + \eta \partial_{xy} K(1) + \varepsilon \partial_x G(1) - \delta \log \frac{1}{\delta} \frac{2\varphi(1)^2}{\varphi(1)^2 + K(1)} \Delta K(1) \right) \\ &\quad + \int_{\mathbb{D}^2} 2K(\xi) W e^V \mathcal{Z}_1 + O\left(\frac{|\varepsilon|^2}{\log^2 \frac{1}{|\varepsilon|}} \right). \end{aligned}$$

Proof. We split $\mathcal{E}^{\text{Int}} = \mathcal{E}_1^{\text{Int}} + \mathcal{E}_2^{\text{Int}} + \mathcal{E}_3^{\text{Int}}$ as in (5.23) and start by estimating the first

term. We can write:

$$\begin{aligned}
\mathcal{E}_1^{\text{Int}} &= 2K(\xi)(W + \tau)e^V + 2(K(f(z)) - K(\xi))(W + \tau)e^V \\
&\quad + 2(K(f(z)) + \varepsilon G(f(z)))(e^{W+\tau} - 1 - (W + \tau))e^V \\
&= 2K(\xi)We^V + 2K(\xi)\tau e^V + O\left(|f(z) - \xi|(|W| + |\tau|) + (|W| + |\tau|)^2(1 + e^{W+\tau})\right).
\end{aligned} \tag{5.45}$$

Since V is radial, $\int_{\mathbb{D}^2} e^V \mathcal{Z}_1 = 0$. Therefore, using Lemma 5.2 and Proposition A.3,

$$\begin{aligned}
\int_{\mathbb{D}^2} \mathcal{E}_1^{\text{Int}} \mathcal{Z}_1 &= 2K(\xi) \int_{\mathbb{D}^2} We^V \mathcal{Z}_1 + 2K(\xi)\tau \int_{\mathbb{D}^2} e^V \mathcal{Z}_1 \\
&\quad + \int_{\mathbb{D}^2} O\left(|f(z) - \xi|(|W| + |\tau|) + (|W| + |\tau|)^2(1 + e^{W+\tau})\right) \\
&= \int_{\mathbb{D}^2} 2K(\xi)We^V \mathcal{Z}_1 + O(\delta^2 + \delta|\tau| + |\tau|^2).
\end{aligned} \tag{5.46}$$

For $\mathcal{E}_2^{\text{Int}}$, we perform a Taylor expansion:

$$\begin{aligned}
\mathcal{E}_2^{\text{Int}} &= 2\langle \nabla K(\xi), f(z) - \xi \rangle e^V + \langle D^2 K(\xi)(f(z) - \xi), f(z) - \xi \rangle e^V + 2\varepsilon \langle \nabla G(\xi), f(z) - \xi \rangle e^V \\
&\quad + O\left(|f(z) - \xi|^3 + |\varepsilon||f(z) - \xi|^2\right) \\
&= 2\delta \left\langle \nabla K(\xi), \frac{z - \xi}{1 + (1 - \delta)\bar{\xi}z} \right\rangle e^V + \delta^2 \left\langle D^2 K(\xi) \frac{z - \xi}{1 + (1 - \delta)\bar{\xi}z}, \frac{z - \xi}{1 + (1 - \delta)\bar{\xi}z} \right\rangle e^V \\
&\quad + 2\delta\varepsilon \left\langle \nabla G(\xi), \frac{z - \xi}{1 + (1 - \delta)\bar{\xi}z} \right\rangle e^V + O\left(\frac{\delta^3}{(\delta + |z + \xi|)^3} + \frac{|\varepsilon|\delta^2}{(\delta + |z + \xi|)^2}\right).
\end{aligned} \tag{5.47}$$

The terms involving the first derivatives can be handled by making a rotation $w = \bar{\xi}z$ and then using Proposition A.5:

$$\begin{aligned}
&\int_{\mathbb{D}^2} \left\langle \nabla K(\xi), \frac{z - \xi}{1 + (1 - \delta)\bar{\xi}z} \right\rangle e^V \mathcal{Z}_1 \\
&= \int_{\mathbb{D}^2} \left\langle \nabla K(\xi), \xi \frac{w - 1}{1 + (1 - \delta)w} \right\rangle \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^3} w_1 dw \\
&= \int_{\mathbb{D}^2} \left\langle \nabla K(\xi), \xi \frac{(1 - \delta)|w|^2 - 1 + \delta w_1 + i(2 - \delta)w_2}{|1 + (1 - \delta)w|^2} \right\rangle \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^3} w_1 dw
\end{aligned}$$

$$\begin{aligned}
&= \langle \nabla K(\xi), \xi \rangle \int_{\mathbb{D}^2} \frac{(1-\delta)|w|^2 - 1 + \delta w_1}{|1 + (1-\delta)w|^2} \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^3} w_1 dw \\
&\quad + \langle \nabla K(\xi), \xi^\perp \rangle \int_{\mathbb{D}^2} \frac{(2-\delta)w_2}{|1 + (1-\delta)w|^2} \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^3} w_1 dw \\
&= \left(\frac{2\pi}{(\varphi(\xi)^2 + K(\xi))^2} + O(\delta) \right) \langle \nabla K(\xi), \xi \rangle \\
&= \left(\frac{2\pi}{(\varphi(\xi)^2 + K(\xi))^2} + O(\delta) \right) \langle \nabla K(1) + \eta(\partial_{xy}K(1) + i\partial_{yy}K(1)) + O(|\eta|^2), 1 + O(|\eta|) \rangle \\
&= \frac{2\pi}{(\varphi(\xi)^2 + K(\xi))^2} (\partial_x K(1) + \eta \partial_{xy}K(1)) + O(\delta + |\eta|^2).
\end{aligned}$$

Likewise,

$$\int_{\mathbb{D}^2} \langle \nabla G(\xi), f(z) - \xi \rangle e^V \mathcal{Z}_1 = \frac{2\pi}{(\varphi(\xi)^2 + K(\xi))^2} \partial_x G(1) + O(\delta + |\eta|).$$

To estimate the second order integral, observe that

$$\frac{z - \xi}{1 + (1-\delta)\bar{\xi}z} = \frac{-2\xi}{1 + (1-\delta)\bar{\xi}z} + O\left(\frac{|z + \xi|}{\delta + |z + \xi|}\right),$$

therefore

$$\begin{aligned}
&\int_{\mathbb{D}^2} \left\langle D^2 K(\xi) \frac{z - \xi}{1 + (1-\delta)\bar{\xi}z}, \frac{z - \xi}{1 + (1-\delta)\bar{\xi}z} \right\rangle e^V \mathcal{Z}_1 \\
&= \int_{\mathbb{D}^2} \left(\left\langle D^2 K(\xi) \frac{-2\xi}{1 + (1-\delta)\bar{\xi}z}, \frac{-2\xi}{1 + (1-\delta)\bar{\xi}z} \right\rangle + O\left(\frac{|z + \xi|}{(\delta + |z + \xi|)^2}\right) \right) \\
&\quad \cdot \left(\frac{-4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi))^3} + O(|z + \xi|) \right) dz \\
&= -\frac{16\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi))^3} \int_{\mathbb{D}^2} \left(\left\langle D^2 K(\xi) \frac{\xi}{1 + (1-\delta)\bar{\xi}z}, \frac{\xi}{1 + (1-\delta)\bar{\xi}z} \right\rangle + O\left(\frac{|z + \xi|}{(\delta + |z + \xi|)^2}\right) \right) dz \\
&= -\frac{16\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi))^3} \frac{1}{(1-\delta)^2} \int_{\Omega_\delta} \left\langle D^2 K(\xi) \frac{\xi}{1+w}, \frac{\xi}{1+w} \right\rangle dw + O(1),
\end{aligned}$$

where $w = \frac{1-\delta}{\delta}(\bar{\xi}z + 1)$ transforms $\mathbb{D}^2$ into

$$\Omega_\delta := \left\{ \left| w - \frac{1-\delta}{\delta} \right| \leq \frac{1-\delta}{\delta} \right\}.$$

Using

$$\frac{\xi}{1+w} = \frac{e^{i\eta}(1+\bar{w})}{|1+w|^2},$$

$$\langle D^2 K(\xi)(z_1 + iz_2), z_1 + iz_2 \rangle = \frac{\Delta K(\xi)}{2} |z|^2 + \frac{\partial_{xx} K(\xi) - \partial_{yy} K(\xi)}{2} (z_1^2 - z_2^2) + 2\partial_{xy} K(\xi) z_1 z_2,$$

and Proposition A.6, we obtain

$$\int_{\Omega_\delta} \left\langle D^2 K(\xi) \frac{\xi}{1+w}, \frac{\xi}{1+w} \right\rangle = \frac{\pi}{2} \Delta K(\xi) \log \frac{1}{\delta} + O(1).$$

Hence,

$$\begin{aligned} & \int_{\mathbb{D}^2} \left\langle D^2 K(\xi) \frac{z - \xi}{1 + (1 - \delta)\bar{\xi}z}, \frac{z - \xi}{1 + (1 - \delta)\bar{\xi}z} \right\rangle e^V \mathcal{Z}_1 \\ &= -\frac{16\pi\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi))^3} (1 + O(\delta)) \left(\frac{\pi}{2} \Delta K(\xi) \log \frac{1}{\delta} + O(1) \right) + O(1) \\ &= -\frac{8\pi^2\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi))^3} \Delta K(1) \log \frac{1}{\delta} + O(1). \end{aligned}$$

Finally, since

$$\int_{\mathbb{D}^2} \frac{\delta^3}{(\delta + |z + \xi|)^3} \mathcal{Z}_1 = O(\delta^2), \quad \int_{\mathbb{D}^2} \frac{|\varepsilon|\delta^2}{(\delta + |z + \xi|)^2} \mathcal{Z}_1 = O(\delta^2 \log \frac{1}{\delta} |\varepsilon|),$$

equation (5.47) gives

$$\begin{aligned} \int_{\mathbb{D}^2} \mathcal{E}_2^{\text{Int}} \mathcal{Z}_1 &= \frac{4\pi}{(\varphi(\xi)^2 + K(\xi))^2} \delta \left(\partial_x K(1) + \eta \partial_{xy} K(1) + \varepsilon \partial_x G(1) - \delta \log \frac{1}{\delta} \frac{2\varphi(1)^2}{\varphi(1)^2 + K(1)} \Delta K(1) \right) \\ &\quad + O\left(\delta(\delta + |\eta|^2 + \delta \log \frac{1}{\delta} |\eta| + \delta \log \frac{1}{\delta} |\varepsilon| + |\eta||\varepsilon|) \right). \end{aligned} \tag{5.48}$$

Finally, since e^V is radially symmetric and $\mathcal{Z}_1$ is odd with respect to ξ , we have

$$\int_{\mathbb{D}^2} \mathcal{E}_3^{\text{Int}} \mathcal{Z}_1 = 0,$$

and putting together (5.46) and (5.48) concludes the proof. $\square$

Proposition 5.11.

$$\begin{aligned} \int_{\partial\mathbb{D}^2} \mathcal{E}^\partial \mathcal{Z}_1 &= \frac{8\pi\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \delta \left((-\Delta)^{\frac{1}{2}} h(1) + \eta(-\Delta)^{\frac{1}{2}} h'(1) + \varepsilon(-\Delta)^{\frac{1}{2}} I(1) \right. \\ &\quad \left. - \delta \log \frac{1}{\delta} \frac{4\varphi(1)}{\varphi(1)^2 + K(1)} \left(\left((-\Delta)^{\frac{1}{2}} h(1) \right)^2 + h'(1)^2 \right) \right) \\ &\quad - 2K(\xi) \int_{\mathbb{D}^2} W e^V \mathcal{Z}_1 + O\left(\frac{|\varepsilon|^2}{\log^2 \frac{1}{|\varepsilon|}}\right). \end{aligned}$$

Proof. We split $\mathcal{E}^\partial = \mathcal{E}_x^\partial + \mathcal{E}_y^\partial + \mathcal{E}_3^\partial$ as in (5.25).

To estimate the first term we see that

$$\begin{aligned} \mathcal{E}_x^\partial &= h(\xi)(W + \tau)e^{\frac{V}{2}} + (h(f(z)) - h(\xi))(W + \tau)e^{\frac{V}{2}} + 2h(f(z))\left(e^{\frac{W+\tau}{2}} - 1 - \frac{W + \tau}{2}\right)e^{\frac{V}{2}} \\ &= h(\xi)W e^{\frac{V}{2}} + \tau h(\xi)e^{\frac{V}{2}} + (h(f(z)) - h(\xi))W e^{\frac{V}{2}} + \tau(h(f(z)) - h(\xi))e^{\frac{V}{2}} \\ &\quad + O\left((|W| + |\tau|)^2(1 + e^{W+\tau})\right). \end{aligned} \tag{5.49}$$

Since $W, \mathcal{Z}_1$ are solutions to (5.14) and (5.33) respectively, an integration by parts gives:

$$\begin{aligned} \frac{2\varphi(\xi)h(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} W \mathcal{Z}_1 &= \int_{\partial\mathbb{D}^2} W \partial_\nu \mathcal{Z}_1 \\ &= \int_{\partial\mathbb{D}^2} \mathcal{Z}_1 \partial_\nu W + \int_{\mathbb{D}^2} (W \Delta \mathcal{Z}_1 - \mathcal{Z}_1 \Delta W) \\ &= \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} 2(h(f(z)) - h(\xi)) \mathcal{Z}_1 \\ &\quad - \frac{1}{\pi} \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} (h(f(w)) - h(\xi)) dw \int_{\partial\mathbb{D}^2} \mathcal{Z}_1 \\ &\quad - 2K(\xi) \int_{\mathbb{D}^2} W e^V \mathcal{Z}_1 \\ &= \frac{4\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} (h(f(z)) - h(\xi)) \mathcal{Z}_1 - 2K(\xi) \int_{\mathbb{D}^2} W e^V \mathcal{Z}_1. \end{aligned}$$

The first integral can be estimated by means of Proposition A.7:

$$\begin{aligned}
\frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} (h(f(z)) - h(\xi)) \mathcal{Z}_1 &= \frac{2\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \int_{\partial\mathbb{D}^2} (h(f(z)) - h(\xi)) \langle z, \xi \rangle dz \\
&= \frac{4\pi\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \delta(-\Delta)^{\frac{1}{2}} h(\xi) + O(\delta^2) \\
&= \frac{4\pi\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \delta \left((-\Delta)^{\frac{1}{2}} h(1) + \eta(-\Delta)^{\frac{1}{2}} h'(1) \right) \\
&\quad + O(\delta(\delta + |\eta|^2)).
\end{aligned}$$

On the other hand, for the estimates involving W we will use (5.19) and then Proposition A.8:

$$\begin{aligned}
&\frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} (h(f(z)) - h(\xi)) W \mathcal{Z}_1 \\
&= \frac{2\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \int_{\partial\mathbb{D}^2} (h(f(z)) - h(\xi)) W(z) \langle z, \xi \rangle dz \\
&= \frac{16\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi))^3} \delta \left((-\Delta)^{\frac{1}{2}} h(\xi) \int_{\partial\mathbb{D}^2} (h(f(z)) - h(\xi)) \log |z + \xi| \langle z, \xi \rangle dz \right. \\
&\quad \left. + h'(\xi) \int_{\partial\mathbb{D}^2} (h(f(z)) - h(\xi)) \arctan \frac{\langle z, \xi^\perp \rangle}{1 + \langle z, \xi \rangle} \langle z, \xi \rangle dz \right) \\
&\quad + \frac{2\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \int_{\partial\mathbb{D}^2} (h(f(z)) - h(\xi)) O \left(\frac{\delta^2}{\delta + |z + \xi|} \left(1 + \left| \log \frac{|z + \xi|}{\delta} \right| \right) \right) \langle z, \xi \rangle dz \\
&= \frac{-32\pi\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi))^3} \delta \left(\left((-\Delta)^{\frac{1}{2}} h(\xi) \right)^2 + h'(\xi)^2 \right) \delta \log \frac{1}{\delta} + O(\delta) \\
&\quad + O \left(\int_{\partial\mathbb{D}^2} \frac{\delta^3}{(\delta + |z + \xi|)^2} \left(1 + \left| \log \frac{|z + \xi|}{\delta} \right| \right) dz \right) \\
&= \frac{-32\pi\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi))^3} \delta^2 \log \frac{1}{\delta} \left(\left((-\Delta)^{\frac{1}{2}} h(\xi) \right)^2 + h'(\xi)^2 \right) + O(\delta^2) \\
&\quad + O \left(\int_0^{+\infty} \frac{\delta^2}{(1+t)^2} (1 + |\log t|) dt \right) \\
&= \frac{-32\pi\varphi(\xi)^2}{(\varphi(1)^2 + K(1))^3} \delta^2 \log \frac{1}{\delta} \left(\left((-\Delta)^{\frac{1}{2}} h(1) \right)^2 + h'(1)^2 \right) + O \left(\delta^2 + \delta^2 \log \frac{1}{\delta} |\eta| \right).
\end{aligned}$$

Moreover, since $e^{\frac{V}{2}}$ is constant on $\partial\mathbb{D}^2$, then

$$\int_{\partial\mathbb{D}^2} e^{\frac{V}{2}} \mathcal{Z}_1 = 0,$$

and therefore the decomposition (5.49) gives:

$$\begin{aligned} \int_{\partial\mathbb{D}^2} \mathcal{E}_x^\partial \mathcal{Z}_1 &= (2 + \tau) \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} (h(f(z)) - h(\xi)) \mathcal{Z}_1 \\ &+ \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} (h(f(z)) - h(\xi)) W \mathcal{Z}_1 \\ &- 2K(\xi) \int_{\mathbb{D}^2} W e^V \mathcal{Z}_1 + \int_{\partial\mathbb{D}^2} O\left((|W| + |\tau|)^2 (1 + e^{W+\tau})\right) \mathcal{Z}_1 \\ &= (2 + O(|\tau|)) \frac{4\pi\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \delta \left((-\Delta)^{\frac{1}{2}} h(1) + \eta(-\Delta)^{\frac{1}{2}} h'(1) \right) \\ &- \frac{-32\pi\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi))^3} \delta^2 \log \frac{1}{\delta} \left(\left((-\Delta)^{\frac{1}{2}} h(1) \right)^2 + h'(1)^2 \right) \\ &+ O\left(\delta^2 + \delta^2 \log \frac{1}{\delta} |\eta| \right) - 2K(\xi) \int_{\mathbb{D}^2} W e^V \mathcal{Z}_1 \\ &+ \int_{\partial\mathbb{D}^2} \left(\delta \left(1 + \log \frac{1}{|z + \xi|} \right) + |\tau| \right)^2 \left(1 + \frac{1}{|z + \xi|^{O(\delta)}} \right) \\ &= \frac{8\pi\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \delta \left((-\Delta)^{\frac{1}{2}} h(1) + \eta(-\Delta)^{\frac{1}{2}} h'(1) \right) \\ &- \delta \log \frac{1}{\delta} \frac{4\varphi(1)^2}{\varphi(1)^2 + K(1)} \left(\left((-\Delta)^{\frac{1}{2}} h(\xi) \right)^2 + h'(\xi)^2 \right) \\ &- 2K(\xi) \int_{\mathbb{D}^2} W e^V \mathcal{Z}_1 + O\left(\delta^2 + \delta|\tau| + \delta|\eta|^2 + \delta^2 \log \frac{1}{\delta} |\eta| + |\tau|^2 \right) \end{aligned} \quad (5.50)$$

To deal with $\mathcal{E}_y^\partial$, we use Proposition A.7 similarly as before:

$$\begin{aligned}
\int_{\partial\mathbb{D}^2} \mathcal{E}_y^\partial \mathcal{Z}_1 &= 2\varepsilon \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} (I(f(z)) - I(\xi)) \mathcal{Z}_1 \\
&+ 2\varepsilon \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} I(\xi) \int_{\partial\mathbb{D}^2} \mathcal{Z}_1 \\
&= 2\varepsilon \left(\frac{4\pi\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \delta(-\Delta)^{\frac{1}{2}} I(\xi) + O(\delta^2) \right) \\
&+ \frac{4\varepsilon I(\xi)\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \int_{\partial\mathbb{D}^2} \langle z, \xi \rangle dz \\
&= \frac{8\pi\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \delta\varepsilon(-\Delta)^{\frac{1}{2}} I(1) + O(\delta|\varepsilon|(\delta + |\eta|)). \quad (5.51)
\end{aligned}$$

Finally, since $\mathcal{E}_3^\partial$ is constant, then

$$\int_{\partial\mathbb{D}^2} \mathcal{E}_3^\partial \mathcal{Z}_1 = 0,$$

therefore the proof follows by putting together (5.50) and (5.51). $\square$

By analogous computations, we obtain the following two propositions.

Proposition 5.12.

$$\begin{aligned}
\int_{\mathbb{D}^2} \mathcal{E}^{\text{Int}} \mathcal{Z}_2 &= \frac{4\pi}{(\varphi(\xi)^2 + K(\xi))^2} \delta(\partial_y K(1) + \eta \partial_{yy} K(1) + \varepsilon \partial_y G(1)) + 2K(\xi) \int_{\mathbb{D}^2} W e^V \mathcal{Z}_2 \\
&+ O\left(\frac{|\varepsilon|^2}{\log^2 \frac{1}{|\varepsilon|}}\right).
\end{aligned}$$

Proposition 5.13.

$$\begin{aligned}
\int_{\partial\mathbb{D}^2} \mathcal{E}^\partial \mathcal{Z}_2 &= \frac{8\pi\varphi(\xi)}{(\varphi(\xi)^2 + K(\xi))^2} \delta(h'(1) + \eta h''(1) + \varepsilon I'(1)) - 2K(\xi) \int_{\mathbb{D}^2} W e^V \mathcal{Z}_2 \\
&+ O\left(\frac{|\varepsilon|^2}{\log^2 \frac{1}{|\varepsilon|}}\right).
\end{aligned}$$

Now, we state and prove the analogous propositions for the estimate of c_0 .

Proposition 5.14.

$$\int_{\mathbb{D}^2} \mathcal{E}^{\text{Int}} = \frac{8\pi}{\varphi(\xi)^2 + K(\xi)} (-\delta \partial_x K(1) + K(1)\tau) + O\left(\frac{|\varepsilon|^2}{\log \frac{1}{|\varepsilon|}}\right).$$

Proof. We split $\mathcal{E}^{\text{Int}}$ as in (5.23), and $\mathcal{E}_1^{\text{Int}}$ again following (5.45). To deal with the term with W we integrate by parts and then use the fact that V is constant along $\partial\mathbb{D}^2$ and Proposition A.9:

$$\begin{aligned}
2K(\xi) \int_{\mathbb{D}^2} W e^V &= - \int_{\mathbb{D}^2} V \Delta W + \int_{\partial\mathbb{D}^2} (V \partial_\nu W - W \partial_\nu V) \\
&= V(\xi) \int_{\partial\mathbb{D}^2} \partial_\nu W - 2h(\xi) \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} W \\
&= \frac{2\varphi(\xi)h(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \left(-\frac{2}{\pi} \frac{2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} \log|z-w|(h(f(w)) - h(\xi)) dw \right) dz \\
&= -\frac{8\varphi(\xi)^2 h(\xi)}{\pi(\varphi(\xi)^2 + K(\xi))^2} \int_{\partial\mathbb{D}^2} (h(f(w)) - h(\xi)) \left(\int_{\partial\mathbb{D}^2} \log|z-w| dz \right) dw \\
&= 0.
\end{aligned}$$

Applying Proposition A.9 again gives

$$2K(\xi)\tau \int_{\mathbb{D}^2} e^V = \frac{8\pi K(\xi)}{\varphi(\xi)^2 + K(\xi)} \tau = \frac{8\pi K(1)}{\varphi(\xi)^2 + K(\xi)} \tau + O(|\eta||\tau|),$$

therefore

$$\begin{aligned}
\int_{\mathbb{D}^2} \mathcal{E}_1^{\text{Int}} &= 2K(\xi) \int_{\mathbb{D}^2} W e^V + 2K(\xi)\tau \int_{\mathbb{D}^2} e^V \\
&\quad + \int_{\mathbb{D}^2} O\left(|f(z) - \xi|(|W| + |\tau|) + (|W| + |\tau|)^2(1 + e^{W+\tau})\right) \\
&= \frac{8\pi K(1)}{\varphi(\xi)^2 + K(\xi)} \tau + O\left(\delta^2 + |\tau|^2\right).
\end{aligned} \tag{5.52}$$

Concerning the second term, we write

$$\begin{aligned}
\mathcal{E}_2^{\text{Int}} &= 2 \langle \nabla K(\xi), f(z) - \xi \rangle e^V + O\left(|f(z) - \xi|^2 + |\varepsilon||f(z) - \xi|\right) \\
&= 2\delta \left\langle \nabla K(\xi), \frac{z - \xi}{1 + (1 - \delta)\bar{\xi}z} \right\rangle e^V + O\left(\frac{\delta^2}{(\delta + |z + \xi|)^2} + \frac{|\varepsilon|\delta}{\delta + |z + \xi|}\right).
\end{aligned}$$

Then, we argue as in the proof of (5.10), (5.12) and apply Proposition A.10:

$$\begin{aligned}
\int_{\mathbb{D}^2} \left\langle \nabla K(\xi), \frac{z - \xi}{1 + (1 - \delta)\bar{\xi}z} \right\rangle e^V &= \langle \nabla K(\xi), \xi \rangle \int_{\mathbb{D}^2} \frac{(1 - \delta)^2 |w|^2 - 1 + \delta w_1}{|1 + (1 - \delta)w|^2} \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^2} dw \\
&+ \langle \nabla K(\xi), \xi^\perp \rangle \int_{\mathbb{D}^2} \frac{(2 - \delta)w_2}{|1 + (1 - \delta)w|^2} \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^2} dw \\
&= -\frac{4\pi}{\varphi(\xi)^2 + K(\xi)} \langle \nabla K(\xi), \xi \rangle \\
&= -\frac{4\pi}{\varphi(\xi)^2 + K(\xi)} \partial_x K(1) + O(|\eta|).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{D}^2} \mathcal{E}_2^{\text{Int}} &= 2\delta \int_{\mathbb{D}^2} \left\langle \nabla K(\xi), \frac{z - \xi}{1 + (1 - \delta)\bar{\xi}z} \right\rangle e^V + \int_{\mathbb{D}^2} O\left(\frac{\delta^2}{(\delta + |z + \xi|)^2} + \frac{|\varepsilon|\delta}{\delta + |z + \xi|}\right) \\
&= -\frac{8\pi\partial_x K(1)}{\varphi(\xi)^2 + K(\xi)}\delta + O\left(\delta\left(\delta \log \frac{1}{\delta} + |\eta| + |\varepsilon|\right)\right). \tag{5.53}
\end{aligned}$$

Finally, since $G(\xi) = O(|\eta|)$, then

$$\int_{\mathbb{D}^2} \mathcal{E}_3^{\text{Int}} = O(|\eta||\varepsilon|),$$

and the result follows by (5.52) and (5.53). $\square$

Proposition 5.15.

$$\int_{\partial\mathbb{D}^2} \mathcal{E}^\partial = \frac{4\pi\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \left(-2\delta(-\Delta)^{\frac{1}{2}}h(1) + \tau h(1)\right) + O\left(\frac{|\varepsilon|^2}{\log \frac{1}{|\varepsilon|}}\right).$$

Proof. We split again as in (5.25), and we write

$$\begin{aligned}
\mathcal{E}_x^\partial &= \frac{2h(\xi)\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} W + \frac{2h(\xi)\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \tau \\
&+ O(|f(z) - \xi|(|W| + \tau)) + O((|W| + |\tau|)^2(1 + e^{W+\tau})).
\end{aligned}$$

The integral of the first term vanishes because, as in the proof of Proposition 5.14,

$$\frac{2h(\xi)\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}^2} W = -\frac{4\varphi(\xi)^2 h(\xi)}{\pi(\varphi(\xi)^2 + K(\xi))^2} \int_{\partial\mathbb{D}^2} (h(f(w)) - h(\xi)) \left(\int_{\partial\mathbb{D}^2} \log |z - w| dz\right) dw = 0.$$

On the other hand, the second term is constant,

$$\frac{2h(\xi)\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)}\tau \int_{\partial\mathbb{D}^2} 1 = \frac{4\pi\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)}\tau h(\xi) = \frac{4\pi\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)}\tau h(1) + O(|\eta||\tau|).$$

Therefore,

$$\begin{aligned} \int_{\partial\mathbb{D}^2} \mathcal{E}_x^\partial &= \frac{4\pi\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)}\tau h(1) + O(|\eta||\tau|) + \int_{\partial\mathbb{D}^2} \frac{\delta}{\delta + |z + \xi|} \delta \left(1 + \log \frac{1}{|z + \xi|}\right) \\ &+ \int_{\partial\mathbb{D}^2} \left(\delta \left(1 + \log \frac{1}{|z + \xi|}\right) + |\tau| \right)^2 \left(1 + \frac{1}{|z + \xi|^{O(\delta)}}\right) \\ &= \frac{4\pi\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)}\tau h(1) + O\left(\delta^2 \log^2 \frac{1}{\delta} + |\eta||\tau| + |\tau|^2\right). \end{aligned}$$

To estimate $\mathcal{E}_y^\partial$, we use that $I(f(z)) = O(|f(z) - \xi| + |\eta|)$, hence

$$\int_{\partial\mathbb{D}^2} \mathcal{E}_y^\partial = \int_{\partial\mathbb{D}^2} O(|\varepsilon|(|f(z) - \xi| + |\eta|)e^W) = O\left(|\varepsilon| \left(\delta \log \frac{1}{\delta} + |\eta|\right)\right).$$

Finally, since $\mathcal{E}_3^\partial$ is constant, we can use Proposition A.4 to conclude the proof:

$$\begin{aligned} \int_{\partial\mathbb{D}^2} \mathcal{E}_3^\partial &= -\frac{8\pi\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)}\delta(-\Delta)^{\frac{1}{2}}h(\xi) + O(\delta^2) \\ &= -\frac{8\pi\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)}\delta(-\Delta)^{\frac{1}{2}}h(1) + O(\delta(\delta + |\eta|)). \end{aligned}$$

□

We are now in a position to derive the precise expansions of the constants c_0 , c_1 , and c_2 .

