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ON THE EXISTENCE OF COMPONENTS OF THE NOETHER-LEFSCHETZ LOCUS WITH GIVEN CODIMENSION

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It is known that the codimension c of a component of the Noether-Lefschetz locus NL(d) satisfies $d-3 \leq c \leq \binom{d-1}{3}$. We prove that for $d \geq 47$ and for every integer $c \in [\frac{9}{2}d^{\frac{3}{2}}, \binom{d-1}{3}]$ there exists a component of NL(d) with codimension c. This is done with families of surfaces of degree d in $\mathbb{P}^3$ containing a curve lying on a cubic or on a quartic surface or a curve with general moduli. Moreover we produce an explicit example, for every $d \geq 4$, of components of maximum codimension $\binom{d-1}{3}$, thus giving a new proof of the fact that these components are dense in the locus of smooth surfaces (density theorem).

1. Introduction and statement of the main results

Let $\mathbb{P}^3$ be the projective space of dimension 3 over the complex numbers. For $d \geq 4$ we denote by $\mathbb{P}^N = \mathbb{P}^{\binom{d+3}{3}-1}$ the projective space whose points correspond to surfaces of degree d in $\mathbb{P}^3$ and by $S(d) \subseteq \mathbb{P}^N$ the open subset consisting of points corresponding to smooth surfaces.

By the Noether-Lefschetz theorem, there is a countable set of proper irreducible closed subvarieties of $S(d)$ such that for every point s outside the union of these subvarieties, the corresponding surface S has $\operatorname{Pic} S \cong \mathbb{Z}$ generated by $\mathcal{O}_S(1)$. The union of the mentioned subvarieties, i.e., the locus of surfaces with Picard group different from $\mathbb{Z}$, is called the Noether-Lefschetz locus and denoted $NL(d)$.

This paper concerns the study of the geometry of components of the Noether-Lefschetz locus started by M. Green, C. Voisin, C. Ciliberto, J. Harris and R. Miranda.

Let W be an irreducible component of $NL(d)$ and denote by c its codimension in $S(d)$. Then one has the inequalities

$$d-3 \leq c \leq p_g(d)$$

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where $p_g(d) = \binom{d-1}{3}$ is the geometric genus of any smooth surface of degree d in $\mathbb{P}^3$ (see [CGGH],[Gr1]). If $c \leq 2d-7$ for $d \geq 5$, then either $c = 2d-7$ (and W is the component of surfaces containing a plane conic) or $c = d-3$ (and W is the component of surfaces containing a line) (see [Gr2],[V1]). On the other hand, there are infinitely many components with $c = p_g(d)$ and their union is dense, in both the Zariski and the natural topology, in $S(d)$ (see [CHM]). We consider here the following question: Which integers $c \in [d-3, p_g(d)]$ actually occur as codimensions of a component?

It is the purpose of this paper to give a partial answer to the above question. We will prove the following theorem.

Theorem 1.1. *For any $d \geq 8$ there exists a component of the Noether-Lefschetz locus $NL(d)$ of codimension c for every integer c such that*

$$\min\left\{\frac{3}{4}d^2 - \frac{17}{4}d + \frac{19}{3}, \frac{9}{2}d^{\frac{3}{2}}\right\} \leq c \leq p_g(d).$$

Note that this result leaves the question open for lower codimensions, namely $c < \frac{9}{2}d^{\frac{3}{2}}$ for $d \geq 47$, in which case it is heuristically reasonable to expect gaps in the range of possible codimensions: To have small codimension means to have curves of small degree and hence few components of the Hilbert scheme.

By C. Voisin's result [V1], there is indeed a gap between $d-3$ and $2d-7$ for $d \geq 6$. The next expected gap is between $2d-7$ and $3d-12$ for $d \geq 7$ ($3d-12$ is the codimension of the locus of surfaces containing a plane cubic).

For any real number x, denote by $\lceil x \rceil$ the smallest integer $k \geq x$, and by $[x]$ the largest integer $k \leq x$. Now define $c_0(d) = \min\{k \in [d-3, p_g(d)] \cap \mathbb{Z} : \forall c \in [k, p_g(d)] \cap \mathbb{Z}$ there exists a component of codimension $c\}$. By Theorem 1.1 and [V1] we have

$$2d-7 \leq c_0(d) \leq \left\lceil \frac{9}{2}d^{\frac{3}{2}} \right\rceil \qquad \text{for } d \geq 47.$$

One weaker version of the question of finding the integers that can occur as codimensions would be to give $c_0(d)$ explicitly as a function of d. We have not attempted to address this question.

In this circle of ideas, it may also be worthwhile to recall the following *conjecture:* For $d \geq 5$ there are finitely many components of codimension strictly smaller than $p_g(d)$.

This is known to be true for $d = 5$ (see [V1]) and for $d = 6,7$ for reduced components (see [V2]). As for the proof of Theorem 1.1, the idea is to introduce some sufficient conditions for a component of the Hilbert scheme to give rise to a component of the Noether-Lefschez locus. Let us recall that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is said to be m-regular if $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0 \ \forall i > 0$. By Castelnuovo-Mumford's lemma ([Mu], p. 99) if $\mathcal{F}$ is m-regular, then the map

$H^0(\mathbb{P}^n, \mathcal{F}(k-1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^n, \mathcal{F}(k))$ is surjective for $k > m$ and $H^i(\mathbb{P}^n, \mathcal{F}(k)) = 0$ whenever $i > 0, k+i \geq m$. In particular $\mathcal{F}$ is $(m+1)$-regular. We will use this notion in the following basic lemma.

Lemma 1.2. *Let W be a component of Hilb$\mathbb{P}^3$ and consider the incidence correspondence*

$$\begin{array}{ccc} & \{(S,C) : C \subset S\} \subseteq \mathbb{P}^N \times W & \\ & {}^{\pi_1}\swarrow \quad \searrow^{\pi_2} & \\ & \mathbb{P}^N \qquad W & \end{array}$$

Let $W(d) = \operatorname{Im} \pi_1$ and let C be a curve representing the generic point of W. Suppose that C is smooth irreducible, that the ideal sheaf $\mathfrak{I}_C$ of C is $(d-1)$-regular and $H^1(\mathfrak{I}_C(d-4)) = 0$. Then $W(d)$ is a component of the Noether-Lefschetz locus and

$$\operatorname{codim}_{S(d)} W(d) = h^0(\mathcal{O}_C(d-4)) - \dim W + 4 \deg C.$$

Once we have this, a simple study of well-known components of Hilb$\mathbb{P}^3$, namely components of curves of general moduli (§2), of curves lying on a smooth cubic surface (§3) and on a smooth quartic surface (§4), together with the analysis of their codimensions, leads to the result.

