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On the positivity of Ulrich bundles

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Abstract

We study the positivity properties of Ulrich bundles defined with respect to an ample and globally generated polarization. First we prove a generalization of a theorem by Lopez on the first Chern class. Then, under some additional assumptions on the polarization, we give a description of its augmented base locus, which consequently leads to a characterization of the V-bigness and of the ampleness of an Ulrich bundle in this setting. Finally we study the normal generation of an Ulrich bundle focusing on curves, on surfaces with $q = p_g = 0$ and on hypersurfaces of dimension 2 and 3.

Contents

Introduction

A nowadays classical way to describe the geometry of a given smooth projective variety X is to study the class of vector bundles supported on it. The essential motivation is the fact that vector bundles enjoy some of those fundamental properties which gave line bundles a prominent role in the field of algebraic geometry. For example, as for line bundles and maps towards projective spaces, a (generically) globally generated vector bundle induces a (rational) map, called *Kodaira map*, from X towards a Grassmannian; or, similarly to the bijection between divisors and line bundles, the celebrated *Hartshorne-Serre correspondence* states that a closed Cohen-Macaulay subscheme $Z \subset X$ of pure codimension 2 can be obtained as the degeneracy locus of $r-1$ global sections of a rank $r \geq 2$ vector bundle on X with a fixed determinant L if (and only if) $\omega_Z(-K_X - L)$ is generated by $r-1$ global sections, provided that $H^2(X, -L) = 0$ (and $H^1(X, -L) = 0$), see [Mae90; Arr07]. Furthermore, completing the parallel with line bundles, it is well-known that we get more properties once we add some positivity assumptions on the vector bundles. For instance, the Kodaira map induced by a globally generated vector bundle $\mathcal{E}$ is finite if and only if $\det \mathcal{E}$ is ample, see [LN10]; or, all degeneracy loci $D_k(\varphi)$ of a sheaf morphism $\varphi\colon \mathcal{E} \to \mathcal{F}$ between locally free sheaves are connected as soon as $\dim X > (\mathrm{rk}(\mathcal{E}) - k)(\mathrm{rk}(\mathcal{F}) - k)$ (resp. $\dim X > (\mathrm{rk}(\mathcal{E}) - k)(\mathrm{rk}(\mathcal{F}) - k) + q$) provided that $\mathcal{E}^* \otimes \mathcal{F}$ is ample (resp. q-ample), see [FL81; Tu90]. Conversely, again just like for line bundles, the positivity properties of a given vector bundle are ruled by certain subschemes associated to it: the so-called *asymptotic base loci,* see [Bau+15]. It is therefore a common strategy to produce and to study vector bundles with some positivity properties in order to shed light on the geometry of a given smooth projective variety. A very special class of vector bundles which both strongly determine the geometry of the underlying variety and also possess some, very often much, positivity is the one of *Ulrich bundles.*

Given an embedded smooth projective variety $X \subset \mathbf{P}^N$, a vector bundle $\mathcal{E}$ on X is *Ulrich* if $H^i(X, \mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq \dim X$. Ulrich bundles were originally introduced in the framework of commutative algebra by Ulrich in [Ulr84] and started to be studied from an algebro-geometric point of view with Eisenbud and Schreyer after their extraordinary papers [ES03; ES11]. Besides the several nice properties enjoyed by these bundles (such as being globally generated, aCM and semistable), the existence of an Ulrich bundle has profound consequences on the geometry of the underlying variety, above all the determinantal representation of the Chow form (see [ES03, Theorem 0.3]) and having the same cone of cohomology tables of vector bundles as the projective space (see [ES11, Theorem 4.2]). Regardless of all these constraints, Eisenbud and Schreyer raised the question in [ES11], and later the conjecture in [ES11], on the existence of Ulrich bundles on any smooth embedded variety. (We refer to Appendix C for a short note on the history of Ulrich bundles). Despite all the attention received afterwards, this conjecture is still widely open.

In this work we are going to focus on Ulrich bundles on smooth projective varieties, but in a slightly more general setting: we allow the polarization to be just ample and globally generated. (We refer to [Vac25] for a study of Ulrich sheaves on projective schemes endowed with a globally generated ample line bundle).

Definition A (Definition-Theorem 1.1.2)**.** Let X be a complex smooth projective variety of dimension $n \geq 1$ and let B be a globally generated ample line bundle on X with $B^n = d$. A vector bundle $\mathcal{E}$ of rank r on X is *B-Ulrich,* or *Ulrich for* (X, B), if it satisfies one of the following equivalent conditions:

1. There exists a linear resolution
$$0 \to \mathcal{O}_{\mathbf{P}^N}(-c)^{\oplus b_c} \to \mathcal{O}_{\mathbf{P}^N}(-c+1)^{\oplus b_{c-1}} \to \cdots \to \mathcal{O}_{\mathbf{P}^N}^{\oplus b_0} \to (\varphi_B)_*\mathcal{E} \to 0$$
where $\varphi_B\colon X \to \mathbf{P}^N$ is the morphism associated to $|B|$ and $c = N - n$.
2. $H^i(X, \mathcal{E}(-pB)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$.
3. For all finite morphisms $\pi\colon X \to \mathbf{P}^n$ such that $\pi^*\mathcal{O}_{\mathbf{P}^n}(1) \cong B$ we have $\pi_*\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^n}^{\oplus rd}$.

Clearly we recover the usual definition of Ulrich bundles as soon as B is very ample. (In addition to this, observe that, letting $\varphi_B = i \circ \varphi\colon X \to \overline{X} \subset \mathbf{P}^N$ be the factorization onto the schematic image of φ_B, as $\Gamma_*(X, \mathcal{E}) \cong \Gamma_*(\overline{X}, \varphi_*\mathcal{E})$, by taking the associated modules in the resolution in Definition A.1 and invoking Proposition C.3.2, we see that $\mathcal{E}$ is Ulrich for (X, B) if and only if $\varphi_*\mathcal{E}$ is an Ulrich sheaf on $\overline{X} \subset \mathbf{P}^N$ in the sense of Definition C.3.1.) Furthermore, despite [ES03, Theorem 0.3] (Theorem C.3.3) no longer makes sense because one can't properly define a Chow form without an embedding, it is still easy to prove [ES11, Theorem 4.2] (Theorem C.3.4) in this setting (namely $C_{vb}(X, B) = C_{vb}(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ if and only if there exists an Ulrich bundle for (X, B)). Therefore it is worth to extend Eisenbud-Schreyer conjecture (Conjecture C.3.5) to Ulrich bundles defined with respect to a non-necessarily very ample globally generated polarization.

The choice of this setting has the following main motivations. First of all, globally generated polarizations, unlike very ample ones, are preserved under finite pullbacks and all the properties of the "usual Ulrich bundles" continue to hold even in this setting (see Section 1.1). Secondly, many smooth projective varieties, such as Del Pezzo manifolds of degree 2 and cyclic coverings of projective spaces, come with a natural base-point-free polarization which is not very ample. For example, in this direction it has been proved that all smooth double and triple covers of $\mathbf{P}^n$ support an Ulrich bundle [ST22; MNP25; Vac25]. Finally, as we are going to see, the theory obtained by relaxing this hypothesis on the polarization is different from the one of Ulrich bundles defined with respect to a hyperplane section. For example, we will find two Ulrich bundles respectively on a surface and on a threefold defined with respect to a non-very ample base-point-free polarization which are non-big (Eaxmples 3.0.27 - 3.0.28) but not ascribable to the classification of non-big Ulrich bundles on embedded surfaces and threefolds in [LM21].

The central theme of this thesis is the study of the positivity of Ulrich bundles defined with respect to a globally generated ample line bundle B. The first result is a generalization to this setting of a result by Lopez on the first Chern class [Lop22, Theorem 1]. The strategy is trying to follow the same arguments. In this way one gets aware of the differences with the usual setting. The first issue we face is the loss of the separation properties of a hyperplane section. In order to get the analogous features, it becomes necessary to study the local positivity of the polarization. To this end, we will consider Seshadri constants $\varepsilon(B; -)$ of B along certain sets of points (Definition B.2.1) which will help to understand what are the obstructions arising in this setting. It turns out that these obstructions are certain curves associated to B which are known as *Seshadri curves* (see Lemmas 3.0.4 - 3.0.5 and Definition B.2.1). At the end we will show that if the variety is not (generically) covered by such curves, then the first Chern class is positive and the B-Ulrich bundle is (V-)big.

Theorem B (Theorem 3.0.16)**.** *Let X be a smooth projective variety of dimension $n \geq 1$ and let B be a globally generated ample line bundle on X with $B^n = d$. Let $\varphi_B \colon X \longrightarrow \mathbf{P}^N$ be the finite morphism induced by $|B|$ and denote by* $\mathrm{Ram}(\varphi_B)$ *its ramification locus. Let $\mathcal{E}$ be a vector bundle of rank r which is 0-regular with respect to B. Then*

$$c_1(\mathcal{E})^k \cdot Z \geq r^k \mathrm{mult}_x(Z)$$

holds for every $x \in X$ and for every subvariety $Z \subset X$ of dimension $k \geq 1$ passing through x provided that the following conditions are satisfied:

(a) $x \notin \mathrm{Ram}(\varphi_B)$,

(b) $\varepsilon(B; \varphi_B^{-1}(\varphi_B(x))) > 1$.

In particular, if X is not generically covered by 1-Seshadri curves for φ_B, then $\mathcal{E}$ is V-big and

$$c_1(\mathcal{E})^n \geq r^n.$$

Moreover, if $\mathcal{E}$ is B-Ulrich of rank $r \geq 2$, then

$$c_1(\mathcal{E})^n \geq r(d-1).$$

The second main result of this work is the characterization of one of the asymptotic base loci (Definition A.3.3) of an Ulrich bundle $\mathcal{E}$. Thanks to the global generation, the *stable base locus* $\mathbf{B}(\mathcal{E})$ and the *restricted base locus* $\mathbf{B}_{-}(\mathcal{E})$ of $\mathcal{E}$, which respectively measure the semiampleness and the nefness of $\mathcal{E}$, are always empty. Under some additional assumptions on the polarization B, which still (strictly) include all very ample line bundles, we can completely describe the *augmented base locus* $\mathbf{B}_{+}(\mathcal{E})$ of $\mathcal{E}$ in terms of *B-lines*, which are nothing but curves $\Gamma \subset X$ with $B \cdot \Gamma = 1$ (Definition 3.0.9): provided the existence of a linear series $|V| \subseteq |B|$ inducing a morphism which is étale onto its schematic image, $\mathbf{B}_{+}(\mathcal{E})$ is the union of all B-lines on which $\mathcal{E}$ is not ample. As soon as we weaken this hypothesis on B, specifically if we do not suppose the morphism to be unramified, this characterization may no longer hold (see Remarks 4.0.17 - 4.0.18). The main tool to get this result will be the Seshadri constants $\varepsilon(\mathcal{E}; x)$ of $\mathcal{E}$ at a point x (Definition B.2.15), systematically formalized in the maximum generality for the first time in [FM21], and the consequent characterization of the augmented base locus $\mathbf{B}_{+}(\mathcal{F})$ of a nef vector bundle $\mathcal{F}$ as the set of points $y \in X$ where $\varepsilon(\mathcal{F}; y) = 0$ (see Remark B.2.17).

Theorem C (Theorem 4.0.11)**.** *Let X be a smooth projective variety and let B be a globally generated ample line bundle such that there is a linear series $|V| \subseteq |B|$ inducing a morphism* *which is étale onto its schematic image. Let $\mathcal{E}$ be a B-Ulrich bundle on X and let $x \in X$ be a point. Then $\varepsilon(\mathcal{E}; x) = 0$ if and only if there exists a B-line $\Gamma \subset X$ passing through x such that $\mathcal{E}_{|\Gamma}$ is not ample on Γ. In particular,*

$$\mathbf{B}_+(\mathcal{E}) = \bigcup_{\Gamma \subset X} \Gamma$$

where Γ ranges over all B-lines in X such that $\mathcal{E}_{|\Gamma}$ is not ample on Γ.

Since the augmented base locus of a vector bundle rules V-bigness (Definition A.3.6) and ampleness, a simple consequence of Theorem C is the characterization of these properties for a B-Ulrich bundle in this setting.

Corollary D (Corollary 4.0.12)**.** *Let X be a smooth projective variety and let B be a globally generated ample line bundle such that there is a linear series $|V| \subseteq |B|$ inducing a morphism which is étale onto its schematic image. Let $\mathcal{E}$ be a B-Ulrich bundle. Then:*

(a) $\mathcal{E}$ is V-big if and only if X is not covered by B-lines $\Gamma \subset X$ on which $\mathcal{E}_{|\Gamma}$ is not ample.

(b) $\mathcal{E}$ is ample if and only if either X contains no B-lines or $\mathcal{E}_{|\Gamma}$ is ample on every B-line $\Gamma \subset X$.

Theorem C and Corollary D match the expectations. Indeed, as suggested by [Lop22, Theorem 1] and by [LS23, Theorem 1] for Ulrich bundles on embedded smooth projective varieties, by Remark 3.0.18 already Theorem B tells that the main obstructions to V-bigness, hence parts of the augmented base locus, are represented by B-lines.

Regarding the ampleness of an Ulrich bundle defined with respect to a polarization as above, we can be more precise. All the technical results about the "separation properties" of the bundle, needed for Theorem 4.0.11, lead to a (very slight) generalization of [LS23, Theorem 1] in this setting. We point out that also the following result may fail without the aforementioned assumption on B (see Remark 4.0.19).

Theorem E (Theorem 4.0.20)**.** *Let X be a smooth projective variety and let B be a globally generated ample line bundle such that there is a linear series $|V| \subseteq |B|$ inducing a morphism which is étale onto its schematic image. Let $\mathcal{E}$ be a B-Ulrich bundle on X. Then the following are equivalent:*

(1) $\mathcal{E}$ is 1-very ample.

(2) $\mathcal{E}$ is very ample.

(3) $\mathcal{E}$ is ample.

(4) Either X contains no B-lines or $\mathcal{E}_{|\Gamma}$ is ample on every B-line $\Gamma \subset X$.

Theorem E as well as already Lopez-Sierra theorem [LS23, Theorem 1] tell that an Ulrich bundle is likely to be very ample. Then it is natural to understand the embedding of the corresponding projective bundle through the (complete) linear system of the tautological line bundle. As is already a very classical and relevant question to determine when a very ample linear system embeds a variety as a projectively normal scheme in some projective space, the last part of this thesis is devoted to the study of the projective normality of an Ulrich bundle, that is, by definition, the normal generation of its tautological line bundle

(Definitions B.3.1 - 5.0.5). As we will see in Chapter 5, an unified result as Lopez-Sierra theorem for very ampleness appears out of reach. Ulrich bundles on curves are projectively normal if the degree of the polarization is big with respect to the genus, with an optimal bound in some cases. However this is no longer true in higher dimension, for instance on hypersurfaces where the behaviour of projective normality of Ulrich bundles suggests that it is unlikely to get a general criterion. The main results, which will be on low-dimensional varieties where at least Castelnuovo-Mumford regularity is well-behaved with respect to tensor operations, are the following.

Theorem F (Theorem 5.0.1)**.** *Let C be a smooth projective curve of genus g and let B be a globally generated ample line bundle of degree d on C. Let $\mathcal{E}$ be a B-Ulrich bundle on C. Then:*

(a) *$\mathcal{E}$ is projectively normal if $d > g + 1$.*

(b) *$\mathcal{E}$ satisfies (N_1) and $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is Koszul if $d > g + 2$.*

(c) *$\mathcal{E}$ satisfies (N_p) for $p \geq 2$ if $d > \frac{1}{2}\left((g + p + 1) + \sqrt{g^2 + 2g(3p + 1) + (p - 1)^2}\right)$.*

(d) *If there exists a linear series $|V| \subseteq |B|$ which induces a morphism which is étale onto the schematic image, the general B-Ulrich bundle of rank r on C is projectively normal as soon as C supports a non-special normally generated line bundle of degree d. This holds in particular if $d \geq g + 2 - \mathrm{Cliff}(C)$.*

(e) *If C is general of genus $g \geq 3$ and B is a general very ample line bundle of degree*

$$d \geq \frac{3 + \sqrt{8g + 1}}{2},$$

then the general B-Ulrich bundle of rank r is projectively normal. Moreover this bound is sharp for $r = 1$.

Theorem G (Theorem 5.0.2)**.** *Let $S \subset \mathbf{P}^N$ be a smooth projective surface with $q(S) = p_g(S) = 0$ and let $\mathcal{E}$ be an ample 0-regular vector bundle of rank $r \geq 2$ on S such that $h = h^0(S, \mathcal{E}) \geq r + 3$. Let $E = \det(\mathcal{E})$ be the determinant bundle and let $\ell = \binom{h-r}{2} - 1$. The following are equivalent:*

(1) *$\varphi_{\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)} : \mathbf{P}(\mathcal{E}) \subset \mathbf{P}(H^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)))$ is not aCM.*

(2) *$\mathcal{E}$ is not projectively normal.*

(3) *There exist a closed subscheme $Z \subset S$ and a non-zero divisor $D \subset S$ such that:*

  (a) *Z is smooth of dimension 0.*

  (b) *Z is the degeneracy locus of ℓ general sections $s_1, \ldots, s_\ell \in H^0(S, \Lambda^2 M_{\mathcal{E}}^*)$.*

  (c) *$[Z] = \frac{1}{2}(h - r - 2)\left((h - r + 1)c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\right)$.*

  (d) *$D \in |K_S + (h - r - 1)E|$.*

  (e) *$Z \subset D$.*

(4) *There exist a closed subscheme $Z \subset S$ and a curve $C \subset S$ such that:*

(f) *Z is the degeneracy locus of ℓ general sections $\sigma_1, \dots, \sigma_\ell \in H^0(S, \Lambda^2 M^*_{\mathcal{E}})$.*
(g) *C is the degeneracy locus of the $(\ell + 1)$ general sections $\sigma_1, \dots, \sigma_\ell, \sigma_{\ell+1} \in H^0(S, \Lambda^2 M^*_{\mathcal{E}})$.*
(h) *$C \in |(h - r - 1)E|$ is smooth and irreducible.*
(i) *$Z \subset C$ is a special (effective) divisor.*

Theorem H (Theorem 5.0.3)**.** *Let $X \subset \mathbf{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 3$ with $2 \leq n \leq 3$ and let $\mathcal{E}$ be an Ulrich bundle of rank r on X. Let*

$$\mu_{\mathcal{E}} \colon H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{E}) \to H^0(X, \mathcal{E} \otimes \mathcal{E})$$

denote the multiplication of sections. Then:

(a) *If $n = 2$ and $\det(\mathcal{E}) = \mathcal{O}_X(\frac{r}{2}(d-1))$, then $\mu_{\mathcal{E}}$ cannot be surjective and $\mathcal{E}$ cannot be projectively normal if $d \geq 5$, or $d = 4$ and $r \leq 5$, or $d = 3$ and $r \leq 2$.*

(b) *If $n = 3$ and $d \geq 4$, then $\mu_{\mathcal{E}}$ is never surjective and $\mathcal{E}$ cannot be projectively normal if $r > \frac{d+4}{3}$.*

As mentioned above, the behavior of the projective normality on (low-dimensional) hypersurfaces is the most unexpected: a general hypersurface contains no lines if its degree is greater than the double of its dimension, therefore Ulrich bundles are expected to be always very positive (at least in this situation). Theorem H and Remark 5.3.5 seem to suggest that just low-rank Ulrich bundles are (potentially) projectively normal. However, by Buchweitz-Greuel-Schreyer conjecture C.1.3 the rank of Ulrich bundles is expected to be very large (see also [LR24b] for the non-existence of low-rank Ulrich bundles on hypersurfaces). Therefore, against the expectations, it seems that Ulrich bundles on hypersurfaces are rarely projectively normal.

Finally, despite the main goal of this work is not the construction of Ulrich bundles on a given projective variety, we mention a result of existence in a very special case. As mentioned at beginning, producing Ulrich bundles is a very challenging problem. However rational homogeneous varieties offer a large test class of vector bundles: the equivariant bundles. Irreducible equivariant Ulrich bundles on $X = G/P$ with $\mathrm{Pic}(X) = \mathbf{Z} \cdot \mathcal{O}_X(1)$ have been fully classified in [CM15; Fon16; LP21] for any G. Except for Grassmannians, i.e. for $G = A_n$, such bundles are very rarely Ulrich. Indeed, on most of the varieties of type $G = B_n, C_n, D_n, E, F_4, G_2$ such bundles are never Ulrich. However, equivariant bundles are not necessarily irreducible, so there is still the chance that such varieties support an equivariant Ulrich bundle. However, merely three such examples have been found, specifically on $G_2/P_1, F_4/P_4, E_6/P_1$, and they are obtained in a non-constructive way as restrictions to hyperplane sections [LP21, Remark 4.2 & §6.1 & Corollary 7.4]. Moreover $G_2/P_1 \cong Q_5$ is well-known to support Ulrich bundles as it is a quadric, and E_6/P_1 already has an irreducible equivariant Ulrich bundle [LP21, Proposition 5.1]. In this direction, we prove the following existence result where we explicitly find reducible equivariant Ulrich bundles on rational homogeneous varieties where there are no irreducible equivariant Ulrich bundles (and known Ulrich bundles). This result will help to prove that almost all prime Mukai varieties support an Ulrich bundle (Corollary 2.0.3).

Proposition I (Proposition 2.0.2)**.** *The spinor tenfold $\mathbb{S}_{10} \subset \mathbf{P}^{15}$ and the Lagrangian grassmannian $\mathrm{LGr}(3,6) \subset \mathbf{P}^{13}$ support a μ-stable reducible equivariant Ulrich bundle.*

This thesis is divided in two parts plus an appendix.

Part I contains facts of general interest on Ulrich bundles. Chapter 1 is a summary of the main properties of Ulrich bundles defined with respect to an ample globally generated line bundle. Chapter 2 deals with equivariant Ulrich bundles on rational homogeneous varieties.

Part II presents the main results of this thesis on the positivity of Ulrich bundles. In Chapter 3 and in Chapter 4, both of them based on [But24b], we study respectively the positivity of the first Chern class and the augmented base locus of an Ulrich bundle. In Chapter 5, based on [But24a], we investigate the projective normality of Ulrich bundles.

The Appendix is dedicated to a recollection of background notions, in Appendix A, and of technical results, in Appendix B, and to a short historical note about the origin of Ulrich bundles, in Appendix C.

Notations and conventions

In the following we will mostly adopt the notations introduced in [Har77] and the following more specific conventions:

- All schemes are separated and of finite type over the field of complex numbers $\mathbf{C}$.
- A variety is an integral scheme and subvarieties are always closed.
- Except for generic points of a scheme, points are exclusively closed.
- We use $\subset$ and $\subseteq$ as synonyms.
- A scheme X is *smooth* if it is regular. This is equivalent to saying that the structural morphism $X \to \operatorname{Spec}(\mathbf{C})$ is smooth ([Liu02, Example 3.2.3 & Corollary 4.3.33 & Proposition 6.2.2] and [Sta23, Tag 04QN & Tag 056S]).
- Divisors are always Cartier divisors. On irreducible varieties we identify every divisor D with its associated line bundle $\mathcal{O}_X(D)$.
- Given a line bundle L on a scheme X, the morphism induced by a linear system $\delta = |V|$, for a non-zero sub-vector space $V \subset H^0(X, L)$, will be denoted by $\varphi_V = \varphi_\delta \colon X \to \mathbf{P}^N$. If $\delta = |L|$, we will simply write φ_L.
- Given a linear series $|V|$ of a line bundle L on a variety X, its *base locus* $\operatorname{Bs}(|V|)$ is the closed subset cut out by the *base ideal* $\mathfrak{b}(|V|) = \operatorname{Im}(V \otimes L^* \to \mathcal{O}_X)$. If we want to emphasize the scheme structure, we will refer to $\operatorname{Bs}(|V|)$ as the *base scheme* of $|V|$.
- For a projective scheme X, the line bundle $\mathcal{O}_X(1)$ will always be associated to the hyperplane section H of an embedding $X \subset \mathbf{P}^N$.
- Given a line bundle A, we will write $\mathcal{F}(pA) = \mathcal{F} \otimes A^{\otimes p}$ for every sheaf $\mathcal{F}$ and for every $p \in \mathbf{Z}$.
- A *polarization* is meant to be an ample line bundle.
- A line bundle L is *strictly nef* if $L \cdot C > 0$ for every irreducible curve C.
- We write $h^i(X, -), \operatorname{ext}^i(-,-)$ to indicate respectively $\dim_{\mathbf{C}} H^i(X, -), \dim_{\mathbf{C}} \operatorname{Ext}^i(-,-)$.
- Given two schemes X_1, X_2, we will always write $\pi_i \colon X_1 \times X_2 \to X_i$ for the projection onto the i-th factor. If $\mathcal{F}_i$ is a sheaf on X_i, we will write $\mathcal{F}_1 \boxtimes \mathcal{F}_2 = \pi_1^*\mathcal{F}_1 \otimes \pi_2^*\mathcal{F}_2$.
- The ideal sheaf of a (closed) point x in a scheme X will be denoted by $\mathfrak{m}_x$. The ideal sheaf of a closed subscheme $Y \subset X$ will be denoted by $\mathcal{I}_{Y/X}$.

- Given a closed subscheme $Y \subset X$, by *"blow-up of X along Y"* we will mean the blow-up of X along the ideal sheaf $\mathcal{I}_{Y/X}$.
- A *generic* property is a property of the generic point. A property is *general* if it holds in the complement of a proper (Zariski) closed subset. A property is *very general* if it is satisfied off a countable union of proper (Zariski) closed subsets.
- Let $X \subset \mathbf{P}^N$ be an embedded projective scheme. We denote by

  $$I_{X/\mathbf{P}^N} = \bigoplus_{t \in \mathbf{Z}} H^0(\mathbf{P}^N, \mathcal{I}_{X/\mathbf{P}^N}(t))$$

  the *homogeneous saturated ideal* of X in $\mathbf{P}^N$ and by

  $$R_X = \mathbf{C}[x_0, \ldots, x_N]/I_{X/\mathbf{P}^N}$$

  the *homogeneous coordinate ring* of X.
- For a vector bundle $\mathcal{E}$ on a scheme X we set $\mathbf{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}))$ for the projective bundle of $\mathcal{E}$ and we denote by $\pi\colon \mathbf{P}(\mathcal{E}) \to X$ the natural projection.
- Given a globally generated vector bundle $\mathcal{E}$ on a scheme X, we call

  $$M_{\mathcal{E}} = \ker\left(H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E}\right)$$

  the *syzygy bundle* of $\mathcal{E}$ and we call the exact sequence

  $$0 \to M_{\mathcal{E}} \to H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E} \to 0,$$

  the *syzygy exact sequence* of $\mathcal{E}$.

Part I

Generalities on Ulrich bundles

Chapter 1

Ulrich vector bundles

1.1 Definitions and first properties

This section contains a summary of the general properties of Ulrich bundles. All the following results can be found in [ES03; Cas+12; CKM13; Cos17a; Bea18; CMP21; Vac25].

We will need the following technical lemma of general nature. It states some important properties which will belong to Ulrich bundles. See Appendix B.1 for a brief introduction to Castelnuovo-Mumford regularity.

Lemma 1.1.1. *Let X be a smooth projective variety of dimension $n \geq 1$ and let B be a globally generated ample line bundle with $B^n = d$. Let $\mathcal{E}$ be a coherent sheaf on X of $\operatorname{rk}(\mathcal{E}) = r$ satisfying $H^i(X, \mathcal{E}(-p)) = 0$ for $i \geq 0$ and $1 \leq p \leq n$. Then:*

(i) $\mathcal{E}$ is 0-regular with respect to B and is generated by global sections.

(ii) The Hilbert polynomial of $\mathcal{E}$ with respect to B is

$$P(\mathcal{E}, m) := \chi(X, \mathcal{E}(mB)) = \frac{rd}{n!}(m+1)\cdots(m+n),$$

and $h^0(X, \mathcal{E}) = rd$.

In particular, if $(X, B) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$, then $\mathcal{E}$ is isomorphic to a trivial vector bundle.

Proof. The cohomology groups $H^i(X, \mathcal{E}(-i))$ clearly vanish for $i > n$ by Grothendieck's theorem, and also for $0 < i \leq n$ by assumption. Then $\mathcal{E}$ is 0-regular, so globally generated by Castelnuovo-Mumford theorem B.1.3. By the same theorem, we also know that $\mathcal{E}$ is k-regular for any $k \geq 0$. Consequently, by Remark B.1.4, we have $H^i(X, \mathcal{E}) = 0$. For the second part of the claim, recall that the Hilbert polynomial has degree less or equal than n in m with

$$P(\mathcal{E}, m) := \chi(X, \mathcal{E}(mB)) = r\frac{d}{n!}m^n + O(m^{n-1}),$$

see for instance [Laz04a, Theorem 1.1.24]. Our hypothesis implies that $P(\mathcal{E}, m)$ has exactly n roots in $m = -1, -2, \ldots, -n$, so that $P(\mathcal{E}, m)$ factors as the desired product. Finally, evaluating $P(\mathcal{E}, m)$ in $m = 0$ immediately yields $rd = P(\mathcal{E}, 0) = \chi(X, \mathcal{E}) = h^0(X, \mathcal{E})$.

For the last assertion, observe that, in this case, $d = 1$ and $h^0(\mathbf{P}^n, \mathcal{E}) = r$. Being globally generated means that there is a short exact sequence

$$0 \to \mathcal{K} = M_{\mathcal{E}} \to H^0(\mathbf{P}^n, \mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^n} \to \mathcal{E} \to 0.$$

If $\mathcal{K}$ is nonzero, then it is 0-regular: tensoring the above sequence through by $\mathcal{O}_{\mathbf{P}^n}(-i)$ for any $0 < i \le n$ and then taking the cohomology, we get

$$0 = H^{i-1}(\mathbf{P}^n, \mathcal{E}(-i)) \to H^i(\mathbf{P}^n, \mathcal{K}(-i)) \to H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-i))^{\oplus r} = 0,$$

which gives the claim. Therefore $\mathcal{K}$ is globally generated by Castelnuovo-Mumford theorem B.1.3. However one has $H^0(\mathbf{P}^n, \mathcal{K}) = 0$ given that $H^0(\mathbf{P}^n, \mathcal{E}) \otimes H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) \to H^0(\mathbf{P}^n, \mathcal{E})$ is an isomorphism by construction. Therefore $\mathcal{K}$ must be 0, so $\mathcal{E}$ is isomorphic to the trivial vector bundle of rank r. □

We are now ready to introduce Ulrich bundles.

Definition - Theorem 1.1.2. *Let X be a smooth projective variety of dimension $n \ge 1$ and let B be a globally generated ample line bundle. Set $d = B^n$, and let $\mathcal{E}$ be a vector bundle of rank r. The following conditions are equivalent:*

1. *There exists a linear resolution*

$$0 \to L_c \to L_{c-1} \to \cdots \to L_0 \to \varphi_*\mathcal{E} \to 0,$$

where $\varphi = \varphi_B \colon X \to \varphi(X) = \overline{X} \subset \mathbf{P}^N$, and $L_i = \mathcal{O}_{\mathbf{P}^N}(-i)^{\oplus b_i}$ and $c = N - n$.

2. *The cohomology groups $H^i(X, \mathcal{E}(-pB))$ vanish for $i \ge 0$ and $1 \le p \le \dim X$.*

3. *For all finite surjective morphism $\pi \colon X \to \mathbf{P}^n$ such that $\pi^*\mathcal{O}_{\mathbf{P}^n}(1) \cong B$, the sheaf $\pi_*\mathcal{E}$ is isomorphic to $\mathcal{O}_{\mathbf{P}^n}^{\oplus rd}$.*

A vector bundle satisfying these equivalent conditions is said to be Ulrich with respect to B, *or simply* B-Ulrich*, or* Ulrich for (X, B).

Proof. Suppose condition 3 holds and take π as the composition of $\varphi = \varphi_B \colon X \to \varphi(X) = \overline{X} \subset \mathbf{P}^N$ with a finite linear projection $\overline{X} \to \mathbf{P}^n$ from a point in $\mathbf{P}^N - \overline{X}$ onto a linear subspace of dimension n. Note that by the projection formula we have, for all k,

$$\pi_*\left(\mathcal{E}(kB)\right) \cong \pi_*\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^n}(k) = (\pi_*\mathcal{E})(k).$$

Since π is affine, there is a natural isomorphism $H^i(X, \mathcal{E}(-pB)) \cong H^i(\mathbf{P}^n, (\pi_*\mathcal{E})(-p))$ for all $i \ge 0$ [Har77, Exercise III.8.2]. Since $\pi_*\mathcal{E}$ is trivial, the cohomology group on the left vanishes for $i \ge 0$ and $1 \le p \le n$, hence 2.

Conversely, assume 2 holds, and let π be a finite and surjective morphism from X onto $\mathbf{P}^n$. By the so-called miracle flatness [Har77, Exercise III.9.3(a)], π is flat, hence $\pi_*\mathcal{E}$ is a vector bundle of rank $r \cdot \deg(\pi) = s$. On the other hand, also the vector bundle $\pi_*\mathcal{E}$ satisfies the vanishings in 2 due to the finiteness of π. Therefore $\pi_*\mathcal{E}$ is trivial (Lemma 1.1.1) of rank s. On the other hand, part 2 of Lemma 1.1.1 gives that $h^0(X, \mathcal{E}) = rd$. Since $H^0(\mathbf{P}^n, \pi_*\mathcal{E}) = H^0(X, \mathcal{E})$ are vector spaces of dimension s and rd respectively, we must have $s = rd$.

Now, suppose there exists a linear resolution as in 1. Write $\varphi_*\mathcal{E} = \overline{\mathcal{E}}$. Then

$$\varphi_*(\mathcal{E} \otimes B^{\otimes k}) \cong \varphi_*\left(\mathcal{E} \otimes \varphi^*\mathcal{O}_{\mathbf{P}^N}(k)\right) \cong \varphi_*\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^N}(k) = \overline{\mathcal{E}}(k)$$

for all k, and the finiteness of φ, the coherent sheaves $\mathcal{E}(kB)$ and $\overline{\mathcal{E}}(k)$ have the same cohomology. To prove 2, we will use [Laz04a, Proposition B.1.2]. So, fix $0 \leq i \leq n$ and $1 \leq p \leq n$, twist the resolution by (the flat sheaf) $\mathcal{O}_{\mathbf{P}^N}(-p)$. Then it is enough to show that $H^{i+j}(\mathbf{P}^N, L_j(-p)) \cong H^{i+j}(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(-j-p))^{\oplus b_j}$ vanishes for each $0 \leq j \leq c$. This is clearly true for $i + j \neq N$; for $i + j = N$, noting that

$$j - N + p - 1 \leq c - N + p - 1 = -n + p - 1 \leq -p + p - 1 = -1,$$

we get the conclusion from the equality $h^N(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(-j-p)) = h^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(j - N + p - 1))$ due to Serre duality.

Finally, let us prove the converse. We define inductively a sequence of 0-regular coherent sheaves $\mathcal{K}_i$ on $\mathbf{P}^N$ for $0 \leq i \leq c$ such that:

(A) $\mathcal{K}_0 = \overline{\mathcal{E}}$,

(B) $\mathcal{K}_{i+1}(-1) = \ker\left(\mathrm{ev}\colon H^0(\mathbf{P}^N, \mathcal{K}_i) \otimes \mathcal{O}_{\mathbf{P}^N} \to \mathcal{K}_i\right)$,

(C) $H^s(\mathbf{P}^N, \mathcal{K}_i(-t)) = 0$ for $1 \leq t \leq n + i$ and all $s \geq 0$.

By hypothesis, the finiteness of φ and Lemma 1.1.1, the sheaf $\mathcal{K}_0 = \overline{\mathcal{E}}$ is 0-regular and satisfies (C). Suppose $\mathcal{K}_i$ are defined for $0 \leq i \leq \ell$, and set $\mathcal{K}_{\ell+1}$ as in (B). Since $\mathcal{K}_\ell$ is 0-regular, it is globally generated by Castelnuovo-Mumford theorem B.1.3. Hence, by construction, there is a short exact sequence

$$0 \to \mathcal{K}_{\ell+1}(-1) \to H^0(\mathbf{P}^N, \mathcal{K}_\ell) \otimes \mathcal{O}_{\mathbf{P}^N} \to \mathcal{K}_\ell \to 0. \tag{1.1}$$

We need to verify that $\mathcal{K}_{\ell+1}$ is 0-regular and satisfies (C). To this end, take $1 \leq t \leq n + \ell + 1$ and tensor the above sequence by $\mathcal{O}_{\mathbf{P}^N}(-t+1)$. From the associated long exact sequence we can extract, for $s \geq 1$,

$$0 \to H^0(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-t)) \to H^0(\mathbf{P}^N, \mathcal{K}_\ell) \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(-t+1)) \to H^0(\mathbf{P}^N, \mathcal{K}_\ell(-t+1)), \tag{1.2}$$

$$H^{s-1}(\mathbf{P}^N, \mathcal{K}_\ell(-t+1)) \to H^s(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-t)) \to H^s(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(-t+1))^{\oplus h^0(\mathbf{P}^N, \mathcal{K}_\ell)}. \tag{1.3}$$

If $t \geq 2$, since $1 \leq t-1 \leq n+\ell$ and $-t+1 \geq -n-c+1 = 1-N > -N-1$, the terms on the left, by induction, and on the right, by the cohomology of projective spaces, in (1.3) are both zero. So $H^s(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-t)) = 0$ for $s \geq 1$. The same holds for $H^0(\mathbf{P}^N, \mathcal{K}_\ell(-t+1))$ if $t \geq 2$ because $\mathcal{O}_{\mathbf{P}^N}(-t+1)$ has no global sections in these cases. Now consider $t = 1$. The map on the right in (1.2) becomes an isomorphism, thus $H^0(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-1)) = H^1(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-1)) = 0$. If $s \geq 2$, using that $H^q(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}) = 0$ for $q \geq 1$, we deduce from the short exact sequence

$$0 \to H^{s-1}(\mathbf{P}^N, \mathcal{K}_\ell) \to H^s(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-1)) \to 0$$

that $H^s(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-1)) \cong H^{s-1}(\mathbf{P}^N, \mathcal{K}_\ell)$ for every $s \geq 2$. But $\mathcal{K}_\ell$ is 0-regular by induction, hence $H^q(X, \mathcal{K}_\ell) = 0$ for all $q > 0$ by Remark B.1.4. As a consequence $H^s(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-1)) = 0$ for $s \geq 2$, saying that $\mathcal{K}_{\ell+1}$ satisfies (C). To complete the inductive step, it remains to verify that $\mathcal{K}_{\ell+1}$ is 0-regular. To do this, consider $0 < q \leq N$ and tensor (1.1) by $\mathcal{O}_{\mathbf{P}^N}(-q+1)$. Using that $\mathcal{K}_\ell$ is 0-regular, we see from the long exact sequence that $H^q(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-q)) = 0$ for $q \geq 2$ since

$$0 = H^{q-1}(\mathbf{P}^N, \mathcal{K}_\ell(-q+1)) \to H^q(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-q)) \to H^0(\mathbf{P}^N, \mathcal{K}_\ell) \otimes H^q(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(-q+1)) = 0.$$

Finally, if $q = 1$, the morphism $H^0(\mathbf{P}^N, \mathcal{K}_\ell) \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}) \to H^0(\mathbf{P}^N, \mathcal{K}_\ell)$ is an isomorphism and $H^1(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}) = 0$. This forces $H^1(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-1))$ to be 0 thanks to the exactness of

$$H^0(\mathbf{P}^N, \mathcal{K}_\ell) \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}) \to H^0(\mathbf{P}^N, \mathcal{K}_\ell) \to H^1(\mathbf{P}^N, \mathcal{K}_{\ell+1}(-1)) \to H^1(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N})^{\oplus h^0(\mathbf{P}^N, \mathcal{K}_\ell)}.$$

This shows that $\mathcal{K}_{\ell+1}$ is 0-regular ending the inductive step.
Now, setting $L_i = \mathcal{O}_{\mathbf{P}^N}(-i)^{\oplus h^0(\mathbf{P}^N, \mathcal{K}_i)}$ for $0 \leq i \leq c-1$, we get short exact sequences

$$0 \to \mathcal{K}_{i+1}(-i-1) \to L_i \to \mathcal{K}_i(-i) \to 0$$

by construction. All together, they give a long exact sequence

$$0 \to \mathcal{K}_c(-c) \to L_{c-1} \to \cdots \to L_0 \to \overline{\mathcal{E}} \to 0.$$

Now, condition (C) on $\mathcal{K}_c$ means that $H^q(\mathbf{P}^N, \mathcal{K}_c(-j)) = 0$ for $1 \leq j \leq n + c = N$ and all $q \geq 0$ forcing $\mathcal{K}_c$ to be a trivial vector bundle $\mathcal{O}_{\mathbf{P}^N}^{\oplus b_c}$ (Lemma 1.1.1). We have then constructed the required linear resolution. □

Notation 1.1.3. Whenever we are dealing with an *embedded* smooth projective variety $X \subset \mathbf{P}^N$, Ulrich bundles will always considered with respect to the line bundle $\mathcal{O}_X(1)$. In this situation, we will simply say that *$\mathcal{E}$ is an Ulrich bundle on X*.

Remark 1.1.4. Henceforth we will use these equivalent definitions without explicit mention.

Before moving on to stating some properties, we show that Ulrich bundles are completely characterized in the case of curves.

Proposition 1.1.5. *Let C be a smooth projective curve of genus g, and let B be a globally generated ample line bundle on C. A vector bundle $\mathcal{E}$ of rank r is B-Ulrich if and only if $\mathcal{E} = \mathcal{F}(B)$ for an acyclic vector bundle $\mathcal{F}$. More precisely, $\mathcal{E}$ is B-Ulrich if and only if $h^0(C, \mathcal{E}(-B)) = 0$ and $\deg(\mathcal{E}) = r(d + g - 1)$.*

Proof. The first part is immediate from condition 2. For the second one, by Grothendieck-Riemann-Roch theorem for curves, we have

$$h^0(C, \mathcal{E}(-B)) - h^1(C, \mathcal{E}(-B)) = c_1(\mathcal{E}(-B)) + \mathrm{rk}(\mathcal{E}(-B))(1-g) = \deg(\mathcal{E}) + r(1-g-d),$$

and the conclusion follows. □

Definition - Proposition 1.1.6. *Let X be a smooth projective variety of dimension $n \geq 1$ and let B be a globally generated ample line bundle on X. A vector bundle $\mathcal{E}$ is B-Ulrich on X if and only if $\mathcal{E}^*((n+1)B \otimes K_X)$ is. For this reason we call*

$$\mathcal{E}^*((n+1)B \otimes K_X)$$

the Ulrich dual of $\mathcal{E}$ (with respect to B).

Proof. If $0 \leq i \leq n$ and $1 \leq p \leq n$, applying Serre duality we get

$$\begin{aligned} h^i(X, \mathcal{E}(-pB)) &= h^{n-i}(X, \mathcal{E}^*(pB \otimes K_X)) \\ &= h^j(X, \mathcal{E}^*((n+1-q)B \otimes K_X)) \qquad j = n-i, q = n+1-p \\ &= h^j(X, \mathcal{E}^*((n+1)B \otimes K_X)(-qB)). \end{aligned}$$

Since j, q range respectively in $0 \leq j \leq n$, $1 \leq q \leq n$ as well, the conclusion follows. □

It is clear from the definition that Ulrich bundles are special cases of vector bundles without intermediate cohomology. The next proposition shows the strict connection between Ulrich bundles and aCM bundles. We refer to Appendix A.2 for the basic definitions and properties about aCM bundles.

Proposition 1.1.7. *Let X be a smooth projective variety of dimension $n \geq 1$ and let B be a globally generated ample line bundle with $d = B^n$. and let $\mathcal{E}$ be a vector bundle of rank r on X. Then the following are equivalent:*

(1) $\mathcal{E}$ is B-Ulrich.

(2) There exists a finite morphism $\pi \colon X \to \mathbf{P}^n$ of degree d with $\pi^*\mathcal{O}_{\mathbf{P}^n}(1) \cong B$ such that $\pi_*\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^n}^{\oplus rd}$.

(3) $H^i(X, \mathcal{E}(-iB)) = H^{i-1}(X, \mathcal{E}(-iB)) = 0$ for $1 \leq i \leq n$.

(4) If $n \geq 2$, $\mathcal{E}$ is aCM with respect to B with Hilbert polynomial

$$P(\mathcal{E}, m) = \frac{rd}{n!}(m+1)\cdots(m+n).$$

Proof. Let's begin with the equivalence between (1) and (2). If $\mathcal{E}$ is B-Ulrich, then such a morphism π is constructed as in the proof of Definition-Theorem 1.1.2: by taking the composition between φ_B and a linear projection from $\overline{X}$ onto an n-linear subspace $\mathbf{P}^n$. The fact that π has degree d follows by observing that $\pi_*\mathcal{E}$ has rank $r \cdot \deg(\pi) = rd$. Conversely, since $\pi_*(\mathcal{E}(kB)) \cong \mathcal{O}_{\mathbf{P}^n}^{\oplus rd}(k)$ and π is finite, $\mathcal{E}$ has the same cohomology of $\mathcal{O}_{\mathbf{P}^n}^{\oplus rd}$, that clearly satisfies the required vanishings.

Next, (1) clearly implies (3). To show the converse, fix $\pi \colon X \to \mathbf{P}^n$ a finite surjective morphism such that $\pi^*\mathcal{O}_{\mathbf{P}^n}(1) \cong B$ chosen as above. As observed so far, $\mathcal{E}(kB)$ and $\pi_*\mathcal{E}(k)$ have isomorphic cohomology, and, by miracle flatness, $\pi_*\mathcal{E}$ is a vector bundle of rank $r \cdot \deg(\pi) = s$. In particular $\pi_*\mathcal{E}$ is 0-regular, as $H^i(\mathbf{P}^n, \pi_*\mathcal{E}(-i)) = 0$ for $i > 0$. Therefore $\pi_*\mathcal{E}$ is globally generated by Castelnuovo-Mumford theorem B.1.3, so we get a short exact sequence

$$0 \to \mathcal{K} = M_{\pi_*\mathcal{E}} \to H^0(\mathbf{P}^n, \pi_*\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^n} \to \pi_*\mathcal{E} \to 0.$$

By construction, the sheaf $\mathcal{K}$ has no global section. Tensoring by $\mathcal{O}_{\mathbf{P}^n}(-i)$ for $1 \leq i \leq n$, and using the other hypothesis, we see from the exact sequence

$$0 = H^{i-1}(\mathbf{P}^n, \pi_*\mathcal{E}(-i)) \to H^i(\mathbf{P}^n, \mathcal{K}(-i)) \to H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-i))^{\oplus r} = 0$$

that $\mathcal{K}$ is 0-regular as well. Castelnuovo-Mumford theorem B.1.3 says that $\mathcal{K}$ is globally generated. However $H^0(\mathbf{P}^n, \mathcal{K}) = 0$ since $H^0(\mathbf{P}^n, \pi_*\mathcal{E}) \otimes H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) \to H^0(\mathbf{P}^n, \pi_*\mathcal{E})$ is an isomorphism by construction. Therefore we have $\mathcal{K} = 0$, which yields that $\pi_*\mathcal{E}$ is trivial.

Now assume $\mathcal{E}$ is B-Ulrich. Its Hilbert polynomial has been already shown in Lemma 1.1.1 to be of the desired form. On the other hand, if π is a morphism as in (2), then $\mathcal{E}$ has the cohomology of a trivial vector bundle on $\mathbf{P}^n$ which is clearly aCM. By Leray spectral sequence we get (4). Finally, suppose (4) holds and let $f\colon X \longrightarrow \mathbf{P}^n$ be a finite surjective morphism such that $f^*\mathcal{O}_{\mathbf{P}^n}(1) \cong B$. Then $f_*\mathcal{E}$ is an aCM vector bundle on $\mathbf{P}^n$ of rank $r \cdot \deg(f) = s$. Horrocks' theorem A.2.3 implies that $f_*\mathcal{E}$ splits as a sum of line bundles $\bigoplus_{j=1}^s \mathcal{O}_{\mathbf{P}^n}(k_j)$. To conclude we need to show that $k_j = 0$ for every $j = 1, \dots, s$. Since $\mathcal{E}(mB)$ and $f_*\mathcal{E}(m)$ have isomorphic cohomologies for all m, Hilbert polynomials coincides:

$$\frac{rd}{n!}(m+1)(m+2)\cdots(m+n) = \sum_{j=1}^{s} \frac{1}{n!}(m+k_j+1)\cdots(m+k_j+n).$$

Top degree coefficients $\frac{rd}{n!}$ and $\sum_{j=1}^s \frac{1}{n!} = \frac{s}{n!}$ must be equal, so $s = rd$. Equating the coefficients of m^{n-1} yields

$$rd\sum_{h=1}^{n} h = \sum_{j=1}^{rd}\sum_{h=1}^{n}(k_j+h) = n\sum_{j=1}^{rd} k_j + rd\sum_{h=1}^{n} h,$$

forcing $\sum_{j=1}^{rd} k_j = 0$. Equating the coefficients of m^{n-2} and using this, we have

$$rd\sum_{1\le p<q\le n} pq = \sum_{j=1}^{rd}\left(\sum_{1\le p<q\le n}(k_j+p)(k_j+q)\right) = \binom{n}{2}\sum_{j=1}^{rd} k_j^2 + rd\sum_{1\le p<q\le n} pq,$$

which implies $\sum_{j=1}^{rd} k_j^2 = 0$. So all k_j are zero and $f_*\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^n}^{\oplus rd}$. □

In [ES03] the authors originally defined the notion of *Ulrich sheaf* on a projective variety.

Definition 1.1.8. Let Y be a projective variety and let B be a base-point-free ample line bundle. A coherent sheaf $\mathcal{E}$ on Y is said to be *Ulrich for B*, or a *B-Ulrich sheaf,* if $H^i(Y, \mathcal{E}(-pB)) = 0$ for $i \geq 0$ and $1 \leq p \leq \dim Y$.

Remark 1.1.9. Given a projective variety Y and a globally generated ample line bundle B, it can be proved exactly as in Definition-Theorem 1.1.2 that a coherent sheaf $\mathcal{E}$ is B-Ulrich if and only if for every finite morphism $\pi\colon Y \longrightarrow \mathbf{P}^{\dim Y}$ such that $\pi^*\mathcal{O}_{\mathbf{P}^n}(1)$ one has $\pi_*\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^n}^{\oplus t}$ for some t.

As a consequence, a B-Ulrich sheaf has no intermediate cohomology: $H^i(X, \mathcal{E}(tB)) = 0$ for $0 < i < \dim Y$ and $t \in \mathbf{Z}$. Moreover, if Y is smooth, then a B-Ulrich sheaf $\mathcal{E}$ is automatically locally free [ES11, §4, p.43]: if $\dim Y > 1$, it follows from [AY08, Lemma 3.2]; if Y is a smooth curve, every coherent sheaf sits in an exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{T}$ is a torsion sheaf and $\mathcal{F}$ is torsion-free, hence locally free [Sta23, Tag 0CC4]. Twisting by $\mathcal{O}_Y(-B)$ and taking the global sections, we immediately see that $H^0(Y, \mathcal{T}(-B)) = 0$. This forces $\mathcal{T} = 0$: if $\operatorname{supp}(\mathcal{T}) = \{y_1, \dots, y_q\}$, one has $\operatorname{supp}(\mathcal{T}(-B)) = \{y_1, \dots, y_q\}$ as well, therefore

$$0 = H^0(Y, \mathcal{T}(-B)) = \bigoplus_{i=1}^{q} \mathcal{T}(-B)_{y_i} \cong \bigoplus_{i=1}^{q} (\mathcal{T}_{y_i} \otimes \mathcal{O}_{Y,y_i}) \cong \bigoplus_{i=1}^{q} \mathcal{T}_{y_i},$$

giving the assertion. Then $\mathcal{E} \cong \mathcal{F}$, and the proof is complete.

We can now list some properties of Ulrich bundles.

Proposition 1.1.10. *Let $f\colon X \to Y$ be a finite surjective morphism of smooth projective varieties and let L be a globally generated ample line bundle on Y. A vector bundle $\mathcal{E}$ on X is f^*L-Ulrich if and only if $f_*\mathcal{E}$ is L-Ulrich.*

Proof. First, f^*L is ample and globally generated, and $f_*\mathcal{E}$ is locally free since f is also flat. Then the projection formula gives

$$f_*\left(\mathcal{E} \otimes f^*L^{\otimes k}\right) \cong f_*\mathcal{E} \otimes L^{\otimes k}$$

for all k. Since f is finite and surjective, $\dim X = \dim Y$ and

$$H^i(X, \mathcal{E} \otimes f^*L^{\otimes k}) \cong H^i(Y, f_*\mathcal{E} \otimes L^{\otimes k})$$

for all $i \geq 0$, the conclusion follows. □

Proposition 1.1.11. *Let (X, B) and (Y, L) be smooth projective varieties of dimension n, m respectively together with a globally generated ample line bundle. Let $\mathcal{E}$ be a B-Ulrich bundle on X, and let $\mathcal{F}$ be a L-Ulrich bundle on Y. Then the vector bundle $\mathcal{E} \boxtimes \mathcal{F}(nL)$ is $B \boxtimes L$-Ulrich on $X \times Y$.*

More generally, if $(X_i, B_i)_{i=0}^k$ are smooth projective varieties of dimension n_i together with globally generated ample line bundles, and $\mathcal{E}_i$ are B_i-Ulrich bundles for each i, then

$$\mathcal{E}_0 \boxtimes \mathcal{E}_1(n_0B_1) \boxtimes \cdots \boxtimes \mathcal{E}_k((n_0 + \cdots + n_{k-1})B_k)$$

is Ulrich for $(X_0 \times \cdots X_k, B_0 \boxtimes \cdots \boxtimes B_k)$.

Proof. Let $i \geq 0$ and $1 \leq p \leq \dim(X \times Y) = n + m$. Then it's immediate that

$$(\mathcal{E} \boxtimes \mathcal{F}(nL))(-p) \cong \mathcal{E}(-pB) \boxtimes \mathcal{F}((n-p)L).$$

Recalling Kunneth's formula

$$H^i(X \times Y, (\mathcal{E} \boxtimes \mathcal{F}(nL))(-p)) \cong \bigoplus_{j+k=i} H^j(X, \mathcal{E}(-pB)) \otimes H^k(Y, \mathcal{F}((n-p)L)),$$

we see that the first factor is 0 for $1 \leq p \leq n$ and the second one vanishes for $n+1 \leq p \leq n+m$.

The final assertion follows by combining the first part and the inductive hypothesis for $k \geq 2$ applied to the pairs $(X, B) = (X_0 \times \cdots X_{k-1}, B_0 \boxtimes \cdots \boxtimes B_{k-1})$ and $(Y, L) = (X_m, B_k)$, with $\mathcal{E} = \mathcal{E}_0 \boxtimes \mathcal{E}_1(n_0B_1) \boxtimes \cdots \boxtimes \mathcal{E}_{k-1}((n_0 + \cdots + n_{k-2})B_{k-1})$ and $\mathcal{F} = \mathcal{E}_k$. □

Lemma 1.1.12. *Let X be a smooth projective variety and let B be a globally generated ample line bundle. Let*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be a short exact sequence of coherent sheaves. If two of them are B-Ulrich, then so is the third.

Proof. The cohomology sequence associated to the exact sequence

$$0 \to \mathcal{E}(-pB) \to \mathcal{F}(-pB) \to \mathcal{G}(-pB) \to 0,$$

for $1 \le p \le \dim X$ yields the following short exact sequences:

$$\begin{aligned} &0 \to H^0(X, \mathcal{E}(-pB)) \to H^0(X, \mathcal{F}(-pB)) \to H^0(X, \mathcal{G}(-pB)) \to H^1(X, \mathcal{E}(-pB)),\\ &H^{i-1}(X, \mathcal{G}(-pB)) \to H^i(X, \mathcal{E}(-pB)) \to H^i(X, \mathcal{F}(-pB)),\\ &H^i(X, \mathcal{E}(-pB)) \to H^i(X, \mathcal{F}(-pB)) \to H^i(X, \mathcal{G}(-pB)),\\ &H^i(X, \mathcal{F}(-pB)) \to H^i(X, \mathcal{G}(-pB)) \to H^{i+1}(X, \mathcal{E}(-pB)) \end{aligned}$$

for $i > 0$. The conclusion easily follows from these by Definition-Theorem 1.1.2.2. □

Proposition 1.1.13. *Let X be a smooth projective variety of dimension $n \ge 2$ and let B be a base-point-free ample line bundle. Let Y be a smooth and irreducible member in $|B|$ and let $\mathcal{E}$ be a vector bundle on X. If $\mathcal{E}$ is B-Ulrich, then $\mathcal{E}_{|Y}$ is $B_{|Y}$-Ulrich on Y. Conversely, if $n \ge 3$ and $\mathcal{E}_{|Y}$ is $B_{|Y}$-Ulrich, then $\mathcal{E}$ is B-Ulrich.*

Proof. Let $\iota\colon Y \hookrightarrow X$ be the inclusion, and set $\mathcal{E}_Y = \iota^*\mathcal{E}$ and $B_Y = \iota^* B$. By projection formula we get

$$\iota_*(\mathcal{E}_Y(kB_Y)) \cong \iota_*\left(\mathcal{O}_Y \otimes \iota^*(\mathcal{E} \otimes B^{\otimes k})\right) = \mathcal{E}(kB) \otimes \iota_*\mathcal{O}_Y \tag{1.4}$$

for all k. Consider the short exact sequence

$$0 \to \mathcal{E} \otimes \mathcal{O}_X(-Y) \cong \mathcal{E}(-B) \to \mathcal{E} \to \mathcal{E} \otimes \iota_*\mathcal{O}_Y \cong \iota_*\mathcal{E}_Y \to 0. \tag{1.5}$$

Suppose that $\mathcal{E}$ is B-Ulrich. Then, if $i \ge 0$ and $1 \le p \le \dim Y = n-1$, the conclusion easily follows from the long exact exact sequence

$$0 = H^i(X, \mathcal{E}(-pB)) \to H^i(Y, \mathcal{E}_Y(-pB_Y)) \to H^{i+1}(X, \mathcal{E}((-p-1)B)) = 0$$

induced by the short exact sequence obtained from the twist of (1.5) by $B^{\otimes p}$, where we have used (1.4).

Now, assume that $\mathcal{E}_Y$ is B_Y-Ulrich and let $\pi\colon X \to \mathbf{P}^n$ be a finite morphism such that $\pi^*\mathcal{O}_{\mathbf{P}^n}(1) \cong B$, for instance the composition of φ_B with a finite projection $\varphi_B(X) \to \mathbf{P}^n$. Set $d = B^n$ and $r = \mathrm{rk}(\mathcal{E})$. Observe that π maps Y onto a hyperplane, whence the restriction

$$\pi_Y := \pi \circ \iota\colon Y \to \pi(Y) =: H \simeq \mathbf{P}^{n-1}$$

is still finite, surjective and such that $\pi_Y^*\mathcal{O}_H(1) \cong B_Y$. Therefore $(\pi_Y)_*\mathcal{E}_Y \cong (\pi_*\mathcal{E})_{|H}$ is trivial of rank rd on $H \simeq \mathbf{P}^{n-1}$. Since $n \ge 3$, Horrocks' theorem for reflexive sheaves A.2.4 says that $\pi_*\mathcal{E}$ splits as a sum of line bundles, namely $\pi_*\mathcal{E} \cong \oplus_{j=1}^{rd}\mathcal{O}_{\mathbf{P}^n}(k_j)$. In particular, the first cohomology group $H^1(\mathbf{P}^n, \pi_*\mathcal{E}(-1)) \cong \oplus_{j=1}^{rd} H^1(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k_j - 1))$ vanishes since $n \ge 3$. Then, by the virtue of [Har77, Proposition III.6.3], we also deduce that

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbf{P}^n}^{\oplus rd}, \pi_*\mathcal{E}(-1)) \cong \bigoplus^{rd} \mathrm{Ext}^1(\mathcal{O}_{\mathbf{P}^n}, \pi_*\mathcal{E}(-1)) \cong \bigoplus^{rd} H^1(\mathbf{P}^n, \pi_*\mathcal{E}(-1)) = 0.$$

Since $(\pi_*\mathcal{E})_{|H} \cong \mathcal{O}_H^{\oplus rd}$, we conclude from [AY08, Theorem 2.2] that $\pi_*\mathcal{E}$ is trivial of rank rd. In conclusion, $\mathcal{E}$ is B-Ulrich by Definition-Theorem 1.1.2.3. □

The following characterization of Ulrich bundles is the original one coming from the commutative algebra. First recall the following definition.

