ON THE CURVES THROUGH A GENERAL POINT OF A SMOOTH SURFACE IN $\mathbb{P}^3$

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1. INTRODUCTION

In the fundamental paper [Mu] on the study of rational equivalence of 0-cycles on surfaces, Mumford showed how Severi's idea of using regular 2-forms could be carried further to prove that the group of 0-cycles of degree 0 modulo rational equivalence is not finite dimensional if $p_g > 0$. This technique has been since then used by others in various instances (see for the latest example the work on the moduli of vector bundles on surfaces [T], [O'G]; see also the note added in proof below). It seemed natural to us to explore the use of this tool in the study of correspondences on surfaces and, as a consequence, on linear series on families of curves on a given surface.

(1.1) *Definitions and notation.* Let S, X be two algebraic surfaces over the complex numbers with S smooth and X integral. A *correspondence of degree n on $X \times S$* is a reduced surface $\Gamma \subset X \times S$ such that the projections $\pi_1 : \Gamma \to X$, $\pi_2 : \Gamma \to S$ are generically finite dominant morphisms and $deg \pi_1 = n$. Let Σ_n be the symmetric group in n elements, $S^{(n)} = S^n/_{\Sigma_n}$ the n-fold symmetric product of S and let $U \subset X_{reg}$ be an open subset such that $dim \pi_1^{-1}(x) = 0$ for every $x \in U$. Associated to Γ there is a map $\gamma : U \to S^{(n)}$ defined by $\gamma(x) = P_1 + \ldots + P_n$, where $\pi_1^{-1}(x) = \{(x, P_i), i = 1, \ldots, n\}$ and a *trace map* $Tr\gamma : H^{2,0}(S) \to H^{2,0}(U)$ defined as in [Mu], section 2 (but see also (2.1)

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below). Γ is said to be a *correspondence with null trace* if $Tr\gamma = 0$. Two correspondences $\Gamma \subset X \times S$, $\Gamma' \subset X' \times S$ are *equivalent* if there exists a birational map $f : X' \to X$ such that $\Gamma' = (f \times id_S)^{-1}(\Gamma)$.

The first task of this paper will be to study the geometry of correspondences with null trace (up to equivalence) of low degree on a smooth surface $S \subset \mathbb{P}^3$ of degree $d \geq 5$. Some quite classical examples are as follows.

Examples **(1.2).** Let C_1, C_2 be two integral curves on S and consider the following varieties : $Sec(C_1, C_2) = \overline{\{< P_1, P_2 > : P_1 \in C_1, P_2 \in C_2, P_1 \neq P_2\}}$ the variety of lines joining C_1 and C_2, $Sec(C_1) = \overline{\{< P, Q > : P, Q \in C_1, P \neq Q\}}$ the variety of secants of C_1 and $T_{C_1}S = \overline{\{< P, Q > : P \in S, Q \in C_1, P \neq Q, < P, Q > \subset T_Q S\}}$ the variety of tangent lines to S along C_1; then let

$$\Gamma_{C_1, C_2} = \overline{\{(L, P) \in Sec(C_1, C_2) \times S : L =< P_1, P_2 >, P \neq P_1, P \neq P_2, P \in L\}}$$

$$\Gamma_{C_1} = \overline{\{(L, P) \in Sec(C_1) \times S : L =< P_1, P_2 >, P \neq P_1, P \neq P_2, P \in L\}}$$

$$\Gamma_{T_{C_1}S} = \overline{\{(L, P) \in T_{C_1}S \times S : L =< P, Q > \subset T_Q S, P \neq Q\}}.$$

We call Γ_{C_1, C_2} the *residual of the secants joining* C_1 *and* C_2, Γ_{C_1} the *residual of the secants of* C_1, and $\Gamma_{T_{C_1}S}$ the *residual of the tangents to* S *along* C_1.

It is not hard to show (Proposition (2.12)) that the above are in fact correspondences with null trace of degree $d-2$ on $X \times S$, where X is $Sec(C_1, C_2)$, $Sec(C_1)$, $T_{C_1}S$ respectively (provided that C_1 and C_2 do not lie in the same plane, C_1 is non degenerate, C_1 is any integral curve respectively). On the other hand the interesting fact is that, as families of 0-cycles, the examples (1.2) are all the correspondences with null trace of minimum degree on a smooth surface in $\mathbb{P}^3$. In fact we have

Theorem (1.3). *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$ and Γ a correspondence of degree n on $X \times S$ with null trace. Then $n \geq d-2$ and if equality holds Γ is equivalent to one of the correspondences (1.2). Moreover if $n \leq 4d - 21$ the only possible values of n are $d - 2 \leq n \leq d$, $2d - 6 \leq n \leq 2d$ or $3d - 12 \leq n \leq 3d$.*

The crucial point in the proof of this theorem is a local computation of Mumford's trace map that allows to show that if Γ is a correspondence with null trace on $X \times S$

and ω is a 2-form on S vanishing on all but one point in a 0-cycle of Γ, then ω vanishes on the remaining point, that is the 0-cycle is a Cayley-Bacharach scheme with respect to $|K_S|$. The latter is a well-known classically studied property ([GH], [R]) and gives strong restrictions on the geometry of the 0-cycle itself in terms of curves passing through it ([EP], Lemma (2.5) below).

