Variational aspects of singular Liouville systems

Luca Battaglia

S.I.S.S.A.

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Singular Liouville systems

I considered **Singular Liouville systems** on a compact surface (Σ, g):

$$-\Delta u_i = \sum_{j=1}^{N} a_{ij}\rho_j \left(\frac{h_j e^{u_j}}{\int_\Sigma h_j e^{u_j} \mathrm{d}V_g} - 1\right) - 4\pi \sum_{m=1}^{M} \alpha_{im}(\delta_{p_m} - 1), \; i = 1, \dots, N.$$

- $A = (a_{ij})_{i,j=1}^N$ symmetric positive definite $N \times N$ matrix,
- $\rho_1, \dots, \rho_N > 0$,
- $0 < h_1, \dots, h_N \in C^\infty(\Sigma)$,
- $p_1, \dots, p_M \in \Sigma$,
- $\alpha_{11}, \dots, \alpha_{NM} > -1$,
- Without loss of generality $|\Sigma| = 1$.

Such systems arise from different fields:

- **Statistical mechanics** (Chern-Simons vortices theory)
- **Physics of particles** (Kinetic plasma models)
- **Algebraic Geometry** (Complex holomorphic curves)
- **Biology** (Chemotaxis)

We re-write the system in an equivalent form:

Consider the solution of

$$\begin{cases} -\Delta G_p = \delta_p - 1 \\ \int_\Sigma G_p dV_g = 0 \end{cases}$$

and apply the change of variable

$$u_i \to u_i + 4\pi \sum_{m=1}^{M} \alpha_{im} G_{p_m}$$

Singular Liouville systems

A change of variables

The new u solve

$$-\Delta u_i = \sum_{j=1}^{N} a_{ij}\rho_j \left(\frac{\widetilde{h}_j e^{u_j}}{\int_\Sigma \widetilde{h}_j e^{u_j} dV_g} - 1 \right) \qquad \widetilde{h}_i := h_i e^{-4\pi \sum_{m=1}^{M} \alpha_{im} G_{p_m}}$$

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Since $G_p = \frac{1}{2\pi} \log \frac{1}{d(\cdot, p)} + O(1)$ around p, then

$$\widetilde{h}_i \sim d(\cdot, p_m)^{2\alpha_{im}} \qquad \text{around } p_m,$$

that is

$$\alpha_{im} > 0 \quad \Rightarrow \quad \widetilde{h}_i \text{ goes to 0 around } p_m$$
$$\alpha_{im} < 0 \quad \Rightarrow \quad \widetilde{h}_i \text{ goes to } +\infty \text{ around } p_m$$

In the last form, the problem has a **variational structure**: solutions are all and only the critical points of

$$J_\rho(u) := \frac{1}{2}\sum_{i,j=1}^{N} a^{ij}\int_\Sigma \nabla u_i \cdot \nabla u_j dV_g - \sum_{i=1}^{N} \rho_i \left(\log \int_\Sigma \widetilde{h}_i e^{u_i} dV_g - \int_\Sigma u_i dV_g \right)$$

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If J_ρ is **coercive** (up to constants), then the system has minimizing solutions.

Existence of minimizing solutions

The scalar case

If $N = 1$,

$$-\Delta u = \rho\left(\frac{\widetilde{h}e^u}{\int_\Sigma \widetilde{h}e^u \mathrm{d}V_g} - 1\right)$$

and we have

$$I_\rho(u) := \frac{1}{2}\int_\Sigma |\nabla u|^2 \mathrm{d}V_g - \rho\left(\log\int_\Sigma \widetilde{h}e^u \mathrm{d}V_g - \int_\Sigma u \mathrm{d}V_g\right).$$

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It is well known that, setting with $\widetilde{\alpha} := \min \left\{ 0, \min_m \alpha_m \right\}$

$\rho < 8\pi(1 + \widetilde{\alpha}) \Rightarrow I_\rho$ coercive

$\rho = 8\pi(1 + \widetilde{\alpha}) \Rightarrow I_\rho$ bounded from below but not coercive

$\rho > 8\pi(1 + \widetilde{\alpha}) \Rightarrow I_\rho$ not bounded from below

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What happens for higher values of ρ?

We take a sequence u_{ρ_n} and discuss its convergence for $\rho_n \to \rho$.
If $u_{\rho_n} \to u_\rho$, then J_ρ must be coercive and u_ρ is a minimizer.

