A AND B OPERATORS

In the sequel we work with projective schemes over an algebraically closed field.

We will often use the following

Lemma 0.1. *Let $f : X \to Y$ be a proper morphism between two nonsingular projective varieties. We have:*

(i) if $u \in N_k(X), v \in N^k(X)$ then, under the isomorphism $\deg : N_0(X) \to \mathbb{R}[p]$ *where p is a point in X and the same in Y, we have $f_*(u \cdot v) = u \cdot v$;*
(ii) if $u \in N^k(X), v \in N^h(Y)$ then $f_*(u \cdot f^*v) = (f_*u) \cdot v$;
(iii) if f is flat or generically separable then $f^*[Z] = [f^{-1}(Z)]$ if $Z \subseteq Y$ is a subvariety of codimension c such that $f^{-1}(Z)$ is generically reduced of codimension c;
(iv) if $v \in N^h(Y)$ then $f_*(f^*v) = \deg f[\mathrm{Im} f] \cdot v$.

Proof. Let $n = \dim X, m = \dim Y$. We have linear maps $f_* : N_k(X) \to N_k(Y)$ for $0 \le k \le n$ and $f^* : N^h(Y) \to N^h(X)$ for $0 \le h \le m$ (see for example [FL, Rmk. 2.4]).

To see (i) observe that $u \cdot v \in N_0(X)$ whence $u \cdot v = \deg(u \cdot v)[p]$ where p is a point in X and $f_*(u \cdot v) = \deg(u \cdot v)[f(p)] = u \cdot v$. To see (ii) note that the same formula holds at the level of Chow groups (it is the push-pull formula [F, Prop. 8.3(c)]), that is $f_*(u \cdot f^*v) = (f_*u) \cdot v \in A^{m-n+k+h}(Y)$. Therefore, for every subvariety $Z \subseteq Y$ of dimension $m - n + k + h$ we have, in $A_0(Y)$,

$$f_*(u \cdot f^*v) \cdot [Z] = (f_*u) \cdot v \cdot [Z]$$

but this means that $f_*(u \cdot f^*v) = (f_*u) \cdot v$ holds in $N^{m-n+k+h}(Y)$. Now (iv) follows by (ii):

$$f_*(f^*v) = f_*([Y] \cdot f^*v) = f_*[Y] \cdot v = \deg f[\mathrm{Im} f] \cdot v.$$

Finally (iii) follows by [EH, Thm. 1.15 and 1.16] and the fact that the product descends from rational to numerical equivalence. □

For any $s \in \mathbb{Z}$ and any $r \in \mathbb{R}$, we set

$$s! = 0 \text{ if } s \le -1; \binom{r}{s} = \begin{cases} \frac{r(r-1)\dots(r-s+1)}{s!} & \text{if } s \ge 1, \\ 1 & \text{if } s = 0 \\ 0 & \text{if } s \le -1. \end{cases}$$

We now fix $d, m, n \in \mathbb{Z}$ such that $d \ge 1, 0 \le m \le d, 0 \le n \le d+1$ and a point $p_0 \in C$.

Define $r(m) := \min\{m, d-m, g\}$.

We first define some standard maps and classes.

For $0 \le \alpha \le d-1$ define the maps

$$i_{\alpha,d} : C_{d-\alpha} \to C_d, \quad \mu_d : C \times C_d \to C_{d+1}$$

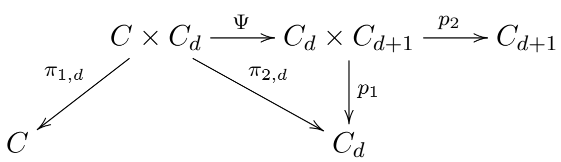
by $i_{\alpha,d}(D) = D + \alpha p_0$ and $\mu_d(p, D) = p + D$. Note that $i_{\alpha,d}$ is injective, $[\mathrm{Im} i_{\alpha,d}] = x^\alpha$ and, with a slight abuse of notation,

$$i_{\alpha,d}^* x = x, i_{\alpha,d}^* \theta = \theta, \quad \mu_d^* x = \pi_{1,d}^* x + \pi_{2,d}^* x$$

where $\pi_{1,d} : C \times C_d \to C$ and $\pi_{2,d} : C \times C_d \to C_d$ are the projections.

Consider the diagram

(0.1)



where p_1, p_2 are the projections and $\Psi(p, D) = (D, p + D)$. Note that

$$\Psi \circ p_1 = \pi_{2,d} \text{ and } \Psi \circ p_2 = \mu_d.$$

When d does not need to be specified we will denote $\pi_1 = \pi_{1,d}, \pi_2 = \pi_{1,d}$ and $\mu = \mu_d$.

