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A variational approach to scalar field equations and Choquard-type equations

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Abstract

The thesis starts from the study of scalar field equations on $\mathbb{R}^N$. The interesting question is to overcome the evident lack of compactness due to the fact that we are not working on bounded domains. We are looking for both ground-states and bound-states solutions, considering both subcritical and critical nonlinearities. In particular, we give original results about multiplicity of solutions in critical case. A wide-studied generalization is the Choquard-type equation. In this case, the significative technical problem that arise is the presence of a nonlocal term, which is the convolution with Riesz's potential.

Contents

Notations

We will make use of the following notations in the whole thesis:

- $\mathbb{N}$ denotes the set of natural numbers, not including 0; we define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
- $\mathbb{R}$ denotes the set of real numbers, while $\mathbb{R}^+$ denotes the set of positive real numbers;
- for $N \in \mathbb{N}$, $\mathbb{R}^N$ denotes the N-dimensional euclidean space;
- given any pair of vectors $x = (x_1, ..., x_N)$ and $y = (y_1, ..., y_N)$ in $\mathbb{R}^N$, we denote the scalar product $x \cdot y := \sum_{i=1}^N x_i y_i$;
- given $x := (x_1, ..., x_N) \in \mathbb{R}^N$ with $x_i \in \mathbb{R}$ for $i = 1, ..., N$, we denote its norm: $|x| := \sqrt{x \cdot x}$;
- given $x := (x_1, ..., x_N) \in \mathbb{R}^N$ with $x_i \in \mathbb{R} \quad \forall\, 1 \leq i \leq N$, we denote its 1-norm: $|x|_1 := \sum_{i=1}^N |x_i|$;
- $\mathbb{C}$ denotes the set of complex numbers; $i \in \mathbb{C}$ denotes the imaginary unit;
- given a V a $\mathbb{R}$-vector space, we denote $\dim(V)$ as its dimension;
- given V a $\mathbb{R}$-vector space equipped with a scalar product $< \cdot, \cdot >$, let us consider W a subspace of V. We denote $W^\perp$ as its orthogonal complement defined by

$$W^\perp = \{v \in V : \quad < v, w >= 0 \quad \forall w \in W\};$$

- given $R > 0$ and $x_0 \in \mathbb{R}^N$, we denote the open ball of radius R centered in x_0: $B_R(x_0) := \{x \in \mathbb{R}^N; |x - x_0| < R\}$. When $x_0 = 0 \in \mathbb{R}^N$, we denote $B_R(0) := B_R$, Furthermore, we denote $\mathbb{S}^{N-1}$ the unit sphere in $\mathbb{R}^N$ with $N \geq 2$ and its measure as ω_N. For the sake of completeness, $\mathbb{S}^0 = \{\pm 1\}$;
- given E a Lebesgue measurable set on $\mathbb{R}^N$, we denote $|E|$ as its Lebesgue measure;
- given two indices i, j, we denote the Kronecker-delta as $\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$;

- given $x \in \mathbb{R}$, we denote its sign as $\text{sign}(x)$, where $\text{sign}(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$;

- given $x > 0$, we denote $\log x$ as its logarithm to base e, where e is the Neper's number;

- given a function u, we denote u^+ as its positive part and u^- as its negative part, respectively $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := \max\{-u(x), 0\}$;

- given a continuous function u, its classical support is denoted by $\text{supp}(u)$;

- given an open subset $\Omega \subseteq \mathbb{R}^N$ and $x_0 \in \Omega$. let us consider $f, g : \Omega \setminus \{x_0\} \to \mathbb{R}$ such that $g(x) \neq 0 \quad \forall\, 0 < |x - x_0| < \delta$, for some $\delta > 0$. Then we say:

  (i) $f = O(g)$ near to $x_0 \Leftrightarrow \exists M > 0$ and $0 < \delta' \leq \delta$ such that $|f(x)| \leq M|g(x)| \quad \forall\, 0 < |x - x_0| < \delta'$;

  (ii) $f = o(g)$ near to $x_0 \Leftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.

  Equivalent definitions may be given for sequences $\{a_n\}_{n \in \mathbb{N}}$ as $n \to +\infty$;

- given an open subset $\Omega \subseteq \mathbb{R}^N$ and $k \in \mathbb{N}$, we will denote $C^k(\Omega)$ the space of functions which are k times differentiable with continuity in Ω. On the other hand, $C^0(\Omega) := C(\Omega)$ denotes the space of continuous functions on Ω and $C^\infty(\Omega) := \cap_{n \in \mathbb{N}_0} C^n(\Omega)$ denotes the space of smooth functions on Ω;

- given an open subset $\Omega \subseteq \mathbb{R}^N$ and $k \in \mathbb{N}_0 \cup \{\infty\}$, we will denote $C_0^k(\Omega)$ the space of functions lying in $C^k(\Omega)$ which have compact support in Ω;

- given an open subset $\Omega \subseteq \mathbb{R}^N$, $k \in \mathbb{N}$ and $0 < \gamma \leq 1$, we denote the space $C^{k,\gamma}(\Omega)$ as

  $$C^{k,\gamma}(\Omega) := \left\{ u \in C^k(\Omega; \mathbb{R}) : \|u\|_{C^{k,\gamma}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{\infty,\Omega} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\Omega)} < +\infty \right\},$$

  where the γ^{th}-Holder semi-norm of $u : \Omega \to \mathbb{R}$ is

  $$[u]_{C^{0,\gamma}(\Omega)} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\gamma},$$

  and the norm $\|\cdot\|_{\infty,\Omega}$ is defined below;

- given an open subset $\Omega \subseteq \mathbb{R}^N$, let us consider $p \in [1, +\infty]$ and $u : \Omega \to \mathbb{R}$ a Lebesgue-measurable function. We denote $L^p(\Omega)$ the usual Lebesgue space endowed with the norm $\|u\|_{L^p(\Omega)} := (\int_\Omega |u|^p dx)^{\frac{1}{p}}$; on the other hand $\|u\|_{L^\infty(\Omega)} := \text{essup}_\Omega |u| = \inf\{a \in \mathbb{R} : |\{x \in \Omega : |u(x)| \geq a\}| = 0\}$. Sometimes we will indicate $\|u\|_{L^p(\mathbb{R}^N)}$ as $\|u\|_{L^p}$ for $1 \leq p < \infty$, and $\|u\|_{L^\infty(\mathbb{R}^N)}$ as $\|u\|_\infty$;

- given an open subset $\Omega \subseteq \mathbb{R}^N$, let us consider any $p, q \in [1, +\infty]$ and $u : \Omega \to \mathbb{R}$ a Lebesgue-measurable function. We say that $u \in L^p(\Omega) + L^q(\Omega)$ if $u = v + w$ for some $v \in L^p(\Omega)$ and $w \in L^q(\Omega)$;

- given an open subset $\Omega \subseteq \mathbb{R}^N$, let us consider $1 \leq p \leq \infty$. We denote $L^p_{loc}(\Omega)$ the space of functions lying in $L^p(\Omega')$ for every $\Omega' \subset \Omega$ compact subset;

- given an open subset $\Omega \subseteq \mathbb{R}^N$, let us consider $1 \leq k < \infty$ and $1 \leq p \leq \infty$. We denote $W^{k,p}(\Omega)$ the usual Sobolev space of functions in $L^p(\Omega)$ whose weak derivatives up to order k are also in $L^p(\Omega)$, endowed with the norm $\|u\|_{W^{k,p}(\Omega)} := \sum_{|\alpha|_1 \leq k} \|D^\alpha u\|_{L^p(\Omega)}$, where $D^\alpha u$ is defined below. In particular, we denote $H^1(\Omega) := W^{1,2}(\Omega)$ with the equivalent norm $\|u\|_{H^1(\Omega)} = (\int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx)^{1/2}$. Sometimes we will indicate $\|u\|_{H^1(\mathbb{R}^N)}$ as $\|u\|_{H^1}$;

- given an open subset $\Omega \subseteq \mathbb{R}^N$, let us consider $1 \leq k < \infty$ and $1 \leq p < \infty$. We denote $W^{k,p}_0(\Omega)$ the space given by the closure of $C^\infty_0(\Omega)$ in $W^{k,p}(\Omega)$. In particular we denote $H^1_0(\Omega) := W^{1,2}_0(\Omega)$;

- given an open subset $\Omega \subseteq \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ a Lebesgue measurable function, we denote $\mathcal{D}^{1,2}(\Omega)$ as the closure of $C^\infty_0(\Omega)$ of the semi-norm $\|u\|_{\mathcal{D}^{1,2}(\Omega)} := (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$ (it is well-known that $\|\cdot\|_{\mathcal{D}^{1,2}}$ is a norm on $H^1(\mathbb{R}^N)$); furthermore we denote $\mathcal{D}^{1,2}_r(\Omega)$ as the subspace of $\mathcal{D}^{1,2}(\Omega)$ formed by the radial functions;

- if $N \geq 3$ we denote $2^* := \frac{2N}{N-2}$ the critical Sobolev exponent and we recall that $2^* - 1 = \frac{N+2}{N-2}$;

- given an open subset $\Omega \subseteq \mathbb{R}^N$, we denote $H^1_r(\Omega)$ as the subspace of $H^1(\Omega)$ formed by the radial functions, endowed with the $H^1(\Omega)$-topology;

- given an open subset $\Omega \subseteq \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ differentiable at $x_0 \in \Omega$, we denote its gradient:
$$\nabla u(x_0) := \left(\frac{\partial u}{\partial x_1}(x_0), ..., \frac{\partial u}{\partial x_N}(x_0) \right)$$
as the vector of the partial derivatives of u in x_0, also denoted by $u_i(x_0)$; while if u is radial, we denote u_r its radial derivative and u_{rr} as its second radial derivative. The same notation will be held when $u \in W^{k,p}(\Omega)$ with $k \geq 1$, intended as "weak" gradient;

- given an open subset $\Omega \subseteq \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ twice differentiable at $x_0 \in \Omega$, we denote its laplacian:
$$\Delta u(x_0) := \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}(x_0).$$
The same notation will be held when $u \in W^{k,p}(\Omega)$ with $k \geq 2$, intended as "weak" laplacian. We denote $u_{ij}(x_0)$ as the second partial derivative of u respect to x_j and x_i respectively;

- given an open subset $\Omega \subseteq \mathbb{R}^N$, consider $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$ and $u : \Omega \to \mathbb{R}$ differentiable $|\alpha|_1$ times at $x_0 \in \Omega$. We denote

$$D^\alpha u(x_0) := \frac{\partial^{|\alpha|_1} u(x_0)}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$$

  as the multi-index derivative of u;

- given E a Banach space, let $\{u_n\}_{n\in\mathbb{N}} \subset E$. We denote as $u_n \rightharpoonup u$ the convergence of u_n in the weak-topology $\sigma(E, E^*)$ as $n \to +\infty$; we denote as $u_n \to u$ the convergence in E-norm as $n \to +\infty$;

- given two Banach spaces E and F, we denote $E \hookrightarrow F$ as the continuous embedding of E into F;

- given E Banach space, we denote E' as its dual; we identify the dual of H^1 with H^{-1}.

Introduction

The study of partial differential equations (PDE's) started in the 18th century in the work of Euler, d'Alembert, Lagrange and Laplace as the principal mode of analytical study of models in the physical science. This duality of viewpoints has been central to the study of PDE's through the 19-th and 20-th century.
The aim of the present issue is to demonstrate some important results on scalar field equation theory on $\mathbb{R}^N$ with $N \geq 2$, and its generalization known as Choquard-type equations. In 1983, Berestycki & Lions gave important results concerning the existence of nontrivial solutions for some semi-linear equations. Such problems are motivated by the search for certain kinds of solitary waves in nonlinear equations of the Klein-Gordon or Schrödinger type.

Consider the following nonlinear Klein-Gordon equation

$$\Phi_{tt} - \Delta_x \Phi + a^2 \Phi = f(\Phi), \tag{0.0.1}$$

where $\Phi(t, x)$ is a complex-valued function defined on $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ and $a \in \mathbb{R}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous odd function satisfying $f(0) = 0$ and

$$f(\rho e^{i\theta}) = f(\rho) f(e^{i\theta}) \quad \forall \rho \geq 0, \forall \theta \in [0, 2\pi). \tag{0.0.2}$$

Then, looking for a "standing wave" in (0.0.1), that is, Φ of the form $\Phi = e^{i\omega t} u(x)$, $w \in \mathbb{R}$ and $u : \mathbb{R}^N \to \mathbb{R}$, one is led to the equation

$$-\Delta u + mu = f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{0.0.3}$$

where $m = a^2 - \omega^2$.

Another classical type is that of travelling waves. Consider a real Klein-Gordon equation (0.0.1). Then, looking for a travelling wave solution of the form $\Phi(t, x) = u(x - ct)$ where $u : \mathbb{R}^N \to \mathbb{R}$ and $c = (c_1, ..., c_N) \in \mathbb{R}^N$ is a fixed vector such that $|c| < 1$, one obtains the following equation

$$-\sum_{i,j=1}^{N} (\delta_{i,j} - c_i c_j) \frac{\partial^2 u}{\partial x_i \partial x_j} + a^2 u = f(u) \quad \text{in} \quad \mathbb{R}^N. \tag{0.0.4}$$

It is easily checked, using the fact that $|c| < 1$, that the constant coefficient operator in the left hand side of (0.0.4) is elliptic. Thus, after a change of coordinates, (0.0.4) can be converted into an equation of type (0.0.3).

Stationary states of nonlinear Schrödinger equations lead to similar problems. Indeed, consider the equation

$$i\Phi_t - \Delta_x \Phi = f(\Phi), \tag{0.0.5}$$

where $\Phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ and f satisfies the simmetry property (0.0.2). Then, looking for standing wave solutions, that is $\Phi(t, x) = e^{-imt}u(x)$, one is again led to problem (0.0.3).

To sum up, we consider the following semi-linear elliptic problem

$$(*) \begin{cases} -\Delta u = g(u) \\ u \in H^1(\mathbb{R}^N), \quad u \not\equiv 0 \end{cases}$$

where $N \geq 3$ and $g : \mathbb{R} \to \mathbb{R}$ is an odd continuous function. The fact that we seek solutions on the Sobolev space $H^1(\mathbb{R}^N)$, gives us the first "restrictions" on g, in view of continuous (not compact) embeddings

$$\begin{cases} H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \quad \forall\, 2 \leq p \leq 2^* \quad \text{if} \quad N \geq 3; \\ H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \quad \forall\, 2 \leq p < \infty \quad \text{if} \quad N = 2. \end{cases}$$

So it comes natural to consider two classes of functions g when $N = 3$:

(i) g subcritical at infinity: $\lim_{s \to +\infty} \frac{g(s)}{s^{2^*-1}} = 0$;

(ii) g critical at infinity: $\lim_{s \to +\infty} \frac{g(s)}{s^{2^*-1}} \in \mathbb{R} \setminus \{0\}$.

Let us consider the energy $\mathcal{S} = \frac{1}{2}T - V$ where

$$T(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad V(u) = \int_{\mathbb{R}^N} G(u) dx, \quad G(s) = \int_0^s g(t) dt.$$

The problem (*) on bounded domains in $\mathbb{R}^N$ is widely studied by using standard variational methods. Evidently, a striking contrast between semi-linear elliptic boundary value problems on a bounded domain and on $\mathbb{R}^N$ is the apparent lack of compactness in treating the latter. Therefore, a first natural approach to (*) would be to approximate a solution of (*) by a solution of an analogous problem on the ball B_R, that is, first solve $-\Delta u_R = g(u_R)$ in B_R, $u_R|_{\partial B_R} = 0$, and then let $R \to +\infty$. One of the difficulties to overcome in such an approach is the absence of uniform (of R) a priori bounds.

We study (*) by using variational methods, working with an appropriate constraint in order to have some compactness. This constraint can be made transparent because of the "autonomous" character of (*) and the fact that one can use scale changes in $\mathbb{R}^N$. A special feature of (*) is its invariance under the group of displacements. That is, if $\mathcal{R}$ is a rotation in $\mathbb{R}^N$ and $C \in \mathbb{R}^N$ is a fixed vector, then for any solution of (*), the function $v(x) = u(\mathcal{R}x + C)$ is also a solution of (*). Such an indeterminacy will not be present in this thesis, since we will be seeking radial solutions of (*). In this case, u, as a function of $r = |x|$, satisfies the ordinary differential equation:

$$-u_{rr} - \frac{N-1}{r} u_r = g(u), \quad r \in (0, +\infty),$$

used in particular to show the regularity of u.

In the first chapter we prove, under suitable conditions, the existence of a ground-state solution in subcritical case using above variational methods. In particular we show, following an argument of [11], that the solution u_0 of (*) which we derive from our variational arguments is a ground-state, that is it has the property of having the least energy among all possible solutions of (*). It can be shown that a ground-state is necessarily a positive and radial solution of (*). On the other hand, the critical case is quite different. Since the compactness is guaranteed under a certain energy level, we require a "growth-boost" hypothesis on g. So in Section 1.4 we show the existence of a ground-state using also a min-max characterization by [11].

Obviously, the case $N = 2$ in Section 2.5 is totally different because $2^* = \infty$. Then, in view of Moser-Trudinger inequality [1], the notions of (sub)criticality of a function g become:

(i) g subcritical at infinity: $\lim_{s\to+\infty} \frac{g(s)}{e^{\theta s^2}} = 0 \quad \forall\theta > 0;$

(ii) g critical at infinity: $\lim_{s\to+\infty} \frac{g(s)}{e^{\theta s^2}} = 0 \quad (+\infty) \quad \text{if} \quad \theta > 4\pi \quad (\theta < 4\pi).$

In the second chapter, we are focused to seek solutions of (*) which are radial but not necessarily positive, and which correspond to higher values of energy. Such solutions are called bound-states. As said above, since the energy of these solutions is arbitrarily large, we study only the subcritical case. We define a sequence $\{b_k\}_k$ by involving Krasnosel'skii genus for $N \geq 2$. A standard variational theorem in [3] ensure that $\{b_k\}$ are critical points of V over a particular constraint. After the proper scale changes, one gets infinitely many distinct solutions $\{u_k\}$ of (*). This will be guaranteed by the fact that $\mathcal{S}(u_k) \to +\infty$ as $k \to +\infty$.
In Section 2.6 we give original multiplicity results for $N \geq 2$, in particular we extend works [26],[27] on the whole $\mathbb{R}^N$. let us consider the problem

$$\begin{cases} -\Delta u + u = f(u) & \text{in} \quad \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \quad u \not\equiv 0, \end{cases}$$

where f is a continuous critical function and $|f(s)| \geq \lambda |s|^{q-1} \quad \forall s \in \mathbb{R}$, for some $q \in (2, 2^*)$ and $\lambda > 0$. We prove that, given any $k \in \mathbb{N}$, there exists $\lambda_k \gg 1$ such that the problem has k pairs of nontrivial solutions for all $\lambda > \lambda_k$.

In the last chapter of the thesis we consider the Choquard problem

$$(**) \begin{cases} -\Delta u + u = (I_\alpha * F(u))F'(u) & \text{in} \quad \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \quad u \not\equiv 0, \end{cases}$$

where $F \in C^1(\mathbb{R};\mathbb{R})$, $\alpha \in (0, N)$ and the Riesz potential I_α is defined on $\mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}} 2^\alpha |x|^{N-\alpha}}.$$

The notion of criticality of a function changes completely here. Indeed, f is critical respect to the Hardy-Littlewood-Sobolev inequality, i.e.,

(i) f subcritical at infinity: $\lim_{s\to+\infty} \frac{f(s)}{s^{\frac{\alpha+2}{N-2}}} = 0$;

(ii) f critical at infinity: $\lim_{s\to+\infty} \frac{f(s)}{s^{\frac{\alpha+2}{N-2}}} \in \mathbb{R} \setminus \{0\}$.

First, we show the existence of a positive radial ground-state in dimension $N \geq 2$ when F' is subritical at infinity. We use a minimax principle in [42] to get a Palais-Smale sequence converging to the mountain pass level associated to $\mathcal{S}$. The novelty is to construct a Palais-Smale sequence which satisfies asymptotically the Pohožaev's identity, in order to get its boundedness.
On the other hand, we prove the existence of a ground-state solution also in critical case. As in critical case for scalar field equations, for $N \geq 3$ we assume the further condition

$$|f(s)| \geq |s|^{\frac{\alpha+2}{N-2}} + \mu|s|^{q-1} \quad \forall s \in \mathbb{R}$$

for some $q \in (2, \frac{N+\alpha}{N-2})$. We consider the functional $\mathcal{S}_\lambda = \frac{1}{2}T - \lambda V$ for $\lambda \in [\frac{1}{2}, 1]$, and we apply a general minimax theorem in [16] to get bounded Palais-Smale sequences at mountain pass level c_λ. The crucial thing is to estimate c_λ at energy level of Sobolev functions. Finally the case $N = 2$ is quite different and we require the Ambrosetti-Rabinowitz condition

$$\exists\, \theta > 2 \quad \text{such that} \quad 0 < \theta F(s) \leq 2F'(s)s \quad \forall s \neq 0$$

to get the boundedness of Palais-Smale sequences to the mountain pass level.

The last section of the thesis is dedicated to the existence of infinitely many bound-states as in the scalar field case. We prove the result in a particular class of subcritical functions. A fountain-like theorem [42, theorem 1.28] gives us an unbounded sequence of critical values c_k which may consider radial. Following the proof of [43], we remove the Ambrosetti-Rabinowitz condition using some arguments involved in previous subcritical case. Finally, as the last contribution, we proved the existence of infinitely many solutions also in the planar case $N = 2$.

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YHUDPHQWHSHQVDYLFKHD YUHLULQJUDCLDWRSXEEO LFDPHQWHOHSHU-
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NUIOTZNMIOTZMNITONMI TONMIZOTNMITNONMI.

Chapter 1

Ground-states

1.1 Main result

In this chapter, we consider the problem:

$$(*)\ \begin{cases} -\Delta u = g(u) \\ u \in H^1(\mathbb{R}^N), \quad u \not\equiv 0 \end{cases}$$

where $N \geq 3$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $g(0) = 0$ and the following conditions:

$$-\infty < \liminf_{s\to 0^+} \frac{g(s)}{s} \leq \limsup_{s\to 0^+} \frac{g(s)}{s} =: -m < 0, \tag{1.1.1}$$

$$-\infty \leq \limsup_{s\to +\infty} \frac{g(s)}{s^{2^*-1}} \leq 0, \tag{1.1.2}$$

$$\exists\, \xi > 0 \text{ such that } G(\xi) = \int_0^\xi g(s)ds > 0. \tag{1.1.3}$$

We will prove the existence of a ground state solution u_0, namely with the property of having the least action among all possible solutions of (*). Also, we will show that a ground state is necessarily a positive and radial solution.

The action

$$S(u) = \frac{1}{2}T(u) - V(u) \quad \text{where} \quad T(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad V(u) = \int_{\mathbb{R}^N} G(u)dx,$$

is defined on the space $H^1(\mathbb{R}^N)$ and after a suitable modification of g (see below), $\mathcal{S}$ is a C^1-functional on $H^1(\mathbb{R}^N)$. So it seems natural to find directly critical points of $\mathcal{S}$ to get a solution of (*).

However, a first difficulty in this approach is that $\mathcal{S}$ is not bounded from above (due to the presence of the gradient term) nor from below. In fact, from hypotheses (1.1.3) there exists $w \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} G(w)dx > 0$ (see below). Now consider a scale change in $\mathbb{R}^N$: $w_\sigma(x) = w(\frac{x}{\sigma})$ for $\sigma > 0$; one readily checks that

$$\mathcal{S}(w_\sigma) = \frac{\sigma^{N-2}}{2} T(w) - \sigma^N V(w).$$

It follows from $V(w) > 0$ that $\mathcal{S}(w_\sigma) \to -\infty$ as $\sigma \to +\infty$. Therefore, rather than looking for global critical points of $\mathcal{S}$, we will consider the following constrained minimization problem:

$$\text{minimize } \{T(w); w \in H^1(\mathbb{R}^N), V(w) = 1\}.$$

The following theorem concerns the existence of a ground state of (*).

Theorem 1.1.1. *Suppose the dimension $N \geq 3$ and that g satisfies conditions (1.1.1)-(1.1.3). Then (*) possesses a solution $u(x) = u(r)$ such that*

(i) $u > 0$ *on* $\mathbb{R}^N$*;*

(ii) $u(x) = u(r)$ *where* $r = |x|$*, and* u *decreases with respect to* r*;*

(iii) $u \in C^2(\mathbb{R}^N)$*;*

(iv) $|D^\alpha u(x)| \leq Ce^{-\delta|x|} \quad \forall x \in \mathbb{R}^N$*, for some* $C, \delta > 0$ *and for multi-index* $|\alpha|_1 \leq 2$*.*

1.2 Necessary conditions

In this section we will present several conditions about general features of a solution to (*). Indeed, a solution u satisfies an identity which is due to Pohožaev. It asserts that, under some assumptions, u necessarily satisfies:

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx = N \int_{\mathbb{R}^N} G(u) dx.$$

This fact will be derived as a corollary of the following more general proposition, valid also for $N = 2$.

Proposition 1.2.1. *Let $g : \mathbb{R} \to \mathbb{R}$ continuous function such that $g(0) = 0$ and consider $G(t) = \int_0^t g(s)ds$. Let u satisfy in a distributional sense*

$$-\Delta u = g(u)$$

and assume that

$$u \in L^\infty_{loc}(\mathbb{R}^N), \quad |\nabla u| \in L^2(\mathbb{R}^N), \quad G(u) \in L^1(\mathbb{R}^N).$$

Then u satisfies:

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx = N \int_{\mathbb{R}^N} G(u) dx.$$

Proof. We want to multiply the equation by $x_i u_i$ and integrate by parts to get the identity on B_R. Then, we will show that the boundary term on ∂B_R approaches to 0 as $R \to +\infty$. So, integrating by parts:

$$\sum_i \int_{B_R} g(u) u_i x_i dx = \sum_i \int_{B_R} \frac{\partial}{\partial x_i}(G(u)) x_i dx = -N \int_{B_R} G(u) dx + \sum_i \int_{\partial B_R} G(u) x_i n_i dS.$$

Observe that all the integrals above on B_R are finite because $u \in L^\infty_{loc}(\mathbb{R}^N)$ implies that $u \in W^{2,q}_{loc}(\mathbb{R}^N)$ for any $1 \le q < +\infty$ due to standard regularity theory. Furthermore we have $-\sum_j u_{jj} = g(u)$ and then

$$-\sum_{i,j} \int_{B_R} u_{jj} u_i x_i dx = \sum_{i,j} \int_{B_R} u_j(\delta_{ij} u_i + x_i u_{ij}) dx - \sum_{i,j} \int_{\partial B_R} u_j n_j x_i u_i dS =$$

$$\int_{B_R} |\nabla u|^2 dx - \frac{N}{2} \int_{B_R} |\nabla u|^2 dx - \frac{1}{2} \int_{\partial B_R} \left| \frac{\partial u}{\partial n} \right|^2 R dS.$$

Thus we deduce:

$$(N-2) \int_{B_R} |\nabla u|^2 dx - 2N \int_{B_R} G(u) dx = -2R \left[\frac{1}{2} \int_{\partial B_R} \left| \frac{\partial u}{\partial n} \right|^2 dS + \int_{\partial B_R} G(u) dS \right].$$

Now it suffices to show that the right hand side of the last equation converges to 0 for at least one suitably sequence $R_n \to +\infty$. In polar coordinates we have

$$\int_{R^N} \left[|G(u)| + |\nabla u|^2 \right] dx = \int_0^{+\infty} \left\{ \int_{\partial B_R} \left[|G(u)| + |\nabla u|^2 \right] dS \right\} dR < +\infty. \tag{1.2.1}$$

Hence by (1.2.1), there exists a sequence $R_n \to +\infty$ such that

$$R_n \int_{\partial B_{R_n}} \left[|G(u)| + |\nabla u|^2 \right] dS \to 0$$

as $n \to +\infty$. In fact, by contradiction if

$$\liminf_{R \to +\infty} R \int_{\partial B_R} \left[|G(u)| + |\nabla u|^2 \right] dS = \alpha > 0,$$

then the function $R \longmapsto \int_{\partial B_R} \left[|G(u)| + |\nabla u|^2 \right] dS$ would not be in $L^1(0, +\infty)$. Finally, by dominated convergence theorem and the fact that $|\nabla u|^2, G(u) \in L^1(\mathbb{R}^N)$, it follows that

$$\int_{B_{R_n}} |\nabla u|^2 dx \to \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad \int_{B_{R_n}} G(u) dx \to \int_{\mathbb{R}^N} G(u) dx$$

as $n \to +\infty$. □

Corollary 1.2.1. *Assume g satisfies (1.1.1) and (1.1.2). Then any solution u of (*) satisfies the Pohožaev's identity.*

Proof. We may consider $u \not\equiv 0$. If u satisfies the assumptions, since $G(u) \in L^1(\mathbb{R}^N)$ by Theorem A.0.1, we suffice to prove that $u \in L^\infty_{loc}(\mathbb{R}^N)$. Indeed, u satisfies the equation

$$-\Delta u = q(x)u \quad \text{in} \quad \mathbb{R}^N$$

where $q(x) = \frac{g(u(x))}{u(x)}$. If g satisfies the strong condition (1.3.1) (see below), one has

$$\left|\frac{g(u)}{u}\right| \leq C|u|^{\frac{4}{N-2}}.$$

Since $u \in H^1(\mathbb{R}^N)$, by Sobolev embedding theorem we also have $u \in L^{2^*}(\mathbb{R}^N)$. Noticing that $2^* = \frac{4}{N-2}\frac{N}{2}$, we see that $q \in L^{\frac{N}{2}}(\mathbb{R}^N)$. Now, using a result of Brezis and Kato (see [8]), we obtain $u \in L^p_{loc}(\mathbb{R}^N)$ for $1 \leq p < +\infty$. A classical bootstrap argument then shows that $u \in L^\infty_{loc}(\mathbb{R}^N)$. □

An important consequence of Pohožaev's identity is a relation between a solution u of (*) and its respective action $\mathcal{S}(u)$. Indeed this type of relation will be used for proving that the solution given by Theorem (1.1.1) is a ground state solution.

Corollary 1.2.2. *If u is any nontrivial solution of (*), then $\mathcal{S}(u) = \frac{1}{N}T(u) > 0$.*

Proof. From previous proposition, it follows that

$$S(u) = \frac{1}{2}T(u) - V(u) = \frac{1}{2}\left[1 - \frac{N-2}{N}\right]T(u) = \frac{1}{N}T(u) > 0.$$

□

1.3 The constrained minimization method

Before starting with the proof of main theorem, we need to modify the function g in order to make V of class $C^1(H^1(\mathbb{R}^N))$ as we said above. Indeed, V is well-defined if g satisfies the condition (see Theorem (A.0.2)):

$$\limsup_{|s|\to+\infty} \frac{|g(s)|}{|s|^{2^*-1}} < +\infty. \tag{1.3.1}$$

So, taking $\xi = \{x : G(x) > 0\}$ (see(1.1.3)), we define a new function $\tilde{g} : \mathbb{R} \to \mathbb{R}$ as follows:

(i) if $g(s) \geq 0$ for all $s \geq \xi$, set $g = \tilde{g}$ for $s \geq 0$;

(ii) otherwise, set $s_0 = \inf\{s \geq \xi : g(s) \leq 0\}$ and

$$\tilde{g}(s) = \begin{cases} g(s) & \text{for} \quad 0 \leq s < s_0 \\ 0 & \text{for} \quad s \geq s_0. \end{cases}$$

(iii) for $s < 0$, $\tilde{g}$ is defined as $\tilde{g}(s) = -g(-s)$.

Note that $\tilde{g}$ satisfies the same conditions as g and also condition (1.3.1). Furthermore, by the strong maximum principle, solutions of problem (*) with $\tilde{g}$ are also solution of the same problem with g. Indeed, in case (ii) above, a solution u satisfies $|u| < s_0$, whence $\tilde{g}(u) = g(u)$. Hence, we will always adopt the convention that g has been replaced by $\tilde{g}$; we keep however the same notation g.

So, the minimization problem:

$$\text{minimize} \quad \{T(w); w \in H^1(\mathbb{R}^N), V(w) = 1\}, \tag{1.3.2}$$

is well-defined since T and V are of class $C^1(H^1(\mathbb{R}^N))$. Minimizers are solutions of (*). In fact, if u solves (*), there exists a Lagrange multiplier θ such that $T'(u) = \theta V'(u)$, namely

$$-\Delta u = \theta g(u) \text{ in } \mathbb{R}^N$$

in a distributional sense. We will show that necessarily $\theta > 0$ and so, letting $u_\sigma(x) = u(\frac{x}{\sigma})$ with $\sigma > 0$, one obtains

$$-\Delta u = \frac{\theta}{\sigma^2} g(u_\sigma) \text{ in } \mathbb{R}^N.$$

Therefore, choosing $\sigma = \sqrt{\theta}$, one get a solution of (*).
In order to proof Theorem 1.1.1, we will give some results concerned the minimization problem summed up in the following

Theorem 1.3.1. *Under the hypotheses of Theorem 1.1.1, the minimization problem (1.3.2) admits a nontrivial solution $u \in H^1(\mathbb{R}^N)$ which is positive, spherically symmetric and decreases with $r = |x|$. Furthermore, there exists a Lagrange multiplier $\theta > 0$ such that u satisfies $-\Delta u = \theta g(u)$ in $\mathbb{R}^N$ in the distributional sense.*

Proof. This will be divided into four steps:

(i) Check that the set $\{w \in H^1(\mathbb{R}^N) : V(w) = 1\}$ is not empty;

(ii) Selection of an appropriate minimizing sequence $\{u_n\}$ and estimates for $\{u_n\}$;

(iii) Passage to the limit;

(iv) Conclusion.

Step 1. It is merely a consequence of hypothesis (1.1.3), which it is used only in this step. Let $\xi > 0$ be such that $G(\xi) > 0$. Now, for $R > 1$ define:

$$w_R(x) = \begin{cases} \xi & \text{for} \quad |x| \leq R \\ \xi(R + 1 - |x|) & \text{for} \quad R < |x| < R + 1 \\ 0 & \text{for} \quad |x| \geq R + 1. \end{cases}$$

In such way, $w_R \in H^1(\mathbb{R}^N)$ and one has:

$$V(w_R) = \int_{\mathbb{R}^N} G(w_R)dx \geq G(\xi)|B_R| - (\max_{s\in[0,\xi]} |G(s)|)|B_{R+1} - B_R| \geq CR^N - C'R^{N-1}$$

for some positive constant C, C'. So for $R > 1$ large enough, we have $V(w_R) > 0$. Then, introducing a scale change in $\mathbb{R}^N$: $w_{R,\sigma}(x) = w_R(\frac{x}{\sigma})$ for $\sigma > 0$, we obtain $V(w_{R,\sigma}) = \sigma^N V(w_R)$. Finally, choosing $\sigma = (V(w_R))^{-\frac{1}{N}}$, we have $V(w_{R,\sigma}) = 1$.

Step 2. Thanks to Step 1, there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$ such that $V(u_n) = 1$ and $\lim_{n\to+\infty} T(u_n) = \inf\{T(w) : w \in H^1(\mathbb{R}^N), V(w) = 1\} =: I \geq 0$. Consider now the Schwarz symmetrization rearrangement u_n^* of $|u_n|$ (see Appendix). Due to $\{u_n\} \subset H^1(\mathbb{R}^N)$, one has $\{u_n^*\} \subset H^1(\mathbb{R}^N), V(u_n^*) = 1$ and $I \leq T(u_n^*) \leq T(u_n)$. This means that $\{u_n^*\}$ is also a minimizing sequence by definition of I. Hence, replacing $\{u_n\}$ by $\{u_n^*\}$, we will consider u_n a nonnegative, radially symmetric and decreasing function with $r = |x|$, for all $n \in \mathbb{N}$.

Now we will prove that $\|u_n\|_{H^1(\mathbb{R}^N)}$ is bounded. For $s \geq 0$, define $g_1(s) := (g(s) + ms)^+$ and $g_2(s) := g_1(s) - g(s)$, where $m > 0$ refers to (1.1.1). While for $s \leq 0$, extend both of them as odd functions. Then we have $g_1, g_2 \geq 0$ on $\mathbb{R}^+$. Furthermore, conditions (1.1.1) and (1.1.2) imply that:

$$g_1(s) = o(s) \quad \text{as} \quad s \to 0; \quad \lim_{|s|\to+\infty} \frac{g_1(s)}{s^{2^*-1}} = 0; \quad g_2(s) \geq ms \quad \forall s \geq 0. \tag{1.3.3}$$

let us consider $G_i(z) = \int_0^z g_i(s)ds$ for $i = 1, 2$. Then, from continuity of $G_i(z)$ and (1.3.3) we obtain that $\forall \epsilon > 0, \exists C_\epsilon > 0$ such that

$$G_1(s) \leq C_\epsilon |s|^{2^*} + \epsilon G_2(s), \quad \forall s \in \mathbb{R}.$$

Since $T(u_n) \to I$, $\|\nabla u_n\|_{L^2}$ is bounded, which implies by Sobolev embedding theorem that $\|u_n\|_{L^{2^*}} \leq C'$ for some constant $C' > 0$ independent of n. Now we will show the boundedness of $\|u_n\|_{L^2}$. In fact, writing the condition $V(u_n) = 1$ in the form:

$$\int_{\mathbb{R}^N} G_1(u_n)dx = \int_{\mathbb{R}^N} G_2(u_n)dx + 1, \tag{1.3.4}$$

and using the last inequality with $\epsilon = \dfrac{1}{2}$, we deduce that

$$\int_{\mathbb{R}^N} G_2(u_n)dx + 1 \leq C' + \frac{1}{2}\int_{\mathbb{R}^N} G_2(u_n)dx.$$

Hence $\int_{\mathbb{R}^N} G_2(u_n)dx \leq C'$ and by (1.3.3) one has:

$$C' \geq \int_{\mathbb{R}^N} G_2(u_n)dx \geq \frac{m}{2}\int_{\mathbb{R}^N} u_n^2 = \frac{m}{2}\|u_n\|_{L^2}^2.$$

Thus $\|u_n\|_{H^1}$ is bounded.

Step 3. First, note that $u_n(x) \to 0$ as $|x| \to +\infty$ uniformly with respect to n. Indeed, since $0 \le u_n$ is radial and decreasing function for all n, it is easily seen that (see Lemma (A.0.1)) $|u_n(x)| \le C|x|^{-\frac{N}{2}}\|u_n\|_{L^2} \quad \forall x \in \mathbb{R}^N \setminus \{0\}$, where $C > 0$ is independent of n. By taking supremum over n in this relation, since $\|u_n\|_{L^2(\mathbb{R}^N)}$ is bounded, we have $\sup_n |u_n(x)| \le C|x|^{-\frac{N}{2}} \quad \forall x \in \mathbb{R}^N \setminus \{0\}$.

Now, since $\|u_n\|_{H^1(\mathbb{R}^N)}$ is bounded, we may extract a subsequence of u_n (again denoted by u_n) such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. It is important to see that $u \in H^1(\mathbb{R}^N)$ remains nonnegative, spherically symmetric and decreasing function with $|x|$ when we pass to the limit a.e. in $\mathbb{R}^N$.

Now we want to deduce conditions about $T(u)$ and $V(u)$ in order to verify that u is a solution of the minimizing problem. Define $Q(s) = s^2 + |s|^{2^*} \quad \forall s \in \mathbb{R}$. From (1.3.3) and $2^* > 2$, we derive:

$$\frac{G_1(s)}{Q(s)} \to 0 \quad \text{as} \quad |s| \to +\infty, \quad s \to 0.$$

Previously we proved that $\sup_n \int_{\mathbb{R}^N} Q(u_n)dx < +\infty$; furthermore, by continuity of G_1, we know that $G_1(u_n) \to G_1(u)$ a.e. in $\mathbb{R}^N$. Then, the compactness lemma due to Strauss (see Theorem (A.0.2)) implies that

$$\int_{\mathbb{R}^N} G_1(u_n)dx \to \int_{\mathbb{R}^N} G_1(u)dx$$

as $n \to +\infty$. Using Fatou's lemma, by continuity of G_2 we have

$$\int_{\mathbb{R}^N} G_1(u)dx \ge \int_{\mathbb{R}^N} G_2(u)dx + 1,$$

that is, $V(u) \ge 1$. On the other hand, by weak semicontinuity of the H^1 norm, one has $T(u) \le \liminf_{n\to+\infty} T(u_n) = I$. Now we want to prove that actually $V(u) = 1$. Suppose by contradiction that $V(u) > 1$; then, by the scale change $u_\sigma(x) = u(\frac{x}{\sigma})$ we have $V(u_\sigma) = \sigma^N V(u) = 1$ if we choose $\sigma := (V(u))^{-\frac{1}{N}} < 1$. Moreover, $T(u_\sigma) = \sigma^{N-2}T(u) \le \sigma^{N-2}I$ but, by definition of I, $T(u_\sigma) \ge I$. This would imply $I = 0$, namely $T(u) = 0$, i.e. $u \equiv 0$, contradicting $V(u) > 0$. This leads to a contradiction and therefore $V(u) = 1$ and $T(u) = I > 0$. Then u is a solution of problem (*).

Step 4. Since T and V are $C^1(H^1(\mathbb{R}^N))$ functionals, there exists a Lagrange multiplier θ such that $\frac{1}{2}T'(u) = \theta V'(u)$. Observe first that $\theta \ne 0$, since if $\theta = 0$ we would have the trivial solution $u = 0$. So, we will prove that $\theta > 0$. Suppose for contradiction that $\theta < 0$. Note that $V'(u) \ne 0$ because $V'(u) = 0$ gives $g(u) = 0$, which implies $u \equiv 0$. Then we can take $w \in H^1(\mathbb{R}^N)$ such that

$$V'(u)[w] = \int_{\mathbb{R}^N} g(u)w dx > 0.$$

Since T and V of class C^1, by Taylor's expansion one has: $V(u+\epsilon w) = V(u)+\epsilon V'(u)[w]+o(\epsilon)$ and $T(u+\epsilon w) = T(u) + 2\epsilon\theta V'(u)[w] + o(\epsilon)$ as $\epsilon \to 0$. We can find $\epsilon > 0$ small enough such that $v := u + \epsilon w$ satisfies $V(v) > V(u) = 1$ and $T(v) < T(u) = I$. Again by a scale change, there exists $0 < \sigma = (V(u))^{-\frac{1}{N}} < 1$ such that $V(v_\sigma) = 1$ and $T(v_\sigma) = \sigma^{N-2}T(v) < I$, which is absurd by definition of I.

Thus u satisfies (in the weak formulation) the desired equation

$$-\Delta u = \theta g(u) \quad \text{in} \quad \mathbb{R}^N$$

and so $u_{\sqrt{\theta}}$ is a solution of problem (*). □

Remark 1.3.1. *In dimension $N = 1$ and $N = 2$, the constrained minimization approach fails because when we try to prove the boundedness of the sequence in $L^{2^*}(\mathbb{R}^N)$, the Sobolev embedding theorem is no longer valide for $N = 1$ and $N = 2$. Indeed, let's try to study separately the cases $N = 1$ and $N = 2$.*

Case 1): $N = 2$. In dimension 2, scale change relations become

$$T(u_\sigma) = T(u), \quad V(u_\sigma) = \sigma^2 V(u).$$

Thus,

$$\inf_{\{V(u)=1\}} T(u) = \inf_{\{V(u)>0\}} T(u).$$

Now, suppose that u_0 is a solution given by constrained minimization problem, namely $V(u_0) = 1$ and $T(u_0) = \min_{\{V(u)>0\}} T(u)$. Hence $T'(u_0) = 0$ implies $u_0 = 0$, a contradiction to $V(u_0) = 1$.

Case 2): $N = 1$. The scaling relations in this case become

$$T(u_\sigma) = \sigma^{-1}T(u), \quad V(u_\sigma) = \sigma V(u).$$

Let $w \in H^1(\mathbb{R})$ such that $V(w) = 1$. Recalling that $\limsup_{s\to 0^+} \frac{g(s)}{s} = -m < 0$ and $G(u) = \int_0^u g(s)ds$, by continuity of G we see that there exists $0 < \theta_0 < 1$ such that $V(\theta_0 w) = 0$ and $V(\theta w) > 0$ for $\theta_0 < \theta \leq 1$.
Clearly, $V(\theta w) \to 0^+$ as $\theta \to \theta_0^+$. For $\theta_0 < \theta < 1$, let $\sigma(\theta) = V(\theta w)^{-1}$, such that $V(\theta w_{\sigma(\theta)}) = 1$. Now, $T(\theta w_{\sigma(\theta)}) = \sigma(\theta)^{-1}T(\theta w) = \theta^2 V(\theta w)T(w)$. Letting $\theta \to \theta_0^+$, this shows that $\inf_{\{V(u)=1\}} T(u) = 0$ because $T(u) \geq 0$ always. Hence, also in this case the minimization approach seems to fail.
The case $N = 2$ will be considered in Section 2.5.

Now we are going to prove that the solution of (*) obtained by the constrained minimization method has the property of minimizing the action among all solutions of (*), namely a ground state solution. The proof is based essentially on Pohožaev's identity; therefore it is crucial to know that any solution of (*) satisfies the identity.

Theorem 1.3.2. *Let u denote the solution of (*) obtained in Theorem (1.3.1). Then, for any other nontrivial solution v of (*), one has*

$$0 < \mathcal{S}(u) \leq \mathcal{S}(v).$$

Proof. Let $\bar{u}$ be the solution obtained in Theorem 1.3.1 such that

$$V(\bar{u}) = 1 \quad \text{and} \quad T(\bar{u}) = \min\{T(w) : w \in H^1(\mathbb{R}^N), V(w) = 1\}.$$

Then, as we seen before, there exists $\theta > 0$ such that $-\Delta \bar{u} = \theta g(\bar{u})$ in H^1- sense, so that $u = \bar{u}_{\sqrt{\theta}}$. By Pohožaev's identity, we have

$$T(u) = \frac{2N}{N-2} V(u).$$

The scale change relations yield

$$T(u) = \theta^{\frac{N-2}{2}} T(\bar{u}), \quad V(u) = \theta^{\frac{N}{2}} V(\bar{u}) = \theta^{\frac{N}{2}}.$$

Hence, we derive

$$\theta = \frac{N-2}{2N} T(\bar{u}).$$

By Corollary (1.2.1), the action for a general solution of (*) has the form $\mathcal{S}(u) = \frac{1}{N} T(u)$. Thus,

$$\mathcal{S}u) = \frac{1}{N} \left(\frac{N-2}{2N} \right)^{\frac{N-2}{2}} [T(\bar{u})]^{\frac{N}{2}}.$$

Now, let v denote another solution of (*); again by Pohožaev's identity: $T(v) = \dfrac{2N}{N-2} V(u)$. Let $\sigma > 0$ be such that $V(v_\sigma) = 1$, that is $\sigma = (V(v))^{-\frac{1}{N}}$ because $V(v) \neq 0$, or equivalently

$$\sigma = \left(\frac{N-2}{2N} \right)^{-\frac{1}{N}} [T(v)]^{-\frac{1}{N}}.$$

Let us express $\mathcal{S}(v)$ in terms of $T(v_\sigma)$. We know that $\mathcal{S}(v) = \frac{T(v)}{N}$; on the other hand $T(v_\sigma) = \sigma^{N-2} T(v)$, so using the preceding expression of σ we obtain

$$T(v_\sigma) = \left(\frac{N-2}{2N} \right)^{-\frac{N-2}{N}} [T(v)]^{\frac{2}{N}}.$$

Hence,

$$\mathcal{S}(v) = \frac{1}{N} T(v) = \frac{1}{N} \left(\frac{N-2}{2N} \right)^{\frac{N-2}{2}} [T(v_\sigma)]^{\frac{N}{2}}.$$

Since $\bar{u}$ solves the minimization problem and $V(v_\sigma) = 1$, we have $T(v_\sigma) \geq T(\bar{u})$. Using the inequalities above, we deduce $\mathcal{S}(v) \geq \mathcal{S}(u)$. □

Remark 1.3.2. *Consider now hypothesis (1.1.1); suppose that g is differentiable at 0 and $g'(0) > 0$. We claim that (*) has no radial solution.*
Indeed, if $u \in H^1(\mathbb{R}^N)$ is radial, then by a result of Strauss (see Lemma (A.0.2)) there exists a constant $C > 0$ such that

$$|u(x)| \leq C \frac{\|u\|_{H^1}}{|x|^{\frac{N-1}{2}}} \quad \forall |x| \geq \alpha_N,$$

where α_N is a positive constant depending on N, hence $|u(x)| = O\left(|x|^{\frac{1-N}{2}}\right)$ as $|x| \to +\infty$. Let $m = g'(0)$ and $q(r) = m - \frac{g(u(r))}{u(r)}$. Then, considering the case $N = 3$ and assuming $g \in C^2$, by Taylor's expansion and the previous inequality one has $q(r) = O(r^{-1})$ as $r \to +\infty$. Now, u satisfies the equation

$$-\Delta u + q(r)u = mu \quad in \quad \mathbb{R}^N.$$

But this is impossible since it violates a result of Kato (see [8]) which states that the linear operator $-\Delta + q(r)$ has no positive eigenvalues associated with eigenfunctions in $L^2(\mathbb{R}^3)$ under the condition $q(r) = O(r^{-1})$.