Corollary 5.16. *The constants c_0, c_1, c_2 appearing in problem (5.41) satisfy*

$$\begin{aligned}
c_1 &= -A_0 \frac{4}{(\varphi(1)^2 + K(1))^2} \delta \left(\underbrace{\eta \left(\partial_{xy} K(1) + 2\varphi(1)(-\Delta)^{\frac{1}{2}} h'(1) \right)}_{=: \mathfrak{a}_{11}} + \varepsilon \underbrace{\left(\partial_x G(1) + 2\varphi(1)(-\Delta)^{\frac{1}{2}} I(1) \right)}_{=: \mathfrak{b}_1} \right) \\
&\quad + \delta \log \frac{1}{\delta} \left(- \underbrace{\frac{2\varphi(1)^2}{\varphi(1)^2 + K(1)} (\Delta K(1) + 4|\nabla H(1)|^2)}_{=: \mathfrak{a}_{12}} \right) + O\left(\frac{|\varepsilon|^2}{\log^2 \frac{1}{|\varepsilon|}} + \|\phi\|^2 e^{O(\|\phi\|^2)} \right), \\
c_2 &= -A_0 \frac{4}{(\varphi(1)^2 + K(1))^2} \delta \left(\underbrace{\eta \left(\partial_{yy} K(1) + 2\varphi(1)h''(1) \right)}_{=: \mathfrak{a}_{21}} + \varepsilon \underbrace{\left(\partial_y G(1) + 2\varphi(1)I'(1) \right)}_{=: \mathfrak{b}_2} \right) \\
&\quad + O\left(\frac{|\varepsilon|^2}{\log^2 \frac{1}{|\varepsilon|}} + \|\phi\|^2 e^{O(\|\phi\|^2)} \right), \\
c_0 &= -\frac{1}{3\pi} \frac{4}{\varphi(1)^2 + K(1)} \left(\delta \underbrace{\left(- \left(2\partial_x K(1) + 2\varphi(1)(-\Delta)^{\frac{1}{2}} h(1) \right) \right)}_{=: \mathfrak{a}_{32}} + \tau \underbrace{\left(2K(1) + \varphi(1)h(1) \right)}_{=: \mathfrak{a}_{33}} \right) \\
&\quad + O\left(\frac{|\varepsilon|^2}{\log \frac{1}{|\varepsilon|}} + \|\phi\|^2 e^{O(\|\phi\|^2)} \right),
\end{aligned}$$

where

$$A_0 := \frac{3\varphi(1)^2(\varphi(1)^2 + K(1))^3}{3\varphi(1)^4 + 3\varphi(1)^2 K(1) + 2K(1)^2}.$$

Proof. We only derive the expansion for c_1 , since the arguments for c_0 and c_2 are completely analogous.

First, recall that $\xi = 1$ is a critical point of φ . In particular,

$$\partial_x K(1) + 2\varphi(1)(-\Delta)^{\frac{1}{2}} h(1) = 0, \quad \varphi(\xi) = \varphi(1) + O(|\eta|^2).$$

Therefore, combining (5.10) and (5.11), we obtain

$$\begin{aligned}
& \int_{\mathbb{D}^2} \mathcal{E}^{\text{Int}} \mathcal{Z}_1 + \int_{\partial \mathbb{D}^2} \mathcal{E}^\partial \mathcal{Z}_1 \\
&= \frac{4\pi}{(\varphi(\xi)^2 + K(\xi))^2} \delta \left(\partial_x K(1) + \eta \partial_{xy} K(1) + \varepsilon \partial_x G(1) - \delta \log \frac{1}{\delta} \frac{2\varphi(1)^2}{\varphi(1)^2 + K(1)} \Delta K(1) \right) \\
&+ \frac{4\pi}{(\varphi(\xi)^2 + K(\xi))^2} \delta \left(2\varphi(\xi) \left((-\Delta)^{\frac{1}{2}} h(1) + \eta (-\Delta)^{\frac{1}{2}} h'(1) + \varepsilon (-\Delta)^{\frac{1}{2}} I(1) \right) \right) \\
&- \frac{4\pi}{(\varphi(\xi)^2 + K(\xi))^2} \delta^2 \log \frac{1}{\delta} \frac{8\varphi(1)^2}{\varphi(1)^2 + K(1)} \left(((-\Delta)^{\frac{1}{2}} h(1))^2 + h'(1)^2 \right) + O\left(\frac{|\varepsilon|^2}{\log \frac{1}{|\varepsilon|}}\right) \\
&= \frac{4\pi}{(\varphi(1)^2 + K(1))^2} \delta \left(\eta \left(\partial_{xy} K(1) + 2\varphi(1) (-\Delta)^{\frac{1}{2}} h'(1) \right) + \varepsilon \left(\partial_x G(1) + 2\varphi(1) (-\Delta)^{\frac{1}{2}} I(1) \right) \right) \\
&- \frac{4\pi}{(\varphi(1)^2 + K(1))^2} \delta^2 \log \frac{1}{\delta} \frac{2\varphi(1)^2}{\varphi(1)^2 + K(1)} \left(\Delta K(1) + 4|\nabla H(1)|^2 \right) + O\left(\frac{|\varepsilon|^2}{\log \frac{1}{|\varepsilon|}}\right). \tag{5.54}
\end{aligned}$$

Next, by Lemmas 5.7 and 5.8, for any $1 < p < \frac{4}{3}$,

$$\begin{aligned}
& \int_{\mathbb{D}^2} \mathcal{L}^{\text{Int}} \phi \mathcal{Z}_1 + \int_{\partial \mathbb{D}^2} \mathcal{L}^\partial \phi \mathcal{Z}_1 + \int_{\mathbb{D}^2} \mathcal{N}^{\text{Int}}(\phi) \mathcal{Z}_1 + \int_{\partial \mathbb{D}^2} \mathcal{N}^\partial(\phi) \mathcal{Z}_1 \\
&= O\left(\|\mathcal{L}^{\text{Int}} \phi\|_{L^p(\mathbb{D}^2)} + \|\mathcal{L}^\partial \phi\|_{L^p(\partial \mathbb{D}^2)} + \|\mathcal{N}^{\text{Int}}(\phi)\|_{L^p(\mathbb{D}^2)} + \|\mathcal{N}^\partial(\phi)\|_{L^p(\partial \mathbb{D}^2)} \right) \\
&= O\left(\frac{|\varepsilon|}{\log \frac{1}{|\varepsilon|}} \|\phi\| + \|\phi\|^2 e^{O(\|\phi\|^2)} \right) \\
&= O\left(\frac{|\varepsilon|^2}{\log^2 \frac{1}{|\varepsilon|}} + \|\phi\|^2 e^{O(\|\phi\|^2)} \right), \tag{5.55}
\end{aligned}$$

where in the last step we used (5.42).

Finally, Proposition A.11 yields

$$2K(\xi) \int_{\mathbb{D}^2} e^V \mathcal{Z}_1^2 + \frac{2h(\xi)\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial \mathbb{D}^2} \mathcal{Z}_1^2 = \frac{\pi}{A_0} + O(|\eta|).$$

Combining this identity with (5.54) and (5.55), the claimed expansion for c_1 follows. $\square$

5.1.5 The finite dimensional reduction

Let $\delta, |\eta|, |\tau|, |\varepsilon|$ be sufficiently small so that Proposition 5.9 applies, yielding a solution to (5.41).

If, for some choice of δ, η, τ , one has

$$c_0 = c_1 = c_2 = 0,$$

then ϕ also solves (5.15), and therefore we obtain a solution to (5.8).

By Corollary 5.16 together with the estimates on $\|\phi\|$, assuming

$$\eta = s\varepsilon, \quad \tau = t\varepsilon \left(\log \frac{1}{\delta} \right)^{-1}, \quad \delta \log \frac{1}{\delta} = d\varepsilon,$$

with $d > 0$ if $\varepsilon > 0$ and $d < 0$ if $\varepsilon < 0$, the vanishing of all the coefficients c_i is equivalent to solving

$$\begin{cases} a_{11}s + a_{12}d + b_1 + o_\varepsilon(1) = 0, \\ a_{21}s + b_2 + o_\varepsilon(1) = 0, \\ a_{32}d + a_{33}t + o_\varepsilon(1) = 0. \end{cases} \quad (5.56)$$

The coefficients a_{ij} are defined in Corollary 5.16.

System (5.56) can be rewritten as

$$F_\varepsilon(s, d, t) = F_0(s, d, t) + o_\varepsilon(1) = 0,$$

where the function $F_0 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by

$$F_0(s, d, t) := A \begin{pmatrix} s \\ d \\ t \end{pmatrix} + B,$$

with

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}.$$

It is immediate that, if

$$\det A = \det \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix} \neq 0,$$

that is,

$$a_{12}a_{21}a_{33} \neq 0 \quad (\text{see (5.1), (5.3), (5.4), and (5.5)}),$$

there exists a unique $(s_0, d_0, t_0) \in \mathbb{R}^3$ such that

$$F_0(s_0, d_0, t_0) = 0.$$

Moreover, one has $d_0 \neq 0$ provided that

$$\det \begin{pmatrix} a_{11} & b_1 & 0 \\ a_{21} & b_2 & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \neq 0,$$

namely,

$$a_{33}(a_{11}b_2 - a_{21}b_1) \neq 0 \quad (\text{see (5.7)}).$$

Since $F_\varepsilon \rightarrow F_0$ uniformly on compact subsets of

$$\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R},$$

and the Brouwer degree of F_ε is nonzero, it follows that there exists

$$(s_\varepsilon, d_\varepsilon, t_\varepsilon) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$$

such that

$$F_\varepsilon(s_\varepsilon, d_\varepsilon, t_\varepsilon) = 0,$$

with

$$(s_\varepsilon, d_\varepsilon, t_\varepsilon) \rightarrow (s_0, d_0, t_0) \quad \text{as } \varepsilon \rightarrow 0.$$

This proves the existence of a solution.

Finally, using the estimates on $\mathcal{E}$, $\mathcal{L}$, $\mathcal{N}$, and $\|\phi\|$, equation (5.15) yields

$$\|\Delta\phi\|_{L^p(\mathbb{D}^2)} + \|\partial_\nu\phi\|_{L^p(\partial\mathbb{D}^2)} = o_\varepsilon(1)$$

for some $p > 1$. Hence,

$$\|\phi\|_{L^\infty(\mathbb{D}^2)} = o_\varepsilon(1).$$

Since both $V(f^{-1}(z))$ and $W(f^{-1}(z))$ concentrate at $\xi = 1$, we conclude that the solution

$$u = V(f^{-1}(z)) + W(f^{-1}(z)) + \tau + \phi(f^{-1}(z))$$

also concentrates at $\xi = 1$. This completes the proof of Theorem 5.1.

5.2 Construction of solution on a generic domain

We consider the following problem:

$$\begin{cases} -\Delta v = \varepsilon^2 K(x) e^{2v} & \text{in } \Omega \\ \frac{\partial v}{\partial n} + \tilde{l}(v) = \varepsilon h(x) e^v & \text{on } \partial\Omega \end{cases} \quad (5.57)$$

where

$$\tilde{l}(v) = \frac{1}{|\partial\Omega|} \left(\int_{\Omega} \varepsilon^2 K(x) e^{2v} dx + \int_{\partial\Omega} \varepsilon h(x) e^v ds \right) \rightarrow \frac{2\pi m}{|\partial\Omega|}$$

Our aim is to construct a family of solutions v_ε which blows-up at m distinct points $\xi_1, \dots, \xi_m \in \partial\Omega$ as $\varepsilon \rightarrow 0$. We assume that K and h are strictly positive smooth functions.

We define the expanded domain $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$, and $\xi'_j = \frac{1}{\varepsilon}\xi_j$. Setting $u(y) = v(\varepsilon y) + 2\log \varepsilon$, the problem (5.57) is equivalent to

$$\begin{cases} -\Delta_g u = K(\varepsilon y) e^{2u} & \text{in } \Omega_\varepsilon \\ \frac{\partial u}{\partial n} + \varepsilon l(u) = h(\varepsilon y) e^u & \text{on } \partial\Omega_\varepsilon \end{cases} \quad (5.58)$$

where

$$l(u) = \frac{1}{|\partial\Omega|} \left(\int_{\Omega_\varepsilon} K(\varepsilon y) e^{2u} dy + \int_{\partial\Omega_\varepsilon} h(\varepsilon y) e^u ds \right)$$

5.2.1 Ansatz and Error Estimates

We define the approximate solution as

$$V(y) = \sum_{i=1}^m (U_i(\varepsilon y) + H_i(\varepsilon y)) + 2\log \varepsilon$$

where the bubbles U_i are given by

$$U_i(x) = \log \left(\frac{2\lambda_i}{\sqrt{K(\xi_i)} (\lambda_i^2 \varepsilon^2 + |x - \xi_i - \mathfrak{D}(\xi_i) \varepsilon \lambda_i n(\xi_i)|^2)} \right)$$

with $\mathfrak{D}(\xi_i) = \frac{h(\xi_i)}{\sqrt{K(\xi_i)}}$.

Note that they solves

$$\begin{cases} -\Delta U_i = K(\xi_i) e^{2U_i} & \text{in } \mathbb{R}_+^2 \\ \frac{\partial U_i}{\partial n} = h(\xi_i) e^{U_i} & \text{on } \partial\mathbb{R}_+^2 \end{cases}$$

Instead, the correction term H_i solves the problem

$$\begin{cases} -\Delta_g H_i = \varepsilon^2 K(x) e^{2U_i} + \Delta_g U_i & \text{in } \Omega, \\ \frac{\partial H_i}{\partial n} = \varepsilon h(x) e^{U_i} - \frac{\partial U_i}{\partial n} - \tilde{l}(U_i) & \text{on } \partial\Omega, \end{cases} \quad (5.59)$$

We define the Green's function of the Neumann problem, denoted by $G(x, y)$, as the unique solution (for any fixed $y \in \Omega$) of

$$\begin{cases} -\Delta_g G(x, y) = 0 & \text{in } \Omega, \\ \frac{\partial G}{\partial n}(x, y) = \delta_y(x) - \frac{1}{|\partial\Omega|} & \text{on } \partial\Omega, \end{cases} \quad (5.60)$$

subject to the normalization condition $\int_{\partial\Omega} G(x, y) ds_x = 0$. We define the regular part of the Green's function by subtracting the standard logarithmic singularity:

$$H(x, y) = G(x, y) + \frac{1}{2\pi} \log(|x - y|^2). \quad (5.61)$$

Lemma 5.17. *Let $H_j(x)$ be the solution of the linear problem (5.59) normalized by $\int_{\partial\Omega} H_j = |\partial\Omega|(\frac{1}{2} \log(K(\xi_j)) - \log(2\lambda_j)) + 2\pi \int_{\partial\Omega} H(x, \xi_j)$. Then, for any $\alpha \in (0, 1)$, the following asymptotic expansion holds uniformly in Ω as $\varepsilon \rightarrow 0$:*

$$H_j(x) = 2\pi H(x, \xi_j) - \log(2\lambda_j) + \frac{1}{2} \log K(\xi_j) + O(\varepsilon^\alpha). \quad (5.62)$$

Proof. The normal derivative of the bubble profile U_j is given by

$$\frac{\partial U_j}{\partial n} = -\frac{2(x - \xi_j - \mathfrak{D}(\xi_j)\varepsilon\lambda_j n(\xi_j)) \cdot n(x)}{|x - \xi_j - \mathfrak{D}(\xi_j)\varepsilon\lambda_j n(\xi_j)|^2 + \lambda_j^2 \varepsilon^2}. \quad (5.63)$$

The boundary condition satisfied by H_j is $\frac{\partial H_j}{\partial n} = \varepsilon h(x) e^{U_j} - \frac{\partial U_j}{\partial n} - \tilde{l}(U_j)$. Substituting the expression of U_j , the boundary source term becomes

$$\varepsilon h(x) e^{U_j} = \frac{2\varepsilon\lambda_j \frac{h(x)}{\sqrt{K(\xi_j)}}}{|x - \xi_j - \mathfrak{D}(\xi_j)\varepsilon\lambda_j n(\xi_j)|^2 + \lambda_j^2 \varepsilon^2}. \quad (5.64)$$

Therefore, the boundary condition for H_j can be written as:

$$\frac{\partial H_j}{\partial n} = \frac{2\varepsilon\lambda_j \frac{h(x)}{\sqrt{K(\xi_j)}} + 2(x - \xi_j - \mathfrak{D}(\xi_j)\varepsilon\lambda_j n(\xi_j)) \cdot n(x)}{|x - \xi_j - \mathfrak{D}(\xi_j)\varepsilon\lambda_j n(\xi_j)|^2 + \lambda_j^2 \varepsilon^2} - \tilde{l}(U_j). \quad (5.65)$$

Recalling that $\mathfrak{D}(\xi_j) = \frac{h(\xi_j)}{\sqrt{K(\xi_j)}}$, it follows that for every $x \neq \xi_j$, the limit as

$\varepsilon \rightarrow 0$ of the fraction approaches $2 \frac{(x-\xi_j) \cdot n(x)}{|x-\xi_j|^2}$. Since $\tilde{l}(U_j) \rightarrow \frac{2\pi}{|\partial\Omega|}$ as $\varepsilon \rightarrow 0$, we get

$$\lim_{\varepsilon \rightarrow 0} \frac{\partial H_j}{\partial n} = 2 \frac{(x-\xi_j) \cdot n(x)}{|x-\xi_j|^2} - \frac{2\pi}{|\partial\Omega|} \quad \forall x \neq \xi_j. \quad (5.66)$$

The regular part of the Green's function satisfies $-\Delta H(x, y) = 0$ in Ω . On the boundary, noting that $\frac{\partial}{\partial n} \log |x - \xi_j|^2 = 2 \frac{(x-\xi_j) \cdot n(x)}{|x-\xi_j|^2}$, we have

$$\frac{\partial H}{\partial n}(x, \xi_j) = \frac{1}{2\pi} \frac{\partial}{\partial n} \log |x - \xi_j|^2 - \frac{1}{|\partial\Omega|} = \frac{1}{\pi} \frac{(x - \xi_j) \cdot n(x)}{|x - \xi_j|^2} - \frac{1}{|\partial\Omega|}. \quad (5.67)$$

Now, we define $z_\varepsilon(x) = H_j(x) + \log(2\lambda_j) - \frac{1}{2} \log K(\xi_j) - 2\pi H(x, \xi_j)$, which satisfies:

$$\begin{cases} -\Delta z_\varepsilon = \varepsilon^2 K(x) e^{2U_j} + \Delta U_j & \text{in } \Omega, \\ \frac{\partial z_\varepsilon}{\partial n} = \frac{\partial H_j}{\partial n} - \frac{2(x-\xi_j) \cdot n(x)}{|x-\xi_j|^2} + \frac{2\pi}{|\partial\Omega|} & \text{on } \partial\Omega. \end{cases} \quad (5.68)$$

We claim that, for any $p > 1$, there exists $C > 0$ such that

$$\left\| \frac{\partial z_\varepsilon}{\partial n} \right\|_{L^p(\partial\Omega)} \leq C \varepsilon^{1/p}. \quad (5.69)$$

To prove (5.69), we fix a small $\delta > 0$ and split the boundary integral into a far-field region $\partial\Omega \setminus B_\delta(\xi_j)$ and a near-field region $\partial\Omega \cap B_\delta(\xi_j)$. In the far-field region, the distance $|x - \xi_j|$ is bounded strictly away from zero ($|x - \xi_j| \geq \delta$). Using the expansion (5.66), uniformly on compact subsets of $\partial\Omega \setminus \{\xi_j\}$, we obtain

$$\left| \frac{\partial z_\varepsilon}{\partial n}(x) \right| \leq C\varepsilon, \quad \forall x \in \partial\Omega \setminus B_\delta(\xi_j).$$

which yields:

$$\int_{\partial\Omega \setminus B_\delta(\xi_j)} \left| \frac{\partial z_\varepsilon}{\partial n} \right|^p ds_x \leq C \varepsilon^p. \quad (5.70)$$

Now we focus on the region $B_\delta(\xi_j) \cap \partial\Omega$. We decompose the normal derivative by adding and subtracting $\mathfrak{D}(\xi_j)$ in the numerator of $\partial_n H_j$:

$$\begin{aligned}
\frac{\partial H_j}{\partial n} - \frac{2(x - \xi_j) \cdot n(x)}{|x - \xi_j|^2} &= \frac{2\varepsilon\lambda_j \left(\frac{h(x)}{\sqrt{K(\xi_j)}} - \mathfrak{D}(\xi_j) \right)}{|x - \xi_j - \varepsilon\lambda_j\mathfrak{D}(\xi_j)n(\xi_j)|^2 + \varepsilon^2\lambda_j^2} \\
&+ \frac{2\varepsilon\lambda_j\mathfrak{D}(\xi_j)(1 - n(\xi_j) \cdot n(x))}{|x - \xi_j - \varepsilon\lambda_j\mathfrak{D}(\xi_j)n(\xi_j)|^2 + \varepsilon^2\lambda_j^2} \\
&+ 2(x - \xi_j) \cdot n(x) \left[\frac{|x - \xi_j|^2 - |x - \xi_j - \varepsilon\lambda_j\mathfrak{D}(\xi_j)n(\xi_j)|^2 - \varepsilon^2\lambda_j^2}{(|x - \xi_j - \varepsilon\lambda_j\mathfrak{D}(\xi_j)n(\xi_j)|^2 + \varepsilon^2\lambda_j^2)|x - \xi_j|^2} \right] \\
&- \tilde{l}(U_j).
\end{aligned}$$

Since n is Lipschitz continuous on $\partial\Omega$ and $|n(x)| = |n(\xi_j)| = 1$, we have

$$1 - n(\xi_j) \cdot n(x) = \frac{1}{2}|n(\xi_j) - n(x)|^2 \leq C|x - \xi_j|^2.$$

Moreover, since $\partial\Omega$ is of class C^2 ,

$$|(x - \xi_j) \cdot n(x)| \leq C|x - \xi_j|^2. \quad (5.71)$$

Furthermore, by the regularity of h ,

$$\left| \frac{h(x)}{\sqrt{K(\xi_j)}} - \mathfrak{D}(\xi_j) \right| \leq C|x - \xi_j|.$$

Expanding the square in the numerator of the third term yields

$$|x - \xi_j|^2 - |x - \xi_j - \varepsilon\lambda_j\mathfrak{D}(\xi_j)n(\xi_j)|^2 = 2\varepsilon\lambda_j\mathfrak{D}(\xi_j)(x - \xi_j) \cdot n(\xi_j) - \varepsilon^2\lambda_j^2\mathfrak{D}(\xi_j)^2. \quad (5.72)$$

Therefore,

$$\left| |x - \xi_j|^2 - |x - \xi_j - \varepsilon\lambda_j\mathfrak{D}(\xi_j)n(\xi_j)|^2 \right| \leq C\varepsilon|x - \xi_j| + C\varepsilon^2. \quad (5.73)$$

Combining the previous estimates and using

$$\left| \tilde{l}(U_j) - \frac{2\pi}{|\partial\Omega|} \right| \leq C\varepsilon,$$

we obtain the pointwise estimate

$$\left| \frac{\partial z_\varepsilon}{\partial n} \right| \leq C\varepsilon + \frac{C\varepsilon|x - \xi_j| + C\varepsilon^2}{|x - \xi_j - \varepsilon\lambda_j\mathfrak{D}(\xi_j)n(\xi_j)|^2 + \varepsilon^2\lambda_j^2}. \quad (5.74)$$

We can estimate the distance at the denominator, using again (5.71):

$$|x - \xi_j - \varepsilon \lambda_j \mathfrak{D}(\xi_j) n(\xi_j)|^2 + \varepsilon^2 \lambda_j^2 \geq C(|x - \xi_j|^2 + \varepsilon^2).$$

Substituting these bounds into (5.74), the integral over the near-field region can be estimated as:

$$\int_{B_\delta(\xi_j) \cap \partial\Omega} \left| \frac{\partial z_\varepsilon}{\partial n} \right|^p ds_x \leq C \int_{B_\delta(\xi_j) \cap \partial\Omega} \left(\varepsilon + \frac{\varepsilon|x - \xi_j| + \varepsilon^2}{|x - \xi_j|^2 + \varepsilon^2} \right)^p ds_x. \quad (5.75)$$

By the convexity inequality $(A + B)^p \leq 2^{p-1}(A^p + B^p)$, so we have a term $O(\varepsilon^p)$. Instead for the other term, we parametrize the boundary piece $B_\delta(\xi_j) \cap \partial\Omega$ by arc-length $t \in (-\delta, \delta)$, with $t = 0$ corresponding to ξ_j . Since $\partial\Omega$ is smooth, we have $c_1|t| \leq |x - \xi_j| \leq c_2|t|$, and the line element is $ds_x = dt$. Therefore, we can bound the integral and perform the scalar change of variables $t = \varepsilon s$:

$$\begin{aligned} \int_{B_\delta(\xi_j) \cap \partial\Omega} \left| \frac{\partial z_\varepsilon}{\partial n} \right|^p ds_x &\leq C\varepsilon^p + C \int_{-\delta}^{\delta} \left(\frac{\varepsilon|t| + \varepsilon^2}{t^2 + \varepsilon^2} \right)^p dt \\ &= C\varepsilon^p + \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \left(\frac{\varepsilon^2|s| + \varepsilon^2}{\varepsilon^2 s^2 + \varepsilon^2} \right)^p \varepsilon ds = C\varepsilon^p + \varepsilon \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \left(\frac{|s| + 1}{s^2 + 1} \right)^p ds \leq C\varepsilon^p + \varepsilon \int_{-\infty}^{\infty} \frac{(|s| + 1)^p}{(s^2 + 1)^p} ds. \end{aligned}$$

Since $p > 1$, the integrand behaves asymptotically as $O(|s|^{-p})$ for large $|s|$. Therefore, the integral is finite. Thus, we obtain:

$$\int_{B_\delta(\xi_j) \cap \partial\Omega} \left| \frac{\partial z_\varepsilon}{\partial n} \right|^p ds_x \leq C\varepsilon. \quad (5.76)$$

Combining (5.70) and (5.76), we find the global boundary estimate:

$$\int_{\partial\Omega} \left| \frac{\partial z_\varepsilon}{\partial n} \right|^p ds_x \leq C\varepsilon.$$

Taking the p -th root yields the desired result:

$$\left\| \frac{\partial z_\varepsilon}{\partial n} \right\|_{L^p(\partial\Omega)} \leq C\varepsilon^{1/p}.$$

This concludes the proof of (5.69).

Now we estimate the interior norm $\|\Delta z_\varepsilon\|_{L^p(\Omega)}$ for $1 < p < 2$. By definition, the interior source is given by

$$-\Delta z_\varepsilon = \varepsilon^2 K(x) e^{2U_j} + \Delta U_j. \quad (5.77)$$

Furthermore, in $\mathbb{R}^2$, the bubble profile U_j satisfies the equation $\Delta U_j = -\varepsilon^2 K(\xi_j) e^{2U_j}$. Therefore, the interior source reduces to a single remainder term:

$$-\Delta z_\varepsilon = \varepsilon^2 (K(x) - K(\xi_j)) e^{2U_j}. \quad (5.78)$$

To estimate this, we must recall that the exponential bubble is not centered at ξ_j , but at the shifted point $x_{j,\varepsilon} = \xi_j + \varepsilon \lambda_j \mathfrak{D}(\xi_j) n(\xi_j)$. The square of the conformal factor takes the form:

$$e^{2U_j} = \frac{C}{(|x - x_{j,\varepsilon}|^2 + \varepsilon^2 \lambda_j^2)^2}. \quad (5.79)$$

Since $K(x)$ is smooth, we use the triangle inequality to bound the difference:

$$|K(x) - K(\xi_j)| \leq C|x - \xi_j| \leq C(|x - x_{j,\varepsilon}| + |x_{j,\varepsilon} - \xi_j|) \leq C(|x - x_{j,\varepsilon}| + \varepsilon).$$

Thus, the term is pointwise bounded by:

$$|\Delta z_\varepsilon| \leq C\varepsilon^2 \frac{|x - x_{j,\varepsilon}| + \varepsilon}{(|x - x_{j,\varepsilon}|^2 + \varepsilon^2 \lambda_j^2)^2}. \quad (5.80)$$

Changing variables centered at the true peak, $x - x_{j,\varepsilon} = \varepsilon y$, the integral over Ω becomes:

$$\int_\Omega |\Delta z_\varepsilon|^p dx \leq C \int_{\mathbb{R}^2} \left(\varepsilon^2 \frac{\varepsilon|y| + \varepsilon}{\varepsilon^4(|y|^2 + \lambda_j^2)^2} \right)^p \varepsilon^2 dy = C\varepsilon^{2-p} \int_{\mathbb{R}^2} \frac{(|y| + 1)^p}{(|y|^2 + \lambda_j^2)^{2p}} dy. \quad (5.81)$$

Since $1 < p < 2$, the integrand decays as $O(|y|^{-3p})$ at infinity, so the integral is finite, yielding a bound of $O(\varepsilon^{2-p})$. Taking the p -th root, we conclude that:

$$\|\Delta z_\varepsilon\|_{L^p(\Omega)} \leq C\varepsilon^{\frac{2-p}{p}} = C\varepsilon^{\frac{2}{p}-1}. \quad (5.82)$$

The standard L^p elliptic theory for the Neumann problem (note that thanks to our choice for the normalisation of H_j , z_ε has zero average on the boundary) yields:

$$\|z_\varepsilon\|_{W^{1+s,p}(\Omega)} \leq C \left(\left\| \frac{\partial z_\varepsilon}{\partial n} \right\|_{L^p(\partial\Omega)} + \|\Delta z_\varepsilon\|_{L^p(\Omega)} \right) \leq C \left(\varepsilon^{1/p} + \varepsilon^{\frac{2}{p}-1} \right) \leq C\varepsilon^{\frac{2}{p}-1}, \quad (5.83)$$

for any fractional order $0 < s < 1/p$.

By the Morrey-Sobolev embedding theorem in dimension two, the fractional space $W^{1+s,p}(\Omega)$ embeds continuously into the Hölder space $C^\gamma(\Omega)$ provided that the exponent satisfies $\gamma = 1 + s - \frac{2}{p} > 0$. Choosing $s < 1/p$ sufficiently close to $1/p$,

we obtain the embedding $W^{1+s,p}(\Omega) \hookrightarrow C^\gamma(\overline{\Omega})$ for every $\gamma < 1 - 1/p$. By choosing $p > 1$ arbitrarily close to 1, the error exponent $\alpha = \frac{2}{p} - 1$ can be made arbitrarily close to 1, while maintaining a strictly positive Hölder exponent $\gamma > 0$. Therefore,

$$\|z_\varepsilon\|_{L^\infty(\Omega)} \leq C\|z_\varepsilon\|_{W^{1+s,p}(\Omega)} \leq C\varepsilon^\alpha, \quad (5.84)$$

for any chosen $\alpha \in (0, 1)$. This completes the proof of the asymptotic expansion. $\square$

We will seek a solution to (5.58) of the form $u = V + \phi$. Thus, the problem can be stated as finding a function ϕ that satisfies

$$\begin{cases} -\Delta\phi - 2K(\varepsilon y)e^{2V}\phi = R_1(y) + N_1(\phi), & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial n} - h(\varepsilon y)e^V\phi + \varepsilon l_1(V)[\phi] = R_2(y) + N_2(\phi), & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where the terms on the right-hand side represent the errors, the non-local linear perturbations, and the non-linear remainders, respectively.