Finally we use Lemma 1.2 to give an explicit construction of components of maximum codimension and therefore a new proof of the density theorem below.

Theorem 1.3 (Density Theorem). *The set of components of maximum codimension $p_g(d)$ is dense in $S(d)$ in the natural topology.*

The proofs of these three statements (Theorem 1.1, Lemma 1.2, Theorem 1.3) will be collected in §5.

2. Components given by curves with general moduli

Let n, g be integers such that $n \geq 3$, $0 \leq g \leq \frac{4n-12}{3}$. By Brill-Noether theory ([EH]) it is known that there is a component $W_{n,g} \subseteq H_{n,g,3}$ dominating $\mathcal{M}_g$ and such that if C is a general curve in $W_{n,g}$ and N_C is the normal bundle of C in $\mathbb{P}^3$, then $H^1(N_C) = 0$ ([Gi]) and C is maximal rank ([BE]). Let $d \geq 8$ such that $n(d-4) - \binom{d-1}{3} + 1 \leq g$; then

$$H^1(\mathfrak{I}_C(d-4)) = H^1(\mathcal{O}_C(d-5)) = 0, \tag{1}$$

i.e., $\mathfrak{I}_C$ is $(d-3)$-regular: In fact, $2g - 2 \leq \frac{2}{3}(4n-12) - 2 < n(d-5)$, hence $H^1(\mathcal{O}_C(d-5)) = 0$ and $\binom{d-1}{3} = h^0(\mathcal{O}_{\mathbb{P}^3}(d-4)) \geq n(d-4) - g + 1 = h^0(\mathcal{O}_C(d-4))$, therefore $H^1(\mathfrak{I}_C(d-4)) = 0$ by maximal rank.

By Lemma 1.2, $W_{n,g}$ gives rise to a component $W_{n,g}(d)$ of $NL(d)$ such that $c_1(n,g) = \operatorname{codim}_{S(d)} W_{n,g}(d) = h^0(\mathcal{O}_C(d-4)) = n(d-4) - g + 1$.

Proposition 2.1. *With the above inequalities $c_1(n,g)$ takes all the integral values in the interval $[\frac{3}{4}d^2 - \frac{17}{4}d + \frac{19}{3}, p_g(d)]$.*

Proof: We have $d \geq 8$, $n \geq 3$ and $\max\{0, n(d-4) - \binom{d-1}{3} + 1\} \leq g \leq \frac{4n-12}{3}$, hence the two domains $A_1 = \{(n,g) : 3 \leq n \leq \frac{1}{6}(d^2 - 2d + 3), 0 \leq g \leq \frac{4n-12}{3}\}$ and $B_1 = \{(n,g) : \frac{1}{6}(d^2 - 2d + 3) \leq n \leq \frac{d^3 - 6d^2 + 11d - 36}{6d-32}, n(d-4) - \binom{d-1}{3} + 1 \leq g \leq \frac{4n-12}{3}\}$. Set $\beta = \frac{1}{6}(d^2 - 2d + 3)$ and $\gamma = \frac{d^3 - 6d^2 + 11d - 36}{6d-32}$. Clearly $c_1(n,g)$ takes all the values in the set $X_{A_1} \cup X_{B_1}$, where

$$X_{A_1} = \bigcup_{3 \leq n \leq \beta} \left[n(d-4) - \frac{4n-12}{3} + 1, n(d-4) + 1 \right] \cap \mathbb{Z}$$

and

$$X_{B_1} = \bigcup_{\beta \leq n \leq \gamma} \left[n(d-4) - \frac{4n-12}{3} + 1, p_g(d) \right] \cap \mathbb{Z}.$$

Let $a_1(n) = n(d-4) - \frac{4n-12}{3} + 1$ and $b_1(n) = n(d-4) + 1$. Since they are both increasing functions of n, and

$$a_1(n+1) \leq b_1(n) \Leftrightarrow n \geq \frac{3}{4}d - 1,$$

we have

$$X_{A_1} \supseteq \left[a_1(\lceil \tfrac{3}{4}d - 1 \rceil), b_1([\beta]) \right] \cap \mathbb{Z}. \tag{2}$$

Write $d = 4k + \varepsilon$, $0 \leq \varepsilon \leq 3$; then $\lceil \frac{3}{4}d - 1 \rceil = \frac{3}{4}d - 1 + \frac{1}{4}\varepsilon \leq \frac{3}{4}d - \frac{1}{4}$, so

$$a_1\left(\left\lceil \frac{3}{4}d - 1 \right\rceil \right) \leq a_1\left(\frac{3}{4}d - \frac{1}{4} \right) = \frac{3}{4}d^2 - \frac{17}{4}d + \frac{19}{3}. \tag{3}$$

Now write instead $d = 6k + \varepsilon$, $0 \leq \varepsilon \leq 5$; then

$$[\beta] = \begin{cases} \beta & \text{if } \varepsilon = 3, 5 \\ \beta - \frac{5}{6} & \text{if } \varepsilon = 4 \\ \beta - \frac{\varepsilon^2 - 2\varepsilon + 3}{6} & \text{if } \varepsilon = 0, 1, 2 \end{cases} \geq \beta - \frac{5}{6},$$

hence

$$b_1([\beta]) \geq b_1(\beta - \frac{5}{6}) = \binom{d-1}{3} - \frac{5}{6}(d-4). \tag{4}$$

Putting together (2), (3) and (4) we get

$$X_{A_1} \supseteq \left[\frac{3}{4}d^2 - \frac{17}{4}d + \frac{19}{3}, p_g(d) - \frac{5}{6}(d-4) \right] \cap \mathbb{Z}. \tag{5}$$