Observe, however, that $g'(0) > 0$ is not exactly the opposite of (1.1.1). The only remaining case is the limiting 'zero mass' case where $g'(0) = 0$. In fact, in this case the approach is always a constrained minimization, but the condition $g'(0) = 0$ does not give the boundedness of the L^2-norm of minimizing sequence u_n as in the previous theorem and also the integrability of $|G(u_n)|$. Then, we will look for a solution u of (*) when $g'(0) = 0$ such that $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. So, the constrained minimization problem becomes

$$minimize \quad \{T(w) : w \in \mathcal{D}^{1,2}_r(\mathbb{R}^N), G(w) \in L^1(\mathbb{R}^N), V(w) = 1\},$$

where G is an integral function of $g : \mathbb{R}^+ \to \mathbb{R}$ continuous function with new hypotheses which generalize the case $g'(0) = 0$:

$$g(0) = 0 \quad and \quad \limsup_{s \to 0^+} \frac{g(s)}{s^{2^*-1}} \leq 0;$$

$$there\ exists \quad \xi > 0 \quad such\ that \quad G(\xi) > 0;$$

$$let \quad \xi_0 = \inf\{\xi > 0; G(\xi) > 0\}; \quad if \quad g(s) > 0 \quad for\ all \quad s > \xi_0, \quad then \quad \limsup_{s \to +\infty} \frac{g(s)}{s^l} = 0.$$

Thus, under these hypotheses, one can prove the following

Theorem 1.3.3. *There exists a positive, spherically simmetric and decreasing (with $|x| = r$) solution u of the equation*

$$-\Delta u = g(u) \quad in \quad \mathbb{R}^N$$

such that $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. Furthermore, u is a classical solution (i.e. $u \in C^2(\mathbb{R}^N)$).

1.4 Critical case

In this section we will complete this study by considering a class of nonlinearities with critical growth and under mild assumptions.
We will assume that $g(s) = -s + f(s)$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with critical growth, in a sense that will be specified later. We thus obtain a ground state solution for the problem

$$(*) \begin{cases} -\Delta u + u = f(u) \\ u \in H^1(\mathbb{R}^N) \\ u > 0, \end{cases}$$

following the minimization problem:

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx; \int_{\mathbb{R}^N} G(u) dx = 1 \right\}, \quad \text{if} \quad N \geq 3 \tag{1.4.1}$$

and

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx; \int_{\mathbb{R}^2} G(u) dx = 0 \right\}, \quad \text{if} \quad N = 2, \tag{1.4.2}$$

where as before $G(s) = \int_0^s g(\tau) d\tau = \int_0^s (f(\tau) - \tau) d\tau = F(s) - \frac{s^2}{2}$ and $F(s) = \int_0^s f(\tau) d\tau$.

Then, the energy funcional $\mathcal{S} : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated to (*) is

$$\mathcal{S}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx - \int_{\mathbb{R}^N} F(u) dx.$$

Remark 1.4.1. *Observe that the original problem (*) corresponds to*

$$(*) \begin{cases} -\Delta u + mu = f(u) \\ u \in H^1(\mathbb{R}^N) \\ u > 0, \end{cases}$$

where $m > 0$ is a parameter. On the other hand, after a proper rescalement, we can assume $m = 1$.

In what follows, the function $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following hypotheses

$$\lim_{s \to 0^+} \frac{f(s)}{s} = 0; \tag{1.4.3}$$

$$\begin{cases} \limsup\limits_{s \to +\infty} \frac{f(s)}{s^{2^*-1}} = 1, \quad \text{if} \quad N \geq 3 \\ \lim_{s \to +\infty} \frac{f(s)}{e^{\alpha s^2}} = 0 \quad (+\infty) \quad \text{if} \quad \alpha > 4\pi \quad (\alpha < 4\pi) \quad \text{when} \quad N = 2; \end{cases} \tag{1.4.4}$$

$$\forall s \geq 0, \quad f(s)s - 2F(s) \geq 0 \quad (>0) \quad \text{if} \quad N \geq 3 \quad (N=2); \tag{1.4.5}$$

$$\exists \lambda > 0 \quad \text{and} \quad q \in \begin{cases} (2, 2^*) & \text{if} \quad N \geq 3 \\ (2, +\infty) & \text{if} \quad N = 2 \end{cases} \quad \text{s.t.} \quad f(s) \geq \lambda s^{q-1} \quad \forall s \geq 0; \tag{1.4.6}$$

Remark 1.4.2. *Under these assumptions, the energy functional $\mathcal{S}$ is well-defined on $H^1(\mathbb{R}^N)$ in view of Sobolev embedding and Moser-Trudinger inequality. Furthermore, condition (1.4.4) is, say, in a normalized form. We have analogue results when*

$$\limsup_{s\to+\infty} \frac{f(s)}{s^{2^*-1}} \in \mathbb{R} \setminus \{0\}, \quad \text{if} \quad N \geq 3,$$

and

$$\lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = 0 \quad (+\infty) \quad \text{if} \quad \alpha > \alpha_0 \quad (\alpha < \alpha_0) \quad \text{if} \quad N = 2, \quad \text{for some} \quad \alpha_0 > 0.$$

Before stating the main results, we need to fix some notations. We will denote in the following, $S, C_q > 0$ as the best constants of Sobolev embedding

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$$

and

$$H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N),$$

for q as in (1.4.6), that is, respectively

$$S\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}} \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad \text{for any} \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

and

$$C_q\left(\int_{\mathbb{R}^N} |u|^q dx\right)^{\frac{2}{q}} \leq \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx, \quad \text{for any} \quad u \in H^1(\mathbb{R}^N).$$

Now we will present the main results for the case $N \geq 3$ and $N = 2$.

Theorem 1.4.1. *If $N \geq 3$ and f satisfies (1.4.3)-(1.4.6) with*

$$\lambda > \lambda_{N,q} := \left[2^{\frac{2-N}{2}} S^{-\frac{N}{2}} N \left(\frac{2N}{N-2}\right)^{\frac{N-2}{2}}\right]^{\frac{q-2}{2}} \left[\frac{q-2}{2q}\right]^{\frac{q-2}{2}} C_q^{\frac{q}{2}},$$

then problem (*) has a minimizing positive solution which is a ground-state.

Theorem 1.4.2. *If $N = 2$ and f satisfies (1.4.3),(1.4.6) with*

$$\lambda > \lambda_{2,q} := \left(\frac{q-2}{2}\right)^{\frac{q-2}{2}} C_q^{\frac{q}{2}},$$

then problem (*) has a minimizing positive solution which is a ground-state.

Before proving them, we are going to fix some notations. First of all, under Schwartz symmetrization and Pólya-Szegö inequality, we can minimize problems (1.4.1) and (1.4.2) on the space $H^1_r(\mathbb{R}^N)$.
In the sequel, since we seek positive solutions, we will assume $f(s) = 0$ for $s \leq 0$ and argue as in subcritical case. Moreover, we will use the following notations

$$m := \inf\{\mathcal{S}(u); \quad u \quad \text{is nontrivial solution of (*)}\},$$

$$A := \inf\left\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx : \quad u \in H^1_r(\mathbb{R}^N) \quad \text{and} \quad \int_{\mathbb{R}^N} G(u)dx = \begin{cases} 1 & \text{if} \quad N \geq 3 \\ 0 & \text{if} \quad N = 2 \end{cases}\right\}.$$

We also need to define the following minimax value

$$b := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{S}(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C([0,1]; H^1_r(\mathbb{R}^N)) : \gamma(0) = 0, \mathcal{S}(\gamma(1)) < 0\}.$$

Define the sets

$$\mathcal{M} := \left\{u \in H^1_r(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} G(u)dx = \begin{cases} 1 & \text{if} \quad N \geq 3 \\ 0 & \text{if} \quad N = 2 \end{cases}\right\},$$

$$\mathcal{P} := \left\{u \in H^1_r(\mathbb{R}^N) \setminus \{0\} : 2N\int_{\mathbb{R}^N} G(u)dx = (N-2)\int_{\mathbb{R}^N} |\nabla u|^2 dx\right\}$$

and

$$\Upsilon := \{u \in H^1_r(\mathbb{R}^N) \setminus \{0\} : \mathcal{S}'(u) = 0\}.$$

From the above definitions, it follows that

$$2A = \inf_{v \in \mathcal{M}} \int_{\mathbb{R}^N} |\nabla v|^2 dx, \quad m = \inf_{v \in \Upsilon} \mathcal{S}(v).$$

Notice that $\mathcal{P}$ is the Pohožaev's identity manifold and $\Upsilon \subset \mathcal{P}$ by Proposition 1.2.1. Moreover, if $p := \inf_{v \in \mathcal{P}} \mathcal{S}(v)$, then $p \leq m$. It is very important to observe that $\mathcal{M}$ is a C^1 manifold for all $N \geq 2$. Indeed, let $V(u) := \int_{\mathbb{R}^N} G(u)dx$, then from (1.4.5) and for every $u \in \mathcal{M}$:

$$V'(u)[u] = \int_{\mathbb{R}^N} (f(u)u - u^2)dx \geq \int_{\mathbb{R}^N} (2F(u) - u^2)dx = 2V(u) = 2 \neq 0$$

if $N \geq 3$ and $V'(u)[u] > 0$ if $N = 2$.

In the following, we will show that A is attained and afterwards we prove that

$$m = A = b \quad \text{if} \quad N = 2,$$

thereby proving that (*) has a ground state solution if $N = 2$.

The case of dimension $N \geq 3$

First of all, by standard arguments involved growth assumptions on f, one shows that any minimizing sequence $\{u_n\}_n$ for (1.4.1) is bounded in $H^1_r(\mathbb{R}^N)$ (see Lemma 3.1 in [4]). So we can assume, up to subsequences, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ for some $u \in H^1_r(\mathbb{R}^N)$.
Now, in the sequel we will prove some estimates involving the levels A and b.

Lemma 1.4.3. $b \geq \frac{1}{N}(\frac{N-2}{2N})^{\frac{N-2}{2}}(2A)^{\frac{N}{2}}$.

Proof. For each $\gamma \in \Gamma$ one has $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$, see [11]. Hence, there exists $t_0 \in [0,1]$ such that $\gamma(t_0) \in \mathcal{P}$, and then $p = \inf_{v\in\mathcal{P}} \mathcal{S}(v)$ satisfies

$$p \leq \mathcal{S}(\gamma(t_0)) \leq \max_{t\in[0,1]} \mathcal{S}(\gamma(t)).$$

Consequently,

$$p \leq \mathcal{S}(\gamma(t_0)) \leq \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{S}(\gamma(t)) = b.$$

Now, due to an idea from Coleman-Glaser-Martin [11], one has

$$p = \inf_{v\in\mathcal{P}} \mathcal{S}(v) = \frac{1}{N}\left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}}(2A)^{\frac{N}{2}}$$

which concludes the proof. □

Now, from Ekeland's Variational Principle (see [42]), there are $\{u_n\} \subset \mathcal{M}$ and $\{\lambda_n\} \subset \mathbb{R}$ Lagrange multipliers such that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \to A$$

and

$$T'(u_n) - \lambda_n V'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N),$$

where $T(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx$. From last condition, one readily checks that $\{\lambda_n\}$ is bounded from above and $\limsup_{n\to+\infty} \lambda_n \leq A$.
Furthermore, the concentration compactness principle of Lions (see [21]) applied to the sequence $\{u_n\}$ guarantees the existence of positive finite measures μ, ν, sequences $\{\mu_i\}, \{\nu_i\} \subset \mathbb{R}$ and $\{x_i\} \subset \mathbb{R}^N$ such that as measure convergence,

(i) $|\nabla u_n|^2 \rightharpoonup d\mu \geq |\nabla u|^2 + \sum_i \delta_{x_i}\mu_i$,

(ii) $|u_n|^{2^*} \rightharpoonup d\nu = |u|^{2^*} + \sum_i \delta_{x_i}\nu_i$,

(iii) $\mu_i \geq S v_i^{\frac{2}{2^*}}$.

One can easily checks that $A > 0$ (see Lemma 3.3 in [4]); then the following lemma is well-posed.

Lemma 1.4.4. *If $\nu_i > 0$ for some index i, then $\nu_i \geq (\frac{S}{A})^{\frac{N}{2}}$.*

Proof. Let ϕ a smooth function with compact support verifying

$$0 \leq \phi(x) \leq 1 \quad \forall x \in \mathbb{R}^N, \quad \phi(x) = 1 \quad \text{in} \quad B_1 \quad \text{with} \quad \text{supp}\phi \subset B_2$$

and $\phi_\epsilon(x) = \phi(\frac{x-x_i}{\epsilon})$, for $\epsilon > 0$.

Then,

$$\int_{\mathbb{R}^N} \nabla u_n \nabla(\phi_\epsilon u_n)dx = \lambda_n \int_{\mathbb{R}^N} (f(u_n)u_n - u_n^2)\phi_\epsilon dx + o_n(1) \quad \text{as} \quad n \to +\infty.$$

The growth assumptions on f imply that, for any $\eta > 1$, there is a constant $C > 0$ and $r \in (2, 2^*)$ such that

$$sf(s) \leq \frac{s^2}{2} + \eta s^{2^*} + Cs^r \quad \text{for} \quad s \geq 0.$$

Hence,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \phi_\epsilon dx + \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \phi_\epsilon dx \leq$$

$$\leq \eta \lambda_n \int_{\mathbb{R}^N} |u_n|^{2^*} \phi_\epsilon dx + \frac{\lambda_n}{2} \int_{\mathbb{R}^N} |u_n|^2 \phi_\epsilon dx + \lambda_n C \int_{\mathbb{R}^N} |u_n|^r \phi_\epsilon dx.$$

By Lions' principle above and standard arguments involving dominate convergence theorem and Corollary A.0.4, first letting $n \to +\infty$ and then letting $\epsilon \to 0^+$, using $\limsup_{n\to+\infty} \lambda_n \leq A$, it follows that $\mu_i \leq \eta A \nu_i$ for all $\eta > 1$. Consequently, $\mu_i \leq A\nu_i$. Using (iii) of Lions' lemma, we get

$$A\nu_i \geq \mu_i \geq S\nu_i^{\frac{2}{2^*}},$$

implying

$$\nu_i \geq \left(\frac{S}{A}\right)^{\frac{N}{2}}.$$

□

Lemma 1.4.5. *If $\nu_i > 0$ for some index i, then $A \geq 2^{-\frac{2}{N}} S$.*

Lemma 1.4.6. *If*

$$\lambda > \lambda_{N,q},$$

then $b < \frac{1}{N}(\frac{N-2}{2N})^{\frac{N-2}{2}} 2^{\frac{N-2}{2}} S^{\frac{N}{2}}$.

Proof. Take $\psi \in H_r^1(\mathbb{R}^N), \psi \geq 0$ verifying

$$\|\psi\|_{L^q}^2 = C_q^{-1} \quad \text{and} \quad \|\psi\|_{H^1} = 1.$$

Now, observe that for any $u \in H^1(\mathbb{R}^N)$ such that $u^+ \not\equiv 0$ and $t > 0$, by condition (1.4.6)

$$\mathcal{S}(tu) \le \frac{t^2}{2}\|u\|^2_{H^1(\mathbb{R}^N)} - \frac{\lambda}{q}t^q\|u\|^q_{L^q(\mathbb{R}^N)} < 0$$

for some $t_u \gg 1$. So we get

$$b = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{S}(\gamma(t)) \le \inf_{u\in H^1(\mathbb{R}^N),u^+\not\equiv 0}\max_{t\in[0,1]} \mathcal{S}(tt_u u) \le \inf_{u\in H^1(\mathbb{R}^N),u^+\not\equiv 0}\max_{t\ge 0} \mathcal{S}(tu) \le$$

$$\le \max_{t\ge 0} \mathcal{S}(t\psi) \le \max_{t\ge 0}\left\{\frac{t^2}{2} - \lambda\frac{t^q}{q}\int_{\mathbb{R}^N}\psi^q dx\right\}.$$

Since $q > 2$, a simple computation shows that

$$\max_{t\ge 0}\left\{\frac{t^2}{2} - \lambda\frac{t^q}{q}\int_{\mathbb{R}^N}\psi^q dx\right\} = \frac{q-2}{2q}\lambda^{-\frac{2}{q-2}}\|\psi\|_q^{-\frac{2q}{q-2}} = \frac{q-2}{2q}\lambda^{-\frac{2}{q-2}}C_q^{\frac{q}{q-2}}.$$

The last inequality combined with the hypothesis on λ finishes the proof of the lemma. □

Lemma 1.4.7. *The weak limit u is non-trivial.*

Proof. Assume $u = 0$. In this case, since $\{u_n\} \subset H^1_r(\mathbb{R}^N)$, there is a ν_i which can be chosen to be positive at the origin. Notice that all other "atoms" are null because $\{u_n\}$ is bounded in $L^\infty\{|x| \ge \delta\}$, for all $\delta > 0$ by Lemma A.0.3. Next, we denote by ν_0 this unique atom.
We claim that $\nu_0 = 0$. Suppose on the contrary that $\nu_0 > 0$; by lemma 1.4.5, $A \ge 2^{-\frac{2}{N}}S$. Combining this inequality with lemma 1.4.6, we get a contradiction with Lemma 1.4.3. Hence $\nu_0 = 0$ and

$$u_n \to 0 \quad \text{in} \quad L^{2^*}_{loc}(\mathbb{R}^N)$$

by compactness' principle of Lions. On the other hand, by Lemma A.0.3

$$u_n \to 0 \quad \text{in} \quad L^{2^*}(\{|x| \ge R\}) \quad \text{for all} \quad R \gg 1 \quad \text{fixed}.$$

Then,

$$u_n \to 0 \quad \text{in} \quad L^{2^*}(\mathbb{R}^N).$$

Now, by definition of $\{u_n\}$ we have

$$\int_{\mathbb{R}^N} F(u_n)dx = \frac{1}{2}\int_{\mathbb{R}^N}|u_n|^2 dx + 1.$$

The growth assumptions on f imply that there is a constant $C > 0$ such that

$$F(s) \le \frac{1}{4}s^2 + Cs^{2^*} \quad \text{for} \quad s \ge 0.$$

Consequently,

$$C\int_{\mathbb{R}^N}|u_n|^{2^*}dx \ge \frac{1}{4}\int_{\mathbb{R}^N}|u_n|^2 dx + 1 \ge 1.$$

The last inequality leads to a contradiction. Therefore $u \ne 0$. □

Now, we are ready to prove Theorem 1.4.1.

Proof. Our goal is to prove that constant A is attained by u. Since $u_n \to u$ in $H^1(\mathbb{R}^N)$, it follows that

$$T(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx \leq \liminf_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx = A.$$

Combining Lemmas 1.4.3, 1.4.5, 1.4.6, we derive that $\nu_i = 0$ for every i. The same argument used in Lemma 1.4.7 shows that

$$u_n \to u \quad \text{in} \quad L^{2^*}_{loc}(\mathbb{R}^N).$$

From this,

$$F(u_n) \to F(u) \quad \text{in} \quad L^1(B_R(0)), \quad \forall R > 0.$$

On the other hand, Strauss' lemma implies

$$F(u_n) \to F(u) \quad \text{in} \quad L^1(\{|x| \geq R\}), \quad \forall R \gg 1 \quad \text{fixed.}$$

Then,

$$F(u_n) \to F(u) \quad \text{in} \quad L^1(\mathbb{R}^N).$$

Recalling that

$$\int_{\mathbb{R}^N} F(u_n) dx = \int_{\mathbb{R}^N} \frac{|u_n|^2}{2} dx + 1,$$

then we have

$$\int_{\mathbb{R}^N} F(u) dx \geq \int_{\mathbb{R}^N} \frac{u^2}{2} dx + 1,$$

that is

$$\int_{\mathbb{R}^N} G(u) dx \geq 1.$$

Now, if $u \notin \mathcal{M}$ one should have

$$\int_{\mathbb{R}^N} G(u) dx > 1.$$

Repeating the same argument used in Step 3 of Theorem 2.5.1 (see Section 2.5 below), one obtains a contradiction. Therefore $u \in \mathcal{M}$ and $T(u) = A$. Finally, we already know from Theorem 1.3.2 that if u minimizes the problem (1.4.1) then it is a ground state solution. □

The case of dimension $N = 2$

In dimension $N = 2$ we have already seen in Proposition 1.2.1 that Pohožaev's identity implies that any solution u of (*) should verify the equality $\int_{\mathbb{R}^2} G(u)dx = 0$.
In dimension $N = 2$, we consider $\mathcal{P} = \mathcal{M}$ and

$$(\star) \quad A = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx; \int_{\mathbb{R}^2} G(u)dx = 0, u \in H^1_r(\mathbb{R}^2) \setminus \{0\}\right\} = \inf_{v\in\mathcal{P}} \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 dx.$$

In what follows, we will consider the following min-max value

$$(\star\star) \quad c := \inf_{0\neq v\in H^1(\mathbb{R}^2)} \max_{t\geq 0} I(tv).$$

The first result of this subsection shows a sufficient condition on a sequence $\{v_n\}_n$ to get a convergence like $F(v_n) \to F(v)$ in $L^1(\mathbb{R}^2)$.

Lemma 1.4.8. *Assume that f satisfies (1.4.3), (1.4.4) and let $\{v_n\}$ be a sequence in $H^1_r(\mathbb{R}^2)$ such that*

$$\limsup_{n\to+\infty} \|\nabla v_n\|^2_{L^2} = \rho < 1 \quad \textit{and} \quad \limsup_{n\to+\infty} \|v_n\|^2_{L^2} = M < +\infty.$$

Then,

$$\int_{\mathbb{R}^2} F(v_n)dx \to \int_{\mathbb{R}^2} F(v)dx$$

where $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$.

Proof. Without loss of generality, we can assume that exists $v \in H^1_r(\mathbb{R}^2)$ such that

$$v_n \rightharpoonup v \quad \text{in} \quad H^1(\mathbb{R}^2), \quad v_n \to v \quad \text{a.e. in} \quad \mathbb{R}^2 \quad \text{and} \quad \lim_{|x|\to+\infty} v_n(x) = 0 \quad \text{uniformly in} \quad n,$$

by Strauss' lemma A.0.2. Using a Trudinger-Moser inequality due to Cao [10], we know that for each $m \in (0, 1)$ and $M > 0$, there exists $C(m, M) > 0$ such that

$$\sup_{u\in\mathcal{B}} \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1)dx \leq C(m, M),$$

where

$$\mathcal{B} := \left\{u \in H^1_r(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\nabla u|^2 dx \leq m, \int_{\mathbb{R}^2} u^2 dx \leq M\right\}.$$

Now, choose $\epsilon > 0$ small enough such that $m = \frac{\rho}{(1-\epsilon)^2} \in (0, 1)$ and set $\alpha = \frac{4\pi}{(1-\epsilon)^2} > 4\pi$.
Then,

$$\int_{\mathbb{R}^2} (e^{\alpha v_n^2} - 1)dx = \int_{\mathbb{R}^2} \left(e^{\alpha(1-\epsilon)^2(\frac{v_n}{1-\epsilon})^2} - 1\right)dx = \int_{\mathbb{R}^2} (e^{4\pi(\frac{v_n}{1-\epsilon})^2} - 1)dx.$$

Since $\frac{v_n}{1-\epsilon} \in \mathcal{B}$ for n large enough, we have that

$$\int_{\mathbb{R}^2} (e^{\alpha v_n^2} - 1)dx \leq C(m, M) \quad \forall n \gg 1.$$

Now, setting $P(s) = F(s)$ and $Q(s) = e^{\alpha s^2} - 1$, from the hypotheses of f and classical Moser-Trudinger inequality, it holds

$$\lim_{s \to 0} \frac{P(s)}{Q(s)} = \lim_{|s| \to +\infty} \frac{P(s)}{Q(s)} = 0, \quad \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2} |Q(v_n)|dx < +\infty$$

and

$$P(v_n(x)) \to P(v(x)) \quad \text{a.e. in} \quad \mathbb{R}^2 \quad \text{as} \quad n \to +\infty.$$

So, Theorem A.0.3 implies that $P(v_n)$ converges to $P(v)$ in $L^1(\mathbb{R}^2)$, that is,

$$\int_{\mathbb{R}^2} F(v_n)dx \to \int_{\mathbb{R}^2} F(v)dx,$$

finishing the proof. □

As in the preceding subsection, we derive two technical lemmas involving the levels A and c, defined as in $(\star)$, $(\star\star)$.

Lemma 1.4.9. *$A \leq c$.*

Proof. For each $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ with $v^+ \neq 0$, we set the continuous function $h : (0, +\infty) \to \mathbb{R}$ by

$$h(t) = \int_{\mathbb{R}^2} G(tv)dx = \int_{\mathbb{R}^2} \left(F(tv) - \frac{t^2 v^2}{2} \right) dx.$$

By virtue of the assumptions on f, one concludes that $h(t) < 0$ for t small enough and $h(t) > 0$ for t large enough. In this way, by the intermediate value theorem, there exists $t_0 > 0$ such that $h(t_0) = 0$, that is, $t_0 v \in \mathcal{P}$. Hence,

$$A \leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla(t_0 v)|^2 dx = \mathcal{S}(t_0 v) \leq \max_{t \geq 0} \mathcal{S}(tv).$$

On the other hand, since $f(s) = 0$ for all $s \leq 0$, if $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ with $v^+ = 0$, we have

$$\max_{t \geq 0} \mathcal{S}(tv) = +\infty.$$

From this, $A \leq c$. □

Lemma 1.4.10. *If*

$$\lambda > \lambda_{2,q}$$

with $\lambda_{2,q}$ as in Theorem 1.4.2, then $c < \frac{1}{2}$.

Proof. By assumption (1.4.5) for f, as in the proof of Lemma 1.4.6, we have

$$c \leq \frac{(q-2)}{2q}\lambda^{-\frac{2}{q-2}}C_q^{\frac{q}{q-2}}$$

and then

$$c < \frac{1}{2}.$$

□

Now, it is well-known that $A > 0$ (see Lemma 5.3 in [4]). Hence, we are ready to prove theorem 1.4.2.

Proof. We need to prove that A is attained, that is, there exists $u \in H_r^1(\mathbb{R}^2) \setminus \{0\}$ such that $A = \int_{\mathbb{R}^2} |\nabla u|^2 dx$ and $\int_{\mathbb{R}^2} G(u)dx = 0$. Let $\{u_n\}$ be a minimizing sequence in $H_r^1(\mathbb{R}^2)$ for A, that is,

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2 dx \to A \quad \text{and} \quad \int_{\mathbb{R}^2} G(u_n)dx = 0. \tag{1.4.7}$$

Arguing as before, we may assume that

$$\int_{\mathbb{R}^2} |u_n|^2 dx = 1.$$

Combining equation (1.4.7) with Lemmas 1.4.10 and 1.4.11, one obtains

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla u_n|^2 dx = 2A \leq 2c < 1.$$

From Lemma 1.4.9,

$$\int_{\mathbb{R}^2} F(u_n)dx \to \int_{\mathbb{R}^2} F(u)dx,$$

where u is the weak limit of $\{u_n\}$ in $H^1(\mathbb{R}^2)$. From last condition,

$$\int_{\mathbb{R}^2} F(u)dx = \frac{1}{2},$$

implying that $u \not\equiv 0$ and

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx \leq A.$$

Now, our goal is to prove that $\int_{\mathbb{R}^2} G(u)dx = 0$. To this end, by weak semicontinuity we have $\int_{\mathbb{R}^2} |u|^2 dx \leq 1$. Consequently,

$$\int_{\mathbb{R}^2} G(u)dx = \int_{\mathbb{R}^2} F(u)dx - \frac{1}{2}\int_{\mathbb{R}^2} |u|^2 dx = \frac{1}{2} - \frac{1}{2}\int_{\mathbb{R}^2} |u|^2 dx \geq 0.$$

Finally, as in step 3 of Section 2.5, we obtain that necessarily $\int_{\mathbb{R}^2} G(u)dx = 0$, from where it follows that A is attained.

Now, we prove that $m = A = b$. First, we recall

$$m = \inf\{\mathcal{S}(u) : u \text{ is nontrivial solution of (*)}\}$$

and

$$b = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{S}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1_r(\mathbb{R}^2)) : \gamma(0) = 0, \mathcal{S}(\gamma(1)) < 0\}$. We proved above that exists $u \in H^1_r(\mathbb{R}^2) \setminus \{0\}$ such that

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx = A \quad \text{and} \quad \int_{\mathbb{R}^2} G(u)dx = 0.$$

By Lagrange multipliers there exists $\theta \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^2} \nabla u \nabla v dx = \theta \int_{\mathbb{R}^2} g(u)v dx \quad \text{for every} \quad v \in H^1(\mathbb{R}^2).$$

The number θ should be positive as we have seen in the proof of Theorem 1.3.1. Define the rescaled function $u_{\sqrt{\theta}}(x) = u(\frac{x}{\sqrt{\theta}})$, which is a nontrivial solution of (*) with

$$\int_{\mathbb{R}^2} |\nabla u_{\sqrt{\theta}}|^2 dx = \int_{\mathbb{R}^2} |\nabla u|^2 dx \quad \text{and} \quad \int_{\mathbb{R}^2} G(u_{\sqrt{\theta}})dx = 0.$$

Thus,

$$m \le \mathcal{S}(u_{\sqrt{\theta}}) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_{\sqrt{\theta}}|^2 dx - \int_{\mathbb{R}^2} G(u_{\sqrt{\theta}})dx = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx = A.$$

To sum up,

$$m \le A.$$

Now, for each $\gamma \in \Gamma$ one has $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$. Hence, there exists $t_0 \in [0,1]$ such that $\gamma(t_0) \in \mathcal{P}$ and then

$$A \le \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \gamma(t_0)|^2 dx = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \gamma(t_0)|^2 dx - \int_{\mathbb{R}^2} G(\gamma(t_0))dx = \mathcal{S}(\gamma(t_0)),$$

by definition of $\mathcal{P}$. Thus

$$A \le \mathcal{S}(\gamma(t_0)) \le \max_{t\in[0,1]} \mathcal{S}(\gamma(t))$$

and so $A \le b$ follows immediately by definition of b.
To sum up, we have proved

$$m \le A \le b.$$

It remains to show that $m \ge b$. For any nontrivial solution $w \in H^1(\mathbb{R}^2)$ of (*), arguing as in [14] we deduce that there exists a path $\gamma_w \in \Gamma$ such that $w \in \gamma_w([0,1])$ and $\max_{t\in[0,1]} \mathcal{S}(\gamma_w(t)) = \mathcal{S}(w)$. Consequently,

$$b \le \mathcal{S}(w).$$

Therefore

$$b \leq m.$$

In conclusion, $m = A = b$ and the function $u_{\sqrt{\theta}}$ is a ground state solution of problem (*). $\square$

Chapter 2

Bound-states

2.1 Introduction and main result

In this chapter, we shall seek solutions of (*) which are radial, but not necessarily positive, and which correspond to higher values of the action $\mathcal{S}$. Such solutions are called "bound-states". Our main result states that under the same conditions as in previous chapter and plus hypothesis that g is odd, the problem (*) possesses infinitely many distinct solutions.
We can now state the main theorem of this chapter:

Theorem 2.1.1. *Let $N \geq 3$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous odd function which satisfies conditions (1.1.1)-(1.1.3). Then, problem (*) possesses an infinite sequence of distinct solutions $\{u_k\}_{k\geq 1}$ with the following properties:*

(i) u_k *is radial and of class* C^2 *on* $\mathbb{R}^N$, $\quad \forall k \geq 1$;

(ii) *there exist constants* $C_k, \delta_k > 0$ *such that*

$$|D^\alpha u_k(x)| \leq C_k e^{-\delta_k |x|}, \quad \forall x \in \mathbb{R}^N,$$

where $|\alpha|_1 \leq 2$ *and* $k \geq 1$;

(iii) $\lim_{k\to+\infty} \mathcal{S}(u_k) = +\infty$.

Heuristically, consider the manifold

$$M = \{u \in H^1_r(\mathbb{R}^N); T(u) = 1\},$$

and recall that

$$T(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad V(u) = \int_{\mathbb{R}^N} G(u) dx,$$

so $\mathcal{S} = \frac{1}{2}T - V$.
As before, by looking for critical points of the constrained functional $V_{|M}$, one can deduce the existence of solutions of (*). Indeed, if $V'_{|M}(v) = 0$ for some $v \in H^1(\mathbb{R}^N)$, then there

is a Lagrange multiplier θ such that $\frac{1}{2}T'(v) = \theta V'(v)$. One can show that $\theta > 0$, and so by scale change

$$u(x) = v_{\sqrt{\theta}}(x) = v\left(\frac{x}{\sqrt{\theta}}\right),$$

one obtains a solution u of (*).

Therefore, the idea will be to prove that $V_{|M}$ has infinitely many distinct critical values $\{c_k\}_{k\geq 1}$ corresponding to infinitely many distinct critical points on M. Finally, we have to ensure that, after the proper scale changes (different for each θ_k), one still gets infinitely many distinct solutions of (*). This will be guaranteed by the fact that $\mathcal{S}(u_k) \nearrow +\infty$ as $k \to +\infty$, which will be a consequence of the condition $c_k \nearrow +\infty$ as $k \to +\infty$. (Indeed, there is a simple relation of the form $\mathcal{S}(u_k) = Cc_k^\tau$ with constants $C, \tau > 0$ depending on N).

Our first task, therefore, will be to derive some results in critical point theory. In fact, we are concerned in general with finding critical points of constrained functionals of the type $J_{|M}$, where $J \in C^1(E, \mathbb{R})$ is even, E is a reflexive Banach space, $M = \{x \in E : \|x\|_H = 1\}$ and H is a Hilbert space such that $E \hookrightarrow H \hookrightarrow E'$. In the application to Theorem (2.1.1), we will consider $E = H_r^1(\mathbb{R}^N), H = \mathcal{D}_r^{1,2}(\mathbb{R}^N)$ and $J = V$.

2.2 Some results in critical point theory

In this section, we will give some general theorems about critical point theory. Let H be a real Hilbert space whose norm and scalar product will be denoted respectively by $\|\cdot\|_H$ and $(\cdot,\cdot)$. Let E a real Banach space with norm $\|\cdot\|_E$ continuously embedding in H. We assume throughout this section that

$$E \hookrightarrow H \hookrightarrow E'$$

by Riesz's duality map. Furthermore, we will suppose (without loss of generality) that $\|x\|_H \leq \|x\|_E$, $\forall x \in E$. We consider the manifold

$$M := \{x \in E : \|x\|_H = 1\}$$

endowed with the topology inherited from E. Moreover M is a submanifold of E of codimension 1 and its tangent space at a given point $x \in M$ can be considered as a subspace of E of codimension 1, namely

$$T_xM = \{v \in E : (x, v) = 0\}.$$

We denote by π_x the orthogonal projection onto T_xM, that is $\pi_x u = u - (u, x)x$ for all $u \in E$. Let us consider a functional $J : E \to \mathbb{R}$ which is of class C^1 on E. Then, $J_{|M}$ is a C^1 functional on M, and for any $x \in M$,

$$J'_{|M}(x)[w] = J'(x)[w] \quad \forall w \in T_xM.$$

Clearly, also $J'_{|M}(x) \in (T_xM)'$. In the sequel, for any $x \in M$ the notation $\|J'_{|M}(x)\|$ is understood to refer to the dual norm induced by the norm of T_xM which is inherited from E.

We recall that J satisfies the Palais-Smale condition (in short (P-S)) if the following condition holds: for any sequence $\{x_n\} \subset M$ such that $J(x_n)$ is bounded and $\|J'_{|M}(x_n)\| \to 0$ as $n \to +\infty$, there exists a subsequence $\{x_{n_k}\}$ which converges strongly in M.
A weaker requirement is the following (P-S$^+$) condition (which we will check instead of previous one): for any $\alpha, C > 0$ and for any sequence $\{x_n\} \subset M$ such that $\alpha \leq J(x_n) \leq C$ and $\|J'_{|M}(x_n)\| \to 0$ as $n \to +\infty$, there exists a subsequence $\{x_{n_k}\}$ which converges in M.
In order to check these conditions, it is useful to have a characterization of the convergence $\|J'_{|M}(x_n)\| \to 0$ as $n \to +\infty$ in terms of $J'(x_n)$.

Lemma 2.2.1. *Let $\{x_n\}$ be a sequence in M which is bounded in E. Then, the following conditions are equivalent:*

(i) $\|J'_{|M}(x_n)\| \to 0$ *as* $n \to +\infty$;

(ii) $J'(x_n) - J'(x_n)[x_n]x_n \to 0$ *in* E' *as* $n \to +\infty$.

Proof. Let $x \in M$; any $v \in E$ has the unique decomposition $v = (v,x)x + \pi_x v$, with $\pi_x v \in T_xM$ as before. Noticing that $|(v,x)| \leq \|v\|_H \leq \|v\|_E$, we have

$$\|\pi_x v\|_E \leq (1 + \|x\|_E)\|v\|_E, \quad \forall v \in E, \quad \forall x \in M.$$

Let $\tilde{J}'(x) := J'(x) - J'(x)[x]x$ in E'. By definition of T_xM, we have

$$\tilde{J}'(x)[w] = J'_{|M}(x)[w], \quad \forall w \in T_xM.$$

Thus, $\|J'_{|M}(x)\| \leq \|\tilde{J}'(x)\|_{E'}, \quad \forall x \in M$, whence (ii) implies (i).

On the other hand, suppose now (i). One has for any $v \in E$:

$$\tilde{J}'(x_n)[v] = J'(x_n)[\pi_{x_n} v].$$

Thus,

$$|\tilde{J}'(x_n)[v]| \leq \|J'_{|M}(x_n)\|(1 + \|x_n\|_E)\|v\|_E \leq C\|J'_{|M}(x_n)\|\|v\|_E$$

for some constant $C > 0$, since $\{x_n\} \subset M$ is bounded. This shows that $\|\tilde{J}'(x_n)\|_{E'} \to 0$ as $n \to +\infty$, that is (ii). □

We recall that a critical point for $J_{|M}$ is a point $x \in M$ such that $J'_{|M}(x) = 0$, and a critical value of $J_{|M}$ is a number $c \in \mathbb{R}$ such that there is an $x \in M$ with $J(x) = c$ and $J'_{|M}(x) = 0$. Let $\Sigma(M)$ denote the set of compact and symmetric, with respect to the origin, subsets of M. We recall that the genus $\gamma(A)$ of a set $A \in \Sigma(M)$, is defined as

$$\gamma(A) := \inf\{n \geq 1 : \exists \phi : A \to \mathbb{S}^{N-1} \quad \text{odd continuous}\}.$$

We set $\gamma(A) = +\infty$ if such integer does not exist, and recall that $\gamma(\mathbb{S}^{n-1}) = n$. For $k \geq 1$, we define

$$\Gamma_k := \{A \in \Sigma(M) : \gamma(A) \geq k\}.$$

Since E is infinite dimensional, we have $\Gamma_k \neq \emptyset$ for all $k \geq 1$.

We can now state the main result about existence of critical values, which will be crucial for the main theorem 2.1.1.

Theorem 2.2.1. *Let $J : E \to \mathbb{R}$ be an even functional of class C^1. We assume that J is bounded from below on M and that $J_{|M}$ satisfies the condition (P-S). Let*

$$c_k = \inf_{A \in \Gamma_k} \sup_{x \in A} J(x).$$

Then for any $k \geq 1$, c_k is a critical value of $J_{|M}$ and $\{c_k\}_k$ is an increasing sequence. Furthermore, if $c_k > 0$ is finite for any $k \geq 1$ and J satisfies (P-S$^+$), then c_k is a critical value of J.

Remark 2.2.2. *We note that, otherwise from previous chapter, we assume that g is odd because J needs to be an even functional. Furthermore, we require that $\inf_M J(x) > -\infty$ due to the fact that $c_1 = \inf_M J(x)$ and $\{c_k\}_k$ is an increasing sequence.*

As in usual in critical point theory, the proof of Theorem 2.2.1 requires a typical "deformation lemma", whose its technical proof is explained in [25] and [3]. In the proof, we consider for semplicity an equivalent relation for c_k, namely

$$b_k = \sup_{A \in \Gamma_k} \inf_{x \in A} J(x).$$

Indeed, it is easy to prove that for $N \geq 3$:

$$c_k = b_k^{-\frac{N-2}{N}}.$$

Lemma 2.2.3. *Suppose $J_{|M} \in C^1$ satisfies condition (P-S) (respectively (P-S$^+$)). Let $b \in \mathbb{R}$ (respectively $b > 0$) be not a critical value of $J_{|M}$, and for $c \in \mathbb{R}$ we put $A_c := \{x \in M : J(x) \geq c\}$. Then, there exist a constant $\bar{\epsilon} > 0$ and a deformation $\eta \in C(M, M)$ such that*

(i) $\eta(x) = x$ *for* $x \in M$, *with* $|J(x) - b| \geq \bar{\epsilon}$;

(ii) η *is a homeomorphism and it is odd if* $J_{|M}$ *is even;*

(iii) $J(\eta(x)) \geq J(x)$ *for* $x \in M$;

(iv) $\eta(A_{b-\epsilon}) \subset A_{b+\epsilon} \quad \forall\, 0 < \epsilon < \bar{\epsilon}$.

Proof. (**of Theorem 2.2.1**) First, b_k is well-defined because $\Gamma_k \neq \emptyset$ for $k \geq 1$. Since $\Gamma_{k'} \subset \Gamma_k$, if $k' \geq k$, it follows that $b_{k'} \leq b_k$. Now, suppose b_k is not a critical value of $J_{|M}$

for some $k \geq 1$. Then let $\epsilon, \bar{\epsilon} > 0$ and η be given by previous lemma. From the definition of b_k, there exists $A \in \Gamma_k$ such that

$$b_k - \epsilon \leq \inf_{x \in A} J(x) \leq b_k.$$

This implies that $A \subset A_{b_k - \epsilon}$. By previous lemma, $\eta(A) \subset A_{b_k + \epsilon}$. Morover, since η is an odd homeomorphism, we have $\eta(A) \in \Sigma(M)$ and $\gamma(\eta(A)) = \gamma(A) \geq k$. Hence $\eta(A) \in \Gamma_k$. But

$$\inf_{x \in \eta(A)} J(x) \geq b_k + \epsilon,$$

which contradicts the definition of b_k. □

Now, in order to get infinitely many distinct solutions of (*), we will show that $c_k \nearrow +\infty$ as $k \to +\infty$.

Theorem 2.2.2. *Let E be an infinite dimensional, separable, reflexive and dense subspace of H. In addition to the hypotheses of Theorem 2.2.1, we assume that $J(0) = 0$ and that J is weakly upper semicontinuous for the H-topology on the set*

$$S = \{x \in E; J(x) \geq 0, \|x\|_H \leq 1\},$$

that is

$$\text{if} \quad \{x_n\} \subset S, \quad x_n \rightharpoonup x \quad \text{in} \quad H \quad \text{and} \quad x \in E, \quad \text{then} \quad J(x) \geq \limsup_{n \to +\infty} J(x_n). \tag{2.2.1}$$

Lastly, we suppose that $c_k > 0$ is finite for all $k \geq 1$. Then we have

$$c_k \nearrow +\infty \quad \text{as} \quad k \to +\infty.$$

Since E is separable, there exists a sequence of finite dimensional subspaces of E, namely $E_1 \subset E_2 \subset ... \subset E_n \subset E_{n+1} \subset ... \subset E$ such that $\dim E_i = i \quad \forall\, i \geq 1$ and the closure of $\cup_{i \in \mathbb{N}} E_i$ in E is equal to E. Note that, since E is dense in H, the closure in H of $\cup_{i \in \mathbb{N}} E_i$ is also equal to H. In the sequel we denote by P_n the orthogonal projection from H onto E_n. Before proving Theorem 2.2.2, we give a technical lemma about orthogonal projections.

Lemma 2.2.4. *Assume the hypotheses of Theorem 2.2.2 hold. Then, for any $\epsilon > 0$, there exists $\rho = \rho_\epsilon > 0$ and $k_\epsilon \in \mathbb{N}$ such that for any $k \geq k_\epsilon$ and any $x \in S$, one has*

$$\|P_k(x)\|_E \leq \rho \quad \textit{implies} \quad J(x) \leq \epsilon.$$

Proof. Let $\epsilon > 0$ be given. Now, we claim that there exists $\rho > 0$ such that for all $x \in S$,

$$\|x\|_H \leq \rho \quad \text{implies} \quad J(x) < \frac{\epsilon}{2}. \tag{2.2.2}$$

Indeed, if it were not true, there would exist a sequence $\{x_n\} \subset S$ such that $x_n \to 0$ in H and $J(x_n) \geq \epsilon$ for n sufficiently bigger. This would contradict (2.2.1), since $0 \in E$

and $J(0) = 0$. The proof now proceeds by contradiction. Suppose there is a sequence of integers $\{n_i\} \nearrow +\infty$ as $i \to +\infty$ and sequences (we can take the same $\{n_i\}$ up to choose a subsequence) $\{x_{n_i}\} \subset S$ and $\{\rho_{n_i}\} \searrow 0$ such that

$$\|P_{n_i} x_{n_i}\|_E \le \rho_{n_i} \quad \text{and} \quad J(x_{n_i}) \ge \epsilon, \quad \forall i \tag{2.2.3}$$

for some $\epsilon > 0$. Then, one can extract a subsequence of $\{x_{n_i}\}$, denoted again by $\{x_{n_i}\}$, such that

$$x_{n_i} \rightharpoonup x \quad \text{in} \quad H, \quad P_{n_i} x_{n_i} \to 0 \quad \text{in} \quad E.$$

We claim that $x = 0$, and thus $x \in E$. Indeed, one has

$$\|x\|_H^2 = \lim_{i \to +\infty} (x_{n_i} - P_{n_i} x_{n_i}, x).$$

Now,

$$(x_{n_i}, x) \to \|x\|_H^2 \quad \text{as} \quad i \to +\infty$$

and

$$(P_{n_i} x_{n_i}, x) = (x_{n_i}, P_{n_i} x) \to \|x\|_H^2$$

since $P_{n_i} x$ converges strongly in H to x (by definition of orthogonal projection in Hilbert spaces). Therefore, $\|x\|_H = 0$, that is $x = 0$ and $x \in E$. Since $x_{n_i} \rightharpoonup x$, by (2.2.1) we have

$$J(0) \ge \limsup_{i \to +\infty} J(x_{n_i}) \ge \epsilon$$

which contradicts $J(0) = 0$. □

Proof. (**of Theorem 2.2.2**) Let $\epsilon > 0$ be given; we shall show that for $k > k_\epsilon$ (k_ϵ given by previous lemma) one has $0 < b_k \le \epsilon$, where b_k's are defined as above. Indeed, suppose for contradiction that $b_k > \epsilon$ for some $k > k_\epsilon$. Then, by definition of b_k there exists $A \in \Gamma_k$ ($A \in \Sigma(M), \gamma(A) \ge k$) such that $b_k \ge \inf_{x \in A} J(x) > \epsilon$. Since $J(x) > \epsilon$ for $x \in A$, we have by Lemma (2.2.4)

$$\|P_{k_\epsilon} x\|_E > \rho \quad \text{for} \quad x \in A,$$

where $\rho = \rho_\epsilon > 0$ given by Lemma (2.2.4). Thus, one can define an odd continuous mapping $\phi : A \to \mathcal{S}^{k_\epsilon - 1}$ as $\phi(x) = \frac{P_{k_\epsilon}(x)}{\|P_{k_\epsilon}(x)\|_E}$. But this implies $\gamma(A) \le k_\epsilon$ by definition of genus, which is a contradiction. Hence $b_k \searrow 0$ as $k \to +\infty$. □

2.3 Proof of the existence of infinitely many bound-states

We now turn to the applications of the previous section to problem (*). Let us first make precise the functional framework which will be used.
For $N \ge 3$, let $H = \mathcal{D}_r^{1,2}(\mathbb{R}^N)$ an Hilbert space with scalar product

$$(\phi, \psi) = \int_{\mathbb{R}^N} \nabla\phi \nabla\psi dx, \quad \forall \phi, \psi \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

We recall that H can be characterized by

$$H = \{\phi \in L^{2^*}(\mathbb{R}^N) \quad \text{radial} : |\nabla\phi| \in L^2(\mathbb{R}^N)\}$$

because of the Sobolev embedding theorem, and we define $E = H^1_r(\mathbb{R}^N)$. Observe that, since E is an infinite dimensional separable and dense subspace of H, we have a suitable functional framework to apply Theorem (2.2.2).

In order to define the functional V, we first need to modify the function g in the same way as we did in the previous chapter. Take ξ as in (1.1.3) and define $\tilde{g} : \mathbb{R} \to \mathbb{R}$ as follows:

(i) If $g(s) \geq 0$ for $s \geq \xi$, then $\tilde{g} = g$;

(ii) Otherwise, set $s_0 = \inf\{s \geq \xi : g(s) \leq 0\}$ and

$$\tilde{g}(s) = \begin{cases} g(s_0) & \text{if} \quad s \geq s_0 \\ g(s) & \text{if} \quad |s| \leq s_0 \\ -g(s_0) & \text{if} \quad s \leq -s_0. \end{cases}$$

Recall that $\tilde{g}$ is odd and satisfies the same conditions as g (in particular it is continuous at $s = s_0$ because g is odd), and that by strong maximum principle, solutions of (*) with $\tilde{g}$ are also solutions of (*) with g. We keep the notation g for the modified function $\tilde{g}$. Then, defining

$$V(u) = \int_{\mathbb{R}^N} G(u)dx,$$

we obtain an even functional $V \in C^1(E)$ with $V(0) = 0$. Now, we consider the submanifold of E

$$M = \{u \in E : T(u) = 1\}.$$

The main Theorem 2.1.1 will be easily derived from the next result.