Concentration-compactness Theorem

Let $\{u_{\rho_n}\}_{\rho_n \to \rho}$ be a sequence of solutions with $\int_\Sigma \widetilde{h}_i e^{u_{i,\rho_n}} \mathrm{d}V_g = 1$.
Then,

$$\mathcal{S}_i := \{x \in \Sigma : \exists\, x_n \to x \text{ such that } u_{i,\rho_n}(x_n) \to +\infty\}$$

is finite for all i's. Moreover,

- Either $\mathcal{S} := \bigcup_{i=1}^{N} \mathcal{S}_i = \emptyset$, and $u_{\rho_n} \to u_\rho$ in $W^{2,q}(\Sigma)^N$;
- Or $\mathcal{S} \neq \emptyset$, for each i, either $u_{i,\rho_n} \to u_i$ in $W^{2,q}_{\mathrm{loc}}(\Sigma \backslash \mathcal{S})$ or $u_{i,\rho_n} \to -\infty$ in $L^\infty_{\mathrm{loc}}(\Sigma \backslash \mathcal{S})$; the latter occurs for at least one i.

Define, for $x \in \mathcal{S}_i$,

$$\sigma_i(x) := \lim_{r \to 0} \lim_{n \to +\infty} \rho_{i,n} \int_{B_r(x)} \widetilde{h}_i e^{u_{i,\rho_n}} dV_g.$$

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Then,

$$\rho_i \geq \sum_{x \in \mathcal{S}_i} \sigma_i(x);$$

$$\rho_i = \sum_{x \in \mathcal{S}_i} \sigma_i(x) \quad \Longleftrightarrow \quad u_{i,\rho_n} \to -\infty \text{ in } L^\infty_{loc}(\Sigma \setminus \mathcal{S}) .$$

If $x \in \mathcal{S}_i$ for $i \in \mathcal{I}$, then

$$\Lambda_{\mathcal{I},x}(\sigma(x)) := 8\pi \sum_{i\in\mathcal{I}}(1+\alpha_i(x))\sigma_i(x) - \sum_{i,j\in\mathcal{I}} a_{ij}\sigma_i(x)\sigma_j(x) = 0$$

where $$\alpha_i(x) = \begin{cases} \alpha_{im} & \text{if } x = p_m \\ 0 & \text{otherwise} \end{cases}$$

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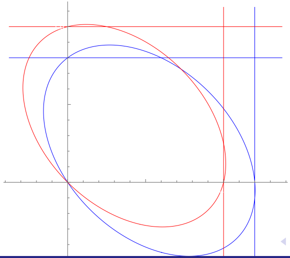

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B.-Malchiodi, 2014 - B., preprint

$$\Lambda(\rho) > 0 \quad \Rightarrow \quad J_\rho \text{ coercive}$$

Existence of minimizing solutions

Conditions for coercivity

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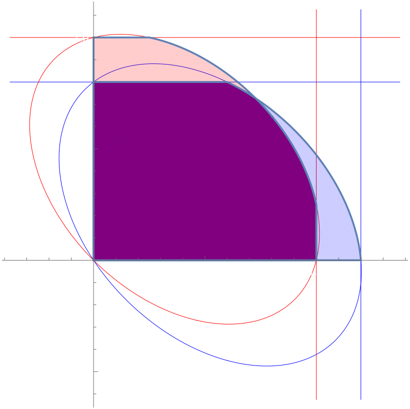
$$\begin{aligned} \Lambda(\rho) > 0 \quad &\Rightarrow \quad J_\rho \text{ coercive} \\ \Lambda(\rho) = 0 \quad &\Rightarrow \quad J_\rho \text{ not coercive} \\ \Lambda(\rho) < 0 \quad &\Rightarrow \quad J_\rho \text{ not coercive nor bounded from below} \end{aligned}$$

Existence of minimizing solutions
Conditions for coercivity

The set $\Lambda > 0$:



Suppose now $a_{ij} \leq 0$ for all $i \neq j$.