Moreover, by the above computation we have $\lceil\beta\rceil = \beta + u$, where

$$u = \begin{cases} 0 & \text{if } \varepsilon = 3,5 \\ \frac{1}{6} & \text{if } \varepsilon = 4 \\ \frac{1}{2} & \text{if } \varepsilon = 0,2 \\ \frac{2}{3} & \text{if } \varepsilon = 1 \end{cases} \leq \frac{2}{3},$$

hence

$$a_1(\lceil\beta\rceil) = a_1(\beta + u) \leq a_1(\beta + \frac{2}{3}) = \frac{1}{18}(3d^3 - 22d^2 + 53d - 22)$$
$$= p_g(d) - \frac{2}{9}(d^2 - 5d + 1).$$

Therefore

$$X_{B_1} = [a_1(\lceil\beta\rceil), p_g(d)] \cap \mathbb{Z} \supseteq [p_g(d) - \frac{2}{9}(d^2 - 5d + 1), p_g(d)] \cap \mathbb{Z}.$$

This, together with (5), proves the proposition. ∎

3. Components given by curves lying on a smooth cubic surface

Let S be a smooth cubic surface in $\mathbb{P}^3$. As is well known, S is isomorphic to the blowing-up of $\mathbb{P}^2$ at six points, $P_1, \dots, P_6$, in general position, and, if $\pi : S \to \mathbb{P}^2$ is the blowing-up morphism, $\ell \in \operatorname{Pic} S$ is the class of $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $e_i \in \operatorname{Pic} S$ $i = 1, \dots, 6$, is the class of the exceptional divisor $\pi^{-1}(P_i)$, then $\operatorname{Pic} S \cong \mathbb{Z}^7$ generated by $\ell, -e_1, \dots, -e_6$ ([H1]). We will denote by $(a, b_1, \dots, b_6)$ the class of $a\ell - \sum_{i=1}^{6} b_i e_i \in \operatorname{Pic} S$ for $a, b_1, \dots, b_6 \in \mathbb{Z}$. For every $n \geq 13$, let us define the function

$$F(n) = \frac{n-9}{6}\sqrt{12n - 134} + 2n - \frac{83}{12}.$$

First we want to show the following.

Proposition 3.1. *For every pair of integers n, g such that $n \geq 13$ and $F(n) < g \leq \frac{1}{6}n(n-3) + 1$, there exists a smooth irreducible curve $C \subset S$ of degree n, genus g and such that $H^1(\mathcal{J}_C(3)) = 0$.*

In the proof of the above proposition, and also later on, we will use the ensuing lemma of Kleppe.

Lemma 3.2. *Let C be an effective divisor on S of type $(a, b_1, \dots, b_6)$ with $b_1 \geq b_2 \geq \dots \geq b_6$ and $a \geq b_1 + b_2 + b_3$. Then*

$$H^1(\mathcal{J}_C(\ell)) \neq 0 \iff \ell \in (b_6, 2a - \sum_{i=2}^{6} b_i - 1)$$

with the following two exceptions:

1. *if C is of type* $(\lambda,t)_1 = (\lambda+3t, \lambda+t, t, t, t, t, t)$ *for* $\lambda \geq 2$, *then*

$$H^1(\mathcal{I}_C(\ell)) \neq 0 \Longleftrightarrow \ell \in [b_6, 2a - \sum_{i=2}^{6} b_i - 1) = [t, 2\lambda + t - 1)$$

and

2. *if C is of type* $(\lambda,t)_2 = (3t, t, t, t, t, t, t-\lambda)$ *for* $\lambda \geq 2$, *then*

$$H^1(\mathcal{I}_C(\ell)) \neq 0 \Longleftrightarrow \ell \in (b_6, 2a - \sum_{i=2}^{6} b_i - 1] = (t-\lambda, t+\lambda-1].$$

Proof of Lemma 3.2: See [K], Prop. 3.1.3. ∎

Proof of Proposition 3.1: By Lemma 3.2 to find curves $C \subset S$ satisfying $H^1(\mathcal{I}_C(3)) = 0$, it is enough to find divisors of type $(a, b_1, \ldots, b_6)$ on S such that

$$(6) \qquad \begin{cases} a \geq b_1 + b_2 + b_3 \\ b_1 \geq b_2 \geq \cdots \geq b_6 \geq 3 \quad (\, b_6 \geq 4 \text{ if type } (\lambda,t)_1\,). \end{cases}$$

These two conditions guarantee that the general divisor of type $(a, b_1, \ldots, b_6)$ is smooth irreducible ([H1], V,4.12 and Ex. 4.8).

Now the proof of Proposition 3.1 is just a simple modification of Gruson and Peskine's proof of existence of smooth irreducible curves on a smooth cubic surface. We will use the same notation here as in [H2], §1.

Set $r = a - b_1$, $\alpha_i = \frac{1}{2}r - b_i$, $i = 2, \ldots, 6$. Since

$$n = 3a - \sum_{i=1}^{6} b_i$$
$$g = \frac{1}{2}(a^2 - \sum_{i=1}^{6} b_i^2 - n) + 1,$$

we have

$$a = \frac{1}{2}(n + \frac{3}{2}r - \sum_{i=2}^{6} \alpha_i)$$
$$b_i = \frac{1}{2}r - \alpha_i, \qquad i = 2, \ldots, 6,$$

with

$$(7) \quad \alpha_i \equiv \frac{1}{2}r \pmod 1,\ i = 2, \ldots, 6 \quad \text{and} \quad n + \frac{3}{2}r - \sum_{i=2}^{6} \alpha_i \equiv 0 \pmod 2.$$

The inequalities (6) translate into

$$(6)' \qquad \begin{cases} |\alpha_2| \le \alpha_3 \le \cdots \le \alpha_6 \le \frac{1}{2}r - 3 \\ -\alpha_2 + \alpha_3 + \cdots + \alpha_6 \le n - \frac{3}{2}r \end{cases}$$

and

$$(8) \qquad g = F_n(r) - \frac{1}{2}\sum_{i=2}^{6} \alpha_i^2$$

where $F_n(r) = \frac{1}{2}\big((r-1)n - \frac{3}{4}r^2\big) + 1$. With this notation setup, it is easy to see that the same proof of Lemma 1.1 of [H2] goes through to show that $\forall n, r, g$ such that $n \ge 13$, $\frac{17+\sqrt{12n-134}}{3} \le r \le \frac{2}{3}n$ and $F_n(r-1) < g \le F_n(r)$, there exist $\alpha_i \in \frac{1}{2}\mathbb{Z}$, $i = 2, \ldots, 6$ satisfying $(6)'$ and (7), such that g is given by (8).