In the interior of the domain Ω_ε , the pieces are defined as:

$$\begin{aligned} R_1(y) &= \Delta V + K(\varepsilon y)e^{2V}, \\ N_1(\phi) &= K(\varepsilon y)e^{2V}(e^{2\phi} - 1 - 2\phi). \end{aligned}$$

On the boundary $\partial\Omega_\varepsilon$, the corresponding pieces are defined as:

$$\begin{aligned} R_2(y) &= -\frac{\partial V}{\partial n} - \varepsilon l(V) + h(\varepsilon y)e^V, \\ N_2(\phi) &= h(\varepsilon y)e^V(e^\phi - 1 - \phi) - \varepsilon N_l(\phi). \end{aligned}$$

The non-local functional $l(u)$ evaluated at the ansatz V , its linearisation $l_1(V)[\phi]$, and its non-linear remainder $N_l(\phi)$ are explicitly given by:

$$\begin{aligned} l(V) &= \frac{1}{|\partial\Omega|} \left(\int_{\Omega_\varepsilon} K(\varepsilon y)e^{2V} dy + \int_{\partial\Omega_\varepsilon} h(\varepsilon y)e^V ds \right), \\ l_1(V)[\phi] &= \frac{1}{|\partial\Omega|} \left(\int_{\Omega_\varepsilon} 2K(\varepsilon y)e^{2V}\phi dy + \int_{\partial\Omega_\varepsilon} h(\varepsilon y)e^V\phi ds \right), \\ N_l(\phi) &= \frac{1}{|\partial\Omega|} \left(\int_{\Omega_\varepsilon} K(\varepsilon y)e^{2V}(e^{2\phi} - 1 - 2\phi) dy + \int_{\partial\Omega_\varepsilon} h(\varepsilon y)e^V(e^\phi - 1 - \phi) ds \right). \end{aligned}$$

In order to uniformly control the error terms near the concentration points, we must choose the parameters $\{\lambda_j\}_{j=1}^m$ such that the interactions between the bubbles

and the Green's function cancel out. We choose $\{\lambda_j\}_{j=1}^m$ such that:

$$\log(2\lambda_j) = 2\pi H(\xi_j, \xi_j) + 2\pi \sum_{i \neq j} G(\xi_i, \xi_j) + \frac{1}{2} \log K(\xi_j) \quad \text{for all } j = 1, \dots, m. \quad (5.85)$$

Lemma 5.18. *Let $\delta > 0$ be small and fixed. In the region $B_{\delta/\varepsilon}(\xi'_j)$ we have:*

$$R_1(y) = \frac{4\lambda_j^2}{(\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2)^2} \left[O(\varepsilon|y - \xi'_j|) + O(\varepsilon^\alpha) \right], \quad y \in \Omega_\varepsilon,$$

and on the boundary $\partial\Omega_\varepsilon$,

$$R_2(y) = \frac{2\mathfrak{D}(\xi_j)\lambda_j}{\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2} \left[O(\varepsilon|y - \xi'_j|) + O(\varepsilon^\alpha) \right] + O(\varepsilon^{1+\alpha}).$$

While in the region $\Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{\delta/\varepsilon}(\xi'_j)$, the errors are:

$$R_1(y) = O(\varepsilon^4), \quad y \in \Omega_\varepsilon,$$

and

$$R_2(y) = O(\varepsilon^{1+\alpha}), \quad y \in \partial\Omega_\varepsilon.$$

Proof. For $y \in B_{\delta/\varepsilon}(\xi'_j)$, we explicitly compute $R_1(y)$ and $R_2(y)$.

$$\begin{aligned} R_1 &= \sum_{i=1}^m \Delta(U_i(\varepsilon y + H_i(\varepsilon y))) + K(\varepsilon y) e^{2(U_j + H_j + \sum_{i \neq j} (U_i + H_i) + 2 \log \varepsilon)} \\ &= K(\varepsilon y) \left(e^{2(U_j + 2 \log \varepsilon)} e^{H_j + \sum_{i \neq j} (U_i + H_i)} - \sum_{i=1}^m e^{2(U_i(\varepsilon y) + 2 \log \varepsilon)} \right) \\ &= \left(\frac{4\lambda_j^2 K(\varepsilon y)}{K(\xi_j)(\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2)^2} \right) (e^{2(H_j + \sum_{i \neq j} (U_i + H_i))} - 1) + O(\varepsilon^4) \\ &= \left(\frac{4\lambda_j^2}{(\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2)^2} \right) (1 + O(\varepsilon|y - \xi'_j|)) (O(\varepsilon|y - \xi'_j|) + O(\varepsilon^\alpha)) \end{aligned}$$

where in the last equality we have used that 5.17 and the choice 5.85 to write:

$$\begin{aligned} H_j + \sum_{i \neq j} (U_i + H_i) &= 2\pi H(\varepsilon y, \xi_j) - \log(2\lambda_j) + \frac{1}{2} \log K(\xi_j) \\ &+ \sum_{i \neq j} \left(U_i(\varepsilon y) + 2\pi H(\varepsilon y, \xi_i) + \frac{1}{2} \log(K(\xi_i)) - \log(2\lambda_i) \right) + O(\varepsilon^\alpha) \\ &= 2\pi H(\xi_j, \xi_j) - \log(2\lambda_j) + \frac{1}{2} \log(K(\xi_j)) + \sum_{i \neq j} \left(\log \left(\frac{1}{|\xi_j - \xi_i|^2} \right) + 2\pi H(\xi_j, \xi_i) \right) \end{aligned}$$

$$+O(\varepsilon|y - \xi'_j|) + O(\varepsilon^\alpha) = O(\varepsilon|y - \xi'_j|) + O(\varepsilon^\alpha)$$

$$R_2 = \sum_{i=1}^m -\frac{\partial(U_i + H_i)}{\partial n} + h(\varepsilon y) e^{U_j + H_j + 2\log \varepsilon + \sum_{i \neq j} (U_i + H_i)} - \varepsilon l(V)$$

Using the boundary conditions satisfied by H_i , the normal derivative expands as $\frac{\partial(U_i + H_i)}{\partial n_y} = \varepsilon^2 h(\varepsilon y) e^{U_i(\varepsilon y)} - \varepsilon \tilde{l}(U_i)$. Therefore, substituting this into $R_2(y)$, we can rearrange the terms as:

$$R_2 = h(\varepsilon y) \left(e^{U_j(\varepsilon y) + 2\log \varepsilon} e^{H_j + \sum_{i \neq j} (U_i + H_i)} - \sum_{i=1}^m e^{U_i(\varepsilon y) + 2\log \varepsilon} \right) + \varepsilon \left(\sum_{i=1}^m \tilde{l}(U_i) - l(V) \right)$$

Isolating the j -th bubble and noting that the external bubbles decay as $e^{U_i(\varepsilon y) + 2\log \varepsilon} = O(\varepsilon^2)$ for $i \neq j$, we get:

$$= h(\varepsilon y) e^{U_j(\varepsilon y) + 2\log \varepsilon} \left(e^{H_j + \sum_{i \neq j} (U_i + H_i)} - 1 \right) + O(\varepsilon^2) + \varepsilon \left(\sum_{i=1}^m \tilde{l}(U_i) - l(V) \right)$$

By definition of the bubble, the factor in front can be written by factoring out $h(\xi_j)$ from $h(\varepsilon y)$:

$$= \left(\frac{2\lambda_j h(\varepsilon y)}{\sqrt{K(\xi_j)}(\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2)} \right) \left(e^{H_j + \sum_{i \neq j} (U_i + H_i)} - 1 \right) + \varepsilon \left(\sum_{i=1}^m \tilde{l}(U_i) - l(V) \right)$$

$$= \left(\frac{2\mathfrak{D}(\xi_j)\lambda_j}{\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2} \right) \left(1 + O(\varepsilon|y - \xi'_j|) \right) \left(e^{H_j + \sum_{i \neq j} (U_i + H_i)} - 1 \right) + \varepsilon \left(\sum_{i=1}^m \tilde{l}(U_i) - l(V) \right)$$

Using the exact same expansion for the exponent as we did for $R_1(y)$, the expression simplifies to:

$$= \left(\frac{2\mathfrak{D}(\xi_j)\lambda_j}{\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2} \right) \left(1 + O(\varepsilon|y - \xi'_j|) \right) \left(O(\varepsilon|y - \xi'_j|) + O(\varepsilon^\alpha) \right) + \varepsilon \left(\sum_{i=1}^m \tilde{l}(U_i) - l(V) \right)$$

$$= \frac{2\mathfrak{D}(\xi_j)\lambda_j}{\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2} \left[O(\varepsilon|y - \xi'_j|) + O(\varepsilon^\alpha) \right] + \varepsilon \left(\sum_{i=1}^m \tilde{l}(U_i) - l(V) \right).$$

Now, we bound the non-local error is exactly equivalent to bounding $l(V) - \sum_{i=1}^m \tilde{l}(U_i)$. Note that, if we call $\mathcal{U}_i(y) = U_i(\varepsilon y) + 2\log \varepsilon$, we have (changing the variable in the integral of $\tilde{l}$): $\tilde{l}(U_i) = l(\mathcal{U}_i)$. We compute the difference of the

interior integrals :

$$\int_{\Omega_\varepsilon} K(\varepsilon y) \left(e^{2V(y)} - \sum_{i=1}^m e^{2\mathcal{U}_i(y)} \right) dy.$$

We split the domain into the near-field regions $\bigcup_{j=1}^m B_{\delta/\varepsilon}(\xi'_j)$ and the far-field. In the far-field $V(y) = 2 \log \varepsilon + O(1)$, so $e^{2V} = O(\varepsilon^4)$ and $e^{2\mathcal{U}_i} = O(\varepsilon^4)$. Integrating these over a domain of area $O(\varepsilon^{-2})$ gives a far-field contribution of $O(\varepsilon^2)$. In the near-field $B_{\delta/\varepsilon}(\xi'_j)$, we isolate the j -th bubble:

$$V(y) = \mathcal{U}_j(y) + H_j(\varepsilon y) + \sum_{i \neq j} (U_i(\varepsilon y) + H_i(\varepsilon y)).$$

From the previous expansion:

$$e^{2V(y)} = e^{2\mathcal{U}_j(y)} \left(1 + O(\varepsilon|y - \xi'_j|) + O(\varepsilon^\alpha) \right).$$

Moreover, for $i \neq j$, $\mathcal{U}_i(y) = U_i(\varepsilon y) + 2 \log \varepsilon = O(1) + 2 \log \varepsilon$, so $e^{2\mathcal{U}_i(y)} = O(\varepsilon^4)$. Therefore, pointwise in $B_{\delta/\varepsilon}(\xi'_j)$, the difference becomes:

$$e^{2V(y)} - \sum_{i=1}^m e^{2\mathcal{U}_i(y)} = e^{2\mathcal{U}_j(y)} \left(O(\varepsilon|y - \xi'_j|) + O(\varepsilon^\alpha) \right) + O(\varepsilon^4).$$

Recalling the exact form of the rescaled bubble profile $\mathcal{U}_j(y) = \log \left(\frac{c_j}{\lambda_j^2 + |y - y_j|^2} \right)$, we have the decay $e^{2\mathcal{U}_j(y)} \sim |y - y_j|^{-4}$ for large $|y|$.

For the interior integral over Ω_ε , passing to polar coordinates:

$$\int_{B_{\delta/\varepsilon}(\xi'_j)} e^{2\mathcal{U}_j(y)} dy \leq C \int_0^\infty \frac{r}{(\lambda_j^2 + r^2)^2} dr = O(1),$$

$$\int_{B_{\delta/\varepsilon}(\xi'_j)} |y - \xi'_j| e^{2\mathcal{U}_j(y)} dy \leq C \int_0^\infty \frac{r^2}{(\lambda_j^2 + r^2)^2} dr = O(1).$$

Hence, the interior near-field remainder evaluates to:

$$\int_{B_{\delta/\varepsilon}} K(\varepsilon y) e^{2\mathcal{U}_j(y)} \left(O(\varepsilon|y - \xi'_j|) + O(\varepsilon^\alpha) \right) dy = O(\varepsilon) + O(\varepsilon^\alpha) = O(\varepsilon^\alpha).$$

For the boundary integral over $\partial\Omega_\varepsilon$, we proceed in the same way. Parametrizing the local boundary curve by arc-length $s \in (-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon})$, the zero-th order term is finite:

$$\int_{-\delta/\varepsilon}^{\delta/\varepsilon} e^{\mathcal{U}_j(s)} ds \leq C \int_{-\infty}^\infty \frac{1}{\lambda_j^2 + s^2} ds = O(1).$$

However, the first-order term possesses a logarithmically divergent tail over $\mathbb{R}$, in-

deed

$$\int_{-\delta/\varepsilon}^{\delta/\varepsilon} |s| e^{\mathcal{U}_j(s)} ds \leq C \int_0^{\delta/\varepsilon} \frac{s}{\lambda_j^2 + s^2} ds = \frac{C}{2} \log \left(1 + \frac{\delta^2}{\lambda_j^2 \varepsilon^2} \right) = O(|\log \varepsilon|).$$

However, multiplying this by the factor ε , the resulting boundary error is $O(\varepsilon |\log \varepsilon|) + O(\varepsilon^\alpha) = O(\varepsilon^\alpha)$.

Summing the interior and boundary estimates, we obtain exactly:

$$\varepsilon \left(\sum_{i=1}^m \tilde{l}(U_i) - l(V) \right) = \varepsilon \cdot O(\varepsilon^\alpha) = O(\varepsilon^{1+\alpha}).$$

which conclude the proof. $\square$

For any positive numbers a, b, λ , let

$$\begin{aligned} z_0(x) &= \frac{1}{2\lambda} - \frac{\lambda + \frac{b}{\sqrt{a}} \left(x_2 + \frac{b}{\sqrt{a}} \lambda \right)}{\lambda^2 + x_1^2 + \left(x_2 + \frac{b}{\sqrt{a}} \lambda \right)^2}, \\ z_1(x) &= \frac{x_1}{\lambda^2 + x_1^2 + \left(x_2 + \frac{b}{\sqrt{a}} \lambda \right)^2}, \quad x = (x_1, x_2) \in \mathbb{R}_+^2. \end{aligned} \tag{5.86}$$

The following classification result is standard and characterizes the kernel of the linearized operator around the standard bubble after stereographic projection.

Lemma 5.19. *Any bounded solution of*

$$\begin{cases} \Delta \phi + \frac{8\lambda^2}{\left(\lambda^2 + x_1^2 + \left(x_2 + \frac{b}{\sqrt{a}} \lambda \right)^2 \right)^2} \phi = 0, & \text{in } \mathbb{R}_+^2, \\ \frac{\partial \phi}{\partial n} - \frac{2b\lambda}{\sqrt{a} \left(\lambda^2 + x_1^2 + \left(\frac{b}{\sqrt{a}} \lambda \right)^2 \right)} \phi = 0, & \text{on } \partial \mathbb{R}_+^2, \end{cases} \tag{5.87}$$

is a linear combination of z_0 and z_1 .

Proof. Let Π be the stereographic projection from the unit sphere in $\mathbb{R}^3$ centered at $(0, -\frac{b}{\sqrt{a}}\lambda, 0)$ onto $\mathbb{R}^2$. More specifically, let $\eta = (\eta_1, \eta_2, \eta_3)$ be the coordinates of $\mathbb{R}^3$ taking $(0, -\frac{b}{\sqrt{a}}\lambda, 0)$ as its origin and $x = (x_1, x_2)$ be the coordinates of $\mathbb{R}^2$, we have

$$\begin{aligned} \eta_1 &= \frac{2x_1\lambda}{\lambda^2 + x_1^2 + \left(x_2 + \frac{b}{\sqrt{a}} \lambda \right)^2}, \\ \eta_2 &= \frac{2\lambda \left(x_2 + \frac{b}{\sqrt{a}} \lambda \right)}{\lambda^2 + x_1^2 + \left(x_2 + \frac{b}{\sqrt{a}} \lambda \right)^2}, \end{aligned}$$

$$\eta_3 = \frac{x_1^2 + \left(x_2 + \frac{b}{\sqrt{a}}\lambda\right)^2 - \lambda^2}{\lambda^2 + x_1^2 + \left(x_2 + \frac{b}{\sqrt{a}}\lambda\right)^2}.$$

Let $C = \Pi^{-1}(\mathbb{R}_+^2)$. It is a spherical cap on S^2 .

Assume ϕ is a bounded solution of problem (5.87). We define a function $\psi(\eta)$ on C by

$$\phi(x) = \psi(\eta) \frac{2\lambda}{\lambda^2 + x_1^2 + \left(x_2 + \frac{b}{\sqrt{a}}\lambda\right)^2},$$

then

$$\begin{cases} \Delta\psi + 2\psi = 0, & \text{in } C, \\ \frac{\partial\psi}{\partial n} - \frac{b}{\sqrt{a}}\psi = 0, & \text{on } \partial C. \end{cases}$$

Proposition 3.2 in [17] points out that the dimension of the eigenspace corresponding to the zero eigenvalue is two, which leads to the desired result. $\square$

5.2.2 Linear Theory

In this section we study the invertibility of the linearized operator associated to our problem: Given $f \in L^\infty(\Omega_\varepsilon)$, $g \in L^\infty(\partial\Omega_\varepsilon)$ we want to find $\phi, c_1, \dots, c_m$ such that

$$\begin{cases} -\Delta\phi - W_1\phi = f + \sum_{j=1}^m c_j \chi_j Z_{1j} & \text{in } \Omega_\varepsilon \\ \frac{\partial\phi}{\partial n} - W_2\phi + \varepsilon l_1(V)[\phi] = g & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \chi_j Z_{1j} \phi = 0 \end{cases} \quad (5.88)$$

Where W_1 and W_2 denote the potentials

$$W_1(y) = 2K(\varepsilon y)e^{2V(y)}, \quad y \in \Omega_\varepsilon, \quad (5.89)$$

$$W_2(y) = h(\varepsilon y)e^{V(y)}, \quad y \in \partial\Omega_\varepsilon. \quad (5.90)$$

and Z_{1j} , $j = 1, \dots, m$, are defined as follows: Consider $z_{0,j}$ and $z_{1,j}$ as in (5.86) with $a = K(\xi_j)$, $b = h(\xi_j)$ and $\lambda = \lambda_j$, which is

$$z_{0,j}(x) = \frac{1}{2\lambda_j} - \frac{\lambda_j + \frac{h(\xi_j)}{\sqrt{K(\xi_j)}} \left(x_2 + \frac{h(\xi_j)}{\sqrt{K(\xi_j)}} \lambda_j\right)}{\lambda_j^2 + x_1^2 + \left(x_2 + \frac{h(\xi_j)}{\sqrt{K(\xi_j)}} \lambda_j\right)^2},$$

$$z_{1,j}(x) = \frac{x_1}{\lambda_j^2 + x_1^2 + \left(x_2 + \frac{h(\xi_j)}{\sqrt{K(\xi_j)}} \lambda_j\right)^2}, \quad x = (x_1, x_2) \in \mathbb{R}_+^2.$$

Around each point $\xi'_j \in \partial\Omega_\varepsilon$ we consider a smooth change of variables

$$F_j^\varepsilon(y) = \frac{1}{\varepsilon} F_j(\varepsilon y),$$

where $F_j : B_\rho(\xi_j) \rightarrow M$ is a conformal diffeomorphism and M is an open neighborhood of the origin such that

$$F_j(B_\rho(\xi_j) \cap \Omega) = \mathbb{R}_+^2 \cap M, \quad F_j(B_\rho(\xi_j) \cap \partial\Omega) = \partial\mathbb{R}_+^2 \cap M.$$

We can select F_j so that it preserves area. Define

$$Z_{ij}(y) = z_{ij} \left(F_j^\varepsilon(y) \right), \quad i = 0, 1, \quad j = 1, \dots, m.$$

Next, we choose a large but fixed number R_0 and a smooth cut-off function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\chi(r) = 1 \text{ for } r \leq R_0, \quad \chi(r) = 0 \text{ for } r \geq R_0 + 1, \quad 0 \leq \chi \leq 1.$$

Then set

$$\chi_j(y) = \chi(|F_j^\varepsilon(y)|).$$

For any $0 < \alpha < 1$, using the same expansion to the one of the previous section, we have:

$$W_1(y) = \sum_{j=1}^m \frac{8\lambda_j^2}{\left(\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2\right)^2} \left(1 + O(\varepsilon^\alpha) + O(\varepsilon|y - \xi'_j|)\right), \quad \forall y \in \Omega_\varepsilon, \quad (5.91)$$

$$W_2(y) = \sum_{j=1}^m \frac{2\mathfrak{D}(\xi_j)\lambda_j}{\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2} \left(1 + O(\varepsilon^\alpha) + O(\varepsilon|y - \xi'_j|)\right), \quad \forall y \in \partial\Omega_\varepsilon, \quad (5.92)$$

Moreover, we call

$$W_1^{(j)} = 2K(\xi_j)e^{2U_j} = \frac{8\lambda_j^2}{\left(\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2\right)^2}$$

and

$$W_2^{(j)} = h(\xi_j)e^{U_j} = \frac{2\mathfrak{D}(\xi_j)\lambda_j}{\lambda_j^2 + |y - \xi'_j - \mathfrak{D}(\xi_j)\lambda_j n(\xi_j)|^2}$$

We choose a small fixed parameter $0 < \sigma < 1$. For a function $f \in L^\infty(\Omega_\varepsilon)$, we define the interior norm:

$$\|f\|_{**, \Omega_\varepsilon} = \sup_{y \in \Omega_\varepsilon} \frac{|f(y)|}{\varepsilon^2 + \sum_{j=1}^m (1 + |y - \xi'_j|)^{-2-\sigma}}. \quad (5.93)$$

For a function $g \in L^\infty(\partial\Omega_\varepsilon)$, we define the boundary norm:

$$\|g\|_{*, \partial\Omega_\varepsilon} = \sup_{y \in \partial\Omega_\varepsilon} \frac{|g(y)|}{\varepsilon + \sum_{j=1}^m (1 + |y - \xi'_j|)^{-1-\sigma}}. \quad (5.94)$$

The result we want to prove is the following

Proposition 5.20. *Let $d > 0$ and $m \in \mathbb{N}$. Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, any family of points $\xi_1, \dots, \xi_m \in \partial\Omega$ satisfying*

$$|\xi_i - \xi_j| \geq d, \quad \forall i \neq j, \quad (5.95)$$

and any $f \in L^\infty(\Omega_\varepsilon)$, $g \in L^\infty(\partial\Omega_\varepsilon)$, there exist a unique solution $\phi \in L^\infty(\Omega_\varepsilon)$ and unique constants $c_1, \dots, c_m \in \mathbb{R}$ solving (5.88).

Moreover,

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \log\left(\frac{1}{\varepsilon}\right) (\|g\|_{*, \partial\Omega_\varepsilon} + \|f\|_{**, \Omega_\varepsilon}).$$

and

$$|c_k| \leq C (\|g\|_{*, \partial\Omega_\varepsilon} + \|f\|_{**, \Omega_\varepsilon}) \quad \forall k = 1, \dots, m.$$

We start by constructing a barrier function

Lemma 5.21. *For sufficiently small $\varepsilon > 0$, there exist $C, R_1 > 0$ independent of ε , and a smooth positive function*

$$\psi : \Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi'_j) \longrightarrow \mathbb{R}$$

such that

$$\begin{aligned} -\Delta\psi - W_1\psi &\geq \sum_{j=1}^m \frac{1}{|y - \xi'_j|^{2+\sigma}} + \varepsilon^2 \quad \text{in } \Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi'_j) \\ \frac{\partial\psi}{\partial n} - W_2\psi + \varepsilon l_1(V)[\psi] &\geq \sum_{j=1}^m \frac{1}{|y - \xi'_j|^{1+\sigma}} + \varepsilon \quad \text{on } \partial\Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi'_j) \\ 0 < \psi &\leq C \quad \text{in } \Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi'_j) \\ \psi &\geq 1 \quad \text{on } \Omega_\varepsilon \cap \left(\bigcup_{j=1}^m \partial B_{R_1}(\xi'_j) \right) \end{aligned}$$

Proof. For $j = 1, \dots, m$, we denote $r_j = |y - \xi'_j - \lambda_j \mathfrak{D}(\xi_j) n(\xi_j)|$ and define

$$\psi_{1j}(y) = \frac{(y - \xi'_j) \cdot n(\xi'_j)}{r_j^{1+\sigma}}$$

By direct computation one can see that

$$\Delta \psi_{1j} = O(r_j^{-2-\sigma})$$

and if $\delta > 0$ is small but fixed and $R_1 > 0$ is large and fixed, then

$$\frac{\partial \psi_{1j}}{\partial n} \geq c r_j^{-1-\sigma} \quad \text{for } R_1 < r_j < \frac{\delta}{\varepsilon},$$

where $c > 0$ is fixed.

To prove the last assertion we may suppose that ξ'_j is at the origin and assume that the normal vector at ξ'_j is $(0, -1)$. Hence

$$\psi_{1j}(y) = -\frac{y_2}{r_j^{1+\sigma}}.$$

Let us write $\partial\Omega_\varepsilon$ near ξ'_j as the graph $\{(y_1, y_2) : y_2 = G_\varepsilon(y_1)\}$, with $G_\varepsilon(y_1) = \frac{1}{\varepsilon}G(\varepsilon y_1)$, where G is a smooth function such that $G(0) = 0, G'(0) = 0$.

Fix $\delta > 0$ small. Then for

$$R_1 < r_j < \frac{\delta}{\varepsilon},$$

we have that

$$G'_\varepsilon(y_1) = O(\varepsilon r_j), \quad G_\varepsilon(y_1) = O(\varepsilon r_j^2).$$

Then

$$\begin{aligned}
\frac{\partial \psi_{1j}}{\partial n} &= \frac{1}{\sqrt{(G'_\varepsilon(y_1))^2 + 1}} \left((1 + \sigma) \frac{G_\varepsilon(y_1) G'_\varepsilon(y_1) y_1}{r_j^{3+\sigma}} + \frac{1}{r_j^{1+\sigma}} \right. \\
&\quad \left. - (1 + \sigma) \frac{G_\varepsilon(y_1)^2}{r_j^{3+\sigma}} - \lambda_j \mathfrak{D}(\xi_j) (1 + \sigma) \frac{G_\varepsilon(y_1)}{r_j^{3+\sigma}} \right) \\
&= \frac{1}{\sqrt{(G'_\varepsilon(y_1))^2 + 1}} \left(+ \frac{1}{r_j^{1+\sigma}} + \frac{O(\varepsilon^2 r_j^4)}{r_j^{3+\sigma}} + \frac{O(\delta)}{r_j^{2+\sigma}} \right) \\
&\quad \text{for } R_1 < r_j < \frac{\delta}{\varepsilon} \\
&= \frac{1}{\sqrt{O(\delta^2) + 1}} \left(+ \frac{1}{r_j^{1+\sigma}} + \frac{O(\delta^2)}{r_j^{1+\sigma}} + \frac{O(\delta)}{r_j^{2+\sigma}} \right) \\
&\quad \text{for } R_1 < r_j < \frac{\delta}{\varepsilon},
\end{aligned}$$

from where the claim follows by taking δ small enough.

Consider also

$$\psi_{2j}(r_j) = 1 - \frac{1}{r_j^\sigma}.$$

Then

$$-\Delta \psi_{2j} = \sigma^2 \frac{1}{r_j^{2+\sigma}}. \quad (5.96)$$

Proceeding analogously as for ψ_{1j} , we find

$$\begin{aligned}
\frac{\partial \psi_{2j}}{\partial n} &= \frac{\sigma}{r_j^{2+\sigma}} \frac{1}{\sqrt{(G'_\varepsilon(y_1))^2 + 1}} \left(-y_1 G'_\varepsilon(y_1) + G_\varepsilon(y_1) + \lambda_j \mathfrak{D}(\xi_j) \right) \\
&= \frac{\sigma}{r_j^{2+\sigma}} \frac{1}{\sqrt{O(\delta^2) + 1}} O(1), \quad R_1 < r_j < \frac{\delta}{\varepsilon} \\
&= O\left(\frac{1}{r_j^{2+\sigma}}\right), \quad R_1 < r_j < \frac{\delta}{\varepsilon}.
\end{aligned}$$

Now let

$$\psi_{3j} = \psi_{1j} + C \psi_{2j}.$$

For C large enough (but independent of ε), using that $W_1 \simeq \frac{1}{r_j^4}$, we have

$$-\Delta \psi_{3j} - W_1 \psi_{3j} \geq \frac{1}{r_j^{2+\sigma}}, \quad R_1 < |y - \xi'_j| < \frac{\delta}{\varepsilon}.$$

For the boundary condition, combining the normal derivatives gives:

$$\begin{aligned}\frac{\partial \psi_{3j}}{\partial n} &= \frac{\partial \psi_{1j}}{\partial n} + C \frac{\partial \psi_{2j}}{\partial n} \\ &\geq \frac{c}{r_j^{1+\sigma}} - \frac{\tilde{C}}{r_j^{2+\sigma}} \\ &\geq \frac{\bar{c}}{r_j^{1+\sigma}}, \quad R_1 < r_j < \frac{\delta}{\varepsilon},\end{aligned}$$

Recalling the asymptotic decay of the boundary potential $W_2(y) \sim \frac{1}{r_j^2}$, we deduce that:

$$\frac{\partial \psi_{3j}}{\partial n} - W_2 \psi_{3j} \geq \frac{c'}{r_j^{1+\sigma}}, \quad R_1 < |y - \xi'_j| < \frac{\delta}{\varepsilon},$$

for some constant $c' > 0$. Note that, since ψ_{3j} is lower-bounded by a non-zero constant, we have $l_1(V)[\psi_{3j}] \geq c > 0$.