Definition 0.2. Let Δ_d be the big diagonal in C_d. Let $\mathcal{D}$ be the universal divisor on $C \times C_d$, that is

$$\mathcal{D} = \{(p, D) \in C \times C_d : p \in \operatorname{Supp}(D)\}.$$

Let

$$\delta = [\mathcal{D}] \in N^1(C \times C_d), \quad \eta := \pi_1^* x, \quad \gamma := g\eta + \pi_2^*\theta - \mu^*\theta^{1} \text{ and } \quad \psi := [\operatorname{Im}\Psi].$$

Lemma 0.3.

(i) $\mu^*[\Delta_{d+1}] = 2\delta + \pi_2^*[\Delta_d]$
(ii) $\delta = d\eta + \pi_2^* x + \gamma$
(iii) $\eta^2 = \gamma\eta = \gamma^3 = 0, \gamma^2 = -2\eta\pi_2^*\theta$
(iv) $(\pi_2)_*\eta = [C_d], (\pi_2)_*\gamma = 0.$

Proof. To see (i) first observe that

$$\mu^{-1}(\Delta_{d+1}) = \{(p, D) \in C \times C_d : p + D \in \Delta_{d+1}\} = \mathcal{D} \cup \pi_2^{-1}(\Delta_d)$$

hence

$$\mu^*[\Delta_{d+1}] = a\delta + b\pi_2^*[\Delta_d] \tag{0.2}$$

for some integers $a \geq 1, b \geq 1$. We claim that $a = 2$ and $b = 1$.

First we prove that $\mu_{|\mathcal{D}} : \mathcal{D} \to \Delta_{d+1}$ has degree 1.

In fact pick a general point $D' = 2p_1 + p_2 + \ldots + p_d \in \Delta_{d+1}$, so that all p_i's are distinct. Now if $(p, D) \in \mathcal{D}$ is such that $p + D = \mu(p, D) = D'$ then $D = p + D''$, hence $2p + D'' = D'$. This forces $p = p_1, D = D' - p_1$ for otherwise $p = p_i$ for some $i \geq 2$ but then $D' \geq 2p_i$, a contradiction.

Next we prove that $\mu_{|\pi_2^{-1}(\Delta_d)} : \pi_2^{-1}(\Delta_d) \to \Delta_{d+1}$ has degree $d - 1$.

In fact pick a general point $D' = 2p_1 + p_2 + \ldots + p_d \in \Delta_{d+1}$, so that all p_i's are distinct. Then $\mu^{-1}(D') = \{(p_i, D' - p_i), 2 \leq i \leq d\}$: if $(p, D) \in \pi_2^{-1}(\Delta_d)$ is such that $p + D = \mu(p, D) = D'$ then either $p = p_i, D = D' - p_i$ for $2 \leq i \leq d$ or $p = p_1$ wich implies that $D = D' - p_1 \in \Delta_d$, a contradiction.

Since $\deg \mu = d + 1$ we find from (0.2), using Lemma 0.1(iv), that

$$(d + 1)[\Delta_{d+1}] = \mu_*(\mu^*[\Delta_{d+1}]) = a\mu_*(\delta) + b\mu_*(\pi_2^*[\Delta_d]) = (a + b(d - 1))[\Delta_{d+1}]$$

that is $a + b(d - 1) = d + 1$ and the only solution is $a = 2, b = 1$. This proves (i).

To see (ii) observe that from (i) we find, using $\mu^* x = \eta + \pi_2^* x$,

$$\delta = \mu^*((d + g)x - \theta) - \pi_2^*((d + g - 1)x - \theta) = (d + g)\eta + \pi_2^* x - \mu^*\theta + \pi_2^*\theta = d\eta + \pi_2^* x + \gamma. \tag{0.3}$$

Now (ii) together with [ACGH, Chap. VIII, §2](in ordinary cohomology or l-adic cohomology) implies that $\gamma^2 = -2\eta\pi_2^*\theta$. Also $\eta^2 = (\pi_1^* x)^2 = \pi_1^*(x^2) = 0$. To see that $\gamma\eta = 0$ observe that $\delta\eta = \eta\pi_2^* x$ whence, from (0.3) we get $(\mu^*\theta)\eta = (\pi_2^*\theta)\eta$ whence

$$\gamma\eta = (g\eta + \pi_2^*\theta - \mu^*\theta)\eta = (\pi_2^*\theta)\eta - (\mu^*\theta)\eta = 0.$$

and then $\gamma^3 = \gamma\gamma^2 = -2\gamma\eta\pi_2^*\theta = 0$. This proves (iii).