Definition 1.1.14. Let X be a projective variety and let B be an ample and globally generated line bundle. A vector bundle $\mathcal{E}$ is *B-initialized* if $H^0(X,\mathcal{E}) \neq 0$ and $H^0(X,\mathcal{E}(-B)) = 0$.

Proposition 1.1.15. *Let X be a smooth projective variety of dimension $n \geq 1$ and let B be a globally generated ample line bundle with $B^n = d$. A vector bundle $\mathcal{E}$ of rank r is B-Ulrich if and only if $\mathcal{E}$ is aCM, B-initialized and has $h^0(X,\mathcal{E}) = rd$.*

Proof. One direction is clear. If $\mathcal{E}$ is B-Ulrich, then it is aCM by Proposition 1.1.7 and $h^0(X,\mathcal{E}) = rd$ by Lemma 1.1.1. Eventually, Proposition 1.1.7(2) says that $\mathcal{E}$ has the cohomology table of $\mathcal{O}_{\mathbf{P}^n}^{\oplus rd}$. Therefore, using Serre duality, one gets that

$$h^0(X,\mathcal{E}(-B)) = rd \cdot h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-1)) = 0.$$

This implies that $\mathcal{E}$ is B-initialized.

Assume the converse. If $n = 1$, we only need to prove that $H^1(X,\mathcal{E}(-B)) = 0$. A general member $Y \in |B|$ consists of d distinct smooth points. The exact sequence

$$0 \to \mathcal{E}((\ell-1)B) \to \mathcal{E}(\ell B) \to \mathcal{E}(\ell B)_{|Y} \to 0 \tag{1.6}$$

yields the cohomology exact sequence

$$\begin{aligned} 0 \to H^0(X,\mathcal{E}((\ell-1)B)) \to H^0(X,\mathcal{E}(\ell B)) \to H^0(Y,\mathcal{E}(\ell B)_{|Y}) \\ \to H^1(X,\mathcal{E}((\ell-1)B)) \to H^1(X,\mathcal{E}(\ell B)) \to 0. \end{aligned}$$

If $\ell = 0$, then $H^0(X,\mathcal{E}) \to H^0(Y,\mathcal{E}_{|Y})$ is an isomorphism, given that $\mathcal{E}$ is B-initialized and $h^0(X,\mathcal{E}) = rd = h^0(Y,\mathcal{E}_{|Y})$. If $\ell > 0$, we obtain the following commutative diagram:

$$\begin{array}{ccc} H^0(X,\mathcal{E}((\ell-1)B)) & \xrightarrow{r_{\ell-1}} & H^0(Y,\mathcal{E}((\ell-1)B)_{|Y}) \\ \downarrow & & \downarrow \cong \\ H^0(X,\mathcal{E}(\ell B)) & \xrightarrow{r_\ell} & H^0(Y,\mathcal{E}(\ell B)_{|Y}). \end{array}$$

The map $r_{\ell-1}$ is surjective by induction, and so is r_ℓ by commutativity. From the surjectivity of r_ℓ for every $\ell \geq 0$, we deduce the inclusions

$$H^1(X,\mathcal{E}(-B)) \subset H^1(X,\mathcal{E}) \subset H^1(X,\mathcal{E}(2B)) \subset \cdots \subset H^1(X,\mathcal{E}(\ell B)) \subset \cdots.$$

On the other hand, Serre theorem implies that $H^1(X,\mathcal{E}(\ell B)) = 0$ for every $\ell \gg 0$. Hence the conclusion follows. Now assume that $n \geq 2$. By hypothesis we have the vanishings $H^i(X,\mathcal{E}(-pB)) = 0$ for $0 \leq i \leq n-1$ and $1 \leq p \leq n$. It remains to check that $H^n(X,\mathcal{E}(-pB)) = 0$ for $1 \leq p \leq n$. Consider the short exact sequence (1.6), where Y is a general smooth irreducible member in $|B|$. Taking again the cohomology, we immediately see that $\mathcal{E}_{|Y}$ is aCM, $B_{|Y}$-initialized with $h^0(Y,\mathcal{E}_{|Y}) = rd$. By induction, $\mathcal{E}_{|Y}$ is $B_{|Y}$-Ulrich on Y. Hence, if $n \geq 3$, the conclusion follows by Proposition 1.1.13. If $n = 2$, then Y is a curve and we saw above that $H^1(Y, \mathcal{E}(\ell B)_{|Y}) = 0$ for $\ell \geq -1$. Therefore, if $\ell \geq -1$, we obtain the following exact sequence:

$$0 = H^1(Y, \mathcal{E}(\ell B)_{|Y}) \to H^2(X, \mathcal{E}((\ell - 1)B)) \to H^2(X, \mathcal{E}(\ell B)) \to 0.$$

In particular we have

$$h^2(X, \mathcal{E}(-2B)) = h^2(X, \mathcal{E}(-B)) = h^2(X, \mathcal{E}) = \cdots = h^2(X, \mathcal{E}(\ell B)) = \cdots .$$

Since these cohomology groups vanish for every $\ell \gg 0$ by Serre theorem, we obtain the claim. □

We now study the stability of Ulrich bundles. For a short introduction to these concepts, we refer to Appendix A.4. First of all we are going to compute the reduced Hilbert polynomial, the degree (for this one, see also [Lop22, Lemma 3.2]) and the slope of an Ulrich bundle.

Lemma 1.1.16. *Let X be a smooth projective variety of dimension $n \geq 1$, and let B be an ample and globally generated line bundle on X with $B^n = d$. Let $\mathcal{E}$ be a B-Ulrich bundle of rank r on X. Then:*

(i) The reduced Hilbert polynomial of $\mathcal{E}$ is

$$p(\mathcal{E}, m) = \frac{1}{n!}(m+1)\cdots(m+n).$$

In particular, all B-Ulrich bundles on X have same reduced Hilbert polynomial.

(ii) The degree and the slope of $\mathcal{E}$ are respectively

$$\deg(\mathcal{E}) = \frac{r}{2}\left((n+1)B^n + K_X \cdot B^{n-1}\right), \quad \mu(\mathcal{E}) = \frac{1}{2}\left((n+1)B^n + K_X \cdot B^{n-1}\right).$$

In particular,

$$c_1(\mathcal{E}) = \frac{r}{2}\left((n+1)B + K_X\right)$$

for varieties having $\mathrm{Pic}(X) \cong \mathbf{Z}$.

Proof. Item (i) immediately follows from Lemma 1.1.1. For (ii), the case $n = 1$ is given by Proposition 1.1.5. If $n \geq 2$, by Bertini's theorem, see e.g. [Har77, Corollary III.10.9 & Exercise III.11.3], we can find a smooth irreducible divisor $Y \in |B|$. It follows from Proposition 1.1.13 that $\mathcal{E}_{|Y}$ is $B_{|Y}$-Ulrich. By induction on dimension we have

$$\deg(\mathcal{E}_{|Y}) = \frac{r}{2}\left(nB_{|Y}^{n-1} + K_Y \cdot B_{|Y}^{n-2}\right).$$

On the other hand K_Y satisfies $K_Y = (K_X + B)_{|Y}$ by adjunction formula. Therefore we obtain

$$\begin{aligned} c_1(\mathcal{E}) \cdot B^{n-1} &= c_1(\mathcal{E}_{|Y}) \cdot B_{|Y}^{n-2} \\ &= \frac{r}{2}\left(nB_{|Y}^{n-1} + (K_X)_{|Y} \cdot B_{|Y}^{n-2} + B_{|Y}^{n-1}\right) \\ &= \frac{r}{2}\left((n+1)B^n + K_X \cdot B^{n-1}\right). \end{aligned}$$

The last assertion for varieties X of Picard rank 1 immediately follows from the above equation by writing $B, K_X, c_1(\mathcal{E})$ in terms of an ample generator A for $\mathrm{Pic}(X)$. □

Proposition 1.1.17. *Let X be a smooth projective variety of dimension $n \geq 1$ together with a globally generated ample line bundle B such that $B^n = d$, and let $\mathcal{E}$ be a B-Ulrich bundle of rank r. Then:*

(1) $\mathcal{E}$ is semistable.

(2) If $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ is an exact sequence of coherent sheaves with $\mathcal{G}$ torsion-free and $\mu(\mathcal{F}) = \mu(\mathcal{E})$, then both $\mathcal{F}$ and $\mathcal{G}$ are B-Ulrich bundles.

(3) If $\mathcal{E}$ is stable, then $\mathcal{E}$ is also μ-stable.

In particular, if $\mathcal{E}$ is not stable, then it is an extension of B-Ulrich bundles of smaller rank.

Proof. Let $\pi\colon X \to \mathbf{P}^n$ be a finite surjective morphism of degree d such that $\pi^*\mathcal{O}_{\mathbf{P}^n}(1) \cong B$, which exists by Proposition 1.1.7. The finiteness implies that π preserves the dimension of the supports of sheaves and that π_*, which keeps injectivity, also preserves Hilbert polynomials. Since $\pi_*\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^n}^{\oplus rd}$ is (pure and) semistable, it is then clear that $\mathcal{E}$ is (pure and) semistable as well, proving (1).

For (2), observe that $\mathrm{supp}(\mathcal{F}) = X$, since $\mathcal{E}$ is pure, and that

$$0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^n}^{\oplus rd} \to \pi_*\mathcal{G} \to 0$$

remains exact because π is finite. Moreover $\pi_*\mathcal{G}$ is torsion-free [Gro60, Proposition 7.4.5, p.163], and $\mu(\pi_*\mathcal{F}) = \mu(\pi_*\mathcal{E})$ as well because

$$0 = \mu(\mathcal{E}) - \mu(\mathcal{F}) = \alpha_n(\mathcal{O}_X)\left(\frac{\alpha_{n-1}(\mathcal{E})}{\alpha_n(\mathcal{E})} - \frac{\alpha_{n-1}(\mathcal{F})}{\alpha_n(\mathcal{F})}\right) = d \cdot \left(\frac{\alpha_{n-1}(\mathcal{E})}{\alpha_n(\mathcal{E})} - \frac{\alpha_{n-1}(\mathcal{F})}{\alpha_n(\mathcal{F})}\right)$$

with $d > 0$, and, recalling that π preserves Hilbert polynomials and supports, we get

$$\mu(\pi_*\mathcal{E}) - \mu(\pi_*\mathcal{F}) = \alpha_n(\mathcal{O}_{\mathbf{P}^n})\left(\frac{\alpha_{n-1}(\mathcal{E})}{\alpha_n(\mathcal{E})} - \frac{\alpha_{n-1}(\mathcal{F})}{\alpha_n(\mathcal{F})}\right) = \left(\frac{\alpha_{n-1}(\mathcal{E})}{\alpha_n(\mathcal{E})} - \frac{\alpha_{n-1}(\mathcal{F})}{\alpha_n(\mathcal{F})}\right) = 0.$$

Hence we may assume $(X, B, \mathcal{E}) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n}^{\oplus r})$, and so $\mu(\mathcal{F}) = \mu(\mathcal{E}) = 0$. If $n = 1$, then $\mathcal{G}$ is locally free [Sta23, Tag 0CC4], and so is $\mathcal{F}$. Then $\mathcal{F}$ is isomorphic to a sum of line bundles $\oplus_{i=1}^k \mathcal{O}_{\mathbf{P}^1}(k_i)$ for some $k_i \in \mathbf{Z}$ [GW20, Theorem 11.53], implying that each $\mathcal{O}_{\mathbf{P}^1}(k_i)$ injects into $\mathcal{O}_{\mathbf{P}^n}^{\oplus r}$. Semistability yields $k_i \geq 0$, and $\mu(\mathcal{F}) = 0$ gives $\sum_{i=1}^k k_i = \deg(\mathcal{F}) = 0$, whence $k_i = 0$ for all i. Therefore $\mathcal{F}$ is a trivial vector bundle, and so is $\mathcal{G}$. Now, assume $n \geq 2$, and take a general hyperplane $H \simeq \mathbf{P}^{n-1}$. We get an exact sequence

$$0 \longrightarrow \mathcal{F}_{|H} \longrightarrow (\mathcal{O}_{\mathbf{P}^n}^{\oplus r})_{|H} \cong \mathcal{O}_H^{\oplus r} \longrightarrow \mathcal{G}_{|H} \longrightarrow 0$$

with $\mathcal{G}_{|H}$ torsion-free and $\mu(\mathcal{F}_{|H}) = \mu(\mathcal{F}) = \mu(\mathcal{E}) = \mu(\mathcal{E}_{|H}) = 0$. By induction $\mathcal{F}_{|H} \cong \mathcal{O}_H^{\oplus k}$ and $\mathcal{G}_{|H} \cong \mathcal{O}_H^{\oplus h}$ are Ulrich bundles on $H \simeq \mathbf{P}^{n-1}$. Taking the cohomology of the exact sequence in 2 we see that

$$h^0(\mathbf{P}^n, \mathcal{F}(t)) = h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t)) = 0, \quad h^1(\mathbf{P}^n, \mathcal{F}(t)) = h^0(\mathbf{P}^n, \mathcal{G}(t)) \quad \text{for all } t < 0. \tag{1.7}$$

Similarly, twisting the exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_{|H} \cong \mathcal{O}_{\mathbf{P}^{n-1}}^{\oplus k} \to 0$$

we deduce that $h^i(\mathbf{P}^n, \mathcal{F}(q-1)) = h^i(\mathbf{P}^n, \mathcal{F}(q))$ for $i > 1$ and all $q \geq -n+1$. Since $h^i(\mathbf{P}^n, \mathcal{F}(q)) = 0$ for sufficiently large q, we see that $h^i(\mathbf{P}^n, \mathcal{F}(q-1)) = 0$ if $i > 1$ and $q \geq -n+1$. In other words

$$0 = h^i(\mathbf{P}^n, \mathcal{F}(-n)) = h^i(\mathbf{P}^n, \mathcal{F}(-n+1)) = h^i(\mathbf{P}^n, \mathcal{F}(-n+2)) = \cdots,$$

and the same holds for $\mathcal{G}$. Finally, for $\ell < 0$ we have that $h^0(\mathbf{P}^n, \mathcal{G}(\ell-1)) = h^0(\mathbf{P}^n, \mathcal{G}(\ell))$. However these vector spaces vanish for sufficiently negative ℓ. Indeed, being torsion-free implies that the natural morphism $\mathcal{G} \to \mathcal{G}^{**}$ is injective. A sufficiently large twist $\mathcal{G}^*(s)$ is globally generated. Taking the dual of the twist by $\mathcal{O}_{\mathbf{P}^n}(-s)$ of the surjection $\mathcal{O}_{\mathbf{P}^n}^{\oplus m} \twoheadrightarrow \mathcal{G}^*(s)$, we obtain an injection $\mathcal{G}^{**} \hookrightarrow \mathcal{O}_{\mathbf{P}^n}(s)^{\oplus m}$. The injective morphism $\mathcal{G} \hookrightarrow \mathcal{O}_{\mathbf{P}^n}(s)^{\oplus m}$ implies the vanishings $H^0(\mathbf{P}^n, \mathcal{G}(\ell)) = 0$ for $\ell \ll 0$. As observed above, this means that $h^0(\mathbf{P}^N, \mathcal{G}(\ell)) = 0$ for all $\ell < 0$. In turn, by (1.7), this yields $h^1(\mathbf{P}^N, \mathcal{F}(t)) = 0$ for every $t < 0$. In conclusion, we have shown that $\mathcal{F}$ is Ulrich for $\mathcal{O}_{\mathbf{P}^n}(1)$, hence trivial. Therefore $\mathcal{G}$ is a trivial vector bundle as well.

To prove (3), suppose $\mathcal{E}$ is stable and assume that there is a non-zero subsheaf $\mathcal{F} \subset \mathcal{E}$ such that $\mu(\mathcal{F}) = \mu(\mathcal{E})$. By Remark A.4.4, we can assume that $\mathcal{E}/\mathcal{F}$ is torsion-free, so that $\mathcal{F}$ is an Ulrich bundle by (2). Then Lemma 1.1.16 implies that $p(\mathcal{F}) = p(\mathcal{E})$ contradicting the stability.

To complete the proof, suppose that $\mathcal{E}$ is not stable. Then, by (1) and by Remark A.4.4, there is a non-zero subsheaf $\mathcal{F} \subset \mathcal{E}$ of smaller rank such that $\mu(\mathcal{F}) = \mu(\mathcal{E})$ and $\mathcal{E}/\mathcal{F}$ is torsion-free. The conclusion follows from (2). □

Proposition 1.1.18. *Let X be a smooth projective variety of dimension and let B be a base-point-free ample line bundle. Let $\mathcal{E}$ be a B-Ulrich bundle. In any Jordan-Hölder filtration of $\mathcal{E}$,*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E},$$

the $\mathcal{E}_i$'s for $1 \leq i \leq \ell$ are B-Ulrich bundles.

Proof. Consider a Jordan-Hölder filtration of $\mathcal{E}$ as above. Then, $\mathcal{E}_i/\mathcal{E}_{i-1}$ is stable, torsion-free with $p(\mathcal{E}_i/\mathcal{E}_{i-1}) = p(\mathcal{E}_i) = p(\mathcal{E})$ for each $1 \leq i \leq \ell$ (Proposition A.4.5). We claim that it is enough to prove that $\mathcal{E}_1$ is B-Ulrich. Indeed, if it so, $\mathcal{F} = \mathcal{E}/\mathcal{E}_1$ is B-Ulrich by Lemma 1.1.12. Then

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 = \mathcal{E}_2/\mathcal{E}_1 \subset \cdots \subset \mathcal{F}_{\ell-1} = \mathcal{F}$$

is a Jordan-Hölder filtration of $\mathcal{F}$: each factor $\mathcal{F}_j/\mathcal{F}_{j-1} \cong \mathcal{E}_j/\mathcal{E}_{j-1}$ is stable, and, since $p(\mathcal{E}_1) = p(\mathcal{E}) = p(\mathcal{E}/\mathcal{E}_1)$ by Lemma 1.1.16(i), we have

$$p(\mathcal{F}_j/\mathcal{F}_{j-1}) = p(\mathcal{E}_j/\mathcal{E}_{j-1}) = p(\mathcal{E}) = p(\mathcal{E}_1/0) = p(\mathcal{E}_1) = p(\mathcal{F}).$$

But $\mathcal{F}$ is a B-Ulrich bundle with $\mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$, hence by induction on the rank we deduce that $\mathcal{E}_i/\mathcal{E}_1$ is B-Ulrich for each $i = 2, \ldots, \ell$. Applying again Lemma 1.1.12, we get the conclusion.

On the other hand, $\mathcal{E}_1$ is stable, torsion-free with same reduced Hilbert polynomial of $\mathcal{E}$, hence it the same slope of $\mathcal{E}$. As $\mathcal{E}/\mathcal{E}_1$ is torsion-free, the assertion follows from Proposition 1.1.17(2). □

We conclude the section with some example.

Example 1.1.19. Any smooth and irreducible hypersurface X in $\mathbf{P}^N$ carries an Ulrich bundle with respect to $\mathcal{O}_X(d)$. Indeed, if X has degree d, then X is a smooth irreducible member of $|\mathcal{O}_{\mathbf{P}^N}(d)|$. The conclusion follows by combining [Bea18, Proposition 3.1] and Proposition 1.1.13.

Proposition 1.1.20. *Let X be a smooth projective variety and let B be an ample and globally generated line bundle. If X admits a B-Ulrich bundle of rank r, then X carries a $B^{\otimes k}$-Ulrich bundle of rank $r \cdot n!$ for any $k > 1$.*

Proof. Let $n = \dim X$ and $d = B^n$. Let $\mathcal{E}$ be a B-Ulrich bundle of rank r, $\pi\colon X \to \mathbf{P}^n$ be a finite surjective morphism such that $\pi^*\mathcal{O}_{\mathbf{P}^n}(1) \cong B$, and $\mathcal{F}$ an Ulrich bundle of rank $n!$ on $\mathbf{P}^n$ for $\mathcal{O}_{\mathbf{P}^n}(k)$ [Bea18, Proposition 3.1]. Then $\mathcal{E} \otimes \pi^*\mathcal{F}$ has rank $r \cdot n!$ which is Ulrich for $(X, B^{\otimes k})$: indeed, if $1 \le p \le n$, as $\pi^*\mathcal{O}_{\mathbf{P}^n}(h) \cong B^{\otimes h}$ for all h, and $\pi_*\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^n}^{\oplus rd}$, we have by projection formula and being B-Ulrich, that

$$\pi_*\left((\mathcal{E} \otimes \pi^*\mathcal{F})(-pB^{\otimes k})\right) \cong \pi_*\mathcal{E} \otimes \mathcal{F}(-pk) \cong \left(\mathcal{F} \otimes \mathcal{O}_{\mathbf{P}^n}(k)^{\otimes(-p)}\right)^{\oplus rd}.$$

Since the pushforward by a finite morphism preserves cohomology, the conclusion follows from the fact that (cohomology commutes with direct sums and that) $\mathcal{F}$ is Ulrich for $\mathcal{O}_{\mathbf{P}^n}(k)$. □

Theorem 1.1.21 ([Kim16, Theorem 0.1]). *Let X be a smooth projective variety and let B be an ample and globally generated line bundle. Let $\mu\colon \widetilde{X} \to X$ be the blow-up of X at x with the exceptional divisor E. Assume that $\widetilde{B} := \mu^*B - E$ is ample and globally generated. Then a vector bundle $\mathcal{F}$ on X is B-Ulrich if and only if $\widetilde{\mathcal{F}} := \mu^*\mathcal{F}(-E)$ is $\widetilde{B}$-Ulrich.*

Proof. Recall the well known fact that $R^i\mu_*\mathcal{O}_{\widetilde{X}}((p-1)E) = 0$ for $i > 0$, and $\mu_*\mathcal{O}_{\widetilde{X}}((p-1)E) = \mathcal{O}_X$ for every $1 \le p \le \dim X = \dim \widetilde{X}$ (see for instance [BEL91, Proof of Lemma 1.4]). Hence, projection formula [Har77, Exercise III.8.3] implies that

$$\begin{aligned} R^i\mu_*\left(\widetilde{\mathcal{F}}(-p\widetilde{B})\right) &= R^i\mu_*\left(\pi^*(\mathcal{F}(-pB)) \otimes \mathcal{O}_{\widetilde{X}}((p-1)E)\right) \\ &\cong \mu^*(\mathcal{F}(-pB)) \otimes R^i\mu_*\mathcal{O}_{\widetilde{X}}((p-1)E) = 0 \end{aligned}$$

for $1 \le p \le \dim X$ and all $i > 0$. Therefore, by [Har77, Exercise III.8.1] and by projection formula, the cohomology groups of $\widetilde{\mathcal{F}}(-p\widetilde{B})$ coincides with those of

$$\mu_*(\widetilde{\mathcal{F}}(-p\widetilde{B})) \cong \mathcal{F}(-pB) \otimes \mu_*\mathcal{O}_{\widetilde{X}}((p-1)E) \cong \mathcal{F}(-pB).$$

In other words, we have the isomorphisms $H^i(\widetilde{X}, \widetilde{\mathcal{F}}(-p\widetilde{B})) \cong H^i(X, \mathcal{F}(-pB))$ for all $i \ge 0$ and $1 \le p \le \dim X$, which proves the theorem. □

Remark 1.1.22. Given a globally generated ample line bundle B on a smooth projective variety X, and said μ the blow-up morphism at $x \in X$ with exceptional divisor E, then $\mu^*B - E$ is ample and base-point-free if and only if $\varepsilon(B; x) > 1$ (see Appendix B.2 for the definition of Seshadri constant) and $B \otimes \mathfrak{m}_x$ is generated by global sections.

Indeed, since $\varepsilon(B; x) \ge 1$ by Remark B.2.4, the claim on ampleness is Lemma 3.0.4 below. About the global generation, the assertion follows from [Laz04a, Lemma 4.3.16] saying that $H^0(\widetilde{X}, \widetilde{B}) \cong H^0(X, B \otimes \mathfrak{m}_x)$

1.2 Ulrich line bundles

In this section we present some example, taken from [Bea18], of smooth projective varieties carrying Ulrich line bundles. These are fairly rare. For instance we have the following restriction.

Remark 1.2.1. If X is a n-dimensional smooth variety and B is an ample globally generated line bundle, then $\mathcal{O}_X(kB)$ is B-Ulrich if and only if $(X, B) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n})$ and $k = 0$.

Indeed, as one direction is clear (e.g. by Lemma 1.1.1), suppose $\mathcal{O}_X(kB)$ is B-Ulrich. Then $h^0(X, \mathcal{O}_X((k-1)B)) = 0$ and $h^0(X, \mathcal{O}_X(kB)) = B^n$ immediately yield $k = 0$. Hence $B^n = 1$ and the claim follows by Lemma 3.0.1.

Another special case is that of curves.

Proposition 1.2.2. *Smooth projective curves carry Ulrich line bundles with respect to any ample and globally generated polarization.*

Proof. Let C be a smooth projective curve and let g be its genus. According to Proposition 1.1.5, we need to find an acyclic line bundle $\mathcal{L}$. By Riemann-Roch, we have

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg(\mathcal{L}) - g + 1,$$

so we reduced to find a line bundle of degree $g-1$ with no global sections. If $g = 0$, then $C \cong \mathbf{P}^1$ and the only choice is $\mathcal{L} = \mathcal{O}_{\mathbf{P}^1}(-1)$. If $g = 1$, the curve is not rational, so it is enough to take $\mathcal{L} = \mathcal{O}_C(P - Q)$ for two distinct points P, Q by [Har77, Example II.6.10.1 & Lemma III.1.2]. If $g \geq 2$, there exists a divisor $\Theta \subset \operatorname{Pic}^{g-1}(C)$ set-theoretically defined as

$$\Theta := \{D \in \operatorname{Pic}^{g-1}(C) \mid H^0(C, D) \neq 0\}.$$

Therefore it is enough to pick $\mathcal{L} \in \operatorname{Pic}^{g-1}(C)$ living outside Θ. □

Proposition 1.2.3. *Let $\pi\colon \mathbf{P}(\mathcal{E}) \to C$ be a n-projective bundle over a smooth projective curve C, let A be a line bundle on C, and let $\mathcal{F}$ be a vector bundle on C. If $B = \pi^*A \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is ample and globally generated, then $\pi^*\mathcal{F}(B)$ is B-Ulrich if and only if $\mathcal{F}$ is acyclic. In particular projective bundles over curves carry Ulrich line bundles.*

Proof. For any $1 \leq p \leq n$, we have

$$\pi^*\mathcal{F}(B)(-pB) \cong (\pi^*\mathcal{F})((-p+1)B) \cong \pi^*(\mathcal{F}((1-p)A)) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1-p).$$

Then, by projection formula [Har77, Exercise III.8.3],

$$R^j\pi_*\left((\pi^*\mathcal{F}(B))(-pB)\right) \cong \mathcal{F}((1-p)A) \otimes R^j\pi^*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1-p)) \tag{1.8}$$

for all $j \geq 0$. However, we have:

(A) $R^j\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1-p)) = 0$ for $0 < j < n$, and $R^n\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1-p)) = 0$ as $-p+1 \geq -n+1 > -n-1$, both following from [Har77, Exercise III.8.4(a)];

(B) if $j = 0$ in (1.8), right-hand-side vanishes for $1 < p \leq n$, and is $\mathcal{F}$ for $p = 1$, both given by [Har77, Proposition II.7.11].

In summary, (A) together with (1.8) tells us that $R^j\pi_*((\pi^*\mathcal{F}(B))(-pB)) = 0$ for $j > 0$. Then, combining this with (B), (1.8) with $j = 0$, and [Har77, Exercise III.8.1], we obtain

$$H^i(\mathbf{P}(\mathcal{E}), \pi^*\mathcal{F}(B)(-pB)) \cong \begin{cases} H^i(C, \mathcal{F}) & \text{for} \quad p = 1 \\ 0 & \text{for} \quad 1 < p \leq n \end{cases}$$

for all $i \geq 0$. So the conclusion follows. □

Another example is given by Del Pezzo surfaces, namely smooth projective surfaces S whose anticanonical divisor $-K_S$ is ample. Recall that the anticanonical divisor of a Del Pezzo surface is globally generated if and only if the degree $K_S^2 \in \{1, 2, \ldots, 9\}$ is greater than or equal to 2, and is very ample if $K_S^2 \geq 3$. The existence of Ulrich line bundles in the case of very ample anticanonical divisor is treated in [Bea18, Proposition 4.1(i)]. So we consider only the case of degree 2 following the same idea.

Proposition 1.2.4. *Let S be a Del Pezzo surface of degree 2, and let L be a line bundle such that $L^2 = -2$ and $L \cdot K_S = 0$. Then $\mathcal{E} = L(-K_S)$ is a $(-K_S)$-Ulrich line bundle on S. Since S is isomorphic to the blow-up of $\mathbf{P}^2$ at seven points in general positions, we can take L as the difference of two distinct exceptional divisors.*

Proof. It's clear that L has no nonzero global sections because it is not trivial, as $L^2 \neq 0$, and if it were effective, then we would have $L \cdot (-K_S) > 0$. The same holds for $K_S - L$ since $-K_S \cdot (K_S - L) < 0$. Using Riemann-Roch theorem and Serre duality, we also see that $h^1(S, L) = 0$. Since $(-L)^2 = -2$ and $-L \cdot K_S = 0$, the same holds for L^*. So L and $L \otimes K_S$ are both acyclic. Applying Serre duality, we get $h^i(S, L \otimes K_S) = h^{2-i}(S, L^*) = 0$ for all $i \geq 0$. As $\mathcal{E}(-(-K_S)) = L$ and $\mathcal{E}(-2(-K_S)) = L \otimes K_S$, we obtain that $\mathcal{E}$ is Ulrich with respect to $-K_S$. □

1.3 Special rank 2 Ulrich bundles on Del Pezzo threefolds of degree 2

A *Del Pezzo manifold of degree d* is a polarized smooth projective variety (X, B) of dimension $n \geq 2$ such that $K_X = -(n-1)B$ and $B^n = d$. These varieties, that generalize Del Pezzo surfaces, are classified in [IP99, Theorem 3.3.1]: if $d \geq 3$, the polarization B determines an embedding in $\mathbf{P}^{d+n-2}$; for $d = 2$, then $\varphi_B \colon X \to \mathbf{P}^n$ is a double cover branched over a hypersurface of degree 4; finally, B has a single base point if $d = 1$ [IP99, Proposition 3.2.4].

Consider the case $n = 3$. It has been proved in [Bea18, Proposition 6.1] that any Del Pezzo threefold (X, B) of degree $d \geq 3$ has a special B-Ulrich bundle of rank 2. In this section we prove the analogous for Del Pezzo threefolds of degree 2.

Definition 1.3.1. A vector bundle $\mathcal{E}$ of rank 2 on a smooth projective variety X of dimension n is *special for B*, with B a globally generated line bundle, if $\det(\mathcal{E}) \cong K_X((n+1)B)$.

The following lemma will play a central role.

Lemma 1.3.2. *A Del Pezzo threefold (X, B) of degree 2 contains a smooth elliptic curve Γ of degree 4 with respect to B such that its ideal sheaf in X satisfies $H^0(X, \mathcal{I}_{\Gamma/X}(B)) = 0$.*

Proof. The following argument is taken from [Fae14, Proof of Theorem D, Step 1]. Fix an embedding $X \subset \mathbf{P}^N$ given by a very ample multiple $\mathcal{O}_X(rB) = \mathcal{O}_X(1)$. Let $\mathrm{Hilb}^{4rm}(X)$ be the Hilbert scheme of X parametrizing closed subschemes having Hilbert polynomial $P(m) = 4rm$. Let $H^4_1(X) \subset \mathrm{Hilb}^{4rm}(X)$ be the open subset parametrizing Cohen-Macaulay curves in $X \subset \mathbf{P}^N$ of degree $4r$ and arithmetic genus 1, and let $V^4_1(X) \subset H^4_1(X)$ be the open subset of smooth irreducible curves of such degree and genus. Note that $H^4_1(X)$ contains all curves C such that $B \cdot C = 4$ having $p_a(C) = 1$. Conversely, if $[D] \in H^4_1(X)$, then $4r = \deg(D) = (rB) \cdot D$ forces $B \cdot D = 4$. Finally, denote by $U_B \subset |B| \simeq \mathbf{P}^3$ the open dense subset $\{S \in |B| \mid S \text{ is smooth and irreducible}\}$.

An element $S \in U_B$ is a smooth Del Pezzo surface of degree $(-K_S)^2 = B^2_{|S} = 2$. We may identify S with the blow-up of $\mathbf{P}^2$ in 7 points $P_1, \dots, P_7$ in general positions, with exceptional divisors $E_1, \dots, E_7$. Denote by $H_S \subset \mathbf{P}^2$ the class of a line and with $\widetilde{H}_S \subset S$ its pullback, so that $B_{|S} = 3\widetilde{H}_S - E_1 - \cdots - E_7$. The linear series $|\mathcal{I}_{\{P_1,\dots,P_5\}/\mathbf{P}^2}(3)|$ has no unassigned base points and has dimension 2 [Har77, Proposition V.4.3 & Corollary V.4.4(a)]. Therefore the linear series $|3\widetilde{H}_S - E_1 - \cdots - E_5|$ is base-point-free. Moreover $(3\widetilde{H}_S - E_1 - \cdots - E_5)^2 = 4 > 0$, hence its linear series is not composite with a pencil. Then, a general member $\Gamma_S \in |3\widetilde{H}_S - E_1 - \cdots - E_5|$ is a smooth irreducible curve by Bertini theorem [Har77, Corollary III.10.9 & Exercise III.11.3] with $g(\Gamma_S) = 1$ [Har77, Corollary V.3.7]. The normal bundle $\mathcal{N}_{\Gamma_S/S}$ in S has degree $\Gamma_S^2 = 4$, and by adjunction formula we see that $\Gamma_S^2 = -K_S \cdot \Gamma_S = B_{|S} \cdot \Gamma_S = B \cdot \Gamma_S$. We deduce that $[\Gamma_S]$ lives in $V^4_1(X) \subset H^4_1(X)$.

For any pair $([\Gamma], S) \in V^4_1(X) \times U_B$ such that $\Gamma \subset S$, we have $(-K_S) \cdot \Gamma = B_{|S} \cdot \Gamma = 4$, whence

$$\deg(\mathcal{N}_{\Gamma/S}) = \Gamma^2 = 2g(\Gamma) - 2 - K_S \cdot \Gamma = 4$$

by adjunction formula. Consider the exact sequences

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(\Gamma) \to \mathcal{O}_\Gamma(\Gamma) = \mathcal{N}_{\Gamma/S} \to 0, \tag{1.9}$$

$$0 \to \mathcal{O}_\Gamma(\Gamma) \to \mathcal{N}_{\Gamma/X} \to \mathcal{O}_\Gamma(B) = (\mathcal{N}_{S/X})_{|\Gamma} \to 0. \tag{1.10}$$

Since Γ is a smooth elliptic curve and $\deg(\mathcal{O}_\Gamma(\Gamma)) = 4 > 0$, we have $h^1(\Gamma, \mathcal{O}_\Gamma(\Gamma)) = 0$. Then, Riemann-Roch theorem on Γ implies $h^0(\Gamma, \mathcal{O}_\Gamma(\Gamma)) = \deg(\mathcal{N}_{\Gamma/S}) = 4$, and the rationality of S says that $H^1(S, \mathcal{O}_S) = 0$. By the cohomology sequence of (1.9) it follows that $h^0(S, \mathcal{O}_S(\Gamma)) = 5$. Again, as $B \cdot \Gamma = 4 > 0$, it follows that $h^1(\Gamma, \mathcal{O}_\Gamma(B)) = 0$. By the exact sequence in cohomology associated to (1.10), we conclude that $H^1(\Gamma, \mathcal{N}_{\Gamma/X}) = 0$ and $h^0(\Gamma, \mathcal{N}_{\Gamma/X}) = 8$. Therefore $\mathrm{Hilb}^{4rm}(X)$ is smooth of dimension 8 at the point $[\Gamma]$.

Now fix a pair $([\Gamma_0], S_0) \in V^4_1(X) \times U_B$ with $\Gamma_0 = \Gamma_{S_0}$. We claim that a general deformation Γ of Γ_0 in $H^4_1(X)$ satisfies $H^0(X, \mathcal{I}_{\Gamma/X}(B)) = 0$, or in other words, there is an open neighborhood $V \subset H^4_1(X)$ of $[\Gamma_0]$ such that $H^0(X, \mathcal{I}_{\Gamma/X}(B)) = 0$ for all $[\Gamma] \in V$. Once proved the assertion, it is enough to take an element $[\Gamma] \in V \cap V^4_1(X)$ to conclude the proof.

To prove the claim, we proceed by contradiction. Consider the incidence correspondence

$$\mathcal{J} = \left\{([\Gamma], S) \in H^4_1(X) \times |B| \mid \Gamma \subset S\right\} \subset H^4_1(X) \times |B|,$$

and let $\pi_1 \colon \mathcal{J} \to H^4_1(X)$ and $\pi_2 \colon \mathcal{J} \to |B| \simeq \mathbf{P}^3$ be the two projections. Assuming the contrary, we can suppose that π_1 is dominant onto an open neighborhood V' of $[\Gamma_0] \in V^4_1(X)$. Up to take $V' \cap V^4_1(X)$, assume $V' \subset V^4_1(X)$. Observe that the fibre of π_1 over $[\Gamma_0]$ consists only of the divisor S_0: if $S \neq S_0$ is another divisor in $|B|$ containing Γ_0, then $(S_0)_{|S} \subset S_0$ is a divisor in S_0 containing Γ_0, hence we must have

$$(S_0)_{|S} = B_{|S} = 3\widetilde{H}_{S_0} - E_1 - \cdots - E_7 \geq \Gamma_0 = 3\widetilde{H}_{S_0} - E_1 - \cdots - E_5,$$

saying that $-E_6 - E_7$ is effective, which is absurd. Let $\mathcal{J}_0 \subset \mathcal{J}$ be an irreducible component which is dominant onto V'. Recalling that $H_1^4(X)$ is smooth of dimension 8 in $[\Gamma_0]$, we have $\dim \mathcal{J}_0 \geq 8$. Set $U_0 = \mathcal{J}_0 \cap \pi_1^{-1}(V_1^4(X))$. Then $([\Gamma_0], S_0) \in U_0$ since $\pi_1^{-1}([\Gamma_0]) = \{S_0\}$. For any $S \in U_B$, the fibre

$$\pi_2^{-1}(S) = \left\{[\Gamma] \in H_1^4(X) \mid \Gamma \subset S\right\}$$

can be identified with the Hilbert scheme of curves in $S \subset \mathbf{P}^N$ having degree $4r$ and arithmetic genus 1.[1] Since $H^1(S, \mathcal{O}_S) = 0$, Fogarty theorem [Fog68, Corollary 2.7] implies that it is a finite disjoint union of projective spaces corresponding to the linear systems of curves in $S \subset \mathbf{P}^N$ of such degree and such arithmetic genus. In particular $\pi^{-1}_{2|\pi_1^{-1}(V_1^4(X))}(S)$ is a finite disjoint union of open subsets of linear systems $|\Gamma|$, with $\Gamma \subset S$ a smooth irreducible elliptic curve of degree 4. We saw above that every such linear series has dimension 4, hence we must have $\dim\left(\pi^{-1}_{2|\pi_1^{-1}(V_1^4(X))}(S)\right) = 4$. Applying [Har77, Exercise II.3.22(b)] to $\pi_{2|U_0} \colon U_0 \longrightarrow |B|$, we get

$$8 \leq \dim \mathcal{J}_0 = \dim U_0 \leq \dim\left(\pi^{-1}_{2|U_0}(S_0)\right) + \dim |B| = 7,$$

which is a contradiction. □

The following result extends [Bea18, Proposition 6.1] to Del Pezzo threefolds of degree 2. See also [Vac25, §4.2.1].

Proposition 1.3.3. *Every Del Pezzo threefold (X, B) of degree 2 carries a stable special B-Ulrich bundle of rank 2.*

Proof. The proof goes as in [Bea18, Proposition 6.1]. Let $\Gamma \subset X$ be as in Lemma 1.3.2. Using that K_Γ is trivial, $\det(\mathcal{N}_{\Gamma/X}) \cong B_{|\Gamma}^{\otimes 2}$ [Har77, Proposition II.8.20]. Since $H^2(X, \mathcal{O}_X(-2B)) = 0$ and $\det(\mathcal{N}_{\Gamma/X}) \otimes \mathcal{O}_X(2B)_{|\Gamma} \cong \mathcal{O}_\Gamma$, there exists by Hartshorne-Serre correspondence [Arr07, Theorem 1.1] a rank 2 vector bundle $\mathcal{E}$ with $\det(\mathcal{E}) \cong \mathcal{O}_X(2B)$ and fitting in the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{\Gamma/X}(2B) \longrightarrow 0. \tag{1.11}$$

Using that $\mathcal{E} \cong \Lambda^{2-1}\mathcal{E}^* \otimes \det(\mathcal{E})$, we have that $\mathcal{E}(-2B) \cong K_X \otimes \mathcal{E}(-2B)^*$ and $\mathcal{E}(-3B) \cong K_X \otimes \mathcal{E}(-B)^*$. Hence, in order to show that $\mathcal{E}$ is Ulrich for B, we reduced to prove that $H^i(X, \mathcal{E}(-B)) = 0$ for $i \geq 0$ and $H^i(X, \mathcal{E}(-B)) = 0$ for $i = 0, 1$ thanks to Serre duality. Since $H^j(X, \mathcal{O}_X) = 0$ for $j > 0$, X being a Fano variety, it is clear from the usual exact sequence

$$0 \longrightarrow \mathcal{I}_{\Gamma/X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_\Gamma \longrightarrow 0. \tag{1.12}$$

that $H^i(X, \mathcal{I}_{\Gamma/X}) = 0$ for $i = 0, 1$. Non-degeneracy means exactly that the restriction map

$$H^0(X, \mathcal{O}_X(B)) \longrightarrow H^0(\Gamma, \mathcal{O}_\Gamma(B))$$

is injective. In addition to this, those vector spaces have dimension 4, so this map is an isomorphism. Twisting (1.12) by $\mathcal{O}_X(B)$ and using that $H^j(X, \mathcal{O}_X(B)) \cong H^j(X, K_X \otimes \mathcal{O}_X(3B)) = 0$ for $j > 0$ by Kodaira vanishing, we obtain from the associated long exact

[1] All divisors in a smooth variety are locally complete intersection, hence they are automatically Cohen-Macaulay [Har77, Proposition II.8.23(a)].

sequence that $H^i(X, \mathcal{I}_{\Gamma/X}(B)) = 0$ for $i \geq 0$. Taking the long exact sequence associated to the suitable twist of (1.11), we obtain the required vanishings. Finally, since $\mathrm{Pic}(X) = \mathbf{Z} \cdot B$ [IP99, Remark 3.3.2(i)], X cannot support B-Ulrich line bundles (Remark 1.2.1). Therefore $\mathcal{E}$ is stable by Proposition 1.1.17. □

As a corollary we obtain the following.

Corollary 1.3.4. *All Del Pezzo threefolds* (X, B) *of degree* $d \geq 2$ *support a* μ*-stable special Ulrich bundle of rank* 2.

Chapter 2

On equivariant Ulrich bundles on rational homogeneous varieties

Finding vector bundles which are also Ulrich is a quite difficult task. However, on rational homogeneous varieties G/P there is a large class of vector bundles worth considering: the one of equivariant vector bundles. Among them, the easiest ones to study from a cohomological point of view are the irreducible ones thanks to Borel-Bott-Weil theorem 2.1.3. Exploiting this tool, irreducible equivariant Ulrich bundles (with respect to the minimal ample class) have been fully classified on all rational homogeneous varieties of Picard rank-1 in [CM15; Fon16; LP21] (respectively for G of type A_n, then for G of type B_n, C_n, D_n, and for G of exceptional type E_6, E_7, E_8, F_4, G_2). For higher Picard ranks, again in the irreducible case, several flag varieties of type A_n are treated in [Cos17b], all isotropic flags, i.e. of type B_n, C_n and D_n, and all flag varieties of exceptional type E_6, F_4, G_2 are studied respectively in [FN24] and in [Nak23].

As said above, in [CM15; Fon16; LP21] the Ulrichness of irreducible equivariant bundle is with respect to the generator of the Picard group. The first result of this chapter is the extension of that classification with respect to all Veronese embeddings.

Proposition 2.0.1. *Let $X = G/P$ be a rational homogeneous variety with $\mathrm{Pic}(X) = \mathbf{Z}\cdot\mathcal{O}_X(1)$. An irreducible equivariant bundle $\mathcal{E}_\lambda$ on $(X, \mathcal{O}_X(d))$ is Ulrich if and only if $\lambda = d\widetilde{\lambda} + (d-1)\rho$ where $\widetilde{\lambda}$ is a $P-$dominant weight such that $\mathcal{E}_{\widetilde{\lambda}}$ is Ulrich on $(X, \mathcal{O}_X(1))$. In particular, all irreducible equivariant Ulrich bundles for $(X, \mathcal{O}_X(d))$ are classified.*

Except for grassmannians, where there are always (several) irreducible equivariant Ulrich bundles (see [CM15]), as it is shown in [Fon16; FN24], irreducible equivariant bundle are rarely Ulrich. Certainly this negative answer does not contradict Eisenbud-Schreyer conjecture (simply because vector bundles on rational homogeneous varieties are not necessarily equivariant), but also it does not prevent the existence of reducible equivariant bundles which are, anyways, Ulrich. The only examples of such bundles can be found on $G_2/P_1 \cong Q_5, F_4/P_4, E_6/P_1$ (see [LP21, Remark 4.2 & §6.1 & Corollary 7.4]) with the last one already supporting an irreducible one (see [LP21, Proposition 5.1]). However these examples are obtained as restriction to the hyperplane section of irreducible equivariant Ulrich bundles on other rational homogeneous varieties. In this chapter we construct in an explicit way (reducible) equivariant Ulrich bundles on $\mathbb{S}_{10} = D_5/P_5 \subset \mathbf{P}^{15}$ and on $\mathrm{LGr}(3,6) = C_3/P_3 \subset \mathbf{P}^{13}$, both of them not supporting irreducible equivariant Ulrich bundles (see [Fon16, Propositions 3.3–6.6]).

Proposition 2.0.2. *The spinor tenfold* $\mathbb{S}_{10} \subset \mathbf{P}^{15}$ *and the Lagrangian grassmannian* $\mathrm{LGr}(3,6) \subset \mathbf{P}^{13}$ *support a* μ*-stable equivariant Ulrich bundle.*

Both of these varieties are of great importance since they are maximal Mukai manifolds of genus $g = 7, 9$ (see Theorem 2.3.3). As a consequence we get the following corollary.

Corollary 2.0.3. *Prime Mukai manifolds of genus* $g \in \{4,5,7,8,9\}$ *and ordinary prime Mukai manifolds of genus* $g = 6$ *support Ulrich bundles for the generator of the Picard group.*

We now set the notations which will be carried throughout the chapter. We denote by G a simple algebraic groupand by $\mathrm{Lie}(G) = \mathfrak{g}$ its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ endowed with the Killing product $(,)$ and consider the corresponding Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

with root system Φ. For $\alpha \in \Phi$ we denote by r_α the reflection with respect to the hyperplane H_α which is orthogonal to α. Let $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Phi$ be a basis of simple roots. This choice determines a decomposition $\Phi = \Phi^- \bigsqcup \Phi^+$ where Φ^+ (resp. Φ^-) is the set of positive (resp. negative) roots, i.e. roots which can be written as a non-negative (resp. non-positive) integral linear combination of simple roots. Let $\{\lambda_1, \dots, \lambda_n\}$ be the fundamental weights (corresponding to Δ), i.e. the dual basis for $\left\{\frac{2\alpha_1}{(\alpha_1,\alpha_1)}, \dots, \frac{2\alpha_n}{(\alpha_n,\alpha_n)}\right\}$ with respect to $(,)$, and let $\Lambda = \mathrm{Span}_{\mathbf{Z}}(\lambda_1, \dots, \lambda_n) \subset \mathfrak{h}^*$ be the weight lattice. Once fixed an orthonormal basis $\{\varepsilon_i\}_{i=1}^n$ for $\mathfrak{h}^*$, for any weight $\lambda \in \Lambda$ we write

$$\lambda = (\ell_1, \dots, \ell_n)$$

for its vector of coordinates with respect to $\{\varepsilon_1, \dots, \varepsilon_n\}$.[1] We denote the sum of all fundamental weights by

$$\rho = \frac{1}{2}\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \lambda_i.$$

The Weyl group $\mathcal{W}$ is generated by the reflections $r_i = r_{\alpha_i}$, for $\alpha_i \in \Delta$ and the fundamental Weyl chamber is

$$D = \left\{ \sum_{i=1}^n x_i\lambda_i \;\middle|\; x_i \geq 0 \text{ for all } 1 \leq i \leq n \right\}.$$

The affine action of the Weyl group on the weight lattice is defined as

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

The weights living in D are called *dominant* and are in bijection with the finite dimensional irreducible $\mathfrak{g}-$modules. For any dominant weight $\lambda = \sum_{i=1}^n p_i\lambda_i \in \Lambda \cap D$ we denote by

$$V_\lambda = V_{p_1,\dots,p_n} = V(p_1, \dots, p_n)$$

the corresponding irreducible $\mathfrak{g}-$module of finite dimension. The dual of V_λ^* is the irreducible $\mathfrak{g}-$module $V_{-w_0(\lambda)}$ where w_0 is the longest element in the Weyl group $\mathcal{W}$.

[1] The coordinates $\ell_1, \dots, \ell_n$ do not need to be integers.

2.1 Preliminaries on rational homogeneous varieties and equivariant bundles

Here we review the main features of rational homogeneous varieties with Pic $= \mathbf{Z}$ and equivariant vector bundles. The main references are [Ott95a; OR06].

Any rational homogeneous variety X with $\mathrm{Pic}(X) = \mathbf{Z}$ is the quotient $X = G/P$ where $P = P(\alpha_k) = P_k < G$ is the parabolic subgroup associated with the set $\Delta \backslash \{\alpha_k\}$. More precisely, if we set

$$\Phi(\alpha_k) = \left\{ \alpha \in \Phi^+ \;\middle|\; \alpha = \sum_{i=1, i\neq k}^{n} k_i \alpha_i \right\},$$

we can define $P = P(\alpha_k)$ as the subgroup of G such that

$$\mathrm{Lie}(P) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(\alpha_k)} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}.$$

In addition to this, we have a splitting $\mathrm{Lie}(P) = \mathrm{Lie}(N) \oplus \mathrm{Lie}(R)$ where $N < P, R < G$ are subgroups such that

$$\mathrm{Lie}(N) = \bigoplus_{\alpha \in \Phi^+ \backslash \Phi(\alpha_k)} \mathfrak{g}_{-\alpha}, \quad \mathrm{Lie}(R) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(\alpha_k)} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi(\alpha_k)} \mathfrak{g}_{-\alpha}.$$

In fact this provides the Levi decomposition $P = N \rtimes R$ with R being the reductive part of P and N being the unipotent radical of P. In this notation, we also have $\dim X = \left|\Phi^+ \backslash \Phi(\alpha_k)\right|$. The reductive part has its own fundamental Weyl chamber $D' \supset D$ defined as

$$D' = \left\{ \sum_{i=1}^{n} x_i \lambda_i \;\middle|\; x_i \geq 0 \text{ for all } 1 \leq i \leq n, i \neq k \right\}.$$

Weights belonging to D' are called $P-$dominant and correspond bijectively to the (finite dimensional) irreducible representations of P. For all the rest of the paper, any parabolic subgroup is of this form.

Definition 2.1.1. A $(G-)$*equivariant vector bundle* on a rational homogeneous variety $X = G/P$ is a vector bundle $E \to X$ such that the diagram

$$\begin{array}{ccc} G \times E & \xrightarrow{\sigma_E} & E \\ \downarrow & & \downarrow \\ G \times X & \longrightarrow & X \end{array}$$

commutes and the maps on fibres $\sigma_E(g, -)\colon E(x) \to E(g.x)$ are linear isomorphisms. A morphism $f\colon E \to F$ between equivariant vector bundles on X is *G-equivariant* if the following diagram commutes

$$\begin{array}{ccc} G \times E & \xrightarrow{\sigma_E} & E \\ {\scriptstyle \mathrm{id}_G \times f}\downarrow & & \downarrow {\scriptstyle f} \\ G \times F & \xrightarrow[\sigma_F]{} & F. \end{array}$$

An exact sequence of equivariant vector bundles $0 \to E' \to E \to E'' \to 0$ is $G-$*equivariant* if all morphisms are equivariant.

By definition, a coherent locally free sheaf $\mathcal{E}$ on X is $(G-)$*equivariant* if its total space $E \to X$ is an equivariant vector bundle. From now on we won't distinguish between equivariant vector bundles and its sheaf of local sections.

For an equivariant vector bundle $\mathcal{E}$ on $X = G/P$, the fibre $\mathcal{E}(z)$ over the lateral class $z = [P] \in X$ is naturally a representation of P. Conversely, given a representation $\eta\colon P \to \mathrm{GL}(V)$, the quotient $G \times_\eta V = (G \times V)/\sim$ via the equivalence $\sim$, where $(g, v) \sim (g', v')$ if there exists $p \in P$ such that $g = g'p$ and $v = \eta^{-1}(p)(v')$, defines an equivariant bundle $\mathcal{E}_\eta$ such that $\mathcal{E}_\eta(z) = V$. It is well known that the correspondence

$$\mathcal{E} \mapsto \mathcal{E}(z)$$

defines an equivalence of categories between the category of equivariant bundles on X and $P-$modules of finite dimension. For instance, the cotangent bundle Ω^1_X is associated with the representation

$$\mathrm{Lie}(N) = \bigoplus_{\alpha \in \Phi^+ \setminus \Phi(\alpha_k)} \mathfrak{g}_{-\alpha}$$

and $-\Phi^+ \setminus \Phi(\alpha_k) = \{\xi_i\}_{i=1}^{\dim X}$ are said the *weights* of Ω^1_X.

Equivariant bundles corresponding to irreducible representations of P are called *irreducible.* For every $P-$dominant weight $\lambda = \sum_{i=1}^n p_i \lambda_i \in D'$, we denote by

$$\mathcal{E}_\lambda = \mathcal{E}_{p_1,\dots,p_n}$$

the corresponding irreducible bundle on $X = G/P$. In particular, $\mathcal{E}_{\lambda_k} = \mathcal{O}_X(1)$ is the (very) ample generator of $\mathrm{Pic}(X) = \mathbf{Z}$ and the canonical bundle $K_X = \mathcal{E}_\kappa$ with

$$\kappa = - \sum_{\alpha \in \Phi^+ \setminus \Phi(\alpha_k)} \alpha = \sum_{i=1}^{\dim X} \xi_i,$$

as it is the determinant representation of $\mathfrak{n}$. Furthermore $\mathcal{E}_{\lambda + \ell \lambda_k} = \mathcal{E}_\lambda(\ell)$ for every $P-$dominant weight λ.

Definition 2.1.2. A rational homogeneous variety $X = G/P$ is called an *irreducible Hermitian symmetric variety* if the cotangent bundle Ω^1_X is an irreducible equivariant bundle.[2]

Every $G-$equivariant bundle $\mathcal{E}$ on $X = G/P$ admits a filtration of G-equivariant subbundles

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = \mathcal{E}$$

such that each quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is irreducible. We define the *graded bundle* of $\mathcal{E}$ as

$$\mathrm{gr}(\mathcal{E}) = \bigoplus_{i=1}^{s} \mathcal{E}_i/\mathcal{E}_{i-1}.$$

This definition does not depend on the filtration. In fact it is given by taking the restriction of the $P-$module defining $\mathcal{E}$ to the reductive part $R < P$.

[2]Irreducible Hermitian symmetric varieties have been classified by Cartan (see also [OR06, Theorem 5.12]). They are Grassmannians $\mathrm{Gr}(k, n) = \mathrm{SL}(n)/P(\alpha_k)$, even and odd quadrics $Q_{2n} = \mathrm{SO}(2n+2)/P(\alpha_1)$ and $Q_{2n-1} = \mathrm{SO}(2n+1)/P(\alpha_1)$, spinor varieties $\mathbb{S}_{2n} = \mathrm{SO}(2n)/P(\alpha_n)$, Lagrangian Grassmannian $\mathrm{LGr}(n, 2n) = \mathrm{Sp}(2n)/P(\alpha_n)$, the Cayley plane $\mathbf{OP}^2 = E_6/P(\alpha_1)$ and the exceptional 27-dimensional variety $X_{27} = E_7/P(\alpha_1)$.

As G acts on equivariant bundles, then it acts on cohomology groups as well. In particular, the morphisms in the cohomology sequence of an equivariant exact sequence are morphisms of $G-$modules. The Borel-Bott-Weyl theorem provides a tool to compute the cohomology of irreducible equivariant bundles. First recall that a weight μ is *singular* if there exists a positive root $\alpha \in \Phi^+$ such that $(\mu, \alpha) = 0$, otherwise it is called *regular.*

Theorem 2.1.3 (Borel-Bott-Weil)**.** *Let $X = G/P$ be a rational homogeneous variety. Given a $P-$dominant weight $\lambda \in D'$, there exists a unique $w \in \mathcal{W}$ such that $w(\lambda + \rho) \in D$. If $w(\lambda + \rho)$ belongs to the boundary of D, which amounts to say that $\lambda + \rho$ is singular, then*

$$H^i(X, \mathcal{E}_\lambda) = 0 \textit{ for all } i \geq 0.$$

If $w(\lambda + \rho)$ lies in the interior of D, or equivalently if $\lambda + \rho$ is regular, then

$$H^i(X, \mathcal{E}_\lambda) = \begin{cases} V_{w\cdot\lambda} & \textit{if } i = \ell(w) \\ 0 & \textit{if } i \neq \ell(w) \end{cases}$$

where $\ell(w)$ is the length of w in the Weyl group $\mathcal{W}$.

The next elementary observation shows that the singularity of a $P-$dominant weight can be checked just on roots in $\Phi^+ \backslash \Phi(\alpha_k)$.

Remark 2.1.4. If $\lambda = \sum_{i=1}^n x_i\lambda_i$ is a $P-$dominant weight, then $\lambda + \rho$ is singular if and only if there exists $\alpha \in \Phi^+ \backslash \Phi(\alpha_k)$ such that $(\lambda + \rho - td, \alpha) = 0$.

Indeed, by definition of fundamental weights one has

$$(\lambda+\rho, \alpha) = \sum_{j=1, j\neq k}^{n} (x_k+1)k_j(\lambda_k, \alpha_j) + \sum_{i=1, i\neq k}^{n} \sum_{j=1, j\neq k}^{n} (x_i+1)k_j(\lambda_i, \alpha_j) = \sum_{i=1, i\neq k}^{n} \frac{1}{2}(\alpha_i, \alpha_i)(x_i+1)k_i$$

which is strictly positive because $x_i \geq 0$ for $i \neq k$ and at least one k_i must be non-zero (as $0 \notin \Phi$ by definition of root system).

Given a $P-$dominant weight λ such that $\lambda+\rho$ is regular, let $w \in \mathcal{W}$ be the unique element such that $w(\lambda+\rho)-\rho \in D$. We define the *inversion set* Φ_w of $w \in \mathcal{W}$ as $\Phi_w = w^{-1}(\Phi^-) \cap \Phi^+$. The following lemma coming from a result due to Dimitrov and Roth involving inversion sets will play a fundamental role.

Lemma 2.1.5. *Let $\mathcal{E}$ and $\mathcal{F}$ be equivariant bundles on a rational homogeneous variety $X = G/P$ such that*

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{F})^G = H^1(\mathcal{E}_\beta)^G = \mathbf{C}\cdot\xi,\ H^i(X, \mathcal{E}) \cong H^i(X, \mathcal{E}_\lambda) = V_\nu,\ H^{i+1}(X, \mathcal{F}) \cong H^{i+1}(X, \mathcal{E}_\mu) = V_\nu$$

for a non-zero dominant weight $\nu \in D$ and for some $P-$dominant weights $\beta, \lambda, \mu \in D'$. Let $w_\beta, w_\lambda, w_\mu \in \mathcal{W}$ be the elements such that $w_\beta \cdot \beta, w_\lambda \cdot \lambda, w_\mu \cdot \mu \in D$ and let

$$\partial \colon H^i(X, \mathcal{E}) \to H^{i+1}(X, \mathcal{F})$$

be the boundary map for the equivariant exact sequence defined by ξ. If $\mu = \lambda + \beta$ and $\Phi_{w_\mu} = \Phi_{w_\beta} \bigsqcup \Phi_{w_\lambda}$, then ∂ is an isomorphism. In particular, ∂ is always an isomorphism for $i = 0$.

Proof. This is just a consequence of [DR17, Theorem I & Corollaries 5.4.1–5.4.2]. □

We conclude this section stating Rohmfeld criterion for the stability of equivariant bundles. For a proof we refer to [OR06, Theorem 7.2] and [Fai06, Theorem 1].