Let $\eta_j \in C_0^\infty(\mathbb{R}^2)$ cut-off function be such that $0 \leq \eta_j \leq 1$, $\eta_j \equiv 1$ in $\Omega_\varepsilon \cap B_{\frac{\delta}{2\varepsilon}}(\xi'_j)$, $\eta_j \equiv 0$ in $\Omega_\varepsilon \setminus B_{\frac{\delta}{\varepsilon}}(\xi'_j)$, $|\nabla \eta_j| \leq C\varepsilon$ in Ω_ε , $|\Delta \eta_j| \leq C\varepsilon^2$ in Ω_ε . Now, we take $\tilde{\Psi}_0$ a strictly positive solution of

$$\begin{cases} -\Delta \tilde{\Psi}_0 = 1 & \text{in } \Omega \\ \frac{\partial \tilde{\Psi}_0}{\partial n} = -\frac{|\Omega|}{|\partial \Omega|} & \text{on } \partial \Omega \end{cases}$$

which is smooth and bounded. We define $\Psi_0(y) = \tilde{\Psi}_0(\varepsilon y)$, note that it solves

$$\begin{cases} -\Delta \Psi_0 = \varepsilon^2 & \text{in } \Omega_\varepsilon \\ \frac{\partial \Psi_0}{\partial n} = -\frac{|\Omega|}{|\partial \Omega|} \varepsilon & \text{on } \partial \Omega_\varepsilon \end{cases}$$

Now we define the barrier

$$\psi(y) = B + A\Psi_0(y) + \sum_{j=1}^m \eta_j \psi_{3j}(y)$$

where A and B are sufficiently large constants such that $B > A$. We denote $\rho_j = |y - \xi'_j|$, so we have

$$-\Delta \psi - W_1 \psi \geq A\varepsilon^2 + \sum_{j=1}^m \left[(-\Delta \eta_j) \psi_{3j} - 2\nabla \eta_j \cdot \nabla \psi_{3j} + \frac{\eta_j}{\rho_j^{2+\sigma}} \right] + O\left(\frac{1}{\rho_j^4}\right) \quad (5.97)$$

In the region where $R_1 < \rho_j < \frac{\delta}{2\varepsilon}$, $\eta_j \equiv 1$, choosing R_1 large enough, we deduce from the inequality (5.97) that

$$-\Delta\psi - W_1\psi \geq A\varepsilon^2 + \sum_{j=1}^m \frac{1}{\rho_j^{2+\sigma}}$$

Instead when $\frac{\delta}{2\varepsilon} < \rho_j < \frac{\delta}{\varepsilon}$, we use that $|\nabla\eta_j| \leq C\varepsilon$ in Ω_ε , $|\Delta\eta_j| \leq C\varepsilon^2$ in Ω_ε and $|\nabla\psi_{3j}| = O(\frac{1}{\rho_j^{1+\sigma}})$, to conclude from (5.97) that

$$-\Delta\psi - W_1\psi \geq c\varepsilon^2 \geq C \left(\varepsilon^2 + \frac{1}{\rho_j^2} \right) \geq C \left(\frac{1}{\rho_j^{2+\sigma}} + \varepsilon^2 \right)$$

In the region where $\rho_j \geq \frac{\delta}{\varepsilon}$, $\psi(y) = A\Psi_0 + B$ and the estimate follows easily.

Now we turn to prove the boundary inequality, using that $W_2 \sim \rho_j^{-2}$ and $l_1(V)$ is a positive functional, we have

$$\begin{aligned} \frac{\partial\psi}{\partial n} - W_2\psi + \varepsilon l_1(V)[\psi] &\geq -\frac{A|\Omega|}{|\partial\Omega|}\varepsilon + \sum_{j=1}^m \left(\frac{\partial\eta_j}{\partial n}\psi_{3j} + \frac{c\eta_j}{\rho_j^{1+\sigma}} \right) + \varepsilon l_1(V)[B] + O\left(\frac{1}{\rho_j^2}\right) \\ &\geq C\varepsilon + \sum_{j=1}^m \left(\frac{\partial\eta_j}{\partial n}\psi_{3j} + \frac{c\eta_j}{\rho_j^{1+\sigma}} \right) + O\left(\frac{1}{\rho_j^2}\right) \end{aligned}$$

where in the last inequality we use that B is fixed sufficiently large after the choice of A . Reasoning separately in the three regions $R_1 < \rho_j < \frac{\delta}{2\varepsilon}$, $\frac{\delta}{2\varepsilon} < \rho_j < \frac{\delta}{\varepsilon}$, $\rho_j > \frac{\delta}{\varepsilon}$ as in the interior case, we conclude. $\square$

Lemma 5.22. *There exist $R_0 > 0$ and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, any solution ϕ of the problem*

$$\begin{cases} -\Delta\phi - W_1\phi = f & \text{in } \Omega_\varepsilon \\ \frac{\partial\phi}{\partial n} - W_2\phi + \varepsilon l_1(V)[\phi] = g & \text{on } \partial\Omega_\varepsilon \end{cases} \quad (5.98)$$

satisfying the orthogonality conditions

$$\int_{\Omega_\varepsilon} Z_{ij} \chi_j \phi = 0, \quad \forall i = 0, 1, \forall j = 1, \dots, m, \quad (5.99)$$

satisfies

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \left(\|g\|_{*,\partial\Omega_\varepsilon} + \|f\|_{**, \Omega_\varepsilon} \right),$$

where C is independent of ε .

Proof. Let $R_0 = 2R_1$, where R_1 is given by the previous lemma. The barrier ψ constructed there yields the following maximum principle: If $\phi \in H^1\left(\Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi'_j)\right)$

satisfies

$$\begin{cases} -\Delta\phi - W_1\phi \leq 0, & \text{in } \Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi'_j), \\ \frac{\partial\phi}{\partial n} - W_2\phi \leq 0, & \text{on } \partial\Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi'_j), \\ \phi \leq 0, & \text{on } \Omega_\varepsilon \cap \left(\bigcup_{j=1}^m \partial B_{R_1}(\xi'_j) \right), \end{cases}$$

then

$$\phi \leq 0 \quad \text{in } \Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi'_j).$$

Let ϕ be a solution of (5.98) satisfying the orthogonality conditions (5.99). We first prove that $\|\phi\|_{L^\infty(\Omega_\varepsilon)}$ can be estimated in terms of $\|\phi\|_i$, $\|f\|_{**, \Omega_\varepsilon}$, and $\|g\|_{*, \partial\Omega_\varepsilon}$, where

$$\|\phi\|_i := \sup_{\Omega_\varepsilon \cap \left(\bigcup_{j=1}^m B_{R_1}(\xi'_j) \right)} |\phi|.$$

Set

$$\tilde{\phi} = C_1 \psi (\|\phi\|_i + \|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}),$$

where $C_1 > 0$ is independent of ε . Choosing C_1 sufficiently large, the definition of ψ gives

$$-\Delta\tilde{\phi} - W_1\tilde{\phi} \geq |f|,$$

and

$$\frac{\partial\tilde{\phi}}{\partial n} - W_2\tilde{\phi} \geq |g|$$

in

$$\Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi'_j).$$

Moreover,

$$\tilde{\phi} \geq |\phi| \quad \text{on } \Omega_\varepsilon \cap \left(\bigcup_{j=1}^m \partial B_{R_1}(\xi'_j) \right).$$

Applying the maximum principle to $\phi - \tilde{\phi}$ and $-\phi - \tilde{\phi}$, we obtain

$$|\phi| \leq \tilde{\phi} \quad \text{in } \Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi'_j).$$

Since ψ is uniformly bounded, it follows that

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C (\|\phi\|_i + \|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}), \quad (5.100)$$

where C is independent of ϕ and ε .

Suppose by contradiction that the conclusion of the lemma does not hold. Then there exist sequences $\varepsilon_n \rightarrow 0$, $\xi_1^n, \dots, \xi_m^n \in \partial\Omega$ satisfying (5.95), and functions ϕ_n , f_n , g_n such that

$$\begin{aligned} \|\phi_n\|_{L^\infty(\Omega_{\varepsilon_n})} &= 1, \\ \|f_n\|_{**, \Omega_{\varepsilon_n}} &\longrightarrow 0, \quad \|g_n\|_{*, \partial\Omega_{\varepsilon_n}} \longrightarrow 0, \end{aligned}$$

and ϕ_n solves (5.98) and satisfies (5.99) for every n .

Estimate (5.100) implies that $\|\phi_n\|_i$ is bounded away from zero. Hence, up to relabeling the concentration points, we may assume that

$$\sup_{B_{R_1}(\xi_j'^n)} |\phi_n| \geq c > 0$$

for some fixed index j and all n . Define

$$\hat{\phi}_n(z) = \phi_n(z + \xi_j'^n).$$

After translating and rotating the coordinates, we may assume that Ω_{ε_n} converges locally to $\mathbb{R}_+^2$ and that $\xi_j'^n = 0$.

Using the asymptotic expansions of W_1 and W_2 , we have

$$W_1 \longrightarrow \frac{8}{(1 + |z|^2)^2}, \quad W_2 \longrightarrow \frac{2}{1 + r^2},$$

uniformly on compact subsets of $\mathbb{R}_+^2$. Moreover,

$$|\varepsilon_n l_1(V)[\phi_n]| \leq C\varepsilon_n \|\phi_n\|_{L^\infty(\Omega_{\varepsilon_n})} = C\varepsilon_n,$$

so that the nonlocal boundary term converges uniformly to zero.

Since

$$\|\hat{\phi}_n\|_{L^\infty} \leq 1,$$

standard elliptic estimates imply that, up to a subsequence,

$$\hat{\phi}_n \longrightarrow \hat{\phi}$$

uniformly on compact subsets of $\overline{\mathbb{R}_+^2}$, where $\hat{\phi}$ is a bounded solution of (5.87).

Moreover,

$$\sup_{B_{R_1}(0)} |\hat{\phi}| \geq c,$$

so that $\widehat{\phi} \neq 0$.

By Lemma 5.19,

$$\widehat{\phi} \in \text{span}\{z_{0,j}, z_{1,j}\}.$$

Passing to the limit in the orthogonality conditions (5.99), and using that

$$Z_{ij}(\cdot + \xi_j'^n) \longrightarrow z_{i,j}$$

uniformly on compact subsets, while χ_j has compact support, we obtain

$$\int_{\mathbb{R}_+^2} \chi_j \widehat{\phi} z_{i,j} = 0, \quad i = 0, 1.$$

Hence $\widehat{\phi}$ is orthogonal to the kernel of the limiting operator. Since

$$\widehat{\phi} \in \text{span}\{z_{0,j}, z_{1,j}\},$$

it follows that

$$\widehat{\phi} \equiv 0,$$

contradicting

$$\sup_{B_{R_1}(0)} |\widehat{\phi}| \geq c.$$

This contradiction proves the lemma. □

Lemma 5.23. *For ε sufficiently small, if ϕ solves*

$$\begin{cases} -\Delta\phi - W_1\phi = f & \text{in } \Omega_\varepsilon \\ \frac{\partial\phi}{\partial n} - W_2\phi + \varepsilon l_1(V)[\phi] = g & \text{on } \partial\Omega_\varepsilon \end{cases}$$

and satisfies

$$\int_{\Omega_\varepsilon} Z_{1j} \chi_j \phi = 0 \quad \forall j = 1, \dots, m$$

then

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \log\left(\frac{1}{\varepsilon}\right) (\|g\|_{*,\partial\Omega_\varepsilon} + \|f\|_{**, \Omega_\varepsilon})$$

where C is independent of ε .

Proof. Let $R > R_0 + 1$ large and fixed. Let $\tilde{\eta}_{1j}$ a radial smooth cut-off function such that $0 \leq \tilde{\eta}_{1j} \leq 1$, $|\nabla \tilde{\eta}| \leq C$ in $\mathbb{R}^2$, $\tilde{\eta}_{1j} \equiv 1$ in $B_R(0)$, $\tilde{\eta}_{1j} \equiv 0$ in $\mathbb{R}^2 \setminus B_{R+1}(0)$ and $\tilde{\eta}_{2j}$ another radial smooth cut-off function such that $0 \leq \tilde{\eta}_{2j} \leq 1$, $|\nabla \tilde{\eta}_{2j}| \leq C \frac{\varepsilon}{\delta}$, $|\nabla^2 \tilde{\eta}_{2j}| \leq C \frac{\varepsilon^2}{\delta^2}$ in $\mathbb{R}^2$, $\tilde{\eta}_{2j} \equiv 1$ in $B_{\frac{\delta}{4\varepsilon}}(0)$, $\tilde{\eta}_{2j} \equiv 0$ in $\mathbb{R}^2 \setminus B_{\frac{\delta}{3\varepsilon}}(0)$. Moreover, we define

.0

$$\tilde{\gamma}(x) = \frac{\log\left(\frac{\delta}{\varepsilon}\right) - \log|x|}{\log\left(\frac{\delta}{\varepsilon}\right) - \log R}$$

note that $\tilde{\gamma}$ is the solution of

$$\begin{cases} \Delta \tilde{\gamma} = 0 & \text{in } B_{\frac{\delta}{\varepsilon}}(0) \setminus \overline{B}_R(0) \\ \tilde{\gamma} = 1 & \text{if } |x| = R \\ \tilde{\gamma} = 0 & \text{if } |x| = \frac{\delta}{\varepsilon} \end{cases}$$

Let $\gamma(y) = \tilde{\gamma}(F_j^\varepsilon(y))$ and $\eta_{ij}(y) = \tilde{\eta}_{ij}(F_j^\varepsilon(y))$ for $i = 1, 2$.

Now we define

$$\tilde{Z}_{0j} = \eta_{1j} Z_{0j} + (1 - \eta_{1j}) \eta_{2j} \gamma Z_{0j}$$

and given ϕ satisfying the hypotesis we consider

$$\tilde{\phi} = \phi + \sum_{j=1}^m d_j \tilde{Z}_{0j} \quad \text{where } d_j = -\frac{\int_{\Omega_\varepsilon} Z_{0j} \chi_j \phi}{\int_{\Omega_\varepsilon} Z_{0j}^2 \chi_j}$$

The lemma's conclusion follows directly from the following claim:

Claim. For all $k = 1, \dots, m$, we have:

$$|d_k| \leq C \log\left(\frac{1}{\varepsilon}\right) (\|g\|_{*, \partial\Omega_\varepsilon} + \|f\|_{**, \Omega_\varepsilon}).$$

To prove the claim, we define $L(\phi) = -\Delta\phi - W_1\phi$ and $B(\phi) = \frac{\partial\phi}{\partial n} - W_2\phi + \varepsilon l_1(V)[\phi]$. By definition of $\tilde{\phi}$, we have:

$$\begin{aligned} L(\tilde{\phi}) &= f + \sum_{j=1}^m d_j L(\tilde{Z}_{0j}) \quad \text{in } \Omega_\varepsilon, \\ B(\tilde{\phi}) &= g + \sum_{j=1}^m d_j B(\tilde{Z}_{0j}) \quad \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

By the previous Lemma, we have

$$\begin{aligned} \|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} &\leq C \left(\|g\|_{*, \partial\Omega_\varepsilon} + \|f\|_{**, \Omega_\varepsilon} + \sum_{j=1}^m |d_j| \left(\left\| \frac{\partial \tilde{Z}_{0j}}{\partial n} - W_2 \tilde{Z}_{0j} \right\|_{*, \partial\Omega_\varepsilon} \right. \right. \\ &\quad \left. \left. + \|\varepsilon l_1(V)[\tilde{Z}_{0j}]\|_{*, \partial\Omega_\varepsilon} + \|L(\tilde{Z}_{0j})\|_{**, \Omega_\varepsilon} \right) \right). \end{aligned} \tag{5.101}$$

Multiplying the equation for $L(\tilde{\phi})$ by $\tilde{Z}_{0k}$ and integrating by parts yields:

$$\begin{aligned} d_k & \left[\int_{\Omega_\varepsilon} L(\tilde{Z}_{0k}) \tilde{Z}_{0k} + \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0k} \left(\frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right) + \varepsilon l_1(V)[\tilde{Z}_{0k}] \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0k} \right] \\ & = - \int_{\partial\Omega_\varepsilon} g \tilde{Z}_{0k} - \int_{\Omega_\varepsilon} f \tilde{Z}_{0k} + \int_{\partial\Omega_\varepsilon} \tilde{\phi} \left(\frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right) + \int_{\Omega_\varepsilon} \tilde{\phi} L(\tilde{Z}_{0k}) + \varepsilon l_1(V)[\tilde{\phi}] \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0k} \end{aligned} \quad (5.102)$$

(Notice that the terms for $j \neq k$ vanish identically since the cut-off functions have disjoint supports for ε small enough).

Combining (5.101) and (5.102) we obtain

$$\begin{aligned} & d_k \left[\int_{\Omega_\varepsilon} L(\tilde{Z}_{0k}) \tilde{Z}_{0k} + \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0k} \left(\frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right) + \varepsilon l_1(V)[\tilde{Z}_{0k}] \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0k} \right] \\ & \leq C(\|g\|_{*,\partial\Omega_\varepsilon} + \|f\|_{**, \Omega_\varepsilon}) + \|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} \left(\left\| \frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right\|_{*,\partial\Omega_\varepsilon} + \|L(\tilde{Z}_{0k})\|_{**, \Omega_\varepsilon} + C\varepsilon \|\tilde{Z}_{0k}\|_{L^1(\partial\Omega_\varepsilon)} \right) \\ & \leq C(\|g\|_{*,\partial\Omega_\varepsilon} + \|f\|_{**, \Omega_\varepsilon}) \left(1 + \left\| \frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right\|_{*,\partial\Omega_\varepsilon} + \|L(\tilde{Z}_{0k})\|_{**, \Omega_\varepsilon} + \varepsilon \|\tilde{Z}_{0k}\|_{L^1(\partial\Omega_\varepsilon)} \right) \\ & + C \sum_{j=1}^m |d_j| \left(\left\| \frac{\partial \tilde{Z}_{0j}}{\partial n} - W_2 \tilde{Z}_{0j} \right\|_{*,\partial\Omega_\varepsilon}^2 + \|L(\tilde{Z}_{0j})\|_{**, \Omega_\varepsilon}^2 + C\varepsilon^2 \|\tilde{Z}_{0j}\|_{L^1(\partial\Omega_\varepsilon)}^2 + \|\varepsilon l_1(V)[\tilde{Z}_{0j}]\|_{*,\partial\Omega_\varepsilon}^2 \right) \end{aligned}$$

The claim follows once we establish the following estimates:

$$l_1(V)[\tilde{Z}_{0k}] = O\left(\frac{1}{\log(\frac{1}{\varepsilon})}\right) \quad (5.103)$$

$$\|\tilde{Z}_{0k}\|_{L^1(\partial\Omega_\varepsilon)} = O\left(\frac{1}{\varepsilon \log(\frac{1}{\varepsilon})}\right) \quad (5.104)$$

$$\int_{\Omega_\varepsilon} L(\tilde{Z}_{0k}) \tilde{Z}_{0k} + \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0k} \left(\frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right) \geq \frac{1}{C \log(\frac{1}{\varepsilon})} \quad (5.105)$$

$$\|L(\tilde{Z}_{0k})\|_{**, \Omega_\varepsilon} = O\left(\frac{1}{\log(\frac{1}{\varepsilon})}\right) \quad (5.106)$$

$$\left\| \frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right\|_{*,\partial\Omega_\varepsilon} = O\left(\frac{1}{\log(\frac{1}{\varepsilon})}\right) \quad (5.107)$$

Proof of (5.103).

By definition

$$l_1(V)[\tilde{Z}_{0j}] = \frac{1}{|\partial\Omega|} \left(\int_{\Omega_\varepsilon} W_1(y) \tilde{Z}_{0j}(y) dy + \int_{\partial\Omega_\varepsilon} W_2(y) \tilde{Z}_{0j}(y) ds_y \right).$$

Recall that $\tilde{Z}_{0j}(y) = \tilde{z}_{0j}(F_j^\varepsilon(y))$, where

$$\tilde{z}_{0j}(x) := \tilde{\eta}_{1j}(x) z_{0j}(x) + (1 - \tilde{\eta}_{1j}(x)) \tilde{\eta}_{2j}(x) \tilde{\gamma}(x) z_{0j}(x),$$

so that, by construction of $\tilde{\eta}_{2j}$, $\tilde{z}_{0j}(x) \equiv 0$ for $|x| \geq \frac{\delta}{3\varepsilon}$. Consequently, $\tilde{Z}_{0j}$ is supported in

$$M_\varepsilon := (F_j^\varepsilon)^{-1} \left(B_{\frac{\delta}{3\varepsilon}}(0) \cap \mathbb{R}_+^2 \right),$$

So

$$l_1(V)[\tilde{Z}_{0j}] = \frac{1}{|\partial\Omega|} \left(\int_{M_\varepsilon} W_1 \tilde{Z}_{0j} dy + \int_{M_\varepsilon \cap \partial\Omega_\varepsilon} W_2 \tilde{Z}_{0j} ds_y \right).$$

We add and subtract $Z_{0j}(y) = z_{0j}(F_j^\varepsilon(y))$, so we write:

$$l_1(V)[\tilde{Z}_{0j}] = \frac{1}{|\partial\Omega|} (\mathcal{I}_1 + \mathcal{I}_2)$$

where

$$\begin{aligned} \mathcal{I}_1 &:= \int_{M_\varepsilon} W_1 Z_{0j} dy + \int_{M_\varepsilon \cap \partial\Omega_\varepsilon} W_2 Z_{0j} ds_y, \\ \mathcal{I}_2 &:= \int_{M_\varepsilon} W_1 (\tilde{Z}_{0j} - Z_{0j}) dy + \int_{M_\varepsilon \cap \partial\Omega_\varepsilon} W_2 (\tilde{Z}_{0j} - Z_{0j}) ds_y. \end{aligned}$$

Estimate of $\mathcal{I}_1$: We use the local conformal change of variables $x = F_j^\varepsilon(y)$. By definition of M_ε , $F_j^\varepsilon(M_\varepsilon) = B_{\frac{\delta}{3\varepsilon}}(0) \cap \mathbb{R}_+^2$. Since F_j^ε is a conformal diffeomorphism, $dy = \det(DF_j^\varepsilon) dx = (1 + O(\varepsilon|x|)) dx$ and $ds_y = (1 + O(\varepsilon|x_1|)) dx_1$.

Moreover, calling $W_1^{(j)} = 2K(\xi_j)e^{2U_j}$ and $W_2^{(j)} = h(\xi_j)e^{U_j}$, we have:

$$\begin{aligned} W_1((F_j^\varepsilon)^{-1}(x)) &= W_1^{(j)}(x) (1 + O(\varepsilon^\alpha) + O(\varepsilon|x|)), \\ W_2((F_j^\varepsilon)^{-1}(x)) &= W_2^{(j)}(x) (1 + O(\varepsilon^\alpha) + O(\varepsilon|x|)). \end{aligned}$$

Substituting the above expansions into the expression for $\mathcal{I}_1$, $O(\varepsilon|x|)$ in the Jacobian factor produces only higher-order contributions. Indeed, integrating against z_{0j} yields an error of order $O(\varepsilon)$ from the interior and $O(\varepsilon \log \frac{1}{\varepsilon})$ from the boundary. Therefore, these contributions are absorbed into the remainder $O(\varepsilon^\alpha)$, and we obtain

$$\mathcal{I}_1 = \int_{B_{\frac{\delta}{3\varepsilon}}(0) \cap \mathbb{R}_+^2} W_1^{(j)}(x) z_{0j}(x) dx + \int_{-\frac{\delta}{3\varepsilon}}^{\frac{\delta}{3\varepsilon}} W_2^{(j)}(x_1) z_{0j}(x_1) dx_1 + O(\varepsilon^\alpha),$$

which holds because at infinity $W_1^{(j)}(x) = O(|x|^{-4})$ and $W_2^{(j)}(x) = O(|x|^{-2})$.

Next, we extend the domain to the whole upper half-plane $\mathbb{R}_+^2$. The error introduced on the exterior region $\{|x| \geq \frac{\delta}{3\varepsilon}\}$ scales as

$$\left| \int_{\mathbb{R}_+^2 \setminus B_{\frac{\delta}{3\varepsilon}}(0)} W_1^{(j)} z_{0j} dx \right| \leq C \int_{\frac{\delta}{3\varepsilon}}^{\infty} \frac{1}{r^4} r dr = O(\varepsilon^2),$$

and a similar bound gives $O(\varepsilon)$ for the boundary contribution. Therefore, we may rewrite

$$\mathcal{I}_1 = \left[\int_{\mathbb{R}_+^2} W_1^{(j)}(x) z_{0j}(x) dx + \int_{\partial \mathbb{R}_+^2} W_2^{(j)}(x) z_{0j}(x) dx_1 \right] + O(\varepsilon^\alpha).$$

By construction, the standard bubble derivative z_{0j} satisfies the homogeneous linear equation $-\Delta z_{0j} = W_1^{(j)} z_{0j}$ in $\mathbb{R}_+^2$, together with the boundary condition $\partial_n z_{0j} = W_2^{(j)} z_{0j}$ on $\partial \mathbb{R}_+^2$. Integrating $-\Delta z_{0j}$ over a large half-disk $B_r \cap \mathbb{R}_+^2$ and applying the divergence theorem, we obtain

$$\int_{B_r \cap \mathbb{R}_+^2} W_1^{(j)} z_{0j} dx = - \int_{-r}^r \partial_n z_{0j} dx_1 - \int_{\partial B_r \cap \mathbb{R}_+^2} \partial_\nu z_{0j} ds.$$

Substituting the boundary condition and letting $r \rightarrow \infty$, we use that $|\nabla z_{0j}(x)| = O(|x|^{-2})$ as $|x| \rightarrow \infty$ (which is immediate from differentiating the explicit formula for z_{0j}), so that

$$\left| \int_{\partial B_r \cap \mathbb{R}_+^2} \partial_\nu z_{0j} ds \right| \leq \pi r \cdot \sup_{|x|=r} |\nabla z_{0j}(x)| = O(r^{-1}) \xrightarrow{r \rightarrow \infty} 0.$$

This yields the cancellation

$$\int_{\mathbb{R}_+^2} W_1^{(j)}(x) z_{0j}(x) dx + \int_{\partial \mathbb{R}_+^2} W_2^{(j)}(x) z_{0j}(x) dx_1 = 0.$$

Hence, we conclude that $\mathcal{I}_1 = O(\varepsilon^\alpha)$.

Estimate of $\mathcal{I}_2$: Since $\tilde{\eta}_{1j} \equiv 1$ in $B_R(0)$, the difference $\tilde{z}_{0j} - z_{0j}$ is supported only in the annular region $\mathcal{A}_R := \{x \in \mathbb{R}_+^2 : R < |x| < \frac{\delta}{3\varepsilon}\}$. Since $z_{0j}(x) = O(1)$, $W_1^{(j)}(x) = O(|x|^{-4})$, and $W_2^{(j)}(x) = O(|x|^{-2})$, we have:

$$|\tilde{z}_{0j} - z_{0j}| \leq C |1 - \tilde{\gamma}(x)| = C \left| 1 - \frac{\log\left(\frac{\delta}{3\varepsilon}\right) - \log|x|}{\log\left(\frac{\delta}{3\varepsilon}\right) - \log R} \right| \leq \frac{C}{\log(1/\varepsilon)} \log|x|.$$

We first estimate the boundary contribution:

$$\left| \int_{\mathcal{A}_R \cap \partial \mathbb{R}_+^2} W_2^{(j)}(\tilde{z}_{0j} - z_{0j}) (1 + O(\varepsilon|x_1|)) dx_1 \right| \leq \frac{C}{\log(1/\varepsilon)} \int_R^{\frac{\delta}{3\varepsilon}} \frac{\log x_1}{x_1^2} dx_1.$$

An explicit integration by parts gives

$$\int_R^{\frac{\delta}{3\varepsilon}} \frac{\log x_1}{x_1^2} dx_1 = \left[-\frac{\log x_1 + 1}{x_1} \right]_R^{\frac{\delta}{3\varepsilon}} \leq C \frac{\log R + 1}{R}.$$

Thus the boundary tail is $O\left(\frac{1}{R \log(1/\varepsilon)}\right)$. A similar computation for the interior term using $dy = (1 + O(\varepsilon|x|))dx$ yields $O\left(\frac{1}{R^2 \log(1/\varepsilon)}\right)$. Since R is fixed and large, independent of ε , we conclude that

$$\mathcal{I}_2 = O\left(\frac{1}{\log(1/\varepsilon)}\right).$$

Finally, combining the above estimates, we obtain

$$l_1(V)[\tilde{Z}_{0j}] = \frac{1}{|\partial \Omega|} \left(O(\varepsilon^\alpha) + O\left(\frac{1}{\log(1/\varepsilon)}\right) \right) = O\left(\frac{1}{\log(1/\varepsilon)}\right),$$

which completes the proof of (5.103).

Proof of (5.104).

Recall that $\tilde{Z}_{0k}(y) = \tilde{z}_{0k}(F_k^\varepsilon(y))$, with $\tilde{z}_{0k}$ vanishing for $|x| \geq \frac{\delta}{3\varepsilon}$ by construction of $\tilde{\eta}_{2k}$. Hence $\tilde{Z}_{0k}$ is supported in $M_\varepsilon := (F_k^\varepsilon)^{-1}(B_{\frac{\delta}{3\varepsilon}}(0) \cap \mathbb{R}_+^2) \subset \Omega_\varepsilon$, and

$$\|\tilde{Z}_{0k}\|_{L^1(\partial \Omega_\varepsilon)} = \int_{M_\varepsilon \cap \partial \Omega_\varepsilon} |\tilde{Z}_{0k}(y)| ds_y.$$

We use the local conformal change of variables $x = F_k^\varepsilon(y)$. Under this diffeomorphism, $M_\varepsilon \cap \partial \Omega_\varepsilon$ maps exactly onto $(-\frac{\delta}{3\varepsilon}, \frac{\delta}{3\varepsilon}) \times \{0\}$. Since F_k^ε is conformal, the boundary length element transforms as $ds_y = (1 + O(\varepsilon|x_1|)) dx_1$. Since $\varepsilon|x_1| \leq \delta/3$ throughout the domain of integration, this geometric factor is uniformly bounded, yielding $ds_y \leq (1 + O(\delta)) dx_1 \leq C dx_1$. Hence

$$\|\tilde{Z}_{0k}\|_{L^1(\partial \Omega_\varepsilon)} \leq C \int_{-\frac{\delta}{3\varepsilon}}^{\frac{\delta}{3\varepsilon}} |\tilde{z}_{0k}(x_1)| dx_1, \quad (5.108)$$

where $\tilde{z}_{0k}(x_1) := \tilde{\eta}_{1k}(x_1) z_{0k}(x_1, 0) + (1 - \tilde{\eta}_{1k}(x_1)) \tilde{\eta}_{2k}(x_1) \tilde{\gamma}(x_1) z_{0k}(x_1, 0)$.