To see (iv) observe that, since on C we have $x = p_0$ then $\eta = \pi_1^* x = \{p_0\} \times C_d$ and $\pi_{2|\{p_0\} \times C_d}$ is an isomorphism onto C_d, whence $(\pi_2)_*\eta = [C_d]$. On the other hand $(\pi_2)_*\delta = (\deg \pi_{2|\mathcal{D}})[\pi_2(\mathcal{D})] = (\deg \pi_{2|\mathcal{D}})[C_d]$. For a general point $D \in C_d$ we have

$$(\pi_{2|\mathcal{D}})^{-1}(D) = \{(p_i, D), p_i \in \operatorname{Supp} D\},$$

[1] N.B. This is different from [ACGH, Chap. VIII, Exercise D.4] that gives the opposite signs. I believe it is a typo there.

that is $\deg \pi_{2|\mathcal{D}} = d$ and therefore $(\pi_2)_*\delta = d[C_d]$ and, using (ii),

$$(\pi_2)_*\gamma = (\pi_2)_*\delta - d(\pi_2)_*\eta - (\pi_2)_*\pi_2^*x = 0$$

because $(\pi_2)_*\pi_2^*x = 0$. This proves (iv). □

We now define the A and B operators.

Definition 0.4. The *push operator* A_d (also denoted only by A if d does not need to be specified) is the following linear map

$$A_d : N^m(C_d) \longrightarrow N^m(C_{d+1}),$$
$$z \mapsto \mu_*\pi_2^*(z).$$

The *pull operator* B_d (also denoted only by B if d does not need to be specified) is the following linear map

$$B_d : N_n(C_{d+1}) \longrightarrow N_n(C_d),$$
$$w \mapsto (\pi_2)_*\mu^*(w).$$

We start with some properties of the A and B operators.

Lemma 0.5. *([ACGH, Chap. VIII, Exercises D.1, D.2 and part of D.7])*

(i) For any $z \in N^m(C_d)$ and any $w \in N_n(C_{d+1})$, we have that

$$A(z) = (p_2)_*\big(\psi \cdot p_1^*z\big), B(w) = (p_1)_*\big(\psi \cdot p_2^*w\big).$$

(ii) For any $z \in N^m(C_d)$ and any $w \in N_m(C_{d+1})$, we have that

$$A(z) \cdot w = z \cdot B(w).$$

(iii) For any $0 \le \alpha \le \min\{m, d-1\}$ and $z \in N^{m-\alpha}(C_d)$, we have

$$A_d(x^\alpha z) = (i_{\alpha,d+1})_*A_{d-\alpha}(i_{\alpha,d}^*z).$$

(iv) $A(x^m) = (d+1-m)x^m$

(v) $A(x^{m-1}\theta) = (d-m)x^{m-1}\theta + gx^m$.

Proof. To see (i) consider the diagram (0.1). Since Ψ is injective we have by Lemma 0.1(iv) that

$$A(z) = \mu_*(\pi_2^*z) = (p_2)_*\Psi_*\Psi^*p_1^*z = (p_2)_*\big(\psi \cdot p_1^*z\big)$$

and similarly

$$B(w) = (\pi_2)_*\big(\mu^*w\big) = (p_1)_*\Psi_*\Psi^*p_2^*w = (p_1)_*\big(\psi \cdot p_2^*w\big).$$

This proves (i).

We give two proofs of (ii), the first is direct, the second one inspired by [ACGH, Chap. VIII, Exercise D.1]. In both we use Lemma 0.1(i) and (ii). We have

$$A(z) \cdot w = \mu_*(\pi_2^*z) \cdot w = \mu_*\big(\pi_2^*z \cdot \mu^*w\big) = \pi_2^*z \cdot \mu^*w = \mu^*w \cdot \pi_2^*z =$$
$$= (\pi_2)_*\big(\mu^*w \cdot \pi_2^*z\big) = (\pi_2)_*(\mu^*w) \cdot z = z \cdot B(w)$$

and this proves (ii). Alternatively, using (i),

$$A(z) \cdot w = (p_2)_*\big(\psi \cdot p_1^*z\big) \cdot w = (p_2)_*\big(\psi \cdot p_1^*z \cdot p_2^*w\big) = (p_2)_*\big(\psi \cdot p_2^*w \cdot p_1^*z\big) =$$
$$= \psi \cdot p_2^*w \cdot p_1^*z = (p_1)_*\big(\psi \cdot p_2^*w \cdot p_1^*z\big) = (p_1)_*\big(\psi \cdot p_2^*w\big) \cdot z = z \cdot B(w)$$

again proving (ii). To see (iii) consider the map

$$j : C \times C_{d-\alpha} \to C \times C_d$$

defined by $j(p, D) = (p, D + \alpha p_0)$ and the commutative diagrams

(0.4)
$$\begin{array}{ccc} C \times C_{d-\alpha} & \xrightarrow{j} & C \times C_d \\ \downarrow{\scriptstyle \mu_{d-\alpha}} & & \downarrow{\scriptstyle \mu} \\ C_{d+1-\alpha} & \xrightarrow{i_{\alpha,d+1}} & C_{d+1} \end{array}$$

and

$$
\begin{array}{ccc}
C \times C_{d-\alpha} & \xrightarrow{j} & C \times C_d \\
\Big\downarrow{\scriptstyle \pi_{2,d-\alpha}} & & \Big\downarrow{\scriptstyle \pi_2} \\
C_{d-\alpha} & \xrightarrow{i_{\alpha,d}} & C_d
\end{array}
\tag{0.5}
$$