Theorem 2.1.6. *Let $X = G/P$ be a rational homogeneous variety and let $\mathcal{E}$ be an equivariant homogeneous bundle. Then the following are equivalent:*

(1) We have $\mu(\mathcal{F}) < \mu(\mathcal{E})$ for every G-equivariant subbundle $\mathcal{F} \subset \mathcal{E}$ (equivariant stability).

(2) There exist an irreducible $G-$module V and a μ-stable equivariant subbundle $\mathcal{E}' \subset \mathcal{E}$ such that $\mathcal{E} \cong V \otimes \mathcal{E}'$

Theorem 2.1.6 together with the Lemma 1.1.16 and Proposition 1.1.17 immediately yields the following observation.

Remark 2.1.7. Let $\mathcal{E}$ be an equivariant Ulrich bundle on a rational homogeneous variety $X = G/P \subset \mathbf{P}^N$. Then the following are equivalent:

(a) $\mathcal{E}$ is equivariantly stable.

(b) $\mathcal{E}$ is stable.

(c) $\mathcal{E}$ is μ-stable.

(d) There are no G-invariant Ulrich subbundles $\mathcal{F} \subset \mathcal{E}$.

2.2 Irreducible equivariant Ulrich bundles for Veronese embeddings of rational homogeneous varieties

This section is devoted to the proof of Proposition 2.0.1.

As in [Fon16], for any $P-$dominant weight λ we are going to set

$$\mathrm{Irr}(\lambda) := \left\{ t \in \mathbf{Z} \;\middle|\; \lambda + \rho - t\lambda_k \text{ is singular} \right\}. \tag{2.1}$$

Analogously to [Fon16, Lemma 2.4], one can prove the following.

Lemma 2.2.1 (Fonarev criterion)**.** *An irreducible equivariant bundle $\mathcal{E}_\lambda$ on a m-dimensional rational homogeneous variety $X = G/P_k$ is Ulrich for the Veronese embedding $(X, \mathcal{O}_X(d))$ if and only if* $\mathrm{Irr}(\lambda) = \{d, 2d, \dots, md\}$.

Proof. This is a straightforward modification of [Fon16, Lemma 2.4] (see also [LP21, Lemma 3.2]). Of course, if $\mathrm{Irr}(\lambda) = \{d, 2d, \dots, md\}$, then $\mathcal{E}_\lambda$ is Ulrich for the d-Veronese embedding by Borel-Bott-Weil theorem. Conversely, if $\mathcal{E}_\lambda$ is Ulrich for $(X, \mathcal{O}_X(d))$, then $H^i(X, \mathcal{E}_\lambda(-\ell d)) = 0$ for all $i \geq 0$ for each $1 \leq \ell \leq m$. So $\{d, 2d, \dots, md\} \subset \mathrm{Irr}(\lambda)$ by definition. Since $|\mathrm{Irr}(\lambda)| \leq \left|\Phi^+ \backslash \Phi(\alpha_k)\right|$ by Remark 2.1.4 and $\left|\Phi^+ \backslash \Phi(\alpha_k)\right| = m$, the claim follows. □

Proof of Proposition 2.0.1. Let $m = \dim X \geq 1$ and let $P = P_k$ for some $1 \leq k \leq n$. As irreducible equivariant Ulrich bundles on $(X, \mathcal{O}_X(1))$ have been classified in [CM15; Fon16; LP21], we just need to prove the first part. One direction is easy. Suppose $\widetilde{\lambda}$ defines an irreducible equivariant Ulrich bundle on $(X, \mathcal{O}_X(1))$. By Remark 2.1.4 and Lemma 2.2.1 for every $1 \leq \ell \leq m$ there exists $\beta_\ell \in \Phi^+ \backslash \Phi(\alpha_k)$ such that $(\widetilde{\lambda} + \rho - \ell\lambda_k, \beta_\ell) = 0$. The weight $\lambda = d\widetilde{\lambda} + (d-1)\rho$ is $P-$dominant and satisfies

$$(\lambda+\rho-\ell d\lambda_k, \beta_\ell) = (d\widetilde{\lambda}+(d-1)\rho+\rho-\ell d\lambda_k, \beta_\ell) = d(\widetilde{\lambda}+\rho-\ell\lambda_k, \beta_\ell) = 0 \text{ for each } 1 \leq \ell \leq m.$$

Then $\mathcal{E}_\lambda$ is Ulrich on $(X, \mathcal{O}_X(d))$ by Lemma 2.2.1.

Conversely, suppose $\lambda = \sum_{i=1}^n p_i\lambda_i$ defines an irreducible equivariant Ulrich bundle for $(X, \mathcal{O}_X(d))$. We set

$$\lambda + \rho = (a_1, \ldots, a_k, b_1, \ldots, b_{n-k})$$

for the vector of coordinates of $\lambda + \rho \in \Lambda$ with respect to the canonical basis of $\mathfrak{h}^*$. Notice that $p_i \geq 0$ for all i, i.e. λ is dominant, because $\mathcal{E}_\lambda$ is generated by global sections. As above, by Remark 2.1.4 and Lemma 2.2.1 for each $1 \leq \ell \leq m$ we can find a positive root $\beta_\ell \in \Phi^+ \backslash \Phi(\alpha_k)$ such that

$$(\lambda + \rho - \ell d\lambda_k, \beta_\ell) = 0.$$

Claim 2.2.2. *It suffices to prove that $\lambda + \rho = d\mu$ for some weight $\mu \in \Lambda$, which is equivalent to prove that $(p_i + 1) = \ell_i d$ for some integer ℓ_i or that $a_i = q_i d, b_j = q'_j d$ for some rational numbers q_i, q'_j.*

Proof. If $\lambda + \rho = d\mu$ for some weight $\mu \in \Lambda$, then μ will automatically be strictly dominant, i.e. $\mu = \sum_{i=1}^n q_i\lambda_i$ with $q_i > 0$ for all i, because $p_i \geq 0$ for all i. The weight $\widetilde{\lambda} = \mu - \rho$ is then $P-$dominant and satisfies

$$(\widetilde{\lambda} + \rho - \ell\lambda_k, \beta_\ell) = (\mu - \ell\lambda_k, \beta_\ell) = \frac{1}{d}(\lambda + \rho - \ell d\lambda_k, \beta_\ell) = 0$$

for every $1 \leq \ell \leq m$. Therefore the irreducible equivariant bundle $\mathcal{E}_{\widetilde{\lambda}}$ is Ulrich on $(X, \mathcal{O}_X(1))$ by Lemma 2.2.1 and $\lambda = \widetilde{\lambda} + (d-1)\rho$ by construction. □

In remaining part of the proof we will simply check that every coordinate of $\lambda + \rho$ either with respect to the fundamental weights (for G of type A, E, F, G) or with respect to the canonical basis (for G of type B, C, D) is a multiple of d.

Let $\boldsymbol{G{=}B_n}$ or $\boldsymbol{G{=}C_n}$. As shown in [Fon16, §§3-4], we have

$$\mathrm{Irr}(\lambda) = \left\{a_i \pm b_j\right\}_{1\leq i\leq k, 1\leq j\leq n-k} \bigcup \left(\mathbf{Z} \cap \left\{\frac{a_i + a_j}{2}\right\}_{1\leq i\leq j\leq k}\right).$$

As $\mathrm{Irr}(\lambda) = \{d, 2d, \ldots, md\}$ and $a_i \in \mathrm{Irr}(\lambda)$, we have $a_i = \ell_i d$ for each $1 \leq i \leq k$ for some $1 \leq \ell_i \leq m$. Since $a_1 - b_j \in \mathrm{Irr}(\lambda)$, we also have $b_j = \ell_1 d + \ell_j d = (\ell_1 + \ell_j)d$ for each $1 \leq j \leq n-k$ and for some $1 \leq \ell_j \leq m$. By Claim 2.2.2, this concludes $G = B_n, C_n$.

Let $\boldsymbol{G}$=$\boldsymbol{D_n}$. Here the weight lattice is identified with $\mathbf{Z}^n \cup \left(\frac{1}{2}+\mathbf{Z}^n\right)$. Then, as in [Fon16, (18)][3], we have

$$\operatorname{Irr}(\lambda)=\left\{a_i \pm b_j\right\}_{1\le i\le k, 1\le j\le n-k} \bigcup \left(\mathbf{Z} \cap \left\{a_i+a_j\right\}_{1\le i<j\le k}\right)=\{d, 2d, \dots, md\}.$$

If $k<n$, then for each $1\le i\le k$ and $1\le j\le n-k$ we have $a_i+b_j=\ell_{ij}d, a_i-b_j=\ell'_{ij}d$ for some $1\le \ell_{ij}, \ell'_{ij}\le m$. Therefore $a_i=(\ell_{ij}+\ell'_{ij})d/2, b_j=(\ell_{ij}-\ell'_{ij})d/2$ and the claim follows. For $k=n$, write $a_i+a_j=\ell''_{ij}$, with $1\le \ell''_{ij}\le m$, for each $1\le i<j\le m$. Then, for $1\le i<j<h\le n$ we have

$$a_i-a_h=(a_i+a_j)-(a_j+a_h)=(\ell''_{ij}-\ell''_{jh})d,\ \ a_i+a_h=\ell''_{ih}d.$$

It follows $a_i=(\ell''_{ij}+\ell''_{ih}-\ell''_{jh})d/2$ for $1\le i\le n-2$. We similarly find that $a_i=(\ell''_{1i}+\ell''_{2i}-\ell''_{12})d/2$ also for $n-1\le i\le n$. By Claim 2.2.2 this concludes $G=D_n$.

Now we go back to the notation by fundamental weights.

Claim 2.2.3 (see also [LP21, Proposition 3.3])**.** *We have*

$$\operatorname{Irr}(\lambda)=\left\{p_k+1+\sum_{i=1, i\ne k}^{n}\frac{(\alpha_i,\alpha_i)x_i}{(\alpha_k,\alpha_k)x_k}(p_i+1),\ \sum_{i=1}^{n}x_i\alpha_i\in\Phi^+\backslash\Phi(\alpha_k)\right\}. \tag{2.2}$$

and $p_k+1=d$.

Proof. Given $\alpha=\sum_{i=1}^n x_i\alpha_i\in\Phi^+\backslash\Phi(\alpha_k)$, namely $x_i\ge 0$ for $1\le i\le n$ and $x_k>0$, we have

$$\begin{aligned}(\lambda+\rho-t\lambda_k,\alpha)&=(p_k+1-t)x_k(\lambda_k,\alpha_k)+\sum_{i=1,i\ne k}^{n}(p_i+1)x_i(\lambda_i,\alpha_i)\\&=\frac{1}{2}\left((p_k+1-t)x_k(\alpha_k,\alpha_k)+\sum_{i=1,i\ne k}^{n}(p_i+1)x_i(\alpha_i,\alpha_i)\right).\end{aligned}$$

Hence

$$t=p_k+1+\sum_{i=1,i\ne k}^{n}\frac{(\alpha_i,\alpha_i)x_i}{(\alpha_k,\alpha_k)x_k}(p_i+1)$$

is a singular value for $\lambda+\rho$. Vice versa, by Remark 2.1.4 all singular values of $\lambda+\rho$ have this form. Hence (2.2) is proved. Obviously the minimum is attained with α_k, so $\min\operatorname{Irr}(\lambda)=p_k+1$. But $\min\operatorname{Irr}(\lambda)=d$ by Lemma 2.2.1, so $p_k+1=d$ as claimed. □

Let $\boldsymbol{G}$=$\boldsymbol{A_n}$. As is well-known $\Phi^+\backslash\Phi(\alpha_k)=\left\{\sum_{t=i}^{j}\alpha_t \text{ for } 1\le i\le k\le j\le n\right\}$, whence

$$\operatorname{Irr}(\lambda)=\{p_i+\cdots+p_j+j+1-i\}_{1\le i\le k, k\le j\le n}$$

by (2.2). Lemma 2.2.1 tells that for every pair $1\le i\le k\le j\le n$ we have

$$p_i+\cdots+p_j+j+1-i=\ell_{ij}d$$

[3] There is a typo in the displayed equation [Fon16, (18)] because we actually have $(\lambda+\rho-t\lambda_k,\varepsilon_i+\varepsilon_j)=a_i+a_j-t$ for $1\le i<j\le k$.

for some $1 \leq \ell_{ij} \leq m$. Since $p_k + 1 = d$, we see that $\ell_{k,k} = 1$. We prove by induction on $0 \leq j \leq n - k$ that $p_{k+j} + 1 = (\ell_{k,k+j}\ell_{k,k+j-1})d$. As we already know that $p_k + 1 = \ell_{k,k}d$, suppose $j > 0$. Using the inductive hypothesis we easily find that

$$\begin{aligned}\ell_{k,k+j}d &= p_k + \cdots + p_{k+j} + j + 1 = \ell_{k,k}d - 1 + \sum_{h=1}^{j-1}[(\ell_{k,k+h} - \ell_{k,k+h-1})d - 1] + p_{k+j} + j + 1 \\ &= \ell_{k,k+j-1}d + p_{k+j} + 1\end{aligned}$$

proving the claim. In a similar way we can check that $p_{k-i} + 1 = (\ell_{k-i,k} - \ell_{k-i+1,k})d$ for every $0 \leq i \leq k - 1$. By Claim 2.2.2 this concludes $G = A_n$.

Let $\boldsymbol{G{=}G_2}$, so that $m = 5$ in both cases, and write $\lambda = p\lambda_1 + q\lambda_2$. We can easily compute the set $\mathrm{Irr}(\lambda)$ (see also [LP21, Proofs of Propositions 4.1–4.2]): for $k = 1$ we have

$$\mathrm{Irr}(\lambda) = \left\{ \begin{array}{l} (p+1), (p+1) + (q+1), (p+1) + \frac{3}{2}(q+1), \\ (p+1) + 2(q+1), (p+1) + 3(q+1) \end{array} \right\}$$

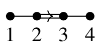
and for $k = 2$ we have

$$\mathrm{Irr}(\lambda) = \left\{ \begin{array}{l} (q+1), \frac{1}{3}(p+1) + (q+1), \frac{1}{2}(p+1) + (q+1), \\ \frac{2}{3}(p+1) + (q+1), (p+1) + (q+1) \end{array} \right\}.$$

For $k = 1$ the equalities $p+1 = d, (p+1)+(q+1) = 2d$ forces $\mathrm{Irr}(\lambda) = \{d, 2d, 5d/2, 3d, 4d\}$. For $k = 2$, the equalities $q+1 = d, (p+1)+(q+1) = 5d$ yield $\mathrm{Irr}(\lambda) = \{d, 7d/3, 3d, 11d/3, 5d\}$. It's clear that there cannot be any irreducible equivariant Ulrich bundle on $(X, \mathcal{O}_X(d))$ as for $d = 1$ [LP21, Propositions 4.1–4.2].

In remaining cases $G = F_4, E_6, E_7, E_8$ the strategy is to find roots $\beta_i \in \Phi^+ \backslash \Phi(\alpha_k)$ for $1 \leq i \neq k \leq n$ whose corresponding singular value is of the form $t_i = d + (p_{j_i}+1) + \cdots + (p_{h_i}+1)$. As each t_i has to be a multiple of d by Lemma 2.2.1, an elementary inductive argument as for A_n will show that each $(p_i + 1)$ is of the form $\ell_i d$ for some integer ℓ_i. Once done that, Claim 2.2.2 will provide the conclusion.

Let $\boldsymbol{G{=}F_4}$. The simple roots satisfy $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2, (\alpha_3, \alpha_3) = (\alpha_4, \alpha_4) = 1$.

(F_4) 

For $X = F_4/P_1$, the singular values corresponding to $\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 \in \Phi^+ \backslash \Phi(\alpha_1)$ are

$$\{d + (p_2 + 1), d + (p_2 + 1) + (p_3 + 1), d + (p_2 + 1) + (p_3 + 1) + (p_4 + 1)\} \subset \mathrm{Irr}(\lambda).$$

For $X = F_4/P_2$, by taking $\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 \in \Phi^+ \backslash \Phi(\alpha_2)$ we get

$$\{d + (p_1 + 1), d + (p_1 + 1) + (p_3 + 1), d + (p_1 + 1) + (p_3 + 1) + (p_4 + 1)\} \subset \mathrm{Irr}(\lambda).$$

For $X = F_4/P_3$, the roots $\alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 \in \Phi^+ \backslash \Phi(\alpha_3)$ give the singular values

$$\{d + p_4 + 1, d + p_2 + 1, d + (p_1 + 1) + (p_2 + 1) + (p_4 + 1)\} \subset \mathrm{Irr}(\lambda).$$

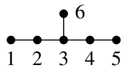
For $X = F_4/P_4$, we get

$$\{d + (p_3 + 1), d + (p_2 + 1) + (p_3 + 1), d + (p_1 + 1) + (p_2 + 1) + (p_3 + 1)\} \subset \mathrm{Irr}(\lambda)$$

from the roots $\alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 \in \Phi^+ \backslash \Phi(\alpha_4)$.

In each case it's clear that $(p_i + 1) = \ell_i d$ for some ℓ_i. So the conclusion follows by Claim 2.2.2. In particular there are no irreducible equivariant Ulrich bundles for $(F_4/P_k, \mathcal{O}_{F_4/P_k}(d))$ by [LP21, Propositions 6.1-6.3-6.4-6.5]. This concludes $G = F_4$.

Let ***G=E$_6$***. In this case we have $(\alpha_j, \alpha_j) = 2$ for all $1 \leq j \leq 6$.

(E_6) 

For $X = E_6/P_1 \cong E_6/P_5$ we get

$$\{d + (p_2 + 1) \cdots + (p_i + 1)\}_{2 \leq i \leq 6} \subset \mathrm{Irr}(\lambda)$$

from the roots $\alpha_1 + \cdots + \alpha_i \in \Phi^+ \backslash \Phi(\alpha_1)$.

For $X = E_6/P_2 \cong E_6/P_4$, the roots $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6 \in \Phi^+ \backslash \Phi(\alpha_2)$ yield

$$\left\{ \begin{array}{l} d + (p_1 + 1), d + (p_3 + 1), d + (p_1 + 1) + (p_3 + 1) + (p_4 + 1), \\ d + (p_3 + 1) + (p_4 + 1) + (p_5 + 1), d + (p_1 + 1) + (p_3 + 1) + (p_6 + 1) \end{array} \right\} \subset \mathrm{Irr}(\lambda).$$

For $X = E_6/P_3$, the singular values

$$\left\{ \begin{array}{l} d + (p_2 + 1), d + (p_4 + 1), d + (p_6 + 1), \\ d + (p_1 + 1) + (p_2 + 1), d + (p_4 + 1), d + (p_5 + 1) \end{array} \right\} \subset \mathrm{Irr}(\lambda)$$

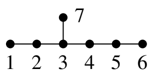
are given by $\alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_3 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3, \alpha_3 + \alpha_4 + \alpha_5 \in \Phi^+ \backslash \Phi(\alpha_3)$.

For $X = E_6/P_6$, we can take $\alpha_3 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \in \Phi^+ \backslash \Phi(\alpha_6)$ to obtain

$$\left\{ \begin{array}{l} d + (p_3 + 1), d + (p_2 + 1) + (p_3 + 1), d + (p_3 + 1) + (p_4 + 1), \\ d + (p_1 + 1) + (p_2 + 1) + (p_3 + 1), d + (p_3 + 1) + (p_4 + 1) + (p_5 + 1) \end{array} \right\} \subset \mathrm{Irr}(\lambda).$$

We conclude that $(p_i + 1)$ is a multiple of d. Claim 2.2.2 provides the conclusion for $G = E_6$.

Let ***G=E$_7$***. Again we have $(\alpha_j, \alpha_j) = 1$ for all $1 \leq j \leq 7$.

(E_7) 

For $X = E_7/P_1$ (resp. $X = E_7/P_2$) we easily get

$$\{d+(p_2+1)+\cdots+(p_i+1)\}_{2 \leq i \leq 7} \subset \mathrm{Irr}(\lambda) \text{ (resp. } \{d + (p_1 + 1) + \cdots + (p_i + 1)\}_{1 \leq i \leq 7, i \neq 2} \subset \mathrm{Irr}(\lambda))$$

from the roots $\alpha_1 + \cdots + \alpha_i \in \Phi^+ \backslash \Phi(\alpha_1)$ for $2 \leq i \leq 7$ (resp. $\alpha_1 + \cdots + \alpha_i \in \Phi^+ \backslash \Phi(\alpha_2)$ for $2 \leq i \leq 7$).

For $X = E_7/P_3$, by taking $\alpha_2+\alpha_3, \alpha_3+\alpha_4, \alpha_3+\alpha_7, \alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6 \in \Phi^+\backslash\Phi(\alpha_3)$ we obtain the singular values

$$\left\{\begin{array}{l} d+(p_2+1), d+(p_4+1), d+(p_7+1), d+(p_1+1)+(p_2+1)+(p_3+1), \\ d+(p_1+1)+(p_2+1)+(p_3+1)+(p_4+1)+(p_5+1), \\ d+(p_1+1)+(p_2+1)+(p_3+1)+(p_4+1)+(p_5+1)+(p_6+1) \end{array}\right\} \subset \mathrm{Irr}(\lambda).$$

For $X = E_7/P_4$, the singular values associated with $\alpha_3+\alpha_4, \alpha_4+\alpha_5, \alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+\alpha_4, \alpha_3+\alpha_4+\alpha_7, \alpha_1+\cdots+\alpha_4 \in \Phi^+\backslash\Phi(\alpha_4)$ are

$$\left\{\begin{array}{l} d+(p_3+1), d+(p_5+1), d+(p_5+1)+(p_6+1), d+(p_2+1)+(p_3+1), \\ d+(p_3+1)+(p_7+1), d+(p_1+1)+(p_2+1)+(p_3+1) \end{array}\right\} \subset \mathrm{Irr}(\lambda).$$

For $X = E_7/P_5$ take $\alpha_4+\alpha_5, \alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5, \alpha_3+\alpha_4+\alpha_5+\alpha_7, \alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\cdots+\alpha_5 \in \Phi^+\backslash\Phi(\alpha_5)$ to get

$$\left\{\begin{array}{l} d+(p_4+1), d+(p_6+1), d+(p_3+1)+(p_4+1), \\ d+(p_3+1)+(p_4+1)+(p_7+1), d+(p_2+1)+(p_3+1)+(p_4+1), \\ d+(p_1+1)+(p_2+1)+(p_3+1)+(p_4+1) \end{array}\right\} \subset \mathrm{Irr}(\lambda).$$

For $X = E_7/P_6$ take $\alpha_5+\alpha_6, \alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\cdots+\alpha_6 \in \Phi^+\backslash\Phi(\alpha_6)$ to get

$$\left\{\begin{array}{l} d+(p_5+1), d+(p_4+1)+(p_5+1), d+(p_3+1)+(p_4+1)+(p_5+1), \\ d+(p_3+1)+(p_4+1)+(p_5+1)+(p_7+1), \\ d+(p_2+1)+(p_3+1)+(p_4+1)+(p_5+1), \\ d+(p_1+1)+(p_2+1)+(p_3+1)+(p_4+1)+(p_5+1) \end{array}\right\} \subset \mathrm{Irr}(\lambda).$$

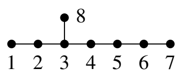
For $X = E_7/P_7$ we find

$$\left\{\begin{array}{l} d+(p_3+1), d+(p_2+1)+(p_3+1), d+(p_3+1)+(p_4+1), \\ d+(p_1+1)+(p_2+1)+(p_3+1), d+(p_3+1)+(p_4+1)+(p_5+1), \\ d+(p_3+1)+(p_4+1)+(p_5+1)+(p_6+1) \end{array}\right\} \subset \mathrm{Irr}(\lambda)$$

as singular values corresponding to $\alpha_3+\alpha_7, \alpha_2+\alpha_3+\alpha_7, \alpha_3+\alpha_4+\alpha_7, \alpha_1+\alpha_2+\alpha_3+\alpha_7, \alpha_3+\alpha_3+\alpha_4+\alpha_5+\alpha_7, \alpha_3+\cdots+\alpha_7 \in \Phi^+\backslash\Phi(\alpha_7)$.

In conclusion, d divides (p_i+1) in each case as required by Claim 2.2.2. This ends $G = E_7$.

Finally let $\boldsymbol{G=E_8}$. We have $(\alpha_j, \alpha_j) = 1$ for all $1 \le j \le 8$.

(E_8) 

For $X = E_8/P_1$ (resp. $X = E_8/P_2$) we easily get

$\{d+(p_2+1)+\cdots+(p_i+1)\}_{2\le i\le 8} \subset \mathrm{Irr}(\lambda)$ (resp. $\{d+(p_1+1)+\cdots+(p_i+1)\}_{1\le i\le 8, i\ne 2} \subset \mathrm{Irr}(\lambda)$)

from the roots $\alpha_1+\cdots+\alpha_i \in \Phi^+\backslash\Phi(\alpha_1)$ (resp. $\alpha_1+\cdots+\alpha_i \in \Phi^+\backslash\Phi(\alpha_2)$).

For $X = E_8/P_3$, from $\alpha_2+\alpha_3, \alpha_3+\alpha_4, \alpha_3+\alpha_8, \alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7 \in \Phi^+\backslash\Phi(\alpha_3)$ we obtain the singular values

$$\left\{\begin{array}{l} d+(p_2+1), d+(p_4+1), d+(p_8+1), \\ d+(p_1+1)+(p_2+1)+(p_3+1), \\ d+(p_1+1)+(p_2+1)+(p_3+1)+(p_4+1)+(p_5+1), \\ d+(p_1+1)+(p_2+1)+(p_3+1)+(p_4+1)+(p_5+1)+(p_6+1) \\ d+(p_1+1)+(p_2+1)+(p_3+1)+(p_4+1)+(p_5+1)+(p_6+1)+(p_7+1). \end{array}\right\} \subset \mathrm{Irr}(\lambda).$$

For $X = E_8/P_4$, take $\alpha_3+\alpha_4, \alpha_4+\alpha_5, \alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+\alpha_4, \alpha_3+\alpha_4+\alpha_8, \alpha_1+\cdots+\alpha_4, \alpha_3+\cdots+\alpha_7 \in \Phi^+\backslash\Phi(\alpha_4)$ to get the singular values

$$\left\{\begin{array}{l} d+(p_3+1), d+(p_5+1), d+(p_5+1)+(p_6+1), d+(p_2+1)+(p_3+1), \\ d+(p_3+1)+(p_8+1), d+(p_1+1)+(p_2+1)+(p_3+1), \\ d+(p_3+1)+(p_5+1)+(p_6+1)+(p_7+1) \end{array}\right\} \subset \mathrm{Irr}(\lambda).$$

For $X = E_8/P_5$, the roots $\alpha_4+\alpha_5, \alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5, \alpha_3+\alpha_4+\alpha_5+\alpha_8, \alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_5+\alpha_6+\alpha_7, \alpha_1+\cdots+\alpha_5 \in \Phi^+\backslash\Phi(\alpha_5)$ provide the singular values

$$\left\{\begin{array}{l} d+(p_4+1), d+(p_6+1), d+(p_3+1)+(p_4+1), \\ d+(p_3+1)+(p_4+1)+(p_8+1), d+(p_6+1)+(p_7+1), \\ d+(p_2+1)+(p_3+1)+(p_4+1), \\ d+(p_1+1)+(p_2+1)+(p_3+1)+(p_4+1) \end{array}\right\} \subset \mathrm{Irr}(\lambda).$$

For $X = E_8/P_6$ the singular values associated to the roots $\alpha_6+\alpha_7, \alpha_5+\alpha_6, \alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_2+\cdots+\alpha_6, \alpha_1+\cdots+\alpha_6 \in \Phi^+\backslash\Phi(\alpha_6)$ are

$$\left\{\begin{array}{l} d+(p_7+1), d+(p_5+1), d+(p_4+1)+(p_5+1), \\ d+(p_3+1)+(p_4+1)+(p_5+1), \\ d+(p_3+1)+(p_4+1)+(p_5+1)+(p_7+1)+(p_8+1), \\ d+(p_2+1)+(p_3+1)+(p_4+1)+(p_5+1), \\ d+(p_1+1)+(p_2+1)+(p_3+1)+(p_4+1)+(p_5+1) \end{array}\right\} \subset \mathrm{Irr}(\lambda).$$

For $X = E_8/P_7$ we get

$$\{d+(p_{7-i}+1)+\cdots+(p_6+1)\}_{1\leq i\leq 6} \cup \{d+(p_1+1)+(p_2+1)+\cdots+(p_6+1)+(p_8+1)\} \subset \mathrm{Irr}(\lambda)$$

as singular values associated to $\alpha_{7-i}+\cdots+\alpha_7, \alpha_1+\alpha_2+\cdots+\alpha_8 \in \Phi^+\backslash\Phi(\alpha_7)$.

For $X = E_8/P_8$ we find

$$\left\{\begin{array}{l} d+(p_3+1), d+(p_2+1)+(p_3+1), d+(p_3+1)+(p_4+1), \\ d+(p_1+1)+(p_2+1)+(p_3+1), d+(p_3+1)+(p_4+1)+(p_5+1), \\ d+(p_3+1)+(p_4+1)+(p_5+1)+(p_6+1), \\ d+(p_3+1)+(p_4+1)+(p_5+1)+(p_6+1)+(p_7+1) \end{array}\right\} \subset \mathrm{Irr}(\lambda)$$

as singular values corresponding to $\alpha_3+\alpha_8, \alpha_2+\alpha_3+\alpha_8, \alpha_3+\alpha_4+\alpha_8, \alpha_1+\alpha_2+\alpha_3+\alpha_8, \alpha_3+\alpha_3+\alpha_4+\alpha_5+\alpha_8, \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_8, \alpha_3+\cdots+\alpha_8 \in \Phi^+\backslash\Phi(\alpha_8)$.

We deduce that all (p_i+1) are multiple of d in each case. By Claim 2.2.2, this concludes $G = E_8$ and the proof. □

2.3 Equivariant Ulrich bundles on Spinor tenfold and on 6-dimensional Lagrangian Grassmannian

In the rest of this section any rational homogeneous varieties $X = G/P_k \subset \mathbf{P}^N$ is embedded through the generator of the Picard group $\mathcal{E}_{\lambda_k} = \mathcal{O}_X(1)$.

The content of Proposition 2.0.2 is just a recollection of the next two Propositions 2.3.1–2.3.2.

2.3.1 Spinor variety $\mathbb{S}_{10}$

Let (V, σ) be a $2n$-dimensional vector space endowed with a non-degenerate symmetric form and let $G = \mathrm{SO}(V) = \mathrm{SO}(2n)$ be the special orthogonal group of V. The dual of the Cartan algebra of $\mathfrak{g}$ of G naturally lives in $\mathbf{R}^n$. Said $\{\varepsilon_i\}_{i=1}^n$ the canonical basis of $\mathbf{R}^n$, the root system is $\Phi = \left\{\pm(\varepsilon_i \pm \varepsilon_j) \text{ for } 1 \le i \neq j \le n\right\}$, the simple roots are

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \text{ for } 1 \le i \le n-1,\ \alpha_n = \varepsilon_{n-1} + \varepsilon_n,$$

giving $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \text{ for } 1 \le i < j \le n\}$, and the fundamental weights are

$$\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i \text{ for } 1 \le i \le n-2,\ \lambda_{n-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n),\ \lambda_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n).$$

The Spinor variety $\mathbb{S}_{2n}$ is one of the two connected components of the Orthogonal grassmannian $\mathrm{OGr}(n, 2n)$, which is the set of all isotropic n-dimensional subspaces in V with respect to the symmetric form σ, and can be interpreted as the rational homogeneous variety $X = \mathrm{SO}(V)/P$ with $P = P(\alpha_n)$. Furthermore, the embedding defined by $\mathcal{E}_{\lambda_n} = \mathcal{O}_{\mathbb{S}_{2n}}(1)$ factors in the closed immersion $\mathbb{S}_{2n} \subset \mathrm{OGr}(n, 2n) \subset \mathrm{Gr}(n, 2n)$ where $\mathrm{OGr}(n, 2n) \subset \mathrm{Gr}(n, 2n)$ is realized as the zero locus of $\sigma \in H^0(\mathrm{Gr}(n, 2n), S^2U^*)$, with U being the rank-n universal subbundle on $\mathrm{Gr}(n, 2n)$. Now, the set $\Phi^+ \backslash \Phi(\alpha_n)$ consists of the roots

$$\begin{aligned} \varepsilon_i + \varepsilon_j &= \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \text{ for } 1 \le i < j \le n-1, \\ \varepsilon_i + \varepsilon_n &= \alpha_i + \cdots + 2\alpha_{n-2} + \alpha_n \text{ for } 1 \le i \le n-1. \end{aligned}$$

Then $\dim \mathbb{S}_{2n} = \frac{n(n-1)}{2}$, the cotangent bundle $\Omega^1_{\mathbb{S}_{2n}}$ is irreducible, as this is an irreducible Hermitian symmetric variety, of highest weight $\xi = -\alpha_n = \lambda_{n-2} - 2\lambda_n$ and the canonical bundle is $K_{\mathbb{S}_{2n}} = \mathcal{O}_{\mathbb{S}_{2n}}(-2n+2)$. For $n = 5$ the spinor variety $\mathbb{S}_{10}$ is in fact a prime Fano 10-fold of index 8.

As is known from [Fon16, (18)], the set of singular values of a $P-$dominant weight λ is

$$\mathrm{Irr}(\lambda) = \left\{a_i + a_j\right\}_{1 \le i < j \le n} \tag{2.3}$$

where $\lambda + \rho = (a_1, \ldots, a_n)$ is the vector of coordinates with respect to the canonical basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$.[4]

In [Fon16, Proposition 6.6] Fonarev proved in particular that there are no irreducible equivariant Ulrich bundles on the Spinor variety $\mathbb{S}_{2n}$ for $n \ge 5$. Nevertheless, we prove that $\mathbb{S}_{10}$ support an equivariant Ulrich bundle which is extension of two irreducible equivariant

[4] There is a typo in the displayed equation [Fon16, (18)]. For our purposes, by Remark 2.1.4 it is enough to note that $(\lambda + \rho - t\lambda_n, \varepsilon_i + \varepsilon_j) = a_i + a_j - t$ for $1 \le i < j \le n$.

bundles.

Proposition 2.3.1. *There exists a* μ*-stable* SO(10)−*equivariant Ulrich bundle of rank* 120 *on the Spinor variety* $\mathbb{S}_{10}$.

Proof. Consider the P−dominant weights $\lambda = 2\lambda_4 + \lambda_5$ and $\mu = \lambda + \xi = \lambda_3 + 2\lambda_4 - \lambda_5$. By Littlewood-Richardson rule the corresponding irreducible equivariant bundles satisfies $\mathrm{Ext}^1(\mathcal{E}_\lambda, \mathcal{E}_\mu)^{\mathrm{SO}(10)} = H^1(\mathbb{S}_{10}, \mathcal{E}_\xi) = \mathbf{C}$ (see also [OR06, Theorem 4.3(i)]), which means that there exists a non-split equivariant exact sequence

$$0 \to \mathcal{E}_\mu \to \mathcal{E} \to \mathcal{E}_\lambda \to 0.$$

We claim that $\mathcal{E}$ is the desired Ulrich bundle. To see this, we can easily see that

$$\lambda + \rho = (\lambda_1 + \lambda_2 + \lambda_3 + 3\lambda_4 + 2\lambda_5) = \left(\frac{11}{2}, \frac{9}{2}, \frac{7}{2}, \frac{5}{2}, -\frac{1}{2}\right),$$
$$\mu + \rho = (\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4) = \left(\frac{11}{2}, \frac{9}{2}, \frac{7}{2}, \frac{3}{2}, -\frac{3}{2}\right)$$

giving $\mathrm{Irr}(\lambda) = \{2, 3, \ldots, 9, 10\}$ and $\mathrm{Irr}(\mu) = \{0, 2, 3, \ldots, 9, 10\}$ by (2.3). As immediate consequence we get the vanishings $H^i(\mathbb{S}_{10}, \mathcal{E}(-t)) = 0$ for all $i \geq 0$ if $2 \leq t \leq 10$. Regarding the cohomology of $\mathcal{E}(-1)$, first observe that $\lambda - \lambda_5 = 2\lambda_4$ is already dominant and that $r_5(\mu - \lambda_5 + \rho) = \lambda_1 + \lambda_2 + \lambda_3 + 3\lambda_4 + \lambda_5$. Therefore $\mathrm{id} \cdot (\lambda - \lambda_5) = \lambda - \lambda_5$, with inversion set $\Phi_{\mathrm{id}} = \emptyset$, and $r_5 \cdot (\mu - \lambda_5) = 2\lambda_4 \in D$. By Lemma 2.1.5 the boundary map

$$\partial\colon H^0(\mathbb{S}_{10}, \mathcal{E}_\lambda(-1)) = V_{0,0,0,2,0} \to V_{0,0,0,2,0} = H^1(\mathbb{S}_{10}, \mathcal{E}_\mu(-1))$$

is an isomorphism. Since $H^i(\mathbb{S}_{10}, \mathcal{E}_\lambda(-1)) = 0$ for $i > 0$ and $H^j(\mathbb{S}_{10}, \mathcal{E}_\mu(-1)) = 0$ for $j \neq i$ by Borel-Bott-Weyl theorem, we deduce that $H^i(\mathbb{S}_{10}, \mathcal{E}(-1)) = 0$ for all $i \geq 0$. We conclude that $\mathcal{E}$ is Ulrich. Now, as

$$h^0(\mathbb{S}_{10}, \mathcal{E}) = h^0(\mathbb{S}_{10}, \mathcal{E}_\lambda) = \dim V_\lambda = 1440$$

and $\deg \mathbb{S}_{10} = 12$, by Proposition 1.1.15 we see that $\mathrm{rk}(\mathcal{E}) = 120$. Finally, the graded bundle of $\mathcal{E}$ is $\mathrm{gr}(\mathcal{E}) = \mathcal{E}_\lambda \oplus \mathcal{E}_\mu$ by construction. Therefore $\mathcal{E}_\lambda$ is the unique equivariant subbundle of $\mathcal{E}$. Since $\mathcal{E}_\lambda$ is not Ulrich, Remark 2.1.7 yields the slope-stability of $\mathcal{E}$ as desired. □

2.3.2 Lagrangian grassmannian LGr(3, 6)

Let (V, ω) be a symplectic $2n$-dimensional vector space let $G = \mathrm{Sp}(V)$ be the symplectic group of V. The root system $\Phi \subset \mathbf{R}^n$ consists of the vectors $\pm(\varepsilon_i \pm \varepsilon_j)$ for $i \neq j$ and $\pm(2\varepsilon_i)$, where $\{\varepsilon_i\}_{i=1}^n$ is the canonical basis. The simple roots are

$$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \text{ for } 1 \leq i \leq n-1,\ \alpha_n = 2\varepsilon_n\},$$

so that $\Phi^+ = \{(\varepsilon_i \pm \varepsilon_j) \text{ for } 1 \leq i < j \leq n,\ 2\varepsilon_i \text{ for } 1 \leq i \leq n\}$, and the fundamental weights are $\{\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i\}_{i=1}^n$.

The Lagrangian grassmannian $\mathrm{LGr}(n, 2n)$ is the Grassmannian of n-dimensional subspace in V which are isotropic with respect to the symplectic form ω and can be interpreted as the rational homogeneous variety $X = \mathrm{Sp}(V)/P$, where $P = P(\alpha_n)$. The set $\Phi^+ \backslash \Phi(\alpha_k)$ consists of the roots

$$\varepsilon_i + \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n \text{ for } 1 \le i < j \le n,$$
$$2\varepsilon_i = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n \text{ for } 1 \le i \le n.$$

In particular $\dim X = \frac{n(n+1)}{2}$, the cotangent bundle $\Omega^1_X = \mathcal{E}_\xi$ is irreducible, X being an irreducible Hermitian symmetric variety, of highest weight $\xi = -\alpha_n = 2\lambda_{n-1} - 2\lambda_n$ and the canonical bundle is $K_X = \mathcal{O}_X(-n-1)$. So for $n = 3$ the Lagrangian grassmannian $\mathrm{LGr}(3,6)$ is in fact a prime Mukai manifold.

By [Fon16, Proposition 3.1], for any $P-$dominant weight λ we can write

$$\mathrm{Irr}(\lambda) = \left\{ \frac{a_i + a_j}{2} \right\}_{1 \le i \le j \le n} \tag{2.4}$$

where $\lambda + \rho = (a_1, a_2, \ldots, a_n)$ is the vector of coordinates with respect to the canonical basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$.

As we know from [Fon16], there are no irreducible equivariant Ulrich bundle on $\mathrm{LGr}(n, 2n)$ for $n \ge 3$. We prove that there exists an equivariant Ulrich bundle on $\mathrm{LGr}(3,6)$ which is extension of two irreducible bundles.

Proposition 2.3.2. *There exists a μ-stable* $\mathrm{Sp}(6)-$*equivariant Ulrich bundle of rank* 32 *on the Lagrangian Grassmannian* $\mathrm{LGr}(3,6)$.

Proof. The proof proceeds as the one of Proposition 2.3.1 Consider the irreducible equivariant bundle $\mathcal{E}_\lambda$ for $\lambda = \rho = \lambda_1 + \lambda_2 + \lambda_3$. Then $\overline{\lambda + \rho} = (6,4,2)$ immediately shows that

$$\mathrm{Irr}(\lambda) = \{2,3,4,5,6\}$$

by (2.4). Letting $\mu = \lambda + \xi$ with $\xi = 2\lambda_2 - 2\lambda_3$ being the highest weight of the cotangent bundle, by Littlewood-Richardson rule we can see that $\mathrm{Ext}^1(\mathcal{E}_\lambda, \mathcal{E}_\mu)^{\mathrm{Sp}(6)} = H^1(\mathrm{LGr}(3,6), \mathcal{E}_\xi) = \mathbf{C}$ (see also[OR06, Theorem 4.3(i)]). This means that there exists a non-split equivariant exact sequence

$$0 \to \mathcal{E}_\mu \to \mathcal{E} \to \mathcal{E}_\lambda \to 0.$$

We claim that $\mathcal{E}$ is our required Ulrich bundle. Indeed, $\mu + \rho = (6,4,0)$ so that $\mathrm{Irr}(\mu) = 0,2,3,4,5,6$ by (2.4). This immediately forces $H^i(\mathrm{LGr}(3,6), \mathcal{E}(-t)) = 0$ for all $i \ge 0$ if $2 \le t \le 6$. As $\lambda - \lambda_3 = \lambda_1 + \lambda_2$ is already dominant for G, we have $H^0(\mathrm{LGr}(3,6), \mathcal{E}_\lambda(-1)) = V_{1,1,0}$ and $H^j(\mathrm{LGr}(3,6), \mathcal{E}_\lambda(-1)) = 0$ for all $j > 0$. On the other hand, $r_3(\mu - \lambda_3 + \rho) = 2\lambda_1 + 2\lambda_2 + \lambda_1$ lies in the interior of the Weyl chamber. Therefore $H^1(\mathrm{LGr}(3,6), \mathcal{E}_\mu(-1)) = V_{r_3 \cdot (\mu - \lambda_3)} = V_{1,1,0}$ and $H^j(\mathrm{LGr}(3,6), \mathcal{E}_\mu(-1)) = 0$ for $j \ne 1$. The boundary map

$$\partial \colon H^0(\mathrm{LGr}(3,6), \mathcal{E}_\lambda(-1)) = V_{1,1,0} \to V_{1,1,0} = H^1(\mathrm{LGr}(3,6), \mathcal{E}_\mu(-1))$$

is an isomorphism by Lemma 2.1.5 and so $H^i(\mathrm{LGr}(3,6), \mathcal{E}(-1))$ vanishes for all $i \ge 0$ as well. This proves that $\mathcal{E}$ is Ulrich. As

$$h^0(\mathrm{LGr}(3,6)) = h^0(\mathrm{LGr}(3,6), \mathcal{E}_\lambda) = \dim V_{1,1,1} = 512$$

and $\deg(\mathrm{LGr}(3,6)) = 16$, Proposition 1.1.15 gives $\mathrm{rk}(\mathcal{E}) = 32$. By construction the graded bundle of $\mathcal{E}$ is just $\mathrm{gr}(\mathcal{E}) = \mathcal{E}_\lambda \oplus \mathcal{E}_\mu$, meaning that $\mathcal{E}_\mu$ is its unique equivariant subbundle. Since $\mathcal{E}_\mu$ is not Ulrich, Remark 2.1.7 implies that $\mathcal{E}$ is μ-stable. □

2.3.3 Prime Mukai manifolds

A *Mukai manifold* is a Fano variety X of dimension n and index $n-2$. We say that X is also *prime* if $\mathrm{Pic}(X) = \mathbf{Z} \cdot H$ and $K_X = -(n-2)H$. In that case, we say that $g = \frac{1}{2}H^n + 1$ is the *genus* of X.

Theorem 2.3.3. *Let X be a prime Mukai manifold of dimension $n \geq 4$ and genus g. Then $2 \leq g \leq 10$ and X can be described as follows:*

(1) If $g = 2$, then X is a double cover of the projective space $X \to \mathbf{P}^n$ branched along a sextic.

(2) If $g = 3$, then X is either a quartic hypersurface $X \subset \mathbf{P}^{n+1}$ or X is a double cover of a quadric $X \to Q_n \subset \mathbf{P}^{n+1}$ branched along a complete intersection of Q_n with a quartic.

(3) If $g = 4$, then $X \subset \mathbf{P}^{n+2}$ is a complete intersection of type $(2,3)$.

(4) If $g = 5$, then $X \subset \mathbf{P}^{n+3}$ is a complete intersection of type $(2,2,2)$.

(5) If $g = 6$, then $4 \leq n \leq 6$ and $X = X_n$ is

- *either a linear section of a quadric section of $\mathrm{Gr}(2,5) \subset \mathbf{P}^{10}$ in the Plücker embedding, i.e. $X_n = \mathrm{Gr}(2,5) \cap \mathbf{P}^{n+4} \cap Q_9 \subset \mathbf{P}^{10}$ with $Q_9 \subset \mathbf{P}^{10}$ being a quadric, and $4 \leq n \leq 5$ (ordinary case);*
- *or a double cover $X_n \to \mathrm{Gr}(2,5) \cap \mathbf{P}^{n+3} \subset \mathbf{P}^{10}$ of a linear section of $\mathrm{Gr}(2,5)$ in the Plücker embedding branched along X_{n-1} (special case).*

(6) For each $g \geq 7$, X is a linear section of a maximal prime Mukai $n(g)$-fold $X^{n(g)}_{2g-2}$ which is one of the following:

(a) If $g = 7$, then $X^{10}_{12} = \mathbb{S}_{10} \subset \mathbf{P}^{15}$ is the Spinor 10-fold.

(b) If $g = 8$, then $X^{8}_{14} = \mathrm{Gr}(2,6) \subset \mathbf{P}^{14}$ is the Grassmannian of planes in a 6-dimensional vector space in its Plücker embedding.

(c) If $g = 9$, then $X^{6}_{16} = \mathrm{LGr}(3,6) \subset \mathbf{P}^{13}$ is the Lagrangian grassmannian of isotropic 3-planes in a 8-dimensional vector space.

(d) If $g = 10$, then $X^{5}_{18} = \mathrm{G}_2/P_2 \subset \mathbf{P}^{13}$ is the G_2-manifold.

Proof. See [IP99, Chapter 5] and [Deb20, Theorem 1.1] for the case $g = 6$. □

Now we can prove Corollary 2.0.3.

Proof of Corollary 2.0.3. Complete intersections support Ulrich bundles by [HUB91] (see also [CMP21, Theorem 4.3.2]), so we get the cases $4 \leq g \leq 5$. Since both $\mathrm{Gr}(2,4) \subset \mathbf{P}^{10}$ and $Q_9 \subset \mathbf{P}^{10}$ support an Ulrich bundle by [CM15; HUB91], then so does the proper intersection $\mathrm{Gr}(2,4) \cap Q_9 \subset \mathbf{P}^{10}$ by [Cas20, Theorem 1.3] (see also [CMP21, Proposition 4.3.1]). As Ulrichness is preserved under hyperplane sections by Proposition 1.1.13, we get the ordinary case for $g = 6$. Finally, all the maximal Mukai manifolds $X^{n(g)}_{2g-2}$ for $7 \leq g \leq 9$ support an (equivariant) Ulrich bundle by [CM15] and by Proposition 2.0.2. The conclusion follows again because restriction of Ulrich bundles to linear sections is still Ulrich (Proposition 1.1.13). □

Part II

Positivity of Ulrich bundles

Chapter 3

Positivity of the first Chern class of an Ulrich bundle

We study the positivity of the first Chern class of an Ulrich bundle on a smooth projective variety. The goal is to extend [Lop22, Theorem 1] in the case of a globally generated polarization. In the first part of the section we will see a characterization in terms of Seshadri constants (see Appendix B.2 for a brief account) of the polarization revealing interesting features about the geometry of the variety.

We will make use of the following technical lemmas of general nature.

Lemma 3.0.1. *Let Y be a projective variety of dimension $m \geq 1$ and let L be a globally generated ample line bundle such that $L^m = 1$. Then $(Y, L) \cong (\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(1))$ via φ_L.*

Proof. Set $\varphi = \varphi_L$ and $\overline{Y} = \varphi(Y) \subset \mathbf{P}^N$, and let $H \subset \mathbf{P}^N$ be a hyperplane. Factor φ as the composition of $\overline{\varphi}\colon Y \longrightarrow \overline{Y}$ followed by the inclusion $\iota\colon \overline{Y} \hookrightarrow \mathbf{P}^N$. Projection formula gives $1 = L^m = \deg(\overline{\varphi}) \cdot (H^m \cdot \overline{Y})$, forcing $H^m \cdot \overline{Y} = \deg(\overline{\varphi}) = 1$. Using Proposition B.2.6(2), we see that $\overline{Y}$ is smooth of degree 1. This means that $\overline{Y} \cong \mathbf{P}^m$ is a m-linear subspace. Note that $\overline{\varphi}$ is birational: it is surjective and finite of degree 1, which means that $\overline{\varphi}^{\#}$ induces an extension of degree 1 of the function fields, namely an isomorphism $\mathbf{C}(Y) \cong \mathbf{C}(\overline{Y})$, which gives the claim. Then $\overline{\varphi}$ is an open immersion by [Liu02, Exexrcise 3.3.17(a)] and Zariski Main Theorem [Liu02, Corollary 4.4.6]. An open immersion which is also surjective is actually an isomorphism. Hence $\overline{\varphi}$ is an isomorphism as required. □

Lemma 3.0.2. *Let X be a projective variety of dimension and let B be an ample and globally generated line bundle. Let $x \in X$ be a smooth point and let $\mu\colon \widetilde{X} \longrightarrow X$ be the blow-up at x with exceptional divisor E. Then the line bundle $\mu^*B - E$ is semiample.*

Proof. Write $\varphi = \varphi_B$ and consider the linear series $|B \otimes \mathfrak{m}_x|$ of divisors in $|B|$ containing x. be the finite morphism determined by $|B|$. Its base scheme coincides with the (schematic) fibre $\varphi^{-1}(\varphi(x))$, which is a finite set by the finiteness of φ. By [Laz04a, Lemma 4.3.16], we have

$$H^0(\widetilde{X}, \mu^*B - E) \cong H^0(X, \mu_*(\mu^*B - E)) \cong H^0(X, B \otimes \mathfrak{m}_x).$$

Hence the base locus of $\mu^*B - E$ is the union of the finite set of points

$$\left\{\mu^{-1}(y) \mid y \in \varphi^{-1}(\varphi(x)), y \neq x\right\}$$

with a subvarietiety $V \subset E$. The restriction of $\mu^*B - E$ to each of those points is trivially ample, as well as the restriction to V because $\mathcal{O}_E(-E) \simeq \mathcal{O}_{\mathbf{P}^{n-1}}(1)$. Therefore the restriction of $\mu^*B - E$ to its base locus is ample by [Laz04a, Proposition 1.2.16(ii)]. The conclusion follows from Zariski-Fujita theorem [Fuj83, Theorem 1.10]. □

Lemma 3.0.3. *A line bundle L on a projective variety X is ample if and only if L is semiample and strictly nef.*

Proof. An ample line bundle is semiample because a sufficiently large multiple defines a closed embedding, and it is strictly nef by Nakai-Moishezon Kleiman criterion [Laz04a, Theorem 1.2.23]. Conversely, suppose that L is strictly nef and that $L^{\otimes m}$ is globally generated for some $m > 0$. Then for every irreducible curve $C \subset X$ we have $(mL) \cdot C = m(L \cdot C) > 0$ by strictly nefness. Then $L^{\otimes m}$ is ample by [Laz04a, Corollary 1.2.15], hence so is L [Har77, Proposition II.7.5]. □

The following result, which is a generalization of [Lop22, Lemma 7.1], shows the existence of a Seshadri curve at a point x for an ample globally generated line bundle B when $\varepsilon(B; x) = 1$.

Lemma 3.0.4. *Let X be a smooth projective variety of dimension $n \geq 1$ and let B be a globally generated ample line bundle on X. Let $x \in X$ be a point and let $\mu \colon \widetilde{X} \to X$ be the blow-up at x with exceptional divisor E. Then the following conditions are equivalent:*

(i) $\varepsilon(B; x) = 1$.

(ii) $\mu^*B - E$ *is not ample on* $\widetilde{X}$.

(iii) $\varepsilon(B; x) = 1$ *and there exists a Seshadri curve $\Gamma \subset X$ for B at x mapping finitely onto a line $L \subset \mathbf{P}^N$ through $(\varphi_B)_{|\Gamma} \colon \Gamma \to L$ with x as point of maximum multiplicity for Γ and being the only point in the fibre over $(\varphi_B)_{|\Gamma}(x)$.*

Proof. Clearly (iii) implies (i). If $\varepsilon(B; x) = 1$, then $\mu^*B - E$ cannot be ample by Lemma B.2.13(i). Hence (i) implies (ii). To conclude the proof, suppose that $\widetilde{B} = \mu^*B - E$ is not ample. It follows from Lemmas 3.0.2 - 3.0.3 that $\widetilde{B}$ cannot be strictly nef on $\widetilde{X}$. Therefore there exists an irreducible curve $\Gamma' \subset \widetilde{X}$ such that $\widetilde{B} \cdot \Gamma' = 0$. Since $-E_{|E}$ is ample, Γ' is not contained in E. Moreover, the irreducible curve $\Gamma = \mu(\Gamma')$ must contain x, otherwise we would have $(\mu^*B - E) \cdot \Gamma' = B \cdot \Gamma > 0$. It follows from [Har77, Corollary II.7.15] that Γ' is blow-up of Γ at x. Therefore, by [Laz04a, Lemma 5.1.10], we get

$$0 = (\mu^*B - E) \cdot \Gamma' = B \cdot \Gamma - \operatorname{mult}_x(\Gamma),$$

which says, together with Remark B.2.4, that $\varepsilon(B; x) = 1$ and that Γ is a Seshadri curve for B at x. Now, consider the restriction $\varphi' = (\varphi_B)_{|\Gamma} \colon \Gamma \to \varphi_B(\Gamma) =: L \subset \mathbf{P}^N$ and let $H \subset \mathbf{P}^N$ be a hyperplane. A consequence of Zariski's formula for finite extensions [ABV20, Equation (2.2)], combined with Remark B.2.4 for H and projection formula, implies

$$\operatorname{mult}_x(\Gamma) \leq \max_{y \in \Gamma} \operatorname{mult}_y(\Gamma) \leq \deg(\varphi') \cdot \max_{z \in L} \operatorname{mult}_z(L) \leq \deg(\varphi') \cdot (H \cdot L) = B \cdot \Gamma = \operatorname{mult}_x(\Gamma).$$

We deduce that all of these are equalities. In particular, one has $H \cdot L = \operatorname{mult}_z(L)$ for some $z \in L$, forcing L to be a line (Remark B.2.7), and

$$B \cdot \Gamma = \operatorname{mult}_x(\Gamma) = \max_{y \in \Gamma} \operatorname{mult}_y(\Gamma) = \deg(\varphi') = \deg(\varphi') \cdot \operatorname{mult}_{\varphi'(x)}(L). \tag{3.1}$$

This forces x to be the only point in the fibre over $\varphi'(x)$: the inverse image of an affine open neighborhood of $\varphi'(x)$ is an affine open neighborhood of x due to the finiteness of φ', hence the claim follows by [ABV20, $(*)$ condition]. This proves (iii) and completes the proof. □

Lemma 3.0.5. *Let X be a smooth projective variety of dimension $n \geq 1$ and let B be a globally generated ample line bundle on X. Let $x \in X$ be a point and let $\rho\colon X' \to X$ be the blow-up of X along the schematic fibre $\varphi_B^{-1}(\varphi_B(x))$ with exceptional divisor E'. Then $\varepsilon(B; \varphi_B^{-1}(\varphi_B(x))) \geq 1$. Moreover, if $\varphi_B^{-1}(\varphi_B(x))$ is smooth, the following are equivalent:*

(i) $\varepsilon(B; \varphi_B^{-1}(\varphi_B(x))) = 1$.

(ii) $\rho^* B - E'$ *is not ample.*

(iii) There exists an irreducible curve $C \subset X$ such that $B \cdot C = \sum_{j=1}^{q} \operatorname{mult}_{x_j}(C)$, where $\varphi_B^{-1}(\varphi_B(x)) = \{x_1, \ldots, x_q\}$.

Proof. The line bundle $\rho^* B - E'$ is generated by global sections because ρ is the blow-up along the base scheme of $|B \otimes \mathfrak{m}_x|$ (see the proof of Lemma 3.0.2). In particular it is nef. As the Seshadri constant is the supremum over all $\varepsilon \geq 0$ such that $\rho^* B - \varepsilon E'$ is nef, we deduce that $\varepsilon(B; \varphi_B^{-1}(\varphi_B(x))) \geq 1$.

We now turn to prove the second part of the statement. Henceforth we suppose that φ_B is unramified at every point of $\varphi_B^{-1}(\varphi_B(x))$, which amounts to say that the schematic fibre over $\varphi_B(x)$ is smooth. If (iii) holds, we get (i) by combining the first part and Proposition B.2.11(1). Moreover, (ii) directly follows from (i) by Lemma B.2.13(i). Now assume (ii). Since $\rho^* B - E'$ is base-point-free, to not get a contradiction, it cannot be strictly nef (Lemma 3.0.3). Therefore there is an irreducible curve $C' \subset X'$ such that $(\rho^* B - E') \cdot C' = 0$. The divisor $-E'_{|E'}$ is ample (see, e.g., the proof of Lemma B.2.13) and every subvariety in E' is contracted to a point, hence $C' \not\subset E'$. Moreover $C = \rho(C')$ passes through some x_i, for otherwise we would have $(\rho^* B - E') \cdot C' = B \cdot C > 0$ by the ampleness of B. Writing $E' = E_1 + \cdots + E_q$, by projection formula and [Laz04a, Lemma 5.1.10] we get

$$0 = (\rho^* B - E') \cdot C' = \rho^* B \cdot C' - \sum_{j=1}^{q} E_j \cdot C' = B \cdot C - \sum_{j=1}^{q} \operatorname{mult}_{x_j}(C).$$

Thus (iii) holds. □

We point out that the condition $\varepsilon(B, \varphi_B^{-1}(\varphi_B(x))) > 1$ is open even without assuming the smoothness of the fibre.

Proposition 3.0.6. *Let B be an ample and globally generated line bundle on a smooth projective variety X. If $\varepsilon(B; \varphi_B^{-1}(\varphi_B(y))) > 1$ for a point $y \in X$, then the locus*

$$\left\{x \in X \mid \varepsilon(B; \varphi_B^{-1}(\varphi_B(x))) > 1\right\}$$

contains a dense open subset.

Proof. The proof follows [EKL95, Lemma 1.4]. Write $\varphi = \varphi_B\colon X \to \mathbf{P}^N$, and, for every point $x \in X$, let $\mu_x\colon \widetilde{X}_x \to X$ be the blow-up along $\varphi^{-1}(\varphi(x))$ and let E_x be the exceptional divisor. Denote by Δ the diagonal of $\mathbf{P}^N \times \mathbf{P}^N$ and let $b\colon Z \to X \times X$ be the blow-up of $X \times X$ along $(\varphi \times \varphi)^{-1}(\Delta)$ with exceptional divisor E. Let $g = \pi_1 \circ b$ and $h = \pi_2 \circ b$ be respectively the composition of b with the first and the second projection from $X \times X$ to X. Denoting by Z_x every schematic fibre $g^{-1}(x)$, we see that $Z_x \cong \widetilde{X}_x$ and $g_{|Z_x} = \mu_x$. In particular, the restrictions $(h^*B)_{|Z_x}$ and $E_{|Z_x}$ are respectively $\mu_x^* B$ and E_x. Recall that $\mu_x^* B - E_x$ is generated by global sections, as μ_x is the blow-up of the base scheme of the linear series $|B \otimes \mathfrak{m}_x|$. It follows that $\varepsilon(B; \varphi^{-1}(\varphi(x))) \geq 1$ (see Lemma 3.0.4).