Region $\{|x_1| \leq R + 1\}$: Here (for $|x_1| \leq R$) $\tilde{z}_{0k}(x_1) = z_{0k}(x_1, 0)$. From the

explicit formula, with $A^2 := \lambda_k^2(1 + D(\xi_k)^2)$,

$$z_{0k}(x_1, 0) = \frac{x_1^2 - A^2}{2\lambda_k(x_1^2 + A^2)},$$

so that $|z_{0k}(x_1, 0)| \leq \frac{1}{2\lambda_k} \leq C$ for all $x_1 \in \mathbb{R}$, since $|x_1^2 - A^2| \leq x_1^2 + A^2$. Hence

$$\int_{-R}^R |\tilde{z}_{0k}(x_1)| dx_1 \leq 2CR = O(1), \quad (5.109)$$

as R is fixed. The transition zone $R < |x_1| < R + 1$, where $\tilde{\eta}_{1k}$ interpolates between 1 and 0, contributes an additional $O(1)$ by the same bound $|\tilde{z}_{0k}| \leq C$ on an interval of length $O(1)$, which we absorb into (5.109).

Region $\{R + 1 \leq |x_1| \leq \frac{\delta}{3\varepsilon}\}$: In this region $\tilde{\eta}_{1k} \equiv 0$, and using $\tilde{\eta}_{2k} \leq 1$ together with $|z_{0k}(x_1, 0)| \leq C$ and $\tilde{\gamma}(x_1) \geq 0$, we obtain the pointwise bound

$$|\tilde{z}_{0k}(x_1)| \leq C \tilde{\gamma}(x_1) = C \frac{\log\left(\frac{\delta}{3\varepsilon}\right) - \log|x_1|}{\log\left(\frac{\delta}{3\varepsilon}\right) - \log R}.$$

By the symmetry of the cutoffs in $|x_1|$, it suffices to estimate the integral over $x_1 > 0$ and double the result. We compute by parts:

$$\begin{aligned} \int_R^{\frac{\delta}{3\varepsilon}} \left(\log\left(\frac{\delta}{3\varepsilon}\right) - \log t \right) dt &= \left[t \left(\log\left(\frac{\delta}{3\varepsilon}\right) - \log t + 1 \right) \right]_R^{\frac{\delta}{3\varepsilon}} \\ &= \frac{\delta}{3\varepsilon} - R \left(\log\left(\frac{\delta}{3\varepsilon}\right) - \log R + 1 \right). \end{aligned}$$

The first term is $O\left(\frac{1}{\varepsilon}\right)$, while the second, with R fixed, is $O\left(\log \frac{1}{\varepsilon}\right)$, of strictly lower order. Hence

$$\int_R^{\frac{\delta}{3\varepsilon}} \left(\log\left(\frac{\delta}{3\varepsilon}\right) - \log t \right) dt = O\left(\frac{1}{\varepsilon}\right).$$

Dividing by $\log\left(\frac{\delta}{3\varepsilon}\right) - \log R = O\left(\log \frac{1}{\varepsilon}\right)$, we conclude

$$\int_R^{\frac{\delta}{3\varepsilon}} \tilde{\gamma}(x_1) dx_1 = \frac{O(1/\varepsilon)}{O(\log(1/\varepsilon))} = O\left(\frac{1}{\varepsilon \log(1/\varepsilon)}\right), \quad (5.110)$$

and by symmetry the same bound holds for $x_1 < 0$, doubling the constant.

Combining (5.109) and (5.110) in (5.108), we obtain

$$\|\tilde{Z}_{0k}\|_{L^1(\partial\Omega_\varepsilon)} \leq C \left[O(1) + O\left(\frac{1}{\varepsilon \log(1/\varepsilon)}\right) \right] = O\left(\frac{1}{\varepsilon \log(1/\varepsilon)}\right),$$

which proves (5.104).

Proof of (5.105).

We write

$$\int_{\Omega_\varepsilon} L(\tilde{Z}_{0k}) \tilde{Z}_{0k} = I_0 + I_1 + I_2 + I_3,$$

where

$$I_\ell = \int_{R_\ell} L(\tilde{Z}_{0k}) \tilde{Z}_{0k}, \quad \ell = 0, 1, 2, 3,$$

and the regions $R_0, \dots, R_3$ are defined in terms of the change of variables F_j^ε . Writing $x = F_j^\varepsilon(y)$, $r = |x|$, we define

$$\begin{aligned} R_0 &= (F_k^\varepsilon)^{-1}(\{r < R\} \cap \mathbb{R}_+^2), \\ R_1 &= (F_k^\varepsilon)^{-1}(\{R < r < R+1\} \cap \mathbb{R}_+^2), \\ R_2 &= (F_k^\varepsilon)^{-1}\left(\left\{R+1 < r < \frac{\delta}{4\varepsilon}\right\} \cap \mathbb{R}_+^2\right), \\ R_3 &= (F_k^\varepsilon)^{-1}\left(\left\{\frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}\right\} \cap \mathbb{R}_+^2\right). \end{aligned}$$

Since F_k is conformal,

$$\Delta_y = \frac{1}{\det(DF_k^\varepsilon(x))} \Delta_x, \quad dy = \det(DF_k^\varepsilon(x)) dx, \quad \det(DF_k^\varepsilon(x)) = 1 + O(\varepsilon|x|)$$

So, we have $W_1(F_k^\varepsilon(x)) \det(DF_k^\varepsilon(x)) = W_1^{(k)}(x)(1 + O(\varepsilon^\alpha) + O(\varepsilon|x|))$.

Writing $\tilde{z}_{0k}(x) := \tilde{\eta}_{1k}(x)z_{0k}(x) + (1 - \tilde{\eta}_{1k}(x))\tilde{\eta}_{2k}(x)\tilde{\gamma}(x)z_{0k}(x)$, so that $\tilde{Z}_{0k}(y) = \tilde{z}_{0k}(F_k^\varepsilon(y))$, we have

$$\begin{aligned} L(\tilde{Z}_{0k})(y) dy &= [-\Delta_x \tilde{z}(x) - W_1(F_k^\varepsilon(y)) \det(DF_k^\varepsilon(x)) \tilde{z}(x)] dx \\ &= [-\Delta_x \tilde{z}_{0k}(x) - W_1^{(k)}(x) \tilde{z}_{0k}(x)] dx + O(\varepsilon^\alpha + \varepsilon|x|) W_1^{(k)}(x) |\tilde{z}_{0k}(x)| dx, \end{aligned}$$

Estimate of I_0 : Here $\tilde{z}_{0k} = z_{0k}$, so

$$I_0 = \int_{\{|x| < R\} \cap \mathbb{R}_+^2} O(\varepsilon^\alpha + \varepsilon|x|) W_1^{(k)}(x) |z_{0k}(x)| z_{0k}(x) dx = O(\varepsilon^\alpha),$$

Estimate of I_1 : Here $\tilde{z}_{0k} = \tilde{\eta}_{1k}z_{0k} + (1 - \tilde{\eta}_{1k})\tilde{\gamma}z_{0k} = (1 - \tilde{\gamma})\tilde{\eta}_{1k}z_{0k} + \tilde{\gamma}z_{0k}$, by direct computation, we have

$$-\Delta_x \tilde{z}_{0k} - W_1^{(k)} \tilde{z}_{0k} = -2(1 - \tilde{\gamma}) \nabla \tilde{\eta}_{1k} \cdot \nabla z_{0k} - 2(1 - \tilde{\eta}_{1k}) \nabla \tilde{\gamma} \cdot \nabla z_{0k} - (1 - \tilde{\gamma}) z_{0k} \Delta \tilde{\eta}_{1k} + 2z_{0k} \nabla \tilde{\eta}_{1k} \cdot \nabla \tilde{\gamma}. \quad (5.111)$$

$$I_1 = \int_{\{R < |x| < R+1\} \cap \mathbb{R}_+^2} \left[-2(1 - \tilde{\gamma}) \nabla \tilde{\eta}_{1k} \cdot \nabla z_{0k} - 2(1 - \tilde{\eta}_{1k}) \nabla \tilde{\gamma} \cdot \nabla z_{0k} \right. \\ \left. - (1 - \tilde{\gamma}) z_{0k} \Delta \tilde{\eta}_{1k} + 2z_{0k} \nabla \tilde{\eta}_{1k} \cdot \nabla \tilde{\gamma} \right] \tilde{z}_{0k} dx + O(\varepsilon^\alpha + \varepsilon R).$$

We integrate by parts the third term, using that $\tilde{\eta}'_1(R) = \tilde{\eta}'_1(R+1) = 0$ and that the cut-off are radial (so the boundary term on the line $\{x_2 = 0\}$ vanish), so

$$I_1 = \int_{\{R < |x| < R+1\} \cap \mathbb{R}_+^2} z_{0k} \tilde{z}_{0k} \nabla \tilde{\eta}_{1k} \cdot \nabla \tilde{\gamma} dx - \int_{\{R < |x| < R+1\} \cap \mathbb{R}_+^2} (1 - \tilde{\gamma}) (\nabla \tilde{\eta}_{1k} \cdot \nabla z_{0k}) \tilde{z}_{0k} dx \\ - 2 \int_{\{R < |x| < R+1\} \cap \mathbb{R}_+^2} (1 - \tilde{\eta}_{1k}) (\nabla \tilde{\gamma} \cdot \nabla z_{0k}) \tilde{z}_{0k} dx + \int_{\{R < |x| < R+1\} \cap \mathbb{R}_+^2} (1 - \tilde{\gamma}) z_{0k} (\nabla \tilde{\eta}_{1k} \cdot \nabla \tilde{z}_{0k}) dx \\ + O(\varepsilon^\alpha + \varepsilon R).$$

Now we concentrate on the first term: Since the functions are radial we have $\nabla \tilde{\eta}_{1k}(x) \cdot \nabla \tilde{\gamma}(x) = \tilde{\eta}'_{1k}(|x|) \tilde{\gamma}'(|x|) = -\frac{\tilde{\eta}'_{1k}(|x|)}{|x|(\log(\frac{\delta}{\varepsilon}) - \log R)}$, passing to polar coordinates, and using that for R large z_{0k} and $\tilde{z}_{0k}$ are lower bounded, we get

$$\int_{\{R < |x| < R+1\} \cap \mathbb{R}_+^2} z_{0k} \tilde{z}_{0k} \nabla \tilde{\eta}_{1k} \cdot \nabla \tilde{\gamma} dx = \frac{1}{\log(\frac{\delta}{\varepsilon}) - \log R} \int_0^\pi \int_R^{R+1} z_{0k} \tilde{z}_{0k} (-\tilde{\eta}'_{1k}) \geq \frac{c_0}{\log(\frac{\delta}{\varepsilon})}$$

For the other three terms, we use that $|z_{0k}|, |\tilde{z}_{0k}|, |\nabla \tilde{\eta}_{1k}|, |\Delta \tilde{\eta}_{1k}| \leq C$, $|\nabla z_{0k}| = O(\frac{1}{R^2})$ and that

$$1 - \tilde{\gamma} = O\left(\frac{1}{R(\log(\frac{\delta}{\varepsilon}) - \log R)}\right), \quad |\nabla \tilde{\gamma}| = O\left(\frac{1}{R(\log(\frac{\delta}{\varepsilon}) - \log R)}\right)$$

which implies

$$|\nabla \tilde{z}_{0k}| = O\left(\frac{1}{R(\log(\frac{\delta}{\varepsilon}) - \log R)}\right) + O\left(\frac{1}{R^2}\right)$$

to conclude that

$$\int_{\{R < |x| < R+1\} \cap \mathbb{R}_+^2} (1 - \tilde{\gamma}) (\nabla \tilde{\eta}_{1k} \cdot \nabla z_{0k}) \tilde{z}_{0k} dx = O\left(\frac{1}{R^2(\log(\frac{\delta}{\varepsilon}) - \log R)}\right)$$

$$2 \int_{\{R < |x| < R+1\} \cap \mathbb{R}_+^2} (1 - \tilde{\eta}_{1k}) (\nabla \tilde{\gamma} \cdot \nabla z_{0k}) \tilde{z}_{0k} dx = O\left(\frac{1}{R^2(\log(\frac{\delta}{\varepsilon}) - \log R)}\right)$$

$$\int_{\{R < |x| < R+1\} \cap \mathbb{R}_+^2} (1-\tilde{\gamma}) z_{0k} (\nabla \tilde{\eta}_{1k} \cdot \nabla \tilde{z}_{0k}) dx = O\left(\frac{1}{R^2(\log(\frac{\delta}{\varepsilon}) - \log R)}\right) + O\left(\frac{1}{R(\log(\frac{\delta}{\varepsilon}) - \log R)^2}\right)$$

Thus, choosing R sufficiently large and ε sufficiently small, we proved

$$I_1 \geq \frac{c_1}{\log(\frac{1}{\varepsilon})}.$$

Estimate of I_2 : On R_2 , $\tilde{z}_{0k} = \tilde{\gamma} z_{0k}$, so

$$-\Delta_x \tilde{z}_{0k} - W_1^{(k)} \tilde{z}_{0k} = -2\nabla \tilde{\gamma} \cdot \nabla z_{0k}.$$

Thus, using again $|\nabla \tilde{\gamma}(x)| = \frac{1}{|x|(\log(\frac{\delta}{\varepsilon}) - \log R)}$ and $|\nabla z_{0k}(x)| = O(|x|^{-2})$,

$$\begin{aligned} |I_2| &\leq \int_R^{\delta/(4\varepsilon)} \int_0^\pi \frac{C}{r^3(\log(\frac{\delta}{\varepsilon}) - \log R)} r dr d\theta + O(\varepsilon^\alpha) \leq \frac{C\pi}{\log(\frac{\delta}{\varepsilon}) - \log R} \int_R^\infty r^{-2} dr + O(\varepsilon^\alpha) \\ &= O\left(\frac{1}{R \log(1/\varepsilon)}\right) + O(\varepsilon^\alpha). \end{aligned}$$

Estimate of I_3 : On R_3 , $\tilde{z}_{0k} = \tilde{\eta}_{2k} \tilde{\gamma} z_{0k}$, so

$$\begin{aligned} -\Delta_x(\tilde{\eta}_{2k} \tilde{\gamma} z_{0k}) - W_1^{(k)} \tilde{\eta}_{2k} \tilde{\gamma} z_{0k} &= -2\tilde{\eta}_{2k} \nabla \tilde{\gamma} \cdot \nabla z_{0k} - 2\nabla \tilde{\eta}_{2k} \cdot \nabla(\tilde{\gamma} z_{0k}) - \tilde{\gamma} z_{0k} \Delta \tilde{\eta}_{2k} \\ &= -\tilde{\gamma} z_{0k} \Delta_x \tilde{\eta}_{2k} - 2z_{0k} \nabla \tilde{\eta}_{2k} \cdot \nabla \tilde{\gamma} - 2\tilde{\gamma} \nabla \tilde{\eta}_{2k} \cdot \nabla z_{0k} - 2\tilde{\eta}_{2k} \nabla \tilde{\gamma} \cdot \nabla z_{0k} \end{aligned}$$

For $\frac{\delta}{4\varepsilon} \leq |x| \leq \frac{\delta}{3\varepsilon}$: $\tilde{\gamma} = O(\frac{1}{\log(1/\varepsilon)})$, $|\nabla \tilde{\gamma}| = O(\frac{\varepsilon}{\log(1/\varepsilon)})$, $|\nabla z_{0k}| = O(\varepsilon^2)$, $|\nabla \tilde{\eta}_{2k}| = O(\varepsilon/\delta)$, $|\Delta \tilde{\eta}_{2k}| = O(\varepsilon^2/\delta^2)$, so

$$|L(\tilde{Z}_{0k})| \leq \frac{C\varepsilon^2}{\delta^2 \log(1/\varepsilon)}.$$

Using $|\tilde{Z}_{0k}| = O(\frac{1}{\log(1/\varepsilon)})$ and $|R_3| = O(\delta^2/\varepsilon^2)$,

$$|I_3| \leq \left(\frac{C\varepsilon^2}{\delta^2 \log(1/\varepsilon)}\right) \cdot \frac{C}{\log(1/\varepsilon)} \cdot \frac{\delta^2}{\varepsilon^2} = O\left(\frac{1}{\log^2(1/\varepsilon)}\right).$$

Now we estimate the boundary integral

$$\int_{\partial\Omega_\varepsilon} \tilde{Z}_{0k} \left(\frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right).$$

Since F_k is conformal,

$$\sqrt{\det(DF_k^\varepsilon(x))} = 1 + O(\varepsilon|x|),$$

and

$$\frac{\partial}{\partial n_y} = \sqrt{\det(DF_k^\varepsilon(x))} \frac{\partial}{\partial n_x}, \quad ds_y = \frac{1}{\sqrt{\det(DF_k^\varepsilon(x))}} dx_1.$$

Therefore, using that

$$W_2(y) = W_2^{(k)}(x) (1 + O(\varepsilon^\alpha) + O(\varepsilon|x|)),$$

we obtain

$$\left(\frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right) (y) ds_y = \left(\frac{\partial \tilde{z}_{0k}}{\partial n_x} - W_2^{(k)} \tilde{z}_{0k} \right) dx_1 + O(\varepsilon^\alpha + \varepsilon|x|) W_2^{(k)} |\tilde{z}_{0k}| dx_1.$$

Since $\tilde{\eta}_{1k}$, $\tilde{\eta}_{2k}$ and $\tilde{\gamma}$ are radial functions,

$$\frac{\partial \tilde{\eta}_{1k}}{\partial n_x} = \frac{\partial \tilde{\eta}_{2k}}{\partial n_x} = \frac{\partial \tilde{\gamma}}{\partial n_x} = 0 \quad \text{on } \partial \mathbb{R}_+^2.$$

Hence,

$$\frac{\partial \tilde{z}_{0k}}{\partial n_x} = (\tilde{\eta}_{1k}(x) + (1 - \tilde{\eta}_{1k}(x))\tilde{\eta}_{2k}(x)\tilde{\gamma}(x)) \frac{\partial z_{0k}}{\partial n_x},$$

Using

$$\frac{\partial z_{0k}}{\partial n_x} = W_2^{(k)} z_{0k} \quad \text{on } \partial \mathbb{R}_+^2,$$

we obtain

$$\frac{\partial \tilde{z}_{0k}}{\partial n_x} - W_2^{(k)} \tilde{z}_{0k} = 0,$$

and therefore

$$\left(\frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right) (y) ds_y = O(\varepsilon^\alpha + \varepsilon|x|) W_2^{(k)} |\tilde{z}_{0k}| dx_1.$$

Thus,

$$\int_{\partial \Omega_\varepsilon} \tilde{Z}_{0k} \left(\frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right) = \int_{-\frac{\delta}{3\varepsilon}}^{\frac{\delta}{3\varepsilon}} O(\varepsilon^\alpha + \varepsilon|x_1|) W_2^{(k)}(x_1) \tilde{z}_{0k}^2(x_1) dx_1.$$

For $0 < x_1 < R + 1$, using that $W_2^{(k)}$ and $\tilde{z}_{0k}$ are bounded,

$$\int_0^{R+1} (\varepsilon^\alpha + \varepsilon x_1) W_2^{(k)}(x_1) \tilde{z}_{0k}^2(x_1) dx_1 = O(\varepsilon^\alpha).$$

For

$$R + 1 < x_1 < \frac{\delta}{4\varepsilon},$$

using $W_2^{(k)}(x_1) = O(x_1^{-2})$,

$$\int_R^{\frac{\delta}{4\varepsilon}} (\varepsilon^\alpha + \varepsilon x_1) W_2^{(k)}(x_1) \tilde{z}_{0k}^2(x_1) dx_1 = O(\varepsilon^\alpha) + O\left(\varepsilon \log \frac{1}{\varepsilon}\right).$$

Finally, for

$$\frac{\delta}{4\varepsilon} < x_1 < \frac{\delta}{3\varepsilon},$$

using $W_2^{(k)}(x_1) = O(\varepsilon^2)$,

$$\int_{\frac{\delta}{4\varepsilon}}^{\frac{\delta}{3\varepsilon}} (\varepsilon^\alpha + \varepsilon x_1) W_2^{(k)}(x_1) \tilde{z}_{0k}^2(x_1) dx_1 = O(\varepsilon).$$

Therefore,

$$\left| \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0k} \left(\frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right) \right| = O(\varepsilon^\alpha),$$

Combining all the estimates,

$$\begin{aligned} & \int_{\Omega_\varepsilon} L(\tilde{Z}_{0k}) \tilde{Z}_{0k} + \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0k} \left(\frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right) \\ & \geq \frac{c_1}{\log(1/\varepsilon)} - O\left(\frac{1}{R \log(1/\varepsilon)}\right) - O\left(\frac{1}{\log^2(1/\varepsilon)}\right) - O(\varepsilon^\alpha) \\ & \geq \frac{1}{C \log(1/\varepsilon)}, \end{aligned}$$

for R chosen sufficiently large (independent of ε) and ε sufficiently small. This proves (5.105).

Proof of (5.106).

We estimate $L(\tilde{Z}_{0k})$ in each of the regions $R_0, \dots, R_3$, using the identities established in the proof of (5.105). Throughout the proof we divide by the weight

$$\varepsilon^2 + (1 + |y - \xi'_k|)^{-2-\sigma},$$

which, on the support of $\tilde{Z}_{0k}$, is comparable to

$$\varepsilon^2 + r^{-2-\sigma},$$

by the separation condition (5.95).

Estimate on R_0 . Since $\tilde{z}_{0k} = z_{0k}$,

$$-\Delta_x \tilde{z}_{0k} - W_1^{(k)} \tilde{z}_{0k} = 0,$$

and therefore

$$|L(\tilde{Z}_{0k})(y)| \leq C(\varepsilon^\alpha + \varepsilon r) W_1^{(k)}(x) |z_{0k}(x)| = O(\varepsilon^\alpha), \quad r < R.$$

Since the weight is bounded from below by a positive constant,

$$\frac{|L(\tilde{Z}_{0k})(y)|}{\varepsilon^2 + (1 + |y - \xi'_k|)^{-2-\sigma}} = O(\varepsilon^\alpha).$$

Estimate on R_1 . Using (5.111),

$$|z_{0k}|, |\tilde{z}_{0k}|, |\nabla \tilde{\eta}_{1k}|, |\Delta \tilde{\eta}_{1k}| \leq C,$$

$$|\nabla z_{0k}| = O(R^{-2}),$$

and

$$1 - \tilde{\gamma} = O\left(\frac{1}{R(\log(\delta/\varepsilon) - \log R)}\right), \quad |\nabla \tilde{\gamma}| = O\left(\frac{1}{R(\log(\delta/\varepsilon) - \log R)}\right),$$

we obtain

$$|L(\tilde{Z}_{0k})(y)| \leq \frac{C}{\log(\delta/\varepsilon) - \log R} + O(\varepsilon^\alpha + \varepsilon R) = O\left(\frac{1}{\log(1/\varepsilon)}\right), \quad R < r < R + 1.$$

As before,

$$\frac{|L(\tilde{Z}_{0k})(y)|}{\varepsilon^2 + (1 + |y - \xi'_k|)^{-2-\sigma}} = O\left(\frac{1}{\log(1/\varepsilon)}\right).$$

Estimate on R_2 . Since

$$-\Delta_x \tilde{z}_{0k} - W_1^{(k)} \tilde{z}_{0k} = -2\nabla \tilde{\gamma} \cdot \nabla z_{0k},$$

using

$$|\nabla \tilde{\gamma}(x)| = \frac{1}{r(\log(\delta/\varepsilon) - \log R)},$$

$$|\nabla z_{0k}(x)| = O(r^{-2}), \quad W_1^{(k)}(x) = O(r^{-4}), \quad |\tilde{z}_{0k}| \leq C,$$

we obtain

$$|L(\tilde{Z}_{0k})(y)| \leq \frac{C}{r^3(\log(\delta/\varepsilon) - \log R)} + C\varepsilon^\alpha r^{-4} + C\varepsilon r^{-3}, \quad R+1 < r < \frac{\delta}{4\varepsilon}.$$

Using the lower bound

$$\varepsilon^2 + r^{-2-\sigma} \geq r^{-2-\sigma},$$

we have

$$\begin{aligned} \frac{r^{-3}}{(\log(\delta/\varepsilon) - \log R) r^{-2-\sigma}} &= \frac{r^{\sigma-1}}{\log(\delta/\varepsilon) - \log R}, \\ \frac{\varepsilon^\alpha r^{-4}}{r^{-2-\sigma}} &= \varepsilon^\alpha r^{\sigma-2}, \quad \frac{\varepsilon r^{-3}}{r^{-2-\sigma}} = \varepsilon r^{\sigma-1}. \end{aligned}$$

Since $\sigma < 1$, all three quantities attain their maximum at $r = R+1$, hence

$$\frac{|L(\tilde{Z}_{0k})(y)|}{\varepsilon^2 + (1 + |y - \xi'_k|)^{-2-\sigma}} = O\left(\frac{1}{\log(1/\varepsilon)}\right).$$

Estimate on R_3 . As proved in the estimate of I_3 ,

$$|L(\tilde{Z}_{0k})(y)| \leq \frac{C\varepsilon^2}{\delta^2(\log(\delta/\varepsilon) - \log R)}, \quad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}.$$

Using

$$\varepsilon^2 + r^{-2-\sigma} \geq \varepsilon^2,$$

we obtain

$$\frac{|L(\tilde{Z}_{0k})(y)|}{\varepsilon^2 + (1 + |y - \xi'_k|)^{-2-\sigma}} \leq \frac{C}{\log(\delta/\varepsilon) - \log R} = O\left(\frac{1}{\log(1/\varepsilon)}\right).$$

Combining the estimates on the four regions,

$$\|L(\tilde{Z}_{0k})\|_{**, \Omega_\varepsilon} = O\left(\frac{1}{\log(1/\varepsilon)}\right),$$

which proves (5.106).

Proof of (5.107).

From the proof of (5.105), since $\tilde{\eta}_{1k}$, $\tilde{\eta}_{2k}$ and $\tilde{\gamma}$ are radial functions,

$$\frac{\partial \tilde{z}_{0k}}{\partial n_x} - W_2^{(k)} \tilde{z}_{0k} = 0 \quad \text{on } \partial \mathbb{R}_+^2,$$

and therefore

$$\left(\frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right) (y) = O(\varepsilon^\alpha + \varepsilon r) W_2^{(k)}(x) |\tilde{z}_{0k}(x)|.$$

We estimate the right-hand side in each of the regions $R_0, \dots, R_3$. Throughout the proof we divide by the weight

$$\varepsilon + (1 + |y - \xi'_k|)^{-1-\sigma},$$

which, on the support of $\tilde{Z}_{0k}$, is comparable to

$$\varepsilon + r^{-1-\sigma},$$

by the separation condition (5.95).

Estimate on R_0 and R_1 . Since $r \leq R + 1$ and $W_2^{(k)}, \tilde{z}_{0k}$ are bounded,

$$\left| \frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right| (y) = O(\varepsilon^\alpha + \varepsilon) = O(\varepsilon^\alpha).$$

As before,

$$\frac{\left| \frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right| (y)}{\varepsilon + (1 + |y - \xi'_k|)^{-1-\sigma}} = O(\varepsilon^\alpha).$$

Estimate on R_2 . Using

$$W_2^{(k)}(x) = O(r^{-2}), \quad |\tilde{z}_{0k}| \leq C,$$

we obtain

$$\left| \frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right| (y) \leq C\varepsilon^\alpha r^{-2} + C\varepsilon r^{-1}, \quad R + 1 < r < \frac{\delta}{4\varepsilon}.$$

Using the lower bound

$$\varepsilon + r^{-1-\sigma} \geq r^{-1-\sigma},$$

we have

$$\frac{\varepsilon^\alpha r^{-2}}{r^{-1-\sigma}} = \varepsilon^\alpha r^{\sigma-1}, \quad \frac{\varepsilon r^{-1}}{r^{-1-\sigma}} = \varepsilon r^\sigma.$$

Since $\sigma < 1$,

$$\varepsilon^\alpha r^{\sigma-1} \leq C\varepsilon^\alpha,$$

while

$$\varepsilon r^\sigma \leq \varepsilon \left(\frac{\delta}{4\varepsilon} \right)^\sigma = O(\varepsilon^{1-\sigma}).$$

Hence

$$\frac{\left| \frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right|(y)}{\varepsilon + (1 + |y - \xi'_k|)^{-1-\sigma}} = O(\varepsilon^\alpha) + O(\varepsilon^{1-\sigma}).$$

Estimate on R_3 . Since

$$W_2^{(k)}(x) = O(\varepsilon^2), \quad |\tilde{z}_{0k}| \leq C,$$

we obtain

$$\left| \frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right|(y) = O(\varepsilon^2)$$

Using the lower bound

$$\varepsilon + r^{-1-\sigma} \geq \varepsilon,$$

we obtain

$$\frac{\left| \frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right|(y)}{\varepsilon + (1 + |y - \xi'_k|)^{-1-\sigma}} = O(\varepsilon)$$

Combining the estimates on the four regions,

$$\left\| \frac{\partial \tilde{Z}_{0k}}{\partial n} - W_2 \tilde{Z}_{0k} \right\|_{*, \partial \Omega_\varepsilon} = O(\varepsilon^\alpha) + O(\varepsilon^{1-\sigma}) = O\left(\frac{1}{\log(1/\varepsilon)} \right),$$

which proves (5.107). □

Proof of Proposition 5.20. Let $(\phi, c_1, \dots, c_m)$ be a solution of (5.88). We first prove the a priori estimate.