Using the fact that j is injective, Lemma 0.1(iv) and that $\mathrm{Im} j = C \times x^\alpha = \pi_2^* x^\alpha$,

$$(i_{\alpha,d+1})_* A_{d-\alpha}(i_{\alpha,d}^* z) = (i_{\alpha,d+1})_* (\mu_{d-\alpha})_* \pi_{2,d-\alpha}^* i_{\alpha,d}^* z = \mu_* j_* j^* \pi_2^* z = \mu_* \big(\pi_2^* x^\alpha \pi_2^* z\big) = A_d(x^\alpha z)$$

and this proves (iii). To see (iv) set, for convenience, $C_0 = \{0\}$ and observe that $\pi_2^*(x^m) = C \times x^m$ whence, in C_{d+1},

$$\mu(C \times x^m) = \{\mu(p, mp_0 + D), p \in C, D \in C_{d-m}\} = \{p + mp_0 + D \in C_{d+1}, p \in C, D \in C_{d-m}\} = x^m$$

and being $C \times x^m$ irreducible we find that

$$\mu_* \pi_2^*(x^m) = \deg(\mu_{|C \times x^m}) x^m.$$

Now pick a general point $(p, mp_0 + D) \in C \times x^m$, so that either $m \le d-1$ and $D = p_1 + \ldots + p_{d-m}$ with $p \ne p_i$ for every $1 \le i \le d-m$ or $m = d$ and $D = 0$. Then either $m \le d-1$ and

$$\mu_{|C \times x^m}^{-1}(\mu_{|C \times x^m}((p, mp_0 + D)) = \{(p, mp_0 + D), (p_i, p + mp_0 + D - p_i), 1 \le i \le d-m\}$$

or $m = d$ and

$$\mu_{|C \times x^m}^{-1}(\mu_{|C \times x^m}((p, mp_0)) = \{(p, mp_0)\}$$

and therefore $\deg(\mu_{|C \times x^m}) = d + 1 - m$, giving (iv).

Alternatively, for $0 \le m \le d-1$, we can use (iii) to prove (iv). In fact, since $\deg \mu_d = d+1$ it is easy to see that $A_d([C_d]) = (d+1)[C_{d+1}]$ whence, setting $\alpha = m$ and $z = [C_d]$ in (iii) we have $A_d(x^m) = (i_{m,d+1})_* A_{d-m}([C_{d-m}]) = (i_{m,d+1})_*(d-m+1)[C_{d-m+1}] = (d+1-m)x^m$, giving again (iv) (for $0 \le m \le d-1$).

To prove (v) we can use the Macdonald's formula $\theta = (d+g-1)x - [\Delta_d]/2$ in C_d. Observe first that

$$\mu_* \pi_2^* x = dx \text{ and } \mu_* \pi_2^* [\Delta_d] = (d-1)[\Delta_{d+1}].$$

Then

$$\mu_* \pi_2^* \theta = (d+g-1)dx - \frac{d-1}{2}[\Delta_{d+1}] = (d-1)\big((d+g)x - \frac{1}{2}[\Delta_{d+1}]\big) + gx = (d-1)\theta + gx.$$

Therefore $A_d(\theta) = (d-1)\theta + gx$ and applying (iii) and Lemma 0.1(iv), we find

$$A_d(x^{m-1}\theta) = (i_{m-1,d+1})_* A_{d-m+1}(i_{m-1,d}^* \theta) = (i_{m-1,d+1})_* A_{d-m+1}(\theta) =$$
$$= (i_{m-1,d+1})_* \big((d-m)\theta + gx\big) = (i_{m-1,d+1})_* \big((d-m) i_{m-1,d+1}^* \theta + g i_{m-1,d+1}^* x\big) =$$
$$= (i_{m-1,d+1})_* i_{m-1,d+1}^* \big((d-m)\theta + gx\big) = x^{m-1}\big((d-m)\theta + gx\big) = (d-m)x^{m-1}\theta + gx^m$$

thus proving (v). □

Lemma 0.6. *([ACGH, Chap. VIII, part of Exercise D.3]) For every $\alpha \ge 0, \beta \ge 0$ such that $\alpha + \beta = m$ we have*

$$A_d(x^\alpha \theta^\beta) \in R^m(C_{d+1}).$$

Proof. First note that the lemma follows by Lemma 0.5(iv) if $\beta = 0$. Therefore we can assume $\alpha \le d-1$. We first reduce to the case $\alpha = 0$. In fact assume that

$$A_d(\theta^m) \in R^m(C_{d+1}) \text{ for every } 0 \le m \le d. \tag{0.6}$$

Now let $\alpha \ge 0, \beta \ge 0$ be such that $\alpha + \beta = m$ (and $\alpha \le d-1$). Then $0 \le \beta \le d - \alpha$, whence we can write, in $R^\beta(C_{d-\alpha+1})$,