Then,

$$\Lambda(\rho) = \min_{i=1,\dots,N} \left(8\pi(1+\widetilde{\alpha}_i)\rho_i - a_{ii}\rho_i^2\right),$$

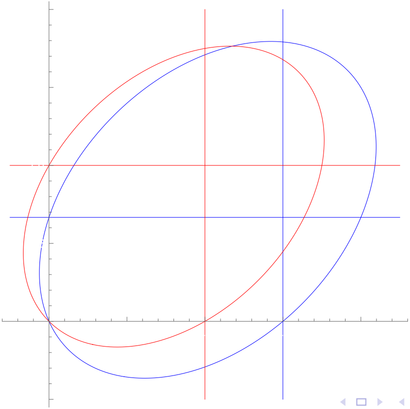
with

$$\widetilde{\alpha}_i = \min_x \alpha_i(x) = \min\left\{0, \min_m \alpha_{im}\right\}$$

Existence of minimizing solutions

Competitive systems

The set $\Lambda(\rho) > 0 = \left\{ \rho_i < \dfrac{8\pi(1+\widetilde{\alpha}_i)}{a_{ii}} \right\}$:



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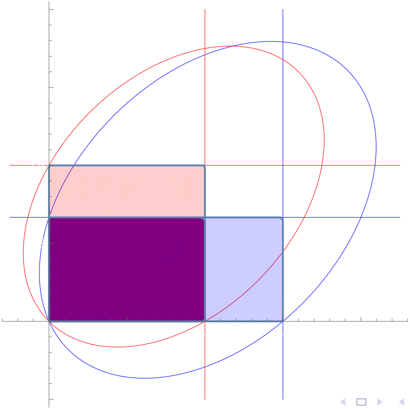

For blowing-up sequences of minimizers for $\rho_{i,n} \to \frac{8\pi(1+\widetilde{\alpha}_i)}{a_{ii}}$, $\mathcal{S}_i = \{x_i\}$ and either $a_{ij} = 0$ or $x_i \neq x_j$ for all $i \neq j$.

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B.-Malchiodi, 2014 - B., preprint

$\Lambda(\rho) > 0 \quad \Rightarrow \quad J_\rho$ coercive
$\Lambda(\rho) = 0 \quad \Rightarrow \quad J_\rho$ not coercive but **bounded from below**
$\Lambda(\rho) < 0 \quad \Rightarrow \quad J_\rho$ not coercive nor bounded from below

Min-max solutions

The role of sub-levels

If $\Lambda(\rho) < 0$, we cannot have minimizers.
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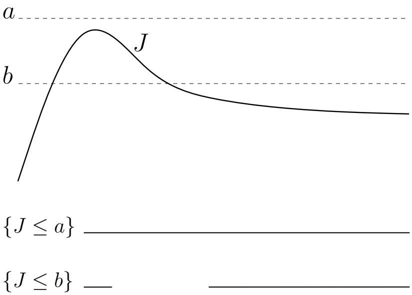
We will study the **topology of sub-levels** $\{J_\rho \leq a\}$:

No critical points with $a \leq J_\rho \leq b \quad \Rightarrow \quad \{J_\rho \leq a\} \simeq \{J_\rho \leq b\}$

$\{J_\rho \leq a\} \not\simeq \{J_\rho \leq b\} \quad \Rightarrow \quad$ Critical points with $a \leq J_\rho \leq b$

Min-max solutions

The role of sub-levels



Min-max solutions

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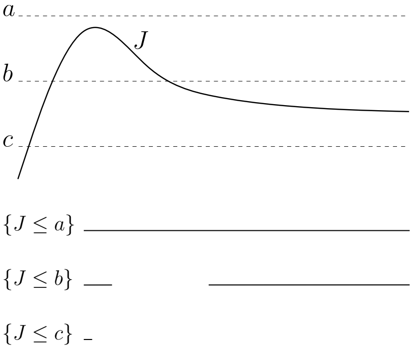

We need some **compactness conditions**.

In general, such compactness conditions are not known, except for some particular systems.

Min-max solutions

Compactness issues

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$$A_2 := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$B_2 := \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \qquad \alpha_{im} \equiv 0$$

$$G_2 := \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \qquad \alpha_{im} \equiv 0$$

Min-max solutions

Compactness issues

Although B_2, G_2 are not symmetric, we can argue in the same way:

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} \mathrm{d}V_g} - 1 \right) - 2 \cdot \frac{\rho_2}{2} \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} \mathrm{d}V_g} - 1 \right) \\ -\Delta u_2 = -2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} \mathrm{d}V_g} - 1 \right) + 4 \cdot \frac{\rho_2}{2} \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} \mathrm{d}V_g} - 1 \right) \end{cases}$$

$$\begin{aligned} J_\rho(u) &= \int_\Sigma \left(\frac{|\nabla u_1|^2}{2} + \frac{\nabla u_1 \cdot \nabla u_2}{2} + \frac{|\nabla u_2|^2}{4} \right) \mathrm{d}V_g \\ &- \rho_1 \left(\log \int_\Sigma h_1 e^{u_1} \mathrm{d}V_g - \int_\Sigma u_1 \mathrm{d}V_g \right) \\ &- \frac{\rho_2}{2} \left(\log \int_\Sigma h_2 e^{u_2} \mathrm{d}V_g - \int_\Sigma u_2 \mathrm{d}V_g \right). \end{aligned}$$

The coercivity threshold is $\rho_1, \rho_2 < 4\pi$.