The equation in the interior can be written as

$$-\Delta \phi - W_1 \phi = \tilde{f} \quad \text{in } \Omega_\varepsilon,$$

where

$$\tilde{f} = f + \sum_{j=1}^m c_j \chi_j Z_{1j}.$$

Since ϕ satisfies

$$\int_{\Omega_\varepsilon} \chi_j Z_{1j} \phi \, dy = 0, \quad j = 1, \dots, m,$$

the estimate of the previous lemma gives

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \log\left(\frac{1}{\varepsilon}\right) \left(\|g\|_{*,\partial\Omega_\varepsilon} + \|\tilde{f}\|_{**, \Omega_\varepsilon}\right). \quad (5.112)$$

By the definition of $\tilde{f}$,

$$\|\tilde{f}\|_{**, \Omega_\varepsilon} \leq \|f\|_{**, \Omega_\varepsilon} + \sum_{j=1}^m |c_j| \|\chi_j Z_{1j}\|_{**, \Omega_\varepsilon}.$$

The support of χ_j is contained in a fixed bounded region in the coordinates F_j^ε , where Z_{1j} is uniformly bounded. Hence

$$\|\chi_j Z_{1j}\|_{**, \Omega_\varepsilon} \leq C,$$

uniformly in ε . Therefore,

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \log\left(\frac{1}{\varepsilon}\right) \left(\|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon} + \sum_{j=1}^m |c_j|\right). \quad (5.113)$$

We next estimate the coefficients c_k . Let

$$\tilde{Z}_{1k}(y) = \tilde{\eta}_{2k}(x) Z_{1k}(y), \quad x = F_k^\varepsilon(y),$$

where $\tilde{\eta}_{2k}$ is a radial cut-off satisfying

$$\tilde{\eta}_{2k} = 1 \quad \text{for } |x| \leq \frac{\delta}{4\varepsilon}, \quad \tilde{\eta}_{2k} = 0 \quad \text{for } |x| \geq \frac{\delta}{3\varepsilon}.$$

Multiplying the equation in (5.88) by $\tilde{Z}_{1k}$ and integrating, we obtain

$$\int_{\Omega_\varepsilon} (-\Delta\phi - W_1\phi) \tilde{Z}_{1k} dy = \int_{\Omega_\varepsilon} f \tilde{Z}_{1k} dy + \sum_{j=1}^m c_j \int_{\Omega_\varepsilon} \chi_j Z_{1j} \tilde{Z}_{1k} dy. \quad (5.114)$$

The separation condition on the points ξ_j implies that, for ε sufficiently small, the supports of χ_j and $\tilde{\eta}_{2k}$ are disjoint whenever $j \neq k$. Consequently,

$$\int_{\Omega_\varepsilon} \chi_j Z_{1j} \tilde{Z}_{1k} dy = \delta_{jk} \int_{\Omega_\varepsilon} \chi_k Z_{1k}^2 dy =: \delta_{jk} \mathcal{I}_k.$$

Moreover,

$$\mathcal{I}_k \geq c_0 > 0$$

uniformly in ε .

Applying Green's formula, we get

$$\begin{aligned} \int_{\Omega_\varepsilon} (-\Delta\phi) \tilde{Z}_{1k} dy &= \int_{\Omega_\varepsilon} \phi(-\Delta_y \tilde{Z}_{1k}) dy - \int_{\partial\Omega_\varepsilon} \frac{\partial\phi}{\partial n_y} \tilde{Z}_{1k} ds_y \\ &\quad + \int_{\partial\Omega_\varepsilon} \phi \frac{\partial \tilde{Z}_{1k}}{\partial n_y} ds_y. \end{aligned}$$

Using the boundary condition satisfied by ϕ , we obtain

$$c_k \mathcal{I}_k = \int_{\Omega_\varepsilon} \phi(-\Delta_y \tilde{Z}_{1k} - W_1 \tilde{Z}_{1k}) dy \quad (5.115)$$

$$+ \int_{\partial\Omega_\varepsilon} \phi \left(\frac{\partial \tilde{Z}_{1k}}{\partial n_y} - W_2 \tilde{Z}_{1k} \right) ds_y \quad (5.116)$$

$$+ \varepsilon \int_{\partial\Omega_\varepsilon} l_1(V)[\phi] \tilde{Z}_{1k} ds_y \quad (5.117)$$

$$- \int_{\Omega_\varepsilon} f \tilde{Z}_{1k} dy - \int_{\partial\Omega_\varepsilon} g \tilde{Z}_{1k} ds_y. \quad (5.118)$$

We estimate the terms on the right-hand side of (5.115). Since

$$z_{1k}(x) = O((1 + |x|)^{-1}),$$

we also have

$$|\tilde{Z}_{1k}(y)| = O((1 + |x|)^{-1}).$$

Therefore, by the definition of the weighted norms,

$$\begin{aligned} \left| \int_{\Omega_\varepsilon} f \tilde{Z}_{1k} dy \right| &\leq C \|f\|_{**, \Omega_\varepsilon} \int_{\mathbb{R}_+^2} \left(\varepsilon^2 + (1 + |x|)^{-2-\sigma} \right) (1 + |x|)^{-1} dx \\ &\leq C \|f\|_{**, \Omega_\varepsilon}, \end{aligned}$$

and similarly

$$\begin{aligned} \left| \int_{\partial\Omega_\varepsilon} g \tilde{Z}_{1k} ds_y \right| &\leq C \|g\|_{*, \partial\Omega_\varepsilon} \int_{\partial\mathbb{R}_+^2} \left(\varepsilon + (1 + |x_1|)^{-1-\sigma} \right) (1 + |x_1|)^{-1} dx_1 \\ &\leq C \|g\|_{*, \partial\Omega_\varepsilon}. \end{aligned}$$

For the non-local term, using

$$|l_1(V)[\phi]| \leq C \|\phi\|_{L^\infty(\Omega_\varepsilon)},$$

we obtain

$$\varepsilon \left| \int_{\partial\Omega_\varepsilon} l_1(V)[\phi] \tilde{Z}_{1k} ds_y \right| \leq C\varepsilon \|\phi\|_{L^\infty(\Omega_\varepsilon)} \int_{|x_1| \leq \frac{\delta}{3\varepsilon}} (1 + |x_1|)^{-1} dx_1.$$

Hence,

$$\varepsilon \left| \int_{\partial\Omega_\varepsilon} l_1(V)[\phi] \tilde{Z}_{1k} ds_y \right| \leq C\varepsilon \log\left(\frac{1}{\varepsilon}\right) \|\phi\|_{L^\infty(\Omega_\varepsilon)}.$$

We now estimate the first term in (5.115). Using the conformal coordinates F_k^ε and the relations

$$\Delta_y = \det(DF_k^\varepsilon)^{-1} \Delta_x, \quad dy = \det(DF_k^\varepsilon) dx,$$

we write

$$\tilde{Z}_{1k}(y) = \tilde{\eta}_{2k}(x) z_{1k}(x).$$

Since

$$-\Delta_x z_{1k} - W_1^{(k)} z_{1k} = 0,$$

the main part cancels and we are left with

$$\begin{aligned} & (-\Delta_y \tilde{Z}_{1k} - W_1 \tilde{Z}_{1k}) dy \\ &= (-z_{1k} \Delta_x \tilde{\eta}_{2k} - 2\nabla \tilde{\eta}_{2k} \cdot \nabla z_{1k}) dx \\ &+ O(\varepsilon^\alpha + \varepsilon|x|) W_1^{(k)}(x) |\tilde{z}_{1k}| dx. \end{aligned}$$

The derivatives of the cut-off are supported in

$$\frac{\delta}{4\varepsilon} \leq |x| \leq \frac{\delta}{3\varepsilon}.$$

In this region,

$$z_{1k} = O(\varepsilon), \quad |\nabla z_{1k}| = O(\varepsilon^2),$$

while

$$|\nabla \tilde{\eta}_{2k}| = O(\varepsilon), \quad |\Delta \tilde{\eta}_{2k}| = O(\varepsilon^2).$$

Therefore the cut-off contribution is $O(\varepsilon^3)$ pointwise and, after integration on a region of size $O(\varepsilon^{-2})$, gives $O(\varepsilon)$.

On the other hand,

$$W_1^{(k)}(x) = O(|x|^{-4}),$$

so the geometric error is bounded by $C\varepsilon^\alpha$. Consequently,

$$\left| \int_{\Omega_\varepsilon} \phi(-\Delta_y \tilde{Z}_{1k} - W_1 \tilde{Z}_{1k}) dy \right| \leq C\varepsilon^\alpha \|\phi\|_{L^\infty(\Omega_\varepsilon)}.$$

It remains to estimate the boundary contribution. We have

$$\partial_{n_y} = \det(DF_k^\varepsilon)^{-1/2} \partial_{n_x}, \quad ds_y = \det(DF_k^\varepsilon)^{1/2} dx_1.$$

Since $\tilde{\eta}_{2k}$ is radial,

$$\partial_{n_x} \tilde{\eta}_{2k} = 0 \quad \text{on } \{x_2 = 0\},$$

and therefore

$$\partial_{n_x} (\tilde{\eta}_{2k} z_{1k}) = \tilde{\eta}_{2k} \partial_{n_x} z_{1k}.$$

Using the boundary kernel relation

$$\partial_{n_x} z_{1k} - W_2^{(k)} z_{1k} = 0,$$

we obtain

$$\left| \int_{\partial\Omega_\varepsilon} \phi \left(\frac{\partial \tilde{Z}_{1k}}{\partial n_y} - W_2 \tilde{Z}_{1k} \right) ds_y \right| \leq C \varepsilon^\alpha \|\phi\|_{L^\infty(\Omega_\varepsilon)}.$$

Combining all the above estimates in (5.115), we get

$$|c_k| \leq C (\|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}) + C \varepsilon^\alpha \|\phi\|_{L^\infty(\Omega_\varepsilon)}. \quad (5.119)$$

Substituting this estimate into (5.113), we find

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \log \left(\frac{1}{\varepsilon} \right) (\|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}) + C \varepsilon^\alpha \log \left(\frac{1}{\varepsilon} \right) \|\phi\|_{L^\infty(\Omega_\varepsilon)}.$$

Since

$$\varepsilon^\alpha \log(1/\varepsilon) \rightarrow 0,$$

the last term can be absorbed into the left-hand side for ε small enough. Hence

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \log \left(\frac{1}{\varepsilon} \right) (\|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}). \quad (5.120)$$

The estimate for c_k follows immediately from (5.119).

It remains to prove existence. Let

$$H := \left\{ \varphi \in H^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} \chi_j Z_{1j} \varphi dy = 0, \quad j = 1, \dots, m \right\}.$$

We consider the bilinear form

$$a(\varphi, \psi) := \int_{\Omega_\varepsilon} \nabla \varphi \cdot \nabla \psi dy - \int_{\Omega_\varepsilon} W_1 \varphi \psi dy - \int_{\partial\Omega_\varepsilon} W_2 \varphi \psi ds_y + \varepsilon l_1(V)[\varphi] \int_{\partial\Omega_\varepsilon} \psi ds_y.$$

Testing the equation

$$-\Delta\varphi - W_1\varphi = \tilde{f}$$

against $\psi \in H^1(\Omega_\varepsilon)$ and using the boundary condition

$$\partial_n\varphi = W_2\varphi - \varepsilon l_1(V)[\varphi] + g$$

we obtain

$$a(\varphi, \psi) = \int_{\Omega_\varepsilon} \tilde{f} \psi \, dy + \int_{\partial\Omega_\varepsilon} g \psi \, ds_y.$$

Since

$$\tilde{f} = f + \sum_{j=1}^m c_j \chi_j Z_{1j},$$

this becomes

$$a(\varphi, \psi) = \int_{\Omega_\varepsilon} f \psi \, dy + \sum_{j=1}^m c_j \int_{\Omega_\varepsilon} \chi_j Z_{1j} \psi \, dy + \int_{\partial\Omega_\varepsilon} g \psi \, ds_y.$$

Hence, for $\psi \in H$, the orthogonality conditions imply that

$$a(\varphi, \psi) = \int_{\Omega_\varepsilon} f \psi \, dy + \int_{\partial\Omega_\varepsilon} g \psi \, ds_y, \quad \forall \psi \in H.$$

Once φ is found, the constants $c_1, \dots, c_m$ are uniquely determined by the equations obtained by testing the interior equation against $\chi_k Z_{1k}$.

Since $W_1, W_2 \in L^\infty(\Omega_\varepsilon)$, the terms involving W_1 , W_2 and $l_1(V)$ are compact perturbations of the bilinear form (for some $\lambda > 0$)

$$(\varphi, \psi) \mapsto \int_{\Omega_\varepsilon} \nabla \varphi \cdot \nabla \psi \, dy + \lambda \int_{\Omega_\varepsilon} \varphi \psi \, dy,$$

which is coercive. Therefore, by the Fredholm alternative, it is enough to prove that the kernel is trivial. Let $\varphi \in H$ solve the homogeneous problem, namely

$$a(\varphi, \psi) = 0, \quad \forall \psi \in H.$$

The corresponding coefficients $c_1, \dots, c_m$ give a solution of (5.88) with $f = g = 0$. The a priori estimate (5.120) yields

$$\|\varphi\|_{L^\infty(\Omega_\varepsilon)} = 0,$$

and therefore $\varphi \equiv 0$. Moreover, by (5.119),

$$c_k = 0, \quad k = 1, \dots, m.$$

Thus the homogeneous problem has only the trivial solution. The Fredholm alternative yields the solvability of the weak formulation for arbitrary f and g , and therefore the existence of a solution to (5.88).

Uniqueness follows from the same argument applied to the difference of two solutions. The estimate (5.120) is precisely the a priori bound proved above. $\square$

Remark 5.24. Recall the bilinear form introduced in the proof of Proposition 5.20,

$$a_s(\varphi, \psi) := \int_{\Omega_\varepsilon} \nabla \varphi \cdot \nabla \psi \, dy - \int_{\Omega_\varepsilon} W_1(\xi'(s)) \varphi \psi \, dy - \int_{\partial\Omega_\varepsilon} W_2(\xi'(s)) \varphi \psi \, ds_y + \varepsilon l_1(V(\xi'(s)))[\varphi] \int_{\partial\Omega_\varepsilon} \psi \, ds_y,$$

which depends on s through $\xi'(s)$. Fix

$$(f, g) \in L^\infty(\Omega_\varepsilon) \times L^\infty(\partial\Omega_\varepsilon)$$

and consider

$$\mathcal{F} : U \times H^1(\Omega_\varepsilon) \times \mathbb{R}^m \longrightarrow H^1(\Omega_\varepsilon)^* \times \mathbb{R}^m,$$

defined by

$$\mathcal{F}(s, \varphi, c) := \left(\begin{aligned} &\psi \mapsto a_s(\varphi, \psi) - \sum_{j=1}^m c_j \int_{\Omega_\varepsilon} \chi_j(\xi'(s)) Z_{1j}(\xi'(s)) \psi \, dy - \int_{\Omega_\varepsilon} f \psi \, dy - \int_{\partial\Omega_\varepsilon} g \psi \, ds_y \\ &\left(\int_{\Omega_\varepsilon} \chi_j(\xi'(s)) Z_{1j}(\xi'(s)) \varphi \, dy \right)_{j=1}^m \end{aligned} \right).$$

The map $\mathcal{F}$ is of class C^1 . Indeed, for fixed s it is affine in (φ, c) , while the dependence on s is C^1 by the regularity of W_1 , W_2 , $\chi_j Z_{1j}$ and $l_1(V(\xi'(s)))[\cdot]$ with respect to the parameters ξ'_j . Moreover, the argument in the proof of Proposition 5.20 shows that

$$D_{(\varphi, c)} \mathcal{F}(s_0, \varphi_0, c_0) : H^1(\Omega_\varepsilon) \times \mathbb{R}^m \longrightarrow H^1(\Omega_\varepsilon)^* \times \mathbb{R}^m$$

is a bounded isomorphism. Hence, by the implicit function theorem,

$$s \longmapsto (\varphi(s), c(s))$$

is of class C^1 in a neighborhood of s_0 .

Fix $k \in \{1, \dots, m\}$ and set

$$Z := \partial_{s_k} \varphi, \quad d_j := \partial_{s_k} c_j.$$

Differentiating the weak formulation of $\mathcal{F}(s, \varphi(s), c(s)) = 0$ with respect to s_k gives a weak equation for Z . By the Proposition 5.20 and elliptic regularity, this equation

can be written as

$$-\Delta Z - W_1(\xi'(s))Z - \sum_{j=1}^m d_j \chi_j(\xi'(s))Z_{1j}(\xi'(s)) = \partial_{s_k}(W_1(\xi'(s)))\varphi + c_k \partial_{s_k}(\chi_k(\xi'(s))Z_{1k}(\xi'(s)))$$

in Ω_ε , and

$$\frac{\partial Z}{\partial n} - W_2(\xi'(s))Z + \varepsilon l_1(V(\xi'(s)))[Z] = \partial_{s_k}(W_2(\xi'(s)))\varphi - \varepsilon \partial_{s_k}(l_1(V(\xi'(s))))[\varphi]$$

on $\partial\Omega_\varepsilon$.

Since χ_j and Z_{1j} depend on s only through the j -th point $\xi'_j = \eta_j(s_j)$, we have

$$\int_{\Omega_\varepsilon} \chi_j(\xi'(s))Z_{1j}(\xi'(s))Z dy = 0, \quad j \neq k,$$

whereas

$$\int_{\Omega_\varepsilon} \chi_k(\xi'(s))Z_{1k}(\xi'(s))Z dy = - \int_{\Omega_\varepsilon} \partial_{s_k}(\chi_k(\xi'(s))Z_{1k}(\xi'(s)))\varphi dy.$$

Set

$$b_k := \frac{\int_{\Omega_\varepsilon} \partial_{s_k}(\chi_k(\xi'(s))Z_{1k}(\xi'(s)))\varphi dy}{\int_{\Omega_\varepsilon} \chi_k(\xi'(s))^2 Z_{1k}(\xi'(s))^2 dy},$$

and define

$$\tilde{Z} := Z + b_k \chi_k(\xi'(s))Z_{1k}(\xi'(s)).$$

Since the supports of the functions χ_j are disjoint for ε sufficiently small, the definition of b_k gives

$$\int_{\Omega_\varepsilon} \chi_j(\xi'(s))Z_{1j}(\xi'(s))\tilde{Z} dy = 0, \quad j = 1, \dots, m.$$

Moreover, using the equation and boundary condition satisfied by $\chi_k Z_{1k}$, we see that $(\tilde{Z}, d)$ solves (5.88) with right-hand sides

$$\tilde{f} = \partial_{s_k}(W_1)\varphi + c_k \partial_{s_k}(\chi_k Z_{1k}) + b_k(-\Delta(\chi_k Z_{1k}) - W_1 \chi_k Z_{1k}),$$

and

$$\tilde{g} = \partial_{s_k}(W_2)\varphi - \varepsilon \partial_{s_k}(l_1(V(\xi'(s))))[\varphi] + b_k \left(\frac{\partial(\chi_k Z_{1k})}{\partial n} - W_2 \chi_k Z_{1k} + \varepsilon l_1(V(\xi'(s)))[\chi_k Z_{1k}] \right).$$

The estimates already proved for φ , c_j , and the functions $\chi_j Z_{1j}$ give

$$|b_k| \leq C \log\left(\frac{1}{\varepsilon}\right) (\|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}),$$

and

$$\|\tilde{f}\|_{**, \Omega_\varepsilon} + \|\tilde{g}\|_{*, \partial\Omega_\varepsilon} \leq C \log\left(\frac{1}{\varepsilon}\right) (\|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}).$$

Applying Proposition 5.20 to $(\tilde{Z}, d)$, we obtain

$$\|\tilde{Z}\|_{L^\infty(\Omega_\varepsilon)} \leq C \left(\log \frac{1}{\varepsilon}\right)^2 (\|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}),$$

and

$$|d_j| \leq C \log\left(\frac{1}{\varepsilon}\right) (\|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}).$$

Finally, since

$$Z = \tilde{Z} - b_k \chi_k(\xi'(s)) Z_{1k}(\xi'(s))$$

and $\chi_k Z_{1k}$ is uniformly bounded, we conclude that

$$\|\partial_{s_k} \varphi\|_{L^\infty(\Omega_\varepsilon)} \leq C \left(\log \frac{1}{\varepsilon}\right)^2 (\|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}).$$

5.2.3 The Non-Linear Problem

We first convert the pointwise Lemma above into estimates in the weighted norms.

We can assume $\alpha < 1 - \sigma$.

Lemma 5.25.

$$\|R_1\|_{**, \Omega_\varepsilon} \leq C \varepsilon^\alpha, \quad \|R_2\|_{*, \partial\Omega_\varepsilon} \leq C \varepsilon^\alpha.$$

Proof. In $B_{\delta/\varepsilon}(\xi'_j)$, by the Lemma 5.18 R_1 is $O((1 + |y - \xi'_j|)^{-4})$, so

$$|R_1(y)| \leq C \varepsilon (1 + |y - \xi'_j|)^{-3} + C \varepsilon^\alpha (1 + |y - \xi'_j|)^{-4}.$$

Dividing by $\varepsilon^2 + \sum_i (1 + |y - \xi'_i|)^{-2-\sigma} \geq (1 + |y - \xi'_j|)^{-2-\sigma}$,

$$\frac{\varepsilon (1 + |y - \xi'_j|)^{-3}}{(1 + |y - \xi'_j|)^{-2-\sigma}} = \varepsilon (1 + |y - \xi'_j|)^{\sigma-1} = O(\varepsilon),$$

$$\frac{\varepsilon^\alpha (1 + |y - \xi'_j|)^{-4}}{(1 + |y - \xi'_j|)^{-2-\sigma}} = \varepsilon^\alpha (1 + |y - \xi'_j|)^{\sigma-2} = O(\varepsilon^\alpha).$$

Outside $\bigcup_j B_{\delta/\varepsilon}(\xi'_j)$, $|R_1| = O(\varepsilon^4)$, and dividing by $\varepsilon^2 + \sum_i (1 + |y - \xi'_i|)^{-2-\sigma}$ gives $O(\varepsilon^2)$. Combining, $\|R_1\|_{**, \Omega_\varepsilon} = O(\varepsilon^\alpha)$.

Similarly, in $B_{\delta/\varepsilon}(\xi'_j) \cap \partial\Omega_\varepsilon$, R_2 is $O((1 + |y - \xi'_j|)^{-2})$, so

$$|R_2(y)| \leq C \varepsilon (1 + |y - \xi'_j|)^{-1} + C \varepsilon^\alpha (1 + |y - \xi'_j|)^{-2} + C \varepsilon^{1+\alpha}.$$

Dividing by the weight $\varepsilon + \sum_i (1 + |y - \xi'_i|)^{-1-\sigma} \geq (1 + |y - \xi'_j|)^{-1-\sigma}$,

$$\frac{\varepsilon (1 + |y - \xi'_j|)^{-1}}{(1 + |y - \xi'_j|)^{-1-\sigma}} = \varepsilon (1 + |y - \xi'_j|)^\sigma = O(\varepsilon^{1-\sigma}),$$

$$\frac{\varepsilon^\alpha (1 + |y - \xi'_j|)^{-2}}{(1 + |y - \xi'_j|)^{-1-\sigma}} = \varepsilon^\alpha (1 + |y - \xi'_j|)^{\sigma-1} = O(\varepsilon^\alpha).$$

$O(\varepsilon^{1+\alpha})$ divided by the weight gives $O(\varepsilon^\alpha)$. Outside, $|R_2| = O(\varepsilon^{1+\alpha})$, dividing again by the weight gives $O(\varepsilon^\alpha)$. Combining, $\|R_2\|_{*,\partial\Omega_\varepsilon} = O(\varepsilon^\alpha)$. $\square$

Now we bound the norm of the non-linear terms.

Lemma 5.26. *There is $C > 0$, independent of ε , such that for $\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq 1$,*

$$\|N_1(\phi)\|_{**, \Omega_\varepsilon} \leq C \|\phi\|_{L^\infty(\Omega_\varepsilon)}^2, \quad \|N_2(\phi)\|_{*, \partial\Omega_\varepsilon} \leq C \|\phi\|_{L^\infty(\Omega_\varepsilon)}^2,$$

and, for $\|\phi_1\|_{L^\infty(\Omega_\varepsilon)}, \|\phi_2\|_{L^\infty(\Omega_\varepsilon)} \leq 1$,

$$\|N_1(\phi_1) - N_1(\phi_2)\|_{**, \Omega_\varepsilon} \leq C \left(\|\phi_1\|_{L^\infty(\Omega_\varepsilon)} + \|\phi_2\|_{L^\infty(\Omega_\varepsilon)} \right) \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)},$$

and likewise for N_2 .

Proof. Since $|e^{2t} - 1 - 2t| \leq Ct^2$ and $|e^t - 1 - t| \leq Ct^2$ for $|t| \leq 1$, $|N_1(\phi)| \leq CW_1\phi^2$ and $|W_2(e^\phi - 1 - \phi)| \leq CW_2\phi^2$ pointwise. Since $W_1(y) = O((1 + |y - \xi'_j|)^{-4})$ and the weight $\varepsilon^2 + \sum (1 + |y - \xi'_j|)^{-2-\sigma} \geq (1 + |y - \xi'_j|)^{-2-\sigma}$, the ratio is $O((1 + |y - \xi'_j|)^{\sigma-2}) = O(1)$; likewise for W_2 and the boundary weight. Hence $\|N_1(\phi)\|_{**} \leq C\|\phi\|_\infty^2$ and $\|W_2(e^\phi - 1 - \phi)\|_* \leq C\|\phi\|_\infty^2$.

For $\varepsilon N_l(\phi)$: since $|e^{2\phi} - 1 - 2\phi| \leq C\phi^2$, $|e^\phi - 1 - \phi| \leq C\phi^2$,

$$|N_l(\phi)| \leq \frac{C\|\phi\|_\infty^2}{|\partial\Omega|} \left(\int_{\Omega_\varepsilon} W_1 + \int_{\partial\Omega_\varepsilon} W_2 \right) = O(\|\phi\|_\infty^2),$$

Since $\varepsilon N_l(\phi)$ is constant in y , dividing by the floor ε of the boundary weight gives $\|\varepsilon N_l(\phi)\|_* = |N_l(\phi)| = O(\|\phi\|_\infty^2)$. Combining, $\|N_2(\phi)\|_* \leq C\|\phi\|_\infty^2$.

For the Lipschitz estimates, since $\frac{d}{dt}(e^{2t} - 1 - 2t) = 2(e^{2t} - 1) = O(t)$ for $|t| \leq 1$,

$$\left| (e^{2\phi_1} - 1 - 2\phi_1) - (e^{2\phi_2} - 1 - 2\phi_2) \right| \leq C(|\phi_1| + |\phi_2|)|\phi_1 - \phi_2|$$

pointwise, and the same weighted-norm computation as above gives the claim for N_1 ; the same argument, together with the analogous bound for $N_l(\phi_1) - N_l(\phi_2)$, gives the claim for N_2 . $\square$

We seek $u = V + \phi$ solving (5.58), which amounts to finding $\phi, c_1, \dots, c_m$ solving

$$\begin{cases} -\Delta\phi - W_1\phi = R_1 + N_1(\phi) + \sum_{j=1}^m c_j \chi_j Z_{1j} & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial n} - W_2\phi + \varepsilon l_1(V)[\phi] = R_2 + N_2(\phi) & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{1j} \phi = 0, & j = 1, \dots, m. \end{cases} \quad (5.121)$$

Let $T(f, g)$ denote the $L^\infty(\Omega_\varepsilon)$ -solution given by Proposition 5.20 for data (f, g) , with associated coefficients $c_j(f, g)$ satisfying, by linearity of the linear theory,

$$\begin{aligned} \|T(f, g)\|_{L^\infty(\Omega_\varepsilon)} &\leq C \log\left(\frac{1}{\varepsilon}\right) (\|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}), \\ \sum_j |c_j(f, g)| &\leq C (\|f\|_{**, \Omega_\varepsilon} + \|g\|_{*, \partial\Omega_\varepsilon}). \end{aligned}$$

Problem (5.121) is then equivalent to the fixed point equation

$$\phi = A(\phi) := T(R_1 + N_1(\phi), R_2 + N_2(\phi)). \quad (5.122)$$

Lemma 5.27. *There exist $\varepsilon_0 > 0$, $C > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and any $\xi_1, \dots, \xi_m \in \partial\Omega$ satisfying (5.95), problem (5.121) admits a unique solution $\phi, c_1, \dots, c_m$ with*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^\alpha |\log \varepsilon|, \quad (5.123)$$

where $\alpha \in (0, 1)$. Moreover, the map $\xi' \mapsto \phi(\xi') \in C(\overline{\Omega_\varepsilon})$ is C^1 , and

$$\left\| \frac{\partial\phi}{\partial\xi'_k} \right\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^\alpha |\log \varepsilon|^2, \quad k = 1, \dots, m. \quad (5.124)$$

Proof. Let C_0 be a fixed constant, to be chosen below, and set

$$\mathcal{F} := \left\{ \phi \in C(\overline{\Omega_\varepsilon}) : \|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C_0\varepsilon^\alpha |\log \varepsilon| \right\}.$$

By Lemma 5.25 and Lemma 5.26, for $\phi \in \mathcal{F}$ (so that $\|\phi\|_\infty \rightarrow 0$ as $\varepsilon \rightarrow 0$, hence $\|\phi\|_\infty \leq 1$ for ε small),

$$\|A(\phi)\|_{L^\infty(\Omega_\varepsilon)} \leq C \log\left(\frac{1}{\varepsilon}\right) \left(\varepsilon^\alpha + C_0^2 \varepsilon^{2\alpha} |\log \varepsilon|^2 \right) = C\varepsilon^\alpha |\log \varepsilon| \left(1 + C_0^2 \varepsilon^\alpha |\log \varepsilon|^3 \right).$$

Since $\varepsilon^\alpha |\log \varepsilon|^3 \rightarrow 0$, choosing $C_0 \geq 2C$ and ε small enough that $C_0^2 \varepsilon^\alpha |\log \varepsilon|^3 \leq 1$, the right-hand side is $\leq C_0 \varepsilon^\alpha |\log \varepsilon|$, so $A(\mathcal{F}) \subset \mathcal{F}$. Similarly, for $\phi_1, \phi_2 \in \mathcal{F}$, Lemma 5.26 gives

$$\|A(\phi_1) - A(\phi_2)\|_{L^\infty(\Omega_\varepsilon)} \leq C \log\left(\frac{1}{\varepsilon}\right) (\|\phi_1\|_\infty + \|\phi_2\|_\infty) \|\phi_1 - \phi_2\|_\infty \leq 2CC_0 \varepsilon^\alpha |\log \varepsilon|^2 \|\phi_1 - \phi_2\|_\infty,$$

which is $\leq \frac{1}{2} \|\phi_1 - \phi_2\|_\infty$ for ε small enough. Hence A is a contraction on $\mathcal{F}$, and the Banach fixed point theorem gives a unique $\phi \in \mathcal{F}$ solving (5.122), satisfying (5.123). The coefficients $c_j := c_j(R_1 + N_1(\phi), R_2 + N_2(\phi))$ then solve (5.121) together with ϕ ; uniqueness of $(\phi, c_1, \dots, c_m)$ among solutions with $\phi \in \mathcal{F}$ follows from uniqueness of the fixed point.