If $\varepsilon(B; \varphi^{-1}(\varphi(y))) > 1$, we claim that $\mu_y^* B - E_y$ is ample. By Lemma 3.0.2, it is enough to prove that $\mu_y^* B - E_y$ is strictly nef. Let $C \subset \widetilde{X}_y$ be an irreducible curve. If $E_y \cdot C < 0$, then $C \subset E_y$, whence C is contracted to a point. Consequently Nakai-Moishezon-Kleiman criterion for mapping [Laz04a, Corollary 1.7.9] gives

$$(\mu_y^* B - E_y) \cdot C = -E_y \cdot C > 0.$$

If $E_y \cdot C \geq 0$, taking a real number $\varepsilon > 1$ such that $\mu_y^* B - \varepsilon E_y$ is ample (Lemma B.2.14), we get

$$(\mu_y^* B - E_y) \cdot C = (\mu_y^* B - \varepsilon E_y) \cdot C + (\varepsilon - 1) E_y \cdot C > 0.$$

This proves the claim. Applying [Laz04a, Theorem 1.2.17] to the pair $(g \colon Z \to X, h^* B - E)$, we can find an open dense subset $U \subset X$ of points such that

$$(h^* B - E)_{|Z_x} = \mu_x^* B - E_x$$

is ample whenever $x \in U$. It follows by Lemma B.2.13(ii') that $\varepsilon(B; \varphi^{-1}(\varphi(x))) > 1$ for every $x \in U$. □

Definition 3.0.7. Let X be a smooth projective variety and let B be a globally generated ample line bundle.

A 1-*Seshadri curve for* B *at a point* x is an irreducible curve $\Gamma \subset X$ such that $B \cdot \Gamma = \operatorname{mult}_x(\Gamma)$. We say that X is *covered by* 1-*Seshadri curves for* B if for every point y there is a 1-Seshadri curve for B at y.

An irreducible curve $C \subset X$ is a 1-*Seshadri curve for* φ_B *at* x if $\varphi_B^{-1}(\varphi_B(x))$ is smooth and $B \cdot C = \sum_{j=1}^{q} \operatorname{mult}_{x_j}(C)$, where $\varphi_B^{-1}(\varphi_B(x)) = \{x = x_1, \ldots, x_q\}$. We say that X is *generically covered by* 1-*Seshadri curves for* φ_B if for a general point $y \in X$ there exists a 1-Seshadri curve for φ_B at y.

Remark 3.0.8. Let X be a smooth projective variety and let B be a globally generated ample line bundle on X. If X is not generically covered by 1-Seshadri curves for φ_B, then there exists a point $x \in X$ such that $\varphi_B^{-1}(\varphi_B(x))$ is smooth and $\varepsilon(B; \varphi_B^{-1}(\varphi_B(x))) > 1$. Actually, there is a dense open subset $V \subset X$ such that the schematic fibre over $\varphi_B(y)$ is smooth and there is no 1-Seshadri curve for φ_B at y, for every $y \in V$.

To see this, let $U = \varphi_B^{-1}(\varphi_B(X) - \operatorname{Ram}(\varphi_B))$ be the (dense) open subset where $\varphi_B^{-1}(\varphi_B(y))$ is smooth for every $y \in U$, and suppose, by contradiction, that $\varepsilon(B; \varphi_B^{-1}(\varphi_B(y))) \leq 1$ for every $y \in U$. Lemma 3.0.5 implies that $\varepsilon(B; \varphi_B^{-1}(\varphi_B(y))) = 1$ and, furthermore, that there exists a 1-Seshadri for φ_B at y, for every $y \in U$. On the other hand, as X is not generically covered by 1-Seshadri curves for φ_B, every non-empty open subset $W \subset X$ contains a point $z \in W$ such that either $\varphi_B^{-1}(\varphi_B(z))$ is singular, or does not admit a 1-Seshadri curve for φ_B at z. As $\varphi_B^{-1}(\varphi_B(y))$ is smooth for every $y \in U$, by taking $W = U$ we get a contradiction. Therefore there must exist $x \in U$ such that $\varepsilon(B; \varphi_B^{-1}(\varphi_B(x))) > 1$ as claimed. The conclusion follows by combining the first part with Proposition 3.0.6 and Lemma 3.0.5.

In a sort of analogy with [Lop22, Lemma 7.1], we will see in Remark 3.0.11 that if φ_B is unramified at x, then the 1-Seshadri curve is a *B-line,* as defined below.

Definition 3.0.9. Let X be a smooth projective variety and let B be a globally generated ample line bundle. A curve $C \subset X$ is said to be a *B-line* if $B \cdot C = 1$.

It's easily shown by Lemma 3.0.1 that B-lines are irreducible smooth rational curves.

Remark 3.0.10. Let X be a smooth projective variety and let B be an ample and globally generated line bundle. Let $C \subset X$ be a B-line and let $V \subset H^0(X, B)$ be any subspace such that $|V|$ is base-point-free. Then C is an irreducible smooth rational curve which is mapped isomorphically onto $\mathbf{P}^1$ by $(\varphi_V)_{|C}$.

Indeed, irreducibility is immediate from Nakai-Moishezon-Kleiman criterion [Laz04a, Theorem 1.2.23]: if $C_1, \dots, C_h$ are the irreducible components, then

$$1 = B \cdot C = \sum_{j=1}^{h} B \cdot C_j \geq h$$

forces $h = 1$, saying that $C = C_1$ is irreducible. Now, $B_{|C}$ is a globally generated ample line bundle on an irreducible curve having $\deg(B_{|C}) = 1$. Therefore $\varphi_{B_{|C}}$ induces an isomorphism with $\mathbf{P}^1$ by Lemma 3.0.1. Consider the subspace $V_C = \operatorname{Im}\left(V \to H^0(C, B_{|C})\right)$ determined by the restriction of sections. We prove that $(\varphi_V)_{|C} = \varphi_{B_{|C}}$. It's clear that $|V_C|$ is base-point-free, for otherwise there would be a point $x \in C$ such that $0 = s_{|C}(x) = s(x)$ for every $s \in V$ yielding a contradiction. The morphism $\varphi_V \colon X \to \mathbf{P}^N$ is finite, for otherwise $B_{|Z} \cong (\varphi_V^* \mathcal{O}_{\mathbf{P}^N}(1))_{|Z}$ would be trivial, hence non-ample, for every subvariety $Z \subset X$ of positive dimension which is contracted to a point. In particular $\dim V_C \geq 2$. As $V_C \subset H^0(C, B_{|C}) \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1)) \cong \mathbf{C}^2$, we must have $V_C = H^0(C, B_{|C})$.

Remark 3.0.11. Let X be a smooth projective variety and let B be a base-point-free ample divisor. Suppose there is a point $x \in X$ such that $\varepsilon(B; x) = 1$ and let $\Gamma \subset X$ be a 1-Seshadri curve for B at x (Lemma 3.0.4). If φ_B is unramified at x, then Γ is a B-line. However the converse does not hold.

To prove this, first set $\varphi = \varphi_B$ and $\varphi' = (\varphi_B)_{|\Gamma}$. If φ is unramified at x, then so is φ' (Remark A.1.3) as the inclusion $\iota \colon \Gamma \hookrightarrow X$ is unramified [Sta23, Tag 02GC]. Therefore, using (3.1), [Sha13, Theorem 2.28] and Lemma 3.0.4.3, we immediately get

$$B \cdot \Gamma = \deg(\varphi') = \#\left((\varphi')^{-1}(\varphi(x))\right) = \#\{x\} = 1.$$

A counterexample for the opposite implication is given by Del Pezzo threefolds of degree 2: for every point there is a B-line passing through [Isk79, Proposition III.1.4(ii)], but φ has non-empty ramification locus (see Section 1.3).

The morphism induced by a very ample line bundle is unramified at all points. Therefore, Remark 3.0.11 shows that Lemma 3.0.4 reduces to [Lop22, Lemma 7.1] when B is very ample.

However, not all 1-Seshadri curves are B-lines.

Remark 3.0.12. A result due to Bogomolov and Mumford [MM83], or see [Huy16, Theorem 13.1.1], says that a general polarized K3 surface (S, B) of genus $g \geq 2$ contains a nodal integral rational curve $\Gamma \subset S$ such that $\Gamma \in |B|$. This says in particular that $|B|$ is base-point-free [Huy16, Proposition 2.3.5]. Letting $g = 2$, then the polarization gives rise to a finite double cover $\varphi_B \colon S \to \mathbf{P}^2$. Now, note that Γ contains 2 singular points: the genus formula implies that $p_a(\Gamma) = 2$, then, using that Γ is rational, we deduce that the number of nodes is $\delta = p_a(\Gamma) - p_g(\Gamma) = 2$. Taking a nodal point $x \in \Gamma$, we have

$$\mathrm{mult}_x(\Gamma) = 2 = 2g - 2 = B^2 = B \cdot \Gamma,$$

saying that $\varepsilon(B; x) = 1$. Therefore Γ is a 1-Seshadri curve for B at x having x as point of maximum multiplicity, Γ being nodal, which is mapped finitely onto a line (by construction), just as said in Lemma 3.0.4(iii). However, if (S, B) is chosen very general, we can suppose $\rho(S) = 1$, which means that $\mathrm{Pic}(S) \cong \mathbf{Z}$ [Huy16, Proposition 1.2.4]. In particular there cannot exist B-lines as 1-Seshadri curves.

It's interesting to observe that, despite $\mathrm{mult}_x(\Gamma) = 2$ and $x \in \mathrm{Ram}(\varphi)$, the exceptional divisor E of the blow-up $\mu \colon \widetilde{S} \to S$ at x is not contained in the base scheme of $\mu^*B - E$. To see this, we need to check $H^0(\widetilde{S}, \mu^*B - 2E) \subsetneq H^0(\widetilde{S}, \widetilde{B})$. We have $h^0(\widetilde{S}, \mu^*B - 2E) = h^0(S, B \otimes \mathfrak{m}_x^2)$ and $h^0(\widetilde{S}, \widetilde{B}) = h^0(S, B \otimes \mathfrak{m}_x) = 3 - 1 = 2$ by [Laz04a, Lemma 4.3.16]. Clearly $h^0(\widetilde{S}, \mu^*B - 2E) \geq 1$ as the strict transform $\widetilde{\Gamma}$ of Γ belongs to $|\mu^*B - 2E|$ [Har77, Proposition V.3.6]. On the other hand, if there was a different irreducible curve $\Gamma' \in |\mu^*B - 2E|$, we would have

$$-2 = (\mu^*B - 2E)^2 = \widetilde{\Gamma} \cdot \Gamma' = \sum_{P \in \widetilde{\Gamma} \cap \Gamma'} (\widetilde{\Gamma}.\Gamma')_P \geq 0$$

by [Har77, Proposition V.1.4], which is absurd. Alternatively, one can say that if $\Gamma' \in |\widetilde{\Gamma}|$, then

$$\widetilde{\Gamma} \cdot \Gamma' = (\mu^*B - 2E)^2 = -2$$

forces $\Gamma' \subset \widetilde{\Gamma}$, whence $\Gamma' = \widetilde{\Gamma}$. In any case we must have $h^0(\widetilde{S}, \mu^*B - 2E) = 1$.

To conclude this part, we see an example of 1-Seshadri curve at $\varphi_B^{-1}(\varphi_B(x))$.

Remark 3.0.13. Let $\varphi \colon S \to \mathbf{P}^2$ be a finite cover of degree q branched over a smooth irreducible plane curve $D \subset \mathbf{P}^2$ and let $B = \varphi^*\mathcal{O}_{\mathbf{P}^2}(1)$. Fix a point $x \in S$ such that $\varphi(x) \notin D$ and set $\varphi^{-1}(\varphi(x)) = \{x_1, \dots, x_q\}$. The pullback via φ of a line $L \subset \mathbf{P}^2$ passing through $\varphi(x)$ yields a divisor $C \in |B|$ containing all x_i's. Combining Lemma 3.0.5 and Proposition B.2.11(1) we immediately see

$$1 \leq \frac{B \cdot C}{\sum_{j=1}^{q} \mathrm{mult}_{x_j}(C)} \leq \frac{B^2}{q} = 1,$$

which says that C is a 1-Seshadri curve for B at $\varphi^{-1}(\varphi(x))$. Observe that C is smooth at each x_i: the morphism φ is unramified at x_i, therefore the map $d_{x_i}\varphi$ is injective (Definition-Theorem A.1.1.5). This amounts to say that B separates 1-jets at x_i [Laz04a, p. 273, lines 16-19], which means that the evaluation maps $H^0(S, B) \to H^0(S, B \otimes (\mathcal{O}_S/\mathfrak{m}_{x_i}^2))$ is surjective. Since $h^0(S, B) = h^0(S, B \otimes (\mathcal{O}_S/\mathfrak{m}_{x_i}^2)) = 3$, it follows that $H^0(S, B \otimes \mathfrak{m}_{x_i}^2) = 0$, saying that there are no divisors in $|B|$ which are singular at x_i.

To prove the main result of the chapter we will need a version of Castelnuovo-Mumford regularity theorem which allows the polarization to have a non-empty base locus. Regularity of a sheaf is typically defined with respect to a polarization which is at least globally generated, so we are going to adopt the following definition of regularity with respect to arbitrary ample line bundles for the rest of the chapter.

Notation 3.0.14. Let X be a projective variety and let A be an ample line bundle on X. A coherent sheaf $\mathcal{F}$ on X is *m-regular with respect to A* if $H^i(X, \mathcal{F}((m-i)A)) = 0$ for $i > 0$.

Proposition 3.0.15 (Castelnuovo-Mumford). *Let X be a projective variety of dimension $n \geq 1$, and let A be an ample line bundle on X. Let $Y = \mathrm{Bs}(|A|)$ be the base scheme of $|A|$ with base ideal $\mathcal{J} = \mathfrak{b}(|A|)$. Suppose that $\dim Y = 0$ and let $\mu\colon \widetilde{X} \to X$ be the blow-up of X along Y with exceptional divisor E. Let $\mathcal{F}$ be a vector bundle on X which is m-regular with respect to A. Then for every $k \geq \ell \geq 0$:*

(1) Assuming that $\mu^* A - E$ is ample on $\widetilde{X}$, then $\mathcal{F}((m+k)A)$ is generated by global sections.

(2) The multiplication map

$$\mu_{m,k}\colon H^0(X, \mathcal{F}(mA)) \otimes H^0(X, kA) \to H^0(X, \mathcal{F}((m+k)A) \otimes \mathcal{J}^{\otimes k})$$

and the map

$$\mu'_{m,k}\colon H^0(X, \mathcal{F}((m+k)A) \otimes \mathcal{J}^{\otimes k}) \to H^0(X, \mathcal{F}((m+k)A) \otimes \mathcal{J}^k)$$

are both surjective.

(3) $\mathcal{F} \otimes \mathcal{J}^{\otimes \ell}$ and $\mathcal{F} \otimes \mathcal{J}^{\ell}$ are $(m+k)$-regular with respect to A.

Observe that if we suppose that A is base-point-free, then $\mathcal{J} = \mathcal{O}_X$ and $\mu^* A - E = A$ and the assumption in (1) is just the ampleness of the polarization. Thus we recover the usual Castelnuovo-Mumford theorem B.1.3.

Proof. Letting $V = H^0(X, A)$, the evaluation map $V_X = V \otimes \mathcal{O}_X \to A$ is surjective off Y. Moreover the image of $V_X(-A) = V \otimes A^* \to \mathcal{O}_X$ is the base ideal $\mathcal{J}$. Denote by U the open subset $X - Y$. Following the construction in [Laz04a, §B.2], we can form the Koszul complex

$$0 \to \Lambda^r V_X(-rA) \to \cdots \to \Lambda^2 V_X(-2A) \to V_X(-A) \xrightarrow{\varepsilon} \mathcal{J} \to 0$$

which is exact off Y, where $r = \dim V$ and with ε being surjective. Tensoring through by $\mathcal{F}(sA) \otimes \mathcal{J}^{\otimes t}$, with $t \geq 0$, we obtain the following complex

$$0 \to \Lambda^r V_X \otimes \mathcal{F}((s-r)A) \otimes \mathcal{J}^{\otimes t} \to \cdots \to V_X \otimes \mathcal{F}((s-1)A) \otimes \mathcal{J}^{\otimes t} \xrightarrow{\delta} \mathcal{F}(sA) \otimes \mathcal{J}^{\otimes (t+1)} \to 0. \tag{3.2}$$

This remains exact off Y, since $(\mathcal{F}(sA) \otimes \mathcal{J}^{\otimes t})_{|U} \cong \mathcal{F}(sA)_{|U}$ is locally free, and δ is still surjective because tensor product is right exact. For every $1 \leq i \leq r$, denote by k_i the dimension of $\Lambda^i V$.

Step 1: we prove (3).

We first show the claim for $\mathcal{F} \otimes \mathcal{J}^{\otimes \ell}$. Let's proceed by induction on $k \geq \ell \geq 0$. The base case $k = \ell = 0$ is the hypothesis. Consider separately the case $\ell = 0$ with $k \geq 1$, and set $(s, t) = (m+k-i, 0)$, for $i > 0$, in (3.2). By induction we know that $\mathcal{F}$ is $(m+k-1)$-regular. Hence, we get the vanishing

$$H^{i+j}(X, \Lambda^{j+1} V_X \otimes \mathcal{F}((m+k-i-j-1)A)) \cong H^{i+j}(X, \mathcal{F}(((m+k-1)-(i+j))A))^{\oplus k_{j+1}} = 0$$

for $0 \leq j \leq r-1$. Applying [Laz04a, Proposition B.1.2] we immediately deduce that $H^i(X, \mathcal{F}((m+k-i)A)\otimes\mathcal{J}) = 0$ for $i > 0$, which amounts to say that $\mathcal{F}\otimes\mathcal{J}$ is $(m+k)$-regular. Twisting the short exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

through by $\mathcal{F}((m+k)A)$ and then taking the associated long exact sequence in cohomology, we obtain short exact sequences

$$0 = H^i(X\mathcal{F}((m+k)A)\otimes\mathcal{J}) \to H^i(X, \mathcal{F}((m+k)A)) \to H^i(Y, \mathcal{F}((m+k)A)_{|Y}) = 0$$

for every $i > 0$. This says that $\mathcal{F}$ is $(m+k)$-regular. Now, let $k \geq \ell \geq 1$, fix $i > 0$ and set $(s,t) = (m+k-i, \ell-1)$ in (3.2). Then for every $0 \leq j \leq r-1$ we have the vanishing

$$\begin{aligned}
&H^{i+j}(X, \Lambda^{j+1}V_X \otimes \mathcal{F}((m+k-i-j-1)A)\otimes\mathcal{J}^{\otimes(\ell-1)}) \\
&\cong H^{i+j}(X, \mathcal{F}(((m+k-1)-(i+j))A)\otimes\mathcal{J}^{\otimes(\ell-1)})^{\oplus k_{j+1}} \\
&= 0
\end{aligned}$$

due to the $(m+k-1)$-regularity of $\mathcal{F}\otimes\mathcal{J}^{\otimes(\ell-1)}$ given by the inductive hypothesis. Then [Laz04a, Proposition B.1.2] implies that $H^i(X, \mathcal{F}((m+k-i)A)\otimes\mathcal{J}^{\otimes\ell}) = 0$, completing the inductive step.

To prove the claim for $\mathcal{F}\otimes\mathcal{J}^{\ell}$, consider the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{J}^{\otimes q} \to \mathcal{J}^q \to 0$$

for $q \geq 1$. For any $x \in U$, the map $\mathcal{O}_{X,x} \cong \mathcal{J}_x^{\otimes q} \to \mathcal{J}_x^q \cong \mathcal{O}_{X,x}$ is an isomorphism, therefore $\mathrm{supp}(\mathcal{K}) = Y$ has dimension 0. As a consequence, $H^i(X, \mathcal{K}\otimes\mathcal{F}(pA)) = 0$ for every $i > 0$. Then the conclusion follows from the $(m+k)$-regularity of $\mathcal{F}\otimes\mathcal{J}^{\otimes\ell}$. In addition to this, the map

$$H^0(X, \mathcal{F}(pA)\otimes\mathcal{J}^{\otimes q}) \to H^0(X, \mathcal{F}(pA)\otimes\mathcal{J}^q) \tag{3.3}$$

is surjective for every $p \in \mathbf{Z}$.

Step 2: $\mathcal{F}((m+k)A)$ is generated by global sections at every point of Y.
Set $(s,t) = (m+k, 0)$ in (3.2). The $(m+k)$-regularity of $\mathcal{F}$ for $k \geq 0$ given by Step 1 implies the vanishing

$$H^{1+j}(X, \Lambda^{j+1}V_X \otimes \mathcal{F}((m+k-1-j)A)) \cong H^{1+j}(X, \mathcal{F}(((m+k)-(1+j))A))^{\oplus k_{j+1}} = 0$$

for each $0 \leq j \leq r$. We deduce from [Laz04a, Proposition B.1.2] that $H^1(X, \mathcal{F}((m+k)A)\otimes\mathcal{J}) = 0$. Taking the cohomology of the short exact sequence

$$0 \to \mathcal{F}((m+k)A)\otimes\mathcal{J} \to \mathcal{F}((m+k)A) \to \mathcal{F}((m+k)A)\otimes\mathcal{O}_Y \to 0,$$

we see that this vanishing forces the restriction map

$$H^0(X, \mathcal{F}((m+k)A)) \to H^0(X, \mathcal{F}((m+k)A)\otimes\mathcal{O}_Y)$$

to be surjective, and hence the claim.

Step 3: we prove (2)
We proceed by induction on $k \geq 0$ with trivial base case $k = 0$. For the inductive step $k \geq 1$,

set $(s,t) = (m+k, k-1)$ in (3.2). We know by Step 1 that $\mathcal{F} \otimes \mathcal{J}^{\otimes(k-1)}$ is $(m+k-1)$-regular, hence we have the vanishing

$$H^j(X, \Lambda^{1+j} V_X \otimes \mathcal{F}((m+k-1-j)A) \otimes \mathcal{J}^{\otimes(k-1)}) = 0$$

for every $j \geq 1$. Applying again [Laz04a, Example B.1.3] we obtain the surjectivity of the map

$$H^0(X, \mathcal{F}((m+k-1)A) \otimes \mathcal{J}^{\otimes(k-1)}) \otimes H^0(X, A) \to H^0(X, \mathcal{F}((m+k)A) \otimes \mathcal{J}^{\otimes k}).$$

By the inductive hypothesis we get a surjective map

$$H^0(X, \mathcal{F}(mA)) \otimes H^0(X, (k-1)A) \otimes H^0(X, A) \to H^0(X, \mathcal{F}((m+k)A) \otimes \mathcal{J}^{\otimes k}).$$

Factoring through the natural multiplication map $H^0(X, (k-1)A) \otimes H^0(X, A) \to H^0(X, kA)$, we obtain a commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{F}(mA)) \otimes H^0(X, (k-1)A) \otimes H^0(X, A) & \longrightarrow & H^0(X, \mathcal{F}((m+k)A) \otimes \mathcal{J}^{\otimes k}) \\ \downarrow & \nearrow & \\ H^0(X, \mathcal{F}(mA)) \otimes H^0(X, kA) & & \end{array}$$

giving the surjectivity of

$$\mu_{m,k} \colon H^0(X, \mathcal{F}(mA)) \otimes H^0(X, kA) \to H^0(X, \mathcal{F}((m+k)A) \otimes \mathcal{J}^{\otimes k}).$$

The claim for $\mu'_{m,k}$ in (2) is finally obtained using the surjectivity of (3.3).

Step 4: $\mathcal{F}(mA)$ is generated by global sections on $X - Y$ assuming the hypothesis in (1). The line bundle $\mu^* A - E$ is ample on $\widetilde{X}$, therefore the vector bundle

$$\mu^*(\mathcal{F}((m+k)A)) \otimes \mathcal{O}_{\widetilde{X}}(-kE) \cong \mu^*(\mathcal{F}(mA)) \otimes \mathcal{O}_{\widetilde{X}}(\mu^* A - E)^{\otimes k}$$

is generated by global sections for every $k \gg 0$. This amounts to say that the morphism

$$\mathrm{ev}_k \colon H^0(\widetilde{X}, \mu^*(\mathcal{F}(mA)) \otimes \mathcal{O}_{\widetilde{X}}(\mu^* A - E)^{\otimes k}) \otimes \mathcal{O}_{\widetilde{X}} \to \mu^*(\mathcal{F}(mA)) \otimes \mathcal{O}_{\widetilde{X}}(\mu^* A - E)^{\otimes k}$$

is surjective for every $k \gg 0$. Moreover, for $k \gg 0$ we also have the isomorphism

$$H^0(\widetilde{X}, \mu^*(\mathcal{F}((m+k)A)) \otimes \mathcal{O}_{\widetilde{X}}(-kE)) \cong H^0(X, \mathcal{F}((m+k)A) \otimes \mathcal{J}^k)$$

obtained by combining [Laz04a, Lemma 5.4.24] and projection formula. Fix $k \gg 0$ such that both properties hold. Since $\mu_U = \mu_{|\mu^{-1}(U)} \colon \mu^{-1}(U) \to U$ is an isomorphism, the functor $(\mu_U)_*$ preserves surjectivity. Therefore, by taking the direct image of ev_k through by μ_U, we can form the following commutative diagram on U:

$$\begin{array}{ccc} H^0(X, \mathcal{F}(mA)) \otimes (kA)_{|U} & \longrightarrow & \mathcal{F}((m+k)A)_{|U} \cong \left(\mathcal{F}((m+k)A) \otimes \mathcal{J}^k\right)_{|U} \\ \uparrow & & \uparrow \beta \\ H^0(X, \mathcal{F}(mA)) \otimes H^0(X, kA) \otimes \mathcal{O}_U & \xrightarrow{\alpha} & H^0(X, \mathcal{F}((m+k)A) \otimes \mathcal{J}^k) \otimes \mathcal{O}_U. \end{array}$$

The map α is surjective by Step 3, as well as $\beta = (\mu_U)_*(\mathrm{ev}_k)$ by the above discussion. Then the commutativity forces the map

$$H^0(X, \mathcal{F}(mA)) \otimes (kA)_{|U} \to \mathcal{F}((m+k)A)_{|U}$$

to be surjective. Twisting by the sheaf $((-k)A)_{|U}$, we finally obtain the claim.

Step 5: we prove (1)

Combining Steps 2 and 4 we get the case $k = 0$. The remaining cases are obtained in the same way using that $\mathcal{F}$ is $(m+k)$-regular for every $k \geq 1$. □

The following is a generalization of [Lop22, Theorem 1 & Theorem 7.2].

Theorem 3.0.16. *Let X be a smooth projective variety of dimension $n \geq 1$ and let B be a globally generated ample line bundle on X with $B^n = d$. Let $\mathcal{E}$ be a vector bundle of rank r which is 0-regular with respect to B. Then*

$$c_1(\mathcal{E})^k \cdot Z \geq r^k \mathrm{mult}_x(Z) \tag{3.4}$$

holds for every $x \in X$ and for every subvariety $Z \subset X$ of dimension $k \geq 1$ passing through x provided that the following conditions are satisfied:

(a) $x \notin \mathrm{Ram}(\varphi_B)$,

(b) $\varepsilon(B; \varphi_B^{-1}(\varphi_B(x))) > 1$.

In particular, if X is not generically covered by 1-Seshadri curves for φ_B, then $\mathcal{E}$ is V-big and

$$c_1(\mathcal{E})^n \geq r^n. \tag{3.5}$$

Moreover, if $\mathcal{E}$ is B-Ulrich of rank $r \geq 2$, then

$$c_1(\mathcal{E})^n \geq r(d-1). \tag{3.6}$$

Proof. Write $\varphi = \varphi_B$ and let $\mu\colon \widetilde{X} \to X$ be the blow-up at x with exceptional divisor E. Set $\widetilde{B} = \mu^* B - E$ and let $Y \subset \widetilde{X}$ be its base scheme.

Suppose x satisfies (a) and (b). We show that the pair $(\widetilde{X}, \widetilde{B})$ satisfies the hypotheses in Proposition 3.0.15. First of all we observe that $\widetilde{B}$ is ample on $\widetilde{X}$: φ is unramified at x, so the schematic fibre over $\varphi(x)$ is the disjoint union $\varphi^{-1}(\varphi(x)) = \{x\} \bigsqcup F$ with x being a smooth point for $\varphi^{-1}(\varphi(x))$ (Definition-Theorem A.1.1.2), whence Proposition B.2.11(3) yields $\varepsilon(B; x) \geq \varepsilon(B; \varphi^{-1}(\varphi(x))) > 1$. Thus Lemma 3.0.4 gives the assertion. Moreover, precisely by Definition-Theorem A.1.1.5, the base scheme Y is the strict transform of F under μ, in particular it is 0-dimensional by the finiteness of φ. Now, consider the blow-up $\rho\colon X' \to \widetilde{X}$ of $\widetilde{X}$ along Y with exceptional divisor E_Y. Then the line bundle $\rho^*\widetilde{B} - E_Y$ is base-point-free by construction. On the other hand, the composition $\pi = \rho \circ \mu\colon X' \to X$ is the blow-up of X along $\varphi^{-1}(\varphi(x))$ and its exceptional divisor is $\overline{E} = E' + E_Y$, where E' is the strict transform of E via ρ. If we show that $\rho^*\widetilde{B} - E_Y = \pi^* B - \overline{E}$ is also strictly nef, then it will be ample by Lemma 3.0.3, completing the claim. For, let $\overline{C} \subset X'$ be an irreducible curve. If $\overline{E} \cdot \overline{C} < 0$, then $\overline{C} \subset \overline{E}$ and it is contracted by π to a point. Therefore we have

$$(\pi^* B - \overline{E}) \cdot \overline{C} = -\overline{E} \cdot \overline{C} > 0.$$

Now suppose $\overline{E} \cdot \overline{C} \geq 0$. As B is ample, by Lemma B.2.14 we can find $\overline{\sigma} > 1$ such that $\pi^*B - \overline{\sigma}\overline{E}$ is ample. Then we get

$$(\pi^*B - \overline{E}) \cdot \overline{C} = (\pi^*B - \overline{\sigma}\overline{E}) \cdot \overline{C} + (\overline{\sigma} - 1)\overline{E} \cdot \overline{C} > 0,$$

hence proving the strictly nefness.

Now, for every integer $0 \leq s \leq n-1$, we have $R^j\mu_*\mathcal{O}_{\widetilde{X}}(sE) = 0$ for $j > 0$ and $\mu_*\mathcal{O}_{\widetilde{X}}(sE) = \mathcal{O}_X$, see e.g. [BEL91, Proof of Lemma 4.1]. Combining 0-regularity with respect to B, projection formula [Har77, Exercise III.8.3] and Leray spectral sequence [Har77, Exercise III.8.1], we obtain

$$H^i(\widetilde{X}, (\mu^*\mathcal{E} \otimes \mathcal{O}_{\widetilde{X}}(-E))(-i\widetilde{B})) = H^i(\widetilde{X}, \mu^*(\mathcal{E}(-iB)) \otimes \mathcal{O}_{\widetilde{X}}((i-1)E)) \cong H^i(X, \mathcal{E}(-iB)) = 0$$

for every $i > 0$. So $\widetilde{\mathcal{E}} = \mu^*\mathcal{E} \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is 0-regular with respect to the ample divisor $\widetilde{B}$. Since $\rho^*\widetilde{B} - E_Y$ is ample, it follows from Proposition 3.0.15(1) that $\widetilde{\mathcal{E}}$ is generated by global sections. Then $c_1(\widetilde{\mathcal{E}}) = \mu^*c_1(\mathcal{E}) - rE$ is base-point-free, so nef. By Kleiman's theorem [Laz04a, Theorem 1.4.9] and [Laz04a, Lemma 5.1.10], we deduce that

$$0 \leq (\mu^*(\mathcal{E}) - rE)^k \cdot \widetilde{Z} = c_1(\mathcal{E})^k \cdot Z + r^k(-1)^k(-1)^{k+1}\mathrm{mult}_x(Z) = c_1(\mathcal{E})^k \cdot Z - r^k\mathrm{mult}_x(Z),$$

where $\widetilde{Z} \subset \widetilde{X}$ is the strict transform of a subvariety $Z \subset X$ of dimension $k \geq 1$ passing through x. This proves (3.4).

To prove the last part of the statement, suppose X is not generically covered by 1-Seshadri curves for φ. By Remark 3.0.8, we can find a point $x \in X$ such that $\varphi^{-1}(\varphi(x))$ is smooth, which means that $x \notin \mathrm{Ram}(\varphi)$ (Definition-Theorem A.1.1.2), having $\varepsilon(B; \varphi^{-1}(\varphi(x))) > 1$. Then, choosing $Z = X$ in (3.4), we immediately get (3.5). The vector bundle $\widetilde{\mathcal{E}} = (\mu^*\mathcal{E})(-E)$ is generated by global sections, hence nef. Then Remark B.2.16(b) tells that $\varepsilon(\mathcal{E}; x) \geq 1$, which means $x \notin \mathbf{B}_+(\mathcal{E})$ by Remark B.2.17. By Theorem A.3.8, $\mathcal{E}$ is V-big as required. To conclude, suppose that $\mathcal{E}$ is B-Ulrich of rank $r \geq 2$. Note that under the assumption (b), it cannot be that $\mathcal{O}_X$ is B-Ulrich: if this was true, then we would have $d = h^0(X, \mathcal{O}_X) = 1$ by Lemma 1.1.1, yielding that $(X, B) \cong (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ (Lemma 3.0.1). However this is a contradiction since $\varepsilon(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), 1) = 1$. Consequently, a decomposition as $\mathcal{E} \cong \mathcal{O}_X^{\oplus(r-1)} \oplus \det(\mathcal{E})$ is not admissible. Therefore [Sie09, Theorem 1] and Lemma 1.1.1 imply that $c_1(\mathcal{E})^n \geq h^0(X, \mathcal{E}) - r = r(d-1)$, completing the proof. □

The hypothesis of Theorem 3.0.16 can be equivalently stated as follows.

Remark 3.0.17. Let $x \in X$ be a point in a smooth projective variety and let B be a globally generated ample line bundle on X. Consider the following conditions:

(a) $x \notin \mathrm{Ram}(\varphi_B)$.

(b) $\varepsilon(X, B; \varphi_B^{-1}(\varphi_B(x))) > 1$.

(c) $\varepsilon(X, B; x) > 1$.

(d) $\varepsilon(\widetilde{X}, \mu^*B - E; \mathrm{Bs}(|\mu^*B - E|)) > 1$, where $\mu\colon \widetilde{X} \to X$ is the blow-up at x with exceptional divisor E.

Then (a)+(b) is equivalent to (a)+(c)+(d).

In order to see this, let's first set $\varphi = \varphi_B$, $\widetilde{B} = \mu^* B - E$ and $Y = \mathrm{Bs}(|\widetilde{B}|)$. Henceforth we suppose that (a) holds. This says that Y is the strict transform under μ of the closed subscheme $Z \subset \varphi^{-1}(\varphi(x)) \subset X$, which does not contain x and such that $\varphi^{-1}(\varphi(x)) = \{x\} \bigsqcup Z$. Let $\rho \colon X' \to \widetilde{X}$ be the blow-up of $\widetilde{X}$ along Y with exceptional divisor E_Y. Set E' to be the strict transform of E via ρ. Then the composition $\pi = \rho \circ \mu \colon X' \to X$ is the blow-up of X along $\varphi^{-1}(\varphi(x))$ with exceptional divisor $\overline{E} = E' + E_Y$. In particular we have $\rho^* \widetilde{B} - E_Y = \pi^* B - \overline{E}$. Recall that $\mathcal{O}_{\overline{E}}(-\overline{E})$ is ample since the centre of the blow-up π is 0-dimensional (see the proof of Lemma B.2.13).

Suppose (c) and (d) hold. Then (c) says that $\widetilde{B}$ is ample (Lemma 3.0.4). We claim that $\rho^* \widetilde{B} - E_Y$ is strictly nef. Let $C' \subset X'$ be an irreducible curve. If $E_Y \cdot C' < 0$, then $C' \subset E_Y$, in particular it is contracted by ρ to a point, whence

$$(\rho^* \widetilde{B} - E_Y) \cdot C' = -E_Y \cdot C' > 0.$$

Suppose $E_Y \cdot C' \geq 0$. Since $\varepsilon(\widetilde{X}, \widetilde{B}; Y) > 1$, by Lemma B.2.14 we can find $\sigma > 1$ such that $\rho^* \widetilde{B} - \sigma E_Y$ is ample. (Here we are using the ampleness of $\widetilde{B}$.) Then we get

$$(\rho^* \widetilde{B} - E_Y) \cdot C' = (\rho^* \widetilde{B} - \sigma E_Y) \cdot C' + (\sigma - 1) E_Y \cdot C' > 0.$$

This proves the claim. Then it follows by Lemma 3.0.3 that $\rho^* \widetilde{B} - E_Y = \pi^* B - \overline{E}$ is ample. This implies that $\varepsilon(X, B; \varphi^{-1}(\varphi(x))) > 1$ (Lemma B.2.13(ii')), which is (b).

Conversely, assume that (b) holds. Then (c) follows from (b) (and (a)) by Proposition B.2.11(3). If we show that $\pi^* B - \overline{E} = \rho^* \widetilde{B} - E_Y$ is ample, then $\varepsilon(\widetilde{X}, \widetilde{B}; Y) \geq 1$ and is actually strictly greater than 1 because $\rho^* \widetilde{B} - \varepsilon(\widetilde{X}, \widetilde{B}; Y) E_Y$ cannot be ample (Lemma B.2.13(i)). Therefore (d) will follow once we have proved the ampleness of $\pi^* B - \overline{E} = \rho^* \widetilde{B} - E_Y$. As it is globally generated, by Lemma 3.0.3 we only need to show it is strictly nef. Let $\overline{C} \subset X'$ be an irreducible curve. If $\overline{E} \cdot \overline{C} < 0$, then $\overline{C} \subset \overline{E}$ and it is contracted to a point via π. Therefore we have

$$(\pi^* B - \overline{E}) \cdot \overline{C} = -\overline{E} \cdot \overline{C} > 0.$$

Now suppose $\overline{E} \cdot \overline{C} \geq 0$. As B is ample, by Lemma B.2.14 we can find $\overline{\sigma} > 1$ such that $\pi^* B - \overline{\sigma} \overline{E}$ is ample. Then we get

$$(\pi^* B - \overline{E}) \cdot \overline{C} = (\pi^* B - \overline{\sigma} \overline{E}) \cdot \overline{C} + (\overline{\sigma} - 1) \overline{E} \cdot \overline{C} > 0,$$

completing the proof.

This observation leads to the following geometrical interpretation.

Remark 3.0.18. Let $x \in X$ be a point in a smooth projective variety and let B be a globally generated ample line bundle on X. Consider conditions (a)-(b)-(c)-(d) in Theorem 3.0.16 and Remark 3.0.17. In light of Lemmas 3.0.4 - 3.0.5 and of Remark 3.0.11, (a)+(c) is equivalent to say that there is no B-line passing through x.

Remark 3.0.19. Let H be a very ample line bundle on a smooth projective variety X. The morphism $\varphi = \varphi_H$ is an embedding, hence $\varphi^{-1}(\varphi(x)) = \{x\}$ is smooth. Then, regarding Theorem 3.0.16: (a) is always satisfied and (b) is equivalent to say that there is no line passing through x (Lemmas 3.0.4 - 3.0.5 and Remark B.2.7). Thus the statement of Theorem 3.0.16 reduces to [Lop22, Theorem 7.2 & Theorem 1].

Remark 3.0.20. Let X be a smooth projective variety of dimension $n \geq 1$ and let B be an ample line bundle which is generated by global sections. In order to satisfy (b) in Theorem 3.0.16, the linear series $|B|$ cannot induce a finite morphism $\varphi = \varphi_B \colon X \to \mathbf{P}^n$ onto the projective space.

Indeed, if this happens, projection formula implies $d := B^n = \deg(\varphi) \cdot \deg(\mathbf{P}^n) = \deg(\varphi)$. Then, if $\varphi(x)$ is not a branch point, the fibre over $\varphi(x)$ consists of $\deg(\varphi) = d$ distinct smooth points, as $\mathbf{P}^n$ is smooth (Lemma A.1.4). Then (B.2) says that

$$\varepsilon(X, B; \varphi^{-1}(\varphi(x))) \leq \left(\frac{B^n}{d}\right)^{\frac{1}{n}} = 1,$$

forcing $\varepsilon(X, B; \varphi^{-1}(\varphi(x))) = 1$ (Lemma 3.0.5).

Let's see an example.

Example 3.0.21. Let (S, B) be a polarized abelian surface of type $(2, 2)$ which we assume to be non-isomorphic to the product of two elliptic curves. In virtue of [BL04, Exercise 8.11(1)], this is the general case. According to [BL04, §10.1, p. 282], the line bundle B is globally generated and $B = L^{\otimes 2}$ for an ample line bundle L. As S does not split, B induces a morphism $\varphi_B \colon S \to K \subset \mathbf{P}^3$ of degree 2 onto its image K, which is a Kummer surface.

Once clarified the setting, we can observe that the argument in [Bea16, Theorem 1] works also in the case of an ample and globally generated polarization. Therefore S supports a B-Ulrich vector bundle $\mathcal{E}$ of rank 2. Since $\varepsilon(L) \geq 4/3$ [Bau+09, Theorem 6.4.4(a)], by [Bau+09, (6.4.7)] and Proposition B.2.6(1) we obtain

$$\varepsilon(B; x_1, x_2) \geq \frac{1}{2}\varepsilon(B) = \varepsilon(L) \geq \frac{4}{3} > 1$$

for every pair of distinct points $x_1, x_2 \in S$. This holds in particular for all pairs of points lying on the fibre over $\varphi_B(x)$ with $x \notin \mathrm{Ram}(\varphi_B)$. Therefore $\mathcal{E}$ is V-big by Theorem 3.0.16.

The conditions in Theorem 3.0.16 are easy to handle when the polarization defines a birational morphism.

Remark 3.0.22. Let (X, B) is a smooth polarized projective variety with B generated by global sections such that $\varphi_B \colon X \to \mathbf{P}^N$ is birational onto its image. If X is not covered by 1-Seshadri curves for B, then every vector bundle which is 0-regular with respect to B is V-big.

Indeed, given a point $x \in X$ not belonging to any 1-Seshadri curve for B at x, that is $\varepsilon(B; x) > 1$ (Lemma 3.0.4), by [EKL95, Lemma 1.4] we can find a dense open subset $U \subset X$ where φ_B is an isomorphism and $\varepsilon(B; y) > 1$ for every $y \in U$. Then the assertion follows from Theorem 3.0.16.

Example 3.0.23. Let (S, B) be polarized abelian surface of type $(1, 4)$ which does not split as the product of two elliptic curves. Then [BL04, Exercise 8.11(1)] says that this is the general case. Then $\varphi_B \colon S \to \overline{S} \subset \mathbf{P}^3$ is a morphism which is birational onto a singular octic surface $\overline{S}$ [BL04, §10.5, p. 302, lines 19-24]. We also know that S supports a B-Ulrich vector bundle $\mathcal{E}$ (see Example 3.0.21). As $\varepsilon(B; 1) = \varepsilon(B) \geq 4/3$ by [Bau+09, Theorem 6.4.4(a)], $\mathcal{E}$ is V-big by the above Remark.

The conditions in Theorem 3.0.16 are not necessary for the bigness of an Ulrich bundle. Combining [LM21, Proof of Theorem 1 & Remark 2.2], we get the following characterization of big B-Ulrich bundles on surfaces.

Remark 3.0.24. Let (S, B) a smooth projective surface together with a base-point-free ample divisor and let $\mathcal{E}$ be a B-Ulrich bundle of rank r. Then $\mathcal{E}$ is big if and only if $c_1(\mathcal{E})^2 > 0$.

To see this, we can suppose that $r \geq 2$ because the claim is clear if $\mathcal{E}$ is a line bundle, $\mathcal{E}$ being nef. Necessity follows by [LM21, Remark 2.2]. For sufficiency, we know by hypothesis that $c_1(\mathcal{E})$ is big and nef. Hence $B^2 > 1$, for otherwise Lemma 3.0.1 would imply $(S, B, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1), \mathcal{O}_{\mathbf{P}^2}^{\oplus r})$ giving a contradiction as the c_1 would not be big. In particular, by Lemma 1.1.1, $h^0(X, \mathcal{E}) = r \cdot B^2 \geq r + 2$, and $H^1(X, \det(\mathcal{E})^*) = 0$ by [Laz04a, Theorem 4.3.1]. Then [BF20, Theorem 3.2] implies that $\mathcal{E}$ is big.

Example 3.0.25. Let $\varphi = \varphi_B \colon S \longrightarrow \mathbf{P}^2$ be a finite double cover branched along a general smooth sextic with $\mathcal{O}_S(B) \cong \varphi^*\mathcal{O}_{\mathbf{P}^2}(1)$. Then (S, B) is a smooth polarized K3 surface of genus 2 which does not satisfies the hypothesis in Theorem 3.0.16 by Remark 3.0.20. On the other hand, S supports a special B-Ulrich bundle $\mathcal{E}$ of rank 2 [ST22, Theorem 1.2]. Since $c_1(\mathcal{E})^2 = (3B)^2 > 0$, then $\mathcal{E}$ is big by Remark 3.0.24.

More generally, for minimal surfaces of non-negative Kodaira dimension we have the following.

Remark 3.0.26. Let S be a minimal smooth projective surface of Kodaira dimension $\kappa(S) \geq 0$ and let B be a globally generated ample line bundle on S. Then every special B-Ulrich bundle of rank 2 is big. In particular, if $\operatorname{Pic}(S) \cong \mathbf{Z}$, every B-Ulrich bundle of rank 2 is big.

Let's prove this. In case $\operatorname{Pic}(S) \cong \mathbf{Z}$, Lemma 1.1.16(ii) tells that every B-Ulrich bundle $\mathcal{F}$ of rank 2 has $c_1(\mathcal{F}) = K_S + 3B$, saying that it is special for B. Thus we only need to show the first part. Since K_S is nef by hypothesis, the Chern class $c_1(\mathcal{E}) = K_S(3B)$ is ample for every special B-Ulrich bundle $\mathcal{E}$ of rank 2. Hence $c_1(\mathcal{E})^2 > 0$ by Nakai-Moishezon-Kleiman criterion [Laz04a, Theorem 1.4.9], and the conclusion follows by Remark 3.0.24.

In this slightly wider setting in which we consider Ulrich bundles with respect to ample and globally generated line bundles, there are more non-big Ulrich bundles. In fact, the following examples are not ascribable to the list of non-big Ulrich bundles (with respect to a very ample divisor) on surfaces and threefolds in [LM21, Theorems 1 & 2].

Example 3.0.27. Let $(S, -K_S)$ be a Del Pezzo surface of degree $d = 2$ and let $\mathcal{E} = L(-K_S)$ be the $-K_S$-Ulrich line bundle of Proposition 1.2.4. Despite $\varepsilon(-K_S; x) = 4/3$ for a general point $x \in S$ [Bau+09, Theorem 6.3.4], we cannot apply Theorem 3.0.16 because we are in the situation of Remark 3.0.20. Indeed, we have $c_1(\mathcal{E})^2 = 0$ by construction. Thus $\mathcal{E}$, which is already nef, is not big.

Example 3.0.28. Consider the Del Pezzo threefold (X, B) of degree $d = 2$ and let $\mathcal{E}$ be the special B-Ulrich bundle of rank 2 constructed in Proposition 1.3.3. We are in the situation of Remark 3.0.20, hence we cannot apply Theorem 3.0.16. Due to $\mathcal{E}$ is globally generated, to show that $\mathcal{E}$ is non-big, it is enough to check that $s_3(\mathcal{E}^*) = 0$ [LM21, Remark 2.2]. Using [Laz04b, Examples 8.3.4 & 8.3.5] and the fact that $\mathcal{E}$ is special of rank 2, we get

$$s_3(\mathcal{E}^*) = s_{(1,1,1)}(\mathcal{E}) = c_1(\mathcal{E})^3 - 2c_1(\mathcal{E}) \cdot c_2(\mathcal{E}) + c_3(\mathcal{E}) = (2B)^3 - 2(2B) \cdot c_2(\mathcal{E}) = 16 - 4c_2(\mathcal{E}) \cdot B.$$

To compute the last intersection product, we are going to use [LR24a, Lemma 3.2(ii)] (which actually does not require the polarization to be very ample):

$$
\begin{aligned}
c_2(\mathcal{E}) \cdot B &= \frac{1}{2}\left[c_1(\mathcal{E})^2 - c_1(\mathcal{E}) \cdot K_X\right] \cdot B + \frac{2}{12}\left[K_X^2 + c_2(X) - \frac{44}{2}B^2\right] \cdot B \\
&= \frac{1}{2}\left[(2B)^2 \cdot B - (2B) \cdot (-2B) \cdot B\right] + \frac{1}{6}\left[(-2B)^2 \cdot B + c_2(X) \cdot B - 22(B)^3\right] \\
&= 8 + \frac{1}{6}\left[-36 + c_2(X) \cdot B\right].
\end{aligned}
$$

It remains to determine $c_2(X) \cdot B$. To this end, we apply Hirzebruch-Riemann-Roch theorem (see, e.g., [Har77, Theorem A.4.1]) on $\mathcal{O}_X$: since X is Fano, for every $i > 0$ we have $h^i(X, \mathcal{O}_X) = h^{3-i}(X, K_X) = 0$ by Kodaira vanishing theorem, therefore

$$
1 = \chi(X, \mathcal{O}_X) = \sum_{i=0}^{3} \mathrm{ch}_i(\mathcal{O}_X) \cdot \mathrm{td}_{3-i}(X) = \mathrm{rk}(\mathcal{O}_X) \cdot \frac{c_1(X) \cdot c_2(X)}{24} = \frac{c_1(K_X^*) \cdot c_2(X)}{24} = \frac{c_2(X) \cdot B}{12},
$$

which gives $c_2(X) \cdot B = 12$. In conclusion, we obtain

$$
s_3(\mathcal{E}^*) = 16 - 4\left[8 + \frac{1}{6}(12 - 36)\right] = 16 - 4 \cdot 4 = 0,
$$

giving the assertion.

Chapter 4

Augmented base locus of an Ulrich bundle

The main goal of this chapter is characterizing the augmented base locus of a B-Ulrich bundle (Theorem 4.0.11). To do this, we will make use of the characterization of the augmented base locus of a vector bundle in terms of Seshadri constants (see Definition B.2.15). Here we will need an additional hypothesis on B: we will suppose that $|B|$ admits a linear subsystem $|V| \subset |B|$ inducing a morphism which is étale onto the schematic image. In light of the previous section the assumption of being unramified is reasonable, in fact it turns out to be fundamental: in Remarks 4.0.17 - 4.0.18 we show that Theorem 4.0.11 and Corollary 4.0.12 no longer apply when φ_V is flat but not unramified (on the schematic image).

A B-Ulrich bundle is globally generated. Hence the stable base locus is clearly empty, as well as the restricted base locus by Proposition A.3.5(1). In view of Remark B.2.17, to determine its augmented base locus, it suffices to characterize the points where the Seshadri constant vanishes.

We begin with a result of general nature.

Lemma 4.0.1. *Let X be a smooth projective variety and let $\mathcal{E}$ be a globally generated vector bundle on X. Suppose there is an integral rational curve $C \subset X$ such that $\mathcal{E}_{|C}$ is not ample. Then $\varepsilon(\mathcal{E}; x) = 0$ for every $x \in C$.*

Proof. Let $\nu\colon C' \to C$ be the normalization of C and let $f = \iota \circ \nu\colon C' \to X$ be the composition of ν with the inclusion $\iota\colon C \hookrightarrow X$. Since $C' \simeq \mathbf{P}^1$, the vector bundle $f^*\mathcal{E} = \nu^*\mathcal{E}_{|C}$ has rank $r = \mathrm{rk}(\mathcal{E})$ and splits as

$$f^*\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_r),$$

where $a_1 \geq \cdots \geq a_r \geq 0$ since $f^*\mathcal{E}$ is globally generated. However, $f^*\mathcal{E} = \nu^*\mathcal{E}_{|C}$ cannot be ample for otherwise $\mathcal{E}_{|C}$ would be so [Laz04b, Proposition 6.1.8(iii)]. In particular, we have (at least) $a_r = 0$ [Laz04b, Example 6.1.3]. Then [FM21, Lemma 3.31] implies that $\varepsilon(f^*\mathcal{E}; y) = 0$ for every $y \in C'$. Consequently

$$\varepsilon(\mathcal{E}_{|C}; x) = \frac{\varepsilon(\nu^*\mathcal{E}_{|C}; y)}{\mathrm{mult}_x(C)} = 0$$

for any $y \in C'$ and every $x \in C$ (see [FM21, p. 12, lines 5-6]). For any $z \in X$, we have

$$\varepsilon(\mathcal{E}; z) = \inf_{z \in F \subset X} \varepsilon(\mathcal{E}_{|F}; z)$$

where F ranges over all irreducible curves in X passing through z (see, e.g., the proof of [FM21, Corollary 3.21]). Therefore, we obtain $0 \leq \varepsilon(\mathcal{E}; x) \leq \varepsilon(\mathcal{E}_{|C}; x) = 0$ for every $x \in C$, which gives the assertion. □

In this section we will use the following convention.

Notation 4.0.2. Given a projective scheme X and a base-point-free linear series $|V| \subset |L|$, let $\overline{X}$ be the schematic image of $\varphi_V \colon X \to \mathbf{P}^N$, that is just the reduced induced subscheme structure on $\varphi_V(X) \subset \mathbf{P}^N$. Let $\iota \colon \overline{X} \hookrightarrow \mathbf{P}^N$ denote the inclusion. By construction there is a morphism $\overline{\varphi_V} \colon X \to \overline{X}$ such that

$$\varphi_V = \iota \circ \overline{\varphi_V} \colon X \to \overline{X} \hookrightarrow \mathbf{P}^N,$$

which then satisfies $\overline{\varphi_V} \mathcal{O}_{\overline{X}}(1) \cong L$. Furthermore $\overline{\varphi_V}$ is finite when L is ample: in this case φ_V is finite (see, e.g., Remark 3.0.10), hence $\overline{\varphi_V}$ must be finite as well being proper and quasi finite. Henceforth we will make no distinction between φ_V and $\overline{\varphi_V}$, and we will use this fact without further reference.

Remark 4.0.3. The hypotheses of Theorems 4.0.11 - 4.0.20 and Corollary 4.0.12 are obviously satisfied by very ample linear series. However these are not the only ones: certainly such line bundles must separate tangents at all point, or equivalently must give rise a local isomorphism, but one can construct several examples of globally generated ample line bundles with a linear system inducing an étale morphism (onto the schematic image) which is not a global isomorphism. Let's proceed as follows. Take a smooth projective variety Y with either $b_1(Y) \neq 0$ or with $H_1(Y, \mathbf{Z})$ containing k-torsion. In both cases we can find a smooth projective variety X and a finite unramified morphism $\pi \colon X \to Y$ which is not an isomorphism [BPV84, Proposition 18.1], but which is étale by [Har77, Exercise III.9.3(a)]. Then the triple (X, B, V), where $B = \pi^* H$ for any very ample line bundle H on Y, and $V = \operatorname{Im}\left(\pi^* \colon H^0(Y, H) \to H^0(X, \pi^* H)\right) \simeq H^0(Y, H)$ [Gro65, Corollaire 2.2.8] satisfies those hypotheses since $\varphi_V = \varphi_H \circ \pi$. However it may happens that B remains very ample, as in the case of the canonical double cover $\pi \colon X \to Y$ of any Enriques surface $Y \subset \mathbf{P}^M$ embedded through any very ample H [GGP08, Lemma 3.4].

Examples (in any dimension) in which such a B is not very ample can be constructed as follows. Let $\overline{C} \subset \mathbf{P}^2$ be a smooth plane curve of degree $2d + 3$, with $d \geq 1$, and take a line bundle ϑ on $\overline{C}$ such that $2\vartheta = K_{\overline{C}}$ and $h^0(\overline{C}, \vartheta) = 0$ (see [Bea00, (4.1) & Remark 4.4]). The canonical bundle is $K_{\overline{C}} = \mathcal{O}_{\overline{C}}(2d)$, hence $L = \vartheta(-d)$ is a 2-torsion element in $\operatorname{Pic}(\overline{C})$. Then L gives rise to an étale double cover $p \colon C \to \overline{C}$ satisfying $p_* \mathcal{O}_C \cong \mathcal{O}_{\overline{C}} \oplus L$ [BPV84, Lemma 17.2]. Setting $B' = p^* \mathcal{O}_{\overline{C}}(d)$ and $V' = p^* H^0(\overline{C}, \mathcal{O}_{\overline{C}}(d))$, by projection formula we have

$$H^0(C, B') \cong H^0(\overline{C}, p_* p^* \mathcal{O}_{\overline{C}}(d)) \cong H^0(\overline{C}, \mathcal{O}_{\overline{C}}(d)) \oplus H^0(\overline{C}, \mathcal{O}_{\overline{C}}(d) \otimes L) = H^0(\overline{C}, \mathcal{O}_{\overline{C}}(d)),$$

saying that $|V'| = |B'|$. Therefore B' is non-very ample such that $\varphi_{B'} = \varphi_{\mathcal{O}_{\overline{C}}(d)} \circ p$ is étale (onto the schematic image). This pair (C, B') gives the desired example in dimension 1. For higher dimensions, take any smooth projective variety $Z \subset \mathbf{P}^N$ and set

$$Y = \overline{C} \times Z, \;\; H = \pi_1^* \left(\mathcal{O}_{\overline{C}}(d)\right) \otimes \pi_2^* \left(\mathcal{O}_Z(1)\right), \;\; M = \pi_1^* \left(\vartheta(-d)\right) \otimes \pi_2^* (T)$$

where $\pi_1 : Y \to \overline{C}, \pi_2 : Y \to Z$ are the projections and T is a (possibly trivial) line bundle on Z such that $T^{\otimes 2} = \mathcal{O}_Z$. Clearly H is very ample and M is of 2-torsion. Therefore we obtain a smooth projective variety X and an étale double cover $\pi : X \to Y$ such that $\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus M$ [BPV84, Lemma 17.2]. Setting $B = \pi^* H$ and $V = \pi^*(H^0(Y, H))$, then, by observing that Kunneth formula yields

$$H^0(Y, H \otimes M) \cong H^0(Y, \pi_1^*(\vartheta) \otimes \pi_2^*(T(1))) \cong H^0(\overline{C}, \vartheta) \otimes H^0(Z, T(1)) = 0,$$

we have

$$H^0(X, B) \cong H^0(Y, \pi_* \pi^* H) \cong H^0(Y, H) \oplus H^0(Y, H \otimes M) = H^0(Y, H).$$

We conclude that $|V| = |B|$, so that B is ample globally generated but non-very ample and such that $\varphi_B = \varphi_H \circ \pi$ is étale. This is the desired example. As a final remark, observe that these provides examples of 1-jet spanned line bundles, i.e. separating 1-jets at every point (see [Laz04a, p. 273, lines 16-19]), which are not very ample.

To prove the main result, we will study the "separation properties" of a B-Ulrich bundle $\mathcal{E}$. More precisely we will consider the restriction map $H^0(X, \mathcal{E}) \to H^0(Z, \mathcal{E}_{|Z})$, where Z is a 0-dimensional closed subscheme of length 2. There are two possibilities: either Z is contained in a B-line or not. Therefore it comes necessary to study also the restriction map to a B-line. To do this, we will use the separation lemmas in [LS23, §3].

Lemma 4.0.4. *Let X be a smooth projective variety and let B be a globally generated ample line bundle on X. Suppose there is a subspace $V \subset H^0(X, B)$ such that $\varphi = \varphi_V : X \to \varphi_V(X) = \overline{X} \subset \mathbf{P}^N$ is étale. Let $\mathcal{E}$ be a B-Ulrich bundle on X. Let $L \subset \overline{X}$ be a line and let $\overline{L} = \varphi^{-1}(L) \subset X$ be the scheme-theoretic inverse image of L. Then the restriction map*

$$H^0(X, \mathcal{E}) \to H^0(\overline{L}, \mathcal{E}_{|\overline{L}})$$

is surjective. In particular, the restriction map

$$H^0(X, \mathcal{E}) \to H^0(\Gamma, \mathcal{E}_{|\Gamma})$$

is surjective for every B-line $\Gamma \subset X$.

