BIGNESS OF NEF $\mathbb{R}$-DIVISORS

In this note we will prove the following

Theorem 0.1. *Let X be a projective variety of dimension n over a field. Let D, D' be two nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X such that $D^n > nD^{n-1} \cdot D'$. Then $D - D'$ is big.*

Proof. Let A be an ample Cartier divisor on X. For $0 < \varepsilon \ll 1$ we have

$$(D + \varepsilon A)^n > n(D + \varepsilon A)^{n-1} \cdot (D' + \varepsilon A)$$

and $D - D' = (D + \varepsilon A) - (D' + \varepsilon A)$, so we can assume that both D and D' are ample.

Hence we can write $D = \sum\limits_{i=1}^{s} c_i A_i, D' = \sum\limits_{j=1}^{s'} c'_j A'_j$ where $c_i, c'_j \in \mathbb{R}^+$, A_i, A'_j ample Cartier divisors. Let $q_{i,m}, q'_{j,m} \in \mathbb{Q}^+$ be two sequences such that $q_{i,m} < c_i, q'_{j,m} > c'_j$ for all $m \in \mathbb{N}$ and $\lim\limits_{m\to\infty} q_{i,m} = c_i, \lim\limits_{m\to\infty} q'_{j,m} = c'_j$ for all i, j.

Set $D_{1,m} = \sum\limits_{i=1}^{s} q_{i,m} A_i, D'_{1,m} = \sum\limits_{j=1}^{s'} q'_{j,m} A'_j, D_{2,m} = D - D_{1,m}, D'_{2,m} = D'_{1,m} - D'$. Note that $D_{1,m}$ and $D'_{1,m}$ are ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors, while $D_{2,m}$ and $D'_{2,m}$ are ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisors. Moreover, in $NS^1_{\mathbb{R}}(X)$, we have

$$\lim_{m\to\infty} [D_{1,m}] = [D], \lim_{m\to\infty} [D'_{1,m}] = [D'].$$

Hence for $m \gg 0$ we have that $D^n_{1,m} > nD^{n-1}_{1,m} \cdot D'_{1,m}$, so that $D_{1,m} - D'_{1,m}$ is big by [L, Thm.2.2.15] (which holds over any field). Therefore also

$$D - D' = D_{1,m} - D'_{1,m} + D_{2,m} + D'_{2,m}$$

is big. □

Corollary 0.2. *Let X be a projective variety of dimension n over a field and let D be a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then D is big if and only if $D^n > 0$.*

Proof. If $D^n > 0$ setting $D' = 0$ we get immediately by Theorem 0.1 that D is big. Now assume that D is big. Let H be a very ample divisor on X, so that, by induction on n, we have $D^{n-1} \cdot H = D^{n-1}_{|H} > 0$. Moreover, as D is big, there is $\alpha \in \mathbb{R}^+$ and an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor E on X such that $D \equiv \alpha H + E$, whence, as D is nef, $D^{n-1} \cdot E \geq 0$. Therefore

$$D^n = D^{n-1} \cdot (\alpha H + E) > 0.$$

□

References

[L] R. Lazarsfeld. *Positivity in algebraic geometry. I.* Ergebnisse der Mathematik und ihrer Grenzgebiete **48**. Springer-Verlag, Berlin, 2004.