$$A_{d-\alpha}(i_{\alpha,d}^* \theta^\beta) = A_{d-\alpha}(\theta^\beta) = \sum_{j=0}^{\beta} b_j x^j \theta^{\beta-j} = i_{\alpha,d+1}^* \big(\sum_{j=0}^{\beta} b_j x^j \theta^{\beta-j}\big)$$

hence, using using Lemma 0.5(iii) and Lemma 0.1(iv),

$$A_d(x^\alpha\theta^\beta)=(i_{\alpha,d+1})_*A_{d-\alpha}(i^*_{\alpha,d}\theta^\beta)=(i_{\alpha,d+1})_*i^*_{\alpha,d+1}\big(\sum_{j=0}^{\beta}b_jx^j\theta^{\beta-j}\big)=$$

$$=x^\alpha\big(\sum_{j=0}^{\beta}b_jx^j\theta^{\beta-j}\big)=\sum_{j=0}^{\beta}b_jx^{j+\alpha}\theta^{\beta-j}\in R^m(C_{d+1}).$$

We now prove (0.6) by induction on m.

The case $m=0$ follows again by Lemma 0.5(iv), whence we suppose $m\geq 1$. Moreover, since $\theta^m=0$ if $m\geq g+1$ we can also assume that $m\leq g$. Let us see that we can assume that $A_d(x^p\theta^q)\in R^m(C_{d+1})$ for every $0\leq p\leq d-1, 0\leq q\leq m-1$ and such that $p+q=m$. In fact, by induction,

$$A_{d-p}(i^*_{p,d}\theta^q)=A_{d-p}(\theta^q)=\sum_{j=0}^{q}b_jx^j\theta^{q-j}=i^*_{p,d+1}\big(\sum_{j=0}^{q}b_jx^j\theta^{q-j}\big)$$

hence, using Lemma 0.1(iv) and Lemma 0.5(iii),

$$A_d(x^p\theta^q)=(i_{p,d+1})_*A_{d-p}(i^*_{p,d}\theta^q)=(i_{p,d+1})_*i^*_{q,d+1}\big(\sum_{j=0}^{q}b_jx^j\theta^{q-j}\big)=$$

$$=x^p\big(\sum_{j=0}^{q}b_jx^j\theta^{q-j}\big)=\sum_{j=0}^{p}b_jx^{j+p}\theta^{q-j}\in R^m(C_{d+1}).$$

Now if $d-m<m$ then $r(m)=d-m$ and [BKLV17, Prop. 2.4(i)] gives the basis $\{x^{m-j}\theta^j, 0\leq j\leq d-m\}$ of $R^m(C_d)$, whence we can write

$$\theta^m=\sum_{j=0}^{d-m}b_jx^{m-j}\theta^j$$

for some real numbers b_j, and therefore, using the above assumptions,

$$A_d(\theta^m)=\sum_{j=0}^{d-m}b_jA_d(x^{m-j}\theta^j)\in R^m(C_{d+1})$$

and (0.6) is proved in this case.

If $m\leq d-m$ then $r(m)=m$. In particular, being $m\geq 1$, we have $d-m\geq 1$. Consider the following partition of d:

$$\underline{a}=(a_1,\dots,a_{d-m}) \text{ with } a_k=2 \text{ for } 1\leq k\leq m \text{ and } a_k=1 \text{ for } m+1\leq k\leq d-m.$$

Let $\Delta_{\underline{a}}$ be the corresponding m-codimensional diagonal. Consider, for $s\in\mathbb{N}$, the polynomials

$$\binom{T}{s}=\begin{cases}\frac{T(T-1)\dots(T-s+1)}{s!} & \text{if } s>0,\\ 1 & \text{if } s=0,\end{cases}$$

$$Q^{\underline{a}}_s(T)=(d-m-s)!s!\binom{g-T}{s}\sigma_s(\underline{a-1}),$$

and

$$P_{\underline{a}}(T)=\sum_{s=0}^{m}Q^{\underline{a}}_s(T)$$

where σ_s is the s-th elementary symmetric function (with the convention that $\sigma_0=1$) and $\underline{a-1}=(a_1-1,\dots,a_{d-m}-1)$. Since $\deg\binom{T}{s}=s$ we have $\deg Q^{\underline{a}}_s(T)\leq s$ and $\deg Q^{\underline{a}}_m(T)=m$ since $\sigma_m(\underline{a-1})=1$, whence the coefficient of T^m is

$$(-1)^m(d-2m)!\sigma_m(\underline{a-1})=(-1)^m(d-2m)!\neq 0.$$

Therefore also $\deg P_{\underline{a}}(T) = m$ and call p_m its leading coefficient. Set

$$c_j = \frac{(-1)^j}{j!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} P_{\underline{a}}(i).$$