Now we want to discuss the differentiability of the map $\xi' \mapsto \phi(\xi')$: $\phi(\xi') \in \mathcal{F}$ is the unique fixed point of

$$A(\phi; \xi') := T(R_1(\xi') + N_1(\phi; \xi'), R_2(\xi') + N_2(\phi; \xi'); \xi'),$$

where $T(\cdot, \cdot; \xi')$ denotes the solution operator given by Proposition 5.20.

Formally, setting

$$Z = \partial_{\xi'_k} \phi,$$

and differentiating the fixed point equation $\phi = A(\phi; \xi')$, we obtain

$$Z = D_\phi A(\phi; \xi')[Z] + D_{\xi'_k} A(\phi; \xi'),$$

where

$$D_\phi A(\phi; \xi')[Z] = T\left(\frac{\partial N_1}{\partial \phi}(\phi)Z, \frac{\partial N_2}{\partial \phi}(\phi)Z; \xi'\right),$$

and

$$D_{\xi'_k} A(\phi; \xi') = \left[\partial_{\xi'_k} T\right](R_1 + N_1(\phi), R_2 + N_2(\phi); \xi') + T\left(\partial_{\xi'_k} R_1 + \frac{\partial N_1}{\partial \xi'_k}, \partial_{\xi'_k} R_2 + \frac{\partial N_2}{\partial \xi'_k}; \xi'\right).$$

We first estimate $D_\phi A$. By Lemma 5.26 and Proposition 5.20,

$$\|D_\phi A(\phi; \xi')[Z]\|_{L^\infty(\Omega_\varepsilon)} \leq C \log\left(\frac{1}{\varepsilon}\right) \|\phi\|_\infty \|Z\|_\infty.$$

Using

$$\|\phi\|_\infty = O(\varepsilon^\alpha |\log \varepsilon|),$$

from (5.123), we deduce

$$\|D_\phi A(\phi; \xi')[Z]\|_{L^\infty(\Omega_\varepsilon)} = O(\varepsilon^\alpha |\log \varepsilon|^2) \|Z\|_\infty.$$

Hence, for ε sufficiently small,

$$\|D_\phi A(\phi; \xi')\| < \frac{1}{2},$$

and therefore $I - D_\phi A(\phi; \xi')$ is invertible on $C(\overline{\Omega_\varepsilon})$.

We next estimate $D_{\xi'_k} A$. By Lemmas 5.25 and 5.26,

$$\|R_1 + N_1(\phi)\|_{**}, \|R_2 + N_2(\phi)\|_* = O(\varepsilon^\alpha).$$

Therefore, using the estimate for $\partial_{\xi'_k} T$ established above,

$$\|[\partial_{\xi'_k} T](R_1 + N_1(\phi), R_2 + N_2(\phi); \xi')\|_{L^\infty(\Omega_\varepsilon)} \leq C \left(\log \frac{1}{\varepsilon}\right)^2 \varepsilon^\alpha.$$

For the second term, we have

$$\|\partial_{\xi'_k} R_i\|_{**} = O(\varepsilon^\alpha),$$

while

$$\frac{\partial N_i}{\partial \xi'_k} = O((\partial_{\xi'_k} W_i) \|\phi\|_\infty^2),$$

using the estimates for $\partial_{\xi'_k} W_i$. Applying Proposition 5.20 once again yields

$$\left\| T\left(\partial_{\xi'_k} R_1 + \frac{\partial N_1}{\partial \xi'_k}, \partial_{\xi'_k} R_2 + \frac{\partial N_2}{\partial \xi'_k}; \xi'\right) \right\|_{L^\infty(\Omega_\varepsilon)} \leq C \log\left(\frac{1}{\varepsilon}\right) \varepsilon^\alpha.$$

It follows that

$$\|D_{\xi'_k} A(\phi; \xi')\|_{L^\infty(\Omega_\varepsilon)} \leq C \left(\log \frac{1}{\varepsilon}\right)^2 \varepsilon^\alpha. \quad (5.125)$$

Since $(I - D_\phi A(\phi; \xi'))^{-1}$ is bounded with norm at most 2, the equation satisfied by Z gives

$$\|Z\|_{L^\infty(\Omega_\varepsilon)} \leq 2 \|D_{\xi'_k} A(\phi; \xi')\|_{L^\infty(\Omega_\varepsilon)} \leq C \varepsilon^\alpha \left(\log \frac{1}{\varepsilon}\right)^2,$$

which proves (5.124).

We now provide a rigorous justification of the previous formal argument. Let $\xi'^0 = (\xi_1'^0, \dots, \xi_m'^0) \in (\partial\Omega_\varepsilon)^m$ satisfy the rescaled separation condition. For each j , let $\eta_j : U_j \subset \mathbb{R} \longrightarrow \partial\Omega_\varepsilon$ be an arc-length parametrization of $\partial\Omega_\varepsilon$ in a neighborhood

of $\xi_j'^0$, with $\eta_j(0) = \xi_j'^0$. Setting

$$\xi_j' = \eta_j(s_j), \quad s = (s_1, \dots, s_m) \in U_1 \times \dots \times U_m,$$

defines local coordinates on $(\partial\Omega_\varepsilon)^m$ near ξ'^0 .

Consider the map

$$G(s, \phi) = \phi - A(\phi; \eta(s)).$$

The differentiability of the linear operator T with respect to ξ' , together with the smooth dependence of R_1 , R_2 , N_1 and N_2 on $\xi' = \eta(s)$, implies that G is of class C^1 in a neighborhood of $(0, \phi(\xi'^0))$ as a map from $\mathbb{R}^m \times \mathcal{F}$ into $C(\overline{\Omega_\varepsilon})$.

Since

$$D_\phi G(0, \phi(\xi'^0)) = I - D_\phi A(\phi(\xi'^0); \xi'^0)$$

is invertible, the implicit function theorem yields a unique C^1 map

$$s \mapsto \widehat{\phi}(s)$$

defined for s sufficiently close to the origin and satisfying

$$G(s, \widehat{\phi}(s)) = 0.$$

By uniqueness of the fixed point in $\mathcal{F}$, it follows that

$$\widehat{\phi}(s) = \phi(\eta(s)).$$

Hence ϕ is of class C^1 with respect to the local coordinates. Since the parametrizations η_j have unit speed, we obtain

$$\partial_{\xi_k'} \phi = \partial_{s_k} \widehat{\phi}.$$

So, estimate (5.124) follows. □

5.2.4 Finite-Dimensional Reduction

As established in the previous sections, for any given configuration of points $\xi' = (\xi'_1, \dots, \xi'_m) \in (\partial\Omega_\varepsilon)^m$ well separated from each other, there exists a unique $\varphi = \varphi_{\xi'} \in H^1(\Omega_\varepsilon)$ and unique constants $c_k = c_k(\xi')$, for $k = 1, \dots, m$, such that $u = V + \varphi$

solves the projected system:

$$\begin{cases} -\Delta u = K(\varepsilon y)e^{2u} + \sum_{j=1}^m c_j \chi_j Z_{1j} & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial n} + \varepsilon l(u) = h(\varepsilon y)e^u & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{1j} \varphi dy = 0 & \text{for } j = 1, \dots, m. \end{cases} \quad (5.126)$$

So, we have to find $\xi' = (\xi'_1, \dots, \xi'_m)$ such that $c_k(\xi') = 0$ for all $k = 1, \dots, m$.

Proposition 5.28. *For every $k = 1, \dots, m$, ε sufficiently small and, choosing $\alpha > \frac{1}{2}$, we have*

$$2A_k c_k = -2\pi \varepsilon a_k(\xi) + o(\varepsilon)$$

where

$$A_k := \int_{\mathbb{R}_+^2} \chi(|x|) z_{1k}(x)^2 dx.$$

and

$$\begin{aligned} a_k(\xi) := & \left(2\pi \partial_{\tau_k}^{(1)} H(\xi_k, \xi_k) + 2\pi \sum_{i \neq k} \partial_{\tau_k}^{(1)} G(\xi_k, \xi_i) + \frac{2 \partial_{\tau_k} h(\xi_k)}{\sqrt{K(\xi_k) + h(\xi_k)^2}} \right. \\ & \left. + \frac{2 \partial_{\tau_k} K(\xi_k)}{K(\xi_k)} \left(\frac{1}{\mathfrak{D}(\xi_k) (\sqrt{1 + \mathfrak{D}(\xi_k)^2})} - \frac{1}{\mathfrak{D}(\xi_k)^2} \log \left(1 + \frac{\mathfrak{D}(\xi_k)}{\sqrt{1 + \mathfrak{D}(\xi_k)^2}} \right) \right) \right) \end{aligned}$$

Proof. We multiply the first equation in (5.126) by $\partial_{\xi'_k} V$ and integrate over Ω_ε . Since $\partial_{\xi'_k} V = 2Z_{1k} + O(\varepsilon)$ on the support of χ_k and is uniformly $O(\varepsilon)$ on the support of χ_j for $j \neq k$, we have

$$\sum_{j=1}^m c_j \int_{\Omega_\varepsilon} \chi_j Z_{1j} \partial_{\xi'_k} V dy = c_k \int_{\Omega_\varepsilon} \chi_k Z_{1k} (2Z_{1k}) dy + o(\varepsilon) = 2A_k c_k + o(\varepsilon).$$

Writing

$$K(\varepsilon y)e^{2(V+\varphi)} = K(\varepsilon y)e^{2V} + W_1 \varphi + N_1(\varphi),$$

$$h(\varepsilon y)e^{V+\varphi} = h(\varepsilon y)e^V + W_2 \varphi + h e^V (e^\phi - \phi - 1),$$

and $l(V + \varphi) = l(V) + l_1(V)[\varphi] + N_l(\varphi)$. Using the definitions of the errors $R_1 = \Delta V + K(\varepsilon y)e^{2V}$ and $R_2 = -\partial_n V + h(\varepsilon y)e^V - \varepsilon l(V)$ and integrating by parts, we have

$$\begin{aligned} 2A_k c_k = & - \int_{\Omega_\varepsilon} R_1 \partial_{\xi'_k} V dy - \int_{\partial\Omega_\varepsilon} R_2 \partial_{\xi'_k} V ds + \int_{\Omega_\varepsilon} (-\Delta(\partial_{\xi'_k} V) - 2K(\varepsilon y)e^{2V} \partial_{\xi'_k} V) \varphi dy \\ & + \int_{\partial\Omega_\varepsilon} (\partial_n(\partial_{\xi'_k} V) - h(\varepsilon y)e^V \partial_{\xi'_k} V) \phi ds + \int_{\partial\Omega_\varepsilon} \varepsilon l_1(V)[\varphi] \partial_{\xi'_k} V ds \end{aligned}$$

$$\begin{aligned}
& - \int_{\Omega_\varepsilon} N_1(\varphi) \partial_{\xi'_k} V \, dy - \int_{\partial\Omega_\varepsilon} N_2(\varphi) \partial_{\xi'_k} V \, ds + o(\varepsilon) = - \int_{\Omega_\varepsilon} R_1 \partial_{\xi'_k} V \, dy - \int_{\partial\Omega_\varepsilon} R_2 \partial_{\xi'_k} V \, ds \\
& \quad - \int_{\Omega_\varepsilon} \phi \partial_{\xi'_k} R_1 - \int_{\partial\Omega_\varepsilon} \phi \partial_{\xi'_k} R_2 + \int_{\partial\Omega_\varepsilon} \varepsilon l_1(V) [\varphi] \partial_{\xi'_k} V \, ds - \int_{\partial\Omega_\varepsilon} \varepsilon l_1(V) [\partial_{\xi'_k} V] \varphi \, ds \\
& - \int_{\Omega_\varepsilon} N_1(\varphi) \partial_{\xi'_k} V \, dy - \int_{\partial\Omega_\varepsilon} N_2(\varphi) \partial_{\xi'_k} V \, ds + o(\varepsilon) = - \int_{\Omega_\varepsilon} R_1 \partial_{\xi'_k} V \, dy - \int_{\partial\Omega_\varepsilon} R_2 \partial_{\xi'_k} V \, ds + o(\varepsilon).
\end{aligned}$$

Where in the last equality, we used the following estimates

$$\begin{aligned}
& \left| \int_{\Omega_\varepsilon} \varphi(\partial_{\xi'_k} R_1) \, dy \right| \leq C \|\varphi\|_{L^\infty} \|\partial_{\xi'_k} R_1\|_{**} = O(\varepsilon^{2\alpha} |\log \varepsilon|), \\
& \left| \int_{\partial\Omega_\varepsilon} \varphi(\partial_{\xi'_k} R_2) \, ds \right| \leq C \|\varphi\|_{L^\infty} \|\partial_{\xi'_k} R_2\|_* = O(\varepsilon^{2\alpha} |\log \varepsilon|), \\
& \left| \int_{\partial\Omega_\varepsilon} \varepsilon l_1(V) [\varphi] \partial_{\xi'_k} V \, ds \right| = \varepsilon |l_1(V) [\varphi]| \left| \int_{\partial\Omega_\varepsilon} \partial_{\xi'_k} V \, ds \right| \leq C \varepsilon \|\varphi\|_{L^\infty} = O(\varepsilon^{1+\alpha} |\log \varepsilon|), \\
& \left| \int_{\partial\Omega_\varepsilon} \varepsilon l_1(V) [\partial_{\xi'_k} V] \varphi \, ds \right| = \varepsilon |l_1(V) [\partial_{\xi'_k} V]| \left| \int_{\partial\Omega_\varepsilon} \varphi \, ds \right| \leq \varepsilon O(\varepsilon^\alpha) (|\partial\Omega_\varepsilon| \|\varphi\|_{L^\infty}) = O(\varepsilon^{2\alpha} |\log \varepsilon|), \\
& \left| \int_{\Omega_\varepsilon} N_1(\varphi) \partial_{\xi'_k} V \, dy \right| \leq C \|\varphi\|_{L^\infty}^2 \int_{\Omega_\varepsilon} e^{2V} |\partial_{\xi'_k} V| \, dy = O(\varepsilon^{2\alpha} |\log \varepsilon|^2), \\
& \left| \int_{\partial\Omega_\varepsilon} N_2(\varphi) \partial_{\xi'_k} V \, ds \right| \leq C \|\varphi\|_{L^\infty}^2 \int_{\partial\Omega_\varepsilon} e^V |\partial_{\xi'_k} V| \, ds = O(\varepsilon^{2\alpha} |\log \varepsilon|^2).
\end{aligned}$$

Therefore

$$\begin{aligned}
2A_k c_k &= - \int_{\Omega_\varepsilon} R_1 \partial_{\xi'_k} V \, dy - \int_{\partial\Omega_\varepsilon} R_2 \partial_{\xi'_k} V \, ds + o(\varepsilon) = \\
& - \int_{\Omega_\varepsilon \cap B_{\frac{\delta}{\varepsilon}}(\xi'_k)} R_1 \partial_{\xi'_k} V \, dy - \int_{\partial\Omega_\varepsilon \cap B_{\frac{\delta}{\varepsilon}}(\xi'_k)} R_2 \partial_{\xi'_k} V \, ds + o(\varepsilon).
\end{aligned}$$

Where, in the last equality, we used that $\partial_{\xi'_k} V = O(\varepsilon)$ outside a ball $B_{\frac{\delta}{\varepsilon}}(\xi'_k)$ and the error expansion in Lemma 5.18. Instead inside the ball $B_{\frac{\delta}{\varepsilon}}(\xi'_k)$, using the same expansions of the proof of Lemma 5.18, the change of variables $x = F_\varepsilon(y)$, the fact that $\partial_{\xi'_k} V = 2z_{1k} + O(\varepsilon)$. Since z_{1k} is odd in x_1 , only the tangential component of the first-order expansion contributes. So, we get

$$\begin{aligned}
- \int_{\Omega_\varepsilon} R_1 \partial_{\xi'_k} V &= -16\varepsilon \lambda_k^2 \left(\partial_\tau H_k(\xi_k) + \sum_{i \neq k} \partial_\tau (U_i + H_i)(\xi_k) \right) \int_{\mathbb{R}_+^2} \frac{x_1^2}{(x_1^2 + (x_2 + \mathfrak{D}(\xi_k) \lambda_k)^2 + \lambda_k^2)^3} dx + o(\varepsilon) \\
&= -2\pi\varepsilon \left(\partial_\tau H_k(\xi_k) + \sum_{i \neq k} \partial_\tau (U_i + H_i)(\xi_k) \right) \left(1 - \frac{\mathfrak{D}(\xi_k)}{\sqrt{1 + \mathfrak{D}(\xi_k)^2}} \right) + o(\varepsilon)
\end{aligned}$$

and

$$\begin{aligned}
-\int_{\partial\Omega_\varepsilon} R_2 \partial_{\xi'_k} V &= -\frac{4h(\xi_k)\lambda_k\varepsilon}{\sqrt{K(\xi_k)}} \left(\partial_\tau H_k(\xi_k) + \sum_{i \neq k} \partial_\tau (U_i + H_i)(\xi_k) \right) \int_{\mathbb{R}} \frac{x_1^2}{(x_1^2 + \lambda_k^2(1 + \mathfrak{D}(\xi_k)^2))^2} dx_1 + o(\varepsilon) \\
&= -2\pi\varepsilon \left(\partial_\tau H_k(\xi_k) + \sum_{i \neq k} \partial_\tau (U_i + H_i)(\xi_k) \right) \frac{\mathfrak{D}(\xi_k)}{\sqrt{1 + \mathfrak{D}(\xi_k)^2}} + o(\varepsilon)
\end{aligned}$$

Thus

$$2A_k c_k = -2\pi\varepsilon \left(\partial_\tau H_k(\xi_k) + \sum_{i \neq k} \partial_\tau (U_i + H_i)(\xi_k) \right) + o(\varepsilon).$$

We now compute $\partial_\tau H_k(\xi_k) + \sum_{i \neq k} \partial_\tau (U_i + H_i)(\xi_k)$ explicitly.

The terms $i \neq k$. Differentiating the explicit formula for U_i and using $|\xi_i - \xi_k| > 0$ fixed, we get, as $\varepsilon \rightarrow 0$,

$$\partial_\tau U_i(\xi_k) \rightarrow -\frac{2\tau_k \cdot (\xi_k - \xi_i)}{|\xi_k - \xi_i|^2}.$$

Since $|\xi_i - \xi_k|$ is bounded below, the difference between H_i and $2\pi H(\cdot, \xi_i) - \log(2\lambda_i) + \frac{1}{2} \log K(\xi_i)$ has interior source term $O(\varepsilon^2)$ and boundary datum $O(\varepsilon)$ in a fixed neighborhood of ξ_k . Standard local C^1 estimates then give

$$\partial_\tau H_i(\xi_k) = 2\pi \partial_{\tau_k}^{(1)} H(\xi_k, \xi_i) + o(1).$$

Using $H(x, y) = G(x, y) + \frac{1}{2\pi} \log(|x - y|^2)$, i.e. $\nabla_1 H(x, y) = \nabla_1 G(x, y) + \frac{x - y}{\pi|x - y|^2}$, we obtain

$$2\pi \partial_{\tau_k}^{(1)} H(\xi_k, \xi_i) - \frac{2\tau_k \cdot (\xi_k - \xi_i)}{|\xi_k - \xi_i|^2} = 2\pi \partial_{\tau_k}^{(1)} G(\xi_k, \xi_i),$$

and therefore

$$\sum_{i \neq k} \partial_\tau (U_i + H_i)(\xi_k) = 2\pi \sum_{i \neq k} \partial_{\tau_k}^{(1)} G(\xi_k, \xi_i) + o(1).$$

The term $\partial_\tau H_k(\xi_k)$. Recall from the proof of Lemma 5.17 that

$$z_\varepsilon(x) = H_k(x) + \log(2\lambda_k) - \frac{1}{2} \log K(\xi_k) - 2\pi H(x, \xi_k),$$

hence

$$\partial_\tau H_k(\xi_k) = \partial_\tau z_\varepsilon(\xi_k) + 2\pi \partial_{\tau_k}^{(1)} H(\xi_k, \xi_k).$$

By (5.68), z_ε solves $-\Delta z_\varepsilon = \varepsilon^2(K(x) - K(\xi_k))e^{2U_k(x)}$ in Ω , $\frac{\partial z_\varepsilon}{\partial n} = g_\varepsilon$ on $\partial\Omega$, with

$g_\varepsilon(x) = \frac{\partial H_k}{\partial n}(x) - \frac{2(x-\xi_k) \cdot n(x)}{|x-\xi_k|^2} + \frac{2\pi}{|\partial\Omega|}$. So, Green's representation formula gives

$$\partial_\tau z_\varepsilon(\xi_k) = \int_\Omega \tau_k \cdot \nabla_x G(\xi_k, y) \varepsilon^2 (K(y) - K(\xi_k)) e^{2U_k(y)} dy + \int_{\partial\Omega} \tau_k \cdot \nabla_x G(\xi_k, y) g_\varepsilon(y) ds_y.$$

Since $G(x, y) = H(x, y) - \frac{1}{2\pi} \log(|x - y|^2)$, we have $\nabla_x G(x, y) = \nabla_x H(x, y) - \frac{x-y}{\pi|x-y|^2}$.

Let's compute the boundary integral. Parametrize $\partial\Omega$ by arc length, $y = \gamma(s)$, $\xi_k = \gamma(0)$, $\tau_k = \gamma'(0)$. Since $\gamma(0) - \gamma(s) = -s\tau_k + O(s^2)$,

$$\tau_k \cdot \nabla_x G(\xi_k, \gamma(s)) = \tau_k \cdot \nabla_1 H(\xi_k, \gamma(s)) + \frac{1}{\pi s} + O(1), \quad s \rightarrow 0,$$

and $\tau_k \cdot \nabla_1 H(\xi_k, \gamma(s))$ is bounded; by the same split as in the proof of Lemma 5.17, this regular part contributes $O(\varepsilon)$ once paired with g_ε .

Substituting $s = \varepsilon\sigma$, using the explicit formula for $\frac{\partial H_k}{\partial n}$ given in the proof of Lemma 5.17, using $\tau' = \kappa n$, $n' = -\kappa\tau$ for γ, n at $s = 0$, and $\mathfrak{D}(\xi_k) = \frac{h(\xi_k)}{\sqrt{K(\xi_k)}}$, we obtain collecting the terms of order ε^0 in σ ,

$$g_\varepsilon(\gamma(\varepsilon\sigma)) = \frac{\kappa(\xi_k)\lambda_k^2(1 + \mathfrak{D}(\xi_k)^2)}{\sigma^2 + \lambda_k^2(1 + \mathfrak{D}(\xi_k)^2)} + \frac{2\lambda_k \partial_\tau h(\xi_k) \sigma}{\sqrt{K(\xi_k)}(\sigma^2 + \lambda_k^2(1 + \mathfrak{D}(\xi_k)^2))} + o(1).$$

The first term is even in σ , so paired with $\frac{1}{\pi\varepsilon\sigma}$, it integrates to 0. To integrate the second term multiplied by $\frac{1}{\pi\varepsilon\sigma}$, we use that $\int_{-\infty}^{\infty} \frac{d\sigma}{\sigma^2 + a^2} = \frac{\pi}{a}$ with $a = \lambda_k \sqrt{1 + \mathfrak{D}(\xi_k)^2}$, thus

$$\int_{\partial\Omega} \tau_k \cdot \nabla_x G(\xi_k, y) g_\varepsilon(y) ds_y = \frac{2 \partial_\tau h(\xi_k)}{\sqrt{K(\xi_k)} \sqrt{1 + \mathfrak{D}(\xi_k)^2}} + o(1) = \frac{2 \partial_\tau h(\xi_k)}{\sqrt{K(\xi_k) + h(\xi_k)^2}} + o(1).$$

Now we pass to the interior integral. Substitute $y = \xi_k + \varepsilon z$, $dy = \varepsilon^2 dz$. Since $\xi_k - y = -\varepsilon z$,

$$\tau_k \cdot \nabla_x G(\xi_k, \xi_k + \varepsilon z) = \tau_k \cdot \nabla_1 H(\xi_k, \xi_k + \varepsilon z) + \frac{\tau_k \cdot z}{\pi\varepsilon|z|^2}.$$

By the same computation as for e^{2U_j} in the proof of Lemma 5.17,

$$e^{2U_k(\xi_k + \varepsilon z)} = \frac{4\lambda_k^2/K(\xi_k)}{\varepsilon^4(\lambda_k^2 + |z - \mathfrak{D}(\xi_k)\lambda_k n(\xi_k)|^2)^2},$$

and $K(\xi_k + \varepsilon z) - K(\xi_k) = \varepsilon z \cdot \nabla K(\xi_k) + O(\varepsilon^2|z|^2)$. Multiplying the singular part of the kernel by the leading order of the source and by $dy = \varepsilon^2 dz$, the powers of ε cancel out, while pairing the regular part of the kernel, or the higher order part of

the source, with the remaining factors gives $O(\varepsilon)$; hence

$$\int_{\Omega} \tau_k \cdot \nabla_x G(\xi_k, y) \varepsilon^2 (K(y) - K(\xi_k)) e^{2U_k(y)} dy =$$

$$\int_{\mathbb{R}_+^2} \frac{\tau_k \cdot z}{\pi |z|^2} (z \cdot \nabla K(\xi_k)) \frac{4\lambda_k^2 / K(\xi_k)}{(\lambda_k^2 + |z - \mathfrak{D}(\xi_k) \lambda_k n(\xi_k)|^2)^2} dz + o(1).$$

Writing $\nabla K(\xi_k) = \partial_\tau K(\xi_k) \tau_k + \partial_n K(\xi_k) n(\xi_k)$ and $z = z_1 \tau_k + z_2 (-n(\xi_k))$: the piece with $\partial_n K(\xi_k)$ is odd in z_1 and its integral vanishes; the piece with $\partial_\tau K(\xi_k)$ gives

$$\frac{\partial_\tau K(\xi_k)}{\pi K(\xi_k)} \int_{\mathbb{R}_+^2} \frac{4\lambda_k^2 z_1^2}{(z_1^2 + z_2^2)(\lambda_k^2 + z_1^2 + (z_2 + \mathfrak{D}(\xi_k) \lambda_k)^2)^2} dz$$

$$= \frac{2 \partial_\tau K(\xi_k)}{K(\xi_k)} \left[\frac{1}{\mathfrak{D}(\xi_k) \sqrt{1 + \mathfrak{D}(\xi_k)^2}} - \frac{1}{\mathfrak{D}(\xi_k)^2} \log \left(1 + \frac{\mathfrak{D}(\xi_k)}{\sqrt{1 + \mathfrak{D}(\xi_k)^2}} \right) \right].$$

Combining the boundary and interior integrals,

$$\partial_\tau z_\varepsilon(\xi_k) = \frac{2 \partial_\tau h(\xi_k)}{\sqrt{K(\xi_k) + h(\xi_k)^2}} +$$

$$\frac{2 \partial_\tau K(\xi_k)}{K(\xi_k)} \left[\frac{1}{\mathfrak{D}(\xi_k) \sqrt{1 + \mathfrak{D}(\xi_k)^2}} - \frac{1}{\mathfrak{D}(\xi_k)^2} \log \left(1 + \frac{\mathfrak{D}(\xi_k)}{\sqrt{1 + \mathfrak{D}(\xi_k)^2}} \right) \right] + o(1),$$

hence

$$\partial_\tau H_k(\xi_k) = 2\pi \partial_{\tau_k}^{(1)} H(\xi_k, \xi_k) + \frac{2 \partial_\tau h(\xi_k)}{\sqrt{K(\xi_k) + h(\xi_k)^2}}$$

$$+ \frac{2 \partial_\tau K(\xi_k)}{K(\xi_k)} \left[\frac{1}{\mathfrak{D}(\xi_k) \sqrt{1 + \mathfrak{D}(\xi_k)^2}} - \frac{1}{\mathfrak{D}(\xi_k)^2} \log \left(1 + \frac{\mathfrak{D}(\xi_k)}{\sqrt{1 + \mathfrak{D}(\xi_k)^2}} \right) \right] + o(1).$$

Adding this to $\sum_{i \neq k} \partial_\tau (U_i + H_i)(\xi_k) = 2\pi \sum_{i \neq k} \partial_{\tau_k}^{(1)} G(\xi_k, \xi_i) + o(1)$ and substituting into

$$2A_k c_k = -2\pi \varepsilon \left(\partial_\tau H_k(\xi_k) + \sum_{i \neq k} \partial_\tau (U_i + H_i)(\xi_k) \right) + o(\varepsilon)$$

gives the desired expansion. $\square$

5.2.5 The Existence Theorem

Now we define $D := \bigcup_{1 \leq i < j \leq m} \{\xi = (\xi_1, \dots, \xi_m) \in (\partial\Omega)^m : \xi_i = \xi_j\}$ and we consider $a(\xi) = (a_1(\xi), \dots, a_m(\xi))$, where a_k is defined in the Proposition 5.28.

Definition 5.29. Let $a : X \rightarrow \mathbb{R}^m$ be continuous and let $\xi^0 \in X$ satisfy $a(\xi^0) = 0$.

We say that ξ^0 is a stable zero of a if, for every neighborhood U of ξ^0 with $\overline{U} \subset X$, there exists $\delta > 0$ such that every continuous map $\tilde{a} : \overline{U} \rightarrow \mathbb{R}^m$ satisfying

$$\|\tilde{a} - a\|_{C^0(\overline{U})} < \delta$$

has a zero in U .