Proposition 2.3.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be an odd continuous function satisfying conditions (1.1.1)-(1.1.3). There exist infinitely many distinct critical values $\{c_k\}_{k\in\mathbb{N}}$ of $T_{|M}$ given by*

$$c_k = \inf_{A\in\Gamma_k} \sup_{x\in A} T_{|M}(x), \quad \forall k \geq 1.$$

Moreover $c_k > 0$ is finite for all $k \geq 1$ and $c_k \nearrow +\infty$ as $k \to +\infty$. For each $k \in \mathbb{N}$, there exists a critical point $v_k \in M$ corresponding to c_k, and there exists $\theta_k > 0$ such that

$$-\Delta v_k = \theta_k g(v_k) \quad in \quad \mathbb{R}^N.$$

Proof. We apply Theorems 2.2.1 and 2.2.2 of the preceding section. The proof of the proposition is based on the following steps:

(i) $T_{|M}$ is bounded from below and satisfies the upper semi-continuity condition (2.2.1);

(ii) $T_{|M}$ satisfies (P-S$^+$);

(iii) $c_k > 0$ is finite for all $k \geq 1$;

(iv) Proof of Theorem 2.1.1.

Step 1. As above, let us consider

$$b_k = \sup_{A \in \Gamma_k} \inf_{x \in A} V_{|M}(x)$$

so that

$$b_k = c_k^{-\frac{N}{N-2}}.$$

Therefore, we will prove that $V_{|M}$ is bounded from below. Using the same notations of previous chapter, we set

$$g_1(s) = (g(s) + ms)^+ \quad \text{and} \quad g_2(s) = g_1(s) - g(s), \quad \forall s \geq 0$$

and

$$g_i(s) = -g_i(-s) \quad \text{for} \quad s \leq 0, \quad i = 1, 2.$$

Then $g = g_1 - g_2$ and $g_1, g_2 \geq 0$ on $\mathbb{R}^+$. Let

$$G_i(z) = \int_0^z g_i(s) ds$$

so that $G_i(z) \geq 0$ for $z \in \mathbb{R}$ and $i = 1, 2$. Recall that, as in the proof of Theorem 1.3.1, for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$G_1(s) \leq C_\epsilon |s|^{2^*} + \epsilon G_2(s), \quad s \in \mathbb{R}.$$

Now, for $u \in H$ such that $\|u\|_H \leq 1$, Sobolev embedding theorem implies $\|u\|_{L^{2^*}(\mathbb{R}^N)} \leq C$ for some constant $C > 0$. Hence, putting $G = G_1 - G_2$ and using the last inequality with $\epsilon = \frac{1}{2}$, one has

$$\|u\|_H \leq 1, \quad V(u) \leq C - \frac{1}{2} \int_{\mathbb{R}^N} G_2(u) dx \leq C, \quad u \in E. \tag{2.3.1}$$

Therefore, in particular $V_{|M}$ is bounded from above. Now, we will prove the upper semi-continuity condition.

Let define

$$S := \{u \in E : V(u) \geq 0, \|u\|_H \leq 1\}.$$

Consider a sequence $\{u_n\} \subset S$ such that $u_n \rightharpoonup u$ in H, with $u \in E$. We want to show that

$$V(u) \geq \limsup_{n \to +\infty} V(u_n).$$

We already know, since $\|u_n\|_H \leq 1$, that for all $n \in \mathbb{N}$

$$\|\nabla u_n\|_{L^2(\mathbb{R}^N)}, \|u_n\|_{L^{2^*}(\mathbb{R}^N)} \leq C,$$

for some constant $C \geq 1$. From $V(u_n) \geq 0$ and (2.3.1), we derive

$$\int_{\mathbb{R}^N} G_2(u_n)dx \leq C.$$

Since by construction we have $g_2(s) \geq ms$ for $s \geq 0$, it follows that

$$G_2(s) \geq \frac{m}{2}s^2, \quad s \in \mathbb{R}.$$

Hence, we obtain $\|u_n\|_{L^2} \leq C$, and so $\|u_n\|_E \leq C$. This shows that $u_n \rightharpoonup u$ in E. We now apply the technique of Step 3 of Section 1.3 to show that

$$\int_{\mathbb{R}^N} G_1(u_n)dx \to \int_{\mathbb{R}^N} G_1(u)dx \quad \text{as} \quad n \to +\infty.$$

Now by Fatou's lemma and continuity of G_2 we have

$$\int_{\mathbb{R}^N} G_2(u)dx \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^N} G_2(u_n)dx.$$

Hence, $V(u) \geq \limsup_{n\to+\infty} V(u_n)$.

Step 2. We will show that for any $\alpha, C > 0$ and for any sequence $\{u_n\} \subset M$ such that $\alpha \leq V(u_n) \leq C$ and $\|V'_{|M}(u_n)\| \to 0$, one can extract a convergent subsequence $\{u_{n_k}\}$. We know by step 1 that if $\{u_n\} \subset M$ and $V(u_n) \geq 0$, then $\|u_n\|_E \leq C$. Thus, applying Lemma 2.2.1, we obtain

$$V'(u_n) - V'(u_n)[u_n]u_n \to 0 \quad \text{in} \quad E',$$

which means

$$\theta_n \Delta u_n + g(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N) \tag{2.3.2}$$

where

$$\theta_n := V'(u_n)[u_n] = \int_{\mathbb{R}^N} g(u_n)u_n dx.$$

Since $\{u_n\}$ is bounded in E, up to a subsequence, we have that $u_n \rightharpoonup u$ in E. As in the Step 1 above, we know that

$$\int_{\mathbb{R}^N} G_1(u_n)dx \to \int_{\mathbb{R}^N} G_1(u)dx,$$

whence, using Fatou's lemma for $\int_{\mathbb{R}^N} G_2(u_n)dx$, we obtain

$$V(u) \geq \limsup_{n\to+\infty} V(u_n) \geq \alpha > 0.$$

Thus, in particular, $u \not\equiv 0$. Since the injection $H^1_r(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $2 < p < 2^*$ (Corollary A.0.4), we have $u_n \to u$ in $L^p(\mathbb{R}^N)$ for those p.

We now show that θ_n is bounded. Indeed, by conditions (1.1.1), (1.1.2) and continuity of g, there exists a constant $C > 0$ such that

$$|g(s)s| \leq C(|s|^2 + |s|^{2^*}), \quad s \in \mathbb{R}.$$

Thus,

$$|\theta_n| \leq C(\|u_n\|_{L^2}^2 + \|u_n\|_{L^{2^*}}^{2^*}) \leq C', \quad n \in \mathbb{N}$$

for some constant $C' > 0$. Therefore, up to a subsequence denoted always by θ_n, one obtains $\theta_n \to \theta$. Now, from (2.3.2) we have for any $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} g(u_n)\phi dx \to -\theta \int_{\mathbb{R}^N} |\nabla u|^2 \phi dx. \tag{2.3.3}$$

Now, applying Theorem A.0.1 with $Q(s) = |s|^{2^*}$ and $P(s) = g(s)$, we have that $g(u_n) \to g(u)$ in $L^1_{loc}(\mathbb{R}^N)$ (recalling that $\frac{|g(s)|}{|s|^{2^*-1}} \to 0$ as $s \to +\infty$).

Since $g(u_n) \to g(u)$ in $L^1_{loc}(\mathbb{R}^N)$, we also have $\int_{\mathbb{R}^N} g(u_n)\phi dx \to \int_{\mathbb{R}^N} g(u)\phi dx \quad \forall \phi \in C_0^\infty(\mathbb{R}^N)$, whence, comparing with condition (2.3.3), one has by density

$$-\theta \Delta u = g(u), \quad u \in H_r^1(\mathbb{R}^N) \setminus \{0\}. \tag{2.3.4}$$

Then by Pohožaev's identity (see Proposition 1.2.1), one obtains

$$\frac{(N-2)}{2}\theta \int_{\mathbb{R}^N} |\nabla u|^2 dx = NV(u) \geq N\alpha > 0.$$

Thus $\theta > 0$. Now, using the same argument to prove that $V(u) \geq \limsup_{n \to +\infty} V(u_n)$, one readily checks that

$$0 < \theta = \lim_{n \to +\infty} \int_{\mathbb{R}^N} g(u_n)u_n dx \leq \int_{\mathbb{R}^N} g(u)u dx.$$

Multiplying (2.3.4) by u and integrating by parts, we have

$$\theta \int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_{\mathbb{R}^N} g(u)u dx.$$

Hence, comparing the above conditions, we deduce $\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq 1$. Since $u_n \rightharpoonup u$ in H, we also have $\int_{\mathbb{R}^N} |\nabla u|^2 dx \leq 1$. Therefore

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} g(u)u dx = \lim_{n \to +\infty} \int_{\mathbb{R}^N} g(u_n)u_n dx. \tag{2.3.5}$$

The first equality shows that $u_n \to u$ in H and $u \in M$. Now we will prove that $u_n \to u$ in E. Indeed, we know that

$$\int_{\mathbb{R}^N} g_1(u_n)u_n dx \to \int_{\mathbb{R}^N} g_1(u)u dx.$$

Hence, by (2.3.5) and definition of g, one has

$$\int_{\mathbb{R}^N} g_2(u_n)u_n dx \to \int_{\mathbb{R}^N} g_2(u)u dx. \tag{2.3.6}$$

Recalling that $g_2(s) = ms + (g(s) + ms)^-$ for $s \leq 0$ and g is odd on $\mathbb{R}$, one obtains $g_2(s)s = ms^2 + q(s)$ with $0 \leq q(s)$ continuous for all $s \in \mathbb{R}$. By Fatou's lemma

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^N} q(u_n)dx \geq \int_{\mathbb{R}^N} q(u)dx, \quad \liminf_{n\to+\infty} \int_{\mathbb{R}^N} u_n^2 dx \geq \int_{\mathbb{R}^N} u^2 dx.$$

Comparing with (2.3.6) we obtain $u_n \to u$ in $L^2(\mathbb{R}^N)$, so $u_n \to u$ in E. $\square$

Step 3. Now we will prove that $b_k > 0$ for all $k \geq 1$. Indeed, since $V_{|M}$ satisfies (P-S$^+$), it is crucial to know a priori that $b_k > 0$ for all $k \geq 1$.
For $k \geq 1$, we consider the polyhedron in $\mathbb{R}^k$ defined by

$$\pi_{k-1} := \left\{ l = (l_1, ..., l_k) \in \mathbb{R}^k : \sum_{i=1}^{k} |l_i| = 1 \right\}.$$

Since π_{k-1} is homeomorphic to $\mathbb{S}^{k-1}$ by an odd homeomorphism, one has $\gamma(\pi_{k-1}) = k$.
We will prove that $b_k > 0$ using the following theorem.

Theorem 2.3.1. *For any $k \geq 1$, there exists a constant $R = R(k) > 1$ and an odd continuous mapping $\tau : \pi_{k-1} \to H_0^1(B_R)$ such that $\tau(l)$ is a radial function for all $l \in \pi_{k-1}$ and*

$$0 \notin \tau(\pi_{k-1}), \tag{2.3.7}$$

$$\exists\, \rho, C > 0 \quad \textit{such that} \quad \forall u \in \tau(\pi_{k-1}), \quad \rho \leq \|\nabla u\|^2_{L^2(B_R)} \leq C, \tag{2.3.8}$$

$$\int_{B_R} G(u)dx \geq 1 \quad \forall u \in \tau(\pi_{k-1}). \tag{2.3.9}$$

The proof can be seen on the Appendix (Theorem A.0.7).
Let us show that Theorem 2.3.1 implies $\beta_k > 0$. Put $\hat{\pi}_{k-1} := \tau(\pi_{k-1})$ and define a mapping $\chi : \hat{\pi}_{k-1} \to M$ in the following way. First, we introduce the canonical injection $H_0^1(B_R) \hookrightarrow H^1(\mathbb{R}^N)$ by setting, for $u \in H_0^1(B_R)$,

$$\tilde{u} = \begin{cases} u & \text{on} \quad B_R \\ 0 & \text{on} \quad \mathbb{R}^N \setminus B_R \end{cases}.$$

Now define $\chi(u) = \tilde{u}_\sigma = \tilde{u}(\frac{\cdot}{\sigma})$ for $0 \not\equiv u \in \hat{\pi}_{k-1}$, where $\sigma = \sigma(u) > 0$ is uniquely determined by the condition $\chi(u) \in M$, that is $T(\tilde{u}_\sigma) = \sigma^{N-2}T(\tilde{u}) = 1$.
Since $T(\tilde{u}) = \|\nabla u\|^2_{L^2(B_R)}$, we have by (2.3.8)

$$0 < \rho' \leq \sigma(u) \leq C' \quad \forall u \in \hat{\pi}_{k-1},$$

for some positive constants ρ', C'. From this and (2.3.9) one has

$$V(\chi(u)) = \sigma^N \int_{B_R} G(u)dx \geq (\rho')^N, \quad \forall u \in \hat{\pi}_{k-1}.$$

Now, define $A_k := \chi(\hat{\pi}_{k-1})$. Since χ is an odd continuous mapping, $A_k \in \Sigma(M)$. Furthermore, as $\chi \circ \tau : \pi_{k-1} \to A_k$ is odd and continuous, we have $\gamma(A_k) \geq \gamma(\pi_{k-1}) = k$. Hence, $A_k \in \Gamma_k$ for all $k \geq 1$. We have seen that

$$V(u) \geq (\rho')^N, \quad \forall u \in A_k.$$

Therefore,

$$b_k = \sup_{A \in \Gamma_k} \inf_{x \in A} V(x) \geq \inf_{x \in A_k} V(x) \geq (\rho')^N > 0.$$

Step 4. Finally, by Theorems 2.2.1 and 2.2.2, we know that $c_k = b_k^{-\frac{N-2}{N}}$ is a critical value for $T_{|M}$ for all $k \geq 1$ and

$$c_k \nearrow +\infty \quad \text{as} \quad k \to +\infty.$$

This last fact shows in particular that there exist infinitely many distinct critical points of $T_{|M}$. Let $v_k \in M$ be a critical point of $V_{|M}$ associated with the critical value c_k. Now, Lemma 2.2.1 implies

$$V'(v_k) = V'(v_k)[v_k]v_k.$$

Thus, defining $\mu_k := V'(v_k)[v_k]$, we have

$$-\mu_k \Delta v_k = g(v_k) \quad \text{in} \quad \mathbb{R}^N.$$

Then, as in the previous chapter, we derive

$$\mu_k = \frac{2N}{N-2} c_k^{-\frac{N}{N-2}} > 0.$$

Hence, letting $\theta_k = \mu_k^{-1}$ we have

$$-\Delta v_k = \theta_k g(v_k) \quad \text{in} \quad \mathbb{R}^N, \qquad \theta_k = \frac{N-2}{2N} c_k^{\frac{N}{N-2}}.$$

Let $u_k = (v_k)_{\sqrt{\theta_k}}$; then u_k is a solution for the problem (*), for each $k \geq 1$. Now, we want to prove that $\{u_k\}_{k \geq 1}$ are actually infinitely many distinct solutions of (*), showing

$$\lim_{k \to +\infty} \mathcal{S}(u_k) = +\infty.$$

We know that $\mathcal{S}(u_k) = \frac{1}{N} T(u_k)$ as above. By the scale change relation, we have

$$\mathcal{S}(u_k) = \frac{1}{N} \left(\frac{N-2}{2N} \right)^{\frac{N-2}{2}} c_k^{\frac{N}{2}}.$$

Thus, since $c_k \nearrow +\infty$ as $k \to +\infty$, we deduce that

$$\mathcal{S}(u_k) \nearrow +\infty.$$

Therefore, one actually has an infinite number of distinct solutions.

2.4 Regularity of solutions and exponential decay

In this section we will discuss the regularity of both ground-states and bound-states. Furthermore, we show a strong decay at infinity of solutions. Indeed, these properties are only connected with solutions u of (*), independent of their energy.

(i) Regularity. In the previous chapter, we showed that u belongs to $L^\infty_{loc}(\mathbb{R}^N)$. Thus, by the L^p- estimates from [1], we know that $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ for any $p < +\infty$. Hence, by Morrey's theorem, $u \in C^{1,\alpha}(\mathbb{R}^N)$ with some $\alpha \in (0,1)$.
Since u is radial, using the laplacian formula in polar coordinates, u satisfies the relation

$$-u_{rr} - \frac{N-1}{r}u_r = g(u), \quad r \in (0,+\infty). \tag{2.4.1}$$

We already know that u_{rr} is continuous on $(0,+\infty)$ (using a bootstrap argument), but we need to be careful at $r = 0$. Indeed, we will show the continuity at this point. Let us define $v(r) := g(u(r))$; v is continuous on $[0,+\infty)$. Rewriting (2.4.1) as

$$-(r^{N-1}u_r)_r = r^{N-1}v(r),$$

integrating from 0 to r, we have

$$r^{N-1}u_r = -\int_0^r s^{N-1}v(s)ds.$$

With a change of variable, one has

$$\frac{u_r}{r} = -\int_0^1 t^{N-1}v(rt)dt.$$

Since, by dominated convergence theorem,

$$\int_0^1 t^{N-1}v(rt)dt \to \frac{v(0)}{N} \quad \text{as} \quad r \to 0^+,$$

we deduce that u_{rr} exists and $u_{rr}(0) = -\frac{v(0)}{N}$. Furthermore, from equation (2.4.1) we note that $u_{rr} \to -\frac{v(0)}{N}$ as $r \to 0^+$. Thus, $u \in C^2(\mathbb{R}^N)$.

(ii) Exponential decay. The exponential decay of u at infinity follows from an argument from ordinary differential equations. We know that $u \in C^2(\mathbb{R}^N)$ and satisfies (2.4.1). Set $v := r^{\frac{N-1}{2}}u$; then v satisfies

$$v_{rr} = \left(q(r) + \frac{b}{r^2}\right)v$$

where $q(r) = -\frac{g(u(r))}{u(r)}$ and $b = \frac{(N-1)(N-3)}{4}$. Recalling that $u(r) \to 0$ as $r \to +\infty$ by Lemma A.0.2, from hypothesis (1.1.1) for g, we have for $r \geq r_0$ large enough:

$$q(r) + \frac{b}{r^2} \geq \frac{m}{2}.$$

Let $w := v^2$; then w satisfies

$$\frac{1}{2} w_{rr} = v_r^2 + \left(q(r) + \frac{b}{r^2} \right) w.$$

Thus, for $r \geq r_0$ one has $w_{rr} \geq mw$. Now let $z := e^{-\sqrt{m}r}(w_r + \sqrt{m}w)$; we have $z_r = e^{-\sqrt{m}r}(w_{rr} - mw) \geq 0$. Hence, z is a nondecreasing function on $(r_0, +\infty)$. If there exists $r_1 > r_0$ such that $z(r_1) > 0$, then $z(r) \geq z(r_1) > 0$ for all $r \geq r_1$. This implies that

$$w_r + \sqrt{m}w \geq z(r_1)e^{\sqrt{m}r},$$

whence $w_r + \sqrt{m}w$ is not integrable on $(r_1, +\infty)$. But v^2 and vv_r are integrable near infinity for radial $u \in H^1(\mathbb{R}^N)$ using polar coordinates, so that w_r and w are also integrable, a contradiction. Hence, $z(r) \leq 0$ for all $r \geq r_0$. This implies that

$$(e^{\sqrt{m}r} w)_r = e^{2\sqrt{m}r} z \leq 0 \quad \text{for} \quad r \geq r_0.$$

Hence, integrating from r_0 to r we have $w(r) \leq Ce^{-\sqrt{m}r}$ for some constant $C > 0$ and

$$|u(r)| \leq Cr^{-\frac{N-1}{2}} e^{-\frac{\sqrt{m}}{2}r} \quad \text{for} \quad r \geq r_0. \tag{2.4.2}$$

To obtain exponential decay of u_r, observe that u_r satisfies

$$(r^{N-1}u_r)_r = -r^{N-1}g(u). \tag{2.4.3}$$

Now, from hypotheses on g we can say that for $r \geq r_0$, $m_1|u| \leq |g(u)| \leq m_2|u|$ for some $m_2 \geq m_1 > 0$. Hence, integrating (2.4.3) on (r, R) and letting $r, R \to +\infty$ using the last inequality and (2.4.2), one has that $r^{N-1}u_r$ has a limit as $r \to +\infty$. This limit can only be zero by (2.4.2). Then, integrating (2.4.3) on $(r, +\infty)$, we have that also u_r has an exponentially decay at infinity. Finally, the exponential decay of u_{rr} (and thus of $|D^\alpha u(x)|$ for $|\alpha|_1 \leq 2$ by polar coordinates) follows immediately from (2.4.1).

(iii) Positivity of ground-state. We note that if $u \in H^1(\mathbb{R}^N)$ then $|u| \in H^1(\mathbb{R}^N)$, so $T(|u|) = T(u)$ and $V(|u|) = V(u)$ by the hypotheses on $\tilde{g}$. Therefore, if u is a ground state solution, so is $|u|$. Since $|u| \geq 0$, by the strong maximum principle we have $|u| > 0$, that is $u > 0$ on $\mathbb{R}^N$.

2.5 Planar case

We want now to discuss the existence of a ground-state solution of (*) and infinitely many bound-states of the same problem in the case of dimension $N = 2$, when some of the previous arguments seem to fail.

We will study the problem

$$-\Delta u = g(u), \quad u \in H^1(\mathbb{R}^2), \quad u \not\equiv 0. \tag{2.5.1}$$

The hypotheses for g are the folllowing:

$$g \in C(\mathbb{R}, \mathbb{R}), \quad g(-s) = -g(s), \quad \forall s \in \mathbb{R}; \tag{2.5.2}$$

$$\exists \xi > 0 \quad \text{such that} \quad G(\xi) = \int_0^\xi g(s)ds > 0; \tag{2.5.3}$$

$$g'(0) = -m < 0; \tag{2.5.4}$$

$$\forall \alpha > 0, \quad \exists C_\alpha > 0 \quad \text{such that} \quad g(s) \leq C_\alpha e^{\alpha s^2}, \quad \forall s \geq 0. \tag{2.5.5}$$

$$g \in C^1(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad \forall \alpha > 0, \quad \exists C_\alpha > 0 \quad \text{such that} \quad |g'(s)| \leq C_\alpha e^{\alpha s^2}, \quad \forall s \in \mathbb{R}. \tag{2.5.6}$$

We note that condition (2.5.5) replaces condition (1.1.2) of previous chapter. These conditions together with a type of Moser-Trudinger inequality from [1], imply that the energy functional

$$\mathcal{S}(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx - \int_{\mathbb{R}^2} G(u)dx$$

is well-defined on $H^1(\mathbb{R}^2)$. We recall that

$$T(u) = \int_{\mathbb{R}^2} |\nabla u|^2 dx, \quad V(u) = \int_{\mathbb{R}^2} G(u)dx.$$

Now, we will present two main results:

Theorem 2.5.1. *Let g satisfying conditions (2.5.2)-(2.5.6); then, there exists a positive u ground-state solution of problem (2.5.1) such that u is radial, non-increasing with exponential decay at infinity.*

Theorem 2.5.2. *Let g satisfying conditions (2.5.2)-(2.5.6); then, there exist infinitely many distinct solutions $\{u_k\}_k \in C^2(\mathbb{R}^2)$ to problem (2.5.1), radial, non-increasing with exponential decay at infinity, for all $k \geq 1$ such that $\mathcal{S}(u_k) \nearrow +\infty$ as $k \to +\infty$.*

An important consequence of Pohožaev's identity (Proposition 1.2.1) for $N = 2$ is the following

Proposition 2.5.1. *Let $g \in C(\mathbb{R}, \mathbb{R})$ satisfying conditions (2.5.4) and (2.5.5). Let u a solution of the problem (2.5.1); then u satisfies*

$$\int_{\mathbb{R}^2} G(u)dx = 0.$$

Remark 2.5.2. *The previous proposition implies that condition (2.5.3) is necessary for non-triviality of the solution, since if we have $G(s) \leq 0$ for all $s \in \mathbb{R}$ and G has zero-average on $\mathbb{R}^2$, then G need to be identically zero.*

Smoothness of solutions and positivity of ground state can be proved as in Theorem (1.1.1), so we will only show existence of solutions. The proof of Theorem 2.5.1 is based on the constrained minimization problem

$$\text{minimize} \quad \left\{ \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx, u \in M \right\},$$

where $M = \left\{ u \in H^1(\mathbb{R}^2) \setminus \{0\}; \int_{\mathbb{R}^2} G(u)dx = 0 \right\}$ by Proposition 2.5.1.

Since the proofs are similar to Theorems (1.1.1) and (2.1.1), we will be sketchy.

Sketch of the proof of Theorem 2.5.1:

Step 1: $M \neq \emptyset$. As in Theorem 1.3.1, it follows from hypothesis (2.5.3).

Step 2: Selection of minimizing sequence. Let $\{u_n\}_n \subset M$ be a minimizing sequence. By Schwartz symmetrization and Pólya-Szegö inequality, we may consider $u_n \geq 0$ and radial non-increasing for all $n \in \mathbb{N}$. Up to rescalement, we may also assume $\|u_n\|_{L^2(\mathbb{R}^2)} = 1$ for all $n \in \mathbb{N}$.

Step 3: Passage to the limit. With the same techniques of Step 2 of previous chapter, we prove that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. Then, up to a subsequence, there exists $u \in H^1(\mathbb{R}^2)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ and $u_n \to u$ a.e. on $\mathbb{R}^2$.
Set $G_1(s) := G(s) + \frac{m}{2}s^2$. Now, due to Moser-Trudinger inequality and Strauss' compactness lemma, it is possibly to see that

$$\int_{\mathbb{R}^2} G_1(u_n)dx \to \int_{\mathbb{R}^2} G_1(u)dx \quad \text{as} \quad n \to +\infty.$$

Since $\{u_n\}_n \subset M$, we have

$$\int_{\mathbb{R}^2} G_1(u)dx = \frac{m}{2} > 0.$$

This implies that $u \not\equiv 0$. Furthermore, by Fatou's lemma we have

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx \leq \frac{1}{2} \inf_{v \in M} \int_{\mathbb{R}^2} |\nabla v|^2 dx =: A, \qquad \int_{\mathbb{R}^2} u^2 dx \leq 1,$$

which implies $\int_{\mathbb{R}^2} G(u)dx \geq 0$.

Now, suppose by contradiction that $\int_{\mathbb{R}^2} G(u)dx > 0$. Define $h : [0,1] \to \mathbb{R}$ as $h(t) := \int_{\mathbb{R}^2} G(tu)dx$. Observe that h is continuous function by Lebesgue's convergence theorem and hypotheses on g. We note that $h(0) = 0$ and $h(1) > 0$. Furthermore, for positive t close to 0, we have that $h(t) < 0$ by hypotheses (2.5.2) and (2.5.3). Then, by intermediate value theorem, there exists $t_0 \in (0,1)$ such that $h(t_0) = 0$. Thus, $t_0 u \in M$ and $\int_{\mathbb{R}^2} |\nabla(t_0 u)|^2 dx \geq 2A$. Finally,

$$2A \leq \int_{\mathbb{R}^2} |\nabla(t_0 u)|^2 dx = t_0^2 \int_{\mathbb{R}^2} |\nabla u|^2 dx \leq 2t_0^2 A < 2A,$$

which is absurd. So, $u \in M$ and it is a solution of the minimization problem.

Step 4: Conclusion It follows that u satisfies in H^1-sense the relation

$$-\Delta u = \theta g(u),$$

where θ is a Lagrange multiplier. As we saw previously, $\theta \geq 0$, that is $\theta > 0$ because $u \not\equiv 0$. Then, $v(x) := u(\frac{x}{\sqrt{\theta}})$ solves problem (2.5.1).

Sketch of the proof of Theorem 2.5.2:

The proof is based on seeking critical points of T over

$$N = \{H_r^1(\mathbb{R}^2) : V(u) \geq 0, \|u\|_{L^2(\mathbb{R}^2)} = 1\}.$$

If $u \in N$ is a critical point of T, then one may have $V(u) = 0$ or $V(u) > 0$. In the first case, there exists $\lambda \in \mathbb{R}$ Lagrange multiplier such that

$$-\Delta u = \lambda u \quad \text{in} \quad H^1(\mathbb{R}^2),$$

which is impossible since $u \not\equiv 0$. On the other hand, if $V(u) = 0$ there exist $\lambda, \mu \in \mathbb{R}$ Lagrange multipliers such that

$$-\Delta u = \lambda g(u) + \mu u.$$

Proposition 2.5.1 implies $\mu = 0$ and so we get a solution of the problem as above.

In order to prove the theorem, it is sufficient to show the existence of a sequence $\{c_k\}_k$ of critical values of T over N, such that $c_k \nearrow +\infty$ as $k \to +\infty$. The existence of $\{c_k\}$ is proved in the same way as in Sections 2.2 and 2.3, but requires two different technical lemma.

Step 1: Modified (P-S) condition.

Lemma 2.5.3. *If c is a critical value of $T_{|N}$, namely $T(u) = c$ for some $u \in N$, there exist $\epsilon, \delta, a > 0$ such that for all $\alpha, \beta \in \mathbb{R}$, it holds*

$$|T(u) - c| \leq \epsilon, \quad 0 \leq V(u) \leq a \quad \Rightarrow \quad \|T'(u) + \alpha g(u) + \beta u\|_{H^{-1}} \geq \delta.$$

Lemma 2.5.4. *For all $R > 0$, there exists $\delta > 0$ such that $\forall \alpha \in \mathbb{R}, u \in N$, it holds*

$$T(u) \leq R \quad \Rightarrow \quad \|T'(u) + \alpha u\|_{H^{-1}} \geq \delta.$$

These two lemmas can be proved by contradiction, using in particular Strauss'compactness lemma. Furthermore, Lemma 2.5.4 implies the boundedness in $H^1(\mathbb{R}^2)$ of Palais-Smale sequence $\{u_n\}_n$. Therefore, there exists $u \in H^1(\mathbb{R}^2)$ such that, up to a subsequence, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ and $u_n \to u$ a.e. in $\mathbb{R}^2$. As in previous sections, we prove that

$$G_1(u_n) \to G_1(u), \quad g_1(u_n)u_n \to g_1(u)u \quad \text{in} \quad L^1(\mathbb{R}^2),$$

since $g_1(s) = g(s) + ms$. Finally, we will prove below that $u_n \to u$ in $H^1(\mathbb{R}^2)$.

Step 2: Deformation lemma. The additional hypothesis (2.5.6) on g is used in this step. Now, using the lemma in Step 1, it is possible to prove that if $c > 0$ is not a critical value for $T_{|N}$, there exists $\epsilon > 0$ and a deformation $\eta : N \to N$ such that

$$\eta(\{u \in N; T(u) \leq c + \epsilon\}) \subset \{u \in N; T(u) \leq c - \epsilon\}.$$

Step 3: Existence of $\{c_k\}$ and behaviour of c_k as $k \to +\infty$. For $k \in \mathbb{N}$, we recall the set

$$\Sigma_k = \{B \subset N : B \text{ compact, symmetric, } \gamma(B) \geq k\},$$

where $\gamma(B)$ denotes the genus of B. Noting that $\Sigma_k \neq \emptyset$ for all $k \in \mathbb{N}$, due to Step 2, we can show that

$$c_k := \inf_{B \in \Sigma_k} \max_{u \in B} T(u) \to +\infty \quad \text{as} \quad k \to +\infty$$

and c_k is a critical value for $T_{|N}$ for $k \geq 1$. This concludes the proof of Theorem 2.5.2.

2.6 New multiplicity results in critical case

Case $N \geq 3$

In this section we will prove original results about the existence of many bound-state solutions in the critical case. More precisely, we consider the problem

$$(*)\ \begin{cases} -\Delta u + u = f(u) & \text{in} \quad \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \quad u \not\equiv 0, \end{cases}$$

where $N \geq 3$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous odd function satisfying conditions (1.4.3)-(1.4.6). In particular f satisfies

$$|f(s)| \geq \lambda |s|^{q-1} \quad \forall s \in \mathbb{R},$$

for some $q \in (2, 2^*)$ and $\lambda > 0$. We want to prove that, given any $k \in \mathbb{N}$, there exists $\lambda_k \gg 1$ such that (*) has k pairs of nontrivial solutions for all $\lambda > \lambda_k$.

Remark 2.6.1. *As in Section 1.4, we may consider the problem*

$$(*)\ \begin{cases} -\Delta u + mu = f(u) & \text{in} \quad \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \quad u \not\equiv 0, \end{cases}$$

with $m > 0$, up to rescalement.

The aim of these multiplicity results is to extend Perera's works for bounded domains ([26] in dimension $N \geq 3$ and [27] in the planar case) on the whole $\mathbb{R}^N$. Let us heuristically explain the idea of the method. Consider the C^1 manifold (see Section 1.4)

$$\mathcal{M} = \left\{ u \in H^1_r(\mathbb{R}^N); \quad \int_{\mathbb{R}^N} G(u)dx = 1 \right\},$$

and recall that

$$G(u) = \int_0^u (f(s) - s)ds = F(u) - \frac{u^2}{2}, \quad T(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad V(u) = \int_{\mathbb{R}^N} G(u)dx,$$

so the energy functional $\mathcal{S} = T - V$ is well-defined on $H^1(\mathbb{R}^N)$. By Schwarz symmetrization, we are looking for critical points of the constrained functional $T_{|\mathcal{M}}$ in order to get solutions of (*) after using Lagrange multipliers and a proper rescalement as in previous sections.
We recall that $\Sigma(\mathcal{M})$ denotes the set of compact and symmetric (with respect to the origin) subsets of $\mathcal{M}$. For $k \geq 1$, let

$$\Gamma_k = \{A \in \Sigma(\mathcal{M}) : \gamma(A) \geq k\},$$

where $\gamma(A)$ denotes the Krasnosel'skii's genus of A.

Our main result is the following:

Theorem 2.6.1. *Let $f : \mathbb{R} \to \mathbb{R}$ an odd continuous critical (for Sobolev embedding) function satisfying conditions (1.4.3)-(1.4.6). Then, given any $k \in \mathbb{N}$, there exists $\lambda_k \gg 1$ such that problem (*) has k pairs of nontrivial solutions $\pm u_1, ..., \pm u_k$ for all $\lambda > \lambda_k$. In particular, the number of solutions of (*) goes to infinity as $\lambda \to +\infty$.*

Remark 2.6.2. *The solutions found in Theorem 2.6.1 are radially decreasing, classical with exponential decay at infinity as in Theorem 1.1.1 which also works in the critical case.*

Remark 2.6.3. *A particularly interesting case of Theorem 2.6.1 is given by*

$$f(u) = \lambda |u|^{q-2}u + |u|^{2^*-2}u,$$

that is,

$$-\Delta u + u = \lambda |u|^{q-2}u + |u|^{2^*-2}u.$$

In order to prove it, we want to apply the well-known result (see Theorem 2.2.1) about critical values related to genus.

Theorem 2.6.2. *Let $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ be an even functional of class C^1 and consider for $k \geq 1$*

$$c_{k,\lambda} := \inf_{A \in \Gamma_k} \sup_{u \in A} J(u).$$

Furthermore, assume that J is bounded from below on $\mathcal{M}$ and that $J_{|\mathcal{M}}$ satisfies $(PS - c_{k,\lambda})$ for every $k \geq 1$. Then, for any $k \geq 1$, $c_{k,\lambda}$ is finite, it is a critical value of $J_{|\mathcal{M}}$ and $-\infty < c_{1,\lambda} \leq c_{2,\lambda} \leq ... \leq c_{k,\lambda} \leq ...$.

Remark 2.6.4. *In our case, we will consider $J = T \geq 0$. Furthermore, we proved in Section 1.4.2 that $c_{1,\lambda} = \inf_{\mathcal{M}} T > 0$, so $c_{k,\lambda} > 0$ for all $k \geq 1$.*

The idea is to prove that for each fixed $k \geq 1$, if $\lambda > \lambda_k$, then $c_{k,\lambda}$ is sufficiently small in order to get Palais-Smale condition. Precisely, we are going to prove

$$c_{k,\lambda} \to 0 \quad \text{as} \quad \lambda \to +\infty,$$

for each fixed $k \geq 1$. We will estimate them from above as in Theorem 2.3.1, in view of constructing a special set with genus equal to k. In particular, Theorem 2.3.1 does not require conditions on g but the only hypothesis is:

$$\exists \xi > 0 \quad \text{such that} \quad G(\xi) > 0,$$

which is verified in our case by condition (1.4.6). Indeed, in view of $\lambda > 1$, we can take by a simple calculation

$$\xi := q^{\frac{1}{q-2}}$$

independent of λ, for every $N \geq 2$.

With the same notations of the theorem, fix $\xi > 0$ as above, $k \in \mathbb{N}$ and let us consider $u \in \tau(\pi_{k-1}) \subset H_0^1(B_R)$, where $R := R_k \geq k+1$. Let define

$$\tilde{u} := \begin{cases} u & \text{on} \quad B_R \\ 0 & \text{on} \quad \mathbb{R}^N \setminus B_R \end{cases} \in H_r^1(\mathbb{R}^N).$$

We identify $T(u)$ and $V(u)$ with $T(\tilde{u})$ and $V(\tilde{u})$ respectively; that is, for $u \in H_0^1(B_R)$

$$T(u) = \int_{B_R} |\nabla u|^2 dx, \quad V(u) = \int_{B_R} G(u) dx.$$

Now, from Theorem 2.3.1

$$V(u) \geq 1 \quad \forall u \in \tau(\pi_{k-1}).$$

So, let us consider $\sigma = \sigma_u > 0$ such that $u_\sigma(x) = u(\frac{x}{\sigma}) \in \mathcal{M}$, i.e. $\sigma > 0$ such that

$$V(u_\sigma) = \sigma^N V(u) = 1 \iff \sigma_u = (V(u))^{-\frac{1}{N}}.$$

Furthermore, again from Theorem 2.3.1, there exists $C_k > 0$ such that

$$\|\nabla u\|_{L^2(B_R)}^2 \leq C_k.$$

Then, in view of $\gamma(\tau(\pi_{k-1})) = k$, $\quad u_\sigma \in M$ and invariance of genus under rescalement, one has the following estimate

$$c_{k,\lambda} \leq \sup_{u \in \tau(\pi_{k-1})} T(u_\sigma) = \sup_{u \in \tau(\pi_{k-1})} (\sigma_u^{N-2} T(u)) \leq C_k \sup_{u \in \tau(\pi_{k-1})} \sigma_u^{N-2},$$

where

$$\sup_{u \in \tau(\pi_{k-1})} \sigma_u^{N-2} = \left(\frac{1}{\inf_{u \in \tau(\pi_{k-1})} V(u)} \right)^{\frac{N-2}{N}}.$$

Now, from growth conditions on f odd and Poincare's inequality, for each fixed $k \geq 1$ and for all $u \in \tau(\pi_{k-1})$,

$$V(u) = \int_{B_R} G(u) dx = \int_{B_R} \left(F(u) - \frac{u^2}{2} \right) dx \geq \int_{B_R} \left(\frac{\lambda}{q} |u|^q - \frac{u^2}{2} \right) dx =$$

$$= \frac{\lambda}{q} \|u\|_{L^q(B_R)}^q - \frac{1}{2} \|u\|_{L^2(B_R)}^2 \geq \frac{\lambda}{q} \|u\|_{L^q(B_R)}^q - \frac{1}{2\lambda_1} \|\nabla u\|_{L^2(B_R)}^2 \geq \frac{\lambda}{q} \|u\|_{L^q(B_R)}^q - \frac{C_k}{2\lambda_1},$$

where $\lambda_1 := \lambda_1(k) > 0$ is the first eigenvalue of $-\Delta$ in Dirichlet's problem on B_R depending on k.

On the other hand, by construction of $\tau(\pi_{k-1})$, one has

$$\int_{B_R} |u|^q dx \geq \xi^q |B_{R-k}| \geq \xi^q |B_1| = \xi^q \frac{\omega_N}{N} \quad \forall u \in \tau(\pi_{k-1}).$$

Finally, the above estimates imply for each fixed $k \geq 1$:

$$c_{k,\lambda} \leq \frac{C_k}{2} \left(\frac{\lambda \xi^q \omega_N}{qN} - \frac{C_k}{2\lambda_1(k)} \right)^{-\frac{N-2}{N}} \to 0 \quad \text{as} \quad \lambda \to +\infty. \tag{2.6.1}$$

Now, we will prove a proposition concerning conditions about Palais-Smale sequences for $T_{|\mathcal{M}}$.

Proposition 2.6.5. *Let $\{u_n\}_n \subset \mathcal{M}$ such that $T(u_n) \to c \in (0, 2^{-\frac{2}{N}} S)$ and $T'_{|\mathcal{M}}(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ where S is the best Sobolev constant defined in Section 1.4. Then, up to subsequences, $u_n \to u$ in $H^1(\mathbb{R}^N)$ for some $u \in H^1_r(\mathbb{R}^N) \setminus \{0\}$.*

Proof. First of all, we will prove that $\|u_n\|_{H^1(\mathbb{R}^N)}$ is bounded. Let $\{u_n\}$ such that

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \to c \quad \text{and} \quad \int_{\mathbb{R}^N} F(u_n) dx = \int_{\mathbb{R}^N} \frac{|u_n|^2}{2} dx + 1 \quad \forall n \in \mathbb{N}.$$

Using the growth assumptions on f, there exists $C > 0$ such that

$$F(s) \leq \frac{1}{4} s^2 + C|s|^{2^*}, \quad \forall s \in \mathbb{R}.$$

Hence

$$C \int_{\mathbb{R}^N} |u_n|^{2^*} dx \geq \frac{1}{4} \int_{\mathbb{R}^N} |u_n|^2 dx + 1, \quad \forall n \in \mathbb{N}.$$

From definition of S,

$$\int_{\mathbb{R}^N} |u_n|^{2^*} dx \leq S^{-\frac{2^*}{2}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^{\frac{2^*}{2}}, \quad \forall n \in \mathbb{N}.$$

Therefore, $\|u_n\|_{H^1(\mathbb{R}^N)}$ is bounded. Then, up to subsequences, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$.
Now, $T'_{|\mathcal{M}}(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ implies that exists a sequence $\{\theta_n\}_n \subset \mathbb{R}$ of Lagrange multipliers such that

$$\theta_n T'(u_n) - V'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N),$$

where

$$\theta_n := \frac{\int_{\mathbb{R}^N} g(u_n) u_n dx}{\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}, \quad \forall n \in \mathbb{N}$$

by a simple calculation, and $g(s) = f(s) - s$. As in Lemma 1.4.7, it is possible to prove that $u \not\equiv 0$ and

$$F(u_n) \to F(u) \quad \text{in} \quad L^1(\mathbb{R}^N). \tag{2.6.2}$$

Last condition and Fatou's lemma imply

$$V(u) \geq 1.$$

By weak semi-continuity,

$$T(u) \leq \frac{1}{2} \liminf_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = c.$$

Now, we want to prove that $u_n \to u$ in $H^1(\mathbb{R}^N)$, i.e.

$$V(u) = 1 \quad \text{and} \quad T(u) = c.$$

By the growth assumptions on f,

$$\limsup_{n\to+\infty} |\theta_n| \leq \frac{C \limsup_{n\to+\infty} \int_{\mathbb{R}^N} (u_n^2 + |u_n|^{2^*}) dx}{\int_{\mathbb{R}^N} |\nabla u|^2 dx} \leq C',$$

which implies, up to subsequences, $\theta_n \to \theta$ in $\mathbb{R}$. Then, by continuity of g,

$$g(u_n) \rightharpoonup -\theta \Delta u$$

in distributional sense, and so

$$-\theta \Delta u = g(u)$$

in H^1-sense. Integrating by parts, we obtain

$$\theta \int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_{\mathbb{R}^N} g(u) u dx. \tag{2.6.3}$$

Furthermore, Pohožaev's identity

$$\frac{(N-2)}{2} \theta \int_{\mathbb{R}^N} |\nabla u|^2 dx = NV(u) \geq N,$$

implies that $\theta > 0$.

Now, with the same arguments used to prove (2.6.2) in Section 1.4, one obtains

$$\int_{\mathbb{R}^N} f(u_n) u_n dx \to \int_{\mathbb{R}^N} f(u) u dx.$$

Hence, by Fatou's lemma and previous inequality one obtains

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^N} g(u_n) u_n dx \leq \int_{\mathbb{R}^N} g(u) u dx,$$

which implies

$$0 < \theta = \liminf_{n\to+\infty} \left(\frac{\int_{\mathbb{R}^N} g(u_n) u_n dx}{\int_{\mathbb{R}^N} |\nabla u_n|^2 dx} \right) \leq \frac{\int_{\mathbb{R}^N} g(u) u dx}{\int_{\mathbb{R}^N} |\nabla u|^2 dx}. \tag{2.6.4}$$

Finally, using (2.6.3), we obtain the equality in (2.6.4) and so $u_n \to u$ in $L^2(\mathbb{R}^N)$ (which implies $V(u) = 1$) and $T(u) = c$. □

Remark 2.6.6. *The upper bound* $2^{-\frac{2}{N}}S$ *for Palais-Smale level c, is given by Lemma 1.4.5. Indeed, assuming that* u_n *does not converge strongly to* u *in* $H^1(\mathbb{R}^N)$*, repeating the same arguments of Lemmas 1.4.4 and 1.4.5, it follows that* $c \geq 2^{-\frac{2}{N}}S$*.*
In conclusion, in order to prove the theorem, it suffices to show that exists a bound $c^* > 0$ *under which the compactness is guaranteed.*

Finally we are ready to prove Theorem 2.6.1.

Proof. (*of Theorem 2.6.1*) We apply Theorem 2.6.2 with $J = T$. In view of Proposition 2.6.6 and $c_{k,\lambda} > 0$ for all $k \geq 1$, we have to check that for each $k \geq 1$ fixed,

$$c_{k,\lambda} < 2^{-\frac{2}{N}}S. \tag{2.6.5}$$

So, from (2.6.1), if we choose

$$\lambda > \lambda_k$$

for a suitable choice of $\lambda_k \gg 1$, one has (2.6.2), hence the thesis.

□

Case $N = 2$

In the case $N = 2$, the previous argument does not work anymore. Indeed, the kinetic part of the functional is invariant under rescalement, namely

$$T(u_\sigma) = T(u).$$

Furthermore, the most essential difference is that the C^1- constraint $\mathcal{M}$ becomes

$$\mathcal{M} = \left\{ u \in H^1_r(\mathbb{R}^2) \setminus \{0\} : \quad \int_{\mathbb{R}^2} G(u)dx = 0 \right\}$$

by Pohožaev's identity (Proposition 1.2.1). However, the idea is to apply a constrained approach, seeking many critical values of $T_{|\mathcal{M}}$ using again Theorem 2.3.1.

Our main result is the following:

Theorem 2.6.3. *Let* $f : \mathbb{R} \to \mathbb{R}$ *an odd continuous critical (for Moser-Trudinger inequality) function satisfying conditions (1.4.3)-(1.4.6). Then, given any* $k \in \mathbb{N}$*, there exists* $\lambda_k \gg 1$ *such that problem (*) has* k *pairs of nontrivial solutions* $\pm u_1, ..., \pm u_k$ *for all* $\lambda > \lambda_k$*. In particular, the number of solutions of (*) goes to infinity as* $\lambda \to +\infty$*.*

As already seen above, we want to apply Theorem 2.6.2. Hence, we are going to estimate

$$c_{k,\lambda} = \inf_{A \in \Gamma_k} \sup_{u \in A} T(u)$$

using appropriated dilated functions $\{tu\}_{t \in \mathbb{R}^+}$ and the fact that genus does not change under translations.

Proposition 2.6.7. *Fix $k \in \mathbb{N}$. Then, for any $u \in \tau(\pi_{k-1})$, there exists $t_{u,\lambda} > 0$ such that $V(t_{u,\lambda}u) = 0$, i.e. $t_{u,\lambda}u \in \mathcal{M}$.*

Proof. We claim that for all $u \in \tau(\pi_{k-1})$, exists $\tilde{t}_{u,\lambda} > 0$ such that $V(\tilde{t}_{u,\lambda}u) < 0$.

Let $t > 0$. By hypotheses on f, we know that

$$\lim_{s\to 0} \frac{F(s)}{s^2} = 0.$$

Then, there exists $\delta > 0$ such that $F(tu) \leq \frac{t^2u^2}{4}$ for all $0 < t < \frac{\delta}{|u|}$ (remember that $0 \notin \tau(\pi_{k-1})$).
So, in view of $\lambda > 1$, define

$$\tilde{t}_{u,\lambda} := \frac{\delta}{\lambda^\alpha |u|} > 0, \tag{2.6.6}$$

where $\alpha := \frac{1}{2} + \frac{1}{q}$. Hence,

$$V(\tilde{t}_{u,\lambda}u) = \int_{B_R} \left(F(\tilde{t}_{u,\lambda}u) - \frac{\tilde{t}^2_{u,\lambda}u^2}{2} \right) dx \leq -\frac{\tilde{t}^2_{u,\lambda}}{4} \inf_{u\in\tau(\pi_{k-1})} \|u\|^2_{L^2(B_R)} \leq -C\frac{\tilde{t}^2_{u,\lambda}}{4}$$

for some positive constant $C > 0$ as in case $N \geq 3$.