One interesting application of this is to the study of linear series on a curve $C \subset S$. It is a classical fact going back to Bertini that the general divisor on a base point free g^1_n on a smooth curve is a Cayley-Bacharach scheme with respect to $|K_C|$ and that the same holds for g^r_n's provided that they are base point free and not composed with an involution. If C moves on S then one can easily show that the Cayley-Bacharach property also holds with respect to $|K_S|$ and interesting conclusions can be drawn on linear series both on C and on its normalization.

Before stating the results in this direction we would like to remark that the ones on the linear series on the curve (Theorems (1.4), (1.5), Remark (3.2)) are in fact independent of the use of Mumford's technique; they just follow from the Cayley-Bacharach property and Lemma (2.5). These results as well as Lemma (2.5) are certainly nothing new, even though we have been unable to find the appropriate reference (however see [R]; [P], [Co] for pencils). On the other hand the results on pencils on the normalization of the curve (Corollaries (1.7) and (1.8)) are in fact new and do make a use of Mumford's technique.

For linear series on the curve we have

Theorem (1.4). *Let $C \subset \mathbb{P}^2$ be an integral curve of degree d and g^r_n a base point free linear series on C that is not composed with an involution if $r \geq 2$. If $n \leq \frac{d^2}{4} - 1$ there exists an integer h such that $h(d-h) \leq n \leq min\{hd, (h+1)(d-h-1)-1\}$ and the g^r_n is contained in the series cut out by curves of degree h.*

Theorem (1.5). *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$, C an integral curve on S such that $|\mathcal{O}_C \otimes \mathcal{O}_S(C)|$ is base point free and g^r_n a base point free special linear series on C that is not composed with an involution if $r \geq 2$. If $n \leq 4d - 21$ there exists an integer h such that $1 \leq h \leq 3$, $h(d-h-1) \leq n \leq min\{hd, (h+1)(d-h-2)-1\}$ and the general divisor of the g^r_n lies on a curve of degree h.*

On the other hand, when studying linear series on the normalization $\widetilde{C}$ of a curve $C \subset S \subset \mathbb{P}^3$ the fact of having the Cayley-Bacharach property with respect to $|K_{\widetilde{C}}|$ does not in general imply that the same holds for $|K_S|$. One interesting case in which this does happen is when the pair (C, g^1_n) moves on S. Observe that it is always possible to cover S with $(d-1)$ or $(d-2)$-gonal curves: smooth plane sections have a g^1_{d-1} and general tangent plane sections have a g^1_{d-2}. On the other hand, if S contains some special curves, we can construct other covers as follows.

Examples **(1.6).** If $C \subset S$ is a rational curve and P is a point in $S - C$, consider the incidence variety $I_P = \overline{\{(Q,R) : R \in < P, Q >, R \neq P, R \neq Q\}} \subset C \times S$ and $C_P = \pi_2(I_P)$; clearly $\bigcup_{P \in S-C} C_P$ covers an open subset of S and $\pi_1 : I_P \to C \cong \mathbb{P}^1$ shows that I_P has a g^1_{d-2}, hence C_P has a g^1_k, $k \leq d-2$; it will follow from Corollary (1.7) that in fact $k = d-2$ in this case. Another example is when S contains an elliptic non degenerate curve E: for every $P_1, P_2 \in E$, $P_1 \neq P_2$, let $I_{P_1,P_2} = \overline{\{(Q+R, P) : P \in < Q, R >, P \neq Q, P \neq R\}} \subset |P_1 + P_2| \times S$ and $C_{P_1,P_2} = \pi_2(I_{P_1,P_2})$; again $\pi_1 : I_{P_1,P_2} \to |P_1 + P_2| \cong \mathbb{P}^1$ is a g^1_{d-2} on I_{P_1,P_2}, hence (by Corollary (1.7)) C_{P_1,P_2} has gonality $d-2$ and of course $\bigcup_{P_1,P_2 \in E} C_{P_1,P_2}$ covers an open subset of S since, as we will see in the proof of Proposition (2.12), $\Gamma_E \to S$ is dominant.