Although B_2, G_2 are not symmetric, we can argue in the same way:

$$\begin{cases} -\Delta u_1 = 2\rho_1\left(\frac{h_1e^{u_1}}{\int_\Sigma h_1e^{u_1}\mathrm{d}V_g}-1\right)-3\cdot\frac{\rho_2}{3}\left(\frac{h_2e^{u_2}}{\int_\Sigma h_2e^{u_2}\mathrm{d}V_g}-1\right)\\ -\Delta u_2 = -3\rho_1\left(\frac{h_1e^{u_1}}{\int_\Sigma h_1e^{u_1}\mathrm{d}V_g}-1\right)+6\cdot\frac{\rho_2}{3}\left(\frac{h_2e^{u_2}}{\int_\Sigma h_2e^{u_2}\mathrm{d}V_g}-1\right)\end{cases}$$

$$\begin{aligned} J_\rho(u) &= \int_\Sigma\left(|\nabla u_1|^2+\nabla u_1\cdot\nabla u_2+\frac{|\nabla u_2|^2}{3}\right)\mathrm{d}V_g \\ &- \rho_1\left(\log\int_\Sigma h_1e^{u_1}\mathrm{d}V_g-\int_\Sigma u_1\mathrm{d}V_g\right) \\ &- \frac{\rho_2}{3}\left(\log\int_\Sigma h_2e^{u_2}\mathrm{d}V_g-\int_\Sigma u_2\mathrm{d}V_g\right). \end{aligned}$$

The coercivity threshold is $\rho_1, \rho_2 < 4\pi$.

Concerning A_2, the coercivity threshold is $\rho_1, \rho_2 < 4\pi(1+\widetilde{\alpha}_i)$:

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\dfrac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} \mathrm{d}V_g} - 1 \right) - \rho_2 \left(\dfrac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} \mathrm{d}V_g} - 1 \right) \\ -\Delta u_2 = -\rho_1 \left(\dfrac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} \mathrm{d}V_g} - 1 \right) + 2\rho_2 \left(\dfrac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} \mathrm{d}V_g} - 1 \right) \end{cases}$$

$$\begin{aligned} J_\rho(u) &= \int_\Sigma \frac{|\nabla u_1|^2 + \nabla u_1 \cdot \nabla u_2 + |\nabla u_2|^2}{3} \mathrm{d}V_g \\ &- \rho_1 \left(\log \int_\Sigma \widetilde{h}_1 e^{u_1} \mathrm{d}V_g - \int_\Sigma u_1 \mathrm{d}V_g \right) \\ &- \rho_2 \left(\log \int_\Sigma \widetilde{h}_2 e^{u_2} \mathrm{d}V_g - \int_\Sigma u_2 \mathrm{d}V_g \right). \end{aligned}$$

Min-max solutions

Compactness issues

Jost-Lin-Wang, 2006 - Lin-Zhang, preprint

Assume $\alpha_{im} \equiv 0$ and $A = A_2, B_2$. Then, $\sigma_1(x), \sigma_2(x) \in 4\pi\mathbb{N}$.
The same holds true for $A = G_2$ if
$\sigma_1(x) < 4\pi\left(2+\sqrt{2}\right)$, $\sigma_2(x) < 4\pi\left(5+\sqrt{7}\right)$.

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Combining with concentration-compactness Theorem, we get

B.-Gabriele Mancini, 2015

Under the same assumptions, if blow-up occurs then
$\rho \in \Gamma_0 := 4\pi\mathbb{N} \times \mathbb{R}_+ \cup \mathbb{R}_+ \times 4\pi\mathbb{N}$.