On the other hand, we say that for all $u \in \tau(\pi_{k-1})$, there exists $\bar{t}_{u,\lambda} > \tilde{t}_{u,\lambda}$ such that $V(\bar{t}_{u,\lambda}u) = \frac{1}{\lambda} > 0$. Indeed, since f is odd

$$F(s) \geq \frac{\lambda}{q}|s|^q, \quad \forall s \in \mathbb{R}$$

for some $q \in (2, +\infty)$. Then, in view of Poincare's inequality, $\lambda \gg 1$ and assuming that $\bar{t}_{u,\lambda} > \tilde{t}_{u,\lambda}$, it holds

$$V(tu) \geq t^2 \left(\frac{\lambda t^{q-2}}{q} \|u\|^q_{L^q(B_R)} - \frac{1}{2}\|u\|_{L^2(B_R)} \right) >$$

$$> \tilde{t}^2_{u,\lambda} \left(\frac{\lambda t^{q-2}}{q} \inf_{u\in\tau(\pi_{k-1})} \|u\|^q_{L^q(B_R)} - \frac{1}{2\lambda_1(k)} \|\nabla u\|^2_{L^2(B_R)} \right) \geq$$

$$\geq \tilde{t}^2_{u,\lambda} \left(C\lambda t^{q-2} - \frac{C_k}{\lambda_1(k)} \right) = \frac{1}{\lambda} \Leftarrow \bar{t}_{u,\lambda} := \left(\left(\frac{1}{\lambda \tilde{t}^2_{u,\lambda}} + \frac{C_k}{\lambda_1(k)} \right) \frac{1}{C\lambda} \right)^{\frac{1}{q-2}} \tag{2.6.7}$$

for some positive constant C and C_k depending on k belonged to Theorem 2.3.1, and $\lambda_1(k)$ as in the previous case. Actually, by the choice of α in (2.6.6), it is possible to check that

$$\bar{t}_{u,\lambda} > \tilde{t}_{u,\lambda}.$$

Now, by continuity of $t \in (0, +\infty) \mapsto V(tu)$ guaranteed by hypotheses on f, the intermediate value theorem implies that exists $\tilde{t}_{u,\lambda} < t_{u,\lambda} < \bar{t}_{u,\lambda}$ such that

$$V(t_{u,\lambda}u) = 0 \iff t_{u,\lambda}u \in \mathcal{M}.$$

□

We are ready to estimate $c_{k,\lambda}$ for each fixed $k \geq 1$. In view of last proposition, $\gamma(\tau(\pi_{k-1})) = k$, Theorem 2.3.1 and invariance of genus under translations, one has by (2.6.7)

$$c_{k,\lambda} \leq \sup_{u\in\tau(\pi_{k-1})} T(t_{u,\lambda}u) = \sup_{u\in\tau(\pi_{k-1})} (t_{u,\lambda}^2 T(u)) \leq C_k \sup_{u\in\tau(\pi_{k-1})} t_{u,\lambda}^2 \leq C_k \sup_{u\in\tau(\pi_{k-1})} \bar{t}_{u,\lambda}^2 =$$

$$= \frac{C_k'}{\lambda^{\frac{2}{q-2}}}\left(\frac{1}{\lambda \inf_{u\in\tau(\pi_{k-1})} \tilde{t}_u^2} + \frac{C_k}{\lambda_1(k)}\right)^{\frac{2}{q-2}} = \frac{C_k'}{\lambda^{\frac{2}{q-2}}}\left(\frac{\lambda^{2\alpha-1}\xi^2}{\delta^2} + \frac{C_k}{\lambda_1(k)}\right)^{\frac{2}{q-2}} \to 0 \quad (2.6.8)$$

as $\lambda \to +\infty$ for each $k \geq 1$ fixed, by (2.6.6) and $\xi > 0$ independent of λ such that $G(\xi) > 0$ (we can take $\xi = q^{\frac{1}{q-2}}$ as in the case $N \geq 3$).

Now, as above we prove a proposition which ensures the Palais-Smale condition.

Proposition 2.6.8. *Let $\{u_n\}_n \subset \mathcal{M}$ such that $T(u_n) \to c \in (0, \frac{1}{2})$ and $T'_{|\mathcal{M}}(u_n) \to 0$ in $H^{-1}(\mathbb{R}^2)$. Then, up to subsequences, $u_n \to u$ in $H^1(\mathbb{R}^2)$ for some $u \in H_r^1(\mathbb{R}^2) \setminus \{0\}$.*

Proof. Let $\{u_n\} \subset H^1(\mathbb{R}^2)$ such that

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2 dx \to c \quad \text{and} \quad \int_{\mathbb{R}^2} F(u_n)dx = \frac{1}{2}\int_{\mathbb{R}^2} |u_n|^2 dx, \quad \forall n \in \mathbb{N}. \qquad (2.6.9)$$

As already seen in dimension $N = 2$, up to rescalement, we can assume

$$\int_{\mathbb{R}^2} |u_n|^2 dx = 1 \quad \forall n \in \mathbb{N}.$$

Hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$ and so, up to subsequences, there exists $u \in H_r^1(\mathbb{R}^2)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$. Now, since $c < \frac{1}{2}$, Lemma 1.4.9 implies

$$\int_{\mathbb{R}^2} F(u_n)dx \to \int_{\mathbb{R}^2} F(u)dx$$

and so $u \not\equiv 0$ by (2.6.9). Fatou's lemma and weak semi-continuity imply

$$T(u) \leq c \quad \text{and} \quad V(u) \geq 0.$$

Now, we want to prove strong convergence of u_n in $H^1(\mathbb{R}^2)$, namely $T(u) = c$ and $V(u) = 0$. Since $T'_{|\mathcal{M}}(u_n) \to 0$ in $H^{-1}(\mathbb{R}^2)$, there exists a sequence of $\{\theta_n\}_n \subset \mathbb{R}$ of Lagrange multipliers such that

$$\theta_n T'(u_n) - V(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^2),$$

where, for any $n \in \mathbb{N}$,

$$\theta_n = \frac{\int_{\mathbb{R}^2} g(u_n)u_n dx}{\int_{\mathbb{R}^2} |\nabla u_n|^2 dx}.$$

First, observe that by condition (1.4.5), one has

$$\int_{\mathbb{R}^2} g(u_n)u_n dx = \int_{\mathbb{R}^2} (f(u_n)u_n - u_n^2)dx > 2\int_{\mathbb{R}^2} \left(F(u_n) - \frac{u_n^2}{2}\right)dx = 0,$$

implying that

$$\int_{\mathbb{R}^2} g(u_n)u_n dx > 0 \quad \forall n \in \mathbb{N}$$

and $\theta_n > 0$. So, by growth assumptions on f,

$$\limsup_{n\to+\infty} \theta_n \le \frac{C \limsup_{n\to+\infty} \int_{\mathbb{R}^2}(u_n^2 + |u_n|(e^{4\pi u_n^2} - 1))dx}{\int_{\mathbb{R}^2} |\nabla u|^2 dx} \le$$

$$\le \frac{C' \limsup_{n\to+\infty} \int_{\mathbb{R}^2}(u_n^2 + e^{\frac{4\pi}{1-\epsilon}u_n^2} - 1)dx}{\int_{\mathbb{R}^2} |\nabla u|^2 dx} \le C''$$

by Moser-Trudinger inequality due to Cao [10] and $\epsilon > 0$ sufficiently small such that $2c < 1 - \epsilon < 1$. Then, up to subsequences, $\theta_n \to \theta \ge 0$. As in the proof of Proposition 2.6.8, we have

$$-\theta \Delta u = g(u) \quad \text{in} \quad H^1(\mathbb{R}^2). \tag{2.6.10}$$

Pohožaev's identity (Proposition 1.2.1) in dimension $N = 2$ implies that

$$V(u) = 0,$$

and consequently

$$u_n \to u \quad \text{in} \quad L^2(\mathbb{R}^2).$$

Now, condition (2.6.10) implies $\theta > 0$. Indeed, if θ would be 0, then

$$g(u) = 0,$$

so u would be 0, which leads to a contradiction. Hence, $\theta > 0$ and $T(u) = c$ as in Proposition 2.6.8. □

Finally we are ready to prove Theorem 2.6.3.

Proof. We apply Theorem 2.6.2 with $J = T$. In view of last proposition and $c_{k,\lambda} > 0$ for all $k \ge 1$, we have to check that for each $k \ge 1$,

$$c_{k,\lambda} < \frac{1}{2}.$$

From (2.6.8), if we choose

$$\lambda > \lambda_k$$

for a suitable choice of $\lambda_k \gg 1$, we conclude the proof. □

Chapter 3

Existence of ground-states for Choquard equations

3.1 Subcritical case

We consider the problem

$$(*)\ \begin{cases} -\Delta u + u = (I_\alpha * F(u))f(u) & \text{in} \quad \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \quad u \not\equiv 0, \end{cases}$$

where $N \geq 3$, $\alpha \in (0, N)$, $F \in C^1(\mathbb{R}; \mathbb{R})$ with $f := F'$ and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined for every $x \in \mathbb{R}^N \setminus \{0\}$ as

$$I_\alpha(x) := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}} 2^\alpha |x|^{N-\alpha}},$$

where $\Gamma(\cdot)$ denotes the Euler's Gamma function. Solutions of (*) are formally critical points of the functional $\mathcal{S} : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{S}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx.$$

We prove the existence of a ground-state solution in the subcritical case, that is we assume that nonlinearity $f \in C(\mathbb{R}; \mathbb{R})$ satisfies the growth assumptions:

(f_1) there exists C>0 such that $\forall s \in \mathbb{R}, \quad |sf(s)| \leq C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}})$,

(f_2) $\lim_{s\to 0} \frac{F(s)}{|s|^{\frac{N+\alpha}{N}}} = 0$ and $\lim_{|s|\to +\infty} \frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}} = 0$,

(f_3) there exists $s_0 \in \mathbb{R}$ such that $F(s_0) \neq 0$.

It is standard to check, using condition (f_1) and Hardy-Littlewood-Sobolev inequality (see Proposition A.0.6 with the choice $f = g = F$, $p = t = \frac{2N}{N+\alpha}$, $\lambda = N - \alpha$), that $\mathcal{S} \in C^1(H^1(\mathbb{R}^N))$.

Remark 3.1.1. *The equation*

$$-\Delta u + u = g(u) \quad in \quad \mathbb{R}^N \tag{3.1.1}$$

studied in previous chapters, can be considered as a limiting problem of (*) when $\alpha \to 0^+$*, with* $g = Ff$*. Indeed, as* $\alpha \to 0^+$*,* I_α *converges to a Dirac delta measure in the vague sense ([20], p.46). So conditions* $(f_1) - (f_3)$ *are in the same spirit of H. Berestycki and P.-L. Lions ([6]).*

The main result is the following theorem.

Theorem 3.1.1. *Assume that* $N \geq 3$ *and* $\alpha \in (0, N)$*. If* $f \in C(\mathbb{R}; \mathbb{R})$ *satisfies* $(f_1)-(f_3)$*, then problem (*) has a nontrivial ground-state.*

Furthermore, in the following we will prove that every solution u of (*) satisfies $u \in W^{2,q}_{loc}(\mathbb{R}^N)$ for all $q \geq 1$. This regularity information allows us to establish a Pohožaev's identity for all solutions of (*) valid also for $N = 2$.

Proposition 3.1.2. *Assume that* $N \geq 2$ *and* $\alpha \in (0, N)$*. If* $f \in C(\mathbb{R}; \mathbb{R})$ *satisfies* (f_1) *and* $u \in H^1(\mathbb{R}^N) \cap W^{2,2}_{loc}(\mathbb{R}^N)$ *solves (*), then*

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} u^2 dx = \frac{N+\alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) dx. \tag{3.1.2}$$

In particular, (3.1.2) implies that if $u \not\equiv 0$ solves (*), then

$$\mathcal{S}(u) = \frac{\alpha + 2}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} u^2 dx > 0.$$

Finally, we obtain qualitative properties of ground-states of (*), summed up in this

Theorem 3.1.2. *Assume that* $N \geq 3$ *and* $\alpha \in (0, N)$*. If* $f \in C(\mathbb{R}; \mathbb{R})$ *satisfies* (f_1) *and, in addition,* f *is odd and has constant sign on* $(0, +\infty)$*, then every ground-state of (*) has constant sign and is radially decreasing and symmetric with respect to the origin up to translation.*

Before explaining the proofs of these results, we make some remarks. With the same notations of Remark 3.1.1, we recall the strategy of H. Berestycki and P.-L. Lions' proof on the existence of a ground-state. They consider the constrained minimization problem

$$\min \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : u \in H^1(\mathbb{R}^N) \quad \text{and} \quad \int_{\mathbb{R}^N} \left(G(u) - \frac{u^2}{2} \right) dx = 1 \right\}.$$

They first show that by Pólya-Szegö inequality, the minimum can be taken among radial and radially decreasing functions. Then they show the existence of minimum $v \in H^1(\mathbb{R}^N)$ satisfying

$$-\Delta v = \theta(g(v) - v) \quad \text{in} \quad \mathbb{R}^N,$$

with a Lagrange multiplier $\theta > 0$. They conclude that $u(x) = v(\frac{x}{\sqrt{\theta}})$ solves (3.1.1).

Unluckily, this approach fails for problem (*). First, the nonlocal term is not preserved or controlled under Schwarz symmetrization unless f satisfies more restrictive assumptions of Theorem 3.1.2. Second, the final scaling argument fails because the three terms in (*) scale differently in space.
On the other hand, in order to prove the existence of a ground-state of (*), we use a mountain pass theorem. We will construct a Palais-Smale sequence at the mountain pass level, that satisfies asymptotically the Pohožaev's identity in order to ensure its boundedness easily. Such sequences will be denoted as Pohožaev-Palais-Smale sequences.

Finally, we will show that the absolute value of a ground-state and its polarization are also ground-states. This leads to a contradiction with the strong maximum principle if the solution is not invariant under these transformations.

Construction and convergence of Palais-Smale sequences

We first prove that there is a sequence of almost critical points at the mountain pass level defined by

$$b := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{S}(\gamma(t)),$$

where the set of paths is defined as

$$\Gamma := \{\gamma \in C([0,1]; H^1(\mathbb{R}^N)) : \gamma(0) = 0, \mathcal{S}(\gamma(1)) < 0\}.$$

We define the Pohožaev functional $\mathcal{P} : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$\mathcal{P}(u) := \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{N+\alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) dx.$$

Furthermore, let us consider

$$c := \inf\{\mathcal{S}(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\} \quad \text{is a solution of} \quad (*)\}.$$

Proposition 3.1.3. *If $f \in C(\mathbb{R};\mathbb{R})$ satisfies (f_1) and (f_3), then there exists a sequence $\{u_n\}_n \subset H^1(\mathbb{R}^N)$ such that, as $n \to +\infty$,*

$$\mathcal{S}(u_n) \to b \in (0, +\infty),$$
$$\mathcal{S}'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N),$$
$$\mathcal{P}(u_n) \to 0.$$

Proof. First, we have to prove that

$$0 < b < +\infty.$$

The case $b < +\infty$ is equivalent to show that $\Gamma \neq \emptyset$. So it is sufficient to construct $u \in H^1(\mathbb{R}^N)$ such that $\mathcal{S}(u) < 0$. Let $s_0 \neq 0$ as in (f_3) and set $w = s_0 \chi_{B_1} \in L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N)$; then

$$\int_{\mathbb{R}^N} (I_\alpha * F(w)) F(w) dx = F(s_0)^2 \int_{B_1} \int_{B_1} I_\alpha(x-y) dx dy > 0.$$

By (f_1) the left-hand side is continuous in $L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, and since $H^1(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, there exists $v \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (I_\alpha * F(v))F(v)dx > 0.$$

Now we define $u_\tau(x) := v(\frac{x}{\tau})$ for every $\tau > 0$. Hence,

$$\mathcal{S}(u_\tau) = \frac{\tau^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{\tau^N}{2}\int_{\mathbb{R}^N} v^2 dx - \frac{\tau^{N+\alpha}}{2}\int_{\mathbb{R}^N} (I_\alpha * F(v))F(v)dx,$$

and observe that for τ large enough, $\mathcal{S}(u_\tau) < 0$.

Now we prove that

$$b > 0.$$

From Hardy-Littlewood-Sobolev inequality, it follows that if $s \in (1, \frac{N}{\alpha})$, then for every $v \in L^s(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |I_\alpha * v|^{\frac{Ns}{N-\alpha s}} dx \leq C\Big(\int_{\mathbb{R}^N} |v|^s dx\Big)^{\frac{N}{N-\alpha s}}. \tag{3.1.3}$$

Respectively, by Holder's inequality, (3.1.3) with $s = \frac{2N}{N+\alpha}$ and condition (f_1), for every $u \in H^1(\mathbb{R}^N)$ we have:

$$\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx \leq \Big(\int_{\mathbb{R}^N} |I_\alpha * F(u)|^{\frac{2N}{N-\alpha}} dx\Big)^{\frac{N-\alpha}{2N}} \Big(\int_{\mathbb{R}^N} |F(u)|^{\frac{2N}{N+\alpha}} dx\Big)^{\frac{N+\alpha}{2N}} \leq$$

$$\leq C\Big(\int_{\mathbb{R}^N} |F(u)|^{\frac{2N}{N+\alpha}} dx\Big)^{1+\frac{\alpha}{N}} \leq C'\Big(\int_{\mathbb{R}^N} (u^2 + |u|^{\frac{2N}{N-2}})dx\Big)^{1+\frac{\alpha}{N}} \leq$$

$$\leq C''\Big(\|u\|_{L^2(\mathbb{R}^N)}^{2(1+\frac{\alpha}{N})} + \|\nabla u\|_{L^2(\mathbb{R}^N)}^{2(1+\frac{\alpha+2}{N-2})}\Big).$$

Hence there exists $\delta > 0$ such that if $\|u\|^2_{H^1(\mathbb{R}^N)} \leq \delta$, then

$$\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx \leq \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)dx,$$

and therefore

$$\mathcal{S}(u) \geq \frac{1}{4}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)dx.$$

In particular, if $\gamma \in \Gamma$, then $\int_{\mathbb{R}^N}(|\nabla\gamma(0)|^2 + |\gamma(0)|^2)dx = 0 < \delta < \int_{\mathbb{R}^N}(|\nabla\gamma(1)|^2 + |\gamma(1)|^2)dx$ and by the intermediate value theorem there exists $\bar{t} \in (0,1)$ such that $\int_{\mathbb{R}^N}(|\nabla\gamma(\bar{t})|^2 + |\gamma(\bar{t})|^2)dx = \delta$. So

$$\max_{t\in[0,1]} \mathcal{S}(\gamma(t)) \geq \mathcal{S}(\gamma(\bar{t})) \geq \frac{\delta}{4}.$$

Since $\gamma \in \Gamma$ is arbitrary, this implies that $b \geq \frac{\delta}{4} > 0$.

Finally, we are ready to construct such a Palais-Smale sequence $\{u_n\}$. Let define a map $\Phi : \mathbb{R} \times H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ by

$$\Phi(\sigma, v)(x) := v(e^{-\sigma} x).$$

So the functional $\mathcal{S} \circ \Phi$ has the form

$$\mathcal{S}(\Phi(\sigma, v)) = \frac{e^{\sigma(N-2)}}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{e^{N\sigma}}{2} \int_{\mathbb{R}^N} v^2 dx - \frac{e^{\sigma(N+\alpha)}}{2} \int_{\mathbb{R}^N} (I_\alpha * F(v))F(v) dx.$$

In view of (f_1) it is possible to check that $\mathcal{S} \circ \Phi \in C^1(\mathbb{R} \times H^1(\mathbb{R}^N))$. Now, we define the following family of paths

$$\tilde{\Gamma} := \Big\{ \tilde{\gamma} \in C([0,1]; \mathbb{R} \times H^1(\mathbb{R}^N)) : \tilde{\gamma}(0) = (0,0) \quad \text{and} \quad (\mathcal{S} \circ \Phi)(\tilde{\gamma}(1)) < 0 \Big\}.$$

Obviously, as $\Gamma = \{\Phi \circ \tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma}\}$,

$$b = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \sup_{t \in [0,1]} (\mathcal{S} \circ \Phi)(\tilde{\gamma}(t)).$$

By the minimax principle [42, theorem 2.9], there exists a sequence $\{(\sigma_n, v_n)\}_n \subset \mathbb{R} \times H^1(\mathbb{R}^N)$ such that as $n \to +\infty$,

$$(\mathcal{S} \circ \Phi)(\sigma_n, v_n) \to b \quad \text{and} \quad (\mathcal{S} \circ \Phi)'(\sigma_n, v_n) \to 0 \quad \text{in} \quad (\mathbb{R} \times H^1(\mathbb{R}^N))^*.$$

Since for every $(h, w) \in \mathbb{R} \times H^1(\mathbb{R}^N)$:

$$(\mathcal{S} \circ \Phi)'(\sigma_n, v_n)[h, w] = \mathcal{S}'(\Phi(\sigma_n, v_n))[\Phi(\sigma_n, w)] + \mathcal{P}(\Phi(\sigma_n, v_n))h,$$

we get the conclusion by taking $u_n := \Phi(\sigma_n, v_n)$.

□

Now we will show how a solution of (*) can be constructed from the sequence given by Proposition 3.1.4.

Proposition 3.1.4. *Let $f \in C(\mathbb{R}; \mathbb{R})$ and $\{u_n\}_n \subset H^1(\mathbb{R}^N)$. If f satisfies (f_1) and (f_2), $\{\mathcal{S}(u_n)\}_n$ is bounded and, as $n \to +\infty$,*

$$\mathcal{S}'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N) \quad \text{and} \quad \mathcal{P}(u_n) \to 0,$$

then, up to subsequences,

(i) either $u_n \to 0$ in $H^1(\mathbb{R}^N)$,

(ii) or there exists $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\mathcal{S}'(u) = 0$ and a sequence $\{x_n\}_n \subset \mathbb{R}^N$ such that $u_n(\cdot - x_n) \rightharpoonup u$ in $H^1(\mathbb{R}^N)$.

Proof. We first establish the boundedness of the sequence. Indeed, for every $n \in \mathbb{N}$,

$$\frac{\alpha+2}{2(N+\alpha)}\int_{\mathbb{R}^N}|\nabla u_n|^2dx+\frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}u_n^2dx=\mathcal{S}(u_n)-\frac{1}{N+\alpha}\int_{\mathbb{R}^N}\mathcal{P}(u_n).$$

As the right-hand side is bounded by our assumptions, the sequence $\{u_n\}_n$ is bounded in $H^1(\mathbb{R}^N)$.

Now we are going to prove the nonvanishing of the sequence. Assume that *(i)* does not hold, that is,

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^N}(|\nabla u_n|^2+u_n^2)dx>0. \tag{3.1.4}$$

We claim that for every $p \in (2, \frac{2N}{N-2})$,

$$\liminf_{n\to+\infty}\sup_{a\in\mathbb{R}^N}\int_{B_1(a)}|u_n|^pdx>0.$$

For every $n \in \mathbb{N}$

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(u_n)dx=\frac{N-2}{N+\alpha}\int_{\mathbb{R}^N}|\nabla u_n|^2dx+\frac{N}{N+\alpha}\int_{\mathbb{R}^N}u_n^2dx-\frac{2}{N+\alpha}\mathcal{P}(u_n),$$

so by (3.1.4) it follows that

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^N}(I_\alpha * F(u_n)F(u_n)dx>0. \tag{3.1.5}$$

The sequence $\{u_n\}$ satisfies the inequality ([21, lemma I.1],[42, lemma 1.21]) for every $n \in \mathbb{N}$

$$\int_{\mathbb{R}^N}|u_n|^pdx\le C\Big(\int_{\mathbb{R}^N}(|\nabla u_n|^2+u_n^2)dx\Big)\Big(\sup_{a\in\mathbb{R}^N}\int_{B_1(a)}|u_n|^pdx\Big)^{1-\frac{2}{p}}.$$

As F is continuous and satisfies (f_2), for every $\epsilon>0$, there exists $C_\epsilon>0$ such that for every $s\in\mathbb{R}$

$$|F(s)|^{\frac{2N}{N+\alpha}}\le\epsilon(s^2+|s|^{\frac{2N}{N-2}})+C_\epsilon|s|^p.$$

Since u_n is bounded in $H^1(\mathbb{R}^N)$, by Sobolev embedding

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^N}|F(u_n)|^{\frac{2N}{N+\alpha}}dx\le C''\epsilon+C'_\epsilon\Big(\liminf_{n\to+\infty}\sup_{a\in\mathbb{R}^N}\int_{B_1(a)}|u_n|^pdx\Big)^{1-\frac{2}{p}}.$$

Now, if $\liminf\limits_{n\to+\infty}\sup_{a\in\mathbb{R}^N}\int_{B_1(a)}|u_n|^pdx=0$, since $\epsilon>0$ is arbitrary

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^N}|F(u_n)|^{\frac{2N}{N+\alpha}}dx=0,$$

and the Hardy-Littlewood-Sobolev inequality implies that

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(u_n)dx = 0,$$

in contradiction with condition (3.1.5).

In conclusion, by definition of supremum, there exists $\{x_n\}\subset\mathbb{R}$ such that $\liminf_{n\to+\infty}\int_{B_1(x_n)}|u_n|^p dx > 0$. Since the problem (*) is invariant by translation, we can assume that $x_n = 0$ for all $n\in\mathbb{N}$. So for some $p\in(2,\frac{2N}{N-2})$,

$$\liminf_{n\to+\infty}\int_{B_1}|u_n|^p dx > 0.$$

By Rellich's theorem, this implies that up to a subsequence, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ to some $u\in H^1(\mathbb{R}^N)\setminus\{0\}$.

Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, using a standard diagonal argument and Rellich's theorem, it converges, up to a subsequence, to u a.e. in $\mathbb{R}^N$. By continuity of F, $F(u_n)$ converges a.e. to $F(u)$ in $\mathbb{R}^N$. Furthermore, since u_n is bounded in $H^1(\mathbb{R}^N)$, by Sobolev embedding and condition (f_1), $F(u_n)$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. So $F(u_n)\rightharpoonup F(u)$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. As the Riesz potential defines a linear continuous map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ by inequality (3.1.3), $I_\alpha * F(u_n)\rightharpoonup I_\alpha * F(u)$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$.

On the other hand, in view of (f_1) and by Rellich's theorem and dominated convergence theorem, $f(u_n)\to f(u)$ in $L^p_{loc}(\mathbb{R}^N)$ for every $p\in[1,\frac{2N}{\alpha+2})$. Hence, using Holder's inequality one can readily checks that, as $n\to+\infty$

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_n))f(u_n)\varphi dx \to \int_{\mathbb{R}^N}(I_\alpha * F(u))f(u)\varphi dx \quad \forall\varphi\in C_0^\infty(\mathbb{R}^N).$$

This implies that for every $\varphi\in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}(\nabla u\cdot\nabla\varphi + u\varphi)dx - \int_{\mathbb{R}^N}(I_\alpha * F(u))f(u)\varphi dx =$$

$$= \lim_{n\to+\infty}\left(\int_{\mathbb{R}^N}(\nabla u_n\cdot\nabla\varphi + u_n\varphi)dx - \int_{\mathbb{R}^N}(I_\alpha * F(u_n))f(u_n)\varphi dx\right) = 0;$$

that is, u is a weak solution of (*).

□

Corollary 3.1.3. *If $f\in C(\mathbb{R};\mathbb{R})$ satisfies conditions $(f_1)-(f_3)$, then problem (*) has a nontrivial solution $u\in H^1(\mathbb{R}^N)$.*

Proof. By Proposition 3.1.3, $\mathcal{S}$ admits a Pohožaev-Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}}$ at the level b. We apply Proposition 3.1.4 to $\{u_n\}_{n\in\mathbb{N}}$. If the first alternative occured, then we would have by continuity $\mathcal{S}(u_n)\to\mathcal{S}(0) = 0$ as $n\to+\infty$, in contradiction with the fact that $b > 0$. Therefore, the second alternative must occur. □

Regularity of solutions and Pohožaev-identity

The assumption (f_1) is not sufficient to apply the standard bootstrap method as in [38, proposition 4.1]. Instead, in order to prove regularity of solutions of (*), we extend the nonlocal Brezis-Kato regularity estimate [8, theorem 2.3] to a class of nonlocal linear equations.

Proposition 3.1.5. *Fix $u \in H^1(\mathbb{R}^N)$ which solves*

$$-\Delta u + u = (I_\alpha * Hu)K, \tag{3.1.6}$$

where $H(u) := \frac{F(u)}{u}$ and $K(u) := F'(u)$. Then, $u \in L^p(\mathbb{R}^N)$ for every $p \in [2, \frac{N}{\alpha}\frac{2N}{N-2})$. Moreover, there exists a constant $C_p > 0$ independent of u such that

$$\Big(\int_{\mathbb{R}^N} |u|^p dx\Big)^{\frac{1}{p}} \leq C_p\Big(\int_{\mathbb{R}^N} u^2 dx\Big)^{\frac{1}{2}}.$$

In order to prove the proposition, we will use a technical lemma whose proof is in the Appendix (Lemma A.0.7).

Lemma 3.1.6. *Let us consider $N \geq 2$, $\alpha \in (0,2)$, $\theta \in (0,2)$ and H, K defined as above. If $\frac{\alpha}{N} < \theta < 2 - \frac{\alpha}{N}$, then, for every $\epsilon > 0$, there exists $C_{\epsilon,\theta} \in \mathbb{R}$ such that for any fixed $u \in H^1(\mathbb{R}^N)$ which solves (3.1.6),*

$$\int_{\mathbb{R}^N} (I_\alpha * (H|u|^\theta))K|u|^{2-\theta} dx \leq \epsilon^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx + C_{\epsilon,\theta}\int_{\mathbb{R}^N} u^2 dx.$$

Now, we are ready to prove Proposition 3.1.5.

Proof. By Lemma 3.1.6 with $\theta = 1$, there exists $\lambda > 0$ such that for every $\varphi \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * |H\varphi|)|K\varphi| dx \leq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\varphi|^2 dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} \varphi^2 dx. \tag{3.1.7}$$

Choose sequences $\{H_n\}_n$ and $\{K_n\}_n$ in $L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$ such that $|H_n| \leq |H|$, $|K_n| \leq |K|$ and $H_n \to H$, $K_n \to K$ a.e. in $\mathbb{R}^N$. For each $n \in \mathbb{N}$, consider the form $a_n : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$a_n(\varphi,\psi) := \int_{\mathbb{R}^N} (\nabla\varphi \cdot \nabla\psi + \lambda\varphi\psi) dx - \int_{\mathbb{R}^N} (I_\alpha * H_n\varphi)K_n\psi dx.$$

Note that a_n is bilinear and coercive by (3.1.7). So, by the Lax-Milgram theorem, there exists a unique solution $u_n \in H^1(\mathbb{R}^N)$ of

$$-\Delta u_n + \lambda u_n = (I_\alpha * (H_n u_n))K_n + (\lambda - 1)u, \tag{3.1.8}$$

where $u \in H^1(\mathbb{R}^N)$ solves (3.1.6). It can be proved that the sequence $\{u_n\}_n$ converges weakly to u in $H^1(\mathbb{R}^N)$ as $n \to +\infty$.

For $\mu > 0$, we define the truncation $u_{n,\mu} : \mathbb{R}^N \to \mathbb{R}$ as

$$u_{n,\mu}(x) := \begin{cases} -\mu & \text{if} \quad u_n(x) \leq -\mu \\ u_n(x) & \text{if} \quad -\mu < u_n(x) < \mu \\ \mu & \text{if} \quad u_n(x) \geq \mu. \end{cases}$$

Since $G(u) = |u|^{q-2}u$ is globally Lipschitz on $\mathbb{R}$ for every $q \geq 2$, then $|u_{n,\mu}|^{q-2}u_{n,\mu} \in H^1(\mathbb{R}^N)$ for those q; therefore we can take it as a test function in (3.1.8):

$$\int_{\mathbb{R}^N} \left(\frac{4(q-1)}{q^2} |\nabla |u_{n,\mu}|^{\frac{q}{2}}|^2 + ||u_{n,\mu}|^{\frac{q}{2}}|^2 \right) dx \leq$$

$$\leq \int_{\mathbb{R}^N} \left((q-1)|u_{n,\mu}|^{q-2}|\nabla u_{n,\mu}|^2 + |u_{n,\mu}|^{q-2}u_{n,\mu}u_n \right) dx =$$

$$= \int_{\mathbb{R}^N} \left((I_\alpha * (H_n u_n))(K_n|u_{n,\mu}|^{q-2}u_{n,\mu}) + (\lambda - 1)u|u_{n,\mu}|^{q-2}u_{n,\mu} \right) dx.$$

If $q < \frac{2N}{\alpha}$, by Lemma 3.1.6 with $\theta = \frac{2}{q}$, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} (I_\alpha * |H_n u_{n,\mu}|)(|K_n||u_{n,\mu}|^{q-2}u_{n,\mu}) dx \leq \int_{\mathbb{R}^N} (I_\alpha * (|H||u_{n,\mu}|))(|K||u_{n,\mu}|^{q-1}) dx \leq$$

$$\leq \frac{2(q-1)}{q^2} \int_{\mathbb{R}^N} |\nabla |u_{n,\mu}|^{\frac{q}{2}}|^2 dx + C \int_{\mathbb{R}^N} ||u_{n,\mu}|^{\frac{q}{2}}|^2 dx.$$

Since the convolution is symmetric, we have

$$\frac{2(q-1)}{q^2} \int_{\mathbb{R}^N} |\nabla |u_{n,\mu}|^{\frac{q}{2}}|^2 dx \leq C' \int_{\mathbb{R}^N} (|u_n|^q + |u|^q) dx + \int_{A_{n,\mu}} (I_\alpha * (|K_n||u_n|^{q-1}))|H_n u_n| dx,$$

where

$$A_{n,\mu} := \{x \in \mathbb{R}^N : |u_n(x)| > \mu\}.$$

Since $q < \frac{2N}{\alpha}$, by the Hardy-Littlewood-Sobolev inequality (Proposition A.0.6 with $f = |K_n||u_n|^{q-1}$ and $g = |H_n u_n|\chi_{A_{n,\mu}}$, where $\chi_{A_{n,\mu}}$ denotes the characteristic function of $A_{n,\mu}$),

$$\int_{A_{n,\mu}} (I_\alpha * (|K_n||u_n|^{q-1}))|H_n u_n| dx \leq C \left(\int_{\mathbb{R}^N} ||K_n||u_n|^{q-1}|^r dx \right)^{\frac{1}{r}} \left(\int_{A_{n,\mu}} |H_n u_n|^s dx \right)^{\frac{1}{s}},$$

with $\frac{1}{r} = \frac{\alpha}{2N} + 1 - \frac{1}{q}$ and $\frac{1}{s} = \frac{\alpha}{2N} + \frac{1}{q}$. By Holder's inequality, if $u_n \in L^q(\mathbb{R}^N)$, then $|K_n||u_n|^{q-1} \in L^r(\mathbb{R}^N)$ and $|H_n u_n| \in L^s(\mathbb{R}^N)$, whence by Lebesgue's dominated convergence theorem for every $n \in \mathbb{N}$

$$\lim_{\mu \to +\infty} \int_{A_{n,\mu}} (I_\alpha * (|K_n||u_n|^{q-1}))|H_n u_n| dx = 0.$$

Now, in view of Sobolev embedding, definition of $u_{n,\mu}$ and $u_n \to u$ a.e., we have

$$\limsup_{n\to+\infty}\left(\int_{\mathbb{R}^N}|u_n|^{\frac{qN}{N-2}}dx\right)^{1-\frac{2}{N}} \leq C'' \limsup_{n\to+\infty}\int_{\mathbb{R}^N}|u_n|^q dx.$$

By iterating over q a finite number of times we cover the range $p \in [2, \frac{N}{\alpha}\frac{2N}{N-2})$.

□

We are finally ready to prove the following theorem which establishes additional regularity of solutions of (*).

Theorem 3.1.4. *Let $N \geq 3$ and $\alpha \in (0, N)$. If $f \in C(\mathbb{R};\mathbb{R})$ is odd, satisfies (f_1) and does not change sign on $(0,+\infty)$, then for every $u \in H^1(\mathbb{R}^N)$ which solves (*), it holds $u \in W^{2,q}_{loc}(\mathbb{R}^N)$ for any $q \geq 1$.*

Proof. Let us consider H, K defined as in Proposition 3.1.5. Observe that H is defined on the set $\{x \in \mathbb{R}^N : u(x) \neq 0\}$; on the other hand we will prove later that $|u| > 0$ on $\mathbb{R}^N$ if f is odd and does not change sign on $(0,+\infty)$. Since u solves (*), by Proposition 3.1.5 it follows that $u \in L^p(\mathbb{R}^N)$ for every $p \in [2, \frac{N}{\alpha}\frac{2N}{N-2})$. In view of (f_1), $F(u) \in L^q(\mathbb{R}^N)$ for every $q \in [\frac{2N}{N+\alpha}, \frac{N}{\alpha}\frac{2N}{N+\alpha})$. Since $\frac{N}{\alpha} < \frac{N}{\alpha}\frac{2N}{N+\alpha}$, by Proposition A.0.7 one has $I_\alpha * (F(u)) \in L^\infty(\mathbb{R}^N)$, and thus

$$|-\Delta u + u| \leq C(|u|^{\frac{\alpha}{N}} + |u|^{\frac{\alpha+2}{N-2}}).$$

Now by the classical bootstrap method for subcritical local problems in bounded domains, we deduce that $u \in W^{2,q}_{loc}(\mathbb{R}^N)$ for any $q \geq 1$. □

The further regularity of solutions allows us to prove Proposition 3.1.2.
The proof of Pohožaev's identity is classical and consists in testing the equation against a suitable cut-off of $x \cdot \nabla u(x)$ and integrating by parts.

Proof. (*of Proposition 3.1.2*) Fix $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi = 1$ in a neighbourhood of 0. Let define a function $v_\lambda : \mathbb{R}^N \to \mathbb{R}$ for every $\lambda \in \mathbb{R}$ as

$$v_\lambda(x) := \varphi(\lambda x)x \cdot \nabla u(x).$$

By Theorem 3.1.4, $u \in W^{2,2}_{loc}(\mathbb{R}^N)$, so $v_\lambda \in H^1(\mathbb{R}^N)$ and it can be used as a test function in the equation to obtain

$$\int_{\mathbb{R}^N}\nabla u \cdot \nabla v_\lambda dx + \int_{\mathbb{R}^N} u v_\lambda dx = \int_{\mathbb{R}^N}(I_\alpha * F(u))(f(u)v_\lambda)dx.$$

The left-hand side can be computed by integration by parts as

$$\int_{\mathbb{R}^N} u v_\lambda dx = \int_{\mathbb{R}^N} u(x)\varphi(\lambda x)x \cdot \nabla u(x)dx = \int_{\mathbb{R}^N}\varphi(\lambda x)x \cdot \nabla\left(\frac{u^2}{2}\right)(x)dx =$$

$$= -\int_{\mathbb{R}^N} (N\varphi(\lambda x) + \lambda x \cdot \nabla\varphi(\lambda x))\frac{u^2(x)}{2}dx.$$

Lebesgue's dominated convergence theorem implies that

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^N} uv_\lambda dx = -\frac{N}{2}\int_{\mathbb{R}^N} u^2 dx.$$

Similarly, as $u \in W^{2,2}_{loc}(\mathbb{R}^N)$, the gradient term can be written as

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda dx = \int_{\mathbb{R}^N} \varphi(\lambda x)\Big(|\nabla u|^2 + x \cdot \nabla\Big(\frac{|\nabla u|^2}{2}\Big)(x)\Big)dx =$$

$$= -\int_{\mathbb{R}^N} ((N-2)\varphi(\lambda x) + \lambda x \cdot \nabla\varphi(\lambda x))\frac{|\nabla u(x)|^2}{2}dx.$$

Again by Lebesgue's dominated convergence theorem

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda dx = -\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Finally the last term can be rewritten by

$$\int_{\mathbb{R}^N} (I_\alpha * F(u))(f(u)v_\lambda)dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (F\circ u)(y)I_\alpha(x-y)\varphi(\lambda x)x\cdot\nabla(F\circ u)(x)dxdy =$$

$$= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y)\Big((F\circ u)(y)\varphi(\lambda x)x\cdot\nabla(F\circ u)(x) + (F\circ u)(x)\varphi(\lambda y)y\cdot\nabla(F\circ u)(y)\Big)dxdy =$$

$$= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(u(y))I_\alpha(x-y)(N\varphi(\lambda x) + x\cdot\nabla\varphi(\lambda x))F(u(x))dxdy +$$

$$+\frac{N-\alpha}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(u(y))I_\alpha(x-y)\frac{(x-y)\cdot(x\varphi(\lambda x) - y\varphi(\lambda y))}{|x-y|^2}F(u(x))dxdy.$$

We can thus apply Lebesgue's dominated convergence theorem to conclude that

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)v_\lambda dx = -\frac{N+\alpha}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx.$$

□

Recovery of the ground-state and its qualitative properties

An important application of the Pohožaev's identity is the possibility to associate to any variational solution of (*) a path. The following proposition is crucial to recover a ground-state solution from Proposition 3.1.4.

Proposition 3.1.7. *Take $f \in C(\mathbb{R};\mathbb{R})$ satisfying (f_1) and $u \in H^1(\mathbb{R}^N)\setminus\{0\}$ solving (*). Then, there exists a path $\gamma_u \in \Gamma$ such that*

$$\gamma_u(1/2) = u \quad \textit{and} \quad \mathcal{S}(\gamma_u(t)) < \mathcal{S}(u) \quad \forall\, t \in [0,1]\setminus\{1/2\}.$$

Proof. We define the path $\tilde{\gamma} : [0,+\infty) \to H^1(\mathbb{R}^N)$ as

$$\tilde{\gamma}(\tau)(x) := \begin{cases} u(\frac{x}{\tau}) & \text{if} \quad \tau > 0 \\ 0 & \text{if} \quad \tau = 0. \end{cases}$$

The function $\tilde{\gamma}$ is continuous on $(0,+\infty)$ by integrability of u; for every $\tau > 0$,

$$\int_{\mathbb{R}^N} (|\nabla\tilde{\gamma}(\tau)|^2 + |\tilde{\gamma}(\tau)|^2)dx = \tau^{N-2}\int_{R^N} |\nabla u|^2 dx + \tau^N \int_{\mathbb{R}^N} u^2 dx,$$

so that $\tilde{\gamma}$ is continuous also at 0. As in Proposition 3.1.3, the functional can be computed for every $\tau > 0$ as

$$\mathcal{S}(\tilde{\gamma}(\tau)) = \frac{\tau^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\tau^N}{2}\int_{\mathbb{R}^N} u^2 dx - \frac{\tau^{N+\alpha}}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx$$

$$= \left(\frac{\tau^{N-2}}{2} - \frac{(N-2)\tau^{N+\alpha}}{2(N+\alpha)}\right)\int_{\mathbb{R}^N} |\nabla u|^2 dx + \left(\frac{\tau^N}{2} - \frac{N\tau^{N+\alpha}}{2(N+\alpha)}\right)\int_{\mathbb{R}^N} u^2 dx.$$

Now, one easily checks that $\mathcal{S}\circ\tilde{\gamma}$ achieves strict global maximum at $\tau = 1$, namely $\mathcal{S}(\tilde{\gamma}(\tau)) < \mathcal{S}(u) \quad \forall\tau \in [0,+\infty)\setminus\{1\}$. Since

$$\lim_{\tau\to+\infty} \mathcal{S}(\tilde{\gamma}(\tau)) = -\infty,$$

there exists $\tau_1 \gg 1$ such that

$$\tilde{\gamma}(0) = 0, \quad \tilde{\gamma}(1) = u, \quad \mathcal{S}(\tilde{\gamma}(\tau)) < \mathcal{S}(u) \quad \forall\tau \in [0,\tau_1]\setminus\{1\} \quad \text{and} \quad \mathcal{S}(\tilde{\gamma}(\tau_1)) < 0.$$

Finally, to get the required γ_u it suffices to take a suitable change of variables $\gamma_u(t) := \tilde{\gamma}(T(t))$ for some function $T \in C([0,1];\mathbb{R})$ satisfying $T(0) = 0$, $T(\frac{1}{2}) = 1$ and $T(1) = \tau_1$. □

We now have all the tools available to show Theorem 3.1.1, namely that the mountain-pass level b coincides with the ground-state energy level c.

Proof. (*of Theorem 3.1.1*) By Propositions 3.1.3 and 3.1.4, we get a Pohožaev-Palais-Smale sequence $\{u_n\}_n \subset H^1(\mathbb{R}^N) \setminus \{0\}$ at level $b > 0$ which, up to subsequences and translations, converges weakly to some $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ that solves (*).
Since $\lim_{n\to+\infty} \mathcal{P}(u_n) = 0$, Pohožaev's identity, Fatou's lemma and weak semi-continuity, we have

$$\mathcal{S}(u) = \mathcal{S}(u) - \frac{\mathcal{P}(u)}{N+\alpha} = \frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} u^2 dx \leq$$

$$\leq \liminf_{n\to+\infty} \left(\frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} u_n^2 dx \right) =$$

$$= \liminf_{n\to+\infty} \left(\mathcal{S}(u_n) - \frac{\mathcal{P}(u_n)}{N+\alpha} \right) = b.$$

By definition of c we have $\mathcal{S}(u) \geq c$, and hence $c \leq b$. Let $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ be another solution of (*) such that $\mathcal{S}(v) \leq \mathcal{S}(u)$. Now, Proposition 3.1.7 implies that $\mathcal{S}(v) \geq b \geq \mathcal{S}(u)$ by definition of b. We have thus proved that $\mathcal{S}(v) = \mathcal{S}(u) = c = b$. □

As a direct consequence of previous theorem, one can prove the strong convergence of Pohožaev-Palais-Smale sequence.

Corollary 3.1.5. *Under the assumptions of Propositions 3.1.3 and 3.1.3, if*

$$\liminf_{n\to+\infty} \mathcal{S}(u_n) \leq c,$$

then there exists $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ *such that* $\mathcal{S}'(u) = 0$ *and up to a subsequence and a translation,* $u_n \to u$ *in* $H^1(\mathbb{R}^N)$.

Proof. We can assume that, up to subsequences and translations, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N) \setminus \{0\}$. By previous theorem

$$\frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} u^2 dx =$$

$$= \liminf_{n\to+\infty} \frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} u_n^2 dx,$$

and hence, up to a subsequence, $u_n \to u$ in $H^1(\mathbb{R}^N) \setminus \{0\}$. □

As conclusion, we now prove some additive properties of ground-states.

Positivity of ground-states. In order to get positivity of any ground-state, we need the following lemma about optimal paths.

Lemma 3.1.8. *Let $f \in C(\mathbb{R};\mathbb{R})$ satisfying (f_1) and $\gamma \in \Gamma$. If there exists $\bar{t} \in (0,1)$ such that for every $t \in [0,1] \setminus \{\bar{t}\}$*

$$b = \mathcal{S}(\gamma(\bar{t})) > \mathcal{S}(\gamma(t)),$$

then $\mathcal{S}'(\gamma(\bar{t})) = 0$.

Proof. The proof of the lemma is standard arguing by contradiction and using a quantitative deformation lemma (see [42, lemma 2.3]). □

Proposition 3.1.9. *Let $f \in C(\mathbb{R};\mathbb{R})$ satisfying (f_1). If f is odd and does not change sign on $(0,+\infty)$, then any ground-state $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ of (*) has constant sign.*

Proof. Without loss of generality, we can assume that $f \geq 0$ on $(0,+\infty)$. By Proposition 3.1.7, there exists an optimal path $\gamma_u \in \Gamma$ on which $\mathcal{S}$ achieves its maximum at $\frac{1}{2}$ equal to $\mathcal{S}(u)$. Since f is odd, F is even and thus for every $v \in H^1(\mathbb{R}^N)$,

$$\mathcal{S}(|v|) = \mathcal{S}(v).$$

Hence, for every $t \in [0,1] \setminus \{\frac{1}{2}\}$,

$$\mathcal{S}(|\gamma(t)|) = \mathcal{S}(\gamma(t)) < \mathcal{S}(\gamma(1/2)) = \mathcal{S}(|\gamma(1/2)|).$$

By Lemma 3.1.8, $|u| = |\gamma(1/2)|$ is also a ground-state and it satisfies

$$-\Delta|u| + |u| = (I_\alpha * F(|u|))f(|u|).$$

By the strong maximum principle we conclude that $|u| > 0$ on $\mathbb{R}^N$ and thus u has constant sign. □

Symmetry of ground-states.