Let us remark a numerical consequence:

(9) if one of the above divisors, i.e., given by $n, r, \alpha_2, \ldots, \alpha_6$, is of type $(\lambda, t)_1$ of Lemma 3.2, then $b_6 \ge 4$.

To see (9) we just have to observe that if $(a, b_1, \ldots, b_6) = (\lambda+3t, \lambda+t, t, t, t, t, t)$ for some $\lambda \ge 2$, then $r = 2t \ge \frac{17+\sqrt{22}}{3} \ge 7.2$, hence $r \ge 8$ and $b_6 = t \ge 4$. Hence (9) shows that even if among the divisors we found there are some of type $(\lambda, t)_1$, we still have $H^1(\mathcal{J}_C(3)) = 0$ (by Lemma 3.2).

To finish the proof of Proposition 3.1, let

$$r_0 = \left\lceil \frac{17+\sqrt{12n-134}}{3} \right\rceil \quad \text{and} \quad r_1 = \frac{1}{3}\sqrt{12n-134} + 7.$$

Then, by the above modification of Lemma 1.1 of [H2], we get that any integer g such that $F_n(r_0 - 1) < g \le F_n(\frac{2}{3}n) = \frac{1}{6}n(n-3) + 1$ is attained as in (8), i.e., as the genus of a smooth irreducible curve of degree n; since $r_1 > r_0$, any integer g such that

$$F(n) = \frac{n-9}{6}\sqrt{12n-134} + 2n - \frac{83}{12} = F_n(r_1) < g \le \frac{1}{6}n(n-3) + 1$$

is also attained. ∎

Corollary 3.3. *For every $n \ge 13$ and g such that $F(n) < g \le \frac{1}{6}n(n-3)+1$ there exists a component $K_{n,g} \subseteq H_{n,g,3}$ of dimension $n + g + 18$ such that its generic curve is as in Proposition 3.1.*

Proof: This is an easy consequence of the fact that $H^1(\mathcal{J}_C(3)) = 0$. One can use for example [K], Corol. 3.1.10. ∎

Our next goal is to construct, as usual with the help of Lemma 1.2, components of the Noether-Lefschetz locus, whose generic surface contains a curve lying on a cubic surface.

Lemma 3.4. *Let $n \geq 13$, g such that $F(n) < g \leq \frac{1}{6}n(n-3)+1$,*

$$d \geq \frac{1}{3}n + 3 + \frac{2}{3}\sqrt{n^2 - 3n + 6 - 6g}$$

and C a curve as in Proposition 3.1. Then

$$H^1(\mathfrak{I}_C(\ell)) = 0 \qquad \forall \ell \geq d-4, \tag{10}$$

and

$$\begin{cases} H^1(\mathcal{O}_C(d-4)) = 0 & \text{unless } C \text{ is a complete intersection of} \\ & S \text{ and } d = \frac{1}{3}n+3. \\ H^1(\mathcal{O}_C(d-3)) = 0 & \text{for every } C. \end{cases} \tag{11}$$

Proof: To show (10) we use Lemma 3.2. Therefore, with the exception of type $(\lambda, t)_2$, we need to show that $d - 4 \geq 2a - \sum_{i=2}^{6} b_i - 1$, i.e., with the notation of the proof of Proposition 3.1, that

$$d \geq 2a - \sum_{i=2}^{6} b_i + 3 = n - a + b_1 + 3 = n - r + 3.$$

By the proof of Proposition 3.1 we have $g \leq F_n(r)$, so $r \geq \frac{2}{3}n - \frac{2}{3}\sqrt{n^2-3n+6-6g}$, hence

$$n - r + 3 \leq \frac{1}{3}n + \frac{2}{3}\sqrt{n^2 - 3n + 6 - 6g} + 3 \leq d,$$

by hypothesis. Now if C is of type $(\lambda, t)_2$, we need to show that $d - 4 \geq t + \lambda$. Here $n = 3t + \lambda$, $g = \frac{1}{2}(3t^2 - \lambda^2 + 2\lambda t - 3t - \lambda + 2)$ and $n^2 - 3n + 6 - 6g = 4\lambda^2$, hence

$$d \geq \frac{1}{3}(3t + \lambda) + 3 + \frac{2}{3}\sqrt{4\lambda^2} = t + \frac{5}{3}\lambda + 3 \geq t + \lambda + 4$$

since $\lambda \geq 2$. As for (11) we notice that

$$n(d-4) \geq n\Big(\frac{1}{3}n - 1 + \frac{2}{3}\sqrt{n^2 - 3n + 6 - 6g}\Big) \geq 2g - 2,$$

with both equalities holding if and only if $g = \frac{1}{6}n(n-3) + 1$ and $d = \frac{1}{3}n + 3$. ∎

Corollary 3.5. *Let n, g, d be three integers such that $n \geq 13$, $F(n) < g \leq \frac{1}{6}n(n-3)+1$ and $d \geq \frac{1}{3}n+3+\frac{2}{3}\sqrt{n^2-3n+6-6g}$. Then there exists a component $K_{n,g}(d)$ of the Noether-Lefschetz locus $NL(d)$ such that*

$$c_3(n,g) = \operatorname{codim}_{S(d)} K_{n,g}(d) = n(d-1) - 2g - 17$$

except if $g = \frac{1}{6}n(n-3)+1, d = \frac{1}{3}n+3$, *in which case* $c_3(n,g) = n(d-1)-2g-16$.