Similarly,

Lin-Wei-Zhang, 2015

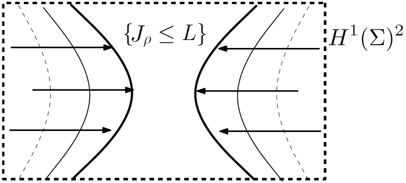
Assume $A = A_2$. If $x \notin \{p_1, \ldots, p_M\}$, then $\sigma_1(x), \sigma_2(x) \in 4\pi\mathbb{N}$, and $(\sigma_1(p_m), \sigma_2(p_m)) \in \Xi_m$ for some finite Ξ_m.

Therefore,

B.-Gabriele Mancini, 2015

Under the same assumptions, if blow-up occurs then $\rho \in \Gamma := \Gamma_1 \times \mathbb{R}_+ \cup \mathbb{R}_+ \times \Gamma_2$ for some discrete $\Gamma_1, \Gamma_2 \subset \mathbb{R}_+$

Moreover, $J_\rho \leq L$ for all solutions, so $\{J_\rho \leq L\}$ is a deformation retract of $H^1(\Sigma)^2$, hence it is **contractible**.



Min-max solutions

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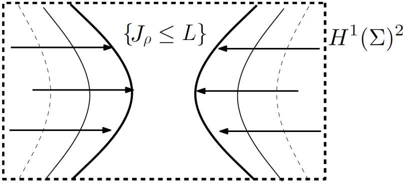

Existence of solutions will follow if $\{J_\rho \leq -L\}$ is **not contractible** for large L.

To prove that low sub-levels are not contractible, we "compare" it with a **not contractible** space $\mathcal{X}$ in the following way:

$$\mathcal{X} \xrightarrow{\Phi} \{J_\rho \le -L\} \xrightarrow{\Psi} \mathcal{X} \qquad \Psi \circ \Phi \simeq \mathrm{Id}_{\mathcal{X}}.$$

$\{J_\rho \le -L\}$ is **dominated** by $\mathcal{X}$.

Min-max solutions

Analysis of sub-levels, $\alpha_{im} \geq 0$

Let us consider the A_2 Toda system in the case $\alpha_{im} \geq 0$:

Min-max solutions

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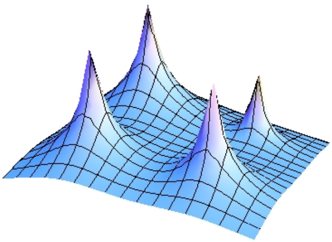
Let us consider the A_2 Toda system in the case $\alpha_{im} \geq 0$:

If $\rho_1 \in (4K_1\pi, 4(K_1+1)\pi)$, $\rho_2 \in (4K_2\pi, 4(K_2+1)\pi)$, then **either** u_1 concentrates at K_1 points **or** u_2 concentrates at K_2 points.



To express the concentration we use the **barycenters** on Σ:

$$(\Sigma)_K := \left\{ \sum_{k=1}^{K} t_k \delta_{x_k};\ x_k \in \Sigma,\ t_k \geq 0,\ \sum_{k=1}^{K} t_k = 1 \right\}.$$

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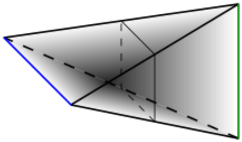
To express the alternative between u_1 and u_2, we use the **join**:

$$X \star Y := \{(1-t)x + ty;\ x \in X,\ y \in Y,\ t \in [0,1]\}.$$



Min-max solutions

Analysis of sub-levels, $\alpha_{im} \geq 0$

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Min-max solutions

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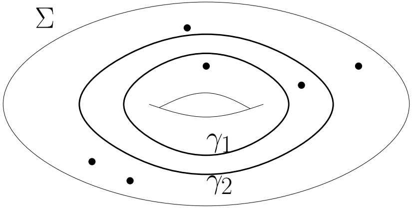
If $\chi(\Sigma) \leq 0$, this can be "by-passed" by a topological trick. There exist two **retractions** $\Pi_i : \Sigma \to \gamma_i$ for $i = 1, 2$ onto disjointed circles not containing any p_m.