Proposition 3.1.10. *Take $f \in C(\mathbb{R};\mathbb{R})$ satisfying (f_1), odd and of constant sign on $(0,+\infty)$. Then, any ground-state $u \in H^1(\mathbb{R}^N)$ of (*) is radially decreasing and symmetric about some point x_0 in $\mathbb{R}^N$.*

The argument of the proof relies on polarizations. In the following, we will recall some necessary results of the theory of polarization.
Assume that $H \subset \mathbb{R}^N$ is a closed half-space and that σ_H is the reflection with respect to ∂H. The polarization $u^H : \mathbb{R}^N \to \mathbb{R}$ of $u : \mathbb{R}^N \to \mathbb{R}$ is defined as

$$u^H(x) := \begin{cases} \max(u(x), u(\sigma_H(x))) & \text{if} \quad x \in H \\ \min(u(x), u(\sigma_H(x))) & \text{if} \quad x \notin H. \end{cases}$$

We will use the following standard property of polarizations [13, lemma 5.3].

Lemma 3.1.11. *If $u \in H^1(\mathbb{R}^N)$, then $u^H \in H^1(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} |\nabla u^H|^2 dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

We shall also use a polarization inequality with equality cases [38, lemma 5.3].

Lemma 3.1.12. *Let $\alpha \in (0, N)$, $u \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ and $H \subset \mathbb{R}^N$ be a closed half-space. If $u \geq 0$, then*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)u(y)}{|x-y|^{N-\alpha}} dxdy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^H(x)u^H(y)}{|x-y|^{N-\alpha}} dxdy,$$

with equality if and only if either $u^H = u$ or $u^H = u \circ \sigma_H$.

The last tool that we need is a characterization of symmetric functions by polarizations [38, lemma 5.4].

Lemma 3.1.13. *let us consider $u \in L^2(\mathbb{R}^N)$ a nonnegative function. Then, there exist $x_0 \in \mathbb{R}^N$ and a decreasing function $v : (0, +\infty) \to \mathbb{R}$ such that for a.e. $x \in \mathbb{R}^N$, $u(x) = v(|x - x_0|)$ if and only if for every closed half-space $H \subset \mathbb{R}^N$, $u^H = u$ or $u^H = u \circ \sigma_H$.*

Proof. (of Proposition 3.1.10). The strategy is to prove that u^H is also a ground-state of (*) and deduce therefrom that $u = u^H$ or $u^H = u \circ \sigma_H$.

Without loss of generality, we can assume that $f \geq 0$ on $(0, +\infty)$. By Proposition 3.1.9, we can further assume that $u > 0$. We first observe that from Lemma 3.1.11 and definition of u^H, for every $u \in H^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} (|\nabla u^H| + |u^H|^2) dx = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx. \tag{3.1.9}$$

Now, in view of Proposition 3.1.7, there exists an optimal path $\gamma \in \Gamma$ such that $\gamma(1/2) = u$ and $\gamma(t) \geq 0$ for every $t \in [0, 1]$ by construction. For every half-space $H \subset \mathbb{R}^N$, let define the path $\gamma^H : [0, 1] \to H^1(\mathbb{R}^N)$ by $\gamma^H(t) := (\gamma(t))^H$. By (3.1.9), $\gamma^H \in C([0, 1]; H^1(\mathbb{R}^N))$.

Note that since F is increasing on $(0, +\infty)$, $F(u^H) = (F \circ u)^H$, and therefore, for every $t \in [0, 1]$, by condition (3.1.9) and Lemma 3.1.12,

$$\mathcal{S}(\gamma^H(t)) \leq \mathcal{S}(\gamma(t))$$

and so $\gamma^H \in \Gamma$. From this,

$$\max_{t \in [0,1]} \mathcal{S}(\gamma^H(t)) \geq b.$$

Since for every $t \in [0, 1] \setminus \{1/2\}$

$$\mathcal{S}(\gamma^H(t)) \leq \mathcal{S}(\gamma(t)) < b,$$

we have

$$\mathcal{S}(u^H) = \mathcal{S}(\gamma^H(1/2)) = \mathcal{S}(\gamma(1/2)) = \mathcal{S}(u) = b.$$

Combining last condition with (3.1.9) and Lemma 3.1.12, we get that $(F \circ u)^H = F(u)$ or $F(u^H) = F(u \circ \sigma_H)$ in $\mathbb{R}^N$. Assume that $(F \circ u)^H = F(u)$. Then, for every $x \in H$,

$$\int_{u(\sigma_H(x))}^{u(x)} f(s) ds = F(u(x) - F(u(\sigma_H(x))) \geq 0.$$

This implies that either $u(\sigma_H(x)) \le u(x)$ or $f = 0$ on the interval $[\min(u(x), u(\sigma_H(x))), \max(u(x), u(\sigma_H(x)))]$, for every $x \in H$. In particular, $f(u^H) = f(u)$ on H. Furthermore, it is possible to repeat the same argument and deduce $f(u^H) = f(u)$ on $\mathbb{R}^N \setminus H$.

Hence, by the previous inequalities and Lemma 3.1.8, we have $\mathcal{S}'(u^H) = 0$ and therefore u^H is a ground-state of (*) which solves

$$-\Delta u^H + u^H = (I_\alpha * F(u^H))f(u^H) = (I_\alpha * F(u))f(u).$$

Since u solves (*), we conclude that $u^H = u$.

If $F(u^H) = F(u \circ \sigma_H)$, we conclude similarly that $u^H = u \circ \sigma_H$. Since this holds for arbitrary H, we conclude by Lemma 3.1.13 that u is radially decreasing and symmetric about some point x_0 in $\mathbb{R}^N$.

□

3.2 Existence of ground-states in subcritical case on the plane

In the present section, we provide a general existence result for ground-state solutions of problem (*) in the planar case $N = 2$, which is a two dimensional counterpart of [37]. We need the following hypotheses on $F \in C^1(\mathbb{R};\mathbb{R})$:

(F_1) there exists $s_0 \in \mathbb{R}$ such that $F(s_0) \neq 0$,

(F_2) $\forall\, \theta > 0 \quad \exists\, C = C_\theta > 0 \quad$ such that $|F'(s)| \le C_\theta \min\{1, |s|^{\frac{\alpha}{2}}\} e^{\theta s^2} \quad \forall s \in \mathbb{R}$,

(F_3) $\lim_{s\to 0} \frac{F(s)}{|s|^{1+\frac{\alpha}{2}}} = 0$.

The main result reads as follows.

Theorem 3.2.1. *Assume $N = 2$ and $F \in C^1(\mathbb{R};\mathbb{R})$ satisfying conditions $(F_1) - (F_3)$. Then problem (*) has a nontrivial ground-state solution $u \in H^1(\mathbb{R}^2)$. Furthermore, if F is even and increasing on $(0, +\infty)$, then every ground-state of (*) has constant sign, radially decreasing and symmetric with respect to some point $x_0 \in \mathbb{R}^2$.*

Let us discuss the assumptions of Theorem 3.2.1. As in previous section, condition (F_1) is necessary for the existence of a nontrivial solution. On the other hand, condition (F_2) ensures that the energy functional $\mathcal{S} : H^1(\mathbb{R}^2) \to \mathbb{R}$ defined as

$$\mathcal{S}(u) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2)dx - \frac{1}{2}\int_{\mathbb{R}^2} (I_\alpha * F(u))F(u)dx,$$

is Frechet-differentiable with continuity on $H^1(\mathbb{R}^2)$ (see [22, proposition 2.3]) . The condition has a different shape, because in dimension $N = 2$ the critical nonlinearity for Sobolev embedding is not anymore a power but rather an exponential-type nonlinearity. Furthermore, by integrating F' from (F_2), it holds

$$\lim_{|s|\to+\infty} \frac{|F(s)| + |F'(s)||s|}{e^{\theta s^2}} = 0$$

for every $\theta > 0$. Finally, a subcriticality condition (F_3) still needs to be imposed in 0.

In order to prove Theorem 3.2.1, we will use a mountain pass construction as in [37]. Let recall some definitions. We start by constructing a Palais-Smale sequence for the mountain pass level

$$b = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{S}(\gamma(t)),$$

where the set of paths is

$$\Gamma = \{\gamma \in C([0,1]; H^1(\mathbb{R}^2)) : \gamma(0) = 0, \mathcal{S}(\gamma(1)) < 0\}.$$

In addition, the sequence satisfies asymptotically the Pohožaev's identity

$$\mathcal{P}(u) = \int_{\mathbb{R}^2} u^2 dx - \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_\alpha * F(u))F(u) dx = 0,$$

which implies the boundedness of the sequence in $H^1(\mathbb{R}^2)$.

We are left with showing that the solution u is actually a ground-state. To prove this, we first show that u satisfies Pohožaev's identity (Proposition 3.1.2) up to ensure further regularity, which turns out to be easier to prove from (F_2) than in dimension $N \geq 3$. The last tool we need is an optimal path $\gamma_v \in \Gamma$ associated to any solution v of (*). The construction of such paths is inspired by [37] but it is more delicate in the plane because dilations $t \longmapsto v(\cdot/t) \in H^1(\mathbb{R}^2)$ are not anymore continuous at $t = 0$.

Before proving Theorem 3.2.1, we need a quantitative estimate of Moser-Trudinger inequality of Adachi and Tanaka [1].

Proposition 3.2.1. *For any $\beta \in (0, 4\pi)$ there exists $C = C_\beta > 0$ such that for every $u \in H^1(\mathbb{R}^2)$ satisfying*

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx \leq 1 \quad \textit{and} \quad \int_{\mathbb{R}^2} u^2 dx \leq M < +\infty,$$

one has

$$\int_{\mathbb{R}^2} \min\{1, u^2\} e^{\beta u^2} dx \leq C_\beta \int_{\mathbb{R}^2} u^2 dx.$$

First, we construct a sequence of almost critical points which asymptotically satisfies (*) and the Pohožaev's identity.

Proposition 3.2.2. *Take $F \in C^1(\mathbb{R}; \mathbb{R})$ satisfying (F_1) and (F_2). Then there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^2)$ such that as $n \to +\infty$,*

$$\mathcal{S}(u_n) \to b \in (0, +\infty),$$

$$\mathcal{S}'(u_n) \to 0 \quad \textit{in} \quad H^{-1}(\mathbb{R}^2),$$

$$\mathcal{P}(u_n) \to 0.$$

Proof. The proof can be adapted from Proposition 3.1.3, using appropriately the growth condition (F_2). ☐

Now, we will construct a nontrivial solution of (*) from the sequence given by previous proposition.

Proposition 3.2.3. *Take $F \in C^1(\mathbb{R};\mathbb{R})$ satisfying (F_2) and (F_3) and let us consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^2)$ satisfying:*

(a) $\mathcal{S}(u_n)$ *is bounded,*

(b) $\mathcal{S}'(u_n) \to 0$ *in* $H^{-1}(\mathbb{R}^2)$ *as* $n \to +\infty$,

(c) $\mathcal{P}(u_n) \to 0$ *as* $n \to +\infty$.

Then, up to subsequences, as $n \to +\infty$

(i) either $u_n \to 0$ *in* $H^1(\mathbb{R}^2)$,

(ii) or there exist $u \in H^1(\mathbb{R}^2) \setminus \{0\}$ *solving (*) and a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ *such that* $u_n(\cdot - x_n) \rightharpoonup u$ *in* $H^1(\mathbb{R}^2)$.

We follow the strategy of [37], proposition 2.2]. Since the gradient does not appear in the Pohožaev's identity, it will be more delicate to show that the nonlocal term does not vanish.

Proof. We assume that the first alternative does not hold, namely

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + u_n^2)dx > 0.$$

For every $n \in \mathbb{N}$,

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2 dx + \frac{\alpha}{2(\alpha+2)}\int_{\mathbb{R}^2} u_n^2 dx = \mathcal{S}(u_n) - \frac{\mathcal{P}(u_n)}{2+\alpha}$$

implies that u_n is bounded in $H^1(\mathbb{R}^2)$.

Now, since $\mathcal{S}'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^2)$ as $n \to +\infty$, clearly $\mathcal{S}'(u_n)[u_n] \to 0$ as $n \to +\infty$, therefore

$$\int_{\mathbb{R}^2} (I_\alpha * F(u_n))F'(u_n)u_n dx = \int_{\mathbb{R}^2} (|\nabla u_n|^2 + u_n^2)dx - \mathcal{S}'(u_n)[u_n] \geq \frac{1}{C} \quad \forall n \gg 1,$$

for some constant $C > 0$. Taking $C_0 \geq \sup_{n\in\mathbb{N}} \|u_n\|^2_{H^1(\mathbb{R}^2)}$, we can apply Proposition 3.2.1 to $\frac{u_n}{\sqrt{C_0}}$ with $\beta = 2\pi$ and we obtain

$$\int_{\mathbb{R}^2} \min\{1, u_n^2\} e^{\frac{2\pi}{C_0}u_n^2} dx \leq C_0 \int_{\mathbb{R}^2} \min\left\{1, \frac{u_n^2}{C_0}\right\} e^{\frac{2\pi}{C_0}u_n^2} dx \leq C_0 C_{2\pi} \frac{\int_{\mathbb{R}^2} u_n^2 dx}{C_0} \leq C_0 C_{2\pi}$$

for each $n \in \mathbb{N}$. Moreover, we also have

$$\int_{\mathbb{R}^2} u_n^2 dx = \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_\alpha * F(u_n))F(u_n)dx + \mathcal{P}(u_n) =$$

$$= \left(1+\frac{\alpha}{2}\right)\int_{\mathbb{R}^2}(I_\alpha * F(u_n))F(u_n)dx + o(1)$$

as $n \to +\infty$. Hence, from conditions (3.1.3) and (F_2) with $\theta = \frac{2\pi}{C_0}$, we get

$$\frac{1}{C}\int_{\mathbb{R}^2}(I_\alpha * F(u_n))F'(u_n)u_n dx \le C'\left(\int_{\mathbb{R}^2}|F(u_n)|^{\frac{4}{2+\alpha}}dx\int_{\mathbb{R}^2}(|F'(u_n)||u_n|)^{\frac{4}{2+\alpha}}dx\right)^{\frac{2+\alpha}{4}} \le$$

$$\le C''\left(\int_{\mathbb{R}^2}\min\{1,u_n^2\}e^{\frac{2\pi}{C_0}u_n^2}dx\right)^{1+\frac{\alpha}{2}} \le C'''\left(\int_{\mathbb{R}^2}u_n^2 dx\right)^{1+\frac{\alpha}{2}} =$$

$$= C'''\left(\left(1+\frac{\alpha}{2}\right)\int_{\mathbb{R}^2}(I_\alpha * F(u_n))F(u_n)dx + o(1)\right)^{1+\frac{\alpha}{2}},$$

namely

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^2}(I_\alpha * F(u_n))F(u_n)dx > 0.$$

We now want to prove that u_n does not vanish. We will use the following inequality (see [42, lemma 1.21]): for every $n \in \mathbb{N}$ and $p > 2$,

$$\int_{\mathbb{R}^N}|u_n|^p dx \le C\left(\int_{\mathbb{R}^N}(|\nabla u_n|^2 + u_n^2)dx\right)\left(\sup_{a\in\mathbb{R}^N}\int_{B_1(a)}|u_n|^p dx\right)^{1-\frac{2}{p}}.$$

By assumptions (F_2) and (F_3), for every $\epsilon > 0$ there exists $C_{\epsilon,\theta} > 0$ such that

$$|F(s)|^{\frac{4}{2+\alpha}} \le \epsilon\min\{1,s^2\}e^{\theta s^2} + C_{\epsilon,\theta}|s|^p, \quad \forall s \in \mathbb{R}.$$

Therefore, $\forall n \gg 1$

$$\left(\sup_{a\in\mathbb{R}^N}\int_{B_1(a)}|u_n|^p dx\right)^{1-\frac{2}{p}} \ge \frac{1}{C}\frac{\int_{\mathbb{R}^2}|u_n|^p dx}{\int_{\mathbb{R}^2}(|\nabla u_n|^2 + u_n^2)dx} \ge$$

$$\ge \frac{1}{CC_0C_\epsilon}\left(\int_{\mathbb{R}^2}|F(u_n)|^{\frac{4}{2+\alpha}}dx - \epsilon\int_{\mathbb{R}^2}\min\{1,u_n^2\}e^{\frac{2\pi}{C_0}u_n^2}dx\right) \ge$$

$$\ge \frac{1}{C'_\epsilon}\left(\left(\int_{\mathbb{R}^2}(I_\alpha * F(u_n))F(u_n)dx\right)^{\frac{2+\alpha}{2}} - \epsilon C\int_{\mathbb{R}^2}u_n^2 dx\right) \ge \frac{1}{C'_\epsilon}\left(\frac{1}{C'} - \epsilon CC_0\right).$$

From the arbitrariness of ϵ and definition of supremum, there exists a sequence $\{x_n\} \subset \mathbb{R}$ such that $\liminf_{n\to+\infty}\int_{B_1(x_n)}|u_n|^p dx > 0$. Since the problem (*) is invariant under translations, we can assume that $x_n = 0$ for all $n \in \mathbb{N}$. Therefore, for every $p > 2$,

$$\liminf_{n\to+\infty}\int_{B_1}|u_n|^p dx > 0.$$

By Rellich's theorem, this implies that up to a subsequence, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ to some $u \in H^1(\mathbb{R}^2) \setminus \{0\}$. Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$, using a standard diagonal argument and Rellich's theorem, it converges up to a subsequence to u a.e. in $\mathbb{R}^2$. So, by the continuity of F, we also have $F(u_n) \to F(u)$ a.e. in $\mathbb{R}^2$ as $n \to +\infty$.
Moreover, (F_2) implies that $\{F(u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^2)$ for every $p \geq \frac{4}{2+\alpha}$. This implies that $F(u_n) \rightharpoonup F(u)$ in $L^p(\mathbb{R}^2)$ for every such p. As the Riesz potential defines a linear continuous map from $L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)$ to $L^{\frac{4}{2-\alpha}}(\mathbb{R}^2)$ by condition (3.1.3), Proposition A.0.9 (since $\frac{2}{\alpha} > \frac{4}{2+\alpha}$) implies that

$$I_\alpha * F(u_n) \rightharpoonup I_\alpha * F(u) \quad \text{in} \quad L^{\frac{4}{2-\alpha}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2).$$

By condition (F_2) and Proposition 3.2.1, the sequence $\{F'(u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^2)$ for every $p \geq \frac{4}{\alpha}$ and by continuity $F'(u_n) \to F'(u)$ a.e. in $\mathbb{R}^2$ as $n \to +\infty$. Now, dominated convergence theorem and condition (F_2) imply that $F'(u_n) \to F'(u)$ in $L^q_{loc}(\mathbb{R}^2)$ for every $q \in [1, +\infty)$. Hence, it is possible to check that

$$\int_{\mathbb{R}^2} (I_\alpha * F(u_n))F'(u_n)\varphi dx \to \int_{\mathbb{R}^2} (I_\alpha * F(u))F'(u)\varphi dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^2).$$

Therefore, for every $\varphi \in C_0^\infty(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla \varphi + u\varphi)dx - \int_{\mathbb{R}^N} (I_\alpha * F(u))F'(u)\varphi dx =$$

$$= \lim_{n\to+\infty} \left(\int_{\mathbb{R}^N} (\nabla u_n \cdot \nabla \varphi + u_n\varphi)dx - \int_{\mathbb{R}^N} (I_\alpha * F(u_n))F'(u_n)\varphi dx \right) = 0;$$

that is, u is a weak solution of (*).

□

Corollary 3.2.2. *If $F \in C^1(\mathbb{R};\mathbb{R})$ satisfies conditions $(F_1) - (F_3)$, then problem (*) has a nontrivial solution $u \in H^1(\mathbb{R}^2)$.*

Proof. By Proposition 3.2.2, $\mathcal{S}$ admits a Pohožaev-Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}}$ at the level b. We apply Proposition 3.2.3 to $\{u_n\}_{n\in\mathbb{N}}$. If the first alternative occured, then we would have by continuity $\mathcal{S}(u_n) \to \mathcal{S}(0) = 0$ as $n \to +\infty$, in contradiction with the fact that $b > 0$. Therefore, the second alternative must occur. □

Now we have to prove a local regularity result for solutions of (*), which is easier than in dimension $N \geq 3$. Indeed, the growth assumption (F_2) gives a good control on $I_\alpha * F(u)$, which permits to apply a standard bootstrap method.

Proposition 3.2.4. *Take $F \in C^1(\mathbb{R};\mathbb{R})$ satisfying condition (F_2) and $u \in H^1(\mathbb{R}^2)$ solving (*). Then $u \in W^{2,p}_{loc}(\mathbb{R}^2)$ for any $p \geq 1$.*

Proof. By (F_2) and Proposition 3.2.1, we deduce that if $v \in H^1(\mathbb{R}^2)$, then $F(v) \in L^p(\mathbb{R}^2)$ for every $p \geq \frac{4}{2+\alpha}$. Since $\frac{2}{\alpha} > \frac{4}{2+\alpha}$, by Proposition A.0.9 we get $I_\alpha * F(v) \in L^\infty(\mathbb{R}^2)$.

Therefore, any solution u of (*) satisfies

$$|-\Delta u + u| \leq C|F'(u)|$$

with $F'(u) \in L^p_{loc}(\mathbb{R}^2)$ for every $p \geq 1$ because of (F_2). By standard (interior) regularity theory on bounded domains, we deduce that $u \in W^{2,p}_{loc}(\mathbb{R}^2)$ for any $p \geq 1$. □

The extra regularity is crucial to say that all the solutions of (*) satisfy the Pohožaev's identity (Proposition 3.1.2), which in dimension $N = 2$ is the following

Proposition 3.2.5. *Take $F \in C^1(\mathbb{R};\mathbb{R})$ satisfying (F_2) and $u \in H^1(\mathbb{R}^2) \cap W^{2,2}_{loc}(\mathbb{R}^2)$ solving (*). Then,*

$$\mathcal{P}(u) = \int_{\mathbb{R}^2} u^2 dx - \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_\alpha * F(u))F(u)dx = 0.$$

As in dimension $N \geq 3$, the Pohožaev's identity allows us to associate to any solution v a path $\gamma_v \in \Gamma$ passing through v. The main difficult here is that the integral of $|\nabla u|^2$ is invariant by dilation. To overcome this difficulty, we will combine properly dilatations and multiplication by constants.

Proposition 3.2.6. *Take $F \in C^1(\mathbb{R};\mathbb{R})$ satisfying (F_2) and $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ solving (*). Then, there exists a path $\gamma_u \in \Gamma$ such that*

$$\gamma_u(1/2) = u \quad \text{and} \quad \mathcal{S}(\gamma_u(t)) < \mathcal{S}(u) \quad \forall t \in [0,1] \setminus \{1/2\}.$$

Proof. We consider the path $\tilde{\gamma} : [0,+\infty) \to H^1(\mathbb{R}^2)$ given for each $\tau \in [0,+\infty)$ by

$$(\tilde{\gamma}(\tau))(x) := \begin{cases} \frac{\tau}{\tau_0} u(x/\tau_0) & \text{if} \quad \tau \leq \tau_0 \\ u(x/\tau) & \text{if} \quad \tau \geq \tau_0, \end{cases}$$

with $\tau_0 \ll 1$ to be chosen later. The function $\tilde{\gamma}$ is clearly continuous on $[0,+\infty)$.

For $\tau \geq \tau_0$, Proposition 3.2.5 implies

$$\mathcal{S}(\tilde{\gamma}(\tau)) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{\tau^2}{2} \int_{\mathbb{R}^2} u^2 dx - \frac{\tau^{2+\alpha}}{2} \int_{\mathbb{R}^2} (I_\alpha * F(u))F(u)dx =$$

$$= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \left(\frac{\tau^2}{2} - \frac{\tau^{2+\alpha}}{2+\alpha}\right) \int_{\mathbb{R}^2} u^2 dx.$$

It is possible to check that $\mathcal{S}(\tilde{\gamma}(\tau))$ attains its strict maximum equal to $\mathcal{S}(u)$ in $\tau = 1$ and is negative for $\tau \geq \tau_1$, for some $\tau_1 \gg 1$.

For $\tau \leq \tau_0$, we use (F_2) and Proposition 3.2.1 (choosing appropriately $\theta > 0$) to the function $\tilde{\gamma}(\tau)/(\int_{\mathbb{R}^2} |\nabla \tilde{\gamma}(\tau)|^2 dx)^{1/2}$ to obtain

$$\int_{\mathbb{R}^2} |F(\tilde{\gamma}(\tau))|^{\frac{4}{2+\alpha}} dx \leq C \int_{\mathbb{R}^2} \min\{1, |\tilde{\gamma}(\tau)|^2\} e^{\frac{4\theta}{2+\alpha}|\tilde{\gamma}(\tau)|^2} dx \leq$$

$$\leq C\frac{\int_{\mathbb{R}^2}|\tilde{\gamma}(\tau)|^2dx}{\int_{\mathbb{R}^2}|\nabla\tilde{\gamma}(\tau)|^2dx} = C\tau_0^2\frac{\int_{\mathbb{R}^2}u^2dx}{\int_{\mathbb{R}^2}|\nabla u|^2dx}. \tag{3.2.1}$$

Therefore, because of inequality (3.1.3) and $\tau \leq \tau_0$, we have

$$\mathcal{S}(\tilde{\gamma}(\tau)) = \frac{\tau^2}{2\tau_0^2}\int_{\mathbb{R}^2}|\nabla u|^2dx + \frac{\tau^2}{2}\int_{\mathbb{R}^2}u^2dx - \frac{1}{2}\int_{\mathbb{R}^2}(I_\alpha * F(\tilde{\gamma}(\tau)))F(\tilde{\gamma}(\tau))dx \leq$$

$$\leq \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2dx + \frac{\tau_0^2}{2}\int_{\mathbb{R}^2}u^2dx + C\Big(\int_{\mathbb{R}^2}|F(\tilde{\gamma}(\tau))|^{\frac{4}{2+\alpha}}dx\Big)^{1+\frac{\alpha}{2}}.$$

Hence, in view of (3.2.1) and Pohožaev's identity

$$\mathcal{S}(\tilde{\gamma}(\tau)) \leq \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2dx + \frac{\tau_0^2}{2}\int_{\mathbb{R}^2}u^2dx + C\tau_0^{2+\alpha}\Big(\frac{\int_{\mathbb{R}^2}u^2dx}{\int_{\mathbb{R}^2}|\nabla u|^2dx}\Big)^{1+\frac{\alpha}{2}} =$$

$$= \mathcal{S}(u) + \Big(\frac{\tau_0^2}{2} - \frac{\alpha}{2(2+\alpha)}\Big)\int_{\mathbb{R}^2}u^2dx + C\tau_0^{2+\alpha}\Big(\frac{\int_{\mathbb{R}^2}u^2dx}{\int_{\mathbb{R}^2}|\nabla u|^2dx}\Big)^{1+\frac{\alpha}{2}},$$

which is strictly less than $\mathcal{S}(u)$ for some $\tau_0 \ll 1$.

Therefore, the function $\tilde{\gamma}$ satisfies:

$$\tilde{\gamma}(0) = 0, \quad \tilde{\gamma}(1) = u, \quad \mathcal{S}(\tilde{\gamma}(\tau)) < \mathcal{S}(u) \quad \forall \tau \in [0, \tau_1] \setminus \{1\} \quad \text{and} \quad \mathcal{S}(\tilde{\gamma}(\tau_1)) < 0.$$

Hence, to get the required γ_u it suffices to take a suitable change of variables $\gamma_u(t) := \tilde{\gamma}(T(t))$ for some function $T \in C([0,1];\mathbb{R})$ satisfying $T(0) = 0$, $T(\frac{1}{2}) = 1$ and $T(1) = \tau_1$. □

Proof. (of Theorem 3.2.1) Take $u \in H^1(\mathbb{R}^2)$ given by Proposition 3.2.3. As in previous section, we may prove that u is a nontrivial ground-state solution to (*). It can be seen easily that positivity and radial symmetry of ground-states hold also in dimension $N = 2$. This concludes the proof of Theorem 3.2.1.

□

3.3 Critical case

In this section we are concerned with the existence of a ground state solution of (*) when $N \geq 3$ in the critical case, namely when the nonlinearity has a critical growth in the sense of Hardy-Littlewood-Sobolev inequality. In order to overcome the lack of compactness of the nonlinear term we require, in the spirit of [4], the following hypotheses on $f \in C(\mathbb{R}^+;\mathbb{R})$:

(F_1) $\lim_{s\to 0^+}\frac{f(s)}{s} = 0$,

(F_2) $\lim_{s\to +\infty}\frac{f(s)}{s^{\frac{\alpha+2}{N-2}}} = 1$,

(F_3) there exists $\mu > 0$ and $q \in (2, \frac{N+\alpha}{N-2})$ such that

$$f(s) \geq s^{\frac{\alpha+2}{N-2}} + \mu s^{q-1}, \quad \forall s > 0.$$

Our first main result is the following:

Theorem 3.3.1. *Assume* $N \geq 3$, $\alpha \in ((N-4)_+, N)$, $q > \max\{1 + \frac{\alpha}{N-2}, \frac{N+\alpha}{2(N-2)}\}$ *and let* $f \in C(\mathbb{R}^+; \mathbb{R})$ *satisfying* $(F_1) - (F_3)$. *Then, problem (*) has a positive nontrivial ground state solution.*

Since we seek a positive solution to (*), we may assume that

$$f(s) = 0 \quad \forall s < 0.$$

Furthermore, it is possible to prove qualitative properties of ground state solutions, namely positivity and radial symmetry, as in Propositions 3.1.12 and 3.1.13.

Brezis-Lieb lemma and splitting lemma

In this subsection, we prove two technical lemmas which involve the nonlocal term of the energy.

Lemma 3.3.1. *(Brezis-Lieb lemma). Assume there exists a constant* $C > 0$ *such that*

$$|f(s)| \leq C(|s|^{\frac{\alpha}{N}} + |s|^{\frac{\alpha+2}{N-2}}), \quad \forall s \in \mathbb{R}.$$

Let $\{u_n\}_{n\in\mathbb{N}}$ *be such that* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^N)$ *and consider* $F(s) := \int_0^s f(t)dt$ *for every* $s \in \mathbb{R}$. *Then, as* $n \to +\infty$,

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)dx = \int_{\mathbb{R}^N} (I_\alpha * F(u_n - u))F(u_n - u)dx + \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx + o(1).$$

Proof. Using Fubini's theorem, by Hardy-Littlewood-Sobolev inequality, it holds

$$\int_{\mathbb{R}^N} ((I_\alpha * F(u_n))F(u_n) - (I_\alpha * F(u_n - u))F(u_n - u) - (I_\alpha * F(u))F(u))dx =$$

$$= \int_{\mathbb{R}^N} ((I_\alpha * [F(u_n) + F(u_n - u)])(F(u_n) - F(u_n - u)) - (I_\alpha * F(u))F(u))dx.$$

Furthermore, there exists $C > 0$ such that

$$|F(s)| \leq C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}}), \quad \forall s \in \mathbb{R},$$

which implies $F(u) \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. For any $\epsilon > 0$ sufficiently small, by the Hardy-Littlewood-Sobolev inequality, there exists $K_1 > 0$ such that

$$\left| \int_{\Omega_1} (I_\alpha * F(u))F(u)dx \right| \leq \frac{\epsilon}{6}, \quad \Omega_1 := \{x \in \mathbb{R}^N : |u(x)| \geq K_1\}.$$

Again by the Hardy-Littlewood-Sobolev inequality,

$$\left|\int_{\Omega_1}(I_\alpha*[F(u_n)+F(u_n-u)])(F(u_n)-F(u_n-u))dx\right|\le$$

$$\le C\bigg(\int_{\mathbb{R}^N}|F(u_n)+F(u_n-u)|^{\frac{2N}{N+\alpha}}dx\bigg)^{\frac{N+\alpha}{2N}}\bigg(\int_{\Omega_1}|F(u_n)-F(u_n-u)|^{\frac{2N}{N+\alpha}}dx\bigg)^{\frac{N+\alpha}{2N}}\le$$

$$\le C'\bigg(\int_{\Omega_1}|F(u_n)-F(u_n-u)|^{\frac{2N}{N+\alpha}}dx\bigg)^{\frac{N+\alpha}{2N}},$$

where we have used the fact that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. It is easy to check there exists $C>0$ such that for $n\in\mathbb{N}$,

$$|F(u_n)-F(u_n-u)|\le C\bigg(|u_n|^{\frac{2\alpha}{N+\alpha}}|u|^{\frac{2N}{N+\alpha}}+|u_n|^{\frac{2+\alpha}{N-2}\frac{2N}{N+\alpha}}|u|^{\frac{2N}{N+\alpha}}+u^2+|u|^{\frac{2N}{N-2}}\bigg).$$

Then, by Holder's inequality

$$\int_{\Omega_1}|u_n|^{\frac{2\alpha}{N+\alpha}}|u|^{\frac{2N}{N+\alpha}}dx\le\bigg(\int_{\Omega_1}u_n^2dx\bigg)^{\frac{\alpha}{N+\alpha}}\bigg(\int_{\Omega_1}u^2dx\bigg)^{\frac{N}{N+\alpha}}$$

and

$$\int_{\Omega_1}|u_n|^{\frac{2+\alpha}{N-2}\frac{2N}{N+\alpha}}|u|^{\frac{2N}{N+\alpha}}dx\le\bigg(\int_{\Omega_1}|u_n|^{\frac{2N}{N-2}}dx\bigg)^{\frac{2+\alpha}{N+\alpha}}\bigg(\int_{\Omega_1}|u|^{\frac{2N}{N-2}}dx\bigg)^{\frac{N-2}{N+\alpha}}.$$

So up to redefine ϵ sufficiently small and K_1 large enough, we get for any n

$$\left|\int_{\Omega_1}(I_\alpha*[F(u_n)+F(u_n-u)])(F(u_n)-F(u_n-u))dx\right|\le\frac{\epsilon}{6}.$$

Similarly, let $\Omega_2:=\{x\in\mathbb{R}^N:|x|\ge R\}\setminus\Omega_1$ with $R>0$ large enough such that

$$\left|\int_{\Omega_1}(I_\alpha*F(u))F(u)dx\right|\le\frac{\epsilon}{6}$$

and for any n,

$$\left|\int_{\Omega_1}(I_\alpha*[F(u_n)+F(u_n-u)])(F(u_n)-F(u_n-u))dx\right|\le\frac{\epsilon}{6}.$$

Now, for $K_2>K_1$, let $\Omega_3(n):=\{x\in\mathbb{R}^N:|u_n(x)|\ge K_2\}\setminus(\Omega_1\cup\Omega_2)$. If $\Omega_3(n)\ne\emptyset$, then $|u(x)|<K_1$ and $|x|<R$ for any $x\in\Omega_3(n)$. By a standard diagonal argument,

$u_n \to u$ a.e. in $\mathbb{R}^N$. So by Egorov's theorem, u_n converges to u in measure in B_R, which implies that $|\Omega_3(n)| \to 0$ as $n \to +\infty$. Similarly, for n large enough we have

$$\left| \int_{\Omega_3(n)} (I_\alpha * F(u))F(u)dx \right| \leq \frac{\epsilon}{6}$$

and

$$\left| \int_{\Omega_3(n)} (I_\alpha * [F(u_n) + F(u_n - u)])(F(u_n) - F(u_n - u))dx \right| \leq \frac{\epsilon}{6}.$$

Finally, let us estimate

$$\int_{\Omega_4(n)} ((I_\alpha * [F(u_n) + F(u_n - u)])(F(u_n) - F(u_n - u)) - (I_\alpha * F(u))F(u))dx,$$

where $\Omega_4(n) := \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2 \cup \Omega_3(n))$. Obviously, $\Omega_4(n) \subset B_R$. By Lebesgue's convergence theorem and Rellich's theorem we have

$$\lim_{n\to+\infty} \int_{\Omega_4(n)} |F(u_n - u))|^{\frac{2N}{N+\alpha}} dx = 0 \quad \text{and} \quad \lim_{n\to+\infty} \int_{\Omega_4(n)} |F(u_n) - F(u)|^{\frac{2N}{N+\alpha}} dx = 0,$$

which implies by the Hardy-Littlewood-Sobolev inequality

$$\left| \int_{\mathbb{R}^N} (I_\alpha * [F(u_n) + F(u_n - u)])F(u_n - u)dx \right| \leq C \bigg(\int_{\Omega_4(n)} |F(u_n - u))|^{\frac{2N}{N+\alpha}} dx \bigg)^{\frac{N+\alpha}{2N}} \to 0$$

as $n \to +\infty$, and

$$\left| \int_{\mathbb{R}^N} (I_\alpha * [F(u_n) + F(u_n - u)])(F(u_n) - F(u))dx \right| \leq$$

$$\leq C \bigg(\int_{\Omega_4(n)} |F(u_n) - F(u)|^{\frac{2N}{N+\alpha}} dx \bigg)^{\frac{N+\alpha}{2N}} \to 0$$

as $n \to +\infty$. Hence, let $H_n := F(u_n) + F(u_n - u) - F(u)$ and we have

$$\limsup_{n\to+\infty} \int_{\Omega_4(n)} ((I_\alpha * F(u_n))F(u_n) - (I_\alpha * F(u_n - u))F(u_n - u) - (I_\alpha * F(u))F(u))dx =$$

$$= \limsup_{n\to+\infty} \int_{\Omega_4(n)} (I_\alpha * H_n)F(u)dx.$$

Noting that H_n is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ and $H_n \to 0$ a.e. in $\mathbb{R}^N$, then $H_n \rightharpoonup 0$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. In view of inequality (3.1.3), $I_\alpha * H_n \rightharpoonup 0$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, yielding

$$\lim_{n\to+\infty} \int_{\Omega_4(n)} (I_\alpha * H_n)F(u)dx = 0.$$

Thus,

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^N}((I_\alpha * F(u_n))F(u_n) - (I_\alpha * F(u_n-u))F(u_n-u) - (I_\alpha * F(u))F(u))dx\right| \le \epsilon$$

and the arbitrary choice of ϵ concludes the proof.

□

Next we prove a splitting property for the nonlocal energy.

Lemma 3.3.2. *(Splitting lemma). Assume $\alpha \in ((N-4)_+, N)$, $(F_1)-(F_2)$ and let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then, up to subsequences, as $n \to +\infty$*

$$\int_{\mathbb{R}^N}((I_\alpha * F(u_n))f(u_n) - (I_\alpha * F(u_n-u))f(u_n-u) - (I_\alpha * F(u))f(u))\phi dx = o(1)\|\phi\|_{L^\infty(\mathbb{R}^N)},$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$.

In order to prove Lemma 3.3.2, we need first to prove Lemmas 3.3.3 and 3.3.4 below.

Lemma 3.3.3. *Let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then the following hold:*

(i) For any $1 < q \le r \le \frac{2N}{N-2}$ and $r > 2$,

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}||u_n|^{q-1}u_n - |u_n-u|^{q-1}(u_n-u) - |u|^{q-1}u|^{\frac{r}{q}}dx = 0.$$

(ii) Assume $h \in C(\mathbb{R};\mathbb{R})$ such that $h(t) = o(t)$ as $t \to 0$ and $|h(t)| \le C(1+|t|^q)$ for any $t \in \mathbb{R}$, where $q \in (1, \frac{N+2}{N-2}]$. The following hold:

(1) For any $r \in [q+1, \frac{2N}{N-2}]$,

$$\lim_{n\to+\infty}\int_{\Omega}|H(u_n) - H(u_n-u) - H(u)|^{\frac{r}{q+1}}dx = 0$$

where $H(t) = \int_0^t h(s)ds$,

(2) If we further assume that $\alpha \in ((N-4)_+, N)$ and $\lim_{|t|\to+\infty}\frac{h(t)}{|t|^{\frac{\alpha+2}{N-2}}} = 0$, then as $n \to +\infty$

$$\int_{\mathbb{R}^N}|h(u_n) - h(u_n-u) - h(u)|^{\frac{2N}{N+\alpha}}|\phi|^{\frac{2N}{N+\alpha}}dx = o(1)\|\phi\|_{L^\infty(\mathbb{R}^N)},$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$.

Proof. The proofs of (i) and (1) are similar to [45, lemma 2.5]. We only give the proof of (2).
For any fixed $\epsilon \in (0,1)$, there exists $s_0 = s_0(\epsilon) \in (0,1)$ such that $|h(t)| \leq \epsilon|t|$ for $|t| \leq 2s_0$. Choose $s_1 = s_1(\epsilon) > 2$ such that

$$|h(t)| \leq \epsilon|t|^{\frac{\alpha+2}{N-2}}$$

for $|t| \geq s_1 - 1$. From the continuity of h, there exists $\delta = \delta(\epsilon) \in (0, s_0)$ such that $|h(t_1) - h(t_2)| \leq s_0\epsilon$ for $|t_1 - t_2| \leq \delta$, $|t_1|, |t_2| \leq s_1 + 1$. Moreover, there exists $c(\epsilon) > 0$ such that

$$|h(t)| \leq c(\epsilon)|t| + \epsilon|t|^{\frac{\alpha+2}{N-2}}$$

for every $t \in \mathbb{R}$. In the following let C denote a positive constant independent of n and ϵ.
Noting that $\alpha \in ((N-4)_+, N)$ (it will be used many times), we have $2 < \frac{4N}{N+\alpha} < \frac{2N}{N-2}$. Then, there exists $R = R(\epsilon) > 0$ large enough such that, by Holder's inequality,

$$\int_{\mathbb{R}^N \setminus B_R} |h(u)\phi|^{\frac{2N}{N+\alpha}} dx \leq c(\epsilon) \int_{\mathbb{R}^N \setminus B_R} (|u|^{\frac{2N}{N+\alpha}} + \epsilon|u|^{\frac{\alpha+2}{N-2}\frac{2N}{N+\alpha}})|\phi|^{\frac{2N}{N+\alpha}} dx \leq$$

$$\leq c(\epsilon)\Big(\int_{\mathbb{R}^N \setminus B_R} |u|^{\frac{4N}{N+\alpha}} dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^N} |\phi|^{\frac{4N}{N+\alpha}} dx\Big)^{\frac{1}{2}} +$$

$$+C\epsilon\Big(\int_{\mathbb{R}^N \setminus B_R} |u|^{\frac{2N}{N-2}} dx\Big)^{\frac{2+\alpha}{N+\alpha}} \Big(\int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{N+\alpha}} \leq C\epsilon\|\phi\|_{\infty}^{\frac{2N}{N+\alpha}}. \tag{3.3.1}$$

Setting $A_n := \{x \in \mathbb{R}^N \setminus B_R : |u_n(x)| \leq s_0\}$, then by Holder's inequality

$$\int_{A_n \cap \{|u| \leq \delta\}} |h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx \leq C\epsilon \int_{\mathbb{R}^N} (|u_n|^{\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2N}{N+\alpha}})|\phi|^{\frac{2N}{N+\alpha}} dx \leq$$

$$\leq C\epsilon\|\phi\|_{\infty}^{\frac{2N}{N+\alpha}}.$$

Let $B_n := \{x \in \mathbb{R}^N \setminus B_R\} : |u_n(x)| \geq s_1\}$. Then,

$$\int_{B_n \cap \{|u| \leq \delta\}} |h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx \leq$$

$$\leq C\epsilon \int_{\mathbb{R}^N} (|u_n|^{\frac{2+\alpha}{N-2}\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2+\alpha}{N-2}\frac{2N}{N+\alpha}})|\phi|^{\frac{2N}{N+\alpha}} dx \leq$$

$$\leq C\epsilon\|\phi\|_{\infty}^{\frac{2N}{N+\alpha}}.$$

Setting $C_n := \{x \in \mathbb{R}^n \setminus B_R : s_0 \leq |u_n(x)| \leq s_1\}$, then $|C_n| < +\infty$ for any n and

$$\int_{C_n \cap \{|u| \leq \delta\}} |h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx \leq (s_0 \epsilon)^{\frac{2N}{N+\alpha}} \int_{C_n \cap \{|u| \leq \delta\}} |\phi|^{\frac{2N}{N+\alpha}} dx \leq$$

$$\leq (s_0\epsilon)^{\frac{2N}{N+\alpha}} |C_n|^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\phi|^{\frac{4N}{N+\alpha}} dx \right)^{\frac{1}{2}} \leq \epsilon^{\frac{2N}{N+\alpha}} \left(\int_{C_n} |u_n|^{\frac{4N}{N+\alpha}} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\phi|^{\frac{4N}{N+\alpha}} dx \right)^{\frac{1}{2}} \leq$$

$$\leq C\epsilon \|\phi\|_\infty^{\frac{2N}{N+\alpha}}.$$

Thus, $(\mathbb{R}^N \setminus B_R) \cap \{|u| \leq \delta\} = A_n \cup B_n \cup C_n$ and

$$\int_{(\mathbb{R}^N \setminus B_R) \cap \{|u| \leq \delta\}} |h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx \leq C\epsilon \|\phi\|_\infty^{\frac{2N}{N+\alpha}}, \quad \forall n \in \mathbb{N}.$$

Now for ϵ given above, there exists $c(\epsilon) > 0$ such that

$$|h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} \leq \epsilon(|u_n|^{\frac{2+\alpha}{N-2}\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2+\alpha}{N-2}\frac{2N}{N+\alpha}}) + c(\epsilon)(|u_n|^{\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2N}{N+\alpha}})$$

and

$$\int_{(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \delta\}} |h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx \leq$$

$$\leq \int_{(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \delta\}} \Bigg(\epsilon(|u_n|^{\frac{2+\alpha}{N-2}\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2+\alpha}{N-2}\frac{2N}{N+\alpha}}) |\phi|^{\frac{2N}{N+\alpha}} +$$

$$+ c(\epsilon)(|u_n|^{\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2N}{N+\alpha}}) |\phi|^{\frac{2N}{N+\alpha}} \Bigg) dx \leq$$

$$\leq C\epsilon \|\phi\|_\infty^{\frac{2N}{N+\alpha}} + c(\epsilon) \int_{(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \delta\}} (|u_n|^{\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2N}{N+\alpha}}) |\phi|^{\frac{2N}{N+\alpha}} dx.$$

Noting that $|(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \delta\}| \to 0$ as $R \to +\infty$, there exists $R = R(\epsilon) > 0$ large enough, such that, by the generalized Holder's inequality,

$$c(\epsilon) \int_{(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \delta\}} (|u_n|^{\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2N}{N+\alpha}}) |\phi|^{\frac{2N}{N+\alpha}} dx \leq$$

$$\leq c(\epsilon) \left[\left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N+\alpha}} + \left(\int_{\mathbb{R}^N} |u_n - u|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N+\alpha}} \right] \left(\int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N+\alpha}} \Bigg) \times$$

$$\times |(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \delta\}|^{\frac{\alpha+4-N}{N+\alpha}} \leq \epsilon \|\phi\|_\infty^{\frac{2N}{N+\alpha}}.$$

Thus, by (3.3.1), for any $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N \setminus B_R} |h(u_n) - h(u_n - u) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx \leq C\epsilon \|\phi\|_\infty^{\frac{2N}{N+\alpha}}. \tag{3.3.2}$$

Finally, for $\epsilon > 0$ given above, there exists $C(\epsilon) > 0$ such that

$$|h(t)|^{\frac{2N}{N+\alpha}} \leq C(\epsilon)|t|^{\frac{2N}{N+\alpha}} + \epsilon|t|^{\frac{2N}{N+\alpha}\frac{2+\alpha}{N-2}}, \quad \forall t \in \mathbb{R}.$$

Recalling that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, up to subsequences, by Rellich's theorem $u_n \to u$ in $L^p(B_R)$ for all $1 \leq p < \frac{2N}{N-2}$. Then, for n large enough

$$\int_{B_R} |h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx \leq \int_{B_R} (C(\epsilon)|u_n - u|^{\frac{2N}{N+\alpha}} + \epsilon|u_n - u|^{\frac{2N}{N+\alpha}\frac{\alpha+2}{N-2}})|\phi|^{\frac{2N}{N+\alpha}} dx \leq$$

$$\leq C\epsilon\|\phi\|_\infty^{\frac{2N}{N+\alpha}}. \tag{3.3.3}$$

Moreover, setting $D_n := \{x \in B_R : |u_n(x) - u(x)| \geq 1\}$, we have $|D_n| = 0$ for n large enough in view of $u_n \to u$ a.e. $x \in B_R$. Hence, noting that $|\{|u| \geq L\}| \to 0$ as $L \to +\infty$, there exists $L = L(\epsilon) > 1$ large enough such that

$$\int_{B_R \cap \{|u| \geq L\}} |h(u_n) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx = \int_{(B_R \setminus D_n) \cap \{|u| \geq L\}} |h(u_n) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx \leq$$

$$\leq \int_{(B_R \setminus D_n) \cap \{|u| \geq L\}} \left(c(\epsilon)(|u|^{\frac{2N}{N+\alpha}} + |u_n|^{\frac{2N}{N+\alpha}}) + \epsilon(|u|^{\frac{2N}{N+\alpha}\frac{\alpha+2}{N-2}} + |u_n|^{\frac{2N}{N+\alpha}\frac{\alpha+2}{N-2}}\right)|\phi|^{\frac{2N}{N+\alpha}} dx \leq$$

$$\leq C\epsilon\|\phi\|_\infty^{\frac{2N}{N+\alpha}}.$$

On the other hand, by Lebesgue's convergence theorem, as $n \to +\infty$

$$\int_{B_R \cap \{|u| \leq L\}} |h(u_n) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx = \int_{(B_R \setminus D_n) \cap \{|u| \leq L\}} |h(u_n) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx =$$

$$= o(1)\|\phi\|_\infty^{\frac{2N}{N+\alpha}}.$$

Thus, by (3.3.3) for n large enough

$$\int_{B_R} |h(u_n) - h(u_n - u) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx \leq C\epsilon\|\phi\|_\infty^{\frac{2N}{N+\alpha}}.$$

Finally, combining the previous estimate with (3.3.2), we conclude the proof. □

Lemma 3.3.4. *Let $\alpha \in (0, N)$, $s \in (1, \frac{N}{\alpha})$ and let $\{g_n\}_{n \in \mathbb{N}} \in L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$ be bounded and such that, up to subsequences, $g_n \to 0$ in $L^s_{loc}(\mathbb{R}^N)$ as $n \to +\infty$. Then, up to a subsequence, $(I_\alpha * g_n)(x) \to 0$ a.e. in $\mathbb{R}^N$ as $n \to +\infty$.*

Proof. Let us prove that for any fixed $k \in \mathbb{N}$, up to subsequences, $(I_\alpha * g_n)(x) \to 0$ a.e. in B_k. Let $k \in \mathbb{N}$ be fixed; due to $\{g_n\}$ bounded in $L^1(\mathbb{R}^N)$, for any $\epsilon > 0$ there exists $R = R(\epsilon) > k$ such that

$$A_\alpha \int_{\mathbb{R}^N \setminus B_R(x)} \frac{|g_n(y)|}{|x-y|^{N-\alpha}} dy \le \epsilon, \quad \text{for any} \quad x \in \mathbb{R}^N, n \in \mathbb{N}$$

where $A_\alpha := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha}$. Obviously, $B_R(x) \subset B_{2R}$ for any $x \in B_R$. Noting that $g_n \chi_{B_{2R}} \in L^s(\mathbb{R}^N)$, by inequality (3.1.3),

$$\|I_\alpha * (|g_n|\chi_{B_{2R}})\|_{L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)} \le C\|g_n\|_{L^s(B_{2R})}.$$

It follows that, up to a subsequence, $I_\alpha * (|g_n|\chi_{B_{2R}}) \to 0$ in $L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)$ and a.e. in B_k. Then, for a.e. $x \in B_k$,

$$\limsup_{n\to+\infty} |(I_\alpha * g_n)(x)| \le A_\alpha \limsup_{n\to+\infty} \left(\int_{B_R(x)} \frac{|g_n(y)|}{|x-y|^{N-\alpha}} dy + \int_{\mathbb{R}^N \setminus B_R(x)} \frac{|g_n(y)|}{|x-y|^{N-\alpha}} dy \right) \le$$

$$\le \epsilon + A_\alpha \limsup_{n\to+\infty} \int_{B_R(x)} \frac{|g_n(y)|}{|x-y|^{N-\alpha}} dy \le \epsilon + A_\alpha \limsup_{n\to+\infty} \int_{B_{2R}} \frac{|g_n(y)|}{|x-y|^{N-\alpha}} dy =$$

$$= \epsilon + \limsup_{n\to+\infty} (I_\alpha * (|g_n|\chi_{B_{2R}}))(x) = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the proof is completed. □

Now we are set to prove Lemma 3.3.2.