Proof: By Corollary 3.3 and Lemma 3.4 we know that there is a component $K_{n,g}$ of the Hilbert scheme whose generic curve C satisfies the hypotheses of Lemma 1.2. In fact (10) and (11) imply that $H^1(\mathcal{J}_C(d-2)) = H^2(\mathcal{J}_C(d-3)) = 0$, i.e., that $\mathcal{J}_C$ is $(d-1)$-regular. By Lemma 1.2 we have

$$\begin{aligned}\operatorname{codim}_{S(d)} K_{n,g}(d) &= n(d-4) - g + 1 - \dim K_{n,g} + 4n + h^1(\mathcal{O}_C(d-4)) \\ &= n(d-1) - 2g - 17 + h^1(\mathcal{O}_C(d-4)).\end{aligned}$$

By (11) of Lemma 3.4 we have that $H^1(\mathcal{O}_C(d-4)) = 0$ unless C is a complete intersection of S and a surface of degree $\frac{n}{3}$ and $d = \frac{n}{3}+3$. In the latter case $\omega_C \cong \mathcal{O}_C(\frac{n}{3}-1) = \mathcal{O}_C(d-4)$, hence $h^1(\mathcal{O}_C(d-4)) = 1$. ∎

We now come to the study of the integral values attained by the function $c_3(n,g)$. To somewhat simplify the result and the proof, we will assume some suitable limitations on the variables n and d. Set

$$\alpha_1 = \frac{281 - 3F(23)}{1587}, \qquad \alpha = \frac{1}{\sqrt{\alpha_1}}$$

and let n, g, d be integers such that

$$\text{(12)} \qquad \begin{cases} d \text{ is even} \\ n \geq 23 \\ \max\{26, n\} \leq d \leq \alpha_1 n^2 \\ F(n) < g \leq \frac{1}{6}n(n-3) + 1. \end{cases}$$

Proposition 3.6. *For* n, g, d *satisfying (12), we have* $\frac{2}{3}\sqrt{n^2-3n+6-6g}+\frac{1}{3}n+3 \leq d$ *and the function* $c_3(n,g) = n(d-1) - 2g - 17$, *representing the codimension of the components of Corollary 3.5, takes at least all the integral values in the interval* $\left[\alpha d^{\frac{3}{2}}+2d-\frac{\alpha^2}{3}d-\frac{4\alpha}{3}d^{\frac{1}{2}}-\frac{61}{3}, d^2-2d-16-2F(d-1)\right)$.

Proof: First observe that for $n \geq 23$, we have

$$d \geq n \geq \frac{1}{3}n + 3 + \frac{2}{3}\sqrt{n^2 - 3n + 6 - 6F(n)} \geq \frac{1}{3}n + 3 + \frac{2}{3}\sqrt{n^2 - 3n + 6 - 6g},$$

therefore we are in the range of Corollary 3.5. Since $c_3(n,g) \equiv n+1 \pmod 2$, we see that $c_3(n,g)$ takes all the values in the set $X = A_3 \cup B_3$ where

$$A_3 = \bigcup_{\substack{n \text{ odd} \\ 23 \leq n \leq d \leq \alpha_1 n^2}} \left[n(d-1) - \frac{1}{3}n(n-3) - 19, n(d-1) - 2F(n) - 17\right) \cap (2\mathbb{Z})$$

and

$$B_3 = \bigcup_{\substack{n \text{ even} \\ 24 \leq n \leq d \leq \alpha_1 n^2}} \left[n(d-1) - \frac{1}{3}n(n-3) - 19, n(d-1) - 2F(n) - 17\right) \cap (\mathbb{Z} \setminus 2\mathbb{Z}).$$

Let us set

$$a_3(n) = n(d-1) - \frac{1}{3}n(n-3) - 19 \quad \text{and}$$
$$b_3(n) = n(d-1) - 2F(n) - 17.$$

From $d \geq n$ follows that $a_3'(n) \geq 0$ and $b_3'(n) \geq 0$, hence that they are increasing functions. Moreover it is easily seen that

$$a_3(n+2) \leq b_3(n) \Longleftrightarrow d \leq \frac{1}{6}(n^2 + n + 10 - 6F(n)) \tag{13}$$

and

$$\alpha_1 = \inf_{n \geq 23} \frac{n^2 + n + 10 - 6F(n)}{6n^2}. \tag{14}$$

Suppose first that n is odd and set $n = 2m+1$ with $\max\left\{11, \frac{\alpha d^{\frac{1}{2}} - 1}{2}\right\} \leq m \leq \frac{d-2}{2}$. By (14) we have $\frac{1}{6}(n^2 + n + 10 - 6F(n)) \geq \alpha_1 n^2 \geq d$, hence

$$a_3(2(m+1)+1) = a_3(n+2) \leq b_3(n) = b_3(2m+1)$$

by (13) and therefore

$$A_3 = \bigcup_{\max\left\{11, \frac{\alpha d^{\frac{1}{2}} - 1}{2}\right\} \leq m \leq \frac{d-2}{2}} \left[a_3(2m+1), b_3(2m+1)\right) \cap (2\mathbb{Z})$$

contains the set $\left[a_3(2\lceil \frac{\alpha d^{\frac{1}{2}} - 1}{2} \rceil + 1), b_3(d-1)\right) \cap (2\mathbb{Z})$. But

$$a_3(2\lceil \frac{\alpha d^{\frac{1}{2}} - 1}{2} \rceil + 1) \leq a_3\left(2(\frac{\alpha d^{\frac{1}{2}} - 1}{2} + 1) + 1\right) = a_3(\alpha d^{\frac{1}{2}} + 2) =$$

$$\alpha d^{\frac{3}{2}} + 2d - \frac{\alpha^2}{3}d - \frac{4\alpha}{3}d^{\frac{1}{2}} - \frac{61}{3}$$

therefore

$$A_3 \supseteq [a_3(\alpha d^{\frac{1}{2}} + 2), b_3(d-1)) \cap (2\mathbb{Z}). \tag{15}$$