Proof. Since $\overline{L} = L \times_{\overline{X}} X$ [GW20, (4.11)], we have the cartesian diagram

$$\begin{array}{ccc} \overline{L} & \overset{\iota}{\hookrightarrow} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \varphi} \\ L & \overset{j}{\hookrightarrow} & \overline{X}, \end{array} \qquad (4.1)$$

where ι and j are the inclusions and f is the restriction of φ. The base change of an étale morphism is étale [Sta23, Tag 02GO], hence f is étale onto $L \simeq \mathbf{P}^1$. Since $\mathbf{P}^1$ is simply connected [Har77, Example IV.2.5.3], $\overline{L}$ decomposes as the disjoint union

$$\overline{L} = \Gamma_1 \bigsqcup \cdots \bigsqcup \Gamma_d,$$

where each Γ_i is isomorphic to $\mathbf{P}^1$ under $f_{|\Gamma_i} = \varphi_{|\Gamma_i}$. Denoting by ι_i the inclusion $\Gamma_i \subset X$ for each $i = 1, \dots, d$, we have $j \circ f_{|\Gamma_i} = \varphi \circ \iota_i$. In particular, we get

$$B_{|\Gamma_i} = \iota_i^* B \cong \iota_i^* \varphi^* \mathcal{O}_{\overline{X}}(1) = f_{|\Gamma_i}^* j^* \mathcal{O}_{\overline{X}}(1) = f_{|\Gamma_i}^* \mathcal{O}_L(1).$$

Then the projection formula $B \cdot \Gamma_i = \deg(f_{|\Gamma_i}) \cdot \deg(L) = 1$ says that Γ_i is a B-line.

The schematic image $\overline{X}$ is smooth [Liu02, Corollary 4.3.24], hence the sheaf $\overline{\mathcal{E}} = \varphi_* \mathcal{E}$ is an Ulrich bundle for $(\overline{X}, \mathcal{O}_{\overline{X}}(1))$ by Proposition 1.1.10. As (4.1) is cartesian and φ is affine, [GW20, Proposition 12.6] yields

$$\overline{\mathcal{E}}_{|L} = j^* \varphi_* \mathcal{E} \cong f_* \iota^* \mathcal{E} = f_*(\mathcal{E}_{|\overline{L}}). \tag{4.2}$$

Using [LS23, Lemma 3.3], we obtain the surjectivity of restriction map $H^0(\overline{X}, \overline{\mathcal{E}}) \to H^0(L, \overline{\mathcal{E}}_{|L})$. On the other hand, we have

$$H^0(X, \mathcal{E}) \cong H^0(\overline{X}, \varphi_* \mathcal{E}) = H^0(\overline{X}, \overline{\mathcal{E}})$$

and, thanks to (4.2),

$$H^0(\overline{L}, \mathcal{E}_{|\overline{L}}) \cong H^0(L, f_*(\mathcal{E}_{|\overline{L}})) \cong H^0(L, \overline{\mathcal{E}}_{|L}).$$

This proves the first part of the assertion. Moreover the space of global sections of $\mathcal{E}_{|\overline{L}}$ splits as

$$H^0(\overline{L}, \mathcal{E}_{|\overline{L}}) = H^0(\Gamma_1, \mathcal{E}_{|\Gamma_1}) \oplus \cdots \oplus H^0(\Gamma_d, \mathcal{E}_{|\Gamma_d}).$$

From this we deduce that the restriction map

$$H^0(\overline{L}, \mathcal{E}_{|\overline{L}}) \to H^0(\Gamma_i, \mathcal{E}_{|\Gamma_i})$$

is surjective for all i. Therefore, if $\Gamma \subset X$ is a B-line, the schematic image $L' = \varphi(\Gamma) \subset \overline{X}$ is a line (Remark 3.0.10) and Γ is one of the connected components of $\varphi^{-1}(L')$. By the previous part, we get the conclusion. □

If the 0-dimensional closed subscheme of length 2 is not contained in any fibre of φ_B, we will use the following.

Lemma 4.0.5. *Let X be a smooth projective variety and let B be a globally generated ample line bundle on X. Suppose there is a subspace $V \subset H^0(X, B)$ such that $\varphi_V \colon X \to \varphi_V(X) = \overline{X} \subset \mathbf{P}^N$ is étale. Let $\mathcal{E}$ be a B-Ulrich bundle on X. Let $Z \subset X$ be a 0-dimensional closed subscheme of length 2 which is not contained in any B-line. If Z satisfies $Z \not\subset \varphi_V^{-1}(\varphi_V(x))$ for every $x \in Z$, then*

$$r \colon H^0(X, \mathcal{E}) \to H^0(Z, \mathcal{E}_{|Z})$$

is surjective.

Proof. Set $\varphi = \varphi_V$ and $\overline{Z} = \varphi(Z) \subset \overline{X}$. Observe that $\overline{X}$ is smooth [Liu02, Corollary 4.3.24] and that the sheaf $\overline{\mathcal{E}} = \varphi_* \mathcal{E}$ is an Ulrich bundle for $(\overline{X}, \mathcal{O}_{\overline{X}}(1))$ by Proposition 1.1.10 (see also the proof of Lemma 4.0.4). Our assumption on Z implies that $\overline{Z}$ is 0-dimensional, closed and of length 2. There are two cases: there is no line in $\overline{X}$ containing $\overline{Z}$ or there exists a line $L \subset \overline{X}$ containing $\overline{Z}$.

Consider the first case. Then the restriction map

$$H^0(\overline{X}, \overline{\mathcal{E}}) \to H^0(\overline{Z}, \overline{\mathcal{E}}_{|\overline{Z}})$$

is surjective by [LS23, Lemma 3.2]. Now, let $Z' = \overline{Z} \times_{\overline{X}} X \subset X$ be the scheme-theoretic inverse image of $\overline{Z}$ [GW20, (4.11)] and consider the cartesian diagram

$$\begin{array}{ccc} Z' & \overset{i}{\hookrightarrow} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \overline{Z} & \overset{h}{\hookrightarrow} & \overline{X}, \end{array} \tag{4.3}$$

where i and h are the inclusions and g is the restriction of φ to Z'. Since φ is affine and (4.3) is cartesian, we can apply [GW20, Proposition 12.6] to get

$$\overline{\mathcal{E}}_{|\overline{Z}} = h^*\varphi_*\mathcal{E} \cong g_* i^*\mathcal{E} = g_*(\mathcal{E}_{|Z'}).$$

Since $H^0(X, \mathcal{E}) \cong H^0(\overline{X}, \overline{\mathcal{E}})$ and $H^0(Z', \mathcal{E}_{|Z'}) \cong H^0(\overline{Z}, g_*(\mathcal{E}_{|Z'}))$, we deduce that

$$H^0(X, \mathcal{E}) \to H^0(Z', \mathcal{E}_{|Z'})$$

is surjective. As $Z \subset Z'$ and $\dim Z' = 0$, also

$$H^0(Z', \mathcal{E}_{|Z'}) \to H^0(Z, \mathcal{E}_{|Z})$$

is surjective. Therefore r is onto as required.

Finally, assume that there exists a line $L \subset \overline{X}$ passing through $\overline{Z}$. Let $\overline{L}$ be the scheme-theoretic inverse image of L. We know that

$$\overline{L} = \Gamma_1 \bigsqcup \cdots \bigsqcup \Gamma_d$$

with Γ_i being a B-line (see the proof of Lemma 4.0.4). Since Z is not contained in any B-line, there must exist $1 \leq a \neq b \leq d$ such that $x \in \Gamma_a$ and $x' \in \Gamma_b$, where $Z = \{x, x'\}$. Observe that x and x' cannot be infinitely near for otherwise $\mathrm{supp}(Z) = \{x\}$ and $\Gamma_a \cap \Gamma_b \neq \emptyset$. The vector bundles $\mathcal{E}_{|\Gamma_a}$ and $\mathcal{E}_{|\Gamma_b}$ are globally generated, therefore

$$H^0(\Gamma_a, \mathcal{E}_{|\Gamma_a}) \to \mathcal{E}(x), \ H^0(\Gamma_b, \mathcal{E}_{|\Gamma_b}) \to \mathcal{E}(x')$$

are surjective. It follows that the restriction map

$$H^0(\overline{L}, \mathcal{E}_{|\overline{L}}) \to \mathcal{E}(x) \oplus \mathcal{E}(x') = H^0(Z, \mathcal{E}_{|Z})$$

is surjective. The conclusion follows from the surjectivity of $H^0(X, \mathcal{E}) \to H^0(\overline{L}, \mathcal{E}_{|\overline{L}})$ (Lemma 4.0.4). □

The next two simple results of general nature will be helpful in case the 0-dimensional closed subscheme of length 2 is contained in a fibre of φ_B.

Lemma 4.0.6. *Let X be a projective variety and let B be a globally generated ample line bundle on X. Let $\mathcal{F}$ be a vector bundle on X which is* 0*-regular with respect to B and let $V \subset H^0(X, B)$ be a non-zero vector subspace such that* $\dim \mathrm{Bs}(|V|) \leq 1$. *Then*

$$H^0(X, \mathcal{F}) \to H^0(\mathrm{Bs}(|V|), \mathcal{F}_{|\mathrm{Bs}(|V|)})$$

is surjective.

Proof. Set $V_X = V \otimes \mathcal{O}_X$ and $Y = \mathrm{Bs}(|V|) \subset X$. Let $s = \dim V$ and let $\mathcal{J} = \mathrm{Im}(V \otimes B^* \to \mathcal{O}_X)$ be the base ideal of $|V|$. As in [Laz04a, §B.2], we can form the Koszul complex

$$0 \to \Lambda^s V_X(-sB) \to \cdots \to \Lambda^2 V_X(-2B) \to V_X(-B) \xrightarrow{\varepsilon} \mathcal{J} \to 0,$$

which is exact off Y and with ε being surjective. Tensoring through by $\mathcal{F}$, we obtain the complex

$$0 \to \Lambda^s V_X \otimes \mathcal{F}(-sB) \to \cdots \to \Lambda^2 V_X \otimes \mathcal{F}(-2B) \to V_X \otimes \mathcal{F}(-B) \xrightarrow{\delta} \mathcal{F} \otimes \mathcal{J} \to 0.$$

This remains exact off Y and δ is still surjective. Let $k_i = \dim \Lambda^i V$ for $1 \leq i \leq s$. The 0-regularity of $\mathcal{F}$ yields the vanishing of the cohomology group

$$H^{1+j}(X, \Lambda^{1+j} V_X \otimes \mathcal{F}((-1-j)B)) \cong H^{1+j}(X, \mathcal{F}((-1-j)B))^{\oplus k_{j+1}} = 0$$

for every $0 \leq j \leq s$. It follows by [Laz04a, Proposition B.1.2] that $H^1(X, \mathcal{F} \otimes \mathcal{J}) = 0$. Then, taking the cohomology of the exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{J} \to \mathcal{F} \to \mathcal{F}_{|Y} \to 0,$$

we get the conclusion. □

Remark 4.0.7. Let $f : X \to Y$ be an affine morphism of schemes and let $\mathcal{F}$ be a coherent sheaf on X such that $f_*\mathcal{F}$ is generated by global sections. Suppose the schematic fibre $X_y = f^{-1}(y)$ over $y \in Y$ is non-empty. Then the restriction map

$$H^0(X, \mathcal{F}) \to H^0(X_y, \mathcal{F}_{|X_y})$$

is surjective.

Indeed, consider the cartesian diagram

$$\begin{array}{ccc} X_y = X \times_Y \mathrm{Spec}\,(\mathbf{C}(y)) & \xhookrightarrow{\;i\;} & X \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle f} \\ \{y\} = \mathrm{Spec}\,(\mathbf{C}(y)) & \xhookrightarrow{\;j\;} & Y \end{array}$$

with i and j being the inclusions and g the base change of f. Since $f_*\mathcal{F}$ is globally generated on Y, the map

$$H^0(Y, f_*\mathcal{F}) \to (f_*\mathcal{F})(y) = H^0(\{y\}, (f_*\mathcal{F})_{|\{y\}})$$

is surjective. On the other hand, [GW20, Proposition 12.6] gives the isomorphism

$$(f_*\mathcal{F})_{|\{y\}} = j^* f_*\mathcal{F} \cong g_* i^* \mathcal{F} = g_*(\mathcal{F}_{|X_y}).$$

As $H^0(X, \mathcal{F}) \cong H^0(Y, f_*\mathcal{F})$ and $H^0(X_y, \mathcal{F}_{|X_y}) \cong H^0(\{y\}, g_*(\mathcal{F}_{|X_y}))$, the assertion follows.

Lemma 4.0.8. *Let X be a smooth projective variety and let B be a globally generated ample line bundle on X. Suppose there is a subspace $V \subset H^0(X, B)$ such that $\varphi_V : X \to \varphi_V(X) = \overline{X} \subset \mathbf{P}^N$ is étale. Let $\mathcal{E}$ be a B-Ulrich bundle on X. Let $Z \subset X$ be a 0-dimensional closed subscheme of length 2 which is not contained in any B-line. Then*

$$r : H^0(X, \mathcal{E}) \to H^0(Z, \mathcal{E}_{|Z})$$

is surjective.

Proof. Write $\varphi = \varphi_V$ and denote by F_y the schematic fibre $\varphi^{-1}(\varphi(y))$ for every $y \in X$. If $Z \not\subset F_x$ for every $x \in Z$, the claim follows by Lemma 4.0.5. On the other hand, since $\varphi_* \mathcal{E}$ is Ulrich on $\overline{X}$ (see the proof of Lemma 4.0.4), hence globally generated, we know from Remark 4.0.7 that

$$r_y : H^0(X, \mathcal{E}) \to H^0(F_y, \mathcal{E}_{|F_y})$$

is surjective for every $y \in X$. As $\mathcal{E}$ is 0-regular and $\mathrm{Bs}(|\mathfrak{m}_x(B)|) = F_x$ is 0-dimensional, the surjectivity of r_x descends also from Lemma 4.0.6. In case $Z \subset F_x$ for $x \in Z$, the schemes F_x and Z consist of distinct points due to φ is unramified. Therefore

$$r' : H^0(F_x, \mathcal{E}_{|F_x}) \to H^0(Z, \mathcal{E}_{|Z})$$

is surjective. Since $r = r' \circ r_x$, the assertion follows. □

Lemma 4.0.9. *Let X be a smooth projective variety and let B be a globally generated ample line bundle on X. Suppose there is a subspace $V \subset H^0(X, B)$ such that $\varphi_V : X \to \varphi_V(X) = \overline{X} \subset \mathbf{P}^N$ is étale. Let $\mathcal{E}$ be a B-Ulrich bundle on X. Let $Z \subset X$ be a 0-dimensional closed subscheme of length 2 and let $r : H^0(X, \mathcal{E}) \to H^0(Z, \mathcal{E}_{|Z})$ be the restriction map. Then r is surjective if one of the following holds:*

(i) There is no B-line containing Z.

(ii) There is a B-line $\Gamma \subset X$ containing Z and $\mathcal{E}_{|\Gamma}$ is ample on Γ.

Proof. In case (i), the claim follows by Lemma 4.0.8. Then suppose that (ii) holds. Lemma 4.0.4 says that $r_\Gamma : H^0(X, \mathcal{E}) \to H^0(\Gamma, \mathcal{E}_{|\Gamma})$ is surjective, and the (very) ampleness of $\mathcal{E}_{|\Gamma}$ on $\Gamma \simeq \mathbf{P}^1$ (Remark 3.0.10) tells that $r_Z : H^0(\Gamma, \mathcal{E}_{|\Gamma}) \to H^0(Z, \mathcal{E}_{|Z})$ is surjective. The map r factors as the composition $r = r_Z \circ r_\Gamma$, hence the assertion follows. □

Now we relate the separation properties of the B-Ulrich bundle $\mathcal{E}$ with the Seshadri constant of the tautological line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Lemma 4.0.10. *Let X be a smooth projective variety and let B be a globally generated ample line bundle on X. Suppose there is a subspace $V \subset H^0(X, B)$ such that $\varphi = \varphi_V : X \to \varphi_V(X) = \overline{X} \subset \mathbf{P}^N$ is étale. Let $\mathcal{E}$ be a B-Ulrich bundle on X. Let $x \in X$ be a point and suppose one of the following conditions holds:*

(i) There are no B-lines passing through x.

(ii) $\mathcal{E}_{|\Gamma}$ is ample on every B-line $\Gamma \subset X$ passing through x.

Then $\varepsilon(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1); y) \geq 1$ for all $y \in \mathbf{P}(\mathcal{E}(x))$.

Proof. Let $\pi\colon \mathbf{P}(\mathcal{E}) \to X$ be the natural projection, and write $\xi = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ and $\xi_x := \xi_{|\mathbf{P}(\mathcal{E}(x))} \simeq \mathcal{O}_{\mathbf{P}^{\mathrm{rk}(\mathcal{E})-1}}(1)$. Fix a point $y \in \mathbf{P}(\mathcal{E}(x))$. According to Remark B.2.9, it suffices to show that the restriction map

$$r_Z\colon H^0(\mathbf{P}(\mathcal{E}), \xi) \to H^0(Z, \xi_{|Z})$$

to every 0-dimensional closed subscheme $Z \subset \mathbf{P}(\mathcal{E})$ of length 2 with $\mathrm{supp}(Z) = \{y\}$ is surjective. If $Z \subset \mathbf{P}(\mathcal{E}(x))$, since $\xi_{|Z} = (\xi_x)_{|Z}$, one has the commutative diagram

$$\begin{array}{ccc} H^0(\mathbf{P}(\mathcal{E}), \xi) & \xrightarrow{r_Z} & H^0(Z, \xi_{|Z}) \\ \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \cong} \\ H^0(\mathbf{P}(\mathcal{E}(x)), \xi_x) & \xrightarrow{r_{Z,x}} & H^0(Z, (\xi_x)_{|Z}). \end{array} \tag{4.4}$$

The map ρ is surjective [LS23, Remark 2.3], as well as $r_{Z,x}$ since ξ_x is very ample on $\mathbf{P}(\mathcal{E}(x))$. Therefore r_Z must be surjective by the commutativity of (4.4). Suppose $Z \not\subset \mathbf{P}(\mathcal{E}(x))$, so that $\pi(Z) \subset X$ is a 0-dimensional closed subscheme of length 2. Thanks to our assumptions, the restriction map $H^0(X, \mathcal{E}) \to H^0(\pi(Z), \mathcal{E}_{|\pi(Z)})$ is surjective by Lemma 4.0.9: if there is no B-line containing $\pi(Z)$, we are in case (i); if $\pi(Z) \subset \Gamma$ for a B-line $\Gamma \subset X$, then we are in the case (ii). It follows that r_Z is surjective as claimed. □

Now we are ready to prove the main result of the section.

Theorem 4.0.11. *Let X be a smooth projective variety and let B be a globally generated ample line bundle such that there is a linear series $|V| \subset |B|$ inducing a morphism which is étale onto its schematic image. Let $\mathcal{E}$ be a B-Ulrich bundle on X and let $x \in X$ be a point. Then $\varepsilon(\mathcal{E}; x) = 0$ if and only if there exists a B-line $\Gamma \subset X$ passing through x such that $\mathcal{E}_{|\Gamma}$ is not ample on Γ. In particular,*

$$\mathbf{B}_+(\mathcal{E}) = \bigcup_{\Gamma \subset X} \Gamma \tag{4.5}$$

where Γ ranges over all B-lines in X such that $\mathcal{E}_{|\Gamma}$ is not ample on Γ.

Proof. The last part of the assertion is a consequence of the characterization of the augmented base locus in Remark B.2.17. Hence we only need to show the first one. The "if part" follows by Lemma 4.0.1 and Remark 3.0.10. Conversely, suppose that $\varepsilon(\mathcal{E}; x) = 0$. By Remark A.3.4 we know that there is a point $z \in \mathbf{P}(\mathcal{E}(x))$ such that $z \in \mathbf{B}_+(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1))$, hence such that $\varepsilon(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1); z) = 0$ (Remark B.2.8). Let $\mathcal{B}_x$ be the set of all B-lines passing through x. Then $\mathcal{B}_x \neq \emptyset$, for otherwise Lemma 4.0.10(i) would imply $\varepsilon(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1); z) \geq 1$. Moreover, if $\mathcal{E}_{|\Gamma_x}$ is ample on every $\Gamma_x \in \mathcal{B}_x$, then $\varepsilon(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1); z) \geq 1$ by Lemma 4.0.10(ii), giving a contradiction. Therefore we conclude that there must exist a B-line $\Gamma \subset X$ such that $\mathcal{E}_{|\Gamma}$ is not ample. □

The following is an immediate consequence.

Corollary 4.0.12. *Let X be a smooth projective variety and let B be a globally generated ample line bundle such that there is a linear series $|V| \subset |B|$ inducing a morphism which is étale onto its schematic image. Let $\mathcal{E}$ be a B-Ulrich bundle. Then:*

(a) *$\mathcal{E}$ is V-big if and only if X is not covered by B-lines $\Gamma \subset X$ on which $\mathcal{E}_{|\Gamma}$ is not ample.*

(b) *$\mathcal{E}$ is ample if and only if either X contains no B-lines or $\mathcal{E}_{|\Gamma}$ is ample on every B-line $\Gamma \subset X$.*

Proof. This is a simple consequence of Theorems A.3.8 - 4.0.11 and of Proposition A.3.5(2). □

Let's see an easy example where the augmented base locus can be computed explicitly.

Example 4.0.13. Consider a Del Pezzo surface $S = S_d$ of degree $3 \leq d \leq 7$. Then $\varphi_{-K_S} : S \hookrightarrow \mathbf{P}^{9-d}$ is an embedding and, as is well known, there are finitely many lines lying on S. Indeed, looking at S as the blow-up of $\mathbf{P}^2$ at $9-d$ points $P_1, \ldots, P_{9-d}$ in general position, the lines are: the exceptional divisors $E_1, \ldots, E_{9-d}$, the strict transforms F_{ij} of the lines through the distinct points P_i, P_j with $1 \leq i < j \leq 9-d$ and, when $d = 3, 4$, the strict transforms $C_0 \subset S_4, C_i \subset S_3$ of the conics passing through 5 distinct points $P_{i_1}, \ldots, P_{i_5}$ with $1 \leq i_1 < \cdots < i_5 \leq 9-d$ and $i \neq i_1, \ldots, i_5$. Conversely, by arguing as in [Har77, Proof of Theorem V.4.9], one sees that any line is linearly equivalent to one of them. By adjunction formula a line is a (-1)-curve, therefore the each linear system $|\Gamma|$ must contain a single representative.

In our situation at least two of them are disjoint, say ℓ_1 and ℓ_2. Then $\mathcal{E} = (\ell_1 - \ell_2)(-K_S)$ is a $(-K_S)$-Ulrich line bundle on S [Bea18, Proposition 4.1(i)]. Since $\mathcal{E}^2 = d-2 > 0$, this is (nef and) big. Hence $\mathbf{B}_+(\mathcal{E}) \neq S$ (Remark A.3.7). We claim that

$$\mathbf{B}_+(\mathcal{E}) = \ell_1 \bigsqcup \left(\bigcup_{\substack{\ell \subset S \text{ line}: \\ \ell \cap \ell_1 = \emptyset,\ \ell \neq \ell_2,\ \ell \cap \ell_2 \neq \emptyset}} \ell \right).$$

It particular, all these $\mathcal{E}$'s are big Ulrich bundles which are not ample (Corollary 4.0.12).

To see this, by Theorem 4.0.11 we only need to find the lines $\ell \subset S$ such that $\mathcal{E} \cdot \ell = 0$. As all lines in S are (-1)-curves and $\mathcal{E} \cdot \ell = \ell_1 \cdot \ell - \ell_2 \cdot \ell + 1$, the claim immediately follows.

For instance, for $\ell_1 = E_1$ and $\ell_2 = E_2$ we have

$$\begin{aligned} &\mathbf{B}_+(\mathcal{E}) = E_1 && \text{if } d = 7, \\ &\mathbf{B}_+(\mathcal{E}) = E_1 \bigsqcup \left(\bigcup_{j=3}^{9-d} F_{2j} \right) && \text{if } 4 \leq d \leq 6, \\ &\mathbf{B}_+(\mathcal{E}) = E_1 \bigsqcup \left(\bigcup_{j=3}^{9-d} F_{2j} \bigcup C_1 \right) && \text{if } d = 3. \end{aligned}$$

Unlike what happens for the ampleness and (1-)very ampleness (see Theorem 4.0.20), bigness and V-bigness for a B-Ulrich bundle, with B as above, are not equivalent.

Remark 4.0.14. Let $Q_n \subset \mathbf{P}^{n+1}$ be a smooth quadric of dimension $n \geq 7, n \neq 10$. Then the spinor bundles $\mathcal{S}, \mathcal{S}'$, if n is even, and $\mathcal{S}''$, when n is odd, are the only indecomposable Ulrich bundles on Q_n [Bea18, Proposition 2.5] and are big in this situation [LMS24, Theorem 1]. However, their restriction to every line in Q_n is not ample [Ott88, Corollary 1.6]. Since Q_n is covered by lines, by Corollary 4.0.12(a) we conclude that $\mathcal{S}, \mathcal{S}'$ and $\mathcal{S}''$ are not V-big.

Thanks to Corollary 4.0.12 and to Theorem 4.0.20 we can find new examples of (V-)big Ulrich bundles which are not ample (and whose c_1 is not very ample). See also [LS23, Remark 4.3].

Example 4.0.15. Let $S \subset \mathbf{P}^3$ be the cubic surface (with hyperplane section $\mathcal{O}_S(1) = -K_S$). Since S is not covered by lines (see Example 4.0.13), all Ulrich bundles on S are trivially V-big by Corollary 4.0.12. Now, looking at S as the blow-up of the plane at 6 points in general position, denote by $\widetilde{H}$ the pullback of a line and by E_i the exceptional divisors, for $i = 1, \dots, 6$. By [Cas+12, Examples 3.6 - 4.7] there exist three Ulrich bundles $\mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_d$ of rank 2 and an Ulrich bundle $\mathcal{E}_h$ of rank 3 on S such that

$$c_1(\mathcal{E}_a) = 2\widetilde{H}, \qquad c_1(\mathcal{E}_b) = 3\widetilde{H} - E_1 - E_2 - E_3,$$
$$c_1(\mathcal{E}_d) = 4\widetilde{H} - 2E_1 - \sum_{i=2}^{5} E_i, \qquad c_1(\mathcal{E}_h) = 6\widetilde{H} - 2\sum_{i=1}^{4} E_i - E_5.$$

Each of the above c_1 is clearly not ample, for instance $c_1 \cdot E_6 = 0$ for all of them. Then $\mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_d, \mathcal{E}_h$ cannot be ample, for otherwise they would be 1-very ample (Theorem 4.0.20) and their first Chern class would be so as well (Remark A.3.2).

We can find big non-ample Ulrich bundles even if its Chern class is very ample and the variety is covered by lines.

Example 4.0.16. Let (X, B) be a Del Pezzo threefold of degree $3 \leq d \leq 5$. Then X is covered by lines [Isk79, Proposition III.1.4(ii)], and in fact the Fano scheme of lines $F(X)$ is a smooth irreducible surface [Isk79, Propositions III.1.3(iii) - III.1.6(i) & Remark III.1.5]. Let $\mathcal{E}$ be any stable special Ulrich bundle of rank 2 on X [CFK23, Theorem 1.1]. In particular $c_1(\mathcal{E}) = 2B$ is very ample. As $\mathcal{E}(-1)$ is still stable and $c_1(\mathcal{E}(-1)) = 0, H^1(X, \mathcal{E}(-2)) = 0$, we see that $\mathcal{E}(-1)$ is an instanton bundle in the sense of [Kuz12, Definition 1.1]. Since $\mathcal{E}_{|L} \cong \mathcal{O}_L(1)^{\oplus 2}$ for a general line $[L] \in F(X)$ [CFK23, (Proof of) Lemma 4.8(i)], by [Kuz12, Theorem 3.17] there exists a non-zero divisor $D_{\mathcal{E}} \subset F(X)$ which parameterizes the lines $L \subset X$ such that $\mathcal{E}(-1)_{|L} \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1)$. Then Corollary 4.0.12 and (4.5) tell that $\mathcal{E}$ is not ample with non-empty augmented base locus

$$\mathbf{B}_+(\mathcal{E}) = \bigcup_{[L] \in D_{\mathcal{E}}} L.$$

Consider the incidence correspondence

$$\begin{array}{ccc} & \mathcal{S} = \{(x, [\Gamma]) \in X \times F(X) \mid x \in \Gamma\} \subset X \times F(X) & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ X & & F(X) \end{array}$$

with projections π_1, π_2. As every fibre $\pi_2^{-1}([L]) \cong L$ is smooth irreducible of dimension 1, a dimension count shows that $\pi_1(\pi_2^{-1}(D_{\mathcal{E}})) \subsetneq X$ is a proper subset. As $\mathbf{B}_+(\mathcal{E}) = \pi_1(\pi_2^{-1}(D_{\mathcal{E}}))$, we conclude that $\mathcal{E}$ is V-big.

We observe that Theorem 4.0.11 and Corollary 4.0.12 no longer hold if we do not assume φ_V to be étale, in particular if we allow the ramification locus to be non-empty.

Remark 4.0.17. The Del Pezzo surface $(S, -K_S)$ of degree $d = 2$, that is finite flat double cover $\varphi_{-K_S}: S \longrightarrow \mathbf{P}^2$ branched over a smooth plane quartic curve, supports a non-big $(-K_S)$-Ulrich line bundle $\mathcal{E}$ (see Proposition 1.2.4 and Example 3.0.27). This means that $\mathbf{B}_+(\mathcal{E}) = S$. However, exactly as already seen in Example 4.0.13 for Del Pezzo surfaces of degree $3 \leq d \leq 7$, S contains finitely many $(-K_S)$-lines: interpreting S as the blow-up of the plane at 7 points in general position, we see there are 56 $(-K_S)$-lines which correspond to the 7 exceptional divisors, to the strict transforms of the 21 lines through two of those points, to the strict transforms of the 21 conics through five of those points and to the strict transforms of the 7 cubics through all of those points with a double point at one of them. Indeed, all these curves are clearly $(-K_S)$-lines; conversely, by arguing as in [Har77, Proof of Theorem V.4.9], one sees that any $(-K_S)$-line is linearly equivalent to one of them. By adjunction formula a $(-K_S)$-line $\Gamma \subset S$ has self-intersection $\Gamma^2 = -1$, therefore the each linear system $|\Gamma|$ must contain a single representative, proving the claim. Thus it's clear that (4.5) cannot hold for $\mathcal{E}$.

The previous example was easy because the variety contains a finite number of B-lines. However the same problem may arise also when the variety is covered by B-lines.

Remark 4.0.18. Let (X, B) be a Del Pezzo 3-fold of degree $d = 2$ and let $\mathcal{E}$ be the stable special B-Ulrich bundle of rank 2 provided by Proposition 1.3.3. As above, we are not in the situation of Theorem 4.0.11 since the morphism $\varphi_B: X \longrightarrow \mathbf{P}^3$ is a finite flat double cover branched over a smooth quartic surface. But, on the contrary of the previous remark, X is covered by B-lines (see Remark 3.0.11). More precisely the Fano scheme of (B-)lines $F(X)$ is an irreducible surface [Isk79, Remark III.1.7]. In Example 3.0.28 we showed that $\mathcal{E}$ is non-big. Therefore $\mathbf{B}_+(\mathcal{E}) = X$. However, in [Fae14, Proof of Theorem D, Step 1] it is proved that $\mathcal{E}_{|\Gamma} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 2}$ for a general B-line $[\Gamma] \in F(X)$. Exactly as in Example 4.0.16, $\mathcal{E}(-B)$ is an instanton bundle on X [Kuz12, Definition 1.1]. Therefore the B-lines $\Gamma \subset X$ such that $\mathcal{E}_{|\Gamma} \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(2)$ form a non-zero divisor $D_{\mathcal{E}} \subset F(X)$ by [Kuz12, Theorem 3.17]. Consider the incidence correspondence

$$\begin{array}{ccc} & \mathcal{S} = \{(x, [\Gamma]) \in X \times F(X) \mid x \in \Gamma\} \subset X \times F(X) & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ X & & F(X), \end{array}$$

with projections π_1, π_2. Every fibre $\pi_2^{-1}([\Gamma]) \cong \Gamma$ is smooth irreducible of dimension 1. For dimensional reasons, $\pi_1(\pi_2^{-1}(D_{\mathcal{E}})) \subsetneq X$ must be a proper subset. By construction, every point $x \in X \backslash \pi_1(\pi_2^{-1}(D_{\mathcal{E}}))$ is crossed only by B-lines $\Gamma \subset X$ such that $\mathcal{E}_{|\Gamma} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 2}$. We conclude that the union over all B-lines $\Gamma \subset X$ such that $\mathcal{E}_{|\Gamma}$ is not ample cannot coincide with X. This says that Theorem 4.0.11 and Corollary 4.0.12 do not apply even in this situation.

We point out that the assumptions on B considered so far are not necessary: there are examples in which φ_B is not étale but the characterization of Theorem 4.0.11 holds.

Remark 4.0.19. First we observe that given any smooth projective curve C of genus $g \geq 0$ with a globally generated ample line bundle B, one can easily see that $\mathcal{E}_L = L(B)$ is B-Ulrich for every line bundle $L \in \mathrm{Pic}^{g-1}(C) \backslash \Theta$, with Θ being the divisor which contains all effective line bundles on C of degree $g - 1$. Note that $\deg(\mathcal{E}_L) = \deg B + g - 1$.

Any pair $(C, B) \neq (\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$ is not a B-line and, in this case, all $\mathcal{E}_L$'s are ample since $\deg(\mathcal{E}_L) > 0$. Therefore $\mathbf{B}_+(\mathcal{E}_L) = \emptyset$ accordingly with the fact that C contains no B-lines.

However, taking for instance (C, B) with $g = 1$ and $\deg B = 2$, we see that $\mathcal{E}_L$ is ample but not (1-)very ample. This shows that also Theorem 4.0.20 does not hold when φ_B is ramified: in fact $\varphi_B : C \to \mathbf{P}^1$ is a finite flat double cover with 2 branch points. Anyway, if $\deg B \geq g + 2$, the B-Ulrich line bundle $\mathcal{E}_L$ is (1-)very ample even if φ_V is not étale for all $|V| \subset |B|$.

Regarding the ampleness of a B-Ulrich bundle (with B as above), thanks to the previous lemmas about the separation properties, we can also prove the analogue of [LS23, Theorem 1] in this setting. Let's recall first another notion of positivity for vector bundles.

Theorem 4.0.20 (Lopez-Sierra)**.** *Let X be a smooth projective variety and let B be a globally generated ample line bundle such that there is a linear series $|V| \subset |B|$ inducing a morphism which is étale onto its schematic image. Let $\mathcal{E}$ be a B-Ulrich bundle on X. Then the following are equivalent:*

(1) $\mathcal{E}$ is 1-very ample.

(2) $\mathcal{E}$ is very ample.

(3) $\mathcal{E}$ is ample.

(4) Either X contains no B-lines or $\mathcal{E}_{|\Gamma}$ is ample on every B-line $\Gamma \subset X$.

Proof. It's clear that (2) implies (3), while the fact that (1) implies (3) follows by Remark A.3.2. Corollary 4.0.12(b) gives the equivalence of (3) and (4). Let $\pi : \mathbf{P}(\mathcal{E}) \to X$ be the natural projection and let $\xi = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ be the tautological line bundle on $\mathbf{P}(\mathcal{E})$. Now assume (4) and let $Z \subset \mathbf{P}(\mathcal{E})$ (resp. $Z' \subset X$) be a 0-dimensional closed subscheme of length 2. To prove (2) (resp. (1)), we show that

$$r_Z : H^0(\mathbf{P}(\mathcal{E}), \xi) \to H^0(Z, \xi_{|Z}) \qquad (\text{resp. } r' : H^0(X, \mathcal{E}) \to H^0(Z', \mathcal{E}_{|Z'}))$$

is surjective. If $Z \subset \mathbf{P}(\mathcal{E}(x))$ for some $x \in X$, we obtain the same commutative diagram of (4.4), which then forces r_Z to be surjective. Suppose $Z \not\subset \mathbf{P}(\mathcal{E}(x))$ for every $x \in X$. Then $\pi(Z) \subset X$ is a 0-dimensional closed subscheme of length 2. Let $r : H^0(X, \mathcal{E}) \to H^0(\pi(Z), \mathcal{E}_{|\pi(Z)})$ be the restriction map. If there is no B-line passing through $\pi(Z)$ (resp. Z'), which happens in particular when X contains no B-lines, then r (resp. r') is surjective by Lemma 4.0.9(i), hence so is r_Z. Analogously, if $\Gamma \subset X$ is a B-line containing $\pi(Z)$ (resp. Z'), then we know that r (resp. r') is surjective thanks to Lemma 4.0.9(ii). Therefore r_Z is surjective as desired. This concludes the proof. □

Chapter 5

Projective normality of Ulrich bundles

Lopez-Sierra theorem 4.0.20 tells that a B-Ulrich bundle $\mathcal{E}$, when $|B|$ contains a linear series $|V|$ inducing an étale morphism onto the schematic image, is ample if and only if it is very ample if and only if either X does not contain B-lines or $\mathcal{E}_{|\Gamma}$ is ample on every B-line $\Gamma \subset X$. It is then natural trying to understand the embedding of the corresponding projective bundle through the (complete) linear system of the tautological line bundle. In this regard, we are going to study the projective normality of an Ulrich bundle, namely the normal generation of its tautological line bundle (see Definitions B.3.1–5.0.5). Unfortunately, this property does not behave as expected: Ulrich bundles on curves are projectively normal when the degree of the polarization is big with respect to the genus, but this is no longer true on (low-dimensional) hypersurfaces where Ulrich bundles, which are likely very ample (it is enough that the hypersurface is general of degree which is at least the double of the dimension), are, instead, surprisingly almost never projectively normal (see also Remark 5.3.5). Therefore it appears very difficult to obtain a general criterion for this property. In addition to this, the techniques we are going to exploit to study projective normality involves Castelnuovo-Mumford regularity which is usually not well-behaved with respect to tensor operations if the dimension is greater than 2 (see Appendix B.1). Therefore the main results, that are stated below, are on curves, surfaces with $q = p_g = 0$ and on hypersurfaces of dimension $n = 2, 3$.

Theorem 5.0.1. *Let C be a smooth projective curve of genus g and let B be a globally generated ample line bundle of degree d on C. Let $\mathcal{E}$ be a B-Ulrich bundle on C. Then:*

(1) $\mathcal{E}$ is projectively normal if $d > g + 1$.

(2) $\mathcal{E}$ satisfies (N_1) and $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is Koszul if $d > g + 2$.

(3) $\mathcal{E}$ satisfies (N_p) for $p \geq 2$ if $d > \frac{1}{2}\left((g + p + 1) + \sqrt{g^2 + 2g(3p + 1) + (p - 1)^2}\right)$.

(4) If there exists a linear series $|V| \subseteq |B|$ which induces a morphism which is étale onto the schematic image, the general B-Ulrich bundle of rank r on C is projectively normal as soon as C supports a non-special normally generated line bundle of degree d. This holds in particular if $d \geq g + 2 - \operatorname{Cliff}(C)$.

(5) If C is general of genus $g \geq 3$ and B is a general very ample line bundle of degree

$$d \geq \frac{3+\sqrt{8g+1}}{2},$$

then the general B-Ulrich bundle of rank r is projectively normal. Moreover this bound is sharp for $r = 1$.

Theorem 5.0.2. *Let $S \subset \mathbf{P}^N$ be a smooth projective surface with $q(S) = p_g(S) = 0$ and let $\mathcal{E}$ be an ample 0-regular vector bundle of rank $r \geq 2$ on S such that $h = h^0(S, \mathcal{E}) \geq r+3$. Let $E = \det(\mathcal{E})$ be its determinant line bundle and let $\ell = \binom{h-r}{2} - 1$. The following are equivalent:*

(1) $\mathbf{P}(\mathcal{E})$ is not aCM.

(2) $\mathcal{E}$ is not projectively normal.

(3) There exist a closed subscheme $Z \subset S$ and a non-zero divisor $D \subset S$ such that:

(a) Z is smooth of dimension 0.

(b) Z is the degeneracy locus of ℓ general sections $s_1, \ldots, s_\ell \in H^0(S, \Lambda^2 M^*_{\mathcal{E}})$.

(c) $[Z] = \frac{1}{2}(h-r-2)\left((h-r+1)c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\right)$.

(d) $D \in |K_S + (h-r-1)E|$.

(e) $Z \subset D$.

(4) There exist a closed subscheme $Z \subset S$ and a curve $C \subset S$ such that:

(f) Z is the degeneracy locus of ℓ general sections $\sigma_1, \ldots, \sigma_\ell \in H^0(S, \Lambda^2 M^*_{\mathcal{E}})$.

(g) C is the degeneracy locus of the $(\ell+1)$ general sections $\sigma_1, \ldots, \sigma_\ell, \sigma_{\ell+1} \in H^0(S, \Lambda^2 M^*_{\mathcal{E}})$.

(h) $C \in |(h-r-1)E|$ is smooth and irreducible.

(i) $Z \subset C$ is a special (effective) divisor.

Theorem 5.0.3. *Let $X \subset \mathbf{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 3$ with $n = 2, 3$ and let $\mathcal{E}$ be an Ulrich bundle of rank r on X. Let*

$$\mu_{\mathcal{E}} \colon H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{E}) \to H^0(X, \mathcal{E} \otimes \mathcal{E})$$

denote the multiplication of sections. Then:

(1) If $n = 2$ and $\det(\mathcal{E}) = \mathcal{O}_X(\frac{r}{2}(d-1))$, then $\mu_{\mathcal{E}}$ cannot be surjective and $\mathcal{E}$ cannot be projectively normal if $d \geq 5$, or $d = 4$ and $r \leq 5$, or $d = 3$ and $r \leq 2$.

(2) If $n = 3$ and $d \geq 4$, then $\mu_{\mathcal{E}}$ is never surjective and $\mathcal{E}$ cannot be projectively normal if $r > \frac{d+4}{3}$.

We start by recalling the main definitions and some properties of projective normality of vector bundles. Standard facts and useful technical results on normal generation of line bundles and of higher rank vector bundles are gathered in Appendix B.3.

Throughout this chapter we will use the following convention.

Notation 5.0.4. For a very ample vector bundle $\mathcal{E}$ on a projective scheme X we will always consider the embedding $\mathbf{P}(\mathcal{E}) \hookrightarrow \mathbf{P}(H^0(X, \mathcal{E}))$ induced by the complete linear system $|\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|$.

The natural generalizations of the notions of projective normality of line bundles and of ACM embedding (Definitions B.3.1–B.3.9) to vector bundles is asking them to hold for the tautological bundle.

Definition 5.0.5. Let $\mathcal{E}$ be a vector bundle on a projective variety X. We say that $\mathcal{E}$ is *projectively normal* (resp. *k-normal* for some $k \geq 1$) if $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is normally generated (resp. k-normal) on $\mathbf{P}(\mathcal{E})$. We say that $\mathbf{P}(\mathcal{E})$ is *aCM* if $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ induces an embedding $\mathbf{P}(\mathcal{E}) \subset \mathbf{P}(H^0(X, \mathcal{E}))$ which is aCM. Finally, $\mathcal{E}$ is *strongly k-normal* (for some $k \geq 1$) if

$$\mu_{\mathcal{E}}^k \colon H^0(X, \mathcal{E})^{\otimes k} \to H^0(X, \mathcal{E}^{\otimes k})$$

is surjective. When $k = 2$, we will write $\mu_{\mathcal{E}} = \mu_{\mathcal{E}}^2$.

By Remarks B.3.2–B.3.10 we know that projective normality is implied both by strongly normality for all $k \geq 1$ and by the ACM property of the projective bundle. This simple observation will be used with no further mention.

Proposition 5.0.6. *Let $X \subset \mathbf{P}^N$ be a projective variety of dimension $n \geq 1$ such that $\mathcal{O}_X(1)$ is $3n$-Koszul. Then every 0-regular vector bundle on X is strongly k-normal for all $k \geq 1$. In particular, all ample 0-regular vector bundles are projectively normal.*

This Proposition applies for instance to all embedded varieties in Example B.1.12.

Proof. Let $\mathcal{E}$ be a 0-regular vector bundle on $X \subset \mathbf{P}^N$ and proceed by induction on $k \geq 1$. As the base case $k = 1$ is trivial, suppose $k > 1$ and consider the commutative diagram

$$\begin{array}{ccc} & H^0(X, \mathcal{E})^{\otimes k} & \\ {}^{\mathrm{id}\otimes\mu_{\mathcal{E}}^{k-1}} \swarrow & & \searrow {}^{\mu_{\mathcal{E}}^k} \\ H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{E}^{\otimes(k-1)}) & \xrightarrow{\quad m_{\mathcal{E}}^k \quad} & H^0(X, \mathcal{E}^{\otimes k}). \end{array}$$

The map $\mathrm{id}\otimes\mu_{\mathcal{E}}^{k-1}$ is surjective by the inductive hypothesis and $m_{\mathcal{E}}^k$ is surjective by Proposition B.1.13. Thus $\mu_{\mathcal{E}}^k = m_{\mathcal{E}}^k \circ (\mathrm{id} \otimes \mu_{\mathcal{E}}^{k-1})$ is surjective as required. □

Strongly 2-normality on 0-regular vector bundles on low dimensional varieties directly implies projective normality.

Remark 5.0.7. Let (X, B) be either an irreducible curve with a globally generated ample line bundle or an irreducible surface with a very ample line bundle. Then a strongly 2-normal 0-regular vector bundle for (X, B) is automatically strongly k-normal for all $k \geq 2$. In particular, all ample 0-regular and strongly 2-normal vector bundles for (X, B) are projectively normal.

Indeed, let $\mathcal{E}$ be a strongly 2-normal 0-regular vector bundle for (X, B) and let $M_{\mathcal{E}}$ be its syzygy bundle. Tensoring the syzygy exact sequence of $\mathcal{E}$ through by $\mathcal{E}$, we immediately see that the strongly 2-normality implies $H^1(X, M_{\mathcal{E}} \otimes \mathcal{E}) = 0$. If $\dim X = 1$, this amounts to say that $M_{\mathcal{E}} \otimes \mathcal{E}$ is 1-regular. If X is a surface, observing that also $\mathcal{E} \otimes \mathcal{E}$ is 0-regular (Corollary B.1.10), the exact sequence

$$0 = H^1(X, \mathcal{E} \otimes \mathcal{E}(-B)) \longrightarrow H^2(X, M_{\mathcal{E}} \otimes \mathcal{E}(-B)) \longrightarrow H^0(X, \mathcal{E}) \otimes H^2(X, \mathcal{E}(-B)) = 0$$

tells that $M_{\mathcal{E}} \otimes \mathcal{E}$ is 1-regular also in this case. Then, by Proposition B.1.9, $M_{\mathcal{E}} \otimes \mathcal{E}^{\otimes j}$ and $\mathcal{E}^{\otimes j}$ are respectively 1-regular and 0-regular for all $j \geq 1$. Proceeding inductively on $k \geq 2$, suppose $k > 2$ and assume that $\mu^h_{\mathcal{E}}$ is surjective for all $1 \leq h \leq k-1$. Using the 1-regularity of $M_{\mathcal{E}} \otimes \mathcal{E}^{\otimes(k-1)}$ we immediately get the surjectivity of

$$H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{E}^{\otimes(k-1)}) \to H^0(X, \mathcal{E}^{\otimes k}).$$

Since $H^0(X, \mathcal{E})^{\otimes k} = H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{E})^{\otimes(k-1)} \to H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{E}^{\otimes(k-1)})$ is onto by the inductive hypothesis, the assertion follows.

We conclude this preliminary part with a result which says that for very ample vector bundles with no first cohomology group over curves and surfaces with $p_g = 0$, the projective normality is equivalent to the 2-normality. This will be crucial for the proofs of Theorems 5.0.1–5.0.2.

Proposition 5.0.8. *Let Y be a smooth projective variety of dimension $m \geq 1$ with $H^2(Y, \mathcal{O}_Y) = 0$ and let $\mathcal{E}$ be a 2-normal very ample vector bundle on Y such that $H^1(Y, \mathcal{E}) = H^0(Y, (K_Y + \det(\mathcal{E})) \otimes S^{m-3}\mathcal{E}) = 0$. Then $\mathcal{E}$ is projectively normal, $\mathcal{I}_{\mathbf{P}(\mathcal{E})/\mathbf{P}(H^0(Y,\mathcal{E}))}$ is 3-regular and $I_{\mathbf{P}(\mathcal{E})/\mathbf{P}(H^0(Y,\mathcal{E}))}$ is generated in degree less than or equal to 3.*

Proof. We just need to verify the hypothesis of Lemma B.3.13. We already know that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is very ample and 2-normal, and also that $H^1(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = H^2(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}) = 0$. Finally, letting $r = \operatorname{rk}(\mathcal{E})$, we have

$$\begin{aligned} H^0(\mathbf{P}(\mathcal{E}), K_{\mathbf{P}(\mathcal{E})} + \mathcal{O}_{\mathbf{P}(\mathcal{E})}(m+r-3)) &= H^0(\mathbf{P}(\mathcal{E}), \pi^*(K_Y + \det(\mathcal{E})) + \mathcal{O}_{\mathbf{P}(\mathcal{E})}(m-3)) \\ &\cong H^0(Y, (K_Y + \det(\mathcal{E})) \otimes \pi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(m-3)) \\ &\cong H^0(Y, (K_Y + \det(\mathcal{E})) \otimes S^{m-3}\mathcal{E}) \\ &= 0 \end{aligned}$$

as required. □

5.1 Projective normality of Ulrich bundles on curves

The goal of this section is to determine under which conditions an Ulrich bundle on a smooth projective curve is projectively normal. Thanks to Proposition 5.0.8 we only need to study the 2-normality of an Ulrich bundle. In this section, a curve is always smooth and projective.

Recall that for a vector bundle $\mathcal{E}$ on a curve, it is defined the quantity

$$\mu^-(\mathcal{E}) = \min \{\mu(Q) \mid Q \text{ is a quotient vector bundle of } \mathcal{E}\},$$

where $\mu(-) = \deg(-)/\operatorname{rk}(-)$ is the slope of a vector bundle. We always have $\mu(\mathcal{E}) \geq \mu^-(\mathcal{E})$ with the equality holding if $\mathcal{E}$ is μ-semistable. We refer to [But94] for more details.

Proposition 5.1.1. *Let $\mathcal{E}$ be a vector bundle on a smooth projective curve C with slope $\mu^-(\mathcal{E}) > 2g(C)$. Then $\mathcal{E}$ is strongly 2-normal, projectively normal, $\mathcal{I}_{\mathbf{P}(\mathcal{E})/\mathbf{P}(H^0(C,\mathcal{E}))}$ is 3-regular and $I_{\mathbf{P}(\mathcal{E})/\mathbf{P}(H^0(X,\mathcal{E}))}$ is generated in degree ≤ 3 in the embedding $\mathbf{P}(\mathcal{E}) \subset \mathbf{P}(H^0(X,\mathcal{E}))$ determined by $|\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|$.*

Proof. The vector bundle $\mathcal{E}$ is very ample with $H^1(C,\mathcal{E}) = 0$ by [But94, Lemma 1.12] and strongly 2-normal by [But94, Theorem 2.1]. The assertion follows from Proposition 5.0.8. □

Corollary 5.1.2. *Let C be a smooth projective curve and let $\mathcal{E}$ be a B-Ulrich bundle of rank r on C, where B is a globally generated ample line bundle of degree d on C. If $d > g(C) + 1$, then $\mathcal{E}$ is strongly k-normal for all $k \geq 2$, projectively normal, and $\mathcal{I}_{\mathbf{P}(\mathcal{E})/\mathbf{P}^{rd-1}}$ is 3-regular and $I_{\mathbf{P}(\mathcal{E})/\mathbf{P}^{rd-1}}$ is generated in degree ≤ 3 in the embedding $\mathbf{P}(\mathcal{E}) \subset \mathbf{P}^{rd-1}$ determined by $|\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|$.*

Proof. By Lemma 1.1.16(ii) and Proposition 1.1.17, $\mathcal{E}$ is semistable with slope $\mu(\mathcal{E}) > 2g(C)$. The claim follows by Proposition 5.1.1 and Remark 5.0.7. □

This condition is certainly not necessary for the projective normality of an Ulrich bundle.

Remark 5.1.3. For a smooth plane $C \subset \mathbf{P}^2$ of degree $d \geq 4$ one has $d \leq g + 1$, but all Ulrich bundles on $C \subset \mathbf{P}^2$ are very ample Theorem 4.0.20 and also strongly 2-normal by [Laz04a, Example B.1.3] applied to the resolution (5.6) in Lemma 5.3.1 below. In particular, by Proposition 5.0.8 every Ulrich bundle $\mathcal{E}$ on $C \subset \mathbf{P}^2$ is projectively normal, $\mathcal{I}_{\mathbf{P}(\mathcal{E})/\mathbf{P}^{rd-1}}$ is 3-regular and $I_{\mathbf{P}(\mathcal{E})/\mathbf{P}^{rd-1}}$ is generated in degree ≤ 3 in the embedding $\varphi_{\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)} : \mathbf{P}(\mathcal{E}) \subset \mathbf{P}^{rd-1}$.

However, the condition in Corollary 5.1.2 is sharp some sense if we consider Ulrich bundles defined with respect to globally generated ample line bundles (and possibly non-very ample).

Remark 5.1.4. Let C be a hyperelliptic curve of genus $g = 3$ and let $\mathcal{L}$ be any K_C-Ulrich line bundle (Proposition 1.2.2). In particular, since $\mathcal{L}$ cannot be of the form $K_C(D)$ for an effective divisor D of degree 2, we know that $\mathcal{L}$ is very ample. However, $\mathcal{L}$ cannot be projectively normal by [GL86, Corollary 1.4].

In light of this, we will focus on the case of Ulrich bundles defined with respect to a very ample polarization.

Remark 5.1.5. Ulrich bundles on a smooth rational curve of degree $d \geq 2$ are always projectively normal by Corollary 5.1.2. Since all very ample line bundles on smooth projective curves of genus $g = 1, 2$ have degree $d \geq g + 2$, the same is true also in this case.

We now recall the definition of Clifford index of a curve.

Definition 5.1.6. Let C be a smooth projective curve of genus $g \geq 2$ and let L be a line bundle on C. The *Clifford index of L* is the quantity

$$\mathrm{Cliff}(L) = \deg(L) - 2\dim|L| = \deg(L) - 2(h^0(C,L) - 1).$$

The *Clifford index of C* is defined as

$$\mathrm{Cliff}(C) = \min\left\{\mathrm{Cliff}(A) \mid A \in \mathrm{Pic}(C),\ h^i(C,A) \geq 2 \text{ for } i = 0, 1\right\}.$$

We now pass to the study of projective normality of the general B-Ulrich bundle on a curve C. We first observe that, as in the case of very ample polarization, they define open subsets in the corresponding moduli space of semistable vector bundles on C. First we state a simple remark.

Remark 5.1.7. Let C be a smooth projective curve of genus g and fix a line bundle A of positive degree $a > 0$. The line bundles of the form $A + D_d$, for D_d being an effective divisor of degree $d \leq g-1$, are contained in a proper closed subset of $\mathrm{Pic}^{a+d}(C)$.

Indeed, an effective divisor of degree $d \leq g-1$ lies in the image of the Abel map $C_d \to \mathrm{Pic}^d(C)$, which is a closed subset of dimension $d < g = \dim \mathrm{Pic}^d(C)$. As the tensor product trough by A determines an isomorphism $\mathrm{Pic}^d(C) \xrightarrow{\cong} \mathrm{Pic}^{a+d}(C)$, the assertion follows.

Remark 5.1.8. Let C be a smooth projective curve of genus g and let B be a globally generated ample line bundle of degree d on C. By semicontinuity B-Ulrich bundles of rank r, which are always semistable, define a non-empty open subset in the good moduli space (resp. moduli stack) of semistable vector bundles of rank r and degree $r(d+g-1)$ (see also [Cas+12, p. 8] and [Cos17a, p. 95]). Moreover, by Remarks 4.0.19–5.1.7, B-Ulrich line bundles form a dense open subset in $\mathrm{Pic}^{d+g-1}(C)$ (see also [Cos17a, (4.1.3)]).

We can now prove one of the main result of this section.

Proposition 5.1.9. *Let C be a smooth projective curve of genus $g \geq 2$ and let B be a globally generated line bundle of degree $d > 1$ such that there exists a linear series $|V| \subset |B|$ which induce a morphism which is étale onto the schematic image.*

If C supports a non-special projectively normal line bundle of degree $d+g-1$, then the general B-Ulrich bundle of rank r is projectively normal, for any $r \geq 1$. In particular, the general rank r Ulrich bundle is projectively normal on C if $d \geq g + 2 - \mathrm{Cliff}(C)$.

For a detailed account of the main properties of the moduli space (resp. moduli stack) of semistable vector bundles on a smooth projective curve we refer to [Alp13; Alp+22].

Proof. The second part of the statement will follow from the first one combined with [GL86, Theorem 1]. For the first part, observe that all Ulrich bundles on C are ample (Corollary 4.0.12), then fix $r \geq 1$ and let $M_C^U(r,e) \subset M_C^{ss}(r,e)$ be the non-empty open subset of B-Ulrich bundles (Remark 5.1.8) of rank r in the irreducible good moduli space of semistable vector bundles of rank r and degree $e = r(d+g-1)$ (see [Alp+22, Theorem 3.12]). Then Lemma B.3.8 says that the subset $\mathcal{P}$ of (semistable) non-special (Notation B.3.7), ample, globally generated, strongly 2-normal vector bundles $\mathcal{F}$ is open in the (irreducible) moduli stack $\mathcal{M}_C^{ss}(r,e)$ of semistable vector bundles of rank r and degree $e = r(d+g-1)$ on C. As the good moduli space

$$f\colon \mathcal{M}_C^{ss}(r,e) \to M_C^{ss}(r,e)$$

is universally closed [Alp+22, Theorem 3.5], the subset

$$P^c = f(\mathcal{M}_C^{ss}(r,e) \setminus \mathcal{P}) \subset M_C^{ss}(r,e)$$

is closed and consists of points $[\mathcal{E}] \in M_C^{ss}(r,e)$ such that there exists a vector bundle $\mathcal{F} \in [\mathcal{E}]$ which either is special either is not ample either is not globally generated or $\mu_{\mathcal{F}}$ is not surjective. The complement $P = M_C^{ss}(r,e) \setminus P^c$ is then open and can be described as

$$\left\{ \begin{array}{l} [\mathcal{E}] \in M_C^{ss}(r,e) \ \mid \ \exists \mathcal{F} \in [\mathcal{E}] \text{ such that} \\ \mathcal{F} \text{ is non-special, ample, globally generated and strongly 2-normal} \end{array} \right\}.$$

Now, if there exists a non-special projectively normal line bundle in $\mathrm{Pic}^{d+g-1}(C)$, by Lemma B.3.8 (or [LM85, Lemma 1.3]) we can find a dense open subset consisting of non-special projectively normal line bundles of degree $d+g-1$. In virtue of the openness of Ulrich line bundles in $\mathrm{Pic}^{d+g-1}(C)$ (Remark 5.1.8), there exists a dense open subset of projectively normal B-Ulrich line bundles on $C \subset \mathbf{P}^N$. Let $\mathcal{L}$ be one of them. Then $\mathcal{L}^{\oplus r}$ lies in $M_C^U(r,e) \cap P$ by Remark B.3.4. Therefore the conclusion follows by combining the irreducibility of $M_C^{ss}(r,e)$ and Proposition 5.0.8. □

As an application of this result, we see that the bound in Corollary 5.1.2 can be lowered for the general rank Ulrich bundle.

Lemma 5.1.10. *Let $C \subset \mathbf{P}^N$ be a smooth projective curve of genus $g \geq 3$ and degree $d > 1$. Then:*

(i) If $d = g, g+1$, all Ulrich line bundles on C are projectively normal.

(ii) If $g \geq g_h$, where

$$g_h = \begin{cases} 15 & \text{if} \quad h = 2 \\ 17 & \text{if} \quad h = 3 \\ 27 & \text{if} \quad h = 4 \\ 33 & \text{if} \quad h = 5 \end{cases}$$

the general non-special line bundle of degree $2g-h$ is projectively normal.

In particular, the general rank r Ulrich vector bundle is projectively normal if $d = g, g+1$ and if $d = g-h+1$ and $g \geq g_h$ for $h = 2,3,4,5$.

Before seeing the proof, we make a couple of simple observations.

Remark 5.1.11. A smooth projective curve $C \subset \mathbf{P}^N$ of genus $g \geq 2$ and degree $d \leq g+1$ is neither hyperelliptic nor elliptic-hyperelliptic, i.e. a double cover of an elliptic curve.

Indeed, very ample line bundles on smooth hyperelliptic curves of genus $g \geq 2$ have degree $d > g+1$ by [Par08, Theorem 3.1(3)], and a very ample line bundle L on an elliptic-hyperelliptic curve must have $h^1(C,L) \leq 1$ [Mar12, (5)] which cannot happen if $\deg(L) \leq g+1$.