Min-max solutions

Analysis of sub-levels, $\alpha_{im} \geq 0$, $\chi(\Sigma) \leq 0$

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$$(\gamma_1)_{K_1} \star (\gamma_2)_{K_2} \simeq \left(\mathbb{S}^1\right)_{K_1} \star \left(\mathbb{S}^1\right)_{K_2} \simeq \mathbb{S}^{2K_1-1} \star \mathbb{S}^{2K_2-1} \simeq \mathbb{S}^{2K_1+2K_2-1}.$$

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$$(\gamma_1)_{K_1} \star (\gamma_2)_{K_2} \simeq \left(\mathbb{S}^1\right)_{K_1} \star \left(\mathbb{S}^1\right)_{K_2} \simeq \mathbb{S}^{2K_1-1} \star \mathbb{S}^{2K_2-1} \simeq \mathbb{S}^{2K_1+2K_2-1}.$$

B.-Jevnikar-Malchiodi-Ruiz, 2015

Suppose $\rho \notin \Gamma$, $\chi(\Sigma) \leq 0$ and $\alpha_{im} \geq 0$ for all m. Then the A_2 Toda system has solutions.

The same results also works for the B_2 and G_2 Toda systems:

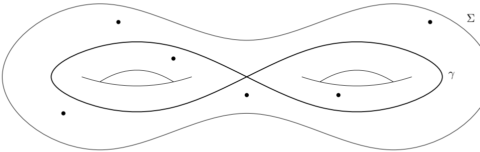
B., in preparation

Suppose $\rho_1, \rho_2 \notin 4\pi\mathbb{N}$, $\chi(\Sigma) \leq 0$. Then the B_2 Toda system has solutions.
The same holds for the G_2 Toda system, provided
$\rho_1 < 4\pi\left(2+\sqrt{2}\right)$, $\rho_2 < 4\pi\left(5+\sqrt{7}\right)$.

Min-max solutions

Analysis of sub-levels, $\alpha_{im} \geq 0$, $\chi(\Sigma) \leq 0$

If Σ has genus $g = \left[\frac{-\chi(\Sigma)}{2}\right] + 1 \geq 2$, we can take γ_1, γ_2 as bouquet of g circles to get a generic multiplicity result via Morse theory:



Min-max solutions

Analysis of sub-levels, $\alpha_{im} \geq 0$, $\chi(\Sigma) \leq 0$

B., 2014 - B., in preparation

If $\rho_1 \in (4K_1\pi, 4(K_2+1)\pi)$, $\rho_2 \in (4K_2\pi, 4(K_2+1)\pi)$, then for a generic choice of g, h_1, h_2 there are at least

$$\binom{K_1 + \left[\frac{-\chi(\Sigma)}{2}\right]}{K_1}\binom{K_2 + \left[\frac{-\chi(\Sigma)}{2}\right]}{K_2}$$

solutions.

If we consider the A_2 Toda system without restrictions on α_{im}, the same argument fails because negative coefficients affect the M-T inequality.

If we consider the A_2 Toda system without restrictions on α_{im}, the same argument fails because negative coefficients affect the M-T inequality.

To take account of this, we introduce the **weighted barycenters**:

$$\omega_i(q) = \begin{cases} 1 + \alpha_{im} & \text{if } q = p_m,\ \alpha_{im} < 0 \\ 1 & \text{otherwise} \end{cases} \qquad \omega_i\left(\bigcup_k q_k\right) = \sum_k \omega_i(q_k)$$

$$(\Sigma)_{\rho_i,\underline{\alpha}_i} := \left\{ \sum_{x_k \in \mathcal{J}} t_k \delta_{x_k};\ x_k \in \Sigma,\ t_k \geq 0,\ \sum_{x_k \in \mathcal{J}} t_k = 1,\ 4\pi\omega_i(\mathcal{J}) < \rho_i \right\}.$$

Min-max solutions

Analysis of sub-levels, $\chi(\Sigma) \leq 0$

The topological argument can be adapted by modifying the retractions to take account of singularities.

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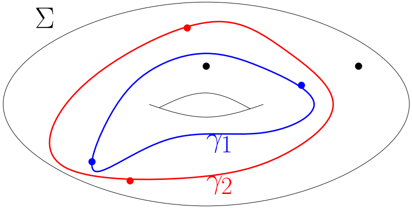
We need $p_m \in \gamma_i$ if $\alpha_{im} < 0$, so we assume $\max\{\alpha_{1m}, \alpha_{2m}\} \geq 0$.