Now let $n = 2m$ be even, with $\max\{12, \frac{\alpha d^{\frac{1}{2}}}{2}\} \leq m \leq \frac{d}{2}$. As above, by (14) and (13) we have $a_3(2(m+1)) = a_3(n+2) \leq b_3(n) = b_3(2m)$, hence

$$B_3 = \bigcup_{\max\left\{12, \frac{\alpha d^{\frac{1}{2}}}{2}\right\} \leq m \leq \frac{d}{2}} [a_3(2m), b_3(2m)) \cap (\mathbb{Z} \setminus 2\mathbb{Z})$$

contains the set $\left[a_3(2\lceil\frac{\alpha d^{\frac{1}{2}}}{2}\rceil), b_3(d)\right) \cap (\mathbb{Z}\setminus 2\mathbb{Z})$.
Since $a_3\left(2\lceil\frac{\alpha d^{\frac{1}{2}}}{2}\rceil\right) \leq a_3\left(2(\frac{\alpha d^{\frac{1}{2}}}{2}+1)\right) = a_3(\alpha d^{\frac{1}{2}}+2)$, we deduce

$$B_3 \supseteq [a_3(\alpha d^{\frac{1}{2}}+2), b_3(d)) \cap (\mathbb{Z}\setminus 2\mathbb{Z}). \tag{16}$$

Finally, since $b_3(d-1) \leq b_3(d)$, combining (15) and (16) we get

$$X \supseteq [a_3(\alpha d^{\frac{1}{2}}+2), b_3(d-1)) \cap \mathbb{Z}. \blacksquare$$

4. Components given by curves lying on a smooth quartic surface

Let n and g be integers such that

$$n \geq 17 \text{ and } 4n-32 \leq g < \frac{1}{8}n^2. \tag{17}$$

Following Mori ([Mo]), we will show the existence of smooth irreducible curves on a smooth quartic surface in $\mathbb{P}^3$ having some extra properties.

Proposition 4.1. *Let ℓ be an integer such that $\ell \geq \frac{1}{4}(n+3\sqrt{n^2-8g})$ or $\ell = 4$. Then for every n, g satisfying (17), there exists a smooth irreducible curve C of degree n, genus g, lying on a smooth quartic surface in $\mathbb{P}^3$ and such that $H^1(\mathcal{J}_C(\ell)) = 0$.*

Proof: Set $i_0 = \left[\frac{n-\sqrt{n^2-8g}}{4}\right] = \max\{i \geq 1 : g-in+2i^2 \geq 0, n-4i \geq 1\}$ and $n' = n-4i_0, g' = g-i_0n+2i_0^2$. Clearly we have $n' > 0$, $g' \geq 0$, $g' < \frac{1}{8}(n')^2$ and $(n', g') \neq (5,3)$: In fact if $n'=5, g'=3$, then $i_0 = \frac{n-5}{4}, 8g-n^2+1=0$, hence $i_0 = \left[\frac{n-1}{4}\right] = \frac{n-1}{4}$ which is a contradiction. Therefore by [Mo], Theorem 1, there exists a smooth irreducible curve C' of degree n', genus g' on a nonsingular quartic surface $X \subseteq \mathbb{P}^3$. Let H be the hyperplane section of X and let C be a generic element of the linear system $|C'+i_0H|$ on X. It is easy to show that $|C'+i_0H|$ is very ample (for example using a theorem of Saint-Donat; see [Mo], Theorem 5) because $i_0 \geq 4$ (since $g \geq 4n-32$). Therefore C is smooth irreducible and $\deg C = n'+4i_0 = n$, genus of $C = g'+2i_0^2+1+i_0n'-1 = g$. For every integer ℓ we have

$$H^1(\mathcal{J}_C(\ell)) = H^1(\mathcal{O}_X(\ell H-C)) = H^1(\mathcal{O}_X((\ell-i_0)H-C')) = H^1(\mathcal{J}_{C'}(\ell-i_0)). \tag{18}$$

Since $i_0 \geq 4$, (18) shows that $H^1(\mathcal{J}_C(4)) = 0$. On the other hand if $\ell \geq \frac{n}{4}+\frac{3}{4}\sqrt{n^2-8g}$, then $\ell-i_0 \geq \deg C'-2$: In fact

$$n-\ell \leq 3\Big(\frac{n-\sqrt{n^2-8g}}{4}\Big) < 3\Big(\Big[\frac{n-\sqrt{n^2-8g}}{4}\Big]+1\Big) = 3i_0+3$$

so $\ell-i_0 \geq n-4i_0-2 = \deg C'-2$. Hence $H^1(\mathcal{J}_C(\ell)) = 0$ follows by (18) and Castelnuovo's completeness theorem. $\blacksquare$

Corollary 4.2. *For every n, g satisfying (17), there exists a component $Q_{n,g} \subseteq H_{n,g,3}$ of dimension $g+33$ and such that its generic curve is as in Proposition 4.1.*

Proof: Again follows easily from $H^1(\mathcal{J}_C(4)) = 0$ and, for example, [K], Remark 3.1.12. ∎

To construct components of the Noether-Lefschetz locus whose generic surface contains a curve lying on a smooth quartic surface, we will use Lemma 1.2 again.

Lemma 4.3. *Let n, g, d be three integers such that $n \geq 17$, $4n - 32 \leq g < \frac{1}{8}n^2$ and $d \geq \frac{1}{4}n + 4 + \frac{3}{4}\sqrt{n^2 - 8g}$ and let C be a curve as in Proposition 4.1. Then*

$$H^1(\mathcal{J}_C(\ell)) = 0 \qquad \forall \ell \geq d-4, \tag{19}$$

$$H^1(\mathcal{O}_C(d-4)) = 0. \tag{20}$$

Proof: Since $d-4 \geq \frac{1}{4}n + \frac{3}{4}\sqrt{n^2 - 8g}$, (19) follows from Proposition 4.1. Moreover, $n(d-4) \geq \frac{1}{4}n^2 + \frac{3}{4}n\sqrt{n^2 - 8g} > 2g - 2$ since $g < \frac{1}{8}n^2$, therefore we get (20). ∎

Corollary 4.4. *For every n, g, d as in Lemma 4.3, there exists a component $Q_{n,g}(d) \subseteq NL(d)$ such that*

$$c_4(n,g) = \operatorname{codim}_{S(d)} Q_{n,g}(d) = nd - 2g - 32.$$

Proof: By Lemma 4.3 and Corollary 4.2 we know that there is a component $Q_{n,g}$ of the Hilbert scheme such that its generic curve C has the ideal sheaf $(d-2)$-regular. So Lemma 1.2 gives

$$\operatorname{codim}_{S(d)} Q_{n,g}(d) = n(d-4) - g + 1 - \dim Q_{n,g} + 4n = nd - 2g - 32. \ \blacksquare$$