Proof of Lemma 5.1.10. By Proposition 5.1.9 we only need to prove (i)-(ii). We observe also that C can be neither hyperelliptic nor elliptic-hyperellitpic (Remark 5.1.11).

Let's show (i). If $d = g+1$, all Ulrich line bundle have degree $2g$ and so projectively normal by [GL86, Corollary 1.4].

Assume $d = g$ and let $H \subset C$ be a hyperplane section. We first show that if $d = g = 6$, then C cannot be a plane quintic. Assuming the contrary, write $H = p_1 + \cdots + p_6$. Since there are no smooth curves of genus 6 and degree 6 in $\mathbf{P}^3$ [Har77, Example IV.6.4.2], we must have

$$h^0(C, K_C - p_1 - \cdots - p_6) = h^1(C,H) = h^0(C,H) - 6 - 1 + 6 = h^0(C,H) - 1 \geq 4.$$

However, $h^0(C, K_C - q_1 - q_2 - q_3) = 3$ for every $q_1, q_2, q_3 \in C$: the effective divisor $D = q_1 + q_2 + q_3$ must have $h^0(C,D) = 1$, for otherwise we would get a g_3^1 on a plane quintic, hence Riemann-Roch gives the claim.

Now, C is not a plane quintic and we immediately see that $N \geq 3$. Using Castelnuovo theorem [Har77, Theorem IV.6.4], we get $d = g \geq 8$. In this case, an Ulrich line bundle $\mathcal{L}$

has degree $\deg(\mathcal{L}) = 2g - 1$. Thus [GL86, Corollary 1.6] tells that $\mathcal{L}$ fails to be projectively normal if and only if C is trigonal and $\mathcal{L} = K_C - E + D$ for an effective divisor D of degree 4 and $E \in W^1_3(C)$. However this cannot happen for otherwise

$$h^1(C, L) = h^1(C, K_C - H + D - E) = h^0(C, H - D + E) > 0.$$

(Indeed, writing $D = \sum_{i=1}^4 p_i$ for some points $p_i \in C$ and using that H is very ample with $h^0(C, H) \geq 4$ and $E \in W^1_3(C)$, we have

$$h^0(C, H - p_1 - p_2 - p_3) \geq 1, \qquad h^0(C, E - p_4) \geq 1,$$

proving the claim.) Therefore $\mathcal{L}$ is projectively normal even in this case. This proves (i).

For (ii), we are going to use [KKO99, Theorems 3.1–3.2] and [Aka04, Theorems 3.1–3.2]: it is proved that a non-special line bundle L of degree $2g - h$ on a curve of genus $g \geq g_h$, for $h = 2, 3, 4, 5$, is not projectively normal if and only if either C has a certain gonality or is a covering of a certain curve and

$$L = K_C - cA + D$$

where $A \in W^j_a = W^j_a(C)$, $D \in W_b = W^0_b(C)$ and with $1 \leq c \leq 5$, $j = 1, 2$, $3 \leq a \leq 9 < g$, $a \geq 2j$, $2 \leq b \leq 12 < g$ such that $2g - 2 - c \cdot a + b = 2g - h$ given in the Theorems in *loc. cit.* This says that non-special non-projectively normal line bundles are contained in the image of the incidence correspondence

$$\begin{array}{ccc} & I^{j,h}_{a,b} = \left\{ \begin{array}{l} (A, L) \in W^j_a \times \mathrm{Pic}^{2g-h}(C) : \\ L = K_C - cA + D \text{ for some } D \in W_b \end{array} \right\} & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ W^j_a & & \mathrm{Pic}^{2g-h}(C) \end{array}$$

through the projection π_2. The fibre of π_1 over each $A \in W^j_a$ can be identified with the image of W_b under the multiplication map $\mathrm{Pic}^b(C) \xrightarrow{\cong} \mathrm{Pic}^{2g-h}(C)$ by $K_C - cA$. As seen in Remark 5.1.7, this set is closed of dimension b. Hence, by [Har77, Exercise II.3.22(b)] and [Arb+85, Theorem IV.5.1], we obtain the bound

$$\dim \pi_2(I^{j,h}_{a,b}) \leq \dim I^{j,h}_{a,b} \leq \dim W^j_a + \dim W_b \leq a - 2j - 1 + b.$$

By [KKO99, Theorems 3.1–3.2] and [Aka04, Theorems 3.1–3.2] we have:

- $(j, a, b) = (1, 3, 6), (1, 4, 4)$ when $h = 2$;
- $(j, a, b) = (1, 3, 8), (1, 4, 3), (1, 5, 4)$ when $h = 3$;
- $(j, a, b) = (1, 3, 10), (1, 4, 6), (1, 5, 3), (1, 6, 4), (2, 8, 6)$ when $h = 4$;
- $(j, a, b) = (1, 3, 12), (1, 4, 5), (1, 5, 2), (1, 6, 3), (1, 7, 4), (2, 8, 5), (2, 9, 6)$ when $h = 5$.

For each (h, j, a, b) listed above, we see that $\dim \pi_2(I^{j,h}_{a,b}) < g_h \leq g$. Therefore we obtain the assertion in (ii). □

Thanks to the proof of the Maximal Rank Conjecture for non-special curves [BE87], we can find a lower bound for the projective normality of the general Ulrich bundle on a general curve. Moreover this is sharp for Ulrich line bundles.

Proposition 5.1.12. *On a general curve of genus $g \geq 3$, the general rank r Ulrich bundle defined with respect to a general very ample polarization of degree*

$$d \geq \frac{3+\sqrt{8g+1}}{2}$$

is projectively normal. Moreover, this bound is sharp for $r = 1$.

Proof. We reduce to the case $r = 1$ thanks to Proposition 5.1.9. By [BE87, Theorem] (and references therein), on a general curve C of genus $g \geq 3$ the general non-special very ample line bundle L of degree $u = g + (d-1)$ inducing an embedding $\varphi_L \colon X \hookrightarrow \mathbf{P}^{d-1}$, for $d \geq \frac{3+\sqrt{8g+1}}{2} \geq 4$, satisfies the Maximal Rank Conjecture (MRC), i.e. the restriction map

$$r_{L,k} \colon \mathrm{S}^k H^0(C, L) \longrightarrow H^0(C, kL)$$

has maximal rank for all $k \geq 1$. Since a general line bundle H of degree d is very ample [EH83, Theorem 5.1.2] and H-Ulrich line bundles form an open subset in $\mathrm{Pic}^u(C)$ [Cos17a, (4.1.3)], we infer that the general H-Ulrich line bundle $\mathcal{L}$ satisfies MRC. Now, Lemma B.3.13 says that $\mathcal{L}$ is projectively normal if and only if it is 2-normal if and only if $r_{\mathcal{L},2} = r_2$ is surjective. Given that r_2 has maximal rank, this holds if and only if

$$\dim \mathrm{S}^2 H^0(C, \mathcal{L}) = \binom{d+1}{2} \geq h^0(C, 2\mathcal{L}) = 2u + 1 - g = 2d + g - 1,$$

where the term on the right is easily computed via Riemann-Roch theorem. The assertion is now clear. □

Remark 5.1.13. Observe that the same bound can be obtained using [BF10, Theorem 1] (together with Proposition 5.1.9).

We conclude the section with some remarks on the (N_p) property for vector bundles.

Definition 5.1.14. Let X be a projective variety and let L be a very ample line bundle defining an embedding

$$\varphi_L \colon X \longrightarrow \mathbf{P}(H^0(X, L)) = \mathbf{P}^N.$$

Then $R(L)$ admits a minimal graded free resolution $E_\bullet \longrightarrow R(L)$ as graded R_X-module:

$$\cdots \longrightarrow E_1 = \bigoplus R_X(-a_{1,j}) \longrightarrow E_0 = R_X \oplus \bigoplus R_X(-a_{0,j}) \longrightarrow R(L) \longrightarrow 0.$$

We say that L satisfies the *Property* (N_p) if $E_0 = R_X$ and $a_{i,j} = i+1$ for all j whenever $1 \leq i \leq p$. In particular, L satisfies (N_0) if and only if L is normally generated. A vector bundle $\mathcal{E}$ on X satisfies the (N_p) *property* if $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ does it on $\mathbf{P}(\mathcal{E})$.

Proposition 5.1.15. *Let C be a smooth projective curve of genus $g \geq 0$ and let B be a globally generated ample line bundle on C of degree $d > 0$. Let $\mathcal{E}$ be a B-Ulrich bundle on C. Then:*

(1) $\mathcal{E}$ *satisfies* (N_1) *and* $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ *is Koszul if* $d > g+2$.

(2) $\mathcal{E}$ *satisfies* (N_p) *for* $p \geq 2$ *if* $d > \frac{1}{2}\left((g+p+1)+\sqrt{g^2+2g(3p+1)+(p-1)^2}\right)$.

Proof. In both situations we have $d > g+1$, therefore $\mathcal{E}$ is very ample (Theorem 4.0.20) and semistable with $\mu(\mathcal{E}) = d+g-1$ (Lemma 1.1.16). We also know from Corollary 5.1.2 that $\mathcal{E}$ is projectively normal. Letting $\pi\colon \mathbf{P}(\mathcal{E}) \to C$ be the natural projection, through [Har77, Exercise III.8.4(a)] it's immediate to see that

$$R^i\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)(-1-i)\right) = R^i\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-i) = 0 \text{ for } i > 0.$$

In other words, $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is (-1)-regular with respect to π in the sense of [Laz04a, Example 1.8.24], or [But94, §3]. Then, if $d > g+2$, [But94, Theorem 6.1] tells that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is Koszul. Since a very ample Koszul line bundle satisfies (N_1), see [But94, Remark 5.2] or [Her10, Remark 7], this proves (1). Finally, (2) is just a rephrasing of [Par06, Theorem 1.3]. □

Proof of Theorem 5.0.1. This is just a recollection of the facts proved in Corollary 5.1.2 and Propositions 5.1.9–5.1.12–5.1.15 □

Analogously to Green theorem for line bundles on curves [Laz04a, Theorem 1.8.53] and to Corollary 5.1.2, the expectation is that an Ulrich bundle on a smooth projective curve of degree d and genus g satisfies (N_p) as soon as $d > g+1+p$.

5.2 Projective normality of Ulrich bundles on surfaces

We study the projective normality of 0-regular vector bundles on smooth regular embedded surfaces. We use a very ample polarization because all the tensor powers remain 0-regular (Corollary B.1.10).

Lemma 5.2.1. *Let* $\mathcal{E}$ *be a* 0*-regular vector bundle on a smooth projective surface* $S \subset \mathbf{P}^N$. *Then:*

(i) $\mathcal{E}$ *is* k*-normal for all* $k \geq 4$*;*

(ii) $\mathcal{E}$ *is* 3*-normal if* $p_g(S) = 0$*;*

(iii) If $q(S) = 0$, $\mathcal{E}$ *is* 2*-normal if and only if* $h^2(S, \Lambda^2 M_{\mathcal{E}}) = p_g(S)\binom{h^0(S,\mathcal{E})}{2}$ *if and only if the map*

$$\Lambda^2 H^0(S,\mathcal{E})^* \otimes H^0(S, K_S) \to H^0(S, \Lambda^2 M_{\mathcal{E}}^* \otimes K_S)$$

is surjective (or, equivalently, an isomorphism).

Observe that this gives another proof for the fact, already know from Proposition 5.0.8, that a 2-normal ample 0-regular vector bundle on an embedded smooth projective surface with $p_g = 0$ is automatically projectively normal.

Proof. We know from (B.3) in Lemma B.3.5 that the syzygy exact sequence $0 \to M_{\mathcal{E}} \to V \otimes \mathcal{O}_S \to \mathcal{E} \to 0$, where $V = H^0(S,\mathcal{E})$, yields the long exact sequence

$$0 \to \Lambda^k M_{\mathcal{E}} \to \Lambda^k V \otimes \mathcal{O}_S \to \Lambda^{k-1} V \otimes \mathcal{E} \to \cdots \to \Lambda^2 V \otimes S^{k-2}\mathcal{E} \to V \otimes S^{k-1}\mathcal{E} \to S^k\mathcal{E} \to 0$$

for every $k \geq 2$. By Remark B.3.3, to prove (i) we have to show the surjectivity on global section of the rightmost map. For $k \geq 4$, the vanishings

$$H^1(S, \Lambda^2 V \otimes S^{k-2}\mathcal{E}) \cong \Lambda^2 V \otimes H^1(S, S^{k-2}\mathcal{E}) = 0, H^2(S, \Lambda^3 V \otimes S^{k-3}\mathcal{E}) \cong \Lambda^3 V \otimes H^2(S, S^{k-3}\mathcal{E}) = 0,$$

which follow from the 0-regularity of the symmetric power of $\mathcal{E}$ (Corollary B.1.10), together with

$$H^i(S, \Lambda^{i+1} V \otimes S^{k-i-1}\mathcal{E}) \cong \Lambda^{i+1} V \otimes H^i(S, S^{k-i-1}\mathcal{E}) = 0 \text{ for } i \geq 3,$$

give the claim by [Laz04a, Example B.1.3].

Now, assuming $p_g(S) = 0$, for $k = 3$ we similarly have

$$H^1(S, \Lambda^2 V \otimes \mathcal{E}) \cong \Lambda^2 V \otimes H^1(S, \mathcal{E}) = 0, H^2(\Lambda^3 V \otimes \mathcal{O}_S) \cong \Lambda^3 H^0(S, \mathcal{E}) \otimes H^2(S, \mathcal{O}_S) = 0$$

and $H^3(S, \Lambda^3 M_{\mathcal{E}}) = 0$. Item (ii) is obtained in the same way.

Assume $q(S) = 0$, so that $H^1(S, M_{\mathcal{E}}) = 0$, then consider $k = 2$ in the above exact sequence and let $K = \ker(V \otimes \mathcal{E} \to S^2\mathcal{E})$. The cohomology of the corresponding exact sequence immediately shows that $H^2(S, K) \cong H^1(S, S^2\mathcal{E}) = 0$ and that $\mathcal{E}$ is 2-normal if and only if $H^1(S, K) = 0$. On the other hand, as $K = \operatorname{Im}(\Lambda^2 M_{\mathcal{E}} \to \Lambda^2 V \otimes \mathcal{O}_S)$, we get the exact sequence of vector spaces

$$0 \to H^1(S, K) \to H^2(S, \Lambda^2 M_{\mathcal{E}}) \xrightarrow{f} \Lambda^2 V \otimes H^2(S, \mathcal{O}_S) \to H^2(S, K) = 0.$$

Thus, f being surjective, $H^1(S, K) = 0$ if and only if f is injective if and only if f is an isomorphism if and only if $h^2(S, \Lambda^2 M_{\mathcal{E}}) = p_g(S) \cdot \dim \Lambda^2 V$. Taking the dual map of f and using Serre duality, we see that f is injective if and only if

$$\Lambda^2 V^* \otimes H^0(S, K_S) \to H^0(S, \Lambda^2 M_{\mathcal{E}}^* \otimes K_S)$$

is surjective. This proves (iii). □

In virtue of this, we will mostly focus on regular smooth surfaces S possibly with $p_g(S) = 0$.

Let's see what happens when we consider 0-regular locally free sheaves with aCM projectivized bundle.

Proposition 5.2.2. *Let $S \subset \mathbf{P}^N$ be smooth projective surface and let $\mathcal{E}$ be a very ample 0-regular bundle on S of rank $r \geq 2$. Then:*

(1) If $\mathbf{P}(\mathcal{E})$ is aCM, then $q(S) = p_g(S) = 0$.

(2) If $q(S) = p_g(S) = 0$ and $\mathcal{E}$ is 2-normal, then $\mathbf{P}(\mathcal{E})$ is aCM, $\mathcal{I}_{\mathbf{P}(\mathcal{E})/\mathbf{P}(H^0(S,\mathcal{E}))}$ is 3-regular and $I_{\mathbf{P}(\mathcal{E})/\mathbf{P}(H^0(S,\mathcal{E}))}$ is generated in degree less than or equal to 3.

Proof. Since $S^k\mathcal{E}$ is 0-regular (Corollary B.1.10), by [Har77, Exercise III.8.4] we have

$$H^i(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(\ell)) = \begin{cases} H^i(S, \mathcal{O}_S) & \text{for } i \geq 0, \ell = 0 \\ 0 & \text{for } i \geq 1, \ell \geq -r+1, \ell \neq 0. \end{cases} \tag{5.1}$$

Therefore, if $\mathbf{P}(\mathcal{E})$ is aCM, then $H^i(S, \mathcal{O}_S) = H^i(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}) = 0$ for $1 \leq i \leq 2$ (Remark B.3.10). This gives (1).

Now assume $q(S) = p_g(S) = 0$ and that $\mathcal{E}$ is 2-normal. Then Proposition 5.0.8 tells that $\mathcal{E}$ is projectively normal with $\mathcal{I}_{\mathbf{P}(\mathcal{E})/\mathbf{P}(H^0(S,\mathcal{E}))}$ which is 3-regular and $I_{\mathbf{P}(\mathcal{E})/\mathbf{P}(H^0(S,\mathcal{E}))}$ that is generated in degree less than or equal to 3. We only need to prove that $\mathbf{P}(\mathcal{E})$ is aCM. To do this, by Lemma B.3.11 it suffices to prove that a smooth sectional curve of $\mathbf{P}(\mathcal{E}) \subset \mathbf{P}(H^0(S,\mathcal{E}))$ is projectively normal. Take r smooth hyperplane sections $H_1, \dots, H_r \in |\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|$ such that $Y_j = H_1 \cap \cdots \cap H_j \subset \mathbf{P}(\mathcal{E})$ is smooth and irreducible. We claim that $H^i(Y_j, \mathcal{O}_{Y_j}(h)) = 0$ for all $i \geq 1$ as soon as $h \geq -r + 1 + j$.
To see this, we proceed by induction on $1 \leq j \leq r$. Thanks to (5.1), we can immediately see from the exact sequence $0 \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(h-1) \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(h) \to \mathcal{O}_{Y_1}(h) \to 0$ that $H^i(Y_1, \mathcal{O}_{Y_1}(h)) = 0$ for all $i \geq 1$ as long as $h - 1 \geq -r + 1$ as desired. For $j > 1$, we get the claim by applying the inductive hypothesis to $0 \to \mathcal{O}_{Y_{j-1}}(h-1) \to \mathcal{O}_{Y_{j-1}}(h) \to \mathcal{O}_{Y_j}(h) \to 0$.
Now, as $H^1(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}) = H^1(Y_j, \mathcal{O}_{Y_j}) = 0$ for all $1 \leq j \leq r-1$, we know that all $Y_1, \dots, Y_r$ are linearly normal. Moreover, since $H^1(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(k-1)) = H^1(Y_j, \mathcal{O}_{Y_j}(k-1)) = 0$ for all $k \geq 2$ and $1 \leq j < r$, Lemma B.3.11 tells that $Y_1, \dots, Y_r$ are projectively normal. Since Y_r is a sectional curve, this proves (2). □

For 0-regular vector bundles with ample determinant on embedded surfaces with $q = p_g = 0$ we can also give a geometric characterization for the non-2-normality.

Proposition 5.2.3. *Let $S \subset \mathbf{P}^N$ be smooth projective surface with $q(S) = p_g(S) = 0$ and let $\mathcal{E}$ be a 0-regular vector bundle of rank $r \geq 2$ on S with ample determinant line bundle bundle $E = \det(\mathcal{E})$. Assume $h = h^0(S,\mathcal{E}) \geq r + 3$ and let $\ell = \binom{h-r}{2} - 1$. The following are equivalent:*

(1) $\mathcal{E}$ is not 2-normal.

(2) There exist a closed subscheme $Z \subset S$ and a non-zero divisor $D \subset S$ such that:

- *(a) Z is smooth of dimension 0.*
- *(b) Z is the degeneracy locus of ℓ general sections $s_1, \dots, s_\ell \in H^0(S, \Lambda^2 M_{\mathcal{E}}^*)$.*
- *(c) $[Z] = \frac{1}{2}(h-r-2)\left((h-r+1)c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})\right)$.*
- *(d) $D \in |K_S + (h-r-1)E|$.*
- *(e) $Z \subset D$.*

(3) There exist a closed subscheme $Z \subset S$ and a curve $C \subset S$ such that:

- *(f) Z is the degeneracy locus of ℓ general sections $\sigma_1, \dots, \sigma_\ell \in H^0(S, \Lambda^2 M_{\mathcal{E}}^*)$.*
- *(g) C is the degeneracy locus of the $(\ell + 1)$ general sections $\sigma_1, \dots, \sigma_\ell, \sigma_{\ell+1} \in H^0(S, \Lambda^2 M_{\mathcal{E}}^*)$.*
- *(h) $C \in |(h-r-1)E|$ is smooth and irreducible.*
- *(i) $Z \subset C$ is a special (effective) divisor.*

Proof. First of all, from the syzygy exact sequence $0 \to M_{\mathcal{E}} \to H^0(S,\mathcal{E}) \otimes \mathcal{O}_S \to \mathcal{E} \to 0$ and its dual we can see that:

(A) $H^0(S, \Lambda^2 M_{\mathcal{E}}) \subset H^0(S, M_{\mathcal{E}} \otimes M_{\mathcal{E}}) \subset H^0(S,\mathcal{E}) \otimes H^0(S, M_{\mathcal{E}}) = 0$;

(B) $c_1(M_{\mathcal{E}}) = -c_1(\mathcal{E})$ and $c_2(M_{\mathcal{E}}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E})$;

(C) the map $\Lambda^2 H^0(S,\mathcal{E})^* \otimes \mathcal{O}_S \to \Lambda^2 M^*_{\mathcal{E}}$ is surjective (thanks to (B.4) for $k = 2$).

Now assume (1). Since (C) tells in particular that $\Lambda^2 M^*_{\mathcal{E}}$ is generated by global sections, we can take a general subspace $V \subset H^0(S, \Lambda^2 M^*_{\mathcal{E}})$ of dimension

$$\ell = \mathrm{rk}(\Lambda^2 M^*_{\mathcal{E}}) - 1 = \binom{h-r}{2} - 1$$

and we let $Z = D_{\ell-1}(v) \subset S$ be the degeneracy locus of the evaluation map $v\colon V \otimes \mathcal{O}_S \to \Lambda^2 M^*_{\mathcal{E}}$. As $\Lambda^2 M^*_{\mathcal{E}}$ is generated by global sections and $\dim S = 2$, it is well known, for instance from [Bǎn91, Statement (folklore), §4.1], that Z is either empty or is smooth of (the expected) codimension 2 and that there is a short exact sequence

$$0 \to V \otimes \mathcal{O}_S \to \Lambda^2 M^*_{\mathcal{E}} \to \mathcal{I}_{Z/S}(M) \to 0,$$

with $M = \det(\Lambda^2 M^*_{\mathcal{E}}) = (h - r - 1)E$ (Lemma B.4.2(i)) and $\mathcal{I}_{Z/S} = \mathcal{O}_S$ in case $Z = \emptyset$. However Z cannot be empty, for otherwise, by dualizing the above sequence and using Kodaira vanishing on the line bundle M, which is ample by the hypothesis, we would have $H^0(S, \Lambda^2 M_{\mathcal{E}}) \cong V^*$ which contradicts (A). This gives (a)-(b). As $Z \neq \emptyset$, by construction and (B) (and by Lemma B.4.2) we get

$$[Z] = c_2(\Lambda^2 M^*_{\mathcal{E}}) = \frac{1}{2}(h - r + 1)(h - r - 2)c_1(\mathcal{E})^2 - (h - r - 2)c_2(\mathcal{E}),$$

that is (c). Now, twist the above sequence through by K_S and take the cohomology. We immediately see that $H^0(S, \Lambda^2 M^*_{\mathcal{E}} \otimes K_S) \cong H^0(S, \mathcal{I}_{Z/S}(M + K_S))$. By Lemma 5.2.1, we finally get (d)-(e), proving (2).

Conversely, if $Z \subset S$ satisfies (a) and (b), then (c) holds by construction. Furthermore, by [Ott95b, Teorema 2.14 & Esercizio, p. 23], the ideal sheaf $\mathcal{I}_{Z/S}$ fits into the same exact sequence as above. If there is $D \subset S$ satisfying (d)-(e), we have $H^0(S, \mathcal{I}_{Z/S}(M + K_S)) \neq 0$. Repeating the above argument backwards, we obtain the equivalence between (1) and (2).

Now we observe that (a) and (b), which imply (c) as seen above, is equivalent to (f) and (g), which in turn yield (h) by construction. One direction is obvious, hence we assume (b). Let $W \subset H^0(S, \Lambda^2 M^*_{\mathcal{E}})$ be the subspace generated by the sections in (b) and choose a subspace $W' \subset H^0(S, \Lambda^2 M^*_{\mathcal{E}})$ of dimension $\ell + 1$ that contains W. The degeneracy locus $D_\ell(w)$ of the evaluation map $w\colon W' \otimes \mathcal{O}_S \to \Lambda^2 M^*_{\mathcal{E}}$ contains Z, is supported on a smooth member $C \in |M|$, which is then irreducible by [Har77, Corollary III.7.9], and $\mathrm{coker}(w) = L$ is a line bundle on C (see [Bǎn91, Statement (folklore), §4.1] and [BT24, §4.1]). This gives the claim. In addition to this, from the Snake lemma applied to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W \otimes \mathcal{O}_S & \longrightarrow & \Lambda^2 M^*_{\mathcal{E}} & \longrightarrow & \mathcal{I}_{Z/S}(M) & \longrightarrow & 0, \\
 & & \big\downarrow & & \big\downarrow \cong & & \big\downarrow & & \\
0 & \longrightarrow & W' \otimes \mathcal{O}_S & \longrightarrow & \Lambda^2 M^*_{\mathcal{E}} & \longrightarrow & L & \longrightarrow & 0,
\end{array}$$

we get the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{I}_{Z/S}(M) \to L \to 0$$

which yields $L(-M) = i_*\mathcal{O}_C(-Z)$, where $i: C \hookrightarrow S$ is the inclusion.

To prove the equivalence (2)-(3), it's enough to see that

$$H^1(C, Z) \neq 0 \text{ if and only if } H^0(S, \mathcal{I}_{Z/S}(M + K_S)) \neq 0.$$

To do this, twist the above sequence through by K_S and take the cohomology. It's immediate to see that $H^0(S, \mathcal{I}_{Z/S}(M+K_S)) \cong H^0(C, \mathcal{O}_C(K_S+M-Z))$. On the other hand, by adjunction and Serre duality, we have $h^0(C, \mathcal{O}_C(K_S+M-Z)) = h^1(C, Z)$. Then the assertion follows. □

In turn we get a characterization for ample 0-regular bundle on embedded surfaces with $q = p_g = 0$ for non-being projectively normal and aCM.

Proof of Theorem 5.0.2. Remark B.3.10 immediately tells that (2) implies (1). For the converse, we just need to observe that if $\mathcal{E}$ was projectively normal, then it would be automatically very ample and 2-normal, but then Proposition 5.2.2 would imply that $\mathbf{P}(\mathcal{E})$ is aCM, contradicting the assumption. On the other hand, since $\mathcal{E}$ is ample, by Remark B.3.2 and Lemma 5.2.1 we know that $\mathcal{E}$ is not projectively normal if and only if $\mathcal{E}$ is not 2-normal. As E is ample as well by [Laz04b, Corollary 6.1.16], the equivalence between (2), (3) and (4) follows from Proposition 5.2.3. □

Then we can provide a numerical criterion for the non projective normality of an ample 0-regular bundle on a surface with $q = p_g = 0$.

Corollary 5.2.4. *Let $S \subset \mathbf{P}^N$ be a smooth projective surface with $q(S) = p_g(S) = 0$. Let $\mathcal{E}$ be an ample 0-regular vector bundle on $S \subset \mathbf{P}^N$ of rank $r \geq 2$ with $h = h^0(S, \mathcal{E}) \geq r + 3$. Then $\mathcal{E}$ is not aCM, or equivalently not projectively normal, if*

$$(h - r - 1)c_1(\mathcal{E}) \cdot K_S + 2(h - r - 2)c_2(\mathcal{E}) + h(h - 1) > (h - r - 3)c_1(\mathcal{E})^2 + r(2h - r - 1).$$

We first state a simple remark involving Grothendieck-Verdier duality.

Remark 5.2.5. Let $f: X \to Y$ be an affine morphism of smooth projective varieties of relative dimension $-k = \dim X - \dim Y \leq 0$, and let $\mathcal{F}, \mathcal{G}$ be locally free sheaves on X, Y respectively. Then

$$\mathcal{E}xt^k_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G}) \cong \mathcal{G}(-K_Y) \otimes f_*(\mathcal{F}^*(K_X)).$$

In particular, if f is the inclusion of a subvariety of codimension $k \geq 1$ with normal bundle $\mathcal{N}_{X/Y}$, then

$$\mathcal{E}xt^k_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G}) \cong \mathcal{G} \otimes f_*(\mathcal{F}^* \otimes \Lambda^k \mathcal{N}_{X/Y}). \tag{5.2}$$

To see this, we are going to use Grothendieck-Verdier duality [Huy06, Theorem 3.34]. First we observe that the direct image $\mathbf{R}f_*\mathcal{E} = f_*\mathcal{E}$ does not need to be derived for any coherent sheaf $\mathcal{E}$ on X and $\mathbf{L}f^*(\mathcal{G}) = f^*\mathcal{G}$ as well. Indeed, for the second assertion, since $\mathcal{G}$ is locally free on Y, it is enough apply the definition (see [GW23, Proposition/Definition 21.110 & Remark 21.92(1)]). For the first one, take any bounded below quasi-coherent f_*-acyclic resolution $\mathcal{E} \to \mathcal{I}^\bullet$, for instance any quasi-coherent injective resolution [Huy06, Lemma 3.24] which exists by [Har77, Exercise III.3.6(a)]. In this way $\mathbf{R}f_*\mathcal{E} = f_*(\mathcal{I}^\bullet)$ in the bounded below derived category $\mathrm{D}^+(\mathbf{Qcoh}(Y))$ of quasi-coherent sheaves on Y. Since we have

$$\mathcal{H}^i(f_*(\mathcal{I}^\bullet)) = \mathcal{H}^i(\mathbf{R}f_*\mathcal{E}) = \mathbf{R}^i(f_*\mathcal{E}) = 0 \text{ for } i < 0$$

and

$$\mathcal{H}^0(f_*(\mathcal{I}^\bullet)) = \mathcal{H}^0(\mathbf{R}f_*\mathcal{E}) = \mathbf{R}^0(f_*\mathcal{E}) = f_*\mathcal{E}$$

by [Huy06, Remark 2.49] and also

$$\mathcal{H}^j(f_*(\mathcal{I}^\bullet)) = \mathcal{H}^j(\mathbf{R}f_*\mathcal{E}) = \mathbf{R}^j(f_*\mathcal{E}) = 0 \text{ for } j > 0$$

by [Har77, Exercise III.8.2], we know from [KS06, Proposition 13.1.12(i)-(iii)] that the object $\mathbf{R}f_*\mathcal{E}$ belongs to the full subcategory $\mathbf{Qcoh}(Y) \subset \mathrm{D}^+(\mathbf{Qcoh}(Y)) \subset \mathrm{D}(\mathbf{Qcoh}(Y))$ and that corresponds to

$$\mathbf{R}f_*\mathcal{E} \cong f_*(\mathcal{I}^\bullet) \cong \mathcal{H}^0(f_*(\mathcal{I}^\bullet))[0] = \mathcal{H}^0(f_*(\mathcal{I}^\bullet)) \cong f_*\mathcal{E}.$$

Since $\mathbf{R}f_*\mathcal{E} = f_*\mathcal{E}$ is in fact coherent [Har77, Corollary II.5.20], this proves the observation. Now, since the relative dualizing bundle of f is $\omega_f = K_X(-f^*K_S)$, Grothendieck-Verdier duality yields the isomorphisms

$$\begin{aligned} \mathbf{R}\mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{F},\mathcal{G}) &\cong \mathbf{R}\mathscr{H}om_{\mathcal{O}_Y}(\mathbf{R}f_*\mathcal{F},\mathcal{G}) \\ &\cong \mathbf{R}f_*\mathbf{R}\mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{F},\mathbf{L}i^*(\mathcal{G})\otimes\omega_f[-k]\right) \\ &\cong \mathbf{R}f_*\mathbf{R}\mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{F},f^*\mathcal{G}\otimes\omega_f[-k]\right) \\ &\cong \mathbf{R}f_*\mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{F},f^*\mathcal{G}\otimes\omega_f[-k]\right) \\ &\cong \mathbf{R}f_*(\mathcal{F}^*\otimes f^*\mathcal{G}\otimes\omega_f[-k]) \\ &= \mathbf{R}f_*(\mathcal{F}^*(K_X)\otimes f^*(\mathcal{G}(-K_Y))[-k]). \end{aligned} \tag{5.3}$$

Here we performed the identifications

$$\mathbf{R}\mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{F},f^*\mathcal{G}\otimes\omega_f[-k]\right) \cong \mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{F},f^*\mathcal{G}\otimes\omega_f[-k]\right) \cong \mathcal{F}^*\otimes f^*\mathcal{G}\otimes\omega_f[-k]$$

as complexes concentrated in degree k which comes from the locally freeness on X of $\mathcal{F}$ (see [Huy06, §3.3, p. 84] and [Har77, Exercise II.5.1(b)]). Taking the k-th cohomology sheaf $\mathcal{H}^k\colon \mathrm{D^b}(\mathbf{Coh}(Y)) \to \mathbf{Coh}(Y)$ on both sides of (5.3) and adopting the usual sign conventions for translations and cohomology [Con00, (1.3.4)], by definition [GW23, (21.21.2)] and by [GW23, Remark 21.26(3)] we have the claimed isomorphism of coherent sheaves

$$\begin{aligned} \mathscr{E}xt^k_{\mathcal{O}_Y}(f_*\mathcal{F},\mathcal{G}) &\cong \mathcal{H}^k(\mathbf{R}\mathscr{H}om_{\mathcal{O}_Y}(f_*\mathcal{F},\mathcal{G})) \\ &\cong \mathcal{H}^k(\mathbf{R}f_*(\mathcal{F}^*(K_X)\otimes f^*(\mathcal{G}(-K_Y))[-k])) \\ &\cong \mathbf{R}^0(f_*(\mathcal{F}^*(K_X)\otimes f^*(\mathcal{G}(-K_Y)))) \\ &\cong f_*(\mathcal{F}^*(K_X)\otimes f^*(\mathcal{G}(-K_Y))) \\ &\cong \mathcal{G}(-K_Y)\otimes f_*(\mathcal{F}^*(K_X)). \end{aligned}$$

The last part of the claim descends from adjunction formula [Har77, Proposition II.8.20].

We point out that the isomorphism (5.2) can be obtained also using the sheaf version of [Har77, Lemma III.7.4], which can be proved in the exact same way.

Proof of Corollary 5.2.4. Just like in Proposition 5.2.3 and its proof, consider the degeneracy loci $Z \subset C \subset S$ of ℓ and $\ell+1$ general sections of the globally generated bundle $\Lambda^2 M^*_{\mathcal{E}}$, with

$\ell = \binom{h-r}{2} - 1$. Both of them are smooth, Z is 0-dimensional with $[Z]$ given in Proposition 5.2.3(c), and $C \in |M|$ is irreducible, where $M = \det(\Lambda^2 M_{\mathcal{E}}^*) = (h-r-1)\det(\mathcal{E})$. As seen in the proof of Proposition 5.2.3, we have the exact sequence

$$0 \to \mathcal{O}_S^{\oplus(\ell+1)} \to \Lambda^2 M_{\mathcal{E}}^* \to i_* \mathcal{O}_C(M-Z) \to 0$$

where $i\colon C \hookrightarrow S$ is the inclusion. Taking the dual with respect to S, we get the short exact sequence

$$0 \to \Lambda^2 M_{\mathcal{E}} \to \mathcal{O}_S^{\oplus(\ell+1)} \to \mathcal{E}xt^1_{\mathcal{O}_S}(i_*\mathcal{O}_C(M-Z), \mathcal{O}_S) \to 0.$$

But $\mathcal{E}xt^1_{\mathcal{O}_S}(i_*\mathcal{O}_C(M-Z), \mathcal{O}_S) \cong i_*\mathcal{O}_C(Z)$ by Remark 5.2.5. Thus, using that $\Lambda^2 M_{\mathcal{E}}$ has no non-zero global sections, we can see that $h^0(C,Z) \geq \ell+1$. Computing the genus of $C \subset S$ via adjunction formula, applying Riemann-Roch theorem and using this inequality, we see that

$$h^1(C,Z) \geq \frac{1}{2}((h-r-1)c_1(\mathcal{E})\cdot K_S + 2(h-r-2)c_2(\mathcal{E}) + h(h-1) - (h-r-3)c_1(\mathcal{E})^2 - r(2h-r-1)).$$

By Proposition 5.2.3, $\mathcal{E}$ is not projectively normal as soon as $h^1(C,Z) > 0$. Then the claim follows. □

For an Ulrich bundle $\mathcal{E}$ on a smooth embedded surface $S \subset \mathbf{P}^N$ we can calculate the dimensions of the global sections of the second tensor power $\mathcal{E}\otimes\mathcal{E}$ and of the symmetric powers $S^2\mathcal{E}, S^3\mathcal{E}$. As a necessary condition for the k-normality of $\mathcal{E}$ is $\dim S^k H^0(S,\mathcal{E}) \geq h^0(S, S^k\mathcal{E})$, this gives a way to check if $\mathcal{E}$ could potentially be k-normal. This method will be used in the next examples and also in the next section for Ulrich bundles on hypersurfaces.

First we recall that the second Chern class of an Ulrich bundle $\mathcal{E}$ of rank r on a smooth surface $S \subset \mathbf{P}^N$ of degree d is given by Casnati formula [Cas17b, Proposition 2.1]

$$c_2(\mathcal{E}) = \frac{1}{2}\left(c_1(\mathcal{E})^2 - c_1(\mathcal{E})\cdot K_S\right) + r\chi(\mathcal{O}_S) - rd. \tag{5.4}$$

Lemma 5.2.6. *Let $\mathcal{E}$ be a rank r Ulrich bundle on a smooth projective surface $S \subset \mathbf{P}^N$ of degree d. Then:*

(i) $h^0(S,\mathcal{E}\otimes\mathcal{E}) = \chi(S,\mathcal{E}\otimes\mathcal{E}) = c_1(\mathcal{E})^2 + r^2(2d - \chi(S,\mathcal{O}_S))$.

(ii) $h^0(S,S^2\mathcal{E}) = \chi(S,S^2\mathcal{E}) = r(r+2)d + \frac{1}{2}(c_1(\mathcal{E})^2 + c_1(\mathcal{E})\cdot K_S) - \frac{r(r+3)}{2}\chi(S,\mathcal{O}_S)$.

(iii) $h^0(S,S^3\mathcal{E}) = \chi(S,S^3\mathcal{E}) = \frac{1}{6}(r+2)(3c_1(\mathcal{E})^2 + 3c_1(\mathcal{E})\cdot K_S + 3rd(r+3) - 2r\chi(S,\mathcal{O}_S)(r+4))$.

In particular, if $c_1(\mathcal{E}) = \frac{r}{2}(K_S + 3H)$, where H is the class of a hyperplane section, we have:

(a) $h^0(\mathcal{E}\otimes\mathcal{E}) = \chi(S,\mathcal{E}\otimes\mathcal{E}) = \frac{r^2}{4}\left[17d + 6K_S\cdot H + K_S^2 - 4\chi(S,\mathcal{O}_S)\right]$.

(b) $h^0(S,S^2\mathcal{E}) = \chi(S,S^2\mathcal{E})$
$= \frac{r}{8}\left[(17r+16)d + (2+r)K_S^2 + 6(r+1)K_S\cdot H - 4(r+3)\chi(S,\mathcal{O}_S)\right]$.

(c) $h^0(S,S^3\mathcal{E}) = \chi(S,S^3\mathcal{E})$
$= \frac{r}{24}(r+2)\left[3d(13r+12) + 3(r+2)K_S^2 + 18(r+1)H\cdot K_S - 8(r+4)\chi(S,\mathcal{O}_S)\right]$.

Proof. Both $\mathcal{E}\otimes\mathcal{E}$ and the symmetric powers $S^2\mathcal{E}$, $S^3\mathcal{E}$ are 0-regular (Corollary B.1.10), therefore we only need to compute the Euler characteristics $\chi(S,\mathcal{E}\otimes\mathcal{E})$ and $\chi(S,S^2\mathcal{E}),\chi(S,S^3\mathcal{E})$. This is easily done via Hirzebruch-Riemann-Roch theorem [Har77, Theorem A.4.1] which says that

$$\chi(S,\mathcal{F}) = s\chi(S,\mathcal{O}_S) + \frac{1}{2}\Big(c_1(\mathcal{F})^2 - c_1(\mathcal{F})\cdot K_S\Big) - c_2(\mathcal{F})$$

for any rank s vector bundle $\mathcal{F}$ on S. Using (5.4), the conclusion follows from formulae in Lemmas B.4.1-B.4.4 and Corollary B.4.3. □

Example 5.2.7. Let $S \subset \mathbf{P}^3$ be any smooth K3 surface of degree 4 and let $\mathcal{E}$ be an Ulrich bundle of rank $r = 2$ or $r = 4$ with $\det(\mathcal{E}) = \mathcal{O}_S(3r/2)$ (which always exists by [Fae19, Theorem 1] combined with Lemma 1.1.12). Then $\mathcal{E}$ cannot be projectively normal because, by Lemma 5.2.6(b), the inequality

$$\dim S^2H^0(S,\mathcal{E}) = 2r(4r+1) \geq h^0(S,S^2\mathcal{E}),$$

which is necessary for the 2-normality, is never satisfied.

Example 5.2.8. Let $S \subset \mathbf{P}^4$ be a smooth very general complete intersection of type $(2,a)$ for $a \geq 3$. Then no rank r-Ulrich bundle on S, which always exists by [HUB91], can be projectively normal if $a \geq 15$.

To see this, let $\mathcal{E}$ be an Ulrich bundle of rank r on S. By Noether-Lefschetz theorem we can assume $\mathrm{Pic}(S) \cong \mathbf{Z}\cdot\mathcal{O}_S(1)$ (for instance, one can apply [RS09, Theorem 1], or [BGL21, Proposition 3.2], to $\mathcal{O}_Q(a)$ where $Q \subset \mathbf{P}^4$ is a smooth quadric). Letting $d := \deg S = 2a$ be the degree of S and $H \subset S$ be a hyperplane section of $S \subset \mathbf{P}^N$, we have $K_S = (a-3)H = \frac{1}{2}(d-6)H$ and $\chi(S,\mathcal{O}_S) = \frac{d}{24}(d^2-9d+26)$. Moreover, by Lemma 1.1.16 we have $c_1(\mathcal{E}) = \frac{r}{2}(K_S+3H) = \frac{r}{4}dH$. Substituting in Lemma 5.2.6(b) we get

$$h^0(S,S^2\mathcal{E}) = \frac{rd}{96}\left[rd^2 + 18(r+1)d + 44r + 36\right].$$

This means that if $\mathcal{E}$ is 2-normal, then $h^0(S,S^2\mathcal{E}) - rd(rd+1)/2 \leq 0$, which happens if and only if

$$rd^2 - (30r-18)d + 44r - 12 \leq 0.$$

One can check that this forces the following conditions on r and $d = 2a$:

- $6 \leq d \leq 18$ if $r \geq 2$;
- $d = 20, 22$ if $r \geq 3$;
- $d = 24$ if $r \geq 5$;
- $d = 26$ if $r \geq 8$;
- $d = 28$ if $r \geq 41$;

In any case we have $2a = d \leq 28$. Then the claim follows.

5.3 Projective normality of Ulrich bundles on low dimensional hypersurfaces

Ulrich bundles on hypersurfaces always exist [HUB91]. Moreover, Ulrich bundles on the general hypersurface $X \subset \mathbf{P}^{n+1}$ of degree $d \geq 2n$ is very ample as X does not contain lines (Threorem 4.0.20). Therefore the expectation is them to be quite positive, in particular projectively normal. However we will see that this is not the case for hypersurfaces of dimension 2 and 3.

In this section, all hypersurfaces are smooth.

We begin by recalling that every Ulrich bundle on a hypersurface $X \subset \mathbf{P}^{n+1}$ admits a locally free resolution on X.

Lemma 5.3.1. *Let $X \subset \mathbf{P}^{n+1}$ be a smooth hypersurface of dimension $n \geq 1$ and degree $d \geq 2$. An Ulrich bundle $\mathcal{E}$ of rank r on X admits the resolution*

$$0 \to \mathcal{E}(-d) \to \mathcal{O}_X(-1)^{\oplus rd} \to \mathcal{O}_X^{\oplus rd} \cong H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E} \to 0. \tag{5.5}$$

In particular, the tensor power $\mathcal{E} \otimes \mathcal{E}$ is resolved by

$$0 \to \mathcal{E} \otimes \mathcal{E}(-d) \to \mathcal{E}(-1)^{\oplus rd} \to \mathcal{E}^{\oplus rd} \cong H^0(X, \mathcal{E}) \otimes \mathcal{E} \to \mathcal{E} \otimes \mathcal{E} \to 0. \tag{5.6}$$

Proof. This is essentially [Tri16, §2] and [Tri17, §2] applied to the Ulrich bundle resolution

$$0 \to \mathcal{O}_{\mathbf{P}^{n+1}}(-1)^{\oplus rd} \to \mathcal{O}_{\mathbf{P}^{n+1}}^{\oplus rd} \cong H^0(\mathbf{P}^{n+1}, i_*\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^{n+1}} \cong H^0(X, \mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^{n+1}} \to i_*\mathcal{E} \to 0$$

of $i_*\mathcal{E}$ on $\mathbf{P}^{n+1}$, with $i \colon X \hookrightarrow \mathbf{P}^{n+1}$ being the inclusion. More precisely, as in *loc. cit.* it is shown that $\mathscr{T}\!or_1^{\mathcal{O}_{\mathbf{P}^{n+1}}}(i_*\mathcal{E}, i_*\mathcal{O}_X) \cong i_*\mathcal{E}(-d)$, taking the tensor product of the above resolution trough by $i_*\mathcal{O}_X$, by Remark B.1.8 and projection formula we obtain the exact complex

$$0 \to i_*(\mathcal{E}(-d)) \to i_*(\mathcal{O}_X(-1)^{\oplus rd}) \to H^0(\mathbf{P}^{n+1}, i_*\mathcal{E}) \otimes i_*\mathcal{O}_X \cong H^0(X, \mathcal{E}) \otimes i_*\mathcal{O}_X \to i_*\mathcal{E} \to 0.$$

Since i is a closed immersion, we get (5.5). □

Remark 5.3.2. Despite the embedding of every smooth quadric $Q \subset \mathbf{P}^{n+1}$ is Koszul (Example B.1.12), which implies that every 0-regular vector bundle on Q is k-normal for all $k \geq 1$, an Ulrich bundle on Q cannot be projectively normal if $n \geq 3$ because it is not ample by Theorem 4.0.20 in virtue of [Ott88, Corollary 1.6].

Lemma 5.3.3. *Let $S \subset \mathbf{P}^3$ be a smooth hypersurface of degree $d \geq 2$ and let $\mathcal{E}$ be an Ulrich bundle of rank r on X such that* $\det(\mathcal{E}) = \mathcal{O}_X(\frac{r}{2}(d-1))$. *Then:*

(i) $h^0(S, \mathcal{E} \otimes \mathcal{E}) = \chi(S, \mathcal{E} \otimes \mathcal{E}) = \frac{r^2 d}{12}(d+1)(d+5)$.

(ii) $h^0(S, S^2\mathcal{E}) = \chi(S, S^2\mathcal{E}) = \frac{rd}{24}(d+1)((d+5)r+6)$.

(iii) $h^0(S, S^3\mathcal{E}) = \chi(S, S^3\mathcal{E}) = \frac{rd}{72}(d+1)(r+2)(r+4+d(5r+2))$.

In particular, if $\mathcal{E}$ is 2-normal, then either $d = 2$, or $d = 3$ and $r \geq 3$, or $d = 4$ and $r \geq 6$.

Note that, by Lemma 1.1.16, the set of 2-dimensional smooth hypersurfaces supporting such Ulrich bundles contains the subset of hypersurfaces $S \subset \mathbf{P}^3$ with $\mathrm{Pic}(S) \cong \mathbf{Z} \cdot \mathcal{O}_S(1)$, which are very general in $|\mathcal{O}_{\mathbf{P}^3}(d)|$ by Noether-Lefschetz theorem [Lef50; GH85].

Proof. By hypothesis we have $c_1(\mathcal{E}) = \frac{r}{2}(d-1)H$, for $H \in |\mathcal{O}_S(1)|$. Since

$$\chi(S, \mathcal{O}_S) = 1 + \binom{d-1}{3} = \frac{d(d^2-6d+11)}{6},$$

formulae (i)-(ii)-(iii) are obtained by substituting these in Lemma 5.2.6(a-b-c). For the last part, observe that the 2-normality of $\mathcal{E}$ implies the inequality

$$\dim \mathrm{S}^2 H^0(S, \mathcal{E}) = \frac{rd(rd+1)}{2} < h^0(X, \mathrm{S}^2\mathcal{E}),$$

which is satisfied in the claimed ranges. □

On the contrary, in the situation of the above Lemma, the difference

$$\dim \mathrm{S}^3 H^0(S, \mathcal{E}) - h^0(S, \mathrm{S}^3\mathcal{E}) = \frac{rd}{72}(d-1)(r-2)(d(7r+2)+r+8)$$

is negative if and only if $r = 1$ (which happens very rarely), it's zero for rank 2 (that is for Pfaffian surfaces), and it is always positive for $r \geq 3$. Therefore the 3-normality of $\mathcal{E}$ usually does not imposes any restriction.

We now move to hypersurfaces in $\mathbf{P}^4$. We begin with the calculations of Chern classes and Euler characteristics of tensor powers and symmetric powers. Note that it is no longer granted that the tensor operations of Ulrich bundles are again 0-regular.

Lemma 5.3.4. *Let $X \subset \mathbf{P}^4$ be a smooth hypersurface of degree $d \geq 1$ and let $\mathcal{E}$ be a rank r Ulrich bundle on X. Then:*

(i) $c_3(\mathcal{E}) = \frac{rd}{48}(d-1)^2(r-2)(rd-r+2)$.

(ii) $c_3(\mathcal{E}\otimes\mathcal{E}) = \frac{r^2d}{12}(d-1)^2\left(r^2-2\right)\left(2r^2(d-1)+3-d\right)$.

(iii) $c_3(\mathrm{S}^2\mathcal{E}) = \frac{rd}{48}(d-1)^2(r+2)\left(r^2+r-4\right)\left(r^2(d-1)+2\right)$.

(iv) $\chi(X, \mathcal{E}\otimes\mathcal{E}) = h^0(X, \mathcal{E}\otimes\mathcal{E}) - h^1(X, \mathcal{E}\otimes\mathcal{E}) = \frac{r^2d}{8}(d+1)(d+3)$.

(v) $\chi(X, \mathrm{S}^2\mathcal{E}) = h^0(X, \mathrm{S}^2\mathcal{E}) - h^1(X, \mathrm{S}^2\mathcal{E}) = \frac{rd}{48}(d+1)(d+3)(3r+4-d)$.

Proof. For (i), see [Ben+23, Proposition 3.7] or just apply Hirzebruch-Riemann-Roch theorem to $\chi(X, \mathcal{E}) = rd$. Lefschetz theorem tells that the restriction map $\mathrm{Pic}(\mathbf{P}^4) \to \mathrm{Pic}(X)$ induces an isomorphism [Laz04a, Example 3.1.25]. Hence, letting H be the class of a hyperplane section of $X \subset \mathbf{P}^4$, we have $c_1(\mathcal{E}) = \frac{r}{2}(d-1)H$ (Lemma 1.1.16). Using [LR24a, Lemma 3.2(ii)] and (i) on Lemma B.4.1(iii) and Corollary B.4.3(iii) one obtains (ii) and (iii) respectively. Now, observe that

$$H^2(X, \mathcal{E}\otimes\mathcal{E}(p)) = H^3(X, \mathcal{E}\otimes\mathcal{E}(p)) = 0 \qquad \text{for } p \geq -2 \tag{5.7}$$

by [Laz04a, Proposition B.1.2(i)] applied to the resolution (5.6) twisted by $\mathcal{O}_X(p)$. Then (iv) and (v) follow from Hirzebruch-Riemann-Roch theorem for $\mathcal{E}\otimes\mathcal{E}$ and $\mathrm{S}^2\mathcal{E}$ respectively. □

Proof of Theorem 5.0.3. Item (1) is an immediate consequence of Lemma 5.3.3. For (2), note that we must have $r \geq 2$ by Lefschetz theorem [Laz04a, Example 3.1.25] and Remark 1.2.1. Thanks to Lemma 5.3.4(iv), we immediately see that the inequality

$$h^0(X, \mathcal{E} \otimes \mathcal{E}) \geq \chi(X, \mathcal{E} \otimes \mathcal{E}) > \dim H^0(X, \mathcal{E})^{\otimes 2} = r^2 d^2$$

holds for every pair (r, d). Therefore $\mu_{\mathcal{E}}$ is never surjective. Finally, by Lemma 5.3.4(v), we can check that

$$h^0(X, \mathrm{S}^2\mathcal{E}) \geq \chi(X, \mathrm{S}^2\mathcal{E}) > \dim \mathrm{S}^2 H^0(X, \mathcal{E}) = \frac{rd(rd+1)}{2}$$

holds for if $r > \frac{d+4}{3}$. In particular $\mathcal{E}$ cannot be projectively normal for all such ranks. □

The bound on the rank, possibly not optimal, is necessary. As we are going to see, Theorem 5.0.3(2) is sharp for the 2-normality of Ulrich bundles on 3-dimensional hypersurfaces of degree 5.

Remark 5.3.5. Let $X \subset \mathbf{P}^{n+1}$ be a hypersurface of dimension $n = 3$ or 4 and degree $d \geq 3$. Suppose X supports an Ulrich bundle $\mathcal{E}$ of rank $r = 2$ or $r = 3$. Then $\mathcal{E}$ is 2-normal. If $n = 4$ and $r = 3$, then $\mathcal{E}$ is also 3-normal.

Indeed, as $\Lambda^2\mathcal{E}$, being isomorphic either to $\mathcal{O}_X(d-1)$ for $n = 3$ or to $\mathcal{E}^*(3(d-1)/2)$ when $n = 4$, is aCM, we know from [Tri17, Proposition 5.1] that $\Lambda^2 M_{\mathcal{E}}$ is aCM as well. Since $H^2(X, \Lambda^2 M_{\mathcal{E}}) = 0$, the first part is provided by Example B.3.6. Assuming $n = 4$ and $r = 3$, by applying [Laz04a, Example B.1.3] to the long exact sequence

$$0 \to \Lambda^3 M_{\mathcal{E}} \to \Lambda^3 H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \to \Lambda^2 H^0(X, \mathcal{E}) \otimes \mathcal{E} \to H^0(X, \mathcal{E}) \otimes \mathrm{S}^2\mathcal{E} \to \mathrm{S}^3\mathcal{E} \to 0,$$

given by (B.3), we deduce that $\mathcal{E}$ is 3-normal if (and only if) $H^3(X, \Lambda^3 M_{\mathcal{E}}) = 0$. As $\Lambda^3 M_{\mathcal{E}}$ is aCM [RT19, Theorem 5.7(a)], the claim follows.

In the above situation, except for $r = 2$ with $n = 3$ and $d \leq 5$, these hypersurfaces live in a closed subset of $|\mathcal{O}_{\mathbf{P}^{n+1}}(d)|$ (see [Bea00, Proposition 8.9] and [LR24b, Corollary 1 & Theorem 2]). Moreover, a pfaffian hypersurface of degree d in $\mathbf{P}^4$ or in $\mathbf{P}^5$, which are exactly those supporting Ulrich bundles of rank 2, contains no lines if $d \geq 6$ or $d \geq 8$ respectively. This means that all Ulrich bundles on such hypersurfaces are very ample (Theorem 4.0.20).

The behaviour of 2-normality of rank 2 Ulrich bundles on pfaffian surfaces and paffian threefolds or fourfolds is different: while no (very general) pfaffian surface of degree $d \geq 5$ supports a 2-normal Ulrich bundle of rank 2 (Lemma 5.3.3), all rank 2 Ulrich bundles on pfaffian 3-folds and 4-folds of degree $d \geq 3$ are 2-normal.

Appendix

Appendix A

Background

The main purpose of this chapter is to recollect the basic definitions and results used throughout the work. For proofs and further details we refer to the books [Har77; HL10; Laz04a; Liu02; OSS80].

A.1 Unramified morphisms

In this section we recollect the main characterizations and properties of unramified morphisms.

Definition - Theorem A.1.1. *Let $f\colon X \to Y$ be a morphism of schemes and let $x \in X$ be a point. Let $y = f(x)$ and let $X_y = f^{-1}(y)$ be the schematic fibre over $f(x)$. The morphism f is* unramified at x *if it satisfies one of the following equivalent conditions:*

1. *The homomorphism $f_x^\#\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ satisfies $\mathfrak{m}_y \cdot \mathcal{O}_{X,x} = \mathfrak{m}_x$.*
2. *X_y is smooth at x and x is an isolated point in X_y.*
3. *$\Omega_{X/Y,x} = 0$.*
4. *$\Omega_{X_y,x} = 0$.*
5. *$d_x f\colon T_x X \to T_y Y$ is injective.*

A point $x' \in X$ is a ramification point of f *if f is not unramified at x'. We set* $\mathrm{Ram}(f) := \mathrm{supp}(\Omega_{X/Y})$ *to be the set of ramification points of f. A point $y' \in Y$ is a* branch point of f *if f is not unramified at all points of the fibre over y'. A morphism is* unramified *if it is unramified at all points.*

Proof. The equivalence between 1 and 2 is [Liu02, Lemma 4.3.20]. Items 1-3-4-5 are equivalent by [Sta23, Tag 02GF & Tag 0B2G]. □

Remark A.1.2. We use these equivalent definitions without explicit mention.

Remark A.1.3. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes which are unramified at $x \in X$ and at $y = f(x)$ respectively. Then $g \circ f$ is unramified at x.

Indeed, taking the stalk at x of the exact sequence $f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$ (see, e.g., [Har77, Proposition II.8.11]) and using the identification $(f^*\Omega_{Y/Z})_x \cong \Omega_{Y/Z,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$ [Sta23, Tag 0098], we immediately see that $\Omega_{X/Z,x} = 0$.

Lemma A.1.4. *Let $f : X \to Y$ be a finite surjective morphism of varieties and suppose Y is normal. Then $\#\left(f^{-1}(y)\right) \leq \deg(f)$ for every $y \in Y$, and the equality holds if and only if y is not a branch point of f.*

Proof. Fix $y \in Y$ and set $d = \deg(f)$. The first part of the assertion is [Sha13, Theorem 2.28]. For the last part, observe that the schematic fibre X_y is a non-empty 0-dimensional Noetherian scheme. In particular, $\mathcal{O}_{X_y}(X_y) = A$ is an Artin ring [AM69, Theorem 8.5] and also a $\mathbf{C}$-algebra of $\dim_{\mathbf{C}}(A) = m \geq d$ [Liu02, Exercise 5.1.25(a)]. By the structure theorem of Artin rings [AM69, Theorem 8.7], we have a decomposition $A = A_1 \times \cdots \times A_k$, where $\#\left(f^{-1}(y)\right) = k \leq m$ and $(A_i, \mathfrak{m}_i)$ is a local Artin ring for all $1 \leq i \leq k$. We have to exclude the case $k > d$, for otherwise $\#\left(f^{-1}(y)\right) > d$ contradicting the previous part. So we must have $k \leq d$. If $k = d$, then $(A_i, \mathfrak{m}_i) = (\mathbf{C}, (0))$ for all $1 \leq i \leq k$, whence $\Omega_{X_y, x_i} = 0$, where x_i is the point corresponding to $\mathfrak{m}_i$, for all i (see for instance [Har77, Proof of Theorem II.8.6A]). This says that f is unramified at all points $x_i \in X_y$, that is y is not a branch point of f. If $k < d$, there must exist $j = 1, \ldots, k$ such that $(A_j, \mathfrak{m}_j) \neq (\mathbf{C}, (0))$. Any non-zero element $a \in \mathfrak{m}_j$ is nilpotent, given that A_j is local and Artinian [AM69, Corollary 8.2 & Proposition 8.4], hence it defines a non-zero class da in $\Omega_{A_j/\mathbf{C}}$. In particular $\Omega_{X_y, x_j} \neq 0$, where x_j is the point corresponding to $\mathfrak{m}_j$, which means that y is a branch point of f. □

A.2 Arithmetically Cohen-Macaulay bundles

One way to understand the geometry of a projective variety X is studying the category of the vector bundles that it supports. Following this philosophy, a central role have been played by vector bundles *with no intermediate cohomology*, also called *arithmetically Cohen-Macaulay*.