As we will see to a given covering of an open subset of S with k-gonal curves we can associate a correspondence on S of degree k, and hence use Theorem (1.3) to prove the following

Corollary (1.7). *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$. If there is a family of curves having a base point free g^1_k, passing through a general point of S and $k \leq 4d-21$ then $d-2 \leq k \leq d$, $2d-6 \leq k \leq 2d$ or $3d-12 \leq k \leq 3d$. Moreover if $k = d-2$ the family is equivalent to either the one given by tangent plane sections or to one of the Examples (1.6) if S contains a rational or an elliptic curve.*

Finally using a theorem of Xu ([X]), we can describe the $(d-2)$-gonal coverings (up to equivalence (3.3)) on a general surface.

Corollary (1.8). *Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 5$. Then every covering of S by (d-2)-gonal curves is equivalent to the one given by tangent plane sections.*

2. MUMFORD'S TRACE MAP AND CORRESPONDENCES WITH NULL TRACE

We would like to recall in this section the basic properties of Mumford's induced differentials ([Mu], section 2) and write a local version of the trace map in (1.1) that will be needed for our study of 0-cycles of correspondences. The first two Propositions that we will obtain are in fact nothing new, especially for readers familiar with Mumford's paper and subsequent papers such as Griffiths-Harris' ([GH]) and Tyurin's ([T]), but we will include them for the reader's convenience.

Let S be a smooth surface, $S^{(n)} = S^n/_{\Sigma_n}$ the n-fold symmetric product of S and $p_i : S^n \to S$ the projection, $i = 1, \dots, n$. Given a form $\omega \in H^{2,0}(S)$ we consider the Σ_n-invariant $(2,0)$-form $\omega^{(n)} = \sum_{i=1}^{n} p_i^*\omega$ on S^n; then for every morphism $f : Y \to S^n$, where Y is a nonsingular variety, there is a canonically induced $(2,0)$-form ω_f on Y; in particular if Γ is a correspondence on $X \times S$ and, as in Definition (1.1), $U = \{x \in X_{reg} : dim\pi_1^{-1}(x) = 0\}$, $\gamma : U \to S^{(n)}$ is given by $\gamma(x) = P_1 + \ldots + P_n$, where $\pi_1^{-1}(x) = \{(x, P_i), i = 1, \dots, n\}$, we get an induced differential ω_γ on U that defines the *trace map* $Tr\gamma : H^{2,0}(S) \to H^{2,0}(U)$ as $Tr\gamma(\omega) = \omega_\gamma$. Now let $\pi : S^n \to S^{(n)}$ be the quotient map, $S_0^{(n)} = \pi(S^n - \bigcup_{i \neq j} \Delta_{i,j})$, where $\Delta_{i,j}$ is the (i,j)-diagonal of S^n and $V = \{x \in U : \pi_1^{-1}(x)$ has n distinct points$\}$. Then $Im(\gamma_{|V}) \subset S_0^{(n)}$ and if we define $\delta_n : H^{2,0}(S) \to H^{2,0}(S_0^{(n)})$ with $\delta_n(\omega) = \pi(\omega^{(n)})$ (that is $\omega^{(n)}$ as a form on $S_0^{(n)}$) we have

Proposition (2.1). $Tr\gamma = \gamma_{|V}^* \circ \delta_n$.

Proof: Let $\widetilde{V} = (V \times_{S_0^{(n)}} (S^n - \bigcup_{i \neq j} \Delta_{i,j}))_{red}$ and consider the diagram

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\widetilde{\gamma}} & S^n - \bigcup_{i\neq j} \Delta_{i,j} \\ \downarrow p & & \downarrow \pi \\ V & \xrightarrow{\gamma_{|V}} & S_0^n \end{array}.$$

By definition ([Mu]) we have that ω_γ is the only form on $H^{2,0}(V)$ such that $p^*(\omega_\gamma) - \widetilde{\gamma}^*(\omega^{(n)})$ is torsion in $\Omega^2_{\widetilde{V}}$, hence $\omega_\gamma = \gamma_{|V}^*(\pi(\omega^{(n)}))$ and therefore

$$Tr\gamma(\omega) = \omega_\gamma = \gamma_{|V}^*(\pi(\omega^{(n)})) = \gamma_{|V}^* \circ \delta_n(\omega). \quad \blacksquare$$

Clearly Proposition (2.1) gives a concrete way of computing Mumford's trace map associated to a correspondence. We will use this to see that the fact of having null trace imposes a very strong condition upon a correspondence.