As in the case of cubics, we will put some restriction on the variables n and d to study the values of $c_4(n,g)$. Namely, we will assume

$$\begin{cases} d \ \text{ is odd and } \ \geq 15 \\ 17 \leq n \leq d+8 \\ 4n - 32 \leq g < \frac{1}{8}n^2. \end{cases} \tag{21}$$

Proposition 4.5. *For n, g, d integers satisfying (21), we have $d \geq \frac{1}{4}n + 4 + \frac{3}{4}\sqrt{n^2 - 8g}$ and the function $c_4(n,g) = nd - 2g - 32$, representing the* *codimension of the components of Corollary 4.4, takes all the integral values in the interval*

$$[(2d-16)\sqrt{2d-16}+14d-80, d^2-d-24].$$

Proof: Since $g \geq 4n-32$ we have

$$\frac{1}{4}n+4+\frac{3}{4}\sqrt{n^2-8g} \leq \frac{1}{4}n+4+\frac{3}{4}\sqrt{(n-16)^2} = n-8 \leq d,$$

therefore we are in the range of Corollary 4.4. Now $c_4(n,g) \equiv n \pmod 2$, hence $c_4(n,g)$ takes all the values in the set $X = A_4 \cup B_4$ where

$$A_4 = \bigcup_{\substack{n \text{ odd:} \\ 17 \leq n \leq d+8}} (nd - \frac{1}{4}n^2 - 32, n(d-8)+32] \cap (\mathbb{Z} \setminus 2\mathbb{Z})$$

and

$$B_4 = \bigcup_{\substack{n \text{ even:} \\ 18 \leq n \leq d+7}} (nd - \frac{1}{4}n^2 - 32, n(d-8)+32] \cap (2\mathbb{Z}).$$

Set $a_4(n) = nd - \frac{1}{4}n^2 - 32$ and $b_4(n) = n(d-8)+32$. Then they are both increasing functions and satisfy

$$a_4(n+2) \leq b_4(n) \iff n \geq 14+2\sqrt{2d-16}. \tag{22}$$

If n is odd, set $n = 2m+1$, with $8 \leq m \leq \frac{d+7}{2}$. By (22) we have

$$a_4(2(m+1)+1) \leq b_4(2m+1)$$

if and only if $m \geq \frac{13}{2} + \sqrt{2d-16}$, hence

$$A_4 \supseteq \left(a_4\left(2\left\lceil\frac{13}{2}+\sqrt{2d-16}\right\rceil+1\right), b_4(d+8)\right] \cap (\mathbb{Z} \setminus 2\mathbb{Z}).$$

Since

$$a_4\left(2\left\lceil\frac{13}{2}+\sqrt{2d-16}\right\rceil+1\right) < a_4\left(2\left(\frac{13}{2}+\sqrt{2d-16}+1\right)+1\right) = a_4(16+2\sqrt{2d-16})$$

and $b_4(d+8) = d^2-32$ we get

$$A_4 \supseteq [a_4(16+2\sqrt{2d-16}), d^2-32] \cap (\mathbb{Z} \setminus 2\mathbb{Z}). \tag{23}$$

If n is even, set $n = 2m$, with $9 \leq m \leq \frac{d+7}{2}$. Again by (22) we get $a_4(2(m+1)) \leq b_4(2m)$ if and only if $m \geq 7+\sqrt{2d-16}$, hence

$$B_4 \supseteq (a_4(2\lceil 7+\sqrt{2d-16}\rceil), b_4(d+7)] \cap (2\mathbb{Z}).$$

Now $a_4(2\lceil 7+\sqrt{2d-16}\rceil) < a_4(2(8+\sqrt{2d-16})) = a_4(16+2\sqrt{2d-16})$ and $b_4(d+7) = d^2 - d - 24$; therefore

$$B_4 \supseteq [a_4(16+2\sqrt{2d-16}), d^2-d-24] \cap (2\mathbb{Z}). \tag{24}$$

Putting together (23) and (24) we get $X \supseteq [a_4(16+2\sqrt{2d-16}), d^2-d-24] \cap \mathbb{Z}$.
∎

5. The proofs of the theorems

Proof of Theorem 1.1: For $8 \le d \le 46$ we have $\frac{3}{4}d^2 - \frac{17}{4}d + \frac{19}{3} \le \frac{9}{2}d^{\frac{3}{2}}$, so we are done by Proposition 2.1. Suppose d odd ≥ 47. Then

$$(2d-16)\sqrt{2d-16} + 14d - 80 \le \frac{9}{2}d^{\frac{3}{2}} \le \frac{3}{4}d^2 - \frac{17}{4}d + \frac{19}{3} \le d^2 - d - 24,$$

hence Propositions 2.1 and 4.5 show that there is a component of the Noether-Lefschetz locus $NL(d)$ of any codimension c such that

$$c \in \left[\frac{9}{2}d^{\frac{3}{2}}, p_g(d)\right] \subseteq \left[(2d-16)\sqrt{2d-16} + 14d - 80, d^2-d-24\right] \cup \cup \left[\frac{3}{4}d^2 - \frac{17}{4}d + \frac{19}{3}, p_g(d)\right].$$

If d is even ≥ 48 we have

$$\alpha d^{\frac{3}{2}} + 2d - \frac{\alpha^2}{3}d - \frac{4\alpha}{3}d^{\frac{1}{2}} - \frac{61}{3} \le \frac{9}{2}d^{\frac{3}{2}} \le \frac{3}{4}d^2 - \frac{17}{4}d + \frac{19}{3} < d^2 - 2d - 16 - 2F(d-1),$$

hence Propositions 2.1 and 3.6 show that any c in the interval

$$\left[\frac{9}{2}d^{\frac{3}{2}}, p_g(d)\right] \subseteq \left[\alpha d^{\frac{3}{2}} + 2d - \frac{\alpha^2}{3}d - \frac{4\alpha}{3}d^{\frac{1}{2}} - \frac{61}{3}, d^2 - 2d - 16 - 2F(d-1)\right) \cup \cup \left[\frac{3}{4}d^2 - \frac{17}{4}d + \frac{19}{3}, p_g(d)\right]$$