Write:

$$\{p_1, \dots, p_M\} = \left\{p'_{01}, \dots, p'_{0M'_0}, p'_{11}, \dots, p'_{1M'_1}, p'_{21}, \dots, p'_{2M'_2}\right\}$$

$$p_m = p'_{0m'} \iff \alpha_{1m}, \alpha_{2m} \geq 0 \iff p_m \notin \gamma_1 \cup \gamma_2$$

$$p_m = p'_{1m'} \iff \alpha'_{1m'} := \alpha_{1m} < 0 \iff p_m \in \gamma_1$$

$$p_m = p'_{2m'} \iff \alpha'_{2m'} := \alpha_{2m} < 0 \iff p_m \in \gamma_2$$

Min-max solutions

Analysis of sub-levels, $\chi(\Sigma) \le 0$

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This time, low sub-levels are dominated by the join of weighted barycenters $(\gamma_1)_{\rho_1,\underline{\alpha}_1} \star (\gamma_2)_{\rho_2,\underline{\alpha}_2}$.

Min-max solutions

Analysis of sub-levels, $\chi(\Sigma) \leq 0$

The weighted barycenters, hence their join, **could be contractible**.

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This happens if

$$\sigma \in (\gamma_i)_{\rho_i,\underline{\alpha}_i} \quad \Rightarrow \quad (1-t)\sigma + t\delta_{p'_{i1}} \in (\gamma_i)_{\rho_i,\underline{\alpha}_i} \quad \forall\, t \in [0,1];$$

which means, in terms of ρ,

$$4\pi\left(K + \sum_{m\in\mathcal{M}}(1+\alpha'_{im})\right) < \rho_i \Rightarrow 4\pi\left(k + \sum_{m\in\mathcal{M}\cup\{1\}}(1+\alpha'_{im})\right) < \rho_i.$$

If this does not happen for either i, then $(\gamma_1)_{\rho_1,\underline{\alpha}_1} \star (\gamma_2)_{\rho_2,\underline{\alpha}_2}$ is not contractible.

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B. (2015)

Suppose $\rho \notin \Gamma$, $\chi(\Sigma) \leq 0$, $\max\{\alpha_{1m}, \alpha_{2m}\} \geq 0$ for all m and

$$4\pi\left(K_i + \sum_{m\in\mathcal{M}_i}(1+\alpha'_{im})\right) < \rho_i < 4\pi\left(K_i + \sum_{m\in\mathcal{M}_i\cup\{1\}}(1+\alpha'_{im})\right)$$

for some $K_1, K_2 \in \mathbb{N}$ and $\mathcal{M}_i \subset \{2, \ldots, M'_i\}$.
Then the A_2 Toda system has solutions.

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Roughly speaking, in case of concentration at the same point with the **same rate**, the point must be given a higher weight.

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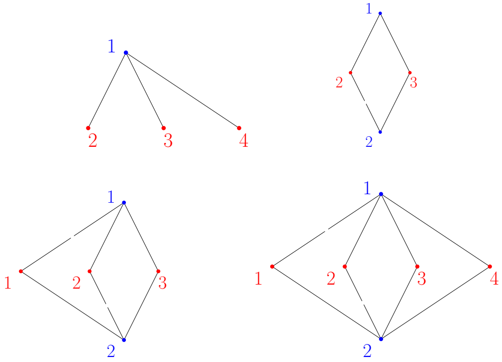
If $\rho_1 < \overline{\rho}_1$, $\rho_2 < \overline{\rho}_2$, where

$$\overline{\rho}_i := 4\pi \min \left\{ 1, \min_{m \neq m'} (2 + \alpha_{im} + \alpha_{im'}) \right\},$$

then low sub-levels are dominated by

$$\mathcal{X} = (\Sigma)_{\rho_1, \underline{\alpha}_1} \star (\Sigma)_{\rho_2, \underline{\alpha}_2} \setminus \left\{ \left(p_m, p_m, \frac{1}{2} \right) : \rho_1, \rho_2 < 4\pi(2 + \alpha_{1m} + \alpha_{2m}) \right\}.$$

Since, for such ρ, both $(\Sigma)_{\rho_i,\underline{\alpha}_i}$ are finite, than $\mathcal{X}$ is easy to study:



We need some assumptions to get a not-contractible space:

B. (2015)

Suppose $\rho \notin \Gamma$, $\rho_i < \overline{\rho}_i$ for both i and

$$(M_1, M_2, M_3) \notin \{(1, m, 0), (m, 1, 0), (2, 2, 1), (2, 3, 2), (3, 2, 2),\ m \in \mathbb{N}\},$$

with M_1, M_2, M_3 defined by

$$M_1 := \#\{m : \ 4\pi(1 + \alpha_{1m}) < \rho_1\},$$

$$M_2 := \#\{m : \ 4\pi(1 + \alpha_{2m}) < \rho_2\},$$

$$M_3 := \#\{m : \ 4\pi(1+\alpha_{im}) < \rho_i,\ \rho_i < 4\pi(2+\alpha_{1m}+\alpha_{2m}) \text{ for both } i\}.$$

Then the A_2 Toda system has solutions.