By [BKLV17, Lemma 3.2] we have

$$c_m = \frac{(-1)^m}{m!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} P_{\underline{a}}(i) = p_m \neq 0.$$

By [BKLV17, Prop. 3.1] we have that

$$[\Delta_{\underline{a}}] = 2^m \sum_{j=0}^{m} c_j x^{m-j} \theta^j. \tag{0.7}$$

Using [BKLV17, Rmk. 2.8] we see that, setting $\underline{a}' = (\underline{a}, 1)$ as a partition of $d+1$,

$$A_d(\Delta_{\underline{a}}) = \{D + p, D \in \Delta_{\underline{a}}, p \in C\} = \Delta_{\underline{a}'}. \tag{0.8}$$

Since $[\Delta_{\underline{a}'}] \in R^m(C_{d+1})$ by [BKLV17, Prop. 3.1] and $A_d([\Delta_{\underline{a}}]) = e[A_d(\Delta_{\underline{a}})]$ for some $e \in \mathbb{Z}$ by [BKLV17, Rmk. 2.8], we find by (0.8) that $A_d([\Delta_{\underline{a}}]) \in R^m(C_{d+1})$ and (0.7) gives

$$\sum_{j=0}^{m} c_j A_d(x^{m-j}\theta^j) \in R^m(C_{d+1}).$$

By induction we know that $A_d(x^{m-j}\theta^j) \in R^m(C_{d+1})$ for $j < m$, whence we deduce that $c_m A_d(\theta^m) \in R^m(C_{d+1})$ and therefore the lemma is proved since $c_m \neq 0$. $\square$

Lemma 0.7. *([ACGH, Chap. VIII, first part of exercise D.7]) For every $\alpha \geq 0, \beta \geq 0, \lambda \geq 0, \delta \geq 0$ such that $\alpha + \beta = m, \lambda + \delta = d + 1 - m$ we have, in C_{d+1}*[2]

$$\begin{aligned} A_d(x^\alpha \theta^\beta) \cdot x^\lambda \theta^\delta &= \lambda x^{\alpha+\lambda} \theta^{\beta+\delta} + \delta(g - \delta + 1) x^{\alpha+\lambda+1} \theta^{\beta+\delta-1} = \\ &= \lambda(\beta+\delta)! \binom{g}{\beta+\delta} + \delta(g-\delta+1)(\beta+\delta-1)! \binom{g}{\beta+\delta-1}. \end{aligned}$$

Proof. The second equality follows from the formula [BKLV17, Lemma 2.2] $x^{d+1-s}\theta^s = s!\binom{g}{s}$ for every $s \in \mathbb{N}$. To see the first equality observe that it is enough to prove the case $\alpha = 0$, that is

$$A_d(\theta^m) \cdot x^\lambda \theta^\delta = \lambda x^\lambda \theta^{m+\delta} + \delta(g-\delta+1) x^{\lambda+1} \theta^{m+\delta-1} = \lambda(m+\delta)! \binom{g}{m+\delta} + \delta(g-\delta+1)(m+\delta-1)! \binom{g}{m+\delta-1}. \tag{0.9}$$

since then we can write, by (0.9),

$$\begin{aligned} A_{d-\alpha}(i^*_{\alpha,d}\theta^\beta) \cdot x^\lambda \theta^\delta = A_{d-\alpha}(\theta^\beta) \cdot x^\lambda \theta^\delta &= \lambda x^\lambda \theta^{\beta+\delta} + \delta(g-\delta+1) x^{\lambda+1} \theta^{\beta+\delta-1} = \\ &= i^*_{\alpha,d+1}\big(\lambda x^\lambda \theta^{\beta+\delta} + \delta(g-\delta+1) x^{\lambda+1} \theta^{\beta+\delta-1}\big) \end{aligned}$$

whence, using Lemma 0.5(iii) and Lemma 0.1(iv) we find

$$\begin{aligned} A_d(x^\alpha \theta^\beta) \cdot x^\lambda \theta^\delta &= \big((i_{\alpha,d+1})_* A_{d-\alpha}(i^*_{\alpha,d}\theta^\beta)\big) \cdot x^\lambda \theta^\delta = (i_{\alpha,d+1})_* \big(A_{d-\alpha}(i^*_{\alpha,d}\theta^\beta) \cdot i^*_{\alpha,d+1}(x^\lambda \theta^\delta)\big) = \\ &= (i_{\alpha,d+1})_* \big(A_{d-\alpha}(i^*_{\alpha,d}\theta^\beta) \cdot x^\lambda \theta^\delta\big) = (i_{\alpha,d+1})_* i^*_{\alpha,d+1}\big(\lambda x^\lambda \theta^{\beta+\delta} + \delta(g-\delta+1) x^{\lambda+1} \theta^{\beta+\delta-1}\big) = \\ &= x^\alpha\big(\lambda x^\lambda \theta^{\beta+\delta} + \delta(g-\delta+1) x^{\lambda+1} \theta^{\beta+\delta-1}\big) = \lambda x^{\alpha+\lambda} \theta^{\beta+\delta} + \delta(g-\delta+1) x^{\alpha+\lambda+1} \theta^{\beta+\delta-1}. \end{aligned}$$