occurs. ∎

Proof of Lemma 1.2: To see that $W(d)$ is a component of $NL(d)$, we observe first that by Castelnuovo-Mumford's lemma, the projective ideal of C in $\mathbb{P}^3$ is generated in degree less than d. If S is the generic surface of degree d containing C, then Corollary II.3.8 of [L] shows that Pic$S \cong \mathbb{Z}^2$ generated by $\mathcal{O}_S(1)$ and $\mathcal{O}_S(C)$. Let V be a component of $NL(d)$ containing $W(d)$ and S' be a surface representing its generic point. We can assume that there is a line bundle $\mathcal{L}'$ on S' that specializes to $\mathcal{L} = \mathcal{O}_S(C)$ when S' specializes (in V) to S. We will be done if we show that $\mathcal{L}'$ is effective and $h^0(\mathcal{L}') = h^0(\mathcal{L})$ (so that $\mathcal{L}'$ corresponds to a deformation of C).

By semicontinuity we have $h^0(\mathcal{L}') \leq h^0(\mathcal{L})$, $h^2(\mathcal{L}') \leq h^2(\mathcal{L})$, and $h^1(\mathcal{L}') \leq h^1(\mathcal{L}) = h^1(\mathcal{O}_S(C)) = h^1(\omega_S(-C)) = h^1(\mathfrak{I}_C(d-4)) = 0$, hence $h^1(\mathcal{L}') = 0$. Therefore

$$h^0(\mathcal{L}') = \mathcal{X}(\mathcal{L}') + h^1(\mathcal{L}') - h^2(\mathcal{L}') = \mathcal{X}(\mathcal{L}) - h^2(\mathcal{L}') \geq \mathcal{X}(\mathcal{L}) - h^2(\mathcal{L}) = h^0(\mathcal{L}).$$

So $h^0(\mathcal{L}') = h^0(\mathcal{L})$ and $\mathcal{L}'$ is thus effective.

Now from the incidence correspondence,

$$\begin{array}{ccc} & I = \{(S,C) : C \subset S\} \subseteq \mathbb{P}^N \times W & \\ & {}^{\pi_1}\swarrow \quad \searrow^{\pi_2} & \\ \operatorname{Im}\pi_1 = W(d) \subseteq \mathbb{P}^N & & W \end{array}$$

since S is the generic element of $W(d)$, we get

$$\begin{aligned} \operatorname{codim}_{S(d)} W(d) &= \binom{d+3}{3} - 1 - \dim \operatorname{Im}\pi_1 \\ &= \binom{d+3}{3} - 1 - \dim I + (h^0(\mathcal{O}_S(C)) - 1) \\ &= \binom{d+3}{3} - 1 - \dim I + h^1(\mathcal{O}_C(d-4)) \\ &= \binom{d+3}{3} - 1 - (\dim W + h^0(\mathfrak{I}_C(d)) - 1) + h^1(\mathcal{O}_C(d-4)) \\ &= h^0(\mathcal{O}_{\mathbb{P}^3}(d)) - \dim W - h^0(\mathfrak{I}_C(d)) + h^1(\mathcal{O}_C(d-4)) \end{aligned}$$

Since $\mathfrak{I}_C$ is $(d-1)$-regular, then $H^1(\mathfrak{I}_C(d)) = H^1(\mathcal{O}_C(d)) = 0$ by Castelnuovo-Mumford's lemma. Therefore the above computation gives

$$\begin{aligned} \operatorname{codim} W(d) &= h^0(\mathcal{O}_C(d)) + h^1(\mathcal{O}_C(d-4)) - \dim W \\ &= (deg C)d - g(C) + 1 + h^1(\mathcal{O}_C(d-4)) - \dim W \\ &= (d-4)\deg C - g(C) + 1 + h^1(\mathcal{O}_C(d-4)) - \dim W + 4\deg C \\ &= h^0(\mathcal{O}_C(d-4)) - \dim W + 4\deg C. \ \blacksquare \end{aligned}$$

As we claimed in the introduction, we proceed now to use Lemma 1.2 in order to explicitly construct a component of $NL(d)$ of maximum codimension for every $d \geq 4$. Let $C \subseteq \mathbb{P}^3$ be the projectively normal curve whose ideal is generated by the maximal minors of a $(d-2) \times (d-1)$ matrix M, with entries linear forms, i.e., the ideal sheaf of C is defined by the minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(1-d)^{d-2} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^3}(2-d)^{d-1} \to \mathfrak{I}_C \to 0.$$

Then it is well known that $H^1(N_C)=0$ where N_C is the normal bundle of C in $\mathbb{P}^3$ (see for example [Ga]) and hence that C belongs to a unique component W_1 of Hilb$\mathbb{P}^3$ of dimension $4\deg C$. By the above resolution, we see that $\mathcal{J}_C$ is $(d-1)$-regular and, of course, $H^1(\mathcal{J}_C(d-4))=0$.

Let $W_1(d)$ be the component of $NL(d)$ coming from W_1 as in Lemma 1.2. Note that the general surface in $W_1(d)$ has equation $\left| \begin{smallmatrix} L_{ij} \\ Q_j \end{smallmatrix} \right| = 0, i=1,\dots,d-2, j=1,\dots,d-1$ where the L_{ij}'s are linear forms and the Q_j's are quadratic forms.

Claim: $\operatorname{codim}_{S(d)} W_1(d) = p_g(d)$.

Proof: Since $H^1(N_C)=0$, then $\dim W_1 = 4\deg C$, hence Lemma 1.2 gives

$$\begin{aligned} p_g(d) - \operatorname{codim} W_1(d) &= \binom{d-1}{3} - h^0(\mathcal{O}_C(d-4)) \\ &= h^0(\mathcal{O}_{\mathbb{P}^3}(d-4)) - h^0(\mathcal{O}_C(d-4)) \\ &= h^0(\mathcal{J}_C(d-4)) = 0. \ \blacksquare \end{aligned}$$

As M. Green pointed out, the existence of just one component of maximum codimension is enough to prove the density of $NL(d)$ in $S(d)$ in the natural topology (see [CHM], §5). Therefore our Claim implies Theorem 1.3.

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