Proof. Set for every $t \in \mathbb{R}$:

$$f_1(t) := f(t) - |t|^{\frac{4+\alpha-N}{N-2}} t \quad \text{and} \quad F_1(t) := \int_0^t f(s) ds.$$

Note that for any $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n) \phi dx = \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f_1(u_n) \phi dx + \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) |u_n|^{\frac{4+\alpha-N}{N-2}} u_n \phi dx.$$

Step 1. We claim as $n \to +\infty$:

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n)) |u_n|^{\frac{4+\alpha-N}{N-2}} u_n \phi dx = \int_{\mathbb{R}^N} (I_\alpha * F(u_n - u)) |u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u) \phi dx +$$

$$+ \int_{\mathbb{R}^N} (I_\alpha * F(u)) |u|^{\frac{4+\alpha-N}{N-2}} u \phi dx + o(1) \|\phi\|_\infty$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$. Noting that $\alpha > (N-4)_+$, by Lemma 3.3.3 (ii) (1) with $h(t) = f(t)$, $q = \frac{2+\alpha}{N-2}$ and $r = \frac{2N}{N-2}$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |F(u_n) - F(u_n - u) - F(u)|^{\frac{2N}{N+\alpha}} dx = 0. \tag{3.3.4}$$

Then for $v_n = |u_n|^{\frac{4+\alpha-N}{N-2}} u_n$, as well as $v_n = |u_n - u|^{\frac{4+\alpha-N}{N-2}}(u_n - u)$ and also $v_n = |u|^{\frac{4+\alpha-N}{N-2}} u$, there exists $C > 0$ such that by Holder's inequality

$$\int_{\mathbb{R}^N} |v_n\phi|^{\frac{2N}{N+\alpha}} dx \le \left(\int_{\mathbb{R}^N} |v_n|^{\frac{2N}{2+\alpha}} dx\right)^{\frac{2+\alpha}{N+\alpha}} \left(\int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N+\alpha}},$$

from which it follows, using Hardy-Littlewood-Sobolev inequality and inequality (3.1.3),

$$\left|\int_{\mathbb{R}^N} (I_\alpha * [F(u_n) - F(u_n - u) - F(u)]) v_n \phi dx\right| \le$$

$$\le C\left(\int_{\mathbb{R}^N} |F(u_n) - F(u_n - u) - F(u)|^{\frac{2N}{N+\alpha}} dx\right)^{\frac{N+\alpha}{2N}} \left(\int_{\mathbb{R}^N} |v_n\phi|^{\frac{2N}{N+\alpha}} dx\right)^{\frac{N+\alpha}{2N}} =$$

$$= o(1)\|\phi\|_\infty, \quad \text{as} \quad n \to +\infty \tag{3.3.5}$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$.

On the other hand, by Lemma 3.3.3 with $q = \frac{2+\alpha}{N-2}$ and $r = \frac{2N}{N-2}$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} ||u_n|^{\frac{4+\alpha-N}{N-2}} u_n - |u_n - u|^{\frac{4+\alpha-N}{N-2}}(u_n - u) - |u|^{\frac{4+\alpha-N}{N-2}} u|^{\frac{2N}{2+\alpha}} dx = 0.$$

For $w_n = F(u_n)$, as well as $w_n = F(u_n - u)$ and also $w_n = F(u)$, one easily checks that $\{w_n\}_n$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. By the Hardy-Littlewood-Sobolev inequality and inequality (3.1.3), we get

$$\left|\int_{\mathbb{R}^N} (I_\alpha * w_n)(|u_n|^{\frac{4+\alpha-N}{N-2}} u_n - |u_n - u|^{\frac{4+\alpha-N}{N-2}}(u_n - u) - |u|^{\frac{4+\alpha-N}{N-2}} u)\phi dx\right| \le$$

$$\le C\left(\int_{\mathbb{R}^N} \left||u_n|^{\frac{4+\alpha-N}{N-2}} u_n - |u_n - u|^{\frac{4+\alpha-N}{N-2}}(u_n - u) - |u|^{\frac{4+\alpha-N}{N-2}} u\right|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} dx\right)^{\frac{N+\alpha}{2N}} \le$$

$$\le C\left(\int_{\mathbb{R}^N} \left||u_n|^{\frac{4+\alpha-N}{N-2}} u_n - |u_n - u|^{\frac{4+\alpha-N}{N-2}}(u_n - u) - |u|^{\frac{4+\alpha-N}{N-2}} u\right|^{\frac{2N}{2+\alpha}} dx\right)^{\frac{2+\alpha}{2N}} \left(\int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{2N}}$$

$$= o(1)\|\phi\|_\infty, \quad \text{as} \quad n \to +\infty \tag{3.3.6}$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$. Then, combining (3.3.5) with (3.3.6) we get

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n))|u_n|^{\frac{4+\alpha-N}{N-2}} u_n \phi dx = \int_{\mathbb{R}^N} (I_\alpha * F(u_n - u))|u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u)\phi dx +$$

$$+ \int_{\mathbb{R}^N} (I_\alpha * F(u))|u|^{\frac{4+\alpha-N}{N-2}} u\phi dx + \int_{\mathbb{R}^N} (I_\alpha * F(u_n - u))|u|^{\frac{4+\alpha-N}{N-2}} u\phi dx +$$

$$+ \int_{\mathbb{R}^N} (I_\alpha * F(u))|u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u)\phi dx + o(1)\|\phi\|_\infty, \quad \text{as} \quad n \to +\infty$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$. Noting that $F(u) \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, by inequality (3.1.3), $|I_\alpha * F(u)|^{\frac{2N}{N+2}} \in L^{\frac{N+2}{N-\alpha}}(\mathbb{R}^N)$. Furthermore, $|u_n - u|^{\frac{2N(2+\alpha)}{(N-2)(N+2)}} \rightharpoonup 0$ in $L^{\frac{N+2}{N+\alpha}}(\mathbb{R}^N)$. This yields

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |I_\alpha * F(u)|^{\frac{2N}{N+2}} |u_n - u|^{\frac{2N(2+\alpha)}{(N-2)(N+2)}} dx = 0, \tag{3.3.7}$$

which implies, by Holder's inequality,

$$\left| \int_{\mathbb{R}^N} (I_\alpha * F(u))|u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u)\phi dx \right| \le$$

$$\le \left(\int_{\mathbb{R}^N} |I_\alpha * F(u)|^{\frac{2N}{N+2}} |u_n - u|^{\frac{2N(2+\alpha)}{(N-2)(N+2)}} dx \right)^{\frac{N+2}{2N}} \left(\int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} = o(1)\|\phi\|_\infty$$

as $n \to +\infty$, for any $\phi \in C_0^\infty(\mathbb{R}^N)$.

At the same time, since $\alpha \in ((N-4)_+, N)$, for $s \in (1, \frac{2N}{N+\alpha}) \subset (1, \frac{N}{\alpha})$, by Rellich's theorem, up to subsequences, $F(u_n - u) \to 0$ in $L^s_{loc}(\mathbb{R}^N)$. By Lemma 3.2.5, $I_\alpha * F(u_n - u) \to 0$ a.e. in $\mathbb{R}^N$. So, inequality (3.1.3) implies

$$\sup_{n\in\mathbb{N}} \||I_\alpha * F(u_n - u)|^{\frac{2N}{N+2}}\|_{L^{\frac{N+2}{N+\alpha}}(\mathbb{R}^N)} \le C \sup_{n\in\mathbb{N}} \|F(u_n - u)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} < +\infty,$$

which yields $|I_\alpha * F(u_n - u)|^{\frac{2N}{N+2}} \rightharpoonup 0$ in $L^{\frac{N+2}{N+\alpha}}(\mathbb{R}^N)$. Noting that $|u|^{\frac{2+\alpha}{N-2}\frac{2N}{N-2}} \in L^{\frac{N+2}{2+\alpha}}(\mathbb{R}^N)$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |I_\alpha * F(u_n - u)|^{\frac{2N}{N+2}} |u|^{\frac{2+\alpha}{N-2}\frac{2N}{N+2}} dx = 0$$

and, by Holder's inequality,

$$\left| \int_{\mathbb{R}^N} (I_\alpha * F(u_n - u))|u|^{\frac{4+\alpha-N}{N-2}} u\phi dx \right| \le$$

$$\le \left(\int_{\mathbb{R}^N} |I_\alpha * F(u_n - u)|^{\frac{2N}{N+2}} |u|^{\frac{2N(2+\alpha)}{(N-2)(N+2)}} dx \right)^{\frac{N+2}{2N}} \left(\int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} = o(1)\|\phi\|_\infty,$$

as $n \to +\infty$, for any $\phi \in C_0^\infty(\mathbb{R}^N)$. The claim is thus proved.

Step 2. We claim

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n))f_1(u_n)\phi dx = \int_{\mathbb{R}^N} (I_\alpha * F(u_n - u))f_1(u_n - u)\phi dx +$$

$$+ \int_{\mathbb{R}^N} (I_\alpha * F(u))f_1(u)\phi dx + o(1)\|\phi\|_\infty, \quad \text{as} \quad n \to +\infty \tag{3.3.8}$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$. The following hold:

(i) $\int_{\mathbb{R}^N} (I_\alpha * [F(u_n) - F(u_n - u) - F(u)])f_1(u_n)\phi dx = o(1)\|\phi\|_\infty$,

(ii) $\int_{\mathbb{R}^N} (I_\alpha * [F(u_n) - F(u_n - u) - F(u)])f_1(u_n - u)\phi dx = o(1)\|\phi\|_\infty$,

(iii) $\int_{\mathbb{R}^N} (I_\alpha * [F(u_n) - F(u_n - u) - F(u)])f_1(u)\phi dx = o(1)\|\phi\|_\infty$,

as $n \to +\infty$, for any $\phi \in C_0^\infty(\mathbb{R}^N)$. Let us only prove the first identity (i), the remaining ones being similar. Observe that there exists $\delta \in (0,1)$ and $C > 0$ such that $|f_1(t)| \le |t|$ for $|t| \le \delta$ and $|f_1(t)| \le C|t|^{\frac{2+\alpha}{N-2}}$ for $|t| \ge \delta$. Noting that $\alpha \in ((N-4)_+, N)$, we have $2 < \frac{4N}{N+\alpha} < \frac{2N}{N-2}$. Then, for any $\phi \in C_0^\infty(\mathbb{R}^N)$ and $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} |f_1(u_n)\phi|^{\frac{2N}{N+\alpha}} dx = \int_{\{|u_n| \le \delta\}} |f_1(u_n)\phi|^{\frac{2N}{N+\alpha}} dx + \int_{\{|u_n| \ge \delta\}} |f_1(u_n)\phi|^{\frac{2N}{N+\alpha}} dx \le$$

$$\le \int_{\{|u_n| \le \delta\}} |u_n\phi|^{\frac{2N}{N+\alpha}} dx + C^{\frac{2N}{N+\alpha}} \int_{\{|u_n| \ge \delta\}} |u_n|^{\frac{2N(2+\alpha)}{(N-2)(N+\alpha)}} |\phi|^{\frac{2N}{N+\alpha}} dx \le$$

$$\le \left(\int_{\mathbb{R}^N} |u_n|^{\frac{4N}{N+\alpha}} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\phi|^{\frac{4N}{N+\alpha}} dx \right)^{\frac{1}{2}} +$$

$$+ C^{\frac{2N}{N+\alpha}} \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2}} dx \right)^{\frac{2+\alpha}{N+\alpha}} \left(\int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N+\alpha}} \le C\|\phi\|_\infty^{\frac{2N}{N+\alpha}}.$$

Then by Hardy-Littlewood-Sobolev inequality, inequality (3.1.3) and (3.3.4),

$$\left| \int_{\mathbb{R}^N} (I_\alpha * [F(u_n) - F(u_n - u) - F(u)])f_1(u_n)\phi dx \right| \le$$

$$\le \left(\int_{\mathbb{R}^N} |F(u_n) - F(u_n - u) - F(u)|^{\frac{2N}{N+\alpha}} dx \right)^{\frac{N+\alpha}{2N}} \left(\int_{\mathbb{R}^N} |f_1(u_n)\phi|^{\frac{2N}{N+\alpha}} dx \right)^{\frac{N+\alpha}{2N}} = o(1)\|\phi\|_\infty,$$

as $n \to +\infty$, for any $\phi \in C_0^\infty(\mathbb{R}^N)$. So (i) holds.

Similarly we prove

(1) $\int_{\mathbb{R}^N}(I_\alpha * F(u_n))[f_1(u_n) - f_1(u_n - u) - f_1(u)]\phi dx = o(1)\|\phi\|_\infty$,

(2) $\int_{\mathbb{R}^N}(I_\alpha * F(u_n - u))[f_1(u_n) - f_1(u_n - u) - f_1(u)]\phi dx = o(1)\|\phi\|_\infty$,

(3) $\int_{\mathbb{R}^N}(I_\alpha * F(u))[f_1(u_n) - f_1(u_n - u) - f_1(u)]\phi dx = o(1)\|\phi\|_\infty$,

as $n \to +\infty$, for any $\phi \in C_0^\infty(\mathbb{R}^N)$. By the Hardy-Littlewood-Sobolev inequality, inequality (3.1.3), (ii) (2) of Lemma 3.3.3 with $h(t) = f_1(t)$ and the fact that $F(u_n)$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^N}(I_\alpha * F(u_n))[f_1(u_n) - f_1(u_n - u) - f_1(u)]\phi dx\right| \le$$

$$\le C\left(\int_{\mathbb{R}^N}|f_1(u_n) - f_1(u_n - u) - f_1(u)|^{\frac{2N}{N+\alpha}}|\phi|^{\frac{2N}{N+\alpha}}dx\right)^{\frac{N+\alpha}{2N}} = o(1)\|\phi\|_\infty, \quad \text{as} \quad n \to +\infty$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$. So the first identity (1) holds and the remaining can be proved in a similar way.

Combining (i)-(iii) with (1)-(3), we get

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_n))f_1(u_n)\phi dx = \int_{\mathbb{R}^N}(I_\alpha * F(u_n - u))f_1(u_n - u)\phi dx + \int_{\mathbb{R}^N}(I_\alpha * F(u))f_1(u)\phi dx +$$

$$+\int_{\mathbb{R}^N}(I_\alpha * F(u_n - u))f_1(u)\phi dx + \int_{\mathbb{R}^N}(I_\alpha * F(u))f_1(u_n - u)\phi dx + o(1)\|\phi\|_\infty,$$

as $n \to +\infty$, for any $\phi \in C_0^\infty(\mathbb{R}^N)$. To conclude the proof of (3.2.8), it remains to prove

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_n - u))f_1(u)\phi dx = o(1)\|\phi\|_\infty$$

and

$$\int_{\mathbb{R}^N}(I_\alpha * F(u))f_1(u_n - u)\phi dx = o(1)\|\phi\|_\infty, \tag{3.3.9}$$

as $n \to +\infty$, for any $\phi \in C_0^\infty(\mathbb{R}^N)$. Notice that for any $\epsilon \in (0,1)$, there exists $\delta(\epsilon) \in (0,1)$ and $C(\epsilon) > 0$ such that $|f_1(t)| \le \epsilon|t|$ for $|t| \le \delta(\epsilon)$ and $|f_1(t)| \le C(\epsilon)|t|^{\frac{2+\alpha}{N-2}}$ for $|t| \ge \delta(\epsilon)$. Then, by Holder's inequality and inequality (3.1.3)

$$\left|\int_{\mathbb{R}^N}(I_\alpha * F(u_n - u))f_1(u)\phi dx\right| \le \epsilon\int_{\{|u|\le\delta(\epsilon)\}}|I_\alpha * F(u_n - u)||u\phi|dx +$$

$$+C(\epsilon)\int_{\{|u|\ge\delta(\epsilon)\}}|I_\alpha * F(u_n - u)||u|^{\frac{2+\alpha}{N-2}}|\phi|dx \le C\epsilon\|F(u_n - u)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \times$$

$$\times\left(\int_{\{|u|\le\delta(\epsilon)\}}|u\phi|^{\frac{2N}{N+\alpha}}dx\right)^{\frac{N+\alpha}{2N}}+$$

$$+C(\epsilon)\left(\int_{\mathbb{R}^N}|I_\alpha * F(u_n-u)|^{\frac{2N}{N+2}}|u|^{\frac{2+\alpha}{N-2}\frac{2N}{N+2}}dx\right)^{\frac{N+2}{2N}}\left(\int_{\mathbb{R}^N}|\phi|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}}.$$

There exists $C>0$ independent of ϵ and ϕ such that

$$\int_{\{|u|\le\delta(\epsilon)\}}|u\phi|^{\frac{2N}{N+\alpha}}dx\le C\|\phi\|_\infty^{\frac{2N}{N+\alpha}}.$$

Then by (3.3.7), there exists $\tilde{C}>0$ independent for ϕ and ϵ such that

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^N}(I_\alpha * F(u_n-u))f_1(u)\phi dx\right|\le\tilde{C}\epsilon\|\phi\|_\infty.$$

It follows that

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_n-u))f_1(u)\phi dx=o(1)\|\phi\|_\infty,\quad \text{as}\quad n\to+\infty$$

for any $\phi\in C_0^\infty(\mathbb{R}^N)$. Similarly, (3.3.9) can be proved and the proof of Lemma 3.3.2 is complete.

□

Proof of Theorem 3.3.1

First of all, let us consider the following family of functionals, for $\lambda\in[\frac{1}{2},1]$:

$$\mathcal{S}_\lambda(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)dx-\frac{\lambda}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)dx,\quad u\in H^1(\mathbb{R}^N).$$

Obviously, if f satisfies the growth assuptions of Theorem 3.3.1, for $\lambda\in[\frac{1}{2},1]$, $\mathcal{S}_\lambda\in C^1(H^1(\mathbb{R}^N);\mathbb{R})$ and every critical point of $\mathcal{S}_\lambda$ is a weak solution of

$$-\Delta u+u=\lambda(I_\alpha * F(u))f(u). \tag{3.3.10}$$

The existence of critical points of $\mathcal{S}_\lambda$ is a consequence of the following result on critical point theory.

Theorem 3.3.2. *(see [16]) Let $(X,\|\cdot\|_X)$ be a Banach space, let $J\subset\mathbb{R}^+$ be an interval and let a family of $C^1(X;\mathbb{R})$-functionals $\{\mathcal{S}_\lambda\}_{\lambda\in J}$ of the form*

$$\mathcal{S}_\lambda(u)=A(u)-\lambda B(u).$$

Assume that $B(u) \geq 0$ for any $u \in X$, at least one between A and B is coercive on X and there exist two points $v_1, v_2 \in X$ such that for any $\lambda \in J$,

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{S}_\lambda(\gamma(t)) > \max\{\mathcal{S}_\lambda(v_1), \mathcal{S}_\lambda(v_2)\},$$

where $\Gamma := \{\gamma \in C([0,1]; X) : \gamma(0) = v_1, \gamma(1) = v_2\}$. Then, for a.e. $\lambda \in J$, $\mathcal{S}_\lambda$ admits a bounded Palais-Smale sequence at level c_λ. Moreover, c_λ is left-continuous with respect to $\lambda \in [\frac{1}{2}, 1]$.

In the following, set $X = H^1(\mathbb{R}^N)$ and

$$A(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx, \quad B(u) = \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) dx.$$

Obviously, $A(u) \to +\infty$ as $\|u\|_{H^1(\mathbb{R}^N)} \to +\infty$. Thanks to (F_3), $B(u) \geq 0$ for any $u \in H^1(\mathbb{R}^N)$. Moreover, by $(F_1)-(F_2)$, there exists $C > 0$ such that $|F(s)| \leq C(|s|^{1+\frac{\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}})$ for any $s \in \mathbb{R}$. Then, as in Proposition 3.1.3, there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) dx \leq \frac{1}{2} \|u\|^2_{H^1(\mathbb{R}^N)} \quad \text{if} \quad \|u\|^2_{H^1(\mathbb{R}^N)} \leq \delta,$$

and therefore for any $\lambda \in J$,

$$\mathcal{S}_\lambda(u) \geq \frac{1}{4} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx > 0 \quad \text{if} \quad 0 < \|u\|^2_{H^1(\mathbb{R}^N)} \leq \delta. \tag{3.3.11}$$

By (F_3), it follows that $F(s) \geq \frac{N-2}{N+\alpha} |s|^{\frac{N+\alpha}{N-2}} + \frac{\mu}{q} |s|^q$ for any $s \in \mathbb{R}$ and for some $\mu > 0$, $q \in (2, \frac{N+\alpha}{N-2})$. On the other hand, for fixed $0 \not\equiv u_0 \in H^1(\mathbb{R}^N)$ and for any $\lambda \in J$, $t > 0$,

$$\mathcal{S}_\lambda(tu_0) \leq \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla u_0|^2 + u_0^2) dx - \frac{t^{\frac{2(N+\alpha)}{N-2}}}{4} \left(\frac{N-2}{N+\alpha}\right)^2 \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{\frac{N+\alpha}{N-2}}) |u_0|^{\frac{N+\alpha}{N-2}} dx \to -\infty$$

as $t \to +\infty$. Then there exists $t_0 = t_0(u_0) > 0$ such that $\mathcal{S}_\lambda(t_0 u_0) < 0$, $\lambda \in J$ and $\|t_0 u_0\|^2_{H^1(\mathbb{R}^N)} > \delta$ by (3.2.11). Furthermore, by (3.3.11) as in Proposition 3.1.3, $c_\lambda \geq \frac{\delta}{4} > 0$ for any $\lambda \in J$. So in order to satisfy the hypotheses of Theorem 3.2.2, we choose

$$\Gamma = \{\gamma \in C([0,1]; H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = t_0 u_0\}.$$

Remark 3.3.5. *Observe that c_λ is independent of u_0. Indeed, let*

$$d_\lambda := \inf_{\gamma \in \Gamma_1} \max_{t \in [0,1]} \mathcal{S}_\lambda(\gamma(t)),$$

where $\Gamma_1 := \{\gamma \in C([0,1]; H^1(\mathbb{R}^N)) : \gamma(0) = 0, \mathcal{S}_\lambda(\gamma(1)) < 0\}$. Clearly, $d_\lambda \leq c_\lambda$. On the other hand, for any $\gamma \in \Gamma_1$, it follows from (3.3.11) that $\|\gamma(1)\|^2_{H^1(\mathbb{R}^N)} > \delta$. Due to the path connectedness of $H^1(\mathbb{R}^N)$, there exists $\tilde{\gamma} \in C([0,1]; H^1(\mathbb{R}^N))$ such that $\tilde{\gamma}(t) = \gamma(2t)$ if $t \in [0, \frac{1}{2}]$, $\|\tilde{\gamma}(t)\|^2_{H^1(\mathbb{R}^N)} > \delta$ if $t \in [\frac{1}{2}, 1]$ and $\tilde{\gamma}(1) = t_0 u_0$. Then $\tilde{\gamma} \in \Gamma$ and

$$\max_{t \in [0,1]} \mathcal{S}_\lambda(\tilde{\gamma}(t)) = \max_{t \in [0,1]} \mathcal{S}_\lambda(\gamma(t)),$$

which implies that $c_\lambda \leq d_\lambda$ and so $c_\lambda = d_\lambda$ for any $\lambda \in J$.

Then, as a consequence of Theorem 3.3.2, we have the following

Lemma 3.3.6. *Take $\alpha \in (0, N)$ and $f \in C(\mathbb{R};\mathbb{R})$ satisfying $(F_1)-(F_3)$. Then, for a.e. $\lambda \in [\frac{1}{2}, 1]$, problem (3.3.10) admits a bounded Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}}$ at the level c_λ.*

Next, in the spirit of [21], we establish a decomposition of such a Palais-Smale sequence $\{u_n\}$, which will play a crucial role in proving Theorem 3.3.1. However, some difficulties with respect to the local case are carried over by the presence of the nonlocal critical (respect to Hardy-Littlewood-Sobolev inequality) term.

Proposition 3.3.7. *With the same assumptions of Theorem 3.3.1, let $\{u_n\}$ be given by previous lemma. Assume $u_n \rightharpoonup u_\lambda$ in $H^1(\mathbb{R}^N)$. Then, up to subsequences, for any $\lambda \in [\frac{1}{2}, 1]$ there exist $k \in \mathbb{N}$, $\{x_n^j\}_{j=1}^k \subset \mathbb{R}^N$ and $\{v_\lambda^j\}_{j=1}^k \subset H^1(\mathbb{R}^N)$ such that*

(i) $\mathcal{S}_\lambda'(u_\lambda) = 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N)$,

(ii) $v_\lambda^j \not\equiv 0 \quad \text{and} \quad \mathcal{S}_\lambda'(v_\lambda^j) = 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N), \quad \forall\, 1 \le j \le k$,

(iii) $c_\lambda = \mathcal{S}(u_\lambda) + \sum_{j=1}^k \mathcal{S}_\lambda(v_\lambda^j)$,

(iv) $\|u_n - u_\lambda - \sum_{j=1}^k v_\lambda^j(\cdot - x_n^j)\|_{H^1(\mathbb{R}^N)} \to 0 \quad \text{as} \quad n \to +\infty$,

Before proving the proposition, we need a few preliminary lemmas.

Lemma 3.3.8. *Take $\alpha \in (0, N)$, $f \in C(\mathbb{R};\mathbb{R})$ satisfying (F_1) and let $u_\lambda \in H^1(\mathbb{R}^N) \cap W^{2,2}_{loc}(\mathbb{R}^N)$ solving problem (3.3.10). Then,*

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} u^2 dx = \frac{(N+\alpha)\lambda}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx. \tag{3.3.12}$$

Moreover, there exist $\beta, \gamma > 0$ independent of $\lambda \in [\frac{1}{2}, 1]$, such that $\|u_\lambda\|_{H^1(\mathbb{R}^N)} \ge \beta$ and $\mathcal{S}_\lambda(u_\lambda) \ge \gamma$ for any nontrivial solution u_λ and $\lambda \in [\frac{1}{2}, 1]$.

Proof. The proof of identity (3.3.12) is the same as Pohožaev's identity.

Now, let $\lambda \in [\frac{1}{2}, 1]$ and let $u_\lambda \in H^1(\mathbb{R}^N)$ be any nontrivial solution of (3.3.10). Then

$$\int_{\mathbb{R}^N} (|\nabla u_\lambda|^2 + u_\lambda^2)dx \le \int_{\mathbb{R}^N} (I_\alpha * F(u_\lambda)f(u_\lambda)u_\lambda dx. \tag{3.3.13}$$

Thanks to $(F_1)-(F_2)$, there exists $C > 0$ such that $F(s), |sf(s)| \le C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}})$ for any $s \in \mathbb{R}$.
Moreover, as in Proposition (3.1.3), there exists $\beta > 0$ such that

$$\int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)udx \le \frac{1}{2}\|u\|^2_{H^1(\mathbb{R}^N)} \quad \text{if} \quad \|u\|_{H^1(\mathbb{R}^N)} \le \beta,$$

which yields by (3.3.13), $\|u_\lambda\|_{H^1(\mathbb{R}^N)} \ge \beta$. By Pohožaev's identity (3.3.12), it holds

$$\mathcal{S}_\lambda(u_\lambda) = \frac{2+\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 dx + \frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N} u_\lambda^2 dx$$

and this concludes the proof. □

Now, for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, combining the Hardy-Littlewood-Sobolev inequality with Sobolev inequality, we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}})|u|^{\frac{N+\alpha}{N-2}} dx \leq A_\alpha C_\alpha \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} dx \right)^{\frac{N+\alpha}{N}}$$

$$\leq A_\alpha C_\alpha S^{-\frac{N+\alpha}{N-2}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{N+\alpha}{N-2}},$$

where $A_\alpha := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha}$, C_α is defined in Proposition A.0.6 and

$$S := \inf_{0 \not\equiv u \in \mathcal{D}^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} dx)^{\frac{N-2}{2N}}}.$$

Then,

$$S_\alpha := \inf_{0 \not\equiv u \in \mathcal{D}^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N} [I_\alpha * |u|^{\frac{N+\alpha}{N-2}}]|u|^{\frac{N+\alpha}{N-2}} dx)^{\frac{N-2}{N+\alpha}}} \geq \frac{S}{(A_\alpha C_\alpha)^{\frac{N-2}{N+\alpha}}}.$$

Minimizers for S_α are explicitly known from [12, theorem 4.3]. Actually,

$$S_\alpha = \frac{S}{(A_\alpha C_\alpha)^{\frac{N-2}{N+\alpha}}}$$

and it is achieved by the "bubble" function

$$U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{4}}}.$$

This information is crucial to prove an upper estimate for c_λ.

Lemma 3.3.9. *Take $\alpha \in (0, N)$, $q > \max\{1 + \frac{\alpha}{N-2}, \frac{N+\alpha}{2(N-2)}\}$, $\lambda \in [\frac{1}{2}, 1]$ and $f \in C(\mathbb{R}; \mathbb{R})$ satisfying $(F_1) - (F_3)$. Then,*

$$c_\lambda < \frac{2+\alpha}{2(N+\alpha)} \left(\frac{N+\alpha}{N-2} \right)^{\frac{N-2}{2+\alpha}} \lambda^{\frac{2-N}{2+\alpha}} S_\alpha^{\frac{N+\alpha}{2+\alpha}}.$$

Proof. The quite technical proof is inspired by pioneering Brezis-Nirenberg's work on critical problem, and it can be seen entirely on [44, Lemma 3.3] (see Appendix). □

Now we are ready to prove Proposition 3.3.7.

Proof. Take $\lambda \in [\frac{1}{2}, 1]$ and assume $u_n \rightharpoonup u_\lambda$ in $H^1(\mathbb{R}^N)$ satisfies $\mathcal{S}_\lambda(u_n) \to c_\lambda$ and $\mathcal{S}'_\lambda(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ as $n \to +\infty$.

Step 1. We claim $\mathcal{S}'_\lambda(u_\lambda) = 0$ in $H^{-1}(\mathbb{R}^N)$. As a consequence of Lemma 3.3.2, it sufficient to show, up to subsequences, that for any $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n - u)) f(u_n - u)\phi dx \to 0 \quad \text{as} \quad n \to +\infty.$$

In fact, by $(F_1) - (F_2)$ we get

$$|f(s)|^{\frac{2N}{N+\alpha}} \le C(|s|^{\frac{2N}{N+\alpha}} + |s|^{\frac{2+\alpha}{N-2}\frac{2N}{N+\alpha}}), \quad \forall s \in \mathbb{R}.$$

Respectively, by inequality (3.1.3), Hardy-Littlewood-Sobolev inequality and $F(u_n - u)$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ and Rellich's theorem, we have, for any $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\left| \int_{\mathbb{R}^N} (I_\alpha * F(u_n - u)) f(u_n - u)\phi dx \right| \le C \bigg(\int_{\mathbb{R}^N} |f(u_n - u)\phi|^{\frac{2N}{N+\alpha}} dx \bigg)^{\frac{N+\alpha}{2N}} \to 0 \quad \text{as} \quad n \to +\infty.$$

Step 2. Set $v_n^1 := u_n - u_\lambda \in H^1(\mathbb{R}^N)$. We claim

$$\liminf_{n \to +\infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |v_n^1|^2 dx > 0. \tag{3.3.14}$$

Indeed, arguing by contradiction, if not, by Lions' lemma [21, lemma I.1], $v_n^1 \to 0$ in $L^t(\mathbb{R}^N)$ for any $t \in (2, \frac{2N}{N-2})$. Noting that $\mathcal{S}'_\lambda(u_n)[v_n^1] \to 0$ as $n \to +\infty$ and $\mathcal{S}'_\lambda(u_\lambda)[v_n^1] = 0$ for any n, respectively by Lemma 3.3.1 and Lemma 3.3.2, we get

$$c_\lambda = \mathcal{S}_\lambda(u_\lambda) + \mathcal{S}_\lambda(v_n^1) + o(1), \quad \|v_n^1\|^2_{H^1(\mathbb{R}^N)} = \lambda \int_{\mathbb{R}^N} (I_\alpha * F(v_n^1)) f(v_n^1) v_n^1 dx + o(1), \tag{3.3.15}$$

as $n \to +\infty$. Next, we show that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} (I_\alpha * F_1(v_n^1)) F_1(v_n^1) dx = 0,$$

where

$$f_1(t) = f(t) - |t|^{\frac{\alpha+4-N}{N-2}} t, \quad F_1(t) = \int_0^s f_1(s) ds \quad t \in \mathbb{R}.$$

Notice that $f_1(t) = o(t)$ as $t \to 0$ and $\lim_{|t| \to +\infty} \frac{|f_1(t)|}{|t|^{\frac{\alpha+2}{N-2}}} = 0$. So for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $|F_1(t)| \le \epsilon(t^2 + |t|^{\frac{N+\alpha}{N-2}}) + C_\epsilon |t|^r$ for some $r \in (2, \frac{N+\alpha}{N-2})$. Using the fact that $v_n^1 \to 0$ in $L^t(\mathbb{R}^N)$ for any $t \in (2, \frac{2N}{N-2})$ and $\frac{4N}{N+\alpha} \in (2, \frac{2N}{N-2})$, it holds

$$\int_{\mathbb{R}^N} |F_1(v_n^1)|^{\frac{2N}{N+\alpha}} dx \le \epsilon \int_{\mathbb{R}^N} (|v_n^1|^{\frac{4N}{N+\alpha}} + |v_n^1|^{\frac{2N}{N-2}}) dx + C'_\epsilon \int_{\mathbb{R}^N} |v_n^1|^{\frac{2Nq}{N+\alpha}} dx \le C\epsilon + o(1)$$

as $n \to +\infty$. From the arbitrariness of $\epsilon > 0$, it follows that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} |F_1(v_n^1)|^{\frac{2N}{N+\alpha}} dx = 0,$$

which yields, by inequality (3.1.3) and Hardy-Littlewood-Sobolev inequality,

$$\left| \int_{\mathbb{R}^N} (I_\alpha * F_1(v_n^1))F_1(v_n^1)dx \right| \leq C\left(\int_{\mathbb{R}^N} |F_1(v_n^1)|^{\frac{2N}{N+\alpha}} dx \right)^{\frac{N+\alpha}{N}} \to 0 \quad \text{as} \quad n \to +\infty.$$

Similarly,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * F_1(v_n^1))|v_n^1|^{\frac{N+\alpha}{N-2}} dx = 0 \quad \text{and} \quad \lim_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * F_1(v_n^1))f_1(v_n^1)v_n^1 dx = 0.$$

Then, by (3.3.15), we get

$$(**) \begin{cases} c_\lambda = \mathcal{S}_\lambda(u_\lambda) + \frac{1}{2}\|v_n^1\|^2_{H^1(\mathbb{R}^N)} - \frac{\lambda}{2}\left(\frac{N-2}{N+\alpha}\right)^2 \int_{\mathbb{R}^N} (I_\alpha * |v_n^1|^{\frac{N+\alpha}{N-2}})|v_n^1|^{\frac{N+\alpha}{N-2}} dx + o(1) \\ \|v_n^1\|^2_{H^1(\mathbb{R}^N)} = \lambda\frac{N-2}{N+\alpha} \int_{\mathbb{R}^N} (I_\alpha * |v_n^1|^{\frac{N+\alpha}{N-2}})|v_n^1|^{\frac{N+\alpha}{N-2}} dx + o(1), \end{cases}$$

as $n \to +\infty$. Now, let us consider

$$\liminf_{n\to+\infty} \|v_n^1\|^2_{H^1(\mathbb{R}^N)} = \lambda \frac{N-2}{N+\alpha} \liminf_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * |v_n^1|^{\frac{N+\alpha}{N-2}})|v_n^1|^{\frac{N+\alpha}{N-2}} dx := b > 0.$$

From

$$\int_{\mathbb{R}^N} |\nabla v_n^1|^2 dx \geq \mathcal{S}_\alpha \left(\int_{\mathbb{R}^N} (I_\alpha * |v_n^1|^{\frac{N+\alpha}{N-2}})|v_n^1|^{\frac{N+\alpha}{N-2}} dx \right)^{\frac{N-2}{N+\alpha}}$$

for any n, we have

$$b \geq \left(\frac{N+\alpha}{N-2}\right)^{\frac{N-2}{2+\alpha}} \lambda^{\frac{2-N}{2+\alpha}} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}}.$$

By Lemma 3.3.8 and (**),

$$c_\lambda \geq \frac{2+\alpha}{2(N+\alpha)} \left(\frac{N+\alpha}{N-2}\right)^{\frac{N-2}{2+\alpha}} \lambda^{\frac{2-N}{2+\alpha}} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}},$$

which is a contradiction with Lemma 3.3.9. Thus (3.3.14) holds true.

Step 3. By (3.3.14), Rellich's theorem and the fact that $v_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, there exists $\{z_n^1\}_n \subset \mathbb{R}^N$ such that $|z_n^1| \to +\infty$ as $n \to +\infty$ and

$$\liminf_{n\to+\infty} \int_{B_1(z_n^1)} |v_n^1|^2 dx > 0.$$

Let $u_n^1 := v_n^1(\cdot + z_n^1)$. Then, up to subsequences, $u_n^1 \rightharpoonup v_\lambda^1$ in $H^1(\mathbb{R}^N)$ for some $v_\lambda^1 \not\equiv 0$. By Lemmas 3.3.1 and 3.3.2, we have

$$\mathcal{S}_\lambda(u_n^1) \to c_\lambda - \mathcal{S}_\lambda(u_\lambda), \quad \mathcal{S}'_\lambda(u_n^1) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N).$$

Similarly as above, $\mathcal{S}'_\lambda(v_\lambda^1) = 0$. Let $v_n^2 := u_n^1 - v_\lambda^1$. Then,

$$u_n = u_\lambda + v_\lambda^1(\cdot - z_n^1) + v_n^2(\cdot - z_n^1).$$

If $v_n^2 \to 0$ in $H^1(\mathbb{R}^N)$, i.e. $u_\lambda^1 \to v_\lambda^1$, then

$$c_\lambda = \mathcal{S}_\lambda(u_\lambda) + \mathcal{S}_\lambda(v_\lambda^1), \quad \|u_n - u_\lambda - v_\lambda^1(\cdot - z_n^1)\|_{H^1(\mathbb{R}^N)} \to 0 \quad \text{as} \quad n \to +\infty,$$

and we are done. Otherwise, if $v_n^2 \nrightarrow 0$ in $H^1(\mathbb{R}^N)$, similarly as above

$$\liminf_{n\to+\infty} \sup_{z\in\mathbb{R}^N} \int_{B_1(z)} |v_n^2|^2 dx > 0.$$

Then there exists $\{z_n^2\}_n \subset \mathbb{R}^N$ such that by Rellich's theorem $|z_n^2| \to +\infty$ and

$$\liminf_{n\to+\infty} \int_{B_1(z_n^2)} |v_n^2|^2 dx > 0.$$

Let $u_n^2 := v_n^2(\cdot + z_n^2)$. Then, up to subsequences, $u_n^2 \rightharpoonup v_\lambda^2$ in $H^1(\mathbb{R}^N)$ for some $0 \not\equiv v_\lambda^2$.

We have $\mathcal{S}'_\lambda(v_\lambda^2) = 0$ and

$$\mathcal{S}_\lambda(u_n^2) \to c_\lambda - \mathcal{S}_\lambda(u_\lambda) - \mathcal{S}_\lambda(v_\lambda^1), \quad \mathcal{S}'_\lambda(u_n^2) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N).$$

Let $v_n^3 := u_n^2 - v_\lambda^2$. Then

$$u_n = u_\lambda + v_\lambda^1(\cdot - z_n^1) + v_\lambda^2(\cdot - z_n^1 - z_n^2) + v_n^3(\cdot - z_n^1 - z_n^2).$$

If $v_n^3 \to 0$ in $H^1(\mathbb{R}^N)$, i.e. $u_n^2 \to v_\lambda^2$, then

$$c_\lambda = \mathcal{S}_\lambda(u_\lambda) + \mathcal{S}_\lambda(v_\lambda^1) + \mathcal{S}_\lambda(v_\lambda^2), \quad \|u_n - u_\lambda - v_\lambda^1(\cdot - z_n^1) - v_\lambda^2(\cdot - z_n^1 - z_n^2)\|_{H^1(\mathbb{R}^N)} \to 0,$$

and we are done. Otherwise, we can iterate the above procedure and by Lemma 3.3.8, we will end up in a finite number k of steps. Namely, let $x_n^j := \sum_{i=1}^j z_n^i$ for $1 \le j \le k$. Then,

$$c_\lambda = \mathcal{S}_\lambda(u_\lambda) + \sum_{j=1}^k \mathcal{S}_\lambda(v_\lambda^j), \quad \left\| u_n - u_\lambda - \sum_{j=1}^k v_n^j(\cdot - x_n^j) \right\|_{H^1(\mathbb{R}^N)} \to 0 \quad \text{as} \quad n \to +\infty.$$

□

Proof. (*of Theorem 3.3.1*) As a consequence of Lemma 3.3.6, Proposition 3.3.7 and Lemma 3.3.8, one has that for a.e. $\lambda \in [\frac{1}{2}, 1]$, problem (3.3.10) admits a nontrivial solution u_λ (by condition (3.3.14)) satisfying $\|u_\lambda\|_{H^1(\mathbb{R}^N)} \ge \beta$, $\gamma \le \mathcal{S}_\lambda(u_\lambda) \le c_\lambda$, where $\beta, \gamma > 0$ are independent of λ. Then there exist $\{\lambda_n\}_n \subset [\frac{1}{2}, 1]$ and $\{u_n\}_n \subset H^1(\mathbb{R}^N)$ such that, as $n \to +\infty$,

$$\lambda_n \to 1^-, \quad \gamma \le \mathcal{S}_{\lambda_n}(u_n) \le c_{\lambda_n}, \quad \mathcal{S}'_{\lambda_n}(u_n) = 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N). \tag{3.3.16}$$

By Pohožaev's identity (3.3.12) we have

$$\mathcal{S}_{\lambda_n}(u_n) = \frac{2+\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} u_n^2 dx,$$

and so $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Notice that for any $\lambda \in [\frac{1}{2}, 1]$ and $u \in H^1(\mathbb{R}^N)$,

$$\mathcal{S}(u) = \mathcal{S}_\lambda(u) + \frac{1}{2}(\lambda - 1) \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) dx.$$

Then by (3.3.16) and boundedness of $\{u_n\}$, up to subsequences, there exists $c_0 \in [\gamma, c_1]$ such that

$$c_0 := \lim_{n\to+\infty} \mathcal{S}(u_n) = \lim_{n\to+\infty} \mathcal{S}_{\lambda_n}(u_n) \le \lim_{n\to+\infty} c_{\lambda_n} = c_1,$$

where we used the fact that c_λ is continuous from the left at λ. Moreover, by (3.3.16), for any $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\mathcal{S}'(u_n)[\phi] = (\lambda_n - 1) \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n)\phi dx.$$

Similarly as above, there exists $C > 0$ independent for ϕ such that

$$\left(\int_{\mathbb{R}^N} |f(u_n)\phi|^{\frac{2N}{N+\alpha}} dx \right)^{\frac{N+\alpha}{2N}} \le C\|\phi\|_\infty.$$

By Hardy-Littlewood-Sobolev inequality and inequality (3.1.3),

$$|\mathcal{S}'(u_n)[\phi]| = (1 - \lambda_n) \left| \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n)\phi dx \right|$$

$$\le C(1-\lambda_n) \left(\int_{\mathbb{R}^N} |F(u_n)|^{\frac{2N}{N+\alpha}} dx \right)^{\frac{N+\alpha}{2N}} \left(\int_{\mathbb{R}^N} |f(u_n)\phi|^{\frac{2N}{N+\alpha}} dx \right)^{\frac{N+\alpha}{2N}} = o(1)\|\phi\|_\infty$$

as $n \to +\infty$, for any $\phi \in C_0^\infty(\mathbb{R}^N)$. Namely, by density $\mathcal{S}'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$. Finally, we obtain

$$\|u_n\|_{H^1(\mathbb{R}^N)} \ge \beta, \quad \mathcal{S}(u_n) \to c_0 \le c_1, \quad \mathcal{S}'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N) \quad \text{as} \quad n \to +\infty.$$

If $u_n \to u_0$ in $H^1(\mathbb{R}^N)$, then $\|u_0\|_{H^1(\mathbb{R}^N)} \ge \beta$, $\mathcal{S}(u_0) = c_0 \le c_1$ and $\mathcal{S}'(u_0) = 0$ in $H^{-1}(\mathbb{R}^N)$. Let define

$$E := \inf\{\mathcal{S}(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\} \quad \text{s.t.} \quad \mathcal{S}'(u) = 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N)\}.$$

So we obtained $E \le c_1$. As in the subcritical case, Proposition 3.1.7 implies $c_1 \le E$ and so $\mathcal{S}(u_0) = E = c_1$, namely u_0 is a ground state solution of (*).
Otherwise, as a consequence of Proposition 3.3.7 with $\lambda = 1$, $c_\lambda = c_0$, $u_\lambda = u_0$, there

exists $k \in \mathbb{N}$ and $\{v^j\}_{j=1}^k \subset H^1(\mathbb{R}^N)$ such that $v^j \not\equiv 0$, $\mathcal{S}'(v^j) = 0$ in $H^{-1}(\mathbb{R}^N)$ for all $1 \leq j \leq k$ and $c_0 = \mathcal{S}(u_0) + \sum_{j=1}^k \mathcal{S}(v^j)$. We know that $E \in [\gamma, c_1]$.