Theorem 5.30. *Let $\Omega \subset \mathbb{R}^2$ be smooth and bounded and let $K, h \in C^2(\overline{\Omega})$ satisfy $K, h > 0$. Let $m \geq 1$, and let*

$$\xi^0 = (\xi_1^0, \dots, \xi_m^0) \in (\partial\Omega)^m \setminus D$$

be a stable zero of a . Then there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, problem (5.57) admits a solution v_ε concentrating at m boundary points $\xi_1(\varepsilon), \dots, \xi_m(\varepsilon) \in \partial\Omega$, with

$$\xi_k(\varepsilon) \longrightarrow \xi_k^0, \quad k = 1, \dots, m.$$

Proof. Let ξ^0 be as in the statement. Since $\xi^0 \in (\partial\Omega)^m \setminus D$, there exists a neighborhood U of ξ^0 such that

$$\overline{U} \subset (\partial\Omega)^m \setminus D.$$

By Proposition 5.28,

$$c_k(\xi/\varepsilon) = -\frac{\pi\varepsilon}{A_k(\xi)}a_k(\xi) + o(\varepsilon),$$

uniformly for $\xi \in \overline{U}$. Define

$$\widehat{c}_\varepsilon(\xi) := \frac{1}{\varepsilon} \left(c_1(\xi/\varepsilon), \dots, c_m(\xi/\varepsilon) \right).$$

Then

$$\widehat{c}_\varepsilon(\xi) \longrightarrow M(\xi)a(\xi) \quad \text{uniformly on } \overline{U},$$

where

$$M(\xi) := -\pi \operatorname{diag} \left(\frac{1}{A_1(\xi)}, \dots, \frac{1}{A_m(\xi)} \right).$$

Since $A_k(\xi) > 0$, the matrix $M(\xi)$ is invertible for every $\xi \in \overline{U}$. Moreover, M^{-1} is continuous on $\overline{U}$. Hence the maps

$$\tilde{a}_\varepsilon(\xi) := M(\xi)^{-1}\widehat{c}_\varepsilon(\xi)$$

are continuous on $\overline{U}$ and satisfy

$$\tilde{a}_\varepsilon \longrightarrow a \quad \text{uniformly on } \overline{U}.$$

Since ξ^0 is a stable zero of a , there exists $\delta_U > 0$ such that every continuous map $\tilde{a} : \overline{U} \rightarrow \mathbb{R}^m$ satisfying $\|\tilde{a} - a\|_{C^0(\overline{U})} < \delta_U$ has a zero in U . Since $\tilde{a}_\varepsilon \rightarrow a$ uniformly on $\overline{U}$, for ε sufficiently small this applies to $\tilde{a}_\varepsilon$, so the set

$$Z_\varepsilon := \{\xi \in \overline{U} : \tilde{a}_\varepsilon(\xi) = 0\}$$

is non-empty; being closed in the compact set $\overline{U}$, it is compact. Choose $\xi(\varepsilon) \in Z_\varepsilon$ with $|\xi(\varepsilon) - \xi^0| = \min_{\xi \in Z_\varepsilon} |\xi - \xi^0|$.

We claim that $\xi(\varepsilon) \rightarrow \xi^0$. Let $V \subset U$ be any neighborhood of ξ^0 with $\overline{V} \subset U$. By the stability of ξ^0 , there exists $\delta_V > 0$ such that every continuous map $\tilde{a} : \overline{V} \rightarrow \mathbb{R}^m$ satisfying $\|\tilde{a} - a\|_{C^0(\overline{V})} < \delta_V$ has a zero in V . Since $\tilde{a}_\varepsilon \rightarrow a$ uniformly on $\overline{U} \supset \overline{V}$, for ε sufficiently small there exists $\eta(\varepsilon) \in V$ with $\tilde{a}_\varepsilon(\eta(\varepsilon)) = 0$; thus $\eta(\varepsilon) \in Z_\varepsilon$, and by minimality of $\xi(\varepsilon)$,

$$|\xi(\varepsilon) - \xi^0| \leq |\eta(\varepsilon) - \xi^0|.$$

Since V can be taken arbitrarily small around ξ^0 , it follows that $\xi(\varepsilon) \rightarrow \xi^0$.

Since $\xi(\varepsilon) \in Z_\varepsilon$, we have $\tilde{a}_\varepsilon(\xi(\varepsilon)) = 0$, and since $M(\xi(\varepsilon))$ is invertible,

$$\widehat{c}_\varepsilon(\xi(\varepsilon)) = 0, \quad \text{i.e.} \quad c_k(\xi(\varepsilon)/\varepsilon) = 0, \quad k = 1, \dots, m.$$

Finally, setting

$$\xi'(\varepsilon) = \frac{\xi(\varepsilon)}{\varepsilon},$$

we have

$$c_k(\xi'(\varepsilon)) = 0, \quad k = 1, \dots, m.$$

Hence the solution

$$u_{\xi'(\varepsilon)} = V_{\xi'(\varepsilon)} + \varphi_{\xi'(\varepsilon)}$$

of the projected problem solves the rescaled problem. Scaling back to the original variables, we set

$$v_\varepsilon(x) = u_{\xi'(\varepsilon)}(x/\varepsilon) - 2 \log \varepsilon.$$

Then v_ε solves (5.57) and concentrates at the boundary points

$$\xi_1(\varepsilon), \dots, \xi_m(\varepsilon),$$

which satisfy

$$\xi_k(\varepsilon) \longrightarrow \xi_k^0, \quad k = 1, \dots, m.$$

□

Remark 5.31. A non-degenerate zero of a is a stable zero. More generally, an isolated zero with non-zero local degree is stable under sufficiently small C^0 perturbations.

Remark 5.32. There are cases in which the reduced vector field a admits a zero in $(\partial\Omega)^m \setminus D$. Suppose, for instance, that

$$K \equiv K_0 > 0 \quad \text{on } \partial\Omega.$$

Then the last term in the definition of a_k vanishes and

$$a_k(\xi) = 2\pi \partial_{\tau_k}^{(1)} H(\xi_k, \xi_k) + 2\pi \sum_{i \neq k} \partial_{\tau_k}^{(1)} G(\xi_k, \xi_i) + \frac{2\partial_{\tau_k} h(\xi_k)}{\sqrt{K_0 + h(\xi_k)^2}}.$$

In this case a is the tangential gradient of the function

$$\begin{aligned} \Phi(\xi) := & \pi \sum_{j=1}^m H(\xi_j, \xi_j) + 2\pi \sum_{1 \leq i < j \leq m} G(\xi_i, \xi_j) \\ & + 2 \sum_{j=1}^m \log \left(h(\xi_j) + \sqrt{K_0 + h(\xi_j)^2} \right), \end{aligned}$$

namely

$$a_k(\xi) = \partial_{\tau_k} \Phi(\xi), \quad k = 1, \dots, m.$$

Moreover, since

$$G(x, y) = -\frac{1}{2\pi} \log |x - y|^2 + H(x, y),$$

we have

$$\Phi(\xi) \longrightarrow +\infty \quad \text{as } \xi \rightarrow D.$$

Fix a connected component Γ of $\partial\Omega$. Then there exists $d > 0$ such that

$$\Phi(\xi) > \Phi(\bar{\xi}) \quad \text{whenever} \quad \min_{i \neq j} |\xi_i - \xi_j| < d,$$

for some fixed $\bar{\xi} \in \Gamma^m \setminus D$. Therefore Φ attains its minimum on the compact set

$$\{\xi \in \Gamma^m : |\xi_i - \xi_j| \geq d \text{ for every } i \neq j\},$$

and such a minimum ξ^0 belongs to $\Gamma^m \setminus D$.

Consequently,

$$a(\xi^0) = 0.$$

Moreover, for a generic choice of h , the function Φ is Morse. Indeed, view $a = a(\xi; h)$ as depending on the parameter h in the Banach space $C_+^2(\partial\Omega)$ of positive functions. Since $a_j(\xi; h)$ depends on h only through $h(\xi_j)$ and $\partial_{\tau_j} h(\xi_j)$, for any zero (ξ, h) of a and any index k a perturbation of h supported near ξ_k alone, with $\delta h(\xi_k) = 0$ and $\partial_{\tau_k}(\delta h)(\xi_k) = 1$, changes only the k -th component of a , by $\frac{2}{\sqrt{K_0 + h(\xi_k)^2}} \neq 0$; varying $k = 1, \dots, m$ shows $D_h a(\xi; h) : C_+^2(\partial\Omega) \rightarrow \mathbb{R}^m$ is onto. Hence 0 is a regular value of $a(\cdot; \cdot)$, and by the Sard–Smale theorem it is a regular value of $a(\cdot; h)$. Hence all critical points of Φ are non-degenerate. In particular, the minimum ξ^0 obtained above is a non-degenerate critical point, and therefore is a stable zero of a , so we can apply Theorem 5.30.

Appendix A

Useful Tools and Computations

A.1 Test functions

Consider a point $p \in \partial\Sigma$. Let $q = p + q_2 n(p)$, where n denotes the outward unit normal to $\partial\Sigma$, and choose $q_2 > 0$ small enough so that q lies in a smooth extension of Σ . For a parameter μ satisfying $\mu d(x, q) > 1$ for every $x \in \Sigma$, define

$$\varphi_{\mu,p} : \Sigma \rightarrow \mathbb{R}, \quad \varphi_{\mu,p}(x) = \log \frac{4\mu^2}{(\mu^2 d^2(x, q) - 1)^2},$$

and

$$\tilde{\varphi}_{\mu,p} : \Sigma \rightarrow \mathbb{R}, \quad \tilde{\varphi}_{\mu,p}(x) = \varphi_{\mu,p}(x) - \log |K(x)|. \quad (\text{A.1})$$

Lemma A.1. *Let $p \in \partial\Sigma$ satisfy $\mathfrak{D}(p) > 1$, and let I be the functional defined in (3.3). With $\tilde{\varphi}_{\mu,p}$ as in (A.1), one has*

$$I(\tilde{\varphi}_{\mu,p}) \rightarrow -\infty, \quad \oint_{\partial\Sigma} e^{\frac{\tilde{\varphi}_{\mu,p}}{2}} \rightarrow +\infty \quad \text{as } \mu \rightarrow \frac{1}{d(p, q)} = \frac{1}{q_2}.$$

Proof. Observe that

$$I(\tilde{\varphi}_{\mu,p}) = \int_{\Sigma} \left(\frac{1}{2} |\nabla \tilde{\varphi}_{\mu,p}|^2 + 2\tilde{K} \tilde{\varphi}_{\mu,p} + 2e^{\varphi_{\mu,p}} \right) - 4 \oint_{\partial\Sigma} \mathfrak{D} e^{\varphi_{\mu,p}/2}.$$

Fix $\varepsilon > 0$. By Young's inequality the gradient term may be estimated as

$$\frac{1}{2} \int_{\Sigma} |\nabla \tilde{\varphi}_{\mu,p}|^2 \leq \left(\frac{1}{2} + \varepsilon \right) \int_{\Sigma} |\nabla \varphi_{\mu,p}|^2 + O(1).$$

We treat the remaining contributions separately and claim the following estimates:

$$\int_{\Sigma} |\nabla \varphi_{\mu,p}|^2 \leq \frac{8\pi}{\sqrt{\mu^2 q_2^2 - 1}} + o\left(\frac{1}{\sqrt{\mu^2 q_2^2 - 1}}\right), \quad (\text{A.2})$$

$$\int_{\Sigma} e^{\varphi_{\mu,p}} \leq \frac{2\pi\mu q_2}{\sqrt{\mu^2 q_2^2 - 1}} + o\left(\frac{1}{\sqrt{\mu^2 q_2^2 - 1}}\right), \quad (\text{A.3})$$

$$\oint_{\partial\Sigma} \mathfrak{D} e^{\varphi_{\mu,p}/2} \geq \min_{B_p(r) \cap \partial\Sigma} \mathfrak{D} \frac{2\pi}{\sqrt{\mu^2 q_2^2 - 1}} + o\left(\frac{1}{\sqrt{\mu^2 q_2^2 - 1}}\right), \quad (\text{A.4})$$

$$\int_{\Sigma} \tilde{K} \tilde{\varphi}_{\mu,p} \leq O(1). \quad (\text{A.5})$$

From these estimates the conclusion of the lemma follows immediately.

Proof of (A.2).

Fix $r > q_2$ and split the integral as

$$\int_{\Sigma} |\nabla \varphi_{\mu,p}|^2 dV_g = \int_{\Sigma \setminus B_q(r)} |\nabla \varphi_{\mu,p}|^2 dV_g + \int_{B_q(r) \cap \Sigma} |\nabla \varphi_{\mu,p}|^2 dV_g.$$

Using the inequality $|\nabla d(x, z)|^2 \leq 2d(x, z)$, we obtain the pointwise bound

$$|\nabla \varphi_{\mu,p}(x)| = 2\mu^2 \frac{|\nabla d^2(x, q)|}{\mu^2 d^2(x, q) - 1} \leq 4\mu^2 \frac{d(x, q)}{\mu^2 d^2(x, q) - 1} \quad \text{for all } x \in \Sigma.$$

Working in normal coordinates centered at q , and using

$$dV_g = (1 + o_r(1)) dx, \quad d(q, x) = |x - q| \quad \text{for } x \in B_q(r) \cap \Sigma,$$

one finds, for r small,

$$\int_{B_q(r) \cap \Sigma} |\nabla \varphi_{\mu,p}|^2 dV_g \leq 16\mu^4 \int_{B_q^+(r)} (1 + o_r(1)) \frac{|x - q|^2 dx}{(\mu^2 |x - q|^2 - 1)^2},$$

where $B_q^+(r) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 + q_2)^2 < r^2, x_2 > 0\}$.

Write

$$(A) := \int_{B_q^+(r)} \frac{\mu^4 |x - q|^2}{(\mu^2 |x - q|^2 - 1)^2} dx = \int_{B_q^+(r)} \frac{\mu^2 (1 + o_r(1))}{(\mu^2 |x - q|^2 - 1)^2} dx.$$

Passing to polar coordinates centered at $(0, -q_2)$, with (ρ, θ) , and using the

change of variables $t = \mu^2 \rho^2 - 1$, we get

$$\begin{aligned}
\int_{B_q^+(r)} \frac{\mu^2 dx}{(\mu^2 |x - q|^2 - 1)^2} &= 2 \int_{\arcsin(\frac{q_2}{r})}^{\pi/2} \int_{\frac{q_2}{\sin \theta}}^r \frac{\mu^2 \rho}{(\mu^2 \rho^2 - 1)^2} d\rho d\theta \\
&= \int_{\arcsin(\frac{q_2}{r})}^{\pi/2} \int_{\frac{\mu^2 q_2^2}{\sin^2 \theta} - 1}^{\mu^2 r^2 - 1} \frac{dt}{t^2} d\theta \\
&= \int_{\arcsin(\frac{q_2}{r})}^{\pi/2} \frac{\sin^2 \theta}{\mu^2 q_2^2 \sin^2 \theta - \sin^2 \theta} d\theta + O(1) \\
&= \frac{\mu q_2}{\sqrt{\mu^2 q_2^2 - 1}} \left[\arctan \left(\frac{\sqrt{\mu^2 q_2^2 - 1}}{\mu q_2} \tan \theta \right) \right]_{\arcsin(\frac{q_2}{r})}^{\pi/2} + O(1) \\
&= \frac{\mu \pi q_2}{2\sqrt{\mu^2 q_2^2 - 1}} + O(1).
\end{aligned} \tag{A.6}$$

On the complement $\Sigma \setminus B_q(r)$ we have the uniform bound

$$|\nabla \varphi_{\mu,p}(x)| \leq \frac{\text{diam}(\Sigma)}{\mu^2 r^2 - 1} \quad \text{for all } x \in \Sigma \setminus B_q(r),$$

and since $\mu r > \mu q_2 > 1$ this yields

$$\int_{\Sigma \setminus B_q(r)} |\nabla \varphi_{\mu,p}|^2 = O(1). \tag{A.7}$$

Combining (A.6) and (A.7) proves (A.2).

Proof of (A.3).

Split

$$\int_{\Sigma} e^{\varphi_{\mu,p}} = \int_{B_q(r) \cap \Sigma} e^{\varphi_{\mu,p}} + \int_{\Sigma \setminus B_q(r)} e^{\varphi_{\mu,p}}.$$

In normal coordinates

$$\int_{B_q^+(r)} e^{\varphi_{\mu,p}} dx = 4 \int_{B_q^+(r)} \frac{\mu^2 dx}{(\mu^2 |x - q|^2 - 1)^2} = 4(A),$$

where (A) is defined above.

Moreover, for $x \in \Sigma \setminus B_q(r)$ we have the uniform bound

$$e^{\varphi_{\mu,p}} \leq \frac{4\mu^2}{(\mu^2 r^2 - 1)^2},$$

hence

$$\int_{\Sigma \setminus B_q(r)} e^{\varphi_{\mu,p}} = O(1). \tag{A.8}$$

Using (A.6) and (A.8) yields (A.3).

Proof of (A.4).

Write

$$\oint_{\partial\Sigma} \mathfrak{D} e^{\varphi_{\mu,p}/2} = \oint_{B_p(r) \cap \partial\Sigma} \mathfrak{D} e^{\varphi_{\mu,p}/2} + \oint_{\partial\Sigma \setminus B_p(r)} \mathfrak{D} e^{\varphi_{\mu,p}/2},$$

and estimate the first term from below as

$$\oint_{B_p(r) \cap \partial\Sigma} \mathfrak{D} e^{\varphi_{\mu,p}/2} \geq \min_{B_p(r) \cap \partial\Sigma} \mathfrak{D}(x) \oint_{B_p(r) \cap \partial\Sigma} e^{\varphi_{\mu,p}/2}.$$

Taking r sufficiently small and using normal coordinates along the boundary, we compute

$$\oint_{B_p(r) \cap \partial\Sigma} e^{\varphi_{\mu,p}/2} = 2 \int_0^r \frac{\mu dx_1}{\mu^2(x_1^2 + q_2^2) - 1} = \frac{\pi}{\sqrt{\mu^2 q_2^2 - 1}} + O(1). \quad (\text{A.9})$$

On $\partial\Sigma \setminus B_p(r)$ the function $e^{\varphi_{\mu,p}/2}$ is uniformly bounded, so

$$\left| \oint_{\partial\Sigma \setminus B_p(r)} \mathfrak{D} e^{\varphi_{\mu,p}/2} \right| \leq O(1). \quad (\text{A.10})$$

Combining (A.9) and (A.10) establishes (A.4).

Proof of (A.5).

By the definition of $\tilde{\varphi}_{\mu,p}$ one checks directly that

$$\int_{\Sigma} \tilde{K} \tilde{\varphi}_{\mu,p} \leq -C \int_{\Sigma} \varphi_{\mu,p} + O(1).$$

Finally, since the pointwise inequality

$$\log \frac{4\mu^2}{(\mu^2 q_2^2 - 1)^2} \leq \varphi_{\mu,p}(x) \quad \text{holds for all } x \in \Sigma,$$

the estimate (A.5) follows. □

A.2 An Harnack type inequality

Lemma A.2. *Let $f \in L^p(B_r^+)$ for some $1 < p \leq +\infty$ and $g \in L^q(\partial B_r^+ \cap \{t = 0\})$ for some $1 < q \leq +\infty$. Suppose that u satisfies*

$$\begin{cases} -\Delta u = f & \text{in } B_r^+, \\ \frac{\partial u}{\partial t} = g & \text{on } \partial B_r^+ \cap \{t = 0\}, \\ u \leq 0 & \text{on } \partial B_r^+ \cap \{t > 0\}. \end{cases}$$

Then, for any $0 < \theta < 1$, there exist a constant $\beta \in (0, 1)$ depending only on r, θ , and a constant $\gamma > 0$ depending only on r, p, q , such that

$$\sup_{B_{\theta r}^+} u \leq \beta \inf_{B_{\theta r}^+} u + (1 + \beta)\gamma \left(\|f\|_{L^p(B_r^+)} + \|g\|_{L^q(\partial B_r^+ \cap \{t=0\})} \right).$$

Proof. Let w solve

$$\begin{cases} -\Delta w = f & \text{in } B_r^+, \\ \frac{\partial w}{\partial t} = g & \text{on } \partial B_r^+ \cap \{t = 0\}, \\ w = 0 & \text{on } \partial B_r^+ \cap \{t > 0\}, \end{cases}$$

and set $v = w - u$. Then v satisfies

$$\begin{cases} -\Delta v = 0 & \text{in } B_r^+, \\ \frac{\partial v}{\partial t} = 0 & \text{on } \partial B_r^+ \cap \{t = 0\}, \\ v \geq 0 & \text{on } \partial B_r^+ \cap \{t > 0\}. \end{cases}$$

By the maximum principle and Hopf's lemma, $v \geq 0$ in B_r^+ . By even reflection across $\{t = 0\}$, v becomes a harmonic function in B_r with $v \geq 0$. Hence, by the Harnack inequality for nonnegative harmonic functions, for any $0 < \theta < 1$ there exists $\beta \in (0, 1)$, depending only on r, θ , such that

$$\sup_{B_{\theta r}^+} v \leq \frac{1}{\beta} \inf_{B_{\theta r}^+} v. \tag{A.11}$$

Next, decompose $w = w_1 + w_2$, where w_1 and w_2 solve respectively

$$\begin{cases} -\Delta w_1 = f & \text{in } B_r^+, \\ \frac{\partial w_1}{\partial t} = 0 & \text{on } \partial B_r^+ \cap \{t = 0\}, \\ w_1 = 0 & \text{on } \partial B_r^+ \cap \{t > 0\}, \end{cases} \quad \begin{cases} -\Delta w_2 = 0 & \text{in } B_r^+, \\ \frac{\partial w_2}{\partial t} = g & \text{on } \partial B_r^+ \cap \{t = 0\}, \\ w_2 = 0 & \text{on } \partial B_r^+ \cap \{t > 0\}. \end{cases}$$

By even extension, w_1 can be regarded as the solution of a Poisson problem in B_r , and standard L^p -estimates yield

$$\sup_{B_r^+} |w_1| \leq \gamma_1 \|f\|_{L^p(B_r^+)} \quad (\text{A.12})$$

for some constant γ_1 depending only on r, p .

For w_2 , define for $x \in B_r^+$

$$\varphi(x) = \frac{1}{2\pi} \int_{\partial B_r^+ \cap \{t=0\}} \left(\log \frac{2r}{|y-x|} + \log \frac{2r}{|y-\bar{x}|} \right) |g(y)| dy,$$

where $\bar{x}$ denotes the reflection of x about $\{t = 0\}$. Then $\varphi \geq 0$ in B_r^+ and satisfies

$$\begin{cases} -\Delta \varphi = 0 & \text{in } B_r^+, \\ \frac{\partial \varphi}{\partial t} = -|g| & \text{on } \partial B_r^+ \cap \{t = 0\}. \end{cases}$$

It is clear that

$$\sup_{B_r^+} \varphi \leq \gamma_2 \|g\|_{L^q(\partial B_r^+ \cap \{t=0\})} \quad (\text{A.13})$$

for some γ_2 depending only on r, q .

Now consider

$$\begin{cases} -\Delta(w_2 - \varphi) = 0 & \text{in } B_r^+, \\ \frac{\partial(w_2 - \varphi)}{\partial t} = g + |g| & \text{on } \partial B_r^+ \cap \{t = 0\}, \\ w_2 - \varphi = 0 & \text{on } \partial B_r^+ \cap \{t > 0\}. \end{cases}$$

By the maximum principle and Hopf's lemma, $w_2 \leq \varphi$ in B_r^+ . A similar argument applied to $w_2 + \varphi$ gives $w_2 \geq -\varphi$, hence

$$|w_2| \leq \varphi \quad \text{in } B_r^+, \quad \text{and thus} \quad \sup_{B_r^+} |w_2| \leq \gamma_2 \|g\|_{L^q(\partial B_r^+ \cap \{t=0\})}. \quad (\text{A.14})$$

Combining (A.12) and (A.14), we obtain

$$\sup_{B_r^+} |w| \leq \gamma \left(\|f\|_{L^p(B_r^+)} + \|g\|_{L^q(\partial B_r^+ \cap \{t=0\})} \right), \quad (\text{A.15})$$

for some $\gamma > 0$ depending only on r, p, q .

Finally, since $u = w - v$, from (A.11) and (A.15) it follows that

$$\sup_{B_{\theta r}^+} u \leq \beta \inf_{B_{\theta r}^+} u + (1 + \beta) \gamma \left(\|f\|_{L^p(B_r^+)} + \|g\|_{L^q(\partial B_r^+ \cap \{t=0\})} \right),$$

which completes the proof. $\square$

A.3 Integrals and Taylor Expansions for the disk case

Here we only state some technical propositions; their proofs can be found in the appendices of [4].

Proposition A.3. *Let $\xi \in \partial\mathbb{D}$. For any $z \in \mathbb{D}$ one has*

$$|f(z) - \xi| = O \left(\frac{\delta}{\delta + |z + \xi|} \right), \quad (\text{A.16})$$

and in particular

$$\|f(z) - \xi\|_{L^p(\mathbb{D})} = \begin{cases} O(\delta) & 1 \leq p < 2, \\ O\left(\delta \sqrt{\log \frac{1}{\delta}}\right) & p = 2, \\ O\left(\delta^{\frac{2}{p}}\right) & p > 2, \end{cases} \quad |f(z) - \xi|_{L^p(\partial\mathbb{D})} = \begin{cases} O\left(\delta \log \frac{1}{\delta}\right) & p = 1, \\ O\left(\delta^{\frac{1}{p}}\right) & p > 1. \end{cases}$$

Moreover, if $z \in \partial\mathbb{D}$ and $h \in C^2(\partial\mathbb{D})$, then

$$h(f(z)) - h(\xi) = \delta h'(\xi) \Theta(z) + O \left(\frac{\delta^2}{(\delta + |z + \xi|)^2} \right),$$

with

$$\Theta(z) = \Theta_{\delta, \xi}(z) := \frac{2 \langle z, \xi^\perp \rangle}{1 + (1 - \delta)^2 + 2(1 - \delta) \langle z, \xi \rangle}.$$

Proposition A.4. *Given $\xi \in \partial\mathbb{D}$,*

$$\int_{\partial\mathbb{D}} (h(f(z)) - h(\xi)) dz = -2\pi \delta (-\Delta)^{\frac{1}{2}} h(\xi) + O(\delta^2).$$

Proposition A.5. *The following identities hold true:*

1.

$$\int_{\mathbb{D}} \frac{(1-\delta)|w|^2 - 1 + \delta w_1}{|1 + (1-\delta)w|^2} \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^3} w_1 dw = \frac{2\pi}{(\varphi(\xi)^2 + K(\xi))^2} + O(\delta).$$

2.

$$\int_{\mathbb{D}} \frac{(2-\delta)w_2}{|1 + (1-\delta)w|^2} \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^3} w_1 dw = 0.$$

3.

$$\int_{\mathbb{D}} \frac{(1-\delta)|w|^2 - 1 + \delta w_1}{|1 + (1-\delta)w|^2} \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^3} w_2 dw = 0.$$

4.

$$\int_{\mathbb{D}} \frac{(2-\delta)w_2}{|1 + (1-\delta)w|^2} \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^3} w_2 dw = \frac{2\pi}{(\varphi(\xi)^2 + K(\xi))^2} + O(\delta).$$

Proposition A.6. *The following identities hold true:*

1.

$$\int_{\Omega_\delta} \frac{dw}{(1+w_1)^2 + w_2^2} = \pi \log \frac{1}{\delta} + O(1).$$

2.

$$\int_{\Omega_\delta} \frac{(1+w_1)^2 - w_2^2}{((1+w_1)^2 + w_2^2)^2} dw = O(1).$$

3.

$$\int_{\Omega_\delta} \frac{(1+w_1)w_2}{((1+w_1)^2 + w_2^2)^2} dw = 0.$$

where

$$\Omega_\delta := \left\{ \left| w - \frac{1-\delta}{\delta} \right| \leq \frac{1-\delta}{\delta} \right\}.$$

Proposition A.7. *The following identities hold true:*

1.

$$\int_{\partial\mathbb{D}} (h(f(z)) - h(\xi)) \langle z, \xi \rangle dz = 2\pi\delta(-\Delta)^{\frac{1}{2}}h(\xi) + O(\delta^2).$$

2.

$$\int_{\partial\mathbb{D}} (h(f(z)) - h(\xi)) \langle z, \xi^\perp \rangle dz = 2\pi\delta h'(\xi) + O(\delta^2).$$

Proposition A.8. *The following identities hold true:*

1.

$$\int_{\partial\mathbb{D}} \log |z + \xi| (h(f(z)) - h(\xi)) \langle z, \xi \rangle dz = -2\pi\delta \log \frac{1}{\delta} (-\Delta)^{\frac{1}{2}} h(\xi) + O(\delta).$$

2.

$$\int_{\partial\mathbb{D}} \arctan \frac{\langle z, \xi^\perp \rangle}{1 + \langle z, \xi \rangle} (h(f(z)) - h(\xi)) \langle z, \xi \rangle dz = -2\pi\delta \log \frac{1}{\delta} h'(\xi) + O(\delta).$$

3.

$$\int_{\partial\mathbb{D}} \log |z + \xi| (h(f(z)) - h(\xi)) \langle z, \xi^\perp \rangle dz = O(\delta).$$

4.

$$\int_{\partial\mathbb{D}} \arctan \frac{\langle z, \xi^\perp \rangle}{1 + \langle z, \xi \rangle} (h(f(z)) - h(\xi)) \langle z, \xi^\perp \rangle dz = O(\delta).$$

Proposition A.9. *The following identities hold true:*

1.

$$\int_{\mathbb{D}} e^{V(z)} dz = \frac{4\pi}{\varphi(\xi)^2 + K(\xi)}.$$

2.

$$\int_{\partial\mathbb{D}} \log |z - w| dz = 0 \quad \text{for every } w \in \partial\mathbb{D}.$$

Proposition A.10. *The following identities hold true:*

1.

$$\int_{\mathbb{D}} \frac{(1 - \delta)|w|^2 - 1 + \delta w_1}{|1 + (1 - \delta)w|^2} \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^2} dw = -\frac{4\pi}{\varphi(\xi)^2 + K(\xi)}.$$

2.

$$\int_{\mathbb{D}} \frac{(2 - \delta)w_2}{|1 + (1 - \delta)w|^2} \frac{4\varphi(\xi)^2}{(\varphi(\xi)^2 + K(\xi)|w|^2)^3} dw = 0.$$

Proposition A.11. *The following identities hold true:*

1.

$$2K(\xi) \int_{\mathbb{D}} e^V \mathcal{Z}_1^2 = 2K(\xi) \int_{\mathbb{D}} e^V \mathcal{Z}_2^2 = \pi \frac{2K(\xi) (3\varphi(\xi)^2 + K(\xi))}{3\varphi(\xi)^2 (\varphi(\xi)^2 + K(\xi))^3}.$$

2.

$$\frac{h(\xi)2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}} \mathcal{Z}_1^2 = \frac{h(\xi)2\varphi(\xi)}{\varphi(\xi)^2 + K(\xi)} \int_{\partial\mathbb{D}} \mathcal{Z}_2^2 = \pi \frac{\varphi(\xi)^2 - K(\xi)}{(\varphi(\xi)^2 + K(\xi))^3}.$$

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