Now to prove (0.9), using Lemma 0.5(i) and and Lemma 0.1(iv), we find (note that the products are all 0-cycles)

$$\begin{aligned} A_d(\theta^m) \cdot x^\lambda \theta^\delta &= (p_2)_*(\psi \cdot p_1^*\theta^m) \cdot x^\lambda \theta^\delta = (p_2)_*\big(\psi \cdot p_1^*\theta^m \cdot p_2^*(x^\lambda \theta^\delta)\big) = \psi \cdot p_1^*\theta^m \cdot p_2^*(x^\lambda \theta^\delta) = \\ &= \Psi_*\big(\Psi^*\big(p_1^*\theta^m \cdot p_2^*(x^\lambda \theta^\delta)\big)\big) = \Psi^* p_1^*\theta^m \cdot \Psi^* p_2^*(x^\lambda \theta^\delta) = \pi_2^*\theta^m (\mu^* x)^\lambda (\mu^*\theta)^\delta = \end{aligned}$$

[2] There is another typo in [ACGH, Chap. VIII, Exercise D.7], since the intersection takes place in C_{d+1}, not in C_{d-1}

$$(0.10) \qquad = \pi_2^*\theta^m(\eta+\pi_2^*x)^\lambda(g\eta+\pi_2^*\theta-\gamma)^\delta$$

Using Lemma 0.3(iii) we find

$$(\eta+\pi_2^*x)^\lambda = \pi_2^*x^\lambda+\lambda\pi_2^*x^{\lambda-1}\eta,$$
$$(\pi_2^*\theta-\gamma)^q = \pi_2^*\theta^q - q\pi_2^*\theta^{q-1}\gamma - q(q-1)\pi_2^*\theta^{q-1}\eta$$

and

$$(g\eta+\pi_2^*\theta-\gamma)^\delta = \pi_2^*\theta^\delta - \delta\pi_2^*\theta^{\delta-1}\gamma+\delta(g-\delta+1)\pi_2^*\theta^{\delta-1}\eta$$

and multiplying as in (0.10) we get

$$A_d(\theta^m)\cdot x^\lambda\theta^\delta = \pi_2^*\theta^m(\eta+\pi_2^*x)^\lambda(g\eta+\pi_2^*\theta-\gamma)^\delta =$$
$$= \pi_2^*(x^\lambda\theta^{m+\delta}) - \delta\pi_2^*(x^\lambda\theta^{m+\delta-1})\gamma+\delta(g-\delta+1)\pi_2^*(x^\lambda\theta^{m+\delta-1})\eta+\lambda\pi_2^*(x^{\lambda-1}\theta^{m+\delta})\eta =$$
$$(0.11) \quad = (\pi_2)_*\big(\pi_2^*(x^\lambda\theta^{m+\delta})-\delta\pi_2^*(x^\lambda\theta^{m+\delta-1})\gamma+\delta(g-\delta+1)\pi_2^*(x^\lambda\theta^{m+\delta-1})\eta+\lambda\pi_2^*(x^{\lambda-1}\theta^{m+\delta})\eta\big)$$

and using Lemma 0.3(iv), Lemma 0.1(ii) and the fact that $x^\lambda\theta^{m+\delta}=0$ in C_d since $m+\delta=d+1$, (0.11) becomes

$$\delta(g-\delta+1)x^\lambda\theta^{m+\delta-1}+\lambda x^{\lambda-1}\theta^{m+\delta} = \lambda(m+\delta)!\binom{g}{m+\delta}+\delta(g-\delta+1)(m+\delta-1)!\binom{g}{m+\delta-1}.$$

This proves (0.9) and therefore also the lemma. □

Lemma 0.8. *([ACGH, Chap. VIII, second part of exercise D.7]) For every $\alpha\geq 0,\beta\geq 0$ such that $\alpha+\beta=m$, we have*

$$A_d(x^\alpha\theta^\beta) = (d+1-\alpha-2\beta)x^\alpha\theta^\beta+\beta(g-\beta+1)x^{\alpha+1}\theta^{\beta-1}.$$