We conclude the proof of Theorem 3.3.1 by showing that E is achieved. Clearly by definition of infimum, there exists $\{v_n\}_n \subset H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\mathcal{S}(v_n) \to E$ and $\mathcal{S}'(v_n) = 0$ in $H^{-1}(\mathbb{R}^N)$. By Pohožaev's identity, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and so $v_n \rightharpoonup v_0 \not\equiv 0$ in $H^1(\mathbb{R}^N)$. As in Proposition 3.1.4, $\mathcal{S}'(v_0) = 0$ in $H^{-1}(\mathbb{R}^N)$. If $v_n \to v_0$ in $H^1(\mathbb{R}^N)$, then $\mathcal{S}(v_0) = E$ and so v_0 is a ground state of (*).
Otherwise, by Proposition 3.3.7 there exists $k \in \mathbb{N}$ and $\{v^j\}_{j=1}^k \subset H^1(\mathbb{R}^N)$ such that $v^j \not\equiv 0$, $\mathcal{S}'(v^j) = 0$ in $H^{-1}(\mathbb{R}^N)$ for all $1 \leq j \leq k$ and $E = \mathcal{S}(v_0) + \sum_{j=1}^k \mathcal{S}(v^j)$. By definition of E, Lemma 3.3.8 and $v^j \not\equiv 0$, it holds $v_0 = 0$, we can assume $k = 1$ and so $E = \mathcal{S}(v^1)$, which yields v^1 as a ground state solution of (*). □

3.4 Existence of ground-states in critical case on the plane

The aim of this section is to prove an existence result for ground-states solutions to (*) in dimension $N = 2$, assuming that the nonlinearity has an exponential critical growth at infinity.
Therefore, let us consider $f \in C(\mathbb{R}^+;\mathbb{R})$ satisfying

(f_1) $\lim_{s\to 0^+} \frac{f(s)}{s^{\frac{\alpha}{2}}} = 0$,

(f_2) $\lim_{s\to +\infty} \frac{f(s)}{e^{\beta s^2}} = 0 \quad (+\infty) \quad \text{if} \quad \beta > 4\pi \quad (\beta < 4\pi)$,

(f_3) *(Ambrosetti-Rabinowitz condition):* $\exists\, \theta > 2$ s.t $0 < \theta F(s) \leq 2f(s)s$ $\forall s > 0$, where $F(s) = \int_0^s f(t)dt$,

(f_4) $\exists p > \frac{2+\alpha}{2}$, s.t. $F(s) \geq C_p s^p \quad \forall s \geq 0$, where $C_p > \frac{\left(\frac{4\theta(p-1)}{\alpha(\theta-2)}\right)^{\frac{p-1}{2}} S_p^p}{p^{\frac{p}{2}}}$ and $S_p := \inf_{u\in H^1(\mathbb{R}^2)\setminus\{0\}} \frac{\|u\|_{H^1(\mathbb{R}^2)}}{\left(\int_{\mathbb{R}^2}(I_\alpha * |u|^p)|u|^p dx\right)^{\frac{1}{2p}}}$.

The main result is the following:

Theorem 3.4.1. *Assume $N = 2$, $\alpha \in (0, 2)$ and $f \in C(\mathbb{R}^+;\mathbb{R})$ satisfying $(f_1) - (f_4)$. Then, problem (*) admits a nontrivial ground state solution.*

First of all, let us introduce the following Moser-Trudinger inequality due to Cao [10], which will be crucial for our variational methods.

Lemma 3.4.1. *If $\beta > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} (e^{\beta u^2} - 1)dx < +\infty.$$

Moreover, if $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$, $\|u\|_{L^2(\mathbb{R}^2)} \leq M < +\infty$ *and* $\beta \in (0, 4\pi)$, *then there exists* $C > 0$, *which depends only on* M *and* β, *such that*

$$\int_{\mathbb{R}^2} (e^{\beta u^2} - 1)dx \leq C(M, \beta).$$

Since we are going to study the existence of positive solutions, we will assume that

$$f(s) = 0 \quad \forall s < 0.$$

Now let us consider the well-defined energy functional (thanks to Lemma 3.4.1) $\mathcal{S} : H^1(\mathbb{R}^2) \to \mathbb{R}$, given by

$$\mathcal{S}(u) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2)dx - \frac{1}{2}\int_{\mathbb{R}^2} (I_\alpha * F(u))F(u)dx.$$

Next, we will show that $\mathcal{S}$ verifies the mountain pass geometry.

Lemma 3.4.2. *Let* $f \in C(\mathbb{R}^+; \mathbb{R})$ *satisfying* $(f_1) - (f_3)$. *Then,*

(i) There exists $\rho, \delta_0 > 0$ *such that* $\mathcal{S}_{|S_\rho} \geq \delta_0$, $\forall u \in S_\rho := \{u \in H^1(\mathbb{R}^2) : \|u\|_{H^1(\mathbb{R}^2)} = \rho\}$.

(ii) There is $e \in H^1(\mathbb{R}^2)$ *with* $\|e\|_{H^1(\mathbb{R}^2)} > \rho$ *such that* $\mathcal{S}(e) < 0$.

Proof. (i). Conditions $(f_1) - (f_3)$ imply that for any $p > 1$, there exists $C = C(p) > 0$ such that

$$|F(s)| \leq C(p)(|s|^{\frac{2+\alpha}{2}} + |s|^p[e^{4\pi s^2} - 1]) \quad \forall s \in \mathbb{R},$$

from which it follows by Minkowski's inequality

$$\|F(u)\|_{L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)} \leq C\left(\|u\|^{\frac{2+\alpha}{2}}_{L^2(\mathbb{R}^2)} + \||u|^p[e^{4\pi u^2} - 1]\|_{L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)}\right).$$

Since $p > 1$, Sobolev embedding and Holder's inequality, there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^2} |u|^{\frac{4p}{2+\alpha}}[e^{4\pi u^2} - 1]^{\frac{4}{2+\alpha}}dx \leq \left(\int_{\mathbb{R}^2} |u|^{\frac{8p}{2+\alpha}}dx\right)^{\frac{1}{2}}\left([e^{4\pi u^2} - 1]^{\frac{8}{2+\alpha}}dx\right)^{\frac{1}{2}} \leq$$

$$\leq C_1\|u\|^{\frac{4p}{2+\alpha}}_{H^1(\mathbb{R}^2)}\left(\int_{\mathbb{R}^2} [e^{\frac{8}{2+\alpha}4\pi u^2} - 1]dx\right)^{\frac{1}{2}}.$$

Noting that

$$\int_{\mathbb{R}^2} [e^{\frac{8}{2+\alpha}4\pi u^2} - 1]dx = \int_{\mathbb{R}^2}\left[e^{\frac{8}{2+\alpha}\|u\|^2_{H^1}4\pi\frac{u^2}{\|u\|^2_{H^1}}} - 1\right]dx,$$

fixing $\xi \in (0,1)$ such that $\frac{8}{2+\alpha}\|u\|^2_{H^1(\mathbb{R}^2)} := \xi < 1$, Lemma 3.4.1 implies

$$\int_{\mathbb{R}^2}\left[e^{\xi 4\pi \frac{u^2}{\|u\|^2_{H^1}}} - 1\right]dx \le C_2 \quad \text{for} \quad \|u\|_{H^1(\mathbb{R}^2)} = \left(\frac{\xi(2+\alpha)}{8}\right)^{\frac{1}{2}}.$$

Then,

$$\|F(u)\|_{L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)} \le C\|u\|^{\frac{2+\alpha}{2}}_{H^1(\mathbb{R}^2)} + C_3\|u\|^p_{H^1(\mathbb{R}^2)} \quad \text{for} \quad \|u\|_{H^1(\mathbb{R}^2)} = \left(\frac{\xi(2+\alpha)}{8}\right)^{\frac{1}{2}}.$$

Thereby, by Hardy-Littlewood-Sobolev inequality and inequality (3.1.3),

$$\int_{\mathbb{R}^2}(I_\alpha * F(u))F(u)dx \le C\|u\|^{2+\alpha}_{H^1(\mathbb{R}^2)} + C_4\|u\|^{2p}_{H^1(\mathbb{R}^2)} \quad \text{for} \quad \|u\|_{H^1(\mathbb{R}^2)} = \left(\frac{\xi(2+\alpha)}{8}\right)^{\frac{1}{2}},$$

and so

$$\mathcal{S}(u) \ge \frac{1}{2}\|u\|^2_{H^1(\mathbb{R}^2)} - C\|u\|^{2+\alpha}_{H^1(\mathbb{R}^2)} - C_4\|u\|^{2p}_{H^1(\mathbb{R}^2)}.$$

Since $\alpha > 0$ and $p > 1$, (i) follows choosing $\rho = \left(\frac{\xi(2+\alpha)}{8}\right)^{\frac{1}{2}}$ with ξ sufficiently small.

(ii). Fixing $u_0 \in H^1(\mathbb{R}^2)$ with $u_0^+ \not\equiv 0$, we set

$$\mathcal{A}(t) = \psi\left(\frac{tu_0}{\|u_0\|_{H^1}}\right) > 0 \quad \forall t > 0,$$

where

$$\psi(u) = \frac{1}{2}\int_{\mathbb{R}^2}(I_\alpha * F(u))F(u)dx.$$

Now, (f_3) implies

$$\frac{\mathcal{A}'(t)}{\mathcal{A}(t)} \ge \frac{\theta}{t}, \quad \forall t > 0.$$

Then, integrating over $[1, s\|u_0\|_{H^1}]$ with $s > \frac{1}{\|u_0\|_{H^1}}$, we find

$$\psi(su_0) \ge \psi\left(\frac{u_0}{\|u_0\|_{H^1}}\right)\|u_0\|^\theta_{H^1}s^\theta.$$

Therefore

$$\mathcal{S}(su_0) \le C_1s^2 - C_2s^\theta \quad \text{for} \quad s > \frac{1}{\|u_0\|_{H^1}},$$

and (ii) holds for $e = su_0$ with s large enough. □

By the mountain pass theorem without (PS) condition from [42], there is a (PS)-c sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$, where the mountain pass level c is defined by

$$\delta_0 \le c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{S}(\gamma(t))$$

with

$$\Gamma := \{\gamma \in C([0,1]; H^1(\mathbb{R}^2)) : \gamma(0) = 0, \mathcal{S}(\gamma(1)) = e < 0\}.$$

The next lemma is crucial because it establishes an important estimate involving the level c.

Lemma 3.4.3. *The mountain pass level c satisfies* $c \in [\delta_0, \frac{\alpha^2(\theta-2)}{8\theta(2+\alpha)})$*. Moreover, the (PS)-c sequence is bounded in* $H^1(\mathbb{R}^2)$ *and its weak limit u satisfies* $\mathcal{S}'(u) = 0$ *in* $H^{-1}(\mathbb{R}^2)$*.*

Proof. First, note that in the proof of (i) we can choose $\rho > 0$ sufficiently small such that $\delta_0 < \frac{\alpha^2(\theta-2)}{8\theta(2+\alpha)}$. From (f_3),

$$c = \limsup_{n\to+\infty} \left(\mathcal{S}(u_n) - \frac{1}{\theta}\mathcal{S}'(u_n)[u_n]\right) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \limsup_{n\to+\infty} \|u_n\|^2_{H^1(\mathbb{R}^2)}$$

which means

$$\limsup_{n\to+\infty} \|u_n\|^2_{H^1(\mathbb{R}^2)} \leq \frac{2\theta}{\theta-2}c. \tag{3.4.1}$$

Let $u \in H^1(\mathbb{R}^2)$ such that $u^+ \not\equiv 0$ and $t > 0$. By (f_4),

$$\mathcal{S}(tu) \leq \frac{t^2}{2}\|u\|^2_{H^1(\mathbb{R}^2)} - \frac{C_p^2}{2}t^{2p}\int_{\mathbb{R}^2}(I_\alpha * u^p)u^p dx < 0 \tag{3.4.2}$$

for some $t_u \gg 1$. Now, it is well-known that exists a positive radial function $u_p \in H^1(\mathbb{R}^2)$ such that S_p is achieved by u_p. By (f_4), it easy to see that

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{S}(\gamma(t)) \leq \inf_{u\in H^1(\mathbb{R}^2), u^+\not\equiv 0}\max_{t\in[0,1]} \mathcal{S}(tt_u u) \leq \inf_{u\in H^1(\mathbb{R}^2), u^+\not\equiv 0}\max_{t\geq 0} \mathcal{S}(tu) \leq$$

$$\leq \max_{t\geq 0} \mathcal{S}(tu_p) \leq \max_{t\geq 0}\left(\frac{t^2}{2}\|u_p\|^2_{H^1(\mathbb{R}^2)} - \frac{C_p^2}{2}t^{2p}\int_{\mathbb{R}^2}(I_\alpha * u_p^p)u_p^p dx\right) =$$

$$= \frac{(p-1)S_p^{\frac{2p}{p-1}}}{2p^{\frac{p}{p-1}}C_p^{\frac{2}{p-1}}} < \frac{\alpha^2(\theta-2)}{8\theta(2+\alpha)}.$$

Consequently, from (3.4.1),

$$\limsup_{n\to+\infty} \|u_n\|^2_{H^1(\mathbb{R}^2)} < \frac{\alpha^2}{4(2+\alpha)}.$$

So without loss of generality, we may assume that

$$\|u_n\|^2_{H^1(\mathbb{R}^2)} \leq m \quad \forall n \in \mathbb{N},$$

for some $m \in (0, \frac{\alpha^2}{4(2+\alpha)})$. Furthermore, we claim that

$$\|I_\alpha * F(u_n)\|_{L^\infty(\mathbb{R}^2)} \leq C \quad \forall n \in \mathbb{N}. \tag{3.4.3}$$

In fact, observe that for any $\frac{4}{2+\alpha} \leq p \leq \frac{4}{\alpha}$, in view of Sobolev embedding and Holder's inequality,

$$\int_{\mathbb{R}^2} |F(u_n)|^p dx \leq C_1 \bigg(\int_{\mathbb{R}^2} |u_n|^{\frac{p(2+\alpha)}{2}} dx + \int_{\mathbb{R}^2} |u_n|^p [e^{4\pi u_n^2} - 1]^p dx \bigg) \leq$$

$$\leq C_2 \|u_n\|_{H^1(\mathbb{R}^2)}^{\frac{p(\alpha+2)}{2}} + C_1 \bigg(\int_{\mathbb{R}^2} |u_n|^{\frac{p(\alpha+2)}{2}} dx \bigg)^{\frac{2}{2+\alpha}} \bigg(\int_{\mathbb{R}^2} [e^{4\pi u_n^2} - 1]^{\frac{p(2+\alpha)}{\alpha}} dx \bigg)^{\frac{\alpha}{2+\alpha}} \leq$$

$$\leq C_3 + C_4 \bigg(\int_{\mathbb{R}^2} \bigg[e^{\frac{p(2+\alpha)m}{\alpha} 4\pi \frac{u_n^2}{\|u_n\|_{H^1}^2}} - 1 \bigg] dx \bigg)^{\frac{\alpha}{2+\alpha}} \leq C_5 \quad \forall n \in \mathbb{N}$$

by Lemma 3.4.1 and definition of m. Since $\frac{4}{2+\alpha} < \frac{2}{\alpha} < \frac{4}{\alpha}$, Proposition A.0.9 and the above estimate imply (3.4.3).

Now, since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$, let us consider its weak limit $u \in H^1(\mathbb{R}^2)$. We are going to prove that $\mathcal{S}'(u) = 0$ in $H^{-1}(\mathbb{R}^2)$. Indeed, we will prove that for any $\phi \in C_0^\infty(\mathbb{R}^2)$, as $n \to +\infty$

$$\int_{\mathbb{R}^2} (I_\alpha * F(u_n) f(u_n) \phi dx \to \int_{\mathbb{R}^2} (I_\alpha * F((u)) f(u) \phi dx.$$

First, observe that as in previous inequality, it is easy to show that $\{f(u_n)\}_n$ is bounded in $L^{\frac{4}{\alpha}}(\mathbb{R}^2)$. So, for any $\phi \in C_0^\infty(\mathbb{R}^2)$,

$$\bigg| \int_{\mathbb{R}^2} \bigg((I_\alpha * F(u_n)) f(u_n) - (I_\alpha * F(u)) f(u) \bigg) \phi dx \bigg| \leq$$

$$\leq \bigg| \int_{\mathbb{R}^2} (I_\alpha * F(u_n))(f(u_n) - f(u)) \phi dx \bigg| + \bigg| \int_{\mathbb{R}^2} (I_\alpha * [F(u_n) - F(u)]) f(u) \phi dx \bigg|. \quad (3.4.4)$$

For the above first term, we recall that $I_\alpha * F(u_n)$ is bounded in $L^\infty(\mathbb{R}^2)$. Then,

$$\bigg| \int_{\mathbb{R}^2} \bigg((I_\alpha * F(u_n)) f(u_n) - (I_\alpha * F(u)) f(u) \bigg) \phi dx \bigg| \leq C \bigg| \int_{\mathbb{R}^2} (f(u_n) - f(u)) \phi dx \bigg|.$$

Since $u_n \to u$ a.e. in $\mathbb{R}^2$, the continuity of f implies $f(u_n) \to f(u)$ a.e. in $\mathbb{R}^2$. This fact, combined with the boundedness of $f(u_n)$ in $L^{\frac{4}{\alpha}}(\mathbb{R}^2)$, leads to

$$f(u_n) \rightharpoonup f(u) \quad \text{in} \quad L^{\frac{4}{\alpha}}(\mathbb{R}^2),$$

from where it follows that

$$\int_{\mathbb{R}^2} (f(u_n) - f(u)) \phi dx \to 0$$

as $n \to +\infty$.

For the second term of (3.4.4), notice that by Fubini-Tonelli's theorem

$$\left| \int_{\mathbb{R}^2} (I_\alpha * [F(u_n) - F(u)]) f(u) \phi dx \right| = \left| \int_{\mathbb{R}^2} (F(u_n) - F(u))(I_\alpha * (f(u)\phi)) dx \right|.$$

Since $u_n \to u$ a.e. in $\mathbb{R}^2$, the continuity of F implies $F(u_n) \to F(u)$ a.e. in $\mathbb{R}^2$. Using the boundedness of $F(u_n)$ in $L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)$ leads to

$$F(u_n) \rightharpoonup F(u) \quad \text{in} \quad L^{\frac{4}{2+\alpha}}(\mathbb{R}^2).$$

As, by Holder's inequality,

$$I_\alpha * (f(u)\phi) \in L^{\frac{4}{2-\alpha}}(\mathbb{R}^2),$$

we get

$$\left| \int_{\mathbb{R}^2} (I_\alpha * [F(u_n) - F(u)]) f(u) \phi dx \right| \to 0$$

for any $\phi \in C_0^\infty(\mathbb{R}^2)$. So the lemma is proved.

□

Proof. (*of Theorem 3.4.1*) Let $\{u_n\}_n \subset H^1(\mathbb{R}^2)$ be the (PS)-c sequence and u its weak limit. We are left to prove that $u \not\equiv 0$ and it is actually a ground state solution to (*).

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$, we have either $\{u_n\}$ is vanishing, i.e.,

$$\liminf_{n \to +\infty} \sup_{y \in \mathbb{R}^2} \int_{B_1(y)} u_n^2 dx = 0,$$

or non-vanishing, i.e., there exists a sequence $\{y_n\} \subset \mathbb{R}^2$ such that

$$\liminf_{n \to +\infty} \int_{B_1(y_n)} u_n^2 dx > 0.$$

If $\{u_n\}$ is vanishing, then, by Lions' result [21, lemma I.1], we have that

$$u_n \to 0 \quad \text{in} \quad L^s(\mathbb{R}^2), \quad 2 < s < +\infty. \tag{3.4.5}$$

Using inequality (3.1.3), Hardy-Littlewood-Sobolev inequality and (f_3), we get

$$\left| \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) f(u_n) u_n dx \right| \le C \|F(u_n)\|_{L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)} \|f(u_n)u_n\|_{L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)} \le$$

$$\le C' \|f(u_n)u_n\|^2_{L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)}.$$

For any $\epsilon > 0$, there exists $C = C(\epsilon) > 0$ such that

$$|f(s)| \le \epsilon |s|^{\frac{\alpha}{2}} + C(\epsilon)(e^{4\pi s^2} - 1) \quad \forall s \in \mathbb{R},$$

from which we derive, using Minkowski's inequality,

$$\|f(u_n)u_n\|_{L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)} \leq \epsilon\|u_n\|^{\frac{2+\alpha}{2}}_{L^2(\mathbb{R}^2)} + C(\epsilon)\left(\int_{\mathbb{R}^2} |u_n|^{\frac{4}{2+\alpha}}[e^{4\pi u_n^2}-1]^{\frac{4}{2+\alpha}}dx\right)^{\frac{2+\alpha}{4}}.$$

Now, from Holder's inequality and for some $p > 1$ to be chosen later, we have

$$\int_{\mathbb{R}^2} |u_n|^{\frac{4}{2+\alpha}}[e^{4\pi u_n^2}-1]^{\frac{4}{2+\alpha}}dx \leq \left(\int_{\mathbb{R}^2}|u_n|^{2p}dx\right)^{\frac{2}{(2+\alpha)p}}\left(\int_{\mathbb{R}^2}\left[e^{4\pi u_n^2}-1\right]^{\frac{4p}{(2+\alpha)p-2}}dx\right)^{\frac{(2+\alpha)p-2}{(2+\alpha)p}}$$

$$\leq \left(\int_{\mathbb{R}^2}|u_n|^{2p}dx\right)^{\frac{2}{(2+\alpha)p}}\left(\int_{\mathbb{R}^2}\left[e^{\frac{4pm}{(2+\alpha)p-2}4\pi\frac{u_n^2}{\|u_n\|^2_{H^1}}}-1\right]dx\right)^{\frac{(2+\alpha)p-2}{(2+\alpha)p}} \leq C_1\left(\int_{\mathbb{R}^2}|u_n|^{2p}dx\right)^{\frac{2}{(2+\alpha)p}}$$

by Lemma 3.4.1 if we choose $p > \frac{2+\alpha}{1+\alpha} > 1$. Then,

$$\left|\int_{\mathbb{R}^2}(I_\alpha * F(u_n))f(u_n)u_n dx\right| \leq \epsilon\|u_n\|^{2+\alpha}_{L^2(\mathbb{R}^2)} + C_2(\epsilon)\left(\int_{\mathbb{R}^2}|u_n|^{2p}dx\right)^{\frac{1}{p}} \leq$$

$$\leq C_3\epsilon + C_2(\epsilon)\left(\int_{\mathbb{R}^2}|u_n|^{2p}dx\right)^{\frac{1}{p}}.$$

Since $p > 1$, (3.4.5) and arbitrariness of $\epsilon > 0$ imply

$$\int_{\mathbb{R}^2}(I_\alpha * F(u_n))f(u_n)u_n dx \to 0,$$

from which we derive

$$u_n \to 0 \quad \text{in} \quad H^1(\mathbb{R}^2),$$

since $\mathcal{S}'(u_n)[u_n] \to 0$ as $n \to +\infty$. Recalling that $\mathcal{S}$ is a continuous functional, we have

$$\mathcal{S}(u_n) \to 0,$$

from where it follows that $c = 0$, which is a contradiction. Thereby, vanishing does not hold.

From now on, we set $v_n := u_n(\cdot - y_n)$. Therefore, $\|v_n\|_{H^1} = \|u_n\|_{H^1}$ for all n and

$$\liminf_{n\to+\infty}\int_{B_1}|v_n|^2dx > 0. \tag{3.4.6}$$

Since $\mathcal{S}$ and $\mathcal{S}'$ are both invariant by translations,

$$\mathcal{S}(v_n) \to c \quad \text{and} \quad \mathcal{S}'(v_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^2).$$

Since $\{v_n\}$ is also bounded in $H^1(\mathbb{R}^2)$, we may assume $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$ and $v_n \to v$ in $L^2_{loc}(\mathbb{R}^2)$ by Rellich's theorem. From (3.4.6) we get $v \not\equiv 0$ and by the same arguments in Lemma 3.4.3 we can assume that $\mathcal{S}'(v) = 0$ in $H^{-1}(\mathbb{R}^2)$.

Finally, let $\mathcal{N}$ be the Nehari manifold by

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^2) \setminus \{0\} : \mathcal{S}'(u)[u] = 0\}.$$

From [30, Proposition 3.11], the mountain pass level can be characterized by

$$c = \inf_{u \in H^1(\mathbb{R}^2)\setminus\{0\}} \max_{t\geq 0} \mathcal{S}(tu) = \inf_{u\in\mathcal{N}} \mathcal{S}(u) \leq \mathcal{S}(v). \tag{3.4.7}$$

On the other hand, by weak lower-semicontinuity of H^1-norm, Fatou's lemma and (f_3), we get

$$c = \liminf_{n\to+\infty} \left((\frac{1}{2}-\frac{1}{\theta})\|v_n\|^2_{H^1(\mathbb{R}^2)} - \frac{1}{2}\int_{\mathbb{R}^2} (I_\alpha * F(v_n))F(v_n)dx + \frac{1}{\theta}\int_{\mathbb{R}^2} (I_\alpha * F(v_n))f(v_n)v_n dx \right) \geq$$

$$\geq \mathcal{S}(v) - \frac{1}{\theta}\mathcal{S}'(u)[u] = \mathcal{S}(v),$$

since $\mathcal{S}'(v) = 0$. So, last inequality combined with (3.4.7) give $\mathcal{S}(v) = c$, showing that v is a ground state solution to (*).

□

3.5 Existence of infinitely many pairs of radial solutions

Case $N \geq 3$

In this section we will discuss the existence of infinitely many radial solutions to problem (*), assuming the nonlinearity belongs to a particular class of subcritical functions.
In Section 2, we proved that the equation

$$-\Delta u + u = f(u), \quad u \in H^1(\mathbb{R}^N)$$

has a sequence of radial solutions $\{u_k\}_k$ with the energies going to infinity as $k \to +\infty$ when $N \geq 2$. On this way, we are interested on the existence of radial solutions to Choquard equation (*) for $N \geq 3$ assuming the following conditions:

(f_1) $f \in C(\mathbb{R};\mathbb{R})$ and there exist $C > 0$ and $\frac{N+\alpha}{N} < q_1 \leq q_2 < \frac{N+\alpha}{N-2}$ such that $|f(s)| \leq C(|s|^{q_1-1} + |s|^{q_2-1}), \quad \forall s \in \mathbb{R}$.

(f_2) $\lim_{|s|\to+\infty} \frac{F(s)}{|s|} = +\infty$, where $F(s) = \int_0^t f(t)dt \quad \forall s \in \mathbb{R}$.

(f_3) f is odd.

Now we state the main result.

Theorem 3.5.1. *Let (f_1)–(f_3) hold and $N \geq 3$. Then, problem (*) admits an unbounded sequence in $H^1(\mathbb{R}^N)$ of radial solutions $\{\pm u_k\}_{k\in\mathbb{N}}$ such that $\mathcal{S}(u_k) = \mathcal{S}(-u_k) \to +\infty$ as $k \to +\infty$.*

First, let explain why it is possible to seek solutions on the space $H^1_r(\mathbb{R}^N)$.
Let us consider the orthonormal group of dimension $N \geq 2$

$$G := O(N),$$

and its action on $H^1(\mathbb{R}^N)$ defined as

$$gu(x) := u(g^{-1}x), \quad \text{for every} \quad g \in G \quad \text{and} \quad x \in \mathbb{R}^N.$$

So

$$Fix(G) := \{u \in H^1(\mathbb{R}^N) : gu = u, \forall g \in G\}$$

consists actually of the space $H^1_r(\mathbb{R}^N)$.

To prove our result, we need the principle of symmetric criticality theorem.

Lemma 3.5.1. *([42], theorem 1.28). Assume that the action of a topological group G on a Hilbert space X is isometric. If $\mathcal{S} \in C^1(X, \mathbb{R})$ is invariant and if u is a critical point of $\mathcal{S}$ restricted to $Fix(G)$, then u is a critical point of $\mathcal{S}$.*

From the above lemma, it suffices to look for critical points of $\mathcal{S}$ on $H^1(\mathbb{R}^N)$.
To prove Theorem 3.5.1, we need the following fountain theorem [42, theorem 3.6].

Theorem 3.5.2. *Let $(E, \|\cdot\|)$ be a Hilbert space with $\{e_j\}_{j\in\mathbb{N}}$ an orthonormal space, and set $E_k := span(e_1, ..., e_k)$ for any fixed $k \in \mathbb{N}$. Consider an even C^1-functional $\phi : E \to \mathbb{R}$ which satisfies (PS) condition. If, for every $k \in \mathbb{N}$, there exists $R_k > r_k > 0$ such that*

(i) $\max_{u\in E_k, \|u\|=R_k} \phi(u) \leq 0$*;*

(ii) $\inf_{u\in E^{\perp}_{k-1}, \|u\|=r_k} \phi(u) \to +\infty$ *as* $k \to +\infty$*.*

Then ϕ possesses an unbounded sequence of critical values c_k characterized as

$$c_k = \inf_{h\in\Gamma_k} \sup_{u\in B_k} \phi(h(u)),$$

where $B_k := \{u \in E_k : \|u\| \leq R_k\}$ with R_k large enough so that (i) holds, and

$$\Gamma_k := \{h : B_k \to E : \quad h \text{ is odd}, \quad h_{|\partial B_{R_k}} = id\}.$$

Now, we will give some important lemmas.

Lemma 3.5.2. *There exist $\rho, \xi > 0$ such that $\mathcal{S}_{|\partial B_\rho} \geq \xi$.*

Proof. By (f_1), Hardy-Littlewood-Sobolev inequality and Sobolev embedding, we have for any $u \in H^1(\mathbb{R}^N)$

$$\mathcal{S}(u) \geq \frac{1}{2}\|u\|^2_{H^1} - C_1(\|u\|^{2q_1}_{L^{r_0q_1}} + \|u\|^{2q_2}_{L^{r_0q_2}}) \geq \frac{1}{2}\|u\|^2_{H^1} - C_2(\|u\|^{2q_1}_{H^1} + \|u\|^{2q_2}_{H^1}).$$

Since $q_2 \geq q_1 > 1$ we can choose constants $\rho, \alpha > 0$ such that $\mathcal{S}_{|\partial B_\rho} \geq \alpha$. ∎

Lemma 3.5.3. *For each finite dimensional space $\tilde{E} \subset H^1_r(\mathbb{R}^N)$, there exists $R = R(\tilde{E}) > 0$ such that $\mathcal{S}_{|\tilde{E}\setminus B_R} \leq 0$.*

Proof. We argue by contradiction. Supppose that, for some finite dimensional subspace $\tilde{E} \subset H^1_r(\mathbb{R}^N)$, there exists $\{v_n\}_n \subset \tilde{E}$ satisfying $\|v_n\|_{H^1(\mathbb{R}^N)} \to +\infty$ and $\mathcal{S}(v_n) > 0$ for any $n \in \mathbb{N}$. Set $w_n := \frac{v_n}{\|v_n\|_{H^1}}$, we may assume $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$, $w_n \to w$ in $L^s(\mathbb{R}^N)$ for $2 < s < \frac{2N}{N-2}$ and $w_n \to w$ a.e. in $\mathbb{R}^N$. If $w \not\equiv 0$, there exists a Lebesgue-measurable set $\Lambda \subset \mathbb{R}^N$ with $|\Lambda| > 0$, such that $|v_n(x)| \to +\infty$ a.e. $x \in \Lambda$. Then, by (f_2) and Fatou's lemma

$$0 < \frac{\mathcal{S}(v_n)}{\|v_n\|^2_{H^1}} \leq \frac{1}{2} - \frac{C}{2} \int_\Lambda \left(\int_\Lambda \frac{1}{|x-y|^{N-\alpha}} \frac{F(v_n(y))}{|v_n(y)|} |w_n(y)| dy \right) \frac{F(v_n(x))}{|v_n(x)|} |w_n(x)| dx \to -\infty$$

as $n \to +\infty$. This is a contradiction. So $w \equiv 0$ and $w_n \to 0$ in L^s for $2 < s < \frac{2N}{N-2}$. Since all norms are equivalent on finite dimensional spaces, we have $\|w_n\|_{H^1(\mathbb{R}^N)} \leq C\|w_n\|_{L^s(\mathbb{R}^N)} \to 0$ which is an absurd since $\|w_n\|_{H^1} = 1$ for any n. □

Lemma 3.5.4. *$\mathcal{S}$ satisfies the (PS)-c_k for every $k \in \mathbb{N}$.*

Proof. Set $r_0 := \frac{2N}{N+\alpha}$. First, observe that by Lemma 3.5.2, $c_k > 0$ for any k. Assume that $\{u_n\}_n \subset H^1_r(\mathbb{R}^N)$ satisfies $\mathcal{S}(u_n) \to c_k$ and $\mathcal{S}'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ as $n \to +\infty$. Arguing as in Propositions 3.1.3 and 3.1.4 using [42, theorem 2.9] (involving Lemma 3.5.3), it follows that $\{u_n\}_n$ is bounded in $H^1(\mathbb{R}^N)$. So we may suppose $u_n \rightharpoonup u$ in $H^1_r(\mathbb{R}^N)$, $u_n \to u$ in $L^{r_0 q_1}(\mathbb{R}^N)$ and $L^{r_0 q_2}(\mathbb{R}^N)$ by Corollary A.0.4, and $u_n \to u$ a.e. in $\mathbb{R}^N$. By Hardy-Littlewood-Sobolev inequality, inequality (3.1.3), Holder's inequality and (f_1) we deduce

$$\left| \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n)(u_n - u) dx \right| \leq$$

$$\leq C(\|u_n\|^{q_1}_{L^{r_0 q_1}} + \|u_n\|^{q_2}_{L^{r_0 q_2}})(\|u_n\|^{q_1-1}_{L^{r_0 q_1}} \|u_n - u\|_{L^{r_0 q_1}} + \|u_n\|^{q_2-1}_{L^{r_0 q_2}} \|u_n - u\|_{L^{r_0 q_2}}) = o(1)$$

as $n \to +\infty$. Similarly,

$$\left| \int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u)(u_n - u) dx \right| = o(1)$$

as $n \to +\infty$. Then,

$$o(1) = (\mathcal{S}'(u_n) - \mathcal{S}'(u))[u_n - u] = \|u_n - u\|^2_{H^1(\mathbb{R}^N)} + o(1)$$

as $n \to +\infty$. So $u_n \to u$ in $H^1(\mathbb{R}^N)$. □

Lemma 3.5.5. *Let $E = H^1_r(\mathbb{R}^N)$. Then there exists $r_k > 0$ such that*

$$\inf_{u \in E^\perp_{k-1}, \|u\|_{H^1} = r_k} \mathcal{S}(u) \to +\infty \quad as \quad k \to +\infty.$$

Proof. Set $r_0 = \frac{2N}{N+\alpha}$. We previously proved that for any $u \in H^1(\mathbb{R}^N)$,

$$\mathcal{S}(u) \geq \frac{1}{2}\|u\|_{H^1}^2 - C(\|u\|_{L^{r_0 q_1}}^{2q_1} + \|u\|_{L^{r_0 q_2}}^{2q_2})$$

for some constant $C > 0$. Let define us, for any $k \in \mathbb{N}$,

$$\alpha_k := \sup_{u \in E_{k-1}^{\perp}, \|u\|_{H^1}=1} \|u\|_{L^{r_0 q_1}}, \quad \beta_k := \sup_{u \in E_{k-1}^{\perp}, \|u\|_{H^1}=1} \|u\|_{L^{r_0 q_2}}.$$

So, for $u \in E_{k-1}^{\perp}$, we have, by homogeneity of the norm, for any $k \in \mathbb{N}$,

$$\mathcal{S}(u) \geq \frac{1}{2}\|u\|_{H^1}^2 - C\alpha_k^{2q_1}\|u\|_{H^1}^{2q_1} - C\beta_k^{2q_2}\|u\|_{H^1}^{2q_2}.$$

Using the fact that $q_2 \geq q_1 > 1$, it holds

$$\|u\|_{H^1}^{2q_1} \leq \|u\|_{H^1}^2 + \|u\|_{H^1}^{2q_2},$$

so that

$$\mathcal{S}(u) \geq \|u\|_{H^1}^2(\frac{1}{2} - C\alpha_k^{2q_1}) - C\|u\|_{H^1}^{2q_2}(\alpha_k^{2q_1} + \beta_k^{2q_2}). \tag{3.5.1}$$

Now, by Lemma 3.5.6, $\alpha_k, \beta_k \to 0$ as $k \to +\infty$. So, for $k \gg 1$, relation (3.5.1) becomes

$$\mathcal{S}(u) \geq \frac{1}{4}\|u\|_{H^1}^2 - C\|u\|_{H^1}^{2q_2}(\alpha_k^{2q_1} + \beta_k^{2q_2}).$$

Choosing $r_k := \left(4q_2 C(\alpha_k^{2q_1} + \beta_k^{2q_2})\right)^{\frac{1}{2(1-q_2)}} \to +\infty$ as $k \to +\infty$, we obtain for $u \in E_{k-1}^{\perp}$

$$\mathcal{S}(u) \geq \frac{1}{4}\left(1 - \frac{1}{q_2}\right)r_k^2 \to +\infty \quad \text{as} \quad k \to +\infty.$$

□

Lemma 3.5.6. *Let $E = H_r^1(\mathbb{R}^N)$ with $2 < p < 2^*$ ($p = 2$ if $N = 2$). Then we have that*

$$d_k := \sup_{u \in E_{k-1}^{\perp}, \|u\|_{H^1}=1} \|u\|_{L^p} \to 0 \quad \text{as} \quad k \to +\infty.$$

Proof. Let $N \geq 3$. It is clear that $0 < d_{k+1} \leq d_k$ for every $k \in \mathbb{N}$, so that $d_k \to d \geq 0$ as $k \to +\infty$. By definition of sup, for any $k \in \mathbb{N}$, there exists $u_k \in E_{k-1}^{\perp}$ such that $\|u_k\|_{H^1} = 1$ and $\|u_k\|_{L^p} > \frac{d_k}{2}$. By definition of $E_{k-1}^{\perp}$ and weak-convergence, $u_k \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$. By Corollary A.0.4, $u_k \to 0$ in $L^p(\mathbb{R}^N)$ for those p. Thus we have proved that $d = 0$.
The case $N = 2$ is equivalent to the case $N \geq 3$, in view of Corollary A.0.4 on the plane. □

Proof. (of Theorem 3.5.1) Using Theorem 3.5.2 and (f_3) with $\{e_i\}_{n\in\mathbb{N}}$ orthonormal basis of $E = H_r^1(\mathbb{R}^N)$, $\phi = \mathcal{S}$ and choosing $R_k > r_k$ large enough characterized by Lemma 3.5.3, we deduce that $\mathcal{S}$ possesses a sequence of radial critical points $\{\pm u_k\}_k \subset H^1(\mathbb{R}^N)$ such that $c_k = \mathcal{S}(u_k) \to +\infty$, and so unbounded in $H^1(\mathbb{R}^N)$.

□

Case $N = 2$

We now want to extend Theorem 3.5.1 giving a new result on the plane. As in previous subsection, we seek infinitely many radial solutions to problem (*) assuming the nonlinearity has subcritical growth conditions in the same spirit of Section 3.2, but a little bit different. Consider $f \in C(\mathbb{R};\mathbb{R})$ satisfying

(f_1) $\forall \theta > 0, \exists C = C_\theta > 0$ such that $|f(s)| \le C_\theta \min\{1, |s|^q\} e^{\theta s^2} \quad \forall s \in \mathbb{R}$, for some $q > \frac{\alpha}{2}$.

(f_2) $\lim_{|s|\to+\infty} \frac{F(s)}{|s|} = +\infty$, where $F(s) = \int_0^t f(t)dt \quad \forall s \in \mathbb{R}$.

(f_3) f is odd.

Our main result is the following

Theorem 3.5.3. *Take $f \in C(\mathbb{R};\mathbb{R})$ and let $(f_1) - (f_3)$ hold on the plane. Then, problem (*) admits an unbounded sequence in $H^1(\mathbb{R}^2)$ of radial solutions $\{\pm u_k\}_{k\in\mathbb{N}}$ such that $\mathcal{S}(u_k) = \mathcal{S}(-u_k) \to +\infty$ as $k \to +\infty$.*

With the same notations above, thanks to Lemma 3.5.1, we use the fountain theorem to get infinitely many radial solution. So we need to verify the conditions of Theorem 3.5.2.

Lemma 3.5.7. *There exist $\rho, \xi > 0$ such that $\mathcal{S}_{|\partial B_\rho} \ge \xi$.*

Proof. Condition (f_1) implies that for any $\theta > 0$ and $p > 1$, there exists $C = C_{p,\theta} > 0$ such that

$$|F(s)| \le C_{p,\theta}(|s|^{\frac{2+\alpha}{2}} + |s|^p[e^{\theta s^2} - 1]) \quad \forall s \in \mathbb{R}.$$

Now, arguing as in (i) of Lemma 3.4.2, we get for any fixed $u \in H^1(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} (I_\alpha * F(u))F(u)dx \le C\|u\|^{2+\alpha}_{H^1(\mathbb{R}^2)} + C_1\|u\|^{2p}_{H^1(\mathbb{R}^2)}$$

and so

$$\mathcal{S}(u) \ge \frac{1}{2}\|u\|^2_{H^1(\mathbb{R}^2)} - C\|u\|^{2+\alpha}_{H^1(\mathbb{R}^2)} - C_1\|u\|^{2p}_{H^1(\mathbb{R}^2)}.$$

Since $\alpha > 0$ and $p > 1$, there exist constants $\xi, \rho > 0$ such that $\mathcal{S}_{|\partial B_\rho} \ge \xi$. □

Lemma 3.5.8. *$\mathcal{S}$ satisfies the (PS)-c_k for any $k \in \mathbb{N}$.*

Proof. Assume that $\{u_n\}_n \subset H^1_r(\mathbb{R}^2)$ satisfies $\mathcal{S}(u_n) \to c_k$ and $\mathcal{S}'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^2)$ as $n \to +\infty$. As in Lemma 3.5.4, we get that $\{u_n\}_n$ is bounded in $H^1(\mathbb{R}^2)$ and we may suppose $u_n \rightharpoonup u$ in $H^1_r(\mathbb{R}^2)$, $u_n \to u$ in $L^s(\mathbb{R}^2)$ for any $s > 2$ by Corollary A.0.4, and $u_n \to u$ a.e. in $\mathbb{R}^2$.

By Hardy-Littlewood-Sobolev inequality and inequality (3.1.3), we have

$$\left| \int_{\mathbb{R}^2} (I_\alpha * F(u_n))f(u_n)(u_n - u)dx \right| \le C\|F(u_n)\|_{L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)} \|f(u_n)(u_n - u)\|^2_{L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)}.$$

It is easy to check (as in Section 3.2) that

$$\|F(u_n)\|_{L^{\frac{4}{2+\alpha}}(\mathbb{R}^2)} \leq C', \quad \forall n \in \mathbb{N}.$$

Now, by (f_1), Holder's inequality and Proposition 3.2.1 we get

$$\int_{\mathbb{R}^2} |f(u_n)|^{\frac{4}{2+\alpha}} |u_n - u|^{\frac{4}{2+\alpha}} dx \leq C_\theta \int_{\mathbb{R}^2} \min\{1, |u_n|^{\frac{4q}{2+\alpha}}\} e^{\theta u_n^2} |u_n - u|^{\frac{4}{2+\alpha}} dx \leq$$

$$\leq C_\theta \left(\int_{\mathbb{R}^2} \min\{1, u_n^2\} e^{\frac{\theta(2+\alpha)}{2q} u_n^2} dx \right)^{\frac{2q}{2+\alpha}} \left(\int_{\mathbb{R}^2} |u_n - u|^{\frac{4}{2+\alpha-2q}} dx \right)^{\frac{2+\alpha-2q}{4}} \leq$$

$$\leq C'_\theta \left(\int_{\mathbb{R}^2} |u_n - u|^{\frac{4}{2+\alpha-2q}} dx \right)^{\frac{2+\alpha-2q}{4}} \to 0$$

as $n \to +\infty$, since $\frac{\alpha}{2} < q < 1 + \frac{\alpha}{2}$ and Corollary A.0.4 (it is not restrictive to assume $q < 1 + \frac{\alpha}{2}$). Arguing as in Lemma 3.5.2, we conclude the proof. □

Lemma 3.5.9. *Let $E = H^1_r(\mathbb{R}^2)$. Then there exists $r_k > 0$ such that*

$$\inf_{u \in E_{k-1}^\perp, \|u\|_{H^1} = r_k} \mathcal{S}(u) \to +\infty \quad \text{as} \quad k \to +\infty.$$

Proof. Condition (f_1) implies that for any $\theta > 0$ and $p > 1$, there exists $C = C_{p,\theta} > 0$ such that

$$|F(s)| \leq C_{p,\theta}(|s|^q + |s|^p(e^{\theta s^2} - 1)) \quad \forall s \in \mathbb{R},$$

for some $q > 1 + \frac{\alpha}{2}$. So for any $u \in H^1(\mathbb{R}^2)$,

$$\|F(u)\|^2_{L^{\frac{4}{2+\alpha}}} \leq C(\|u\|^{2q}_{L^{\frac{4q}{2+\alpha}}} + \|u\|^{2p}_{L^{\frac{8p}{2+\alpha}}}).$$

Then, by Hardy-Littlewood-Sobolev inequality and inequality (3.1.3), we have

$$\mathcal{S}(u) \geq \frac{1}{2}\|u\|^2_{H^1} - C\|u\|^{2q}_{L^{\frac{4q}{2+\alpha}}} - C\|u\|^{2p}_{L^{\frac{8p}{2+\alpha}}}.$$

Choosing $p > q > 1$, arguing as in Lemma 3.5.5, we conclude the proof. □

Proof. (of Theorem 3.5.3) We use Theorem 3.5.2 since Lemmas 3.5.6, 3.5.7, 3.5.9 and Lemma 3.5.3 (valid also for $N = 2$) hold. So we obtain a sequence of radial solutions $\{\pm u_k\}_k$ with their energy unbounded as in the proof of Theorem 3.5.1, and so unbounded in $H^1(\mathbb{R}^2)$. □

Appendix A

Technical results and useful tools

In this chapter we will prove some technical results used in the thesis.

Theorem A.0.1. *Let Ω be an open bounded set with Lipschitz boundary in $\mathbb{R}^N$ with $N \geq 3$. Let $g \in C(\mathbb{R})$ satisfying $g(0) = 0$ and*

$$\limsup_{s \to 0} \frac{|g(s)|}{|s|} < +\infty, \quad \limsup_{|s| \to +\infty} \frac{|g(s)|}{|s|^{2^*-1}} < +\infty. \tag{A.0.1}$$

Let denote $G(u) = \int_0^u g(s)ds$. Then, the functional $V(u) = \int_\Omega G(u(x))dx$ is well-defined and of class C^1 on $H^1(\Omega)$. Moreover:

$$V'(u)[v] = \int_\Omega g(u(x))vdx, \quad \forall u, v \in H^1(\Omega).$$

Proof. The fact that $G(u) \in L^1(\Omega)$ follows from (A.0.1) and Sobolev embedding theorem if $u \in H^1(\Omega)$. Now, it suffices to show:

(i) $\left|\frac{1}{t}\left(V(u+tv) - V(u) - t\int_\Omega g(u)vdx\right)\right| \to 0$ as $t \to 0$, $\forall u, v \in H^1(\Omega)$;

(ii) If $u_n \to u$ in $H^1(\Omega)$, then: $\sup_{\|v\|_{H^1(\Omega)} \leq 1} \left|\int_\Omega (g(u_n) - g(u))vdx\right| \to 0$ as $n \to +\infty$.

To prove the first statement, we have that

$$\left|\frac{1}{t}\left(V(u+tv) - V(u) - t\int_\Omega g(u)vdx\right)\right| \leq \int_\Omega \left|(G(u+tv) - G(u) - tg(u)v)\frac{1}{t}\right| dx.$$

Now, by the Mean Value Theorem:

$$\left|(G(u+tv) - G(u) - tg(u)v)\frac{1}{t}\right| \leq \left(\sup_{r\in[0,1]} |g(u+rv)| + |g(u)|\right)|v| \leq$$

$$\leq (C + C|u|^{2^*-1} + C|v|^{2^*-1})|v| =: h,$$

using that $|g(s)| \leq C + C|s|^{2^*-1}$ for $s \in \mathbb{R}$, for some constant $C > 0$. By Holder's inequality and Sobolev embedding theorem, one has $h \in L^1(\Omega)$. Next, by continuity of g, we have

$$(G(u+tv) - G(u) - tg(u)v)\frac{1}{t} \to 0 \quad \text{a.e.} \quad x \in \Omega, \quad \text{as} \quad t \to 0.$$

The conclusion now follows applying Lebesgue's dominated convergence theorem.

On the other hand, to prove the second statement, we also know that $u_n \to u$ in $L^{2^*}(\Omega)$ by the Sobolev embedding theorem. Hence, by a standard result in integration theory, there exists $0 \leq \bar{u} \in L^{2^*}(\Omega)$ such that (up to a subsequence)

$$|u|, |u_n| \leq \bar{u} \quad \text{a.e. in} \quad \Omega, \quad \forall n \in \mathbb{N}.$$

Therefore we have

$$|g(u_n) - g(u)|^{\frac{2N}{N+2}} \leq C(1 + \bar{u}^{2^*}).$$

Due to the boundedness of Ω, by continuity of g and dominated convergence theorem, we have $g(u_n) \to g(u)$ in $L^{\frac{2N}{N+2}}(\Omega)$. Finally, by Holder's inequality:

$$\sup_{\|v\|_{H^1(\Omega)}\leq 1} \left| \int_\Omega (g(u_n)-g(u))v dx \right| \leq \left(\int_\Omega |g(u_n)-g(u)|^{\frac{2N}{N+2}} dx \right)^{\frac{N+2}{2N}} \sup_{\|v\|_{H^1(\Omega)}\leq 1} \left(\int_\Omega |v|^{2^*} dx \right)^{\frac{1}{2^*}}.$$

□

Theorem A.0.2. *Let $N \geq 3$ and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying*

$$g(0) = 0, \quad \limsup_{s\to 0} \frac{|g(s)|}{|s|} < +\infty, \quad \limsup_{|s|\to+\infty} \frac{|g(s)|}{|s|^{2^*-1}} < +\infty. \tag{A.0.2}$$

Let denote $G(u) = \int_0^u g(s)ds$. Then, the functional $V(u) = \int_{\mathbb{R}^N} G(u(x))dx$ is well-defined and of class C^1 on $H^1(\mathbb{R}^N)$. Moreover:

$$V'(u)[v] = \int_{\mathbb{R}^N} g(u(x))v dx, \quad \forall u, v \in H^1(\mathbb{R}^N).$$

Proof. The fact that $G(u) \in L^1(\mathbb{R}^N)$ follows from (A.0.2) and Sobolev embedding theorem if $u \in H^1(\mathbb{R}^N)$. Now, we follow the proof of previous theorem. For any $u, v \in H^1(\mathbb{R}^N)$, one has

$$\left| \frac{1}{t} \left(V(u+tv) - V(u) - t \int_{\mathbb{R}^N} g(u)v dx \right) \right| \to 0 \quad \text{as} \quad t \to 0, \quad \forall u, v \in H^1(\mathbb{R}^N).$$

Indeed, using now the inequality

$$|g(s)| \leq C|s| + C|s|^{2^*-1} \quad \forall s \in \mathbb{R},$$

for some positive constant C, we have (with the same notation above)

$$h = (C|u| + C|v| + C|u|^{2^*} + C|v|^{2^*})|v|,$$

which is in $L^1(\mathbb{R}^N)$ due to Holder's inequality and Sobolev embedding theorem.