Definition A.2.1. A coherent sheaf $\mathcal{E}$ on a projective variety X is called *arithmetically Cohen-Macaulay*, or *aCM* for short, with respect to an ample line bundle L if it is locally Cohen-Macaulay and

$$H^i(X, \mathcal{E}(jL)) = 0$$

for $0 < i < n$ and $j \in \mathbf{Z}$.

Example A.2.2. The line bundle $\mathcal{O}_{\mathbf{P}^n}(d)$ is aCM with respect to $\mathcal{O}_{\mathbf{P}^n}(1)$ on $\mathbf{P}^n$.

A seminal result due to Horrocks asserts that on projective spaces vector bundles which are aCM with respect to $\mathcal{O}(1)$ split as sum of line bundles.

Theorem A.2.3 (Horrocks' theorem). *A vector bundle $\mathcal{E}$ on a projective space $\mathbf{P}^n$ splits as sum of line bundles if and only if $\mathcal{E}$ is aCM.*

Proof. See [OSS80, Theorem 2.3.1]. □

In [AY08] the authors proved a similar splitting theorem for reflexive sheaves.

Theorem A.2.4. *Let $\mathbf{F}$ be an algebraically closed field, and let $\mathcal{E}$ be a reflexive sheaf of rank $r \geq 1$ on $\mathbf{P}^n_{\mathbf{F}}$, with $n \geq 3$. Then $\mathcal{E}$ splits as a sum of line bundles if and only if there exists a hyperplane $H \subset \mathbf{P}^n_{\mathbf{F}}$ such that $\mathcal{E}_{|H}$ splits as a sum of line bundles.*

Proof. See [AY08, Theorem 0.2]. □

A.3 Positivity of vector bundles

Positivity of line bundles is nowadays well understood and has played a major role in the study of projective geometry. For vector bundles of higher rank, the situation is quite different. While the usual notions of positivity (nefness, ampleness) have been generalized by using the tautological line bundle on the projectivized bundle (see, e.g., [Laz04b, §6-§8]), at the same time other positivity notions appeared and are different from the previous generalizations. Here we recollect some of these notions and their interplay with the asymptotic base loci. We mainly refer to [Bau+15].

Definition A.3.1. A vector bundle $\mathcal{F}$ on a projective variety X is *nef* (resp. *big, ample, very ample*) if the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ is nef (resp. big, ample, very ample) on the projectivized bundle $\mathbf{P}(\mathcal{F})$. Given an integer $k \geq 0$, we say that $\mathcal{F}$ is *k-very ample* if the restriction map

$$H^0(X,\mathcal{F}) \to H^0(Z,\mathcal{F}_{|Z})$$

is surjective for every 0-dimensional closed subscheme $Z \subset X$ of length $k+1$.

Remark A.3.2. Observe that a vector bundle is 0-very ample if and only if it is globally generated. For line bundles, 1-very ampleness is equivalent to very ampleness. This is no longer true for higher ranks. In general, if $k > 0$ and $\mathcal{F}$ is a k-very ample vector bundle on a (smooth) projective variety, then $\mathcal{F}$ is ample (and $\det(\mathcal{F})$ is very ample) [Bal94, Remark 1.3 & Lemma 1.4].

One defines the asymptotic base loci also for vector bundles. Rather than on the projectivized bundle, these can be defined on the variety itself.

Definition A.3.3. Let $\mathcal{F}$ be a vector bundle on a projective variety X. The *base locus* of $\mathcal{F}$ is the set

$$\mathrm{Bs}(\mathcal{F}) = \left\{ x \in X \mid H^0(X,\mathcal{F}) \to \mathcal{F}(x) \text{ is not surjective} \right\}.$$

The *stable base locus* of $\mathcal{F}$ is defined as

$$\mathbf{B}(\mathcal{F}) = \bigcap_{k\geq 1} \mathrm{Bs}(\mathrm{Sym}^k\mathcal{F}).$$

The *augmented* (resp. *restricted*) *base locus* of $\mathcal{F}$ is

$$\mathbf{B}_+(\mathcal{F}) = \bigcap_{k\geq 1} \mathbf{B}((\mathrm{Sym}^k\mathcal{F})(-A)) \qquad \left(\text{resp. } \mathbf{B}_-(\mathcal{F}) = \bigcup_{k\geq 1} \mathbf{B}((\mathrm{Sym}^k\mathcal{F})(A))\right),$$

where A is an ample line bundle on X.

Remark A.3.4. One can prove that the definition of augmented (and restricted) base locus of a vector bundle $\mathcal{F}$ does not depend on the choice of the ample line bundle. Moreover, if $\pi\colon \mathbf{P}(\mathcal{F}) \to X$ denotes the natural projection, we have $\pi(\mathbf{B}_+(\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1))) = \mathbf{B}_+(\mathcal{F})$ [FM21, Proposition 6.4].

As for line bundles, the positivity of vector bundle is strictly related to asymptotic base loci. See [Bau+15] for complete treatment.

Proposition A.3.5. *Let X be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on X. Then:*

(1) $\mathcal{F}$ is nef if and only if $\mathbf{B}_-(\mathcal{F}) = \emptyset$.

(2) $\mathcal{F}$ is ample if and only if $\mathbf{B}_+(\mathcal{F}) = \emptyset$.

Proof. See [Bau+15, Propositions 5.2-6.2]. □

It is well known that a line bundle L is big if and only if $\mathbf{B}_+(L) \neq X$, see for instance [Bau+15, Proposition-Definition 4.2]. For vector bundles, this condition produces a notion of positivity which is stronger than bigness.

Definition A.3.6. A vector bundle $\mathcal{F}$ on a projective variety X is said *V-big* if there exist an ample line bundle A and a positive integer $k > 0$ such that $\overline{\mathbf{B}_-((\mathrm{Sym}^k\mathcal{F})(-A))} \neq X$.

Remark A.3.7. On smooth projective varieties, a V-big vector bundle is big [Bau+15, Corollary 6.5]. The converse is typically false for vector bundles of rank strictly greater than 1 (see for instance [Bau+15, Remark 6.6]).

Theorem A.3.8 ([Bau+15, Theorem 6.4]). *Let X be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on X. Then $\mathcal{F}$ is V-big if and only if $\mathbf{B}_+(\mathcal{F}) \neq X$.*

A.4 Stability of coherent sheaves

In this section we recall the notions of stability and of semistability of a coherent sheaf on a projective variety. See [HL10] for more details.

Definition A.4.1. Given a coherent sheaf $\mathcal{E}$ on a scheme X, the *dimension of* $\mathcal{E}$ is defined to be

$$\dim(\mathcal{E}) := \dim(\mathrm{supp}(\mathcal{E})).$$

We say that $\mathcal{E}$ is *pure* if $\dim(\mathcal{F}) = \dim(\mathcal{E})$ for all nonzero coherent subsheaf $\mathcal{F} \subset \mathcal{E}$.

Definition A.4.2. Let (X, B) be a polarized projective variety and let let $\mathcal{E}$ be a coherent sheaf whose support has dimension k. By [HL10, Lemma 1.2.1], we can write the Hilbert polynomial $P(\mathcal{E}, m) := \chi(X, \mathcal{E}(mB))$ with respect to B in the form

$$P(\mathcal{E}, m) = \sum_{i=0}^{k} \alpha_i(\mathcal{E}) \frac{m^i}{i!},$$

where $\alpha_i(\mathcal{E})$ is an integral coefficient for every $i = 0, \ldots k$. The *reduced Hilbert polynomial* is defined by

$$p(\mathcal{E}, m) := \frac{P(\mathcal{E}, m)}{\alpha_k(\mathcal{E})}.$$

The *rank* of $\mathcal{E}$ is

$$\mathrm{rk}(\mathcal{E}) := \frac{\alpha_k(\mathcal{E})}{\alpha_k(\mathcal{O}_X)} = \frac{\alpha_k(\mathcal{E})}{B^{\dim X}}.$$

The coherent sheaf $\mathcal{E}$ is said *(semi)stable* if $\mathcal{E}$ is pure and $p(\mathcal{F}) < p(\mathcal{E})$ (resp. $p(\mathcal{F}) \leq p(\mathcal{E})$) in the lexicographic order of their coefficients for every proper coherent subsehaf $\mathcal{F} \subset \mathcal{E}$.

The *slope* of $\mathcal{E}$ is defined by

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})},$$

where $\deg(\mathcal{E}) := \alpha_{k-1}(\mathcal{E}) - \alpha_{k-1}(\mathcal{O}_X)\,\mathrm{rk}(\mathcal{E})$. If X is smooth, then $\deg(\mathcal{E}) = c_1(\mathcal{E}) \cdot B^{k-1}$ by Hirzebruch-Riemann-Roch theorem. The coherent sheaf $\mathcal{E}$ is *μ-(semi)stable* if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$) for all nonzero coherent subsheaves $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$.

Remark A.4.3. All the quantities defined above, except the rank, depend on the choice of the polarization B. If we want to emphasize the dependence on B, we put a subscript $P_B(\mathcal{E}, m)$, $p_B(\mathcal{E}, m)$ and so on.

The following chain of implications holds [HL10, Lemma 1.2.13]

$$\mathcal{E} \text{ is } \mu\text{-stable} \Rightarrow \mathcal{E} \text{ is stable} \Rightarrow \mathcal{E} \text{ is semistable} \Rightarrow \mathcal{E} \text{ is } \mu\text{-semistable}.$$

Note that on a smooth projective curve, (semi)stability is equivalent to μ-(semi)stability.

Remark A.4.4. Let X be a projective variety and let B be an ample and globally generated line bundle. Consider a μ-semistable coherent sheaf $\mathcal{E}$ on X which is not μ-stable. Then we can always assume that the "μ-destabilizing subsheaf" $\mathcal{F} \subset \mathcal{E}$, i.e. $\mu(\mathcal{F}) = \mu(\mathcal{E})$, is such that the quotient $\mathcal{E}/\mathcal{F}$ is torsion free.

Indeed, if it is not the case, let $\mathcal{G} \subset \mathcal{E}$ be the sheaf

$$\mathcal{G} := \ker\left(\mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{F} \twoheadrightarrow (\mathcal{E}/\mathcal{F})_{\mathrm{tf}}\right),$$

where $(\mathcal{E}/\mathcal{F})_{\mathrm{tf}}$ is the torsion-free part of $\mathcal{E}/\mathcal{F}$. Then: $\mathcal{G}$ is torsion-free, contains $\mathcal{F}$, has the same rank of $\mathcal{F}$, and is such that $\mathcal{G}/\mathcal{F}$ is torsion. As a consequence $\deg(\mathcal{G}/\mathcal{F}) \geq 0$, giving $\deg(\mathcal{F}) \leq \deg(\mathcal{G})$. Therefore, using the μ-semistability of $\mathcal{E}$, we deduce that

$$\mu(\mathcal{E}) = \mu(\mathcal{F}) \leq \mu(\mathcal{G}) \leq \mu(\mathcal{E}).$$

To get the assertion, it is enough to replace $\mathcal{F}$ with $\mathcal{G}$.

Proposition A.4.5. *Let X be a projective variety together with an ample line bundle B. Every semistable sheaf $\mathcal{E}$ admits a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$$

such that, for $1 \leq i \leq \ell$, $\mathrm{gr}_i(\mathcal{E}) = \mathcal{E}_i/\mathcal{E}_{i-1}$ is stable with $p(\mathcal{E}_i/\mathcal{E}_{i-1}) = p(\mathcal{E})$. This is called a Jordan-Hölder filtration of $\mathcal{E}$. *Moreover, non-zero $\mathcal{E}_i$'s are semistable with $p(\mathcal{E}_i) = p(\mathcal{E})$, and $\mathrm{gr}(\mathcal{E}) = \bigoplus_{i=1}^{\ell} \mathrm{gr}_i(\mathcal{E})$ does not depend on the choice of such filtration.*

Proof. See [HL10, §1.5 & Proposition 1.5.2]. □

Appendix B

Preliminary results

This chapter is devoted to a recollection of some standard results for which we add proofs for convenience or because of a lack of references.

B.1 Castelnuovo-Mumford regularity

Let X be a projective variety, and let B be an ample line bundle. If $\mathcal{F}$ is a coherent sheaf on X, Cartan-Serre-Grothendieck theorem [Laz04a, Theorem 1.2.6] and its consequences [Laz04a, Example 1.2.22] states that the following hold for $d \gg 0$:

- $\mathcal{F}(dB)$ is generated by global sections;
- $H^i(X, \mathcal{F}(dB)) = 0$ for $i > 0$;
- the natural multiplication map

$$\mu_{d,k} \colon H^0(X, \mathcal{F}(dB)) \otimes H^0(X, B^{\otimes k}) \to H^0(X, \mathcal{F}((d+k)B))$$

  is surjective for all $k \gg 0$.

Castelnuovo-Mumford regularity provides a quantitative measure of the size of d which is necessary to have these three properties.

Definition B.1.1. Let X be a projective variety and let B be an ample line bundle. A coherent sheaf $\mathcal{F}$ on X is said *m-regular with respect to B (in the sense of Castelnuovo-Mumford)* if

$$H^i(X, \mathcal{F}((m-i)B)) = 0$$

for $i > 0$. The *regularity of $\mathcal{F}$ (with respect to B)* is

$$\mathrm{reg}_B(\mathcal{F}) := \min\{s \in \mathbf{Z} \mid \mathcal{F} \text{ is } s\text{-regular with respect to } B\}.$$

Example B.1.2. The line bundle $\mathcal{O}_{\mathbf{P}^n}(d)$ is $(-d)$-regular with respect $\mathcal{O}_{\mathbf{P}^n}(1)$ on $\mathbf{P}^n$.

Theorem B.1.3 (Castelnuovo-Mumford). *Let X be a projective variety and let B be an ample globally generated line bundle. If $\mathcal{F}$ is an m-regular sheaf with respect to B, then for every $k \geq 0$:*

(1) $\mathcal{F}((m+k)B)$ is generated by global sections.

(2) The natural multiplication maps

$$\mu_{m,k} \colon H^0(X, \mathcal{F}(mB)) \otimes H^0(X, B^{\otimes k}) \to H^0(X, \mathcal{F}((m+k)B))$$

are surjective.

(3) $\mathcal{F}$ is $(m+k)$-regular with respect to B.

Proof. See [Laz04a, Theorem 1.8.5]. □

Remark B.1.4. Let X be a projective variety, and let $\mathcal{F}$ be a m-regular sheaf with respect to the base-point-free ample line bundle B. Then $\mathcal{F}(mB)$ is generated by global sections (Theorem B.1.3(1)) and satisfies $H^p(X, \mathcal{F}(mB)) = 0$ for $p > 0$: Castelnuovo-Mumford theorem implies that $H^i(X, \mathcal{F}((m+p-i)B)) = 0$ for every $i > 0$ and every $p \geq 0$. Taking $i = p$ the assertion follows.

As is well-known, e.g. from [Laz04a, Proposition 1.8.9], the regularity of the tensor product of two vector bundles on the projective space is (at most) the sum of the regularity of each vector bundle. This no longer holds for other varieties, mainly because the polarization is not (-1)-regular (with respect to itself). In [Ara04, §3], Arapura observes that one needs to consider the regularity of the structure sheaf in order to compute the regularity of tensor products.

For any projective variety X endowed with a globally generated ample line bundle B we fix

$$M := \max\{1, \operatorname{reg}_B(\mathcal{O}_X)\}$$

for the rest of the section.

Remark B.1.5. Normal polarized varieties (X, B) of dimension $n \geq 2$ with at worst $\mathbf{Q}$-factorial terminal singularities having $M = 1$ are classified. Indeed, $\operatorname{reg}(\mathcal{O}_X) \leq 1$ implies

$$h^0(X, K_X + (n-1)B) = h^n(X, (1-n)B) = 0 \quad \text{and} \quad h^1(X, \mathcal{O}_X) = 0$$

which, by [BS95, Corollary 7.28 & Table 7.1], force (X, B) to one of the following:

- $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$;
- $(Q^n, \mathcal{O}_{\mathbf{P}^{n+1}}(1)_{|Q^n}))$, where $Q^n \subset \mathbf{P}^{n+1}$ is a quadric;
- $(\mathbf{P}(\mathcal{F}), \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1))$, where $\mathcal{F}$ is an ample vector bundle of rank n over $\mathbf{P}^1$;
- a possibly degenerate generalized cone $C_n(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$ over $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2})$.

The following is a version of [Laz04a, Example 1.8.7 & Proposition 1.8.8] for every polarized variety.

Lemma B.1.6. *Let X be a projective variety of dimension $n \geq 1$ and let B be a globally generated ample line bundle on X.*

A p-regular coherent sheaf $\mathcal{F}$ on X admits a long resolution

$$F_\bullet : \qquad \cdots \to W_\ell \otimes B^{\otimes(-p-\ell M)} \to \cdots \to W_1 \otimes B^{\otimes(-p-M)} \to W_0 \otimes B^{\otimes(-p)} \to \mathcal{F} \to 0 \quad \text{(B.1)}$$

where the W_i's are some finite-dimensional vector spaces and $W_0 = H^0(X, \mathcal{F}(pB))$.

Conversely, if a coherent sheaf $\mathcal{G}$ on X admits a possibly infinite resolution by coherent sheaves

$$G_\bullet: \qquad \cdots \to G_h \to \cdots \to G_1 \to G_0 \to \mathcal{G} \to 0$$

with G_j being q_j-regular for $0 \le j \le n-1$ (resp. $0 \le j \le n$), then $\mathcal{G}$ is q-regular (resp. the map

$$H^0(X, G_0(q'B)) \to H^0(X, \mathcal{G}(q'B))$$

is surjective), with $q = \max_{0\le j\le n-1}\{q_j - j\}$ (resp. $q' = \max_{0\le j\le n}\{q_j - j\}$).

Proof. See [Ara04, Corollary 3.2 & Lemma 3.9]. □

This leads to a generalization of [Laz04a, Proposition 1.8.9].

Corollary B.1.7. *Let X be a projective variety of dimension $n \ge 1$ and let B be a globally generated ample line bundle on X. Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on X such that at every point of X either $\mathcal{E}$ or $\mathcal{F}$ is locally free. If $\mathcal{E}$ is e-regular and $\mathcal{F}$ is f-regular, then $\mathcal{E}\otimes\mathcal{F}$ is $(e+f+(n-1)(M-1))$-regular and the multiplication map*

$$H^0(X, \mathcal{F}(fB)) \otimes H^0(\mathcal{E}((e+n(M-1))B)) \to H^0(X, (\mathcal{E}\otimes\mathcal{F})(e+f+n(M-1))B)$$

is surjective.

Proof. Consider the resolution $F_\bullet$ of $\mathcal{F}$ given in (B.1) and twist it through by $\mathcal{E}$. The resulting complex

$$\cdots \to W_\ell \otimes \mathcal{E}((-f-\ell M)B) \to \cdots \to W_1 \otimes \mathcal{E}((-f-M)B) \to W_0 \otimes \mathcal{E}(-fB) \xrightarrow{\varepsilon} \mathcal{E}\otimes\mathcal{F} \to 0$$

is still exact: as $W_\ell \otimes B^{\otimes(-f-\ell M)}$ is flat, the claim follows by the fact that it remains exact on the right since at stalk level either $\mathcal{E}$ or $\mathcal{F}$ is flat. We immediately see that $W_j \otimes \mathcal{E}((-f-jM)B)$ is $(e+f+jM)$-regular for every $0 \le j \le n$. Indeed, for $i > 0$ we have

$$H^i(X, W_j \otimes \mathcal{E}((-f-jM)B)(e+f+jM-i)B) \cong H^i(X, \mathcal{E}((e-i)B))^{\oplus \dim W_i} = 0$$

by the e-regularity of $\mathcal{E}$. The conclusion follows by the second part of Lemma B.1.6. □

The following is a simple observation, but we add the proof for the sake of completeness.

Remark B.1.8. For a closed immersion $f\colon Y \to Z$ of schemes, the canonical morphism

$$\psi\colon f_*\mathcal{F} \otimes_{\mathcal{O}_Z} f_*\mathcal{G} \to f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G})$$

of $\mathcal{O}_Z$-modules [GW20, (7.8.3)] is an isomorphism for any pair of $\mathcal{O}_Y$-modules $\mathcal{F}, \mathcal{G}$ on Y.

Indeed, the stalk of ψ at $z \in Z$ is either $0 \otimes_{\mathcal{O}_{Z,z}} 0 \to 0$ if $z \notin f(Y)$ or $\mathcal{F}_y \otimes_{\mathcal{O}_{Z,f(y)}} \mathcal{G}_y \to \mathcal{F}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{G}_y$ if $z = f(y)$ for some $y \in Y$. In both cases ψ_z is a bijective map of $\mathcal{O}_{Z,z}$-modules, therefore ψ is an isomorphism.

Proposition B.1.9. *Let X be a projective variety and let B be a globally generated ample line bundle on X. Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on X. If $\mathcal{E}$ is e-regular and $\mathcal{F}$ is f-regular, then $\mathcal{E}\otimes\mathcal{F}$ is $(e+f)$-regular assuming one of the following holds:*

(1) $M = 1$ and at every point of X either $\mathcal{E}$ or $\mathcal{F}$ is locally free.

(2) X is a curve and at every point of X either $\mathcal{E}$ or $\mathcal{F}$ is locally free.

(3) X is a surface and B is very ample.

Part (3) is just [Sid02, Proposition 1.5].

Proof. If (1) or (2) holds, the claim follows immediately from Corollary B.1.7. Assume (3) and consider an embedding $\iota\colon X \hookrightarrow \mathbf{P}^N$ such that $\mathcal{O}_X(1) = B$. The canonical map

$$\iota_*\mathcal{E} \otimes \iota_*\mathcal{F} \to \iota_*(\mathcal{E} \otimes \mathcal{F})$$

is an isomorphism (Remark B.1.8) and, by projection formula, $\iota_*\mathcal{E}$ and $\iota_*\mathcal{F}$ are respectively e-regular and f-regular with respect to $\mathcal{O}_{\mathbf{P}^N}(1)$. By [Laz04a, Proposition 1.8.8], or by Lemma B.1.6, we have a linear resolution

$$F_\bullet\colon \cdots \to \bigoplus \mathcal{O}_{\mathbf{P}^N}(-f-2) \to \bigoplus \mathcal{O}_{\mathbf{P}^N}(-f-1) \to \bigoplus \mathcal{O}_{\mathbf{P}^N}(-f) \to \iota_*\mathcal{F} \to 0.$$

Tensoring the above complex through by $\iota_*\mathcal{E}$ and using projection formula, we obtain a complex

$$Q_\bullet\colon \cdots \to \underbrace{\bigoplus \iota_*(\mathcal{E}(-f-2))}_{=Q_2} \to \underbrace{\bigoplus \iota_*(\mathcal{E}(-f-1))}_{=Q_1} \to \underbrace{\bigoplus \iota_*(\mathcal{E}(-f))}_{=Q_0} \xrightarrow{\varepsilon} \iota_*(\mathcal{E} \otimes \mathcal{F}) \to 0$$

which is still exact on the right, i.e. $Q_1 \to Q_0 \to \iota_*(\mathcal{E} \otimes \mathcal{F}) \to 0$ is exact, thanks to the right-exactness of the tensor operation. In particular, the 0-th homology sheaf is $\mathcal{H}_0(Q_\bullet) \cong \iota_*(\mathcal{E} \otimes \mathcal{F})$. By definition, the higher homology sheaves of $Q_\bullet$ are the Tor sheaves

$$\mathcal{H}_i(Q_\bullet) = \mathcal{T}or_i^{\mathcal{O}_{\mathbf{P}^N}}(\iota_*\mathcal{E}, \iota_*\mathcal{F}), \text{ for } i > 0.$$

As a consequence, the support of $\mathcal{H}_i(Q_\bullet)$ for $i > 0$ is contained in X: the stalks $(\iota_*\mathcal{E})_y, (\iota_*\mathcal{F})_y$ are zero for every $y \in \mathbf{P}^N \backslash X$ because ι is a closed embedding, therefore, e.g. by [GW23, (21.20.3)], we find that

$$\mathcal{H}_i(Q_\bullet)_y = \left(\mathcal{T}or_i^{\mathcal{O}_{\mathbf{P}^N}}(\iota_*\mathcal{E}, \iota_*\mathcal{F})\right)_y \cong \mathrm{Tor}_i^{\mathcal{O}_{\mathbf{P}^N,y}}\left((\iota_*\mathcal{E})_y, (\iota_*\mathcal{F})_y\right) = 0$$

for every $y \in \mathbf{P}^N \backslash X$ as claimed. Furthermore, every Q_j is $(e+f+j)$-regular: indeed, for $i > 0$ we have

$$H^i(\mathbf{P}^N, Q_j(e+f+j-i)) \cong \bigoplus H^i(X, \mathcal{E}(e-i)) = 0$$

by the e-regularity of $\mathcal{E}$. Since $\dim X = 2$, the statement follows by [Sid02, Lemma 1.4]. □

Corollary B.1.10. *Let X be a projective variety of dimension $n \geq 1$ and let B be a globally generated ample line bundle on X. Given an e-regular vector bundle $\mathcal{E}$ on X, the bundles $\mathcal{E}^{\otimes p}, S^p\mathcal{E}, \Lambda^p\mathcal{E}$ are (pe)-regular if one of the following holds:*

(a) $M = 1$.

(b) X is a curve.

(c) X is a surface and B is very ample.

Proof. Under one of those assumptions, the p-fold tensor power $\mathcal{E}^{\otimes p}$ is (pe)-regular by Proposition B.1.9. As we are working over the field of complex numbers, the p-th symmetric power $S^p\mathcal{E}$ and p-th exterior power $\Lambda^p\mathcal{E}$ are direct summands of $\mathcal{E}^{\otimes p}$. Hence they must be (pe)-regular as well. □

Another property of the polarization which allows the regularity to behave well under the tensor operation is the *Koszul* property. We refer to [Tot13; CDR13; Frö99] for more details.

Definition B.1.11. A line bundle L on a n-dimensional projective variety X is K-*Koszul* if L is very ample and its section ring $R(L) = \bigoplus_m H^0(X, mL)$ determines a resolution of $\mathbf{C}$ as graded $R(L)$-module

$$\cdots \to M_K \to \cdots \to M_1 \to M_0 \to \mathbf{C} \to 0$$

with $M_i = \bigoplus R(L)(-i)$ being a free $R(L)$-module generated in degree i for every $i \leq K$. We say that L is *Koszul ample* if L is $2n$-Koszul. We say that L is *Koszul* if it is K-Koszul for all K.

Example B.1.12. Examples of embedded n-dimensional smooth projective varieties $X \subset \mathbf{P}^N$ whose hyperplane section $H = \mathcal{O}_X(1)$ is Koszul are:

- smooth complete intersections of type $(2, 2, \ldots, 2)$ [Frö99, §3.1].
- anticanonically embedded Del Pezzo surfaces of degree $d \geq 4$ [BMR24, Remark 3.22].
- canonically embedded curves which are neither hyperelliptic nor trigonal nor plane quintics [PP97].
- embedded homogeneous varieties $X = G/P$ with G being a simply connected semisimple algebraic group and P a parabolic subgroup of G [Rav95].
- abelian varieties such that $H = L^{\otimes p}$ for some ample line bundle L on X and $p \geq 4$ [Kem89, Theorem 1].
- embedded varieties such that $H = B^{\otimes k}$ for any $k > \mathrm{reg}_B(B)$ with B very ample on X [Han10, Theorem 3.3].
- embedded varieties with trivial canonical bundle such that $H = A^{\otimes(n+1)}$ for a very ample line bundle A on X [Par93, Theorem B].

The following can be found also in [Ray23, Theorem 6.8 & Lemma 6.9 (First Version)].

Proposition B.1.13. *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on X that are respectively e-regular and f-regular. Then:*

(1) $\mathcal{E} \otimes \mathcal{F}$ is $(e + f)$-regular if B is Koszul ample.

(2) $H^0(X, \mathcal{E}(eB)) \otimes H^0(X, \mathcal{F}(fB)) \to H^0(X, (\mathcal{E} \otimes \mathcal{F})((e + f)B))$ is surjective if B is $3n$-Koszul.

Proof. Item (1) is [Tot13, Theorem 3.4]. For (2), up to replace $\mathcal{E}, \mathcal{F}$ with $\mathcal{E}(eB), \mathcal{F}(fB)$, suppose $e = f = 0$. Then consider the long resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset X \times X$ given by [Tot13, Theorem 2.1]

$$\mathcal{R}_{3n-1} \boxtimes B^{\otimes -(3n-1)} \to \cdots \to \mathcal{R}_1 \boxtimes B^{\otimes(-1)} \to \mathcal{O}_X \boxtimes \mathcal{O}_X \to \mathcal{O}_\Delta \to 0,$$

for some vector bundles $\mathcal{R}_i$, with $i = 1, \ldots, 3n$.[1] Tensoring the above complex through by $\mathcal{E} \boxtimes \mathcal{F}$ and truncating it, we obtain a resolution of $\mathcal{E} \otimes \mathcal{F}$ on $X \times X$:

$$0 \to \mathcal{K} \to (\mathcal{R}_{2n} \otimes \mathcal{E}) \boxtimes (\mathcal{F}(-2nB)) \to \cdots \to (\mathcal{R}_1 \otimes \mathcal{E}) \boxtimes (\mathcal{F}(-B)) \to \mathcal{E} \boxtimes \mathcal{F} \to \mathcal{E} \otimes \mathcal{F} \to 0.$$

For dimensional reasons we have $H^{2n+1}(X \times X, \mathcal{K}) = 0$. Letting $1 \le i \le 2n$, by Kunneth formula we have

$$H^i(X \times X, (\mathcal{R}_i \otimes \mathcal{E}) \boxtimes (\mathcal{F}(-iB))) \cong \bigoplus_{p=0}^{i} \left(H^{i-p}(X, \mathcal{R}_i \otimes \mathcal{E}) \otimes H^p(X, \mathcal{F}(-iB)) \right).$$

But $H^i(X, \mathcal{F}(-iB))) = 0$ by the 0-regularity of $\mathcal{F}$, and

$$H^{i-p}(X, \mathcal{R}_i \otimes \mathcal{E}) = 0 \text{ when } 0 \le p < i$$

by [Tot13, Lemma 3.3]. Then [Laz04a, Example B.1.3] implies that

$$H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{F}) \cong H^0(X \times X, \mathcal{E} \boxtimes \mathcal{F}) \to H^0(X, \mathcal{E} \otimes \mathcal{F})$$

is surjective as required. □

B.2 Seshadri constants

In order to study the local positivity of a line bundle, Demailly introduced the Seshadri constants in [Dem92]. More precisely the aim is quantifying how much of the positivity of an ample line bundle can be localized at a given point of a projective variety. See [Bau+09] for a detailed account of the main properties of Seshadri constants.

Definition B.2.1. Let x be a point in an projective variety X and let L be a nef line bundle on X. The *Seshadri constant of L at x* is the non-negative real number

$$\varepsilon(X, L; x) = \varepsilon(L; x) := \max\{\, \varepsilon \ge 0 \mid \mu^* L - \varepsilon E \text{ is nef} \,\},$$

where $\mu \colon \widetilde{X} \to X$ is the blow-up at x, with exceptional divisor E. The real number

$$\varepsilon(L) = \varepsilon(X, L) := \inf_{x \in X} \varepsilon(X, L; x)$$

is the *Seshadri constant of the line bundle L*.

Actually Demailly's original definition of Seshadri constant is the following.

[1] For more details on the $\mathcal{R}_i$'s, we refer to [Tot13, §2]

Proposition B.2.2. *Let $x \in X$ be a point in a projective variety and let L be a nef line bundle. One has*

$$\varepsilon(L;x) = \inf\left\{ \frac{(L\cdot C)}{\mathrm{mult}_x(C)} \mid C \subset X \text{ is an irreducible curve passing through } x \right\}.$$

Proof. See [Laz04a, Proposition 5.1.5]. □

Remark B.2.3. We use these equivalent definitions without explicit mention.

Remark B.2.4. As shown in [Laz04a, Example 5.1.18], it is easy to see that if L is ample and base-point-free, then $\varepsilon(L;x) \geq 1$ for every x.

Definition B.2.5. Let $x \in X$ and L be as in Definition B.2.1. An irreducible curve $\Gamma \subset X$ such that

$$\varepsilon(L;x) = \frac{(L\cdot \Gamma)}{\mathrm{mult}_x(\Gamma)}$$

is called a *Seshadri curve for L at x*.

Generally it is not known the existence of Seshadri curves.

We list some basic properties of Seshadri constants.

Proposition B.2.6. *Let X be a projective variety, let L be a nef line bundle and fix a point $x \in X$. Then:*

(1) $\varepsilon(L;x)$ depends only on the numerical equivalence class of L and satisfies $\varepsilon(mL;x) = m\cdot\varepsilon(L;x)$ for every $m \in \mathbf{N}$.

(2) If $V \subset X$ is a subvariety of dimension $k \geq 1$ passing through x, then

$$\varepsilon(L;x) \leq \left(\frac{(L^k\cdot V)}{\mathrm{mult}_x(V)}\right)^{\frac{1}{k}}.$$

Moreover, equality holds for some V, possibly equal to X, passing through x.

(3) Assume that X is smooth. The Seshadri function

$$\varepsilon(X,-;-) = \varepsilon(-;-)\colon \mathrm{Nef}(X)\times X \ni (L,x) \mapsto \varepsilon(L;x) \in \mathbf{R}$$

is continuous with respect to the first variable and lower semi-continuous with respect to the second variable in the topology for which closed sets are countable union of closed Zariski sets. In particular, for any $L \in \mathrm{Nef}(X)$, the function $\varepsilon(L;-)$ attains its maximal value for a very general point. We denote this value by $\varepsilon(X,L;1) = \varepsilon(L;1)$.

Proof. See [Laz04a, Examples 5.1.3-5.1.4 & Proposition 5.1.9] for items (1)-(2). For the continuity in the first variable and the lower semi-continuity in the second variable, see [Bau+09, Remark 1.17 & (2.2.8)]. To conclude, let $p\colon X\times X \to X$ be the projection onto the first factor, let $\Delta\colon X \to X\times X$ be the diagonal morphism and let $\mathcal{L} = p^*L$. Since X is separated, the restriction of p is smooth along $\Delta(X)$. Then, by [Laz04a, Example 5.1.11], a very general point $x^* \in X$ satisfies

$$\varepsilon(X,L;x) = \varepsilon(X_x, \mathcal{L}_{|X_x}; \Delta(x)) \leq \varepsilon(X_{x^*}, \mathcal{L}_{|X_{x^*}}; \Delta(x^*)) = \varepsilon(X,L;x^*)$$

for all $x \in X$. In conclusion $\varepsilon(L;x^*) = \varepsilon(L;1)$ is the maximal value. □

Remark B.2.7. Let $x \in X \subset \mathbf{P}^n$ be a point in a projective variety, and let H be a hyperplane section. Then $\varepsilon(H; x) = 1$ if and only if there is a line $\ell \subset X$ passing through x. In particular, X is covered by lines if and only if $\varepsilon(H; 1) = 1$.

Indeed, a line $\ell \subset X$ satisfies $(H \cdot \ell) = 1 = \mathrm{mult}_y(\ell)$ for every $y \in \ell$, whence one direction follows from Remark B.2.4. Conversely, $\varepsilon(H; x) = 1$ means that $\mu^* H - E$ is not ample in the blow-up $\mu \colon \widetilde{X} \to X$ centered at x with exceptional divisor E (Lemma B.2.13(i)), hence the conclusion descends from [Lop22, Lemma 7.1].

Given a nef line bundle on a smooth projective variety, Nakamaye theorem leads to a characterization in terms of Seshadri constants of its augmented base locus. See also [Ein+09, Remark 6.5].

Remark B.2.8. Let X be a smooth projective variety of dimension $n \geq 1$ and let L be a nef line bundle on X. Then

$$\mathbf{B}_+(L) = \{x \in X \mid \varepsilon(L; x) = 0\} .$$

To prove this, suppose initially that L is non-big. In this case $\mathbf{B}_+(L) = X$ and $L^n = 0$. The claim immediately follows by Proposition B.2.6(2): for every $x \in X$ one has

$$0 \leq \varepsilon(L; x) \leq (L^n)^{\frac{1}{n}} = 0.$$

Now assume L is big. Nakamaye theorem [Laz04b, Theorem 10.3.5] tells that $\mathbf{B}_+(L)$ coincides with the *null locus* of L which is defined as

$$\mathrm{Null}(L) = \bigcup_{V \subsetneq X} V$$

with V ranging over all non-empty proper subvarieties of X such that $L^{\dim V} \cdot V = 0$. Therefore $\varepsilon(L; x) = 0$ for every $x \in \mathbf{B}_+(L)$ by Proposition B.2.6(2). Conversely, if $\varepsilon(L; y) = 0$, by Proposition B.2.6(2) we can find a subvariety $W \subset X$ of dimension $k > 0$ containing y such

$$0 = \varepsilon(L; y) = \left(\frac{L^k \cdot W}{\mathrm{mult}_y(W)} \right)^{\frac{1}{k}} .$$

The subvariety W must be proper because $L^n \neq 0$ given that L is big. We conclude that y belongs to $\mathrm{Null}(L) = \mathbf{B}_+(L)$.

Remark B.2.9. Let X be a projective variety and let L be a nef line bundle on X. Let $x \in X$ be a point and suppose that L separates tangent vectors at x, namely the restriction map $H^0(X, L) \to H^0(Z, L_{|Z})$ to every 0-dimensional closed subscheme $Z \subset X$ of length 2 with $\mathrm{supp}(Z) = \{x\}$ is surjective (see, for instance, the proofs in [Sta23, Tag 0E8R]). Then $\varepsilon(L; x) \geq 1$.

To prove this we use Proposition B.2.2. Let $C \subset X$ be an irreducible curve and let $v \in T_x C \subset T_x X$ be a non-zero tangent vector. Since L separates tangents at x, we can find a divisor $D \in |L|$ which passes through x with $v \notin T_x D$ [Har77, Remark II.7.8.2]. It follows that $C \not\subset D$, for otherwise $v \in T_x C \subset T_x D$ giving a contradiction. Then

$$(L \cdot C) = (D \cdot C) \geq \mathrm{mult}_x(C).$$

This implies that $\varepsilon(L; x) \geq 1$, as claimed.

The definition of Seshadri constant can be naturally extended to arbitrary subschemes [CEL01].

Definition B.2.10. Let $Y \subset X$ be a closed subscheme in a projective variety X, and let L be a nef line bundle on X. The *Seshadri constant of L at Y* is the non-negative real number

$$\varepsilon(X, L; Y) = \varepsilon(L; Y) = \sup\{\varepsilon \geq 0 \mid \mu^* L - \varepsilon E \text{ is nef}\},$$

where $\mu\colon \widetilde{X} \to X$ is the blow-up of X along Y with exceptional divisor E.

If Y is a reduced subscheme supported at q distinct points $x_1, \dots, x_q$, then

$$\varepsilon(L; Y) = \varepsilon(L; x_1, \dots, x_q)$$

is called the *multi-point Seshadri constant of L at $x_1, \dots, x_q$*.

Analogously to Proposition B.2.2, one can prove the following characterization.

Proposition B.2.11. *Let X be a projective variety of dimension $n \geq 1$ and let L be a nef line bundle.*

(1) If $x_1, \dots, x_q$ are q distinct points, then

$$\varepsilon(L; x_1, \dots, x_q) = \inf\left\{ \frac{(L \cdot C)}{\sum_{i=1}^{q} \operatorname{mult}_{x_i}(C)} \;\middle|\; \begin{array}{c} C \subset X \text{ is an irreducible curve} \\ \text{passing through some } x_i \end{array} \right\}.$$

(2) If $x_1, \dots, x_q$ are smooth points for X, then

$$\varepsilon(L; x_1, \dots, x_q) \leq \left(\frac{L^n}{q}\right)^{\frac{1}{n}}. \tag{B.2}$$

Moreover

$$\varepsilon(L; x_1, \dots, x_q) \leq \varepsilon(L; x_{j_1}, \dots, x_{j_p})$$

for any subset $\{x_{j_1}, \dots, x_{j_p}\} \subset \{x_1, \dots, x_q\}$.

(3) If $x \in X$ is a smooth point and $Z \subset X$ is a 0-dimensional closed subscheme which is smooth at $x \in Z$, then $\varepsilon(L; Z) \leq \varepsilon(L; x)$.

Proof. The blow-up $\mu\colon \widetilde{X} \to X$ of X along $Y = \{x_1, \dots, x_q\}$ can be seen as the composition

$$\mu = \mu_q \circ \cdots \circ \mu_1\colon \widetilde{X} = X_q \longrightarrow X_{q-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X,$$

where μ_j, for $1 \leq j \leq q$, is the blow-up of X_{j-1} centred at the strict transform $x^{(j)} \in X_{j-1}$ of x_j via μ_{j-1}. By a slight abuse of notation, for each $1 \leq p \leq q$, we denote the exceptional divisor $E^{(p)} \subset X_p$ of $\mu^{(p)} = \mu_p \circ \cdots \circ \mu_1$ by the sum $E^{(p)} = E_1 + \cdots + E_p$, where each E_j for $1 \leq j \leq p-1$ is (the strict transform of) the exceptional divisor of μ_j. Here $E = E^{(q)}$ is the exceptional divisor of μ. To study the nefness of $\mu^* L - \varepsilon E$, take an irreducible curve $\widetilde{C} \subset X_q$. If $\widetilde{C} \subset E$, then $(\mu^* L - \varepsilon E) \cdot \widetilde{C} > 0$ by Nakai-Moishezon-Kleiman criterion for mapping [Laz04a, Corollary 1.7.9], which can be invoked since $\mathcal{O}_E(-E)$ is $\mu_{|E}$-ample ([GW20, Proposition 13.96(1)] and [Laz04a, Example 1.7.7]) and $\widetilde{C}$ is contracted to a point.

Hence, we can suppose that $\mu(\widetilde{C}) = C \subset X$ is an irreducible curve. Setting $\mu_q(\widetilde{C}) = C_{q-1}$, then one has

$$(\mu^* L - \varepsilon E) \cdot \widetilde{C} = (\mu^{(q-1)})^*(L - \varepsilon E^{(q-1)}) \cdot C_{q-1} - \varepsilon \cdot \operatorname{mult}_{x^{(q)}}(C_{q-1})$$

by [Laz04a, Lemma 5.1.10]. As $\mu^{(q-1)}$ is an isomorphism around $x^{(q)}$, we see that

$$\operatorname{mult}_{x^{(q)}}(C_{q-1}) = \operatorname{mult}_{x_q}(C).$$

Proceeding inductively, we obtain

$$(\mu^* L - \varepsilon E) \cdot \widetilde{C} = (L \cdot C) - \varepsilon \sum_{i=1}^{q} \operatorname{mult}_{x_i}(C).$$

Clearly, if C does not contain any x_i, the above equation is greater than or equal to 0 as L is nef. Therefore, assuming that C passes through some x_i, we conclude that $(\mu^* L - \varepsilon E) \cdot \widetilde{C} \geq 0$ if and only if

$$\varepsilon \leq \frac{(L \cdot C)}{\sum_{i=1}^{q} \operatorname{mult}_{x_i}(C)},$$

giving (1).

For (2), take any subset $\{x_{j_1}, \ldots, x_{j_p}\} \subset Y$, and write ε_Y for the Seshadri constant $\varepsilon(L; x_1, \ldots, x_q)$. If $C \subset X$ is any irreducible curve passing through some x_{j_k}, in particular it satisfies $C \cap Y \neq \emptyset$. Consequently we get that

$$\varepsilon_Y \leq \frac{(L \cdot C)}{\sum_{i=1}^{q} \operatorname{mult}_{x_i}(C)} \leq \frac{(L \cdot C)}{\sum_{k=1}^{p} \operatorname{mult}_{x_{j_k}}(C)}.$$

Taking the inf over all irreducible curves passing through some x_{j_k}, we obtain the claim.

To prove (B.2), consider let $\mu\colon \widetilde{X} \longrightarrow X$ and $E = E_1 + \cdots + E_q$ be as above. Then E_i's are mutually disjoint with $E_i^n = (-1)^{n+1}$ for all i. Therefore, using the nefness of $\mu^* L - \varepsilon(L; x_1 \ldots, x_q)E$ (Lemma B.2.13(i)), we have

$$0 \leq \left(\mu^* L - \varepsilon(L; x_1 \ldots, x_q)E\right)^n = L^n - \varepsilon(L; x_1 \ldots, x_q)^n \cdot q,$$

thus proving (B.2).

In order to prove (3), let $\rho\colon X' \longrightarrow X$ be the blow-up of X at x with exceptional divisor F, and let $Z' \subset X'$ be the strict transform of $Z_1 = Z - \{x\}$. Let $\rho' : \overline{X} \longrightarrow X'$ be the blow-up of X' along Z' with exceptional divisor $F_{Z'}$. Then the composition $\pi = \rho' \circ \rho$ is the blow-up of X along Z with exceptional divisor $\overline{F} = F' + F_{Z'}$, where F' is the strict transform of F via ρ'. Let $\varepsilon \geq 0$ be such that $\pi^* L - \varepsilon \overline{F}$ is nef. To obtain the conclusion, it is enough to prove that $\varepsilon \leq \frac{L \cdot C}{\operatorname{mult}_x(C)}$ for every irreducible curve $C \subset X$ passing through x. Indeed, if this is true, by taking the infimum over all irreducible curves passing through x we get $\varepsilon \leq \varepsilon(L; x)$. By taking the sup over all $\varepsilon \geq 0$ such that $\pi^* L - \varepsilon \overline{F}$ is nef, we get $\varepsilon(L; Z) \leq \varepsilon(L; x)$.

Given an irreducible curve $C \subset X$ containing x, set $C' \subset X'$ and $\overline{C} \subset \overline{X}$ to be respectively the strict transform of C via ρ and the strict transform of C' via ρ'. In particular $\overline{C}$ is the strict transform of C through π, and $C' \not\subset F$ and $\overline{C} \not\subset \overline{F}$ for otherwise they would be mapped onto a point. As a consequence $F_{Z'} \cdot \overline{C} \geq 0$. Therefore, using also projection formula and [Laz04a, Lemma 5.1.10], we obtain:

$$0 \leq (\pi^* L - \varepsilon \overline{F}) \cdot \overline{C} = \rho'^* (\rho^* L - \varepsilon F) \cdot \overline{C} - \varepsilon F_{Z'} \cdot \overline{C} \leq (\rho^* L - \varepsilon F) \cdot C' = L \cdot C - \varepsilon \cdot \operatorname{mult}_x(C),$$

which gives the claim. □

Definition B.2.12. Given a line bundle L and q distinct smooth points $x_1, \dots, x_q$ on a projective variety X, an irreducible curve realizing the infimum in Proposition B.2.11(1) is called *multi-point Seshadri curve of L at $x_1, \dots, x_q$*. An irreducible curve C such that

$$\frac{(L \cdot C)}{\sum_{i=1}^q \operatorname{mult}_{x_i}(C)} < \left(\frac{L^{\dim X}}{q}\right)^{\frac{1}{\dim X}}$$

is said to be *submaximal for L at $x_1, \dots, x_q$*. If the inequality in (B.2) is strict, then L is said to be *submaximal at $x_1, \dots, x_q$*.

Lemma B.2.13. *Let X be a projective variety of dimension $n \geq 1$ and let $Y \subset X$ be a non-empty proper closed subscheme. Let $\mu\colon \widetilde{X} \to X$ be the blow-up of X along Y with exceptional divisor E, and let L be a nef line bundle on X. Let $\sigma \in \mathbf{R}$. Then:*

(i) $\mu^*L - \varepsilon(L; Y)E$ is nef but not ample.

(ii) If $\mu^*L - \sigma E$ is nef, then $\sigma \leq \varepsilon(L; Y)$.

(ii') If $\mu^*L - \sigma E$ is ample, then $\sigma < \varepsilon(L; Y)$.

(iii) Assuming $n \geq 2$ and $\dim Y = 0$, if $\mu^*L - \sigma E$ is nef, then $\sigma \geq 0$.

(iii') Assuming $n \geq 2$ and $\dim Y = 0$, if $\mu^*L - \sigma E$ is ample, then $\sigma > 0$.

(iv) Assuming $\dim Y = 0$ and $n \geq 2$, we have that $\mu^*L - \sigma E$ is nef if and only if $\sigma \in [0, \varepsilon(L; Y)]$. For $n = 1$, we have that $\mu^*L - \sigma E$ is nef if and only if $\sigma \in (-\infty, \varepsilon(L; Y)]$.

(iv') Assume $\dim Y = 0$. For $n = 1$, then $\mu^*L - \sigma E$ is ample if and only $\sigma \in (-\infty, \varepsilon(L; Y))$. For $n \geq 2$ and L ample, we have that $\mu^*L - \sigma E$ is ample if and only if $\sigma \in (0, \varepsilon(L; Y))$.

Proof. Write $\varepsilon = \varepsilon(L; Y)$ and $\widetilde{L}_\varepsilon = \mu^*L - \varepsilon E$.

Item (i) easily follows from the closedness of the nef cone and the opennes of the ample cone: by definition there exists a sequence of non-negative real numbers $\{\varepsilon_k\}_{k \in \mathbf{N}}$ converging to ε such that $\mu^*L - \varepsilon_k E$ is nef for all k. Therefore $\mu^*L - \varepsilon E = \lim_{k \to \infty}(\mu^*L - \varepsilon_k E)$ must be nef since $\operatorname{Nef}(\widetilde{X})$ is closed. On the other hand, $\mu^*L - \varepsilon E$ cannot be ample, otherwise we could find a sufficiently small real number $\delta > 0$ such that $\mu^*L - (\varepsilon + \delta)E$ is ample [Laz04a, Example 1.3.14], giving a contradiction.

Items (ii) and (ii') are clear by definition and by part (i).

Suppose for the moment that X is a curve. Since $\deg(\widetilde{L}_\varepsilon) \geq 0$ by (i), and $\deg(E) > 0$ as E is a non-trivial effective divisor, from

$$\deg(\mu^*L - \sigma E) = \deg(\widetilde{L}_\varepsilon) + (\varepsilon - \sigma)\deg(E)$$

we immediately get the case $n = 1$ in (iv) and (iv'), the second one by Nakai-Moishezon-Kleiman criterion for $\mathbf{R}$-divisors [Laz04a, Theorem 2.3.18 & Proposition 1.3.13].

Henceforth we assume $n \geq 2$ and $\dim Y = 0$. In this situation, the line bundle $\mathcal{O}_E(-E)$ is ample: it is the restriction of the μ-ample line bundle $\mathcal{O}_{\widetilde{X}}(-E)$ [GW20, Proposition 13.96(1)], hence it is $\mu_{|E}$-ample by [Laz04a, Example 1.7.7]. Since every subvariety in E is contracted by μ to a point, Nakai-Moishezon-Kleiman criterion for mapping [Laz04a, Corollary 1.7.9] says that $\mathcal{O}_E(-E)$ is ample on E, proving the claim.

If $\mu^*L - \sigma E$ is nef (resp. ample), then σ cannot be negative (resp. non-positive) since for any irreducible curve $C \subset E$ we would have

$$(\mu^*L - \sigma E) \cdot C = -\sigma E \cdot C < 0 \qquad \text{(resp. } (\mu^*L - \sigma E) \cdot C = -\sigma E \cdot C \leq 0) .$$

This proves (iii) (resp. (iii')).

Combining (ii) and (iii) (resp. (ii') and (iii')) we immediately obtain the "only if" part of (iv) (resp. (iv')). For (d), it remains to prove that $\mu^*L - \sigma E$ is nef if $\sigma \in [0, \varepsilon]$. To this end, fix such a σ and take an irreducible curve $\widetilde{C} \subset \widetilde{X}$. As $-E_{|E}$ is ample, if $\widetilde{C} \subset E$, we have $(\mu^*L - \sigma E) \cdot \widetilde{C} = -\sigma E \cdot \widetilde{C} \geq 0$. Suppose $\widetilde{C} \not\subset E$, and let $C = \mu(\widetilde{C}) \subset X$ be the image. If $\widetilde{C} \cap E = \emptyset$, we have $(\mu^*L - \sigma E) \cdot \widetilde{C} = L \cdot C > 0$. Hence we may assume $\widetilde{C} \cap E \neq \emptyset$. Using the nefness of $\widetilde{L}_\varepsilon$ given by (i), we have

$$(\mu^*L - \sigma E) \cdot \widetilde{C} = \widetilde{L}_\varepsilon \cdot \widetilde{C} + (\varepsilon - \sigma) E \cdot \widetilde{C} \geq 0.$$

This proves that $\mu^*L - \sigma E$ is nef. To conclude the proof, we need to prove that $\mu^*L - \sigma E$ is ample if $\sigma \in (0, \varepsilon)$ when L is ample. To this end, by Nakai-Moishezon-Kleiman criterion for **R**-divisors [Laz04a, Theorem 2.3.18 & Proposition 1.3.13], it is enough to show that $(\mu^*L - \sigma E)^k \cdot V' > 0$ for every irreducible subvariety $V' \subset \widetilde{X}$ of dimension $k \geq 1$. One easily checks that this holds both for $V' \subset E$ and for V' which is disjoint from E: in the first case one has $(\mu^*L - \sigma E)^k \cdot V' = \sigma^k (-E)^k \cdot V' > 0$ by the ampleness of $-E_{|E}$, in the other one, $V = \mu(V') \subset X$ is an irreducible subvariety of dimension k and the projection formula, together with the ampleness of L, implies $(\mu^*L - \sigma E)^k \cdot V' = L^k \cdot V > 0$. Hence we can assume $V' \cap E \neq \emptyset$ and that $V = \mu(V') \subset X$ is a k-dimensional subvariety. Then, observe that $(\mu^*L)^{k-j} \cdot E^j \cdot V' = 0$ for every $0 < j < k$. Indeed, as $j > 0$, $Z = (\mu^*L)^{k-j-1} \cdot E^j \cdot V'$ is a 1-cycle (see the proof of [Laz04a, Lemma 1.1.18]) which is supported on E. Therefore, writing $Z = \sum_{h=1}^t a_i Z_i$, it follows that $\mu_* Z_h = 0$ for all h, which forces $\mu_* Z = 0$. By projection formula and [Ful98, Proposition 2.5(a)] we deduce the claim:

$$(\mu^*L)^{k-j} \cdot E^j \cdot V' = \mu^*L \cdot Z = 0.$$

On the other hand, $(-E)^k \cdot V' < 0$. To see this, first observe that $E \cap V'$ is a $(k-1)$-cycle on E which is effective. Indeed, $V' \not\subset E$, so $E \cap V' \subset V'$ is a divisor which must be effective for otherwise the restriction map $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(E)) \to H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(E) \otimes \mathcal{O}_{V'})$ would be zero, saying that $V' \subset E$. Therefore, writing $(-E)^k \cdot V' = -(-E)^{k-1} \cdot (E \cap V')$, the assertion follows by the ampleness of $\mathcal{O}_E(-E)$. Putting all together, we obtain

$$(\mu^*L - \sigma E)^k \cdot V' = (L^k \cdot V) + \sigma^k (-E)^k \cdot V' > (L^d \cdot V) + \varepsilon^k (-E)^k \cdot V' = \widetilde{L}_\varepsilon^k \cdot V' \geq 0,$$

with the last inequality coming from (i) and Kleiman theorem [Laz04a, Theorem 1.4.9], as desired. □

Lemma B.2.14. *Let X be a projective variety of dimension $n \geq 1$ and let $Y \subset X$ be a non-empty proper closed subscheme. Let $\mu \colon \widetilde{X} \to X$ be the blow-up of X along Y with exceptional divisor E, and let L be an ample line bundle on X. Then* $\varepsilon(L; Y) = \sup\{\ell \in \mathbf{R}_+ \mid \mu^*L - \ell E \text{ is ample}\}$.

Proof. Write $\varepsilon = \varepsilon(L; Y)$, $\widetilde{L}_\varepsilon = \mu^*L - \varepsilon E$, and let $\varepsilon' = \sup\{\ell > 0 \mid \mu^*L - \ell E \text{ is ample}\}$. Observe that the set $\{\ell > 0 \mid \mu^*L - \ell E \text{ is ample}\}$ is non-empty since we can always find a positive integer $m \gg 0$ such that $m\mu^*L - E$ is ample by [Har77, Exercise II.7.14(b)]. Clearly $\varepsilon' \leq \varepsilon$: if $\mu^*L - \ell E$ is ample, then it is nef, hence $\ell \leq \varepsilon$. Taking the sup over all $\ell > 0$ such that $\mu^*L - \ell E$ is ample, we obtain the claimed inequality. To prove the other direction, consider again an $\ell > 0$ such that $\mu^*L - \ell E$ is ample. Then $v = [\mu^*L - \ell E]$ belongs to the ample cone $\widetilde{A} = \mathrm{Amp}(\widetilde{X})$ of $\widetilde{X}$. As $\widetilde{A}$ is open, v belongs to its interior. On the other hand, $w = [\mu^*L - \varepsilon E]$ belongs to the nef cone $\mathrm{Nef}(\widetilde{X})$, which is the closure of $\widetilde{A}$. Then

$$(1-t)v + tw = [\mu^*L - (\ell - tl + \varepsilon t)E]$$

is in the interior of $\widetilde{A}$ for every $t \in [0, 1)$ by [Roc97, Theorem 6.1]. In other words, $(1-t)v+tw$ is ample for every $0 \leq t < 1$. In particular we have

$$\ell - t\ell + \varepsilon t \leq \varepsilon' \quad \text{for all} \quad 0 \leq t < 1.$$

Taking the limit of the above inequality for $t \to 1$, we finally obtain $\varepsilon \leq \varepsilon'$, proving the assertion. □

The definition of Seshadri constant at a point has been generalized to vector bundles in [FM21]. Actually the setting can be even more general. However, for our purposes, it is enough to stick to the case of locally free sheaves.

Definition B.2.15. Let X be a projective variety and let $\mathcal{F}$ be a vector bundle on X. Let $\pi\colon \mathbf{P}(\mathcal{F}) \to X$ be the natural projection and let $\xi = \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ be the tautological line bundle on $\mathbf{P}(\mathcal{F})$. Let $x \in X$ be a point and let $C_{\mathcal{F},x}$ denote the set of irreducible curves in $\mathbf{P}(\mathcal{F})$ that meet $\mathbf{P}(\mathcal{F}(x))$ but are not contained in the fibre $\mathbf{P}(\mathcal{F}(x))$. The *Seshadri constant of $\mathcal{F}$ at x* is

$$\varepsilon(\mathcal{F}; x) := \inf_{C \in C_{\mathcal{F},x}} \left\{ \frac{\xi \cdot C}{\mathrm{mult}_x(\pi_* C)} \right\}.$$

Whenever the vector bundle is nef, i.e. the tautological line bundle is nef, the definition can be formulated analogously to the one of line bundles.

Remark B.2.16 ([FM21, Remark 3.10(a-c)]). Let X be a projective variety and let $\mathcal{F}$ be a nef vector bundle on X. Let $x \in$ be a point and let $\mu\colon \widetilde{X} \to X$ be the blow-up at x with exceptional divisor E. Then:

(a) $\varepsilon(\mathcal{F}; x) \geq 0$.

(b) $\varepsilon(\mathcal{F}; x) = \sup\left\{\varepsilon \geq 0 \mid \mu^*\mathcal{F}\langle -\varepsilon E\rangle \text{ is nef on } \widetilde{X}\right\}$.

Remark B.2.17. Analogously to Remark B.2.8, for a nef vector bundle $\mathcal{F}$ on a projective variety X, one has the following characterization [FM21, Proposition 6.9]:

$$\mathbf{B}_+(\mathcal{F}) = \{x \in X \mid \varepsilon(\mathcal{F}; x) = 0\}.$$

B.3 Projective normality of vector bundles

The notion of projective normality for line bundles is well known. We refer to [Laz04a, §1.8.D] for more details. We recall the definition for convenience.

Definition B.3.1. A line bundle L on a projective variety X is said k-normal for some $k \geq 1$ if the natural map

$$\mathrm{S}^k H^0(X, L) \to H^0(X, kL)$$

is surjective. We say L is *normally generated* if it is ample and k-normal for all $k \geq 1$.

An embedded projective variety $X \subset \mathbf{P}^N$ is said m-normal for some $m \geq 1$ if the restriction map

$$H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(m)) \to H^0(X, \mathcal{O}_X(m))$$

is surjective, or equivalently if $H^1(\mathbf{P}^N, \mathcal{I}_{X/\mathbf{P}^N}(m)) = 0$. A 1-normal variety is said *linearly normal.* The variety $X \subset \mathbf{P}^N$ is *projectively normal* if it is m-normal for all $m \geq 1$.