Non-existence results

General systems

We made topological assumptions on Σ to get general existence results.

In fact, if Σ has a “simple” topology, general systems could not be solvable.

Non-existence results

General systems

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In fact, if Σ has a "simple" topology, general systems could not be solvable.

On the standard unit disk we get, through a Pohožaev identity, **necessary** algebraic conditions.

B.-Malchiodi, preprint

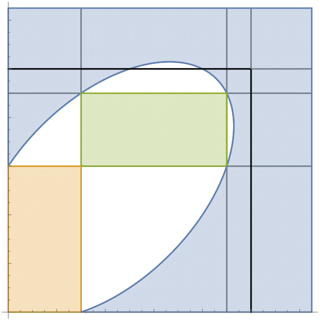
The following problem on the unit disk $\mathbb{B}$:

$$-\Delta u_i = \sum_{j=1}^{N} a_{ij}\rho_j \frac{|\cdot|^{2\alpha_j} e^{u_j}}{\int_{\mathbb{B}} |x|^{2\alpha_j} e^{u_j} \mathrm{d}x} \qquad u_i|_{\partial\mathbb{B}} = 0 \qquad i = 1, \dots, N,$$

has no solutions if ρ satisfies

$$\Lambda_{\{1,\dots,N\},p}(\rho) = 8\pi \sum_{i=1}^{N} (1+\alpha_i)\rho_i - \sum_{i,j=1}^{N} a_{ij}\rho_i\rho_j \leq 0.$$

Comparison with existence results for the A_2 Toda system:



Similar results hold on the unit sphere with antipodal singularities:

B.-Malchiodi, preprint

The following problem on the unit sphere $\mathbb{S}^2$:

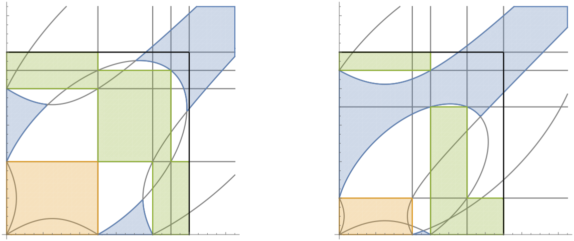
$$-\Delta u_i = \sum_{j=1}^{N} a_{ij}\rho_j \left(\frac{e^{u_j}}{\int_{\mathbb{S}^2} e^{u_j} \mathrm{d}V_g} - \frac{1}{4\pi} \right) - 4\pi \sum_{m=1}^{2} \alpha_{im} \left(\delta_{p_m} - \frac{1}{4\pi} \right),$$

has no solutions if ρ satisfies:

$$\begin{array}{rll} \text{either} & \Lambda_{\mathcal{I},p_1}(\rho) \geq \Lambda_{\{1,\dots,N\}\setminus\mathcal{I},p_2}(\rho) & \forall \mathcal{I} \subset \{1,\dots,N\} \\ \text{or} & \Lambda_{\mathcal{I},p_2}(\rho) \geq \Lambda_{\{1,\dots,N\}\setminus\mathcal{I},p_1}(\rho) & \forall \mathcal{I} \subset \{1,\dots,N\} \end{array}$$

and at least one inequality is strict.

Comparison with existence results for the A_2 Toda system:



We also get a non-existence results for the A_2 Toda system on **any** surface.

Non-existence results

A_2 Toda system

We also get a non-existence results for the A_2 Toda system on **any** surface.

If we take a couple of coefficients $(\alpha_{11}, \alpha_{21})$ close to -1 we show, through a blow-up analysis, that no solutions exist.

B.-Malchiodi, preprint

For any fixed $\alpha_{12}, \ldots, \alpha_{1M}, \alpha_{22}, \ldots, \alpha_{2M}$ and $\rho \notin \Gamma_{\underline{\alpha}_{1\hat{1}}, \underline{\alpha}_{2,\hat{1}}}$ there exists $\alpha^* \in (-1, 0)$ such that the A_2 Toda system has no solutions for $\alpha_{11}, \alpha_{21} \leq \alpha^*$.

THANK YOU FOR YOUR ATTENTION!