In order to prove the second statement (as above), by previous theorem, it suffices to show that $\forall\epsilon > 0$, there exists $R_0 > 0$ such that

$$\sup_{\|v\|_{H^1}\leq 1}\left|\int_{|x|\geq R_0}(g(u_n) - g(u))v dx\right| < \epsilon.$$

Now, we know that $u_n \to u$ in $L^{2^*}(\mathbb{R}^N)$ by Sobolev embedding theorem because $u_n \to u$ in $H^1(\mathbb{R}^N)$. Hence, as in the proof of Theorem A.0.1, there exists (up to a subsequence) $\bar{u} \in L^{2^*}(\mathbb{R}^N)$, $\tilde{u} \in L^2(\mathbb{R}^N)$ such that for all $n \in \mathbb{N}$:

$$|u|, |u_n| \leq \bar{u} \quad \text{a.e. in} \quad \mathbb{R}^N, \qquad |u|, |u_n| \leq \tilde{u} \quad \text{a.e. in} \quad \mathbb{R}^N.$$

Then, for any $R > 0$, using the inequality $\quad |g(s)| \leq C|s| + C|s|^{2^*-1}, \quad \forall s \in \mathbb{R}$, we have

$$\sup_{\|v\|_{H^1}\leq 1}\left|\int_{|x|\geq R}(g(u_n) - g(u))v dx\right| \leq C\|\tilde{u}\|_{L^2(\{|x|\geq R\})}\left(\sup_{\|v\|_{H^1}\leq 1}\|v\|_{L^2(\{|x|\geq R\})}\right) +$$

$$+ C\|\bar{u}\|_{L^{2^*}(\{|x|\geq R\})}\left(\sup\nolimits_{\|v\|_{H^1}\leq 1}\|v\|_{L^{2^*}(\{|x|\geq R\})}\right)$$

by Holder's inequality. Hence, again by Sobolev embedding theorem, one has

$$\sup_{\|v\|_{H^1}\leq 1}\left|\int_{|x|\geq R}(g(u_n) - g(u))v dx\right| \leq C\|\tilde{u}\|_{L^2(\{|x|\geq R\})} + C\|\bar{u}\|_{L^{2^*}(\{|x|\geq R\})}.$$

Since $\tilde{u} \in L^2(\mathbb{R}^N)$ and $\bar{u} \in L^{2^*}(\mathbb{R}^N)$, we derive the existence of $R_0 > 0$ such that

$$\sup_{\|v\|_{H^1}\leq 1}\left|\int_{|x|\geq R_0}(g(u_n) - g(u))v dx\right| < \epsilon.$$

Now, we will give some results about radial functions in Sobolev spaces: □

Lemma A.0.1. *(**Radial Lemma**) Let $N \geq 2$. If $u \in L^p(\mathbb{R}^N)$, with $1 \leq p < +\infty$, is a radial decreasing function (i.e. $|u(x)| \leq |u(y)|$ if $|x| \geq |y|$), then u is a.e. equal to a continuous function for $x \neq 0$ such that*

$$|u(x)| \leq |x|^{-\frac{N}{p}}\left(\frac{N}{|\mathbb{S}^{N-1}|}\right)^{\frac{1}{p}}\|u\|_{L^p(\mathbb{R}^N)}, \quad \forall x \neq 0.$$

Proof. For all $r > 0$, setting $r = |x|$ in polar coordinates, we have

$$\|u\|^p_{L^p(\mathbb{R}^N)} \geq |\mathbb{S}^{N-1}|\int_0^r [u(s)]^p s^{N-1} ds \geq |\mathbb{S}^{N-1}|[u(r)]^p \frac{r^N}{N}.$$

□

Theorem A.0.3. *(Strauss' Compactness Lemma) For $N \geq 1$, let $P, Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\frac{P(s)}{Q(s)} \to 0 \quad as \quad |s| \to +\infty. \tag{A.0.3}$$

Let $\{u_n\}$ be a sequence of measurable functions $: \mathbb{R}^N \to \mathbb{R}$ such that

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))| dx < +\infty \quad and \quad P(u_n(x)) \to v(x) \quad a.e.\ in \quad \mathbb{R}^N, \quad as \quad n \to +\infty.$$

Then, for any bounded Borel set B, one has

$$\int_B |P(u_n(x)) - v(x)| dx \to 0 \quad as \quad n \to +\infty.$$

If one further assumes that

$$\frac{P(s)}{Q(s)} \to 0 \quad as \quad s \to 0 \quad and \quad u_n(x) \to 0 \quad as \quad |x| \to +\infty \quad uniformly\ with\ respect\ to\ n, \tag{A.0.4}$$

then $P(u_n)$ converges to v in $L^1(\mathbb{R}^N)$ as $n \to +\infty$.

Proof. To prove the first part of the theorem, we need to show that $\{P(u_n)\}_n$ is uniformly integrable on B. In fact, uniform integrability on a bounded set B and convergence a.e. for $P(u_n)$ implies $L^1(B)$-convergence by Vitali's convergence theorem.
First of all, from condition (A.0.3) we have

$$|P(u_n(x))| \leq C + C|Q(u_n(x))| \quad \forall x \in \mathbb{R}^N,$$

for some constant $C > 0$, by continuity of P and Q. Thus $P(u_n)$ and v (by Fatou's lemma) are in $L^1(B)$ for all n, because

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))| dx < +\infty.$$

Applying again (A.0.3), one has

$$\int_{\{|P(u_n(x))| \geq K\} \cap B} |P(u_n(x))| dx \leq \epsilon(K) \int_B |Q(u_n(x))| dx \leq C\epsilon(K),$$

where $\epsilon(K) \to 0$ as $K \to +\infty$. This shows the uniform integrability on B.

Now, take $\epsilon > 0$; condition (A.0.4) implies that exists $R_0 > 0$ such that

$$|x| \geq R_0 \quad \Rightarrow \quad |P(u_n(x))| \leq \epsilon |Q(u_n(x))|, \quad \forall n.$$

Therefore, by Fatou's lemma, $v \in L^1(\mathbb{R}^N)$, and

$$\int_{\{|x| \geq R_0\}} |v(x)| dx \leq C\epsilon.$$

Now, from the first part of the theorem, there exists $n_0 \gg 1$ such that for any $n \geq n_0$:

$$\int_{\{|x|<R_0\}} |P(u_n(x)) - v(x)|dx \leq \epsilon.$$

To sum up, we have for $n \geq n_0$,

$$\int_{\mathbb{R}^N} |P(u_n(x)) - v(x)|dx \leq 2\epsilon C + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the proof is finished. □

Corollary A.0.4. *If $N \geq 3$, the embedding $H^1_r(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $2 < p < \frac{2N}{N-2}$. If $N = 2$, the embedding $H^1_r(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ is compact for $2 < p < \infty$.*

Proof. let us consider $N \geq 3$. The embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is continuous from Sobolev embedding theorem for those p. Now let $\{u_n\} \subset H^1_r(\mathbb{R}^N)$ be a sequence of radial functions such that $\|u_n\|_{H^1}$ is bounded. From Lemma A.0.1 we deduce that $|u_n(x)| \to 0$ as $|x| \to +\infty$ uniformly with respect to n. Hence, we can extract a subsequence (always denoted by u_n) which converges a.e. in $\mathbb{R}^N$ and weakly in $H^1(\mathbb{R}^N)$ to a radial function u. More precisely, by Rellich-Kondrachov's theorem we find a subsequence that converges a.e on B_k for each $k \geq 1$. Now, a standard diagonal argument implies the existence of a subsequence convergent a.e. in $\mathbb{R}^N$ to u.

Therefore, applying Theorem A.0.3 with the choice $P(s) = |s|^p$ and $Q(s) = s^2 + |s|^{2^*}$, we have that u_n converges strongly in $L^p(\mathbb{R}^N)$ for $2 < p < \frac{2N}{N-2}$.
Finally, when $N = 2$, it is possible to repeat the same arguments above using Theorem A.0.3 with $Q(s) = s^2 + e^{\alpha s^2} - 1$ and an appropriate $\alpha > 0$ given by classical Moser-Trudinger inequality. □

Lemma A.0.2. *For $N \geq 2$, every radial function $u \in H^1(\mathbb{R}^N)$ is a.e. equal to a continuous function for $x \neq 0$ such that*

$$|u(x)| \leq C_N |x|^{\frac{1-N}{2}} \|u\|_{H^1(\mathbb{R}^N)} \quad \forall |x| \geq \alpha_N,$$

where C_N and α_N are positive constant depending only on the dimension N.

Proof. As $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, it suffices to consider radial $u \in C_0^\infty(\mathbb{R}^N)$. Let $m = \frac{N-1}{2}$ and $u = u(r)$ with $r = |x| > 0$; by a simple calculation we have

$$(r^{2m}u^2)_r = 2(r^m u)_r r^m u \leq [(r^m u)_r]^2 + (r^m u)^2 = r^{N-1}(u_r^2 + u^2) + m(r^{N-2}u^2)_r -$$

$$- \tfrac{(N-1)(N-3)}{4} r^{N-3}u^2 \leq r^{N-1}(u_r^2 + u^2) + m(r^{N-2}u^2)_r.$$

Now, if $N \geq 3$, integrating over $(0, r)$ with r such that $B_r \supset \operatorname{supp}(u)$, we obtain

$$r^{N-1}u^2(r) \leq \int_0^r (u_\rho^2 + u^2)\rho^{N-1}d\rho + mr^{N-2}u^2(r).$$

Using polar coordinates, we have

$$\left(1-\frac{m}{r}\right)r^{N-1}u^2(r) \le C_N\|u\|^2_{H^1(\mathbb{R}^N)}.$$

Choosing $r > m$ fixed, we have concluded the proof.

In the case $N = 2$, the differential inequality becomes

$$-(ru^2)_r \le r(u_r^2+u^2) + \frac{1}{2}(u^2)_r + \frac{1}{4r}u^2.$$

Integrating over $(r, +\infty)$ with $r > 0$ such that the integrals below make sense, we have

$$\left(r+\frac{1}{2}\right)u^2(r) \le \int_r^\infty (u_\rho^2+u^2)\rho d\rho + \int_r^\infty \frac{u^2}{4\rho}d\rho \le \int_0^\infty (u_\rho^2+u^2)\rho d\rho + \int_0^\infty \frac{1}{4}u^2\rho d\rho,$$

if $r \ge 1$. Thus,

$$2\pi\left(r+\frac{1}{2}\right)u^2(r) \le \frac{5}{4}\|u\|^2_{H^1(\mathbb{R}^N)}.$$

The proof is completed as before. □

Lemma A.0.3. *For $N \ge 3$, every radial function u in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is a.e. equal to a continuous function for $x \ne 0$, such that*

$$|u(x)| \le C_N|x|^{\frac{2-N}{2}}\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \quad \forall x \ne 0,$$

where $C_N > 0$ only depends on N.

Proof. As above by density, it suffices to consider $u \in C_0^\infty(\mathbb{R}^N)$ radial. Now, setting $r = e^y$ for $y \in \mathbb{R}$, consider

$$v(y) = u(r)e^{\frac{1}{2}(N-2)y}.$$

Using polar coordinates, by a simple change of variables, one readily checks that

$$\|\nabla u\|^2_{L^2(\mathbb{R}^N)} = |\mathbb{S}^{N-1}|\left(\int_{-\infty}^{+\infty}(v'(y))^2dy + \int_{-\infty}^{+\infty}\frac{(N-2)^2}{4}v^2(y)dy\right),$$

using the fact that also $v \in C_0^\infty(\mathbb{R}^N)$. Now, for any $g \in H^1(\mathbb{R})$ one has

$$g^2(y) \le 2\|g\|_{L^2(\mathbb{R})}\|g'\|_{L^2(\mathbb{R})} \quad \forall y \in \mathbb{R}.$$

Indeed, since $g \in H^1(\mathbb{R})$, we have that g is absolute continuous on $\mathbb{R}$ and vanishes to 0 at infinity. So, by fundamental theorem of calculus and Holder's inequality:

$$g^2(y) = \left|\int_{-\infty}^y \frac{d}{dt}(g^2(t))dt\right| = \left|\int_{-\infty}^y 2g(t)g'(t)dt\right| \le 2\|g\|_{L^2(\mathbb{R})}\|g'\|_{L^2(\mathbb{R})}.$$

Then, in our case we obtain

$$\left|u(r)r^{\frac{N-2}{2}}\right| \le C_N\|\nabla u\|_{L^2(\mathbb{R}^N)}$$

for some positive constant C_N. Finally, by definition of $\mathcal{D}^{1,2}$-norm, we have the desired inequality. □

Some results about Schwarz symmetrization: We recall here, without proofs (see [18]), the basic properties of Schwarz symmetrization.
Let $f \in L^1(\mathbb{R}^N)$; then f^*, the Schwarz symmetrized function of f, is a radial, decreasing in $|x| = r$, measurable function such that for any $\alpha > 0$,

$$|\{f^* \geq \alpha\}| = |\{|f| \geq \alpha\}|.$$

Furthermore, one easily finds that

$$\int_{\mathbb{R}^N} F(f)dx = \int_{\mathbb{R}^N} F(f^*)dx$$

for every continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F(f)$ is integrable.
An important property of Schwarz symmetrization is the following:

Theorem A.0.5. *(**Riesz's inequality**) For $N \geq 1$, let f, g be in $L^2(\mathbb{R}^N)$; then*

$$\int_{\mathbb{R}^N} f(x)g(x)dx = \int_{\mathbb{R}^N} f^*(x)g^*(x)dx.$$

A fundamental fact about Schwarz symmetrization is the following result:

Theorem A.0.6. *(**Pólya-Szegö inequality**) Let u be in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ if $N \geq 3$ (respectively, in $H^1(\mathbb{R}^N)$ for any $N \geq 1$). Then u^* belongs to $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (respectively, to $H^1(\mathbb{R}^N)$ for any N), and we have*

$$\int_{\mathbb{R}^N} |\nabla u^*(x)|^2 dx \leq \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx.$$

Now, we will give the proof of Theorem 2.3.1.

Theorem A.0.7. *For any $k \geq 1$, there exists a constant $R = R(k) > 1$ and an odd continuous mapping $\tau : \pi_{k-1} \to H_0^1(B_R)$ (recalling that $\pi_{k-1} = \{l = (l_1, ..., l_k) \in \mathbb{R}^k : \sum_{i=1}^k |l_i| = 1\}$) such that $\tau(l)$ is a radial function for all $l \in \pi_{k-1}$ and*

$$0 \notin \tau(\pi_{k-1}), \tag{A.0.5}$$

$$\exists \rho, C > 0 \quad \text{depending on } k \text{ such that} \quad \rho \leq \|\nabla u\|^2_{L^2(B_R)} \leq C \quad \forall u \in \tau(\pi_{k-1}), \tag{A.0.6}$$

$$\int_{B_R} G(u)dx \geq 1 \quad \forall u \in \tau(\pi_{k-1}). \tag{A.0.7}$$

Proof. The proof is divided into three steps:

(i) Choice of $R = R(k) > 0$;

(ii) Construction of τ;

(iii) Properties of τ.

Step 1. In the following, we consider $u = u(r)$ for $r \in [0, R]$, $r = |x|$. Recall that there exist $\xi > 0$, given by hypothesis (1.1.3), such that $G(\xi) > 0$. We now define a subset in $H^1(B_R)$ which will be useful later.
For $k \geq 1$ and $R > 1$, we say that $u \in N_k(R)$ if $u \in H_0^1(B_R)$ is radial, continuous and satisfies the following three properties:

$$-\xi \leq u \leq \xi \quad \text{on} \quad [0, R]. \tag{A.0.8}$$

$$u = \pm\xi \quad \text{on} \quad [0, R] \quad \text{except in at most} \quad k \quad \text{subintervals} \quad J_1, ..., J_p \quad (p \leq k) \quad \text{of} \quad [0, R], \\ \text{each of which having length one, such that } u(R) = 0. \tag{A.0.9}$$

$$\text{In each of the intervals} \quad J_j, 1 \leq j \leq p, \quad u \quad \text{is affine with} \quad |u'(r)| = 2\xi. \tag{A.0.9}$$

The choice of $R = R(k) > 1$ is determined by the following lemma.

Lemma A.0.4. *For all $k \geq 1$, there exists $R = R(k) > 1$ such that $V(u) \geq 1$ for all $u \in N_k(R)$.*

Remark A.0.5. *In the preceding statement, and henceforth, we identify $T(u)$ and $V(u)$ with $T(\tilde{u})$ and $V(\tilde{u})$, respectively (as in Theorem 2.3.1); that is, for $u \in H_0^1(B_R)$ we set*

$$T(u) = \int_{B_R} |\nabla u|^2 dx, \quad V(u) = \int_{B_R} G(u) dx.$$

Proof. (*of Lemma A.0.4*) Take $R \geq k + 1$; it is easily seen, using (A.0.8) and (A.0.9), that for $u \in N_k(R)$ one has

$$V(u) \geq G(\xi)|B_{R-k}| - Ck|B_R - B_{R-1}|,$$

where $0 < C = \max_{|z| \leq \xi} |G(z)|$. Since $|B_R - B_{R-1}| \leq C' R^{N-1}$, we deduce

$$V(u) \geq C_k R^N - C'_k R^{N-1}, \quad u \in N_k(R), \quad R \geq k + 1.$$

Therefore, there exist an $R = R(k) > 1$ verifying Lemma A.0.4. □

Step 2. Let $k \geq 1$ and fix $R = R(k) > 1$ as in previous lemma. Now, we are going to construct an odd continuous mapping $\tau : \pi_{k-1} \to N_k(r) \subset H_0^1(B_R)$.
Let $l = (l_1, ..., l_k) \in \pi_{k-1}$; we set

$$\alpha_i = R \sum_{j=1}^{i} |l_j|, \quad \text{or equivalently,} \quad R|l_i| = \alpha_i - \alpha_{i-1}, \quad i \leq p.$$

For $l_i \neq 0$ and $r \in (\alpha_{i-1}, \alpha_i)$, we let $\epsilon_l(r) = \text{sign}(l_i)$. Joining together all adjacent intervals like $(\alpha_{i-1}, \alpha_i), (\alpha_i, \alpha_{i+1}), ...$ on which $\epsilon_l(r)$ has the same sign, we obtain a new subdivision of $[0, R]$ based on endpoints

$$0 = a_0 < a_1 < ... < a_p = R, \quad \text{with} \quad 1 \leq p \leq k,$$

which is coarser than the subdivision $\{\alpha_i\}_i$.

In this way, $\epsilon_l(r)$ has a given sign for all $r \in (a_{i-1}, a_i)$. We denote this sign by ϵ_i, thus $\epsilon_l = \epsilon_l(r)$ for $r \in (\alpha_{i-1}, \alpha_i)$. In particular, $\epsilon_i = \pm 1$ and ϵ_i is alternating, that is $\epsilon_{i+1} = -\epsilon_i$. Let us observe at this point that the subdivision (a_i) is unambiguously determined from $l \in \pi_{k-1}$.

Let $I_i = (a_{i-1}, a_i), \quad 1 \leq i \leq p$. We first define $u = \tau(l)$ on I_p, then on I_{p-1}, and so on, inductively for all I_i's.

(i) In I_p. Set $u(r) = 2\epsilon_p \xi (R - r)$, for $\max\{a_{p-1}, R - \frac{1}{2}\} \leq r \leq R$. There are now two cases:

If $a_{p-1} \geq R - \frac{1}{2}$, then u is defined in all of I_p.

If $a_{p-1} \leq R - \frac{1}{2}$, then we set $u(r) = \epsilon_p \xi$ for $a_{p-1} \leq r \leq R - \frac{1}{2}$.

(ii) In I_{p-1}. Set

$$u(r) = u(a_{p-1}) + 2\epsilon_{p-1}\xi(a_{p-1} - r),$$

for all r such that $a_{p-2} \leq r \leq a_{p-1}$ and $|u(r)| < \xi$; that is for $a_{p-1} \geq r \geq \max\{a_{p-2}, r_{p-1}\}$, where

$$r_{p-1} = a_{p-1} - \frac{\xi - \epsilon_{p-1} u(a_{p-1})}{2\xi}.$$

Again there are two cases:

If $r_{p-1} \leq a_{p-2}$, then u is defined on all of I_{p-1}.

If $r_{p-1} > a_{p-2}$, then we set $u(r) = \epsilon_{p-1}\xi$ for $a_{p-2} \leq r \leq r_{p-1} \leq a_{p-1}$.

(iii) Define u by induction on I_j, $1 \leq j \leq p-1$. Set

$$u(r) = u(a_j) + \epsilon_j 2\xi(a_j - r)$$

for all $r \in [a_{j-1}, a_j]$ for which $|u(r)| < \xi$; that is, for $r \geq \max(a_{j-1}, r_j)$, where

$$r_j = a_j - \frac{\xi - \epsilon_j u(a_j)}{2\xi}.$$

If $r_j \leq a_{j-1}$, u is defined on all of I_j. Otherwise, $a_{j-1} < r_j < a_j$ and we let $u(r) = \epsilon_j \xi$ for r in $a_{j-1} \leq r \leq r_j$.

This construction defines unambiguously, for all $l \in \pi_{k-1}$, a function u; we denote as $u = \tau(l)$. It is immediately checked that $\tau(l) \in N_k(R)$ for all $l \in \tau_{k-1}$. Thus we have constructed a map $\tau : \pi_{k-1} \to N_k(R) \subset H_0^1(B_R)$.

Step 3. We now show that τ has the properties stated in the theorem. First, one readily checks that $\tau : \pi_{k-1} \to H_0^1(B_R)$ is a continuous mapping. Furthermore τ is odd. Indeed, it is easily seen that $\tau(-l) = -\tau(l)$ at each step in the preceding construction. The subdivisions (a_i) associated with l and $-l$ are the same, while the ϵ_i's are of opposite signs for l and $-l$. By construction, $\tau(l)$ is a radial function. Lastly, as (A.0.5) and (A.0.6) are consequences of the definition and continuity of τ, we get(A.0.7).
Since $\tau(l) \in N_k(R)$, we have, by the choice of R in Lemma A.0.4,

$$V(u) \geq 1 \quad \text{for} \quad u \in \tau(\pi_{k-1}).$$

□

Proposition A.0.6. ***(Hardy-Littlewood-Sobolev inequality)*** *Take $N \geq 2$, $f \in L^p(\mathbb{R}^N)$, $g \in L^t(\mathbb{R}^N)$ with $p > 1$, $t < \infty$ such that $\frac{1}{p} + \frac{1}{t} + \frac{\lambda}{N} = 2$ and $\lambda \in (0, N)$. Then, there exists $C > 0$ such that*

$$\int\int_{\mathbb{R}^N \times \mathbb{R}^N} f(x)|x-y|^{-\lambda} g(y)dxdy \leq C_{p,\lambda,N} \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^t(\mathbb{R}^N)}.$$

In particular, if $p = t = \frac{2N}{N+\alpha}$, the best possible constant is given by

$$C_\alpha := \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \left[\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right]^{-\frac{\alpha}{N}}.$$

Proof. The proof of the inequality can be seen on [12, theorem 4.3]. □

Lemma A.0.7. *Let $N \geq 2$, $\alpha \in (0, 2)$, $\theta \in (0, 2)$ and H, K defined as in Proposition 3.1.5. If $\frac{\alpha}{N} < \theta < 2 - \frac{\alpha}{N}$, then, for every $\epsilon > 0$, there exists $C_{\epsilon,\theta} \in \mathbb{R}$ such that for any fixed $u \in H^1(\mathbb{R}^N)$ which solves (3.1.6),*

$$\int_{\mathbb{R}^N} (I_\alpha * (H|u|^\theta))K|u|^{2-\theta} dx \leq \epsilon^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx + C_{\epsilon,\theta} \int_{\mathbb{R}^N} u^2 dx.$$

In order to prove the lemma, we will use several times the following inequality.

Lemma A.0.8. *Let $N \geq 2$, $q, r, s, t \in [1, +\infty)$ and $\lambda \in [0, 2]$ such that*

$$1 + \frac{\alpha}{N} - \frac{1}{s} - \frac{1}{t} = \frac{\lambda}{q} + \frac{2-\lambda}{r}.$$

If $\theta \in (0, 2)$ satisfies

$$\min\{q, r\}\left(\frac{\alpha}{N} - \frac{1}{s}\right) < \theta < \max\{q, r\}\left(1 - \frac{1}{s}\right),$$

$$\min\{q, r\}\left(\frac{\alpha}{N} - \frac{1}{t}\right) < 2 - \theta < \max\{q, r\}\left(1 - \frac{1}{t}\right),$$

then for every $H \in L^s(\mathbb{R}^N)$, $K \in L^t(\mathbb{R}^N)$ and $u \in L^q(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * (H|u|^\theta))K|u|^{2-\theta} dx \leq C\|u\|_{L^s(\mathbb{R}^N)} \|u\|_{L^t(\mathbb{R}^N)} \|u\|^\lambda_{L^q(\mathbb{R}^N)} \|u\|^{2-\lambda}_{L^r(\mathbb{R}^N)}.$$

Proof. First observe that if $\tilde{s} > 1$, $\tilde{t} > 1$ satisfy $\frac{1}{\tilde{t}} + \frac{1}{\tilde{s}} = 1 + \frac{\alpha}{N}$, the Hardy-Littlewood-Sobolev inequality implies

$$\int_{\mathbb{R}^N} (I_\alpha * (H|u|^\theta))K|u|^{2-\theta} dx \leq C\left(\int_{\mathbb{R}^N} (|H||u|^\theta)^{\tilde{s}} dx\right)^{\frac{1}{\tilde{s}}} \left(\int_{\mathbb{R}^N} (|K||u|^{2-\theta})^{\tilde{t}} dx\right)^{\frac{1}{\tilde{t}}}.$$

Let $\mu \in \mathbb{R}$. Note that if

$$0 \le \mu \le \theta \quad \text{and} \quad \frac{1}{\tilde{s}} := \frac{\mu}{q} + \frac{\theta - \mu}{r} + \frac{1}{s} < 1, \tag{A.0.10}$$

then by Holder's inequality

$$\left(\int_{\mathbb{R}^N} (|H||u|^{2-\theta})^{\tilde{s}} dx \right)^{\frac{1}{\tilde{s}}} \le \|H\|_{L^s(\mathbb{R}^N)} \|u\|^{\mu}_{L^q(\mathbb{R}^N)} \|u\|^{\theta-\mu}_{L^r(\mathbb{R}^N)}.$$

Similarly, if

$$\lambda - (2-\theta) \le \mu \le \lambda \quad \text{and} \quad \frac{1}{\tilde{t}} := \frac{\lambda - \mu}{q} + \frac{(2-\theta) - (\lambda - \mu)}{r} + \frac{1}{t} < 1, \tag{A.0.11}$$

then

$$\left(\int_{\mathbb{R}^N} (|K||u|^{2-\theta})^{\tilde{t}} dx \right)^{\frac{1}{\tilde{t}}} \le \|K\|_{L^t(\mathbb{R}^N)} \|u\|^{\lambda-\mu}_{L^q(\mathbb{R}^N)} \|u\|^{2-\theta-(\lambda-\mu)}_{L^r(\mathbb{R}^N)}.$$

It can be checked that (A.0.10) and (A.0.11) may be satisfied for some $\mu \in \mathbb{R}$ if and only if the assumptions of the lemma hold. In particular, $\frac{1}{\tilde{t}} + \frac{1}{\tilde{s}} = \frac{1}{s} + \frac{1}{t} = \frac{\lambda}{q} + \frac{2-\lambda}{r} = 1 + \frac{\alpha}{N}$, so that we can conclude. $\square$

Proof. (of Lemma A.0.7) Fix $u \in H^1(\mathbb{R}^N)$. Let $R > 0$ and $\phi_R \in C_0^\infty(\mathbb{R})$ be such that $0 \le \phi_R \le 1$, $\phi_R(s) = 1$ for $|s| \le R$ and $\phi_R(s) = 0$ for $|s| \ge 2R$. Set

$$H^*(u) := \phi_R(u)H(u), \quad H_*(u) := H(u) - H^*(u)$$

and the same thing for $K := K^* + K_*$. By growth conditions on F, we get that $H^*, K^* \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$ and $H_*, K_* \in L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)$. Applying previous lemma with $q = r = \frac{2N}{N-2}$, $s = t = \frac{2N}{\alpha+2}$ and $\lambda = 0$, we have since $|\theta - 1| < \frac{N-\alpha}{N-2}$,

$$\int_{\mathbb{R}^N} (I_\alpha * (H_*|u|^\theta))(K_*|u|^{2-\theta}) dx \le C\|H_*\|_{L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)} \|K_*\|_{L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)} \|u\|^2_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}.$$

Taking now $s = t = \frac{2N}{\alpha}$ and $q = r = \lambda = 2$, we have since $|\theta - 1| < \frac{N-\alpha}{N}$,

$$\int_{\mathbb{R}^N} (I_\alpha * (H^*|u|^\theta))(K^*|u|^{2-\theta}) dx \le C\|H^*\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)} \|K^*\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)} \|u\|^2_{L^2(\mathbb{R}^N)}.$$

Similarly, with $s = \frac{2N}{\alpha+2}$, $t = \frac{2N}{\alpha}$, $q = 2$, $r = \frac{2N}{N-2}$ and $\lambda = 1$,

$$\int_{\mathbb{R}^N} (I_\alpha * (H_*|u|^\theta))(K^*|u|^{2-\theta}) dx \le C\|H_*\|_{L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)} \|K^*\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)} \|u\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \|u\|_{L^2(\mathbb{R}^N)}$$

and with $s = \frac{2N}{\alpha}$, $t = \frac{2N}{\alpha+2}$, $q = 2$, $r = \frac{2N}{N-2}$ and $\lambda = 1$,

$$\int_{\mathbb{R}^N} (I_\alpha * (H^*|u|^\theta))(K_*|u|^{2-\theta}) dx \le C\|H^*\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)} \|K_*\|_{L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)} \|u\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \|u\|_{L^2(\mathbb{R}^N)}.$$

By Sobolev inequality, we have thus proved that for every $u \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * (H^*|u|^\theta))(K_*|u|^{2-\theta})dx \leq$$

$$\leq C\Big(\|H_*\|_{L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)} \|K_*\|_{L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \|H^*\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)} \|K^*\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)} \int_{\mathbb{R}^N} u^2 dx \Big).$$

The conclusion follows by choosing $R = R(\epsilon) > 0$ sufficiently small such that

$$C\|H_*\|_{L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)} \|K_*\|_{L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)} \leq \epsilon^2.$$

□

Proposition A.0.9. *Take $N \geq 2$, $p \in [1, \frac{N}{\alpha})$, $q \in (\frac{N}{\alpha}, +\infty)$ and $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. Then, there exists $C > 0$ such that*

$$\|I_\alpha * f\|_{L^\infty(\mathbb{R}^N)} \leq C(\|f\|_{L^p(\mathbb{R}^N)} + \|f\|_{L^q(\mathbb{R}^N)}).$$

Proof. By choosing p, q in that range we have $(N-\alpha)\frac{q}{q-1} < N < (N-\alpha)\frac{p}{p-1}$; therefore, after splitting the integral, Holder's inequality and a change of variables, we get, for every $x \in \mathbb{R}^N$

$$|I_\alpha * f(x)| \leq C \int_{\mathbb{R}^N} \frac{|f(x-y)|}{|y|^{N-\alpha}} dy \leq C\Big(\int_{B_1} \frac{dy}{|y|^{\frac{(N-\alpha)q}{q-1}}} \Big)^{1-\frac{1}{q}} \|f\|_{L^q(B_1(x))} +$$

$$+ C\Big(\int_{\mathbb{R}^N \setminus B_1} \frac{dy}{|y|^{\frac{(N-\alpha)p}{p-1}}} \Big)^{1-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^N \setminus B_1(x))} \leq C(\|f\|_{L^p(\mathbb{R}^N)} + \|f\|_{L^q(\mathbb{R}^N)}).$$

□

Finally, we give the proof of Lemma 3.3.9.

Proof. (of Lemma 3.3.9) Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function with support B_2 such that $\phi \equiv 1$ on B_1 and $0 \leq \phi \leq 1$ on B_2. Given $\epsilon > 0$, we set $\psi_\epsilon(x) := \phi(x)U_\epsilon(x)$, where

$$U_\epsilon(x) = \frac{(N(N-2)\epsilon^2)^{\frac{N-2}{4}}}{(\epsilon^2 + |x|^2)^{\frac{N-2}{2}}}.$$

By [42, lemma 1.46], we have the following estimates:

$$\int_{\mathbb{R}^N} |\nabla \psi_\epsilon|^2 dx = S^{\frac{N}{2}} + \begin{cases} O(\epsilon^{N-2}) & \text{if} \quad N \geq 4 \\ K_1\epsilon + O(\epsilon^3) & \text{if} \quad N = 3, \end{cases}$$

$$\int_{\mathbb{R}^N} |\psi_\epsilon|^{\frac{2N}{N-2}} dx = S^{\frac{N}{2}} + O(\epsilon^N) \quad \text{if} \quad N \geq 3,$$

$$\int_{\mathbb{R}^N} \psi_\epsilon^2 dx = \begin{cases} K_2\epsilon^2 + O(\epsilon^{N-2}) & \text{if} \quad N \geq 5 \\ K_2\epsilon^2|\log\epsilon| + O(\epsilon^2) & \text{if} \quad N = 4 \\ K_2\epsilon + O(\epsilon^2) & \text{if} \quad N = 3, \end{cases}$$

where $K_1, K_2 > 0$ and S being the Sobolev constant. Then we get

$$\int_{\mathbb{R}^N} (|\nabla\psi_\epsilon|^2 + \psi_\epsilon^2) dx = S^{\frac{N}{2}} + \begin{cases} K_2\epsilon^2 + O(\epsilon^{N-2}) & \text{if} \quad N \geq 5 \\ K_2\epsilon^2|\log\epsilon| + O(\epsilon^2) & \text{if} \quad N = 4 \\ (K_1 + K_2)\epsilon + O(\epsilon^2) & \text{if} \quad N = 3. \end{cases} \tag{A.0.12}$$

By direct computation, we get

$$\left(\int_{\mathbb{R}^N} |\psi_\epsilon|^{\frac{2Nq}{N+\alpha}} dx\right)^{\frac{N+\alpha}{N}} = K_3\epsilon^{N+\alpha-(N-2)q} + o(\epsilon^{N+\alpha-(N-2)q}),$$

and then by the Hardy-Littlewood-Sobolev inequality,

$$\int_{\mathbb{R}^N} (I_\alpha * |\psi_\epsilon|^{\frac{N+\alpha}{N-2}})|\psi_\epsilon|^q dx \leq C_\alpha \left(\int_{\mathbb{R}^N} |\psi_\epsilon|^{\frac{2N}{N-2}} dx\right)^{\frac{N+\alpha}{2N}} \left(\int_{\mathbb{R}^N} |\psi_\epsilon|^{\frac{2Nq}{N+\alpha}} dx\right)^{\frac{N+\alpha}{2N}} \leq$$

$$\leq K_4\epsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\epsilon^{\frac{N+\alpha-(N-2)q}{2}}), \tag{A.0.13}$$

where $K_3, K_4 > 0$ and C_α defined in Proposition A.0.6. Moreover, similarly as in [40,41], by direct computation, for some $K_5 > 0$,

$$\int_{\mathbb{R}^N} (I_\alpha * |\psi_\epsilon|^{\frac{N+\alpha}{N-2}})|\psi_\epsilon|^{\frac{N+\alpha}{N-2}} dx \geq (A_\alpha C_\alpha)^{\frac{N}{2}} S_\alpha^{\frac{N+\alpha}{2}} - K_5\epsilon^{\frac{N+\alpha}{2}} + o(\epsilon^{\frac{N+\alpha}{2}}), \tag{A.0.14}$$

where $A_\alpha := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha}$ and S_α is defined in Section 3.3. We also have

$$\int_{\mathbb{R}^N} (I_\alpha * |\psi_\epsilon|^{\frac{N+\alpha}{N-2}})|\psi_\epsilon|^q dx \geq A_\alpha \Bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{U_\epsilon^{\frac{N+\alpha}{N-2}}(x) U_\epsilon^q(y)}{|x-y|^{N-\alpha}} dxdy -$$

$$- \int_{\mathbb{R}^N\setminus B_1}\int_{B_1} \frac{U_\epsilon^{\frac{N+\alpha}{N-2}}(x) U_\epsilon^q(y)}{|x-y|^{N-\alpha}} dxdy - \int_{B_1}\int_{\mathbb{R}^N\setminus B_1} \frac{U_\epsilon^{\frac{N+\alpha}{N-2}}(x) U_\epsilon^q(y)}{|x-y|^{N-\alpha}} dxdy$$

$$- \int_{\mathbb{R}^N\setminus B_1}\int_{\mathbb{R}^N\setminus B_1} \frac{U_\epsilon^{\frac{N+\alpha}{N-2}}(x) U_\epsilon^q(y)}{|x-y|^{N-\alpha}} dxdy\Bigg),$$

where for some $\tilde{K}_i > 0$, $i = 1, 2, 3, 4$,

$$\begin{cases} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{U_\epsilon^{\frac{N+\alpha}{N-2}}(x) U_\epsilon^q(y)}{|x-y|^{N-\alpha}} dxdy = \tilde{K_1} \epsilon^{\frac{N+\alpha-(N-2)q}{2}}, \\ \int_{\mathbb{R}^N \setminus B_1} \int_{B_1} \frac{U_\epsilon^{\frac{N+\alpha}{N-2}}(x) U_\epsilon^q(y)}{|x-y|^{N-\alpha}} dxdy \le \tilde{K_2} \epsilon^{N+\alpha-\frac{N-2}{2}q} + o(\epsilon^{N+\alpha-\frac{N-2}{2}q}), \\ \int_{B_1} \int_{\mathbb{R}^N \setminus B_1} \frac{U_\epsilon^{\frac{N+\alpha}{N-2}}(x) U_\epsilon^q(y)}{|x-y|^{N-\alpha}} dxdy \le \tilde{K_3} \epsilon^{\frac{N-2}{2}q} + o(\epsilon^{\frac{N-2}{2}q}), \\ \int_{\mathbb{R}^N \setminus B_1} \int_{\mathbb{R}^N \setminus B_1} \frac{U_\epsilon^{\frac{N+\alpha}{N-2}}(x) U_\epsilon^q(y)}{|x-y|^{N-\alpha}} dxdy \le \tilde{K_4} \epsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\epsilon^{\frac{N+\alpha-(N-2)q}{2}}). \end{cases}$$

Thus, for some $K_6 > 0$, we have

$$\int_{\mathbb{R}^N} (I_\alpha * |\psi_\epsilon|^{\frac{N+\alpha}{N-2}}) |\psi_\epsilon|^q dx \ge K_6 \epsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\epsilon^{\frac{N+\alpha-(N-2)q}{2}}). \tag{A.0.15}$$

Here we used the fact that $q > \frac{N+\alpha}{2(N-2)}$. Then, for any $t > 0$,

$$\mathcal{S}_\lambda(t\psi_\epsilon) \le \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla \psi_\epsilon|^2 + \psi_\epsilon^2) dx - \frac{\mu\lambda}{q} \frac{N-2}{N+\alpha} t^{q+\frac{N+\alpha}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * |\psi_\epsilon|^{\frac{N+\alpha}{N-2}}) |\psi_\epsilon|^q dx -$$

$$- \frac{t^{\frac{2(N+\alpha)}{N-2}}}{2} \left(\frac{N-2}{N+\alpha} \right)^2 \lambda \int_{\mathbb{R}^N} (I_\alpha * \psi_\epsilon^{\frac{N+\alpha}{N-2}}) \psi_\epsilon^{\frac{N+\alpha}{N-2}} dx =: g_\epsilon(t).$$

One has $g_\epsilon(t) \to -\infty$ as $t \to +\infty$ and $g_\epsilon(t) > 0$ for $t > 0$ small. By a simple calculation, g_ϵ has a unique critical point $t_\epsilon \in (0, +\infty)$, which is its maximum point. From $g'_\epsilon(t_\epsilon) = 0$,

$$t_\epsilon \int_{\mathbb{R}^N} (|\nabla \psi_\epsilon|^2 + \psi_\epsilon^2) dx - \left(q + \frac{N+\alpha}{N-2} \right) \frac{\mu\lambda}{q} \frac{N-2}{N+\alpha} t_\epsilon^{q+\frac{N+\alpha}{N-2}-1} \int_{\mathbb{R}^N}) I_\alpha * \psi_\epsilon^{\frac{N+\alpha}{N-2}}) \psi_\epsilon^q dx =$$

$$= t_\epsilon^{\frac{2(N+\alpha)}{N-2}-1} \frac{N-2}{N+\alpha} \lambda \int_{\mathbb{R}^N} (I_\alpha * \psi_\epsilon^{\frac{N+\alpha}{N-2}}) \psi_\epsilon^{\frac{N+\alpha}{N-2}} dx. \tag{A.0.16}$$

Claim. There exist $t_0, t_1 > 0$ (both independent of ϵ) such that $t_\epsilon \in [t_0, t_1]$ for $\epsilon > 0$ small.

Consider first the case $t_\epsilon \to 0$ as $\epsilon \to 0^+$. Then by (A.0.12)-(A.0.14), there exist $c_1, c_2 > 0$ (independent of ϵ) such that for ϵ small,

$$c_1 t_\epsilon \le c_2 \epsilon^{\frac{N+\alpha-(N-2)q}{2}} t_\epsilon^{q+\frac{N+\alpha}{N-2}-1} + t_\epsilon^{q+\frac{N+\alpha}{N-2}-1} \le 2 t_\epsilon^{q+\frac{N+\alpha}{N-2}-1},$$

where we used the fact that $q < \frac{N+\alpha}{N-2}$: hence a contradiction and $t_\epsilon \ge t_0$. By (A.0.16), one has

$$\int_{\mathbb{R}^N} (|\nabla \psi_\epsilon|^2 + \psi_\epsilon^2) dx \ge t_\epsilon^{\frac{2(N+\alpha)}{N-2}-2} \frac{N-2}{N+\alpha} \lambda \int_{\mathbb{R}^N} (I_\alpha * \psi_\epsilon^{\frac{N+\alpha}{N-2}}) \psi_\epsilon^{\frac{N+\alpha}{N-2}} dx,$$

which implies, combining (A.0.12) and (A.0.14), that $t_\epsilon \le t_1$ for some $t_1 > 0$ and ϵ small.

By the claim just proved and (A.0.15), we have for some $K_7 > 0$,

$$\frac{\mu\lambda}{q}\frac{N-2}{N+\alpha}t_\epsilon^{q+\frac{N+\alpha}{N-2}}\int_{\mathbb{R}^N}(I_\alpha * \psi_\epsilon^{\frac{N+\alpha}{N-2}})\psi_\epsilon^q dx \geq K_7\epsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\epsilon^{\frac{N+\alpha-(N-2)q}{2}}),$$

and hence

$$\max_{t\geq 0}\mathcal{S}_\lambda(t\psi_\epsilon) = g_\epsilon(t_\epsilon) \leq \frac{t_\epsilon^2}{2}\int_{\mathbb{R}^N}(|\nabla\psi_\epsilon|^2+\psi_\epsilon^2)dx - K_7\epsilon^{\frac{N+\alpha-(N-2)q}{2}} -$$

$$-t_\epsilon^{\frac{2(N+\alpha)}{N-2}}\left(\frac{N-2}{N+\alpha}\right)^2\lambda\int_{\mathbb{R}^N}(I_\alpha * \psi_\epsilon^{\frac{N+\alpha}{N-2}})\psi_\epsilon^{\frac{N+\alpha}{N-2}}dx + o(\epsilon^{\frac{N+\alpha-(N-2)q}{2}}) \leq$$

$$\leq \max_{t\geq 0}\left[\frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla\psi_\epsilon|^2+\psi_\epsilon^2)dx - \frac{t^{\frac{2(N+\alpha)}{N-2}}}{2}\left(\frac{N-2}{N+\alpha}\right)^2\lambda\int_{\mathbb{R}^N}(I_\alpha * \psi_\epsilon^{\frac{N+\alpha}{N-2}})\psi_\epsilon^{\frac{N+\alpha}{N-2}}dx\right] -$$

$$-K_7\epsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\epsilon^{\frac{N+\alpha-(N-2)q}{2}}) =$$

$$= \frac{2+\alpha}{2(N+\alpha)}\left(\frac{N+\alpha}{N-2}\right)^{\frac{N-2}{2+\alpha}}\lambda^{\frac{2-N}{2+\alpha}}\frac{(\int_{\mathbb{R}^N}(|\nabla\psi_\epsilon|^2+\psi_\epsilon^2)dx)^{\frac{N+\alpha}{2+\alpha}}}{(\int_{\mathbb{R}^N}(I_\alpha * \psi_\epsilon^{\frac{N+\alpha}{N-2}})\psi_\epsilon^{\frac{N+\alpha}{N-2}}dx)^{\frac{N-2}{2+\alpha}}} -$$

$$-K_7\epsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\epsilon^{\frac{N+\alpha-(N-2)q}{2}}).$$

On the other hand, by (A.0.12) and (A.0.14), for some $K_8 > 0$,

$$\frac{(\int_{\mathbb{R}^N}(|\nabla\psi_\epsilon|^2+\psi_\epsilon^2)dx)^{\frac{N+\alpha}{2+\alpha}}}{(\int_{\mathbb{R}^N}(I_\alpha * \psi_\epsilon^{\frac{N+\alpha}{N-2}})\psi_\epsilon^{\frac{N+\alpha}{N-2}}dx)^{\frac{N-2}{2+\alpha}}} \leq S_\alpha^{\frac{N+\alpha}{2+\alpha}} + \begin{cases} K_8\epsilon^2 + o(\epsilon^2) & \text{if } \ N \geq 5 \\ K_8\epsilon^2|\log\epsilon| + o(\epsilon^2|\log\epsilon|) & \text{if } \ N = 4 \\ K_8\epsilon + o(\epsilon) & \text{if } \ N = 3. \end{cases}$$

Then, for some $K_9, K_{10} > 0$,

$$\max_{t\geq 0}\mathcal{S}_\lambda(t\psi_\epsilon) \leq \frac{2+\alpha}{2(N+\alpha)}\left(\frac{N+\alpha}{N-2}\right)^{\frac{N-2}{2+\alpha}}\lambda^{\frac{2-N}{2+\alpha}}S_\alpha^{\frac{N+\alpha}{2+\alpha}} +$$

$$+\begin{cases} K_9\epsilon^2 - K_{10}\epsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\epsilon^{\frac{N+\alpha-(N-2)q}{2}}) & \text{if } \ N \geq 5 \\ K_9\epsilon^2|\log\epsilon| - K_{10}\epsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\epsilon^{\frac{N+\alpha-(N-2)q}{2}}) & \text{if } \ N = 4 \\ K_9\epsilon - K_{10}\epsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\epsilon^{\frac{N+\alpha-(N-2)q}{2}}) & \text{if } \ N = 3. \end{cases} <$$

$$< \frac{2+\alpha}{2(N+\alpha)}\left(\frac{N+\alpha}{N-2}\right)^{\frac{N-2}{2+\alpha}}\lambda^{\frac{2-N}{2+\alpha}}S_\alpha^{\frac{N+\alpha}{2+\alpha}} \quad \text{if } \epsilon > 0 \text{ is sufficiently small,}$$

since $N+\alpha-(N-2)q < 2$. Therefore, for any $\lambda \in [\frac{1}{2}, 1]$ and $\epsilon > 0$ sufficiently small, we get

$$c_\lambda \leq \max_{t\geq 0}\mathcal{S}_\lambda(t\psi_\epsilon) < \frac{2+\alpha}{2(N+\alpha)}\left(\frac{N+\alpha}{N-2}\right)^{\frac{N-2}{2+\alpha}}\lambda^{\frac{2-N}{2+\alpha}}S_\alpha^{\frac{N+\alpha}{2+\alpha}}.$$

□

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