The following observation is very well-known.

Remark B.3.2. Let L be an ample line bundle on a projective variety X. Then the following are equivalent:

(a) L is normally generated.

(b) $H^0(X, L)^{\otimes k} \to H^0(X, kL)$ is surjective for all $k \geq 1$.

(c) $H^0(X, L) \otimes H^0(X, kL) \to H^0(X, (k+1)L)$ is surjective for all $k \geq 0$.

(d) L is very ample and embeds $X \hookrightarrow \mathbf{P}(H^0(X, L))$ as a projectively normal variety.

For the equivalence, let $k \geq 0$ and consider the following diagram:

$$\begin{array}{ccccc} H^0(X, L)^{\otimes (k+1)} & \xrightarrow{p_{k+1}} & H^0(X, L) \otimes H^0(X, kL) & \xrightarrow{\mu_{1,k}} & H^0(X, (k+1)L) \\ & {\scriptstyle q_{k+1}} \searrow & & \nearrow {\scriptstyle s_{k+1}} & \\ & & \mathrm{S}^{k+1} H^0(X, L). & & \end{array}$$

with q_{k+1} being always surjective. If s_{k+1} is surjective, then so are $\mu_{1,k}$ and the composition $\mu_{1,k} \circ p_{k+1}$. This means that (a) implies (b) and (c). Obviously, if (b) holds, then both (a) and (c) do. Assuming (c), in order to prove (b) we can suppose by induction that $H^0(X, L)^{\otimes h} \to H^0(X, hL)$ is surjective for all $1 \leq h \leq k$. In particular this says that

$$p_{k+1} \colon H^0(X, L)^{\otimes (k+1)} = H^0(X, L) \otimes H^0(X, L)^{\otimes k} \to H^0(X, L) \otimes H^0(X, kL)$$

is surjective. Thus the surjectivity of $\mu_{1,k}$ yields that the composition is onto as desired, proving (b). Now, if (a) holds, it is shown in [Mum70, §1] that L is very ample. Letting $X \subset \mathbf{P}(H^0(X, L)) = \mathbf{P}^N$ be the embedding, we have a short exact sequence

$$0 \longrightarrow \mathcal{I}_{X/\mathbf{P}^N}(k) \longrightarrow \mathcal{O}_{\mathbf{P}^N}(k) \longrightarrow L^{\otimes k} \longrightarrow 0$$

for every $k \geq 1$. Since $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(k)) \cong \mathrm{S}^k H^0(X, L)$, (d) immediately follows. Conversely, if we assume (d), and $X \subset \mathbf{P}^N$ is the embedding determined by $|L|$, the above exact sequence shows that the map

$$\mathrm{S}^k H^0(X, L) \cong H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(k)) \to H^0(X, kL)$$

is surjective for every $k \geq 1$, giving (a).

Therefore a normally generated line bundle L on a projective variety X induces an embedding $\varphi_L\colon X \hookrightarrow \mathbf{P}^N$ as a projectively normal scheme. We identify normal generation (of L) and projective normality (of $\varphi_L(X) \subset \mathbf{P}^N$). Moreover in this case the coordinate ring R_X and the section ring $R(L)$ coincide.

We now recollect some standard facts and some technical results on projective normality of vector bundles whose relevant definitions are given in Definition 5.0.5.

Remark B.3.3. Given a vector bundle $\mathcal{E}$ of rank r on a projective variety X, by [Har77, Exercises III.8.1-III.8.4] there are isomorphisms

$$H^i(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(k)) \cong H^i(X, \mathrm{S}^k\mathcal{E})$$

for every $i, k \geq 0$. In particular, by Remark B.3.2, $\mathcal{E}$ is k-normal for some $k \geq 1$ (resp. projectively normal) if and only if

$$\mathrm{S}^k H^0(X, \mathcal{E}) \to H^0(X, \mathrm{S}^k\mathcal{E}) \text{ or, equivalently, } H^0(X, \mathcal{E})^{\otimes k} \to H^0(X, \mathrm{S}^k\mathcal{E})$$

is surjective (resp. is surjective for all $k \geq 1$ and $\mathcal{E}$ is ample).

A stronger condition to get k-normality (resp. projective normality) of $\mathcal{E}$ is requiring that multiplication the map

$$\mu^k_{\mathcal{E}}\colon H^0(X, \mathcal{E})^{\otimes k} \to H^0(X, \mathcal{E}^{\otimes k})$$

is surjective (resp. is surjective for all $k \geq 1$ and $\mathcal{E}$ is ample). Indeed, one has the commutative diagram

$$\begin{array}{ccc} & H^0(X,\mathcal{E})^{\otimes k} & \\ {}^{\mu^k_{\mathcal{E}}}\swarrow & & \searrow \\ H^0(X, \mathcal{E}^{\otimes k}) & \longrightarrow & H^0(X, \mathrm{S}^k\mathcal{E}) \end{array}$$

for every $k \geq 1$. As the h-symmetric product $\mathrm{S}^h\mathcal{E}$ is a direct summand of the h-tensor power $\mathcal{E}^{\otimes h}$ for all $h \geq 0$, the horizontal map is always surjective. Thus the conclusion immediately follows.

The above remarks will be used with no further mention.

Remark B.3.4. Let $\mathcal{E} \cong \mathcal{F}^{\oplus s}$ for a vector bundle $\mathcal{F}$ on a projective variety X and for some $s \geq 1$. Then $\mathcal{E}$ is strongly 2-normal if and only if $\mathcal{F}$ is strongly 2-normal.

This follows by observing that

$$\mu_{\mathcal{E}}\colon H^0(X,\mathcal{E}) \otimes H^0(X,\mathcal{E}) \cong \left(H^0(X,\mathcal{F}) \otimes H^0(X,\mathcal{F})\right)^{\oplus s^2} \to H^0(X, \mathcal{F} \otimes \mathcal{F})^{\oplus s^2} \cong H^0(X, \mathcal{E} \otimes \mathcal{E})$$

is surjective if and only if each $\mu_{\mathcal{F}}\colon H^0(X,\mathcal{F}) \otimes H^0(X,\mathcal{F}) \to H^0(X, \mathcal{F} \otimes \mathcal{F})$ is surjective.

The following result is well-know, see for instance [Mir94, Fact 1.7] or [Tri16, p. 1014]. For more details we refer to [BE75; Eis95; Wey03].

Lemma B.3.5. *Let* $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ *be an exact sequence of vector bundles on a variety* X*. Then we have the following exact sequences of vector bundles for every* $k \geq 1$*:*

$$0 \to \Lambda^k\mathcal{E} \to \Lambda^k\mathcal{F} \to \Lambda^{k-1}\mathcal{F} \otimes \mathcal{G} \to \cdots \to \mathcal{F} \otimes \mathrm{S}^{k-1}\mathcal{G} \to \mathrm{S}^k\mathcal{G} \to 0, \tag{B.3}$$

$$0 \to \mathrm{S}^k\mathcal{E} \to \mathrm{S}^{k-1}\mathcal{E} \otimes \mathcal{F} \to \cdots \to \mathcal{E} \otimes \Lambda^{k-1}\mathcal{F} \to \Lambda^k\mathcal{F} \to \Lambda^k\mathcal{G} \to 0, \tag{B.4}$$

This lemma shows that we can obtain the k-normality through some cohomology vanishings.

Example B.3.6. Let X be a smooth regular projective variety and let $\mathcal{E}$ be a globally generated vector bundle on X such that $H^1(X, \mathcal{E}) = 0$. Then $\mathcal{E}$ is 2-normal if $H^2(X, \Lambda^2 M_{\mathcal{E}}) = 0$. The converse holds if $H^2(X, \mathcal{O}_X) = 0$.

To see this, consider (B.3) with $k = 2$ for the syzygy exact sequence of $\mathcal{E}$ and then split it in the two exact sequences

$$0 \to K \to H^0(X, \mathcal{E}) \otimes \mathcal{E} \to \mathrm{S}^2\mathcal{E} \to 0 \text{ and } 0 \to \Lambda^2 M_{\mathcal{E}} \to \Lambda^2 H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \to K \to 0.$$

Taking the cohomology of the first one, we immediately see that $H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{E}) \to H^0(X, \mathrm{S}^2\mathcal{E})$ is surjective if and only if $H^1(X, K) = 0$. The cohomology of the second one yields the exact sequence

$$0 = \Lambda^2 H^0(X, \mathcal{E}) \otimes H^1(X, \mathcal{O}_X) \to H^1(X, K) \to H^2(X, \Lambda^2 M_{\mathcal{E}}) \to \Lambda^2 H^0(X, \mathcal{E}) \otimes H^2(X, \mathcal{O}_X).$$

The claim is now obvious.

Notation B.3.7. A coherent sheaf $\mathcal{F}$ on a variety X is *special* if $H^1(X, \mathcal{F}) \neq 0$. We say $\mathcal{F}$ is *non-special* if $H^1(X, \mathcal{F}) = 0$.

We observe that projective normality is an open property in proper flat families of non-special vector bundles. This will allow us to consider open subsets of projectively normal Ulrich bundles in the moduli spaces of vector bundles.

Lemma B.3.8. *Let $f\colon Y \to S$ be a proper morphism over a Noetherian scheme S and let $\mathcal{F}$ be a coherent sheaf on Y that is flat over S. Suppose there is $0 \in S$ such that $\mathcal{F}_0 = \mathcal{F}_{|Y_0}$ is locally free, semistable, non-special, ample, globally generated and 2-normal (resp. strongly 2-normal). Then there exists an open neighborhood $U \subset S$ of 0 such that $\mathcal{F}_s = \mathcal{F}_{|Y_s}$ satisfies all the above properties for every $s \in U$.*

Proof. The assertion is local on the target, so we assume S is affine. Thanks to [HL10, Lemma 2.1.8 & Proposition 2.3.1] and to [GW23, Corollary 23.144], up to shrink S, we can suppose that $\mathcal{F}_s$ is locally free, semistable and non-special for every $s \in S$. In particular, $\mathcal{F}$ is locally free on Y by [HL10, Lemma 2.1.7]. Shrinking again, we can assume also $f_*\mathcal{F}$ is locally free as well [GW23, Corollary 23.144]. Up to replace S with a suitable open neighborhood of 0, by [Laz04b, Proposition 6.1.9] we can suppose $\mathcal{F}_s$ is ample for every $s \in S$. Note that the function $s \mapsto h^1(Y_s, \mathcal{F}_s) = 0$ is constant.

Letting $\rho\colon f^*f_*\mathcal{F} \to \mathcal{F}$ be the natural morphism, we can see that $\mathcal{F}_s$ is not globally generated if and only if $\rho_s = \rho_{|Y_s}\colon H^0(Y, \mathcal{F}) \otimes \mathcal{O}_{Y_s} \to \mathcal{F}_s$ is not surjective. To prove this, observe that there is a factorization

$$\rho_s = e_s \circ (r_s \otimes \mathrm{id}_{\mathcal{O}_{Y_s}})$$

where $e_s\colon H^0(Y_s, \mathcal{F}_s) \otimes \mathcal{O}_{Y_s} \to \mathcal{F}_s$ is the evaluation map and $r_s\colon H^0(Y, \mathcal{F}) \to H^0(Y_s, \mathcal{F}_s)$ is the restriction map. If we prove that r_s is surjective, then the claim will be clear. In order to do this, observe that, since $h^1(Y_s, \mathcal{F}_s) = 0$ for all s, [GW23, Corollary 23.144] provide the isomorphism

$$(f_*\mathcal{F})_{|\operatorname{Spec}(\mathbf{C}(s))} \cong (f_{|Y_s})_*(\mathcal{F}_s)$$

for every $s \in S$. Since S is affine, thanks to [Har77, Proposition III.8.5 & Proposition II.5.2(e)] this yields

$$\begin{aligned} H^0(\widetilde{Y, \mathcal{F}) \otimes} \mathbf{C}(s) &\cong H^0(S, \widetilde{f_* \mathcal{F}) \otimes} \mathbf{C}(s) \\ &\cong (H^0(\widetilde{S, f_*}\mathcal{F}))_{|\operatorname{Spec}(\mathbf{C}(s))} \\ &\cong H^0(\operatorname{Spec}(\widetilde{\mathbf{C}(s)), (f_{|Y_s})_*}(\mathcal{F}_s)) \\ &\cong H^0(\widetilde{Y_s, \mathcal{F}_s}). \end{aligned}$$

In virtue of the equivalence of categories between $\mathcal{O}_S$-modules over S and $\Gamma(S, \mathcal{O}_S)$-modules, we thus obtain the isomorphism

$$H^0(Y, \mathcal{F}) \otimes \mathbf{C}(s) \cong H^0(Y_s, \mathcal{F}_s).$$

Since S is affine, the locally free sheaf $f_*\mathcal{F}$ is globally generated. Therefore the restriction of the surjective evaluation map $H^0(S, f_*\mathcal{F}) \otimes \mathcal{O}_S \to f_*\mathcal{F}$ to $\operatorname{Spec}(\mathbf{C}(s))$ yields the surjective map

$$\begin{aligned} r_s \colon H^0(Y, \mathcal{F}) \cong H^0(S, f_*\mathcal{F}) \to H^0(\operatorname{Spec}(\mathbf{C}(s)), (f_*\mathcal{F})_{|\operatorname{Spec}(\mathbf{C}(s))}) \\ \cong H^0(S, (f_{|Y_s})_*(\mathcal{F}_s)) \cong H^0(Y_s, \mathcal{F}_s), \end{aligned}$$

as desired.

As $\mathcal{F}_0$ is generated by global sections, the coherence of $\operatorname{coker}(\rho)$ tells that there is an affine open neighborhood of 0 such that ρ is surjective on every fibre Y_s. Hence, up to shrink S, we suppose that $\mathcal{F}_s$ is globally generated for all s, so that ρ is globally surjective.

Now, $\mathbf{P}(\mathcal{F}_s)$ is a fibre of the composite proper morphism $\mathbf{P}(\mathcal{F}) \to Y \to S$ and $\mathcal{L} = \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ is a line bundle on $\mathbf{P}(\mathcal{F})$ such that $\mathcal{L}_s = \mathcal{O}_{\mathbf{P}(\mathcal{F}_s)}(1)$ for every $s \in S$. The surjectivity of the multiplication map

$$H^0(\mathbf{P}(\mathcal{F}_0), \mathcal{L}_0) \otimes H^0(\mathbf{P}(\mathcal{F}_0), \mathcal{L}_0) \to H^0(\mathbf{P}(\mathcal{F}_0), \mathcal{L}_0 \otimes \mathcal{L}_0)$$

is well-known to be an open condition by semicontinuity (see [LM85, Proof of Lemma 1.3, lines 7-8]), thus the conclusion in case of the 2-normality of $\mathcal{F}_0$ follows.

As a recap, we have that $\mathcal{F}_s$ is locally free, semistable, non-special, ample and globally generated for all $s \in S$. Assuming that $\mathcal{F}_0$ is strongly 2-normal, we need to prove that there is a neighborhood $U \subset S$ of 0 such that $\mathcal{F}_s$ is strongly 2-normal for all $s \in U$. To do this, consider the morphism $\mu \colon H^0(Y, \mathcal{F}) \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$. We claim that μ is surjective on Y. Indeed, as $f_*\mathcal{F}$ is globally generated, then so is $f^* f_* \mathcal{F}$. Thanks to the surjectivity of ρ, we deduce that $\mathcal{F}$ is globally generated as well. The assertion follows by tensoring the syzygy exact sequence of $\mathcal{F}$ through by $\mathcal{F}$ itself. In conclusion, we have the exact sequence

$$0 \to \mathcal{K} = M_{\mathcal{F}} \otimes \mathcal{F} \to H^0(Y, \mathcal{F}) \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F} \to 0.$$

Note also that $M_{\mathcal{F}}$, hence $\mathcal{K}$, is flat over S by [AM69, Exercise 2.25]. Hence the above sequence restricts to the short exact sequence

$$0 \to \mathcal{K}_s \to H^0(Y, \mathcal{F}) \otimes \mathcal{F}_s \xrightarrow{\mu_s = \mu_{|Y_s}} \mathcal{F}_s \otimes \mathcal{F}_s \to 0$$

for each $s \in S$. Taking the cohomology, we obtain the following commutative diagram

$$\begin{array}{ccccccc} H^0(Y,\mathcal{F}) \otimes H^0(Y_s,\mathcal{F}_s) & \xrightarrow{\mu_s(Y_s)} & H^0(Y_s,\mathcal{F}_s \otimes \mathcal{F}_s) & \longrightarrow & H^1(Y_s,\mathcal{K}_s) & \longrightarrow & 0 \\ & {\scriptstyle r_s\otimes \mathrm{id}} \searrow & \nearrow {\scriptstyle \mu_{\mathcal{F}_s}} & & & & \\ & H^0(Y_s,\mathcal{F}_s) \otimes H^0(Y_s,\mathcal{F}_s) & & & & & \end{array}$$

where the row is exact. Since r_s is surjective, we deduce that $\mu_s(Y_s)$ is surjective if and only if $\mu_{\mathcal{F}_s}$ is surjective. Now, $\mathcal{F}_0$ is strongly 2-normal, and this forces $\mu_{\mathcal{F}_0}$, hence $\mu_0(Y_0)$, to be onto. The exactness of the cohomology sequence yields $H^1(Y_0,\mathcal{K}_0) = 0$. As $\mathcal{K}$ is flat over S, the semicontinuity theorem applied to $s \mapsto h^1(Y_s,\mathcal{K}_s)$ provides an open neighborhood $U \subset S$ of 0 where $h^1(Y_s,\mathcal{K}_s) = 0$ for all $s \in U$. We obtain the surjectivity of $\mu_s(Y_s)$, thus of $\mu_{\mathcal{F}_s}$, for all $s \in U$ as required. □

It's well known that a normal projective variety $X \subset \mathbf{P}^N$ is projectively normal if and only if R_X is integrally closed (see for instance [Har77, Exercise II.5.14]). A related property on R_X is being *Cohen-Macaulay.*

Definition B.3.9. An embedded projective variety $X \subset \mathbf{P}^N$ is said *arithmetically Cohen-Macaulay* (*aCM* for short) if the homogeneous coordinate ring R_X is Cohen-Macaulay, i.e. $\dim R_X = \operatorname{depth}(R_X)$.

As is well-known, aCM property is stronger than projective normality.

Remark B.3.10. A projective variety $X \subset \mathbf{P}^N$ of positive dimension is aCM if and only if $H^1(\mathbf{P}^N, \mathcal{I}_{X/\mathbf{P}^N}(k)) = 0$ for all $k \geq 0$ and $\mathcal{O}_X$ is an aCM sheaf with respect to $\mathcal{O}_X(1)$ if and only if $H^i(\mathbf{P}^N, \mathcal{I}_{X/\mathbf{P}^N}(k)) = 0$ for $1 \leq i \leq \dim X$ and $k \in \mathbf{Z}$ [CMP21, Proposition 2.1.9]. In particular, an irreducible projective curve $C \subset \mathbf{P}^N$ is aCM if and only if it is projectively normal.

Let's see how k-normality and aCM property behave with respect to hyperplane sections.

Lemma B.3.11. *Let $X \subset \mathbf{P}^N$ be a linearly normal projective variety of dimension $n \geq 2$ and let $Y = X \cap \mathbf{P}^{N-1}$ be a linearly normal irreducible hyperplane section. Then the following holds:*

(i) For $k > 0$, if the embedding $Y \subset \mathbf{P}^{N-1}$ is h-normal for $2 \leq h \leq k$, then so is the embedding $X \subset \mathbf{P}^N$. Conversely, $Y \subset \mathbf{P}^{N-1}$ is k-normal for $k \geq 0$ if $X \subset \mathbf{P}^N$ is k-normal and $H^1(X, \mathcal{O}_X(k-1)) = 0$.

(ii) If X is aCM, then so is Y. The converse holds if X is locally Cohen-Macaulay.

Proof. Let's consider (i). For every $k \geq 0$ we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(k-1)) & \longrightarrow & H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(k)) & \longrightarrow & H^0(\mathbf{P}^{N-1}, \mathcal{O}_{\mathbf{P}^{N-1}}(k)) & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle r_{k-1}} & & \downarrow {\scriptstyle r_k} & & \downarrow {\scriptstyle r'_k} & & \\ 0 & \longrightarrow & H^0(X, \mathcal{O}_X(k-1)) & \longrightarrow & H^0(X, \mathcal{O}_X(k)) & \xrightarrow{\rho_k} & H^0(Y, \mathcal{O}_Y(k)) & \longrightarrow & H^1(X, \mathcal{O}_X(k-1)). \end{array}$$

For the first part, we proceed by induction on h, with $2 \leq h \leq k$. The base case $h = 2$ is obtained immediately from the Snake lemma since r_1 is surjective, by the 1-normality of X,

as well as r'_2 by hypothesis. The inductive step with $2 < h \leq k$ follows again by the Snake lemma, the inductive hypothesis and the surjectivity of r'_h.
For the converse, our assumption implies the surjectivity of the composite map $\rho_k \circ r_k$. The commutativity of the right square tells that r'_k is onto as desired.

To prove the first part of (ii), consider the short exact sequence

$$0 \to \mathcal{O}_X(j-1) \to \mathcal{O}_X(j) \to \mathcal{O}_Y(j) \to 0$$

for $j \in \mathbf{Z}$. Remark B.3.10 and (i) imply that $Y \subset \mathbf{P}^{N-1}$ is projectively normal. Moreover, it is clear from the cohomology of the above sequence that $H^i(Y, \mathcal{O}_Y(j)) = 0$ for all $j \in \mathbf{Z}$ and $0 < i < n-1$. Therefore $Y \subset \mathbf{P}^{N-1}$ is aCM by Remark B.3.10. For the converse, see [Mig98, Theorem 1.3.3]. □

Remark B.3.12. Let X be a regular smooth projective variety of dimension $n \geq 1$ and let $\mathcal{E}$ be a very ample vector bundle of rank $r \geq 2$ on X. Then $\mathbf{P}(\mathcal{E})$ is aCM as soon as

$$(3-n)s_n(\mathcal{E}^*) \geq 3 + (K_X + \det(\mathcal{E})) \cdot s_{n-1}(\mathcal{E}^*)$$

where $s_k(\mathcal{E}^*)$ is the k-th Segre class of $\mathcal{E}^*$.

To see this, consider a smooth sectional curve $C \subset \mathbf{P}(\mathcal{E}) \subset \mathbf{P}(H^0(X, \mathcal{E})) = \mathbf{P}^M$ for the embedding defined by $|\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|$ and let $g = g(C)$ be its genus. Letting ξ be the class of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$, we know that

$$\deg C = \deg \mathbf{P}(E) = \xi^{n+r-1} = s_n(\mathcal{E}^*).$$

Observe that this is linearly normal since $H^1(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}) = H^1(X, \mathcal{O}_X) = 0$. By Lemma B.3.11 we can reduce to study the projective normality of $C \subset \mathbf{P}^{M+2-n-r}$. A classical result states that $C \subset \mathbf{P}^{M+2-n-r}$ is projectively normal if $\deg C \geq 2g+1$ (see, e.g., [Mum76, §2, Corollary to Theorem 6]). Using the adjuction formula and recalling that $K_{\mathbf{P}(\mathcal{E})} = \pi^*(K_X + \det(\mathcal{E})) - r\xi$, we find that

$$\begin{aligned} g &= 1 + \frac{1}{2}\left(K_{\mathbf{P}(\mathcal{E})} \cdot \xi^{n+r-2} + (n+r-2)\xi^{n+r-1}\right) \\ &= 1 + \frac{1}{2}\left(\pi^*(K_X + \det(\mathcal{E})) \cdot \xi^{n+r-2} + (n-2)\xi^{n+r-1}\right) \\ &= 1 + \frac{1}{2}\left((K_X + \det(\mathcal{E})) \cdot s_{n-1}(\mathcal{E}^*) + (n-2)s_n(\mathcal{E}^*)\right). \end{aligned}$$

The conclusion follows by putting all together.

The last useful result, which is just a rephrasing of [AR02, Proposition 1.2], will be important for Proposition 5.0.8.

Lemma B.3.13. *Let L be a very ample line bundle on a smooth projective variety Y of dimension $n \geq 1$ and let $Y \subset \mathbf{P}^N$ be the embedding determined by $|L|$. If L is 2-normal and*

$$H^0(Y, K_Y + (n-2)L) = H^1(Y, L) = H^2(Y, \mathcal{O}_Y) = 0,$$

then:

(i) $\mathcal{I}_{Y/\mathbf{P}^N}$ is 3-regular and $I_{Y/\mathbf{P}^N}$ is generated in degree less than or equal to 3,

(ii) L is projectively normal.

Proof. Our assumption and Kodaira vanishing immediately yield $H^{i+1}(Y, L^{\otimes(1-i)}) = 0$ for all $0 \leq i \leq \dim Y - 1$. As $Y \subset \mathbf{P}^N$ is linearly normal as embedded through a complete linear system, the statement follows by [AR02, Proposition 1.2] and Remark B.3.2. □

B.4 Chern classes computations

We calculate the first three Chern classes of $\mathcal{E}\otimes\mathcal{E}$, $S^2\mathcal{E}$, $\Lambda^2\mathcal{E}$ and the first two Chern classes of $S^3\mathcal{E}$ for a vector bundle $\mathcal{E}$.

Lemma B.4.1. *Let X be a smooth variety and let $\mathcal{E}$ be a vector bundle of rank r. Then:*

(i) $c_1(\mathcal{E}\otimes\mathcal{E}) = 2rc_1(\mathcal{E})$.

(ii) $c_2(\mathcal{E}\otimes\mathcal{E}) = (2r^2 - r - 1)c_1(\mathcal{E})^2 + 2rc_2(\mathcal{E})$.

(iii) $c_3(\mathcal{E}\otimes\mathcal{E}) = \frac{2}{3}(2r^3 - 3r^2 - 2r + 3)c_1(\mathcal{E})^3 + (4r^2 - 2r - 4)c_1(\mathcal{E})c_2(\mathcal{E}) + 2rc_3(\mathcal{E})$.

Proof. Item (i) is well known, see for instance [EH16, Proposition 5.18]. Recalling that the Chern character

$$\mathrm{ch} = \mathrm{rk} + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$

satisfies $\mathrm{ch}(\mathcal{E})^2 = \mathrm{ch}(\mathcal{E}\otimes\mathcal{E})$ [EH16, §5.5.2], to prove (ii) and (iii) it suffices to equate terms up to degree 2 and 3 respectively in the previous equality. More precisely, for (ii), taking the terms up to degree 2 on the left-hand-side of

$$\left(r + c_1(\mathcal{E}) + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E}))\right)^2 = r^2 + c_1(\mathcal{E}\otimes\mathcal{E})^2 + \frac{1}{2}(c_1(\mathcal{E}\otimes\mathcal{E})^2 - 2c_2(\mathcal{E}\otimes\mathcal{E}))$$

and using (ii), we obtain $2c_1(\mathcal{E})^2 + 2r(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) = 4r^2c_1(\mathcal{E})^2 - 2c_2(\mathcal{E}\otimes\mathcal{E})$. Then (ii) follows. For (iii), the proof is completely analogous: extract the terms of degree 3 from

$$\left(r + c_1(\mathcal{E}) + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) + \frac{1}{6}(c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) + 3c_3(\mathcal{E}))\right)^2,$$

equate the result with the terms of degree 3 in

$$r^2 + c_1(\mathcal{E}\otimes\mathcal{E})^2 + \frac{1}{2}(c_1(\mathcal{E}\otimes\mathcal{E})^2 - 2c_2(\mathcal{E}\otimes\mathcal{E})) + \frac{1}{6}(c_1(\mathcal{E}\otimes\mathcal{E})^3 - 3c_1(\mathcal{E}\otimes\mathcal{E})c_2(\mathcal{E}\otimes\mathcal{E}) + 3c_3(\mathcal{E}\otimes\mathcal{E})),$$

and use (i) and (ii) on the resulting equation

$$\begin{aligned} c_1(\mathcal{E})^3 - 2c_1(\mathcal{E})c_2(\mathcal{E}) + \frac{r}{3}c_1(\mathcal{E})^3 - rc_1(\mathcal{E})c_2(\mathcal{E}) + rc_3(\mathcal{E}) \\ = \frac{1}{6}(c_1(\mathcal{E}\otimes\mathcal{E})^3 - 3c_1(\mathcal{E}\otimes\mathcal{E})c_2(\mathcal{E}\otimes\mathcal{E}) + 3c_3(\mathcal{E}\otimes\mathcal{E}))\end{aligned}$$

to expand as much as possible. In the end, we obtain the desired expression for $c_3(\mathcal{E}\otimes\mathcal{E})$. □

Lemma B.4.2. *Let X be a smooth variety and let $\mathcal{E}$ be a vector bundle of rank r. Then:*

(i) $c_1(\Lambda^2\mathcal{E}) = (r-1)c_1(\mathcal{E})$.

(ii) $c_2(\Lambda^2\mathcal{E}) = \binom{r-1}{2}c_1(\mathcal{E})^2 + (r-2)c_2(\mathcal{E})$.

(iii) $c_3(\Lambda^2\mathcal{E}) = \binom{r-1}{3}c_1(\mathcal{E})^3 + (r-2)^2c_1(\mathcal{E})c_2(\mathcal{E}) + (r-4)c_3(\mathcal{E})$.

Proof. We prove the above formulae by induction on $r \geq 1$ by using the splitting principle. Since they are clearly true for $r = 1$, we suppose $r \geq 2$ and that $\mathcal{E}$ decomposes as $\mathcal{E} = L \oplus \mathcal{F}$ with L being a line bundle and $\mathcal{F}$ being a direct sum of $(r-1)$ line bundles. Then, in virtue of $\Lambda^2\mathcal{E} \cong \Lambda^2\mathcal{F} \oplus (L \otimes \mathcal{F})$, we have

$$\begin{aligned}
c_1(\mathcal{E}) &= c_1(L) + c_1(\mathcal{F}), & c_1(\Lambda^2\mathcal{E}) &= c_1(\Lambda^2\mathcal{F}) + c_1(L \otimes \mathcal{F}),\\
c_2(\mathcal{E}) &= c_1(L)c_1(\mathcal{F}) + c_2(\mathcal{F}), & c_2(\Lambda^2\mathcal{E}) &= c_2(\Lambda^2\mathcal{F}) + c_1(L \otimes \mathcal{F})c_1(\Lambda^2\mathcal{F}) + c_2(L \otimes \mathcal{F}),\\
c_3(\mathcal{E}) &= c_1(L)c_2(\mathcal{F}) + c_3(\mathcal{F}), & c_3(\Lambda^2\mathcal{E}) &= c_3(\Lambda^2\mathcal{F}) + c_2(\Lambda^2\mathcal{F})c_1(L \otimes \mathcal{F})\\
& & &\quad + c_1(\Lambda^2\mathcal{F})c_2(L \otimes \mathcal{F}) + c_3(L \otimes \mathcal{F}),
\end{aligned}$$

with $c_k(L \otimes \mathcal{F}) = \sum_{i=0}^{k} \binom{r-1-k+i}{i} c_1(L)^i c_{k-i}(\mathcal{F})$ (see [EH16, Proposition 5.17]). Using the inductive hypothesis on $\mathcal{F}$ we immediately get the desired formulae. For (i), we immediately see

$$c_1(\Lambda^2\mathcal{E}) = (r-2)c_1(\mathcal{F}) + (r-1)c_1(L) + c_1(\mathcal{F}) = (r-1)(c_1(L) + c_1(\mathcal{F})) = (r-1)c_1(\mathcal{E})$$

as claimed. The other items are proved in the exact same way. Indeed, we have

$$\begin{aligned}
c_2(\Lambda^2\mathcal{E}) = (r-3)c_2(\mathcal{F}) &+ \binom{r-2}{2}c_1(\mathcal{F})^2 + (r-2)c_1(\mathcal{F})^2 + (r-2)(r-1)c_1(L)c_1(\mathcal{F})\\
&+ c_2(\mathcal{F}) + (r-2)c_1(L)c_1(\mathcal{F}) + \binom{r-1}{2}c_1(L)^2\\
&= \frac{(r-1)(r-2)}{2}c_1(\mathcal{F})^2 + 2\binom{r-1}{2}c_1(L)c_1(\mathcal{F}) + \binom{r-1}{2}c_1(L)^2\\
&\quad + (r-2)c_1(L)c_1(\mathcal{F}) + (r-2)c_2(\mathcal{F})\\
&= \binom{r-1}{2}\left(c_1(\mathcal{F})^2 + 2c_1(L)c_1(\mathcal{F}) + c_1(L)^2\right) + (r-2)\left(c_2(\mathcal{F}) + c_1(L)c_1(\mathcal{F})\right)\\
&= \binom{r-1}{2}c_1(\mathcal{E})^2 + (r-2)c_2(\mathcal{E})
\end{aligned}$$

which is (ii). Finally, expanding the expression for $c_3(\Lambda^2\mathcal{E})$ through by the inductive hypothesis and (i-ii), and factoring the result we get

$$\begin{aligned}
c_3(\Lambda^2\mathcal{E}) = &\left(\frac{11}{6}r - 1 - r^2 + \frac{r^3}{6}\right)c_1(\mathcal{F})^3 + \left(1 + \frac{3}{2}r - 2r^2 + \frac{1}{2}r^3\right)c_1(L)c_1(\mathcal{F})^2\\
&+ \left(1 + \frac{3}{2}r - 2r^2 + \frac{1}{2}r^3\right)c_1(L)^2c_1(\mathcal{F}) + \left(\frac{11}{6}r - 1 - r^2 + \frac{r^3}{6}\right)c_1(L)^3\\
&+ \left(r^2 - 3r\right)c_1(L)c_2(\mathcal{F}) + \left(4 - 4r + r^2\right)c_1(\mathcal{F})c_2(\mathcal{F}) + (r-4)\,c_3(\mathcal{F}).
\end{aligned}$$

On the other hand, the expression (iii) can be written as

$$\begin{aligned}
&\binom{r-1}{3}c_1(\mathcal{E})^3 + (r-2)^2 c_1(\mathcal{E})c_2(\mathcal{E}) + (r-4)c_3(\mathcal{E}) \\
&= \frac{(r-1)(r-2)(r-3)}{6}\left(c_1(L)^3 + 3c_1(L)^2 c_1(\mathcal{F}) + 3c_1(L)c_1(\mathcal{F})^2 + c_1(\mathcal{F})^3\right) \\
&\quad + (r-2)^2\left(c_1(L)^2 c_1(\mathcal{F}) + c_1(L)c_2(\mathcal{F}) + c_1(L)c_1(\mathcal{F})^2 + c_1(\mathcal{F})c_2(\mathcal{F})\right) \\
&\qquad + (r-4)\left(c_1(L)c_2(\mathcal{F}) + c_3(\mathcal{F})\right) \\
&= \binom{r-1}{3}c_1(L)^3 + \left(\frac{(r-1)(r-2)(r-3)}{2} + (r-2)^2\right)c_1(L)^2 c_1(\mathcal{F}) \\
&\quad + \binom{r-1}{3}c_1(\mathcal{F})^3 + \left(\frac{(r-1)(r-2)(r-3)}{2} + (r-2)^2\right)c_1(L)c_1(\mathcal{F})^2 \\
&\qquad + \left((r-2)^2 + (r-4)\right)c_1(L)c_2(\mathcal{F}) + (r-2)^2 c_1(\mathcal{F})c_2(\mathcal{F}) + (r-4)c_3(\mathcal{F}) \\
&= \left(\frac{11}{6}r - 1 - r^2 + \frac{r^3}{6}\right)c_1(L)^3 + \left(1 + \frac{3}{2}r - 2r^2 + \frac{1}{2}r^3\right)c_1(L)^2 c_1(\mathcal{F}) \\
&\quad + \left(1 + \frac{3}{2}r - 2r^2 + \frac{1}{2}r^3\right)c_1(L)c_1(\mathcal{F})^2 + \left(\frac{11}{6}r - 1 - r^2 + \frac{r^3}{6}\right)c_1(\mathcal{F})^3 \\
&\qquad + \left(r^2 - 3r\right)c_1(L)c_2(\mathcal{F}) + \left(4 - 4r + r^2\right)c_1(\mathcal{F})c_2(\mathcal{F}) + (r-4)\,c_3(\mathcal{F}).
\end{aligned}$$

This shows (iii) and completes the proof. □

Corollary B.4.3. *Let X be a smooth variety and let $\mathcal{E}$ be a vector bundle of rank r. Then:*

(i) $c_1(S^2\mathcal{E}) = (r+1)c_1(\mathcal{E})$.

(ii) $c_2(S^2\mathcal{E}) = \frac{(r+2)(r-1)}{2}c_1(\mathcal{E})^2 + (r+2)c_2(\mathcal{E})$.

(iii) $c_3(S^2\mathcal{E}) = \frac{(r+3)(r-1)(r-2)}{6}c_1(\mathcal{E})^3 + (r^2+2r-4)c_1(\mathcal{E})c_2(\mathcal{E}) + (r+4)c_3(\mathcal{E})$.

Proof. Thanks to the decomposition $\mathcal{E}\otimes\mathcal{E} \cong S^2\mathcal{E}\oplus\Lambda^2\mathcal{E}$, the above formulae immediately follow from

$$c_k(\mathcal{E}\otimes\mathcal{E}) = \sum_{i=0}^{k} c_i(S^2\mathcal{E})c_{k-i}(\Lambda^2\mathcal{E})$$

and from Lemmas B.4.1–B.4.2. □

Lemma B.4.4. *Let X be a smooth variety and let $\mathcal{E}$ be a vector bundle of rank r on X. Then:*

(i) $c_1(S^k\mathcal{E}) = \binom{r+k-1}{k-1}c_1(\mathcal{E}) = \binom{r+k-1}{r}c_1(\mathcal{E})$.

(ii) $c_2(S^3\mathcal{E}) = \frac{1}{8}(r-1)(r+2)(r^2+5r+8)c_1(\mathcal{E})^2 + \frac{1}{2}(r+2)(r+3)c_2(\mathcal{E})$.

Item (i) is proved also in [Rub13, p. 523].

Proof. As both formulae are clearly true for line bundles, we proceed by induction on $r \geq 2$ using the splitting principle. Then we suppose $\mathcal{E} = L\oplus\mathcal{F}$ where L is a line bundle and $\mathcal{F}$ is sum of $r-1$ line bundles. In this way, we have

$$c_1(\mathcal{E}) = c_1(L) + c_1(\mathcal{F}) \text{ and } c_2(\mathcal{E}) = c_1(L)c_1(\mathcal{F}) + c_2(\mathcal{F}).$$

Before going into the proof, we present the following equalities which can be easily proved by induction:

(A) $\sum_{i=0}^{h} \binom{\ell-2+i}{i-1} = \binom{\ell+h-1}{h-1}$.

(B) $\sum_{i=0}^{h} \binom{\ell+i-2}{i}(h-i) = \binom{\ell+h-1}{h-1}$.

For (i), using (A)-(B) together and the inductive hypothesis, we find that

$$\begin{aligned} c_1(\mathrm{S}^k\mathcal{E}) &= \sum_{i=0}^{k} c_1(L^{\otimes(k-i)} \otimes \mathrm{S}^i\mathcal{F}) = \sum_{i=0}^{k} \left[c_1(\mathrm{S}^i\mathcal{F}) + (k-i)\,\mathrm{rk}(\mathrm{S}^i\mathcal{F})c_1(L)\right] \\ &= \sum_{i=0}^{k} \left[\binom{r-2+i}{i-1}c_1(\mathcal{F}) + \binom{r-2+i}{i}(k-i)c_1(L)\right] \\ &= \binom{r+k-1}{k-1}(c_1(L)+c_1(\mathcal{F})) \\ &= \binom{r+k-1}{k-1}c_1(\mathcal{E}) \end{aligned}$$

as desired. For (ii), expanding through $c_k(\mathcal{H} \oplus \mathcal{G}) = \sum_{i=0}^{k} c_i(\mathcal{H})c_{k-i}(\mathcal{G})$ and using the inductive hypothesis together with (i), Corollary B.4.3(i)-(ii) and [EH16, Proposition 5.17], we have

$$
\begin{aligned}
c_2(\mathrm{S}^3\mathcal{E}) &= c_2(\mathrm{S}^3(L\oplus\mathcal{F})) = c_2\left(L^{\otimes 3}\oplus L^{\otimes 2}\otimes\mathcal{F}\oplus L\otimes \mathrm{S}^2\mathcal{F}\oplus \mathrm{S}^3\mathcal{F}\right)\\
&= c_1(L^{\otimes 3})\left[c_1(L^{\otimes 2}\otimes\mathcal{F}) + c_1(L\otimes \mathrm{S}^2\mathcal{F}) + c_1(\mathrm{S}^3\mathcal{F})\right] + c_2(L^{\otimes 2}\otimes\mathcal{F})\\
&\quad + c_1(L^{\otimes 2})\left[c_1(L\otimes \mathrm{S}^2\mathcal{F}) + c_1(\mathrm{S}^3\mathcal{F})\right] + c_2(L\otimes \mathrm{S}^2\mathcal{F}) + c_1(L\otimes \mathrm{S}^2\mathcal{F})c_1(S^3\mathcal{F}) + c_2(\mathrm{S}^3\mathcal{F})\\
&= 3c_1(L)\left(c_1(\mathcal{F}) + 2(r-1)c_1(L) + rc_1(\mathcal{F}) + \binom{r}{2}c_1(L) + \binom{r+1}{2}c_1(\mathcal{F})\right)\\
&\quad + c_2(\mathcal{F}) + 2(r-2)c_1(\mathcal{F})c_1(L) + 4\binom{r-1}{2}c_1(L)^2\\
&\quad\quad + (c_1(\mathcal{F}) + 2(r-1)c_1(L))\left(rc_1(\mathcal{F}) + \binom{r}{2}c_1(L) + \binom{r+1}{2}c_1(\mathcal{F})\right)\\
&\quad\quad\quad + \frac{1}{2}(r-2)(r+1)c_1(\mathcal{F})^2 + (r+1)c_2(\mathcal{F}) + r\left(\binom{r}{2}-1\right)c_1(F)c_1(L) + \binom{\frac{1}{2}r(r+1)}{2}c_1(L)^2\\
&\quad\quad\quad\quad + \binom{r+1}{2}c_1(\mathcal{F})\left(rc_1(\mathcal{F}) + \binom{r}{2}c_1(L)\right)\\
&\quad\quad\quad\quad\quad + \frac{1}{8}(r-2)(r+1)\left((r-1)^2 + 5(r-1) + 8\right)c_1(\mathcal{F})^2 + \frac{1}{2}(r+1)(r+2)c_2(\mathcal{F})\\
&= \frac{1}{8}(r+2)(r^3+4r^2+3r-8)(c_1(L)^2 + c_1(\mathcal{F})^2)\\
&\quad + \frac{1}{4}(r+2)(r^3+4r^2+5r-2)c_1(L)c_1(\mathcal{F}) + \frac{1}{2}(r+2)(r+3)c_2(\mathcal{F})\\
&= \frac{1}{8}(r-1)(r+2)(r^2+5r+8)(c_1(L)^2 + 2c_1(L)c_1(\mathcal{F}) + c_1(\mathcal{F})^2)\\
&\quad + \frac{1}{2}(r+2)(r+3)(c_1(L)c_1(\mathcal{F}) + c_2(\mathcal{F}))\\
&= \frac{1}{8}(r-1)(r+2)(r^2+5r+8)c_1(\mathcal{E})^2 + \frac{1}{2}(r+2)(r+3)c_2(\mathcal{E})
\end{aligned}
$$

as required. □

Appendix C

Brief history of Ulrich sheaves

We conclude this thesis with a short note about the origin of Ulrich sheaves.

C.1 The class of aCM sheaves

The beginning of the story is the macro-class of sheaves which contains Ulrich sheaves: aCM sheaves.

As already seen in Appendix A.2, aCM bundles form one of the most accomplished classes of vector bundles. The reason behind their success is twofold. From a geometric point of view, the starting point and guiding principle has been Horrocks theorem A.2.3, which appeared for the first time in [Hor64]. This result suggests that aCM bundles are a sort of measurement of the complexity of the underlying variety, meaning that to a simple variety it corresponds a simple category of aCM bundles. In this direction, a classification of aCM varieties (of dimension $n \geq 2$) with a finite number of indecomposable aCM bundles was proved.

Theorem C.1.1. *([EH88]) ACM projective varieties of dimension $n \geq 2$ with a finite number of indecomposable aCM bundles are: projective spaces $\mathbf{P}^n$, quadric hypersurfaces $Q_n \subset \mathbf{P}^{n+1}$, the Veronese surface $V \subset \mathbf{P}^5$, a cubic scroll $S \subset \mathbf{P}^4$.*

On the other hand, aCM bundles are pushed by commutative algebra in virtue of the close relation with the class of *maximal Cohen-Macaulay (MCM) modules* (recall that a (graded) R-module M, where R is a (graded) ring, is MCM if $\operatorname{depth}(M) = \dim M = \dim R$.)

Proposition C.1.2. *(see [CMP21, Proposition 2.1.8]) Let $X \subset \mathbf{P}^N$ be a projective variety with homogeneous coordinate ring R_X. The correspondence*

$$\mathcal{E} \longmapsto \Gamma_*(X, \mathcal{E}) := \bigoplus_{t \in \mathbf{Z}} H^0(X, \mathcal{E}(t))$$

establishes a bijection between aCM sheaves on X and MCM R_X-modules.

MCM modules have been a very active research topic in commutative algebra and helped to prove several nice properties for the geometric counterpart. For example, through matrix factorization techniques it has been proved in [Saw10] that an aCM bundle on a general hypersurface splits as sum of line bundles if the rank is sufficiently low (see also [Erm22]).

This is in line with the famous, but still widely open, *(generic) Buchweitz-Greuel-Schreyer conjecture*.

Conjecture C.1.3. *([BGS87]) A rank r aCM bundle $\mathcal{E}$ on a (general) smooth projective hypersurface $X \subset \mathbf{P}^{n+1}$ splits as sum of line bundles if $r < 2^{\lfloor \frac{n-2}{2} \rfloor}$.*

Since the formulation of the above conjecture, lots of works on the splitting of low-rank aCM bundles on smooth hypersurfaces have been produced. We mention [Kle78; MRR07a; MRR07b; Rav09] and [Tri16; Tri17; RT19] where the authors show respectively the splitting of rank 2 and rank 3 aCM bundles on smooth projective hypersurfaces.

In general, the category of aCM bundles on an arbitrary subvariety is still too big, as showed by the following result.

Theorem C.1.4. *([FP21]) An aCM projective variety $X \subset \mathbf{P}^N$ of dimension $n \geq 2$ is of wild CM-type, namely supporting families of arbitrarily large dimension of indecomposable non-isomorphic aCM sheaves, unless it either one of those in Theorem C.1.1 or a smooth quartic rational normal surface scroll.*

Thus a classification of (indecomposable) aCM bundles, as initially hoped for, is impossible. Moreover, it is difficult to study aCM bundles in families because, generally, we can't consider moduli spaces of aCM bundles as they are not necessarily semistable. Therefore the class of aCM bundles is still to difficult to handle. It becomes necessary to focus the attention on a subfamily of aCM bundles: the *Ulrich bundles.*

C.2 Ulrich modules

Ulrich sheaves were actually originally introduced in commutative algebra by Bernd Ulrich in [Ulr84].

Given a local Cohen-Macaulay ring R, Ulrich found an upper bound for the number of minimal generators $\nu(M)$ of a MCM R-module (improving [Sal76, Theorem 2.1]):

$$\nu(M) \leq e(R)\text{rank}(M) \tag{C.1}$$

where $e(R)$ is the multiplicity of R.

Since he also proved in [Ulr84, Theorem 3.1] that a local Cohen-Macaulay ring R is Gorenstein provided the existence of a finitely generated R-module M satisfying $2\nu(M) > e(R)\text{rank}(M)$ and $\text{Ext}^i_R(M,R) = 0$ for $1 \leq i \leq \dim R$, he was led to raise the following question.

Question C.2.1. *([Ulr84]) Given a local Cohen-Macaulay module R, does there always exist a MCM R-module M of positive rank such that $\nu(M) = e(R)\text{rank}(M)$?*

MCM modules satisfying the above equality, i.e. with maximum number of minimal generators, were thus named *maximally generated MCM modules,* or *Ulrich modules* after him.

Initially, Ulrich found a positive answer just in the cases $\dim R = 1$ and R of minimal multiplicity, see [Ulr84]. Later, using Eisenbud's results on matrix factorization in [Eis80] which imply that the existence of a rank r Ulrich module over an integral hypersurface domain $R = A/(f)$ with A a regular local ring yields a presentation of f^r as determinant of a matrix of linear forms, research on Ulrich modules took a step forward, specifically in the series of papers [BHU87; HK87; BHS88; BH89; HUB91]. Ulrich modules started to be studied also for homogeneous Cohen-Macaulay rings, where (C.1) was showed to hold as well for MCM modules (see [BHU87, Proposition 1.1]), their definition was fixed once for all and Question C.2.1 was extended to this setting.

Definition C.2.2. Let R be a homogeneous Cohen-Macaulay ring, namely $R = \bigoplus_{i\geq 0} R_i$ is a graded Cohen-Macaulay ring over a field k with $R_0 = k$ and which is generated by R_1 as k-algebra. Let M be a finitely generated MCM graded R-module of positive rank. Let $e(R)$ denote the multiplicity and let $\nu(M)$ denote the number of minimal generators of M. Then M is called a *maximally generated maximal Cohen-Macaulay,* or an *Ulrich module,* if $\nu(M) = e(R)\mathrm{rank}(M)$.

In this context they proved a fundamental characterization for Ulrich modules.

Proposition C.2.3. ***([BHU87, Proposition 1.5])*** *Let R be a homogeneous Cohen-Macaulay ring of dimension n which is quotient of a regular homogeneous Cohen-Macaulay ring S of dimension $n + c$. A graded MCM module M over R is Ulrich if and only if M is Cohen-Macaulay over S and admits a minimal free graded linear resolution as S-module*

$$0 \to S(-c)^{\oplus b_c} \to \cdots \to S(-2)^{\oplus b_2} \to S(-1)^{\oplus b_1} \to S^{\oplus b_0} \to M \to 0.$$

In virtue of this result, Ulrich modules were also called *linear MCM modules.*

Finally, research on the existence of Ulrich modules culminated in [HUB91] with the proof that hypersurface rings and, more generally, complete intersection rings support an Ulrich module.

Now a natural question may arise: Ulrich modules are special MCM modules, so what about the geometric aspect of the story? In the middle of this purely algebraic setup we mentioned between lines a couple of results which lead to a connection with a concrete and very classical geometric problem whose roots can be dated back to the mid 1800's: the determinantal representation of the equation of a smooth projective hypersurface (see, for example, [Gra55; Hes55; Sch63; Cay71; Sch81] for some of the first papers addressing to this problem). In fact, even if Ulrich modules were not explicitly mentioned, in [Bea00] Beauville proved that a smooth hypersurface $X \subset \mathbf{P}^{n+1}$ is determinantal or pfaffian (set-theoretically) if and only if there exists an aCM bundle on X with a linear resolution. This result was later improved in [Bea18].

Theorem C.2.4. ***([Bea18, Proposition 2.1])*** *Let $X \subset \mathbf{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 2$ defined by a homogeneous equation $F = 0$. Then the following conditions are equivalent:*

(1) $F^r = \det(L_{ij})$ for some $r \geq 1$, where (L_{ij}) is a $rd \times rd$ matrix of linear forms on $\mathbf{P}^{n+1}$.

(2) There exists a vector bundle $\mathcal{E}$ of rank r on X fitting into an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^{n+1}}(-1)^{\oplus rd} \xrightarrow{L} \mathcal{O}_{\mathbf{P}^{n+1}}^{\oplus rd} \to i_*\mathcal{E} \to 0$$

where $i\colon X \hookrightarrow \mathbf{P}^{n+1}$ is the inclusion.

Passing to modules $S = k[x_0, \dots, x_N], R_X = S/(F), \Gamma_*(X, \mathcal{E}(t)) \cong \Gamma_*(\mathbf{P}^{n+1}, i_*\mathcal{E}(t))$, as R_X is Cohen-Macaulay over S since $X \subset \mathbf{P}^{n+1}$ is aCM, by Propositions C.1.2 - C.2.3 and by [Bea00, Theorem A] we see that (2) in Theorem C.2.4 is equivalent to the existence of an Ulrich module M over R_X (which actually holds by the aforementioned result in [HUB91]!). The connection between Ulrich modules and algebraic geometry is now established.

C.3 Ulrich sheaves

The definitive arrival on the scenes of Ulrich sheaves in algebraic geometry is devoted to Eisenbud and Schreyer with their extraordinary paper [ES03].

Definition C.3.1. Let $X \subset \mathbf{P}^N$ be a projective variety of dimension $n \geq 1$. A coherent sheaf $\mathcal{F}$ on X is *Ulrich* if

$$H^0(X, \mathcal{F}(j)) = 0 \text{ for } j < 0,\ H^i(X, \mathcal{F}(d)) = 0 \text{ for } 0 < i < n \text{ and } \forall d \in \mathbf{Z},\ H^n(X, \mathcal{F}(j)) = 0 \text{ for } j \geq -n.$$

Afterwards Eisenbud and Schreyer proved a characterization of Ulrich sheaves which, in virtue of Proposition C.2.3, shows the connection with Ulrich modules. More precisely, the proposition below tells that *an aCM variety $X \subset \mathbf{P}^N$ supports an Ulrich sheaf if and only if there is an Ulrich module over R_X* in the sense of Definition C.2.2.

Proposition C.3.2. *(**[ES03, Proposition 2.1]**) Let $\mathcal{F}$ be a coherent sheaf $\mathcal{F}$ on a n-dimensional projective variety $i\colon X \subset \mathbf{P}^{n+c}$. The following are equivalent:*

(a) $\mathcal{F}$ is Ulrich on X.

(b) $H^i(X, \mathcal{F}(-i)) = H^{i-1}(X, \mathcal{F}(-i)) = 0$ for $1 \leq i \leq n$.

(c) For all finite linear projections $\pi\colon X \to \mathbf{P}^n$ we have $\pi_*\mathcal{F} \cong \mathcal{O}_{\mathbf{P}^n}^{\oplus t}$.

(d) $\Gamma_*(X, \mathcal{F}) \cong \Gamma_*(\mathbf{P}^{n+c}, i_*\mathcal{F})$ is Cohen-Macaulay over $S = k[x_0, \dots, x_N]$ and admits a S-free linear resolution

$$0 \to S(-c)^{\oplus b_c} \xrightarrow{L_c} \cdots \xrightarrow{L_2} S(-1)^{\oplus b_1} \xrightarrow{L_1} S^{\oplus b_0} \to \Gamma_*(X, \mathcal{F}) \to 0. \tag{C.2}$$

In particular, an Ulrich sheaf $\mathcal{F}$ is an *aCM sheaf*: In fact, $\mathcal{F}$ has no intermediate cohomology by definition and it is also locally Cohen-Macaulay because for any $x \in X \subset \mathbf{P}^N$ we have

$$\begin{aligned}
\operatorname{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) &= \operatorname{depth}_{\mathcal{O}_{\mathbf{P}^N,x}}(i_*\mathcal{F}_x) && \text{(see, e.g., [Sta23, Tag 0AUK])}\\
&= \operatorname{depth}_{\mathcal{O}_{\mathbf{P}^N,x}} \mathcal{O}_{\mathbf{P}^N,x} - \operatorname{pd}_{\mathcal{O}_{\mathbf{P}^N,x}}(i_*\mathcal{F}_x) && \text{(Auslander-Buchsbaum formula)}\\
&\geq \dim \mathcal{O}_{\mathbf{P}^N,x} - c && \text{(sheafification of (C.2))}\\
&= \dim \mathcal{O}_{X,x}.
\end{aligned}$$

Moreover, Ulrich sheaves always possess several nice properties which usually do not hold for arbitrary aCM sheaves. For example: 0-*regularity* (from the definition), being simultaneously *initialized* and *globally generated* (by the linear resolution), and *semistability* (by Proposition C.3.2(c) since finite pushforwards preserve subsheaves and Hilbert polynomials, and $\mathcal{O}_{\mathbf{P}^n}^{\oplus t}$ is semistable) which gives the possibility to consider *moduli spaces of Ulrich bundles* when the variety is smooth.

The motivation behind the introduction of Ulrich sheaves by Eisenbud and Schreyer was the study of the Chow form of a variety. Given an embedded variety $X \subset \mathbf{P}(V)$ of codimension $c \geq 1$, the incidence correspondence

$$\begin{array}{ccc} & \{(x,\Lambda) \in X \times \mathrm{Gr}(c,V) \mid x \in \Lambda\} \subset X \times \mathrm{Gr}(c,V) & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ X & & \mathrm{Gr}(c,V) \end{array}$$

yields a natural divisor $D_X = \pi_2(\pi_1^{-1}(X)) = \{\Lambda \in \mathrm{Gr}(c,V) \mid \Lambda \cap X \neq \emptyset\} \subset \mathrm{Gr}(c,V)$ called the *Chow divisor of* X. By the projective normality of the Plücker embedding, D_X is globally defined by the vanishing of a homogeneous polynomial equation f_X in the Plücker coordinates. This form f_X is the *Chow form of* $X \subset \mathbf{P}(V)$. The importance of the Chow form comes from the very powerful feature that it scheme-theoretically determines the variety X whenever X is smooth or a hypersurface [Cat92, Theorem 1.14]. Therefore it is an important (and quite challenging) question determining the Chow form of a given smooth variety. Eisenbud and Schreyer proved that f_X has a very special form when $X \subset \mathbf{P}^N$ supports a rank r Ulrich sheaf: it is the r-th root of the determinant of a matrix of linear forms in the Plücker coordinates.

Theorem C.3.3. ***([ES03, Theorem 0.3])*** *Let $X \subset \mathbf{P}(V)$ be a projective variety codimension $c \geq 1$. Let $\mathcal{E}$ be a rank r Ulrich sheaf on X and consider the linear resolution* (C.2)*. Then*

$$f_X^r = \det(L)$$

where $L = (1/c!)(L_1 \wedge \cdots \wedge L_c)$ can be interpreted as a square matrix with entries in $\Lambda^c V$, which can be seen as the space of linear forms in the Plücker embedding $\mathrm{Gr}(c,V) \subset \mathbf{P}(\Lambda^c V)$.

In addition to this, Eisenbud and Schreyer proved also that the existence of an Ulrich sheaf on $X \subset \mathbf{P}^N$ completely characterizes the cohomology of the polarized variety $(X, \mathcal{O}_X(1))$. Recall that the *cohomology table* of a sheaf (resp. vector bundle) $\mathcal{F}$ on a polarized variety (Y, A) is the function

$$c_{\mathcal{F}} \colon \mathbf{Z} \times \{0, \ldots, \dim Y\} \longrightarrow \mathbf{Z}, \ (t,i) \mapsto h^i(Y, \mathcal{F}(tA)).$$

By setting $c_{\mathcal{F}} + c_{\mathcal{G}} = c_{\mathcal{F} \oplus \mathcal{G}}$ for any pair of coherent sheaves (resp. vector bundles) $\mathcal{F}$ and $\mathcal{G}$ on Y and by including linear combinations with non-negative rational coefficients, we obtain a cone $C(Y,A)$ (resp. $C_{vb}(Y,A)$) over rational numbers called the *cone of cohomology tables of coherent sheaves* (resp. *of vector bundles*) *of* (Y,A).

Theorem C.3.4. ***([ES11, Theorem 4.2])*** *Let $X \subset \mathbf{P}^N$ be a projective variety. Then $C(X, \mathcal{O}_X(1)) = C(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ (resp. $C_{vb}(X, \mathcal{O}_X(1)) = C_{vb}(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$) if and only if $X \subset \mathbf{P}^N$ supports an Ulrich sheaf (resp. Ulrich bundle).*

We can conclude that the existence of an Ulrich sheaf strongly determines the geometry of the underlying variety, giving another instance of our guiding philosophy on the study of varieties. However, despite all of these constraints, Eisenbud and Schreyer raised the question, analogous to Question C.2.1, whether any embedded variety has an Ulrich sheaf. It soon became a conjecture.

Conjecture C.3.5. *([ES03; ES11]) Any embedded (smooth) projective variety supports an Ulrich sheaf (resp. Ulrich bundle).*

So far, an affirmative answer has been found for several classes of smooth embedded varieties. For example: curves [Ulr84; Eis80], complete intersections [HUB91], grassmann varieties [CM15], minimal surfaces of Kodaira dimension 0 [Cas17a; Bea16; Bea18; Fae19], several regular and irregular surfaces [Cas17a; Cas22; Lop21], Del Pezzo surfaces and threefolds [Bea18; CFK23]. Despite every smooth surface which is embedded through a sufficiently large multiple of a fixed very ample line bundle support an Ulrich bundle [CH18], Conjecture C.3.5 is still widely open, even in dimension 2.

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