Proof. Again it is enough to prove the case $\alpha=0$, that is

$$(0.12) \qquad A_d(\theta^m) = (d+1-2m)\theta^m+m(g-m+1)x\theta^{m-1}$$

since then

$$A_{d-\alpha}(i^*_{\alpha,d}\theta^\beta) = A_{d-\alpha}(\theta^\beta) = (d-\alpha+1-2\beta)\theta^\beta+\beta(g-\beta+1)x\theta^{\beta-1} =$$
$$= i^*_{\alpha,d+1}\big((d-\alpha+1-2\beta)\theta^\beta+\beta(g-\beta+1)x\theta^{\beta-1}\big)$$

whence, using Lemma 0.5(iii) and Lemma 0.1(iv) we find

$$A_d(x^\alpha\theta^\beta) = (i_{\alpha,d+1})_*A_{d-\alpha}(i^*_{\alpha,d}\theta^\beta) = (i_{\alpha,d+1})_*i^*_{\alpha,d+1}\big((d-\alpha+1-2\beta)\theta^\beta+\beta(g-\beta+1)x\theta^{\beta-1}\big) =$$
$$= x^\alpha\big((d-\alpha+1-2\beta)\theta^\beta+\beta(g-\beta+1)x\theta^{\beta-1}\big) = (d+1-\alpha-2\beta)x^\alpha\theta^\beta+\beta(g-\beta+1)x^{\alpha+1}\theta^{\beta-1}.$$

To see (0.12), observing that we can assume that $m\leq g$, set

$$e_m = \begin{cases} 0 & \text{if } m\leq d+1-m,\\ 2m-d-1 & \text{if } m\geq d+1-m.\end{cases}$$

Now [BKLV17, Prop. 2.4(i)] gives the basis

$$\{x^{m-j}\theta^j, e_m\leq j\leq m\}$$

of $R^m(C_{d+1})$, whence, by Lemma 0.6, we can write

$$A_d(\theta^m) = \sum_{j=e_m}^m b_jx^{m-j}\theta^j$$

for some real numbers b_j, and therefore, using Lemma 0.7, we find, for $0\leq\delta\leq d+1-m$,

(0.13)
$$\sum_{j=e_m}^m b_jx^{d+1-j-\delta}\theta^{j+\delta} = (d+1-m-\delta)(m+\delta)!\binom{g}{m+\delta}+\delta(g-\delta+1)(m+\delta-1)!\binom{g}{m+\delta-1}$$

and we want to prove that (0.13) has the unique solution

$$(*) \quad b_j=0 \text{ for } j\neq m-1,m;\ b_{m-1}=m(g-m+1), b_m=d+1-2m.$$

Now the matrix
$$\left(x^{d+1-j-\delta}\theta^{j+\delta}\right)_{e_m\le j\le m, 0\le\delta\le d+1-m} = \left(x^{m-j}\theta^j\cdot x^{d+1-m-\delta}\theta^\delta\right)_{e_m\le j\le m, 0\le\delta\le d+1-m}$$
has rank $m-e_m+1$ equal to the number of rows by the proof of [BKLV17, Prop. 2.4], whence we just need to prove that $(*)$ is a solution. Plugging in we need to verify that, for every $0\le\delta\le d+1-m$,

$$m(g-m+1)(m-1+\delta)!\binom{g}{m-1+\delta}+(d+1-2m)(m+\delta)!\binom{g}{m+\delta} \tag{0.14}$$
$$=(d+1-m-\delta)(m+\delta)!\binom{g}{m+\delta}+\delta(g-\delta+1)(m+\delta-1)!\binom{g}{m+\delta-1}.$$

Now if $m-1+\delta\ge g+1$, (0.14) is just $0=0$. If $m-1+\delta\le g-1$, then (0.14) becomes
$$m(g-m+1)\frac{g!}{(g-m+1-\delta)!}+(d+1-2m)\frac{g!}{(g-m-\delta)!}=$$
$$=(d+1-m-\delta)\frac{g!}{(g-m-\delta)!}+\delta(g-\delta+1)\frac{g!}{(g-m+1-\delta)!}$$
that is
$$m(g-m+1)+(d+1-2m)(g-m+1-\delta)=(d+1-m-\delta)(g-m+1-\delta)+\delta(g-\delta+1)$$
and this is easily verified. If $m-1+\delta=g$, then (0.14) becomes
$$m(g-m+1)g!=\delta(g-\delta+1)g!$$
also verified. This proves (0.12) and therefore the lemma. □

References

[ACGH] A. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris: *Geometry of algebraic curves.* Vol. I. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 267. Springer-Verlag, New York, 1985. 2, 3, 4, 6, 7

[BKLV17] F. Bastianelli, A.F. Lopez, A. Kouvidakis, F. Viviani: *Effective cycles on the symmetric product of a curve, I: the diagonal cone.* Preprint 2017. 5, 6, 7, 8

[EH] D. Eisenbud, J. Harris : *3264 and all that?a second course in algebraic geometry.* Cambridge University Press, Cambridge, 2016. 1

[F] W. Fulton: *Intersection theory.* Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, Berlin, 1998. 1

[FL] M. Fulger, B. Lehmann. *Positive cones of dual cycle classes. Algebr. Geom. 4 (2017), no. 1, 1–28 1*