Proposition (2.2). *With notation as in Definition (1.1) let* Γ *be a correspondence with null trace on* $X \times S$. *Then* $\{P_1, \dots, P_n\}$ *satisfies the Cayley-Bacharach condition with respect to* $|K_S|$, *i.e. for every* $i = 1, \dots, n$ *and for any effective divisor* K_S *containing* $P_1, \dots, \widehat{P}_i, \dots, P_n$, *we have* $P_i \in K_S$.

Proof: We write locally what it means to have null trace. Let $Y = \gamma(V) \overset{i}{\hookrightarrow} S_0^{(n)}$ and $\gamma_1 : V \to Y$ so that $\gamma_{|V} = i \circ \gamma_1$; then we have a diagram

$$\begin{array}{ccccc} H^{2,0}(S) & \xrightarrow{\delta_n} & H^{2,0}(S_0^{(n)}) & \xrightarrow{\gamma_{|V}^*} & H^{2,0}(V) \\ & & \searrow i^* & \nearrow \gamma_1^* & \\ & & & H^{2,0}(Y) & \end{array}$$

hence if $\omega \in H^{2,0}(S)$ we get $\quad 0 = Tr\gamma(\omega) = \gamma_{|V}^*(\delta_n(\omega)) = \gamma_1^*(i^*(\delta_n(\omega))) = \gamma_1^*(\delta_n(\omega)_{|Y})$ so

(2.3) $\quad \delta_n(\omega)_{|Y} = 0$

since γ_1^* is injective because γ_1 is a surjective rational map of surfaces. Now let $y = P_1 + \ldots + P_n$ be a general point of Y and $\omega_{P_i} : \bigwedge^2 T_{P_i}S \to \mathbb{C}$ the linear form induced by ω at P_i.

Claim (2.4). *For every* $u = u_1 \wedge u_2 \in \bigwedge^2 T_y S_0^{(n)} \cong \bigwedge^2 (\bigoplus_{i=1}^n T_{P_i}S)$, *with* $u_j = (v_{j1}, \dots, v_{jn})$, $j = 1, 2$, *we have* $\quad \delta_n(\omega)(y)(u) = \sum_{i=1}^n \omega_{P_i}(v_{1i} \wedge v_{2i})$.

Proof of the Claim: For every $i = 1, \dots, n$ let $\bigwedge^2 (T_{(P_1,\dots,P_n)}S^n) \cong \bigwedge^2 (\bigoplus_{j=1}^n T_{P_j}S) = \bigwedge^2 (T_{P_i}S) \oplus W_i$; since $p_i^* \omega_{|\bigwedge^2 (T_{P_i}S)} = \omega_{P_i}$ and $p_i^* \omega_{|W_i} = 0$ we have

$$\delta_n(\omega)(y)(u_1 \wedge u_2) = \sum_{i=1}^n p_i^* \omega(v_{1i} \wedge v_{2i}) = \sum_{i=1}^n \omega_{P_i}(v_{1i} \wedge v_{2i})$$

■ (for Claim (2.4)).

Now for every $i = 1, \dots, n$ let $\omega \in H^{2,0}(S)$ be such that $\omega_{P_j} = 0$ for $j \neq i$; by (2.3) and (2.4) we have, for any $u \in \bigwedge^2 (\bigoplus_{i=1}^n T_{P_i}S)$,

$$0 = \delta_n(\omega)(y)(u) = \sum_{j=1}^n \omega_{P_j}(v_{1j} \wedge v_{2j}) = \omega_{P_i}(v_{1i} \wedge v_{2i})$$

hence $\omega_{P_i} = 0$ since v_{1i} and v_{2i} are arbitrary. ■

Now let S be a smooth surface of degree $d \geq 5$ in $I\!P^3$ and Γ a null trace correspondence on $X \times S$. By Proposition (2.2) we have that $\{P_1, \ldots, P_n\} = \pi_2(\pi_1^{-1}(x))$ satisfies the Cayley-Bacharach condition with respect to $|K_S| = |\mathcal{O}_S(d-4)|$. This simple fact allows to classify all the Γ's of low degree up to equivalence as we will see later. First we need the following elementary result.

Lemma (2.5). *Let $Z = \{P_1, \ldots, P_n\} \subset I\!P^r, r = 2, 3,$ be a set of $n \geq 1$ distinct points that satisfy the Cayley-Bacharach property with respect to $|\mathcal{O}_{I\!P^r}(k)|$, $k \geq 1$. Then $n \geq k+2$ and if $n \leq h(k-h+3)-1$ for some h such that $2 \leq h \leq \frac{k+3}{2} + \frac{(r-2)(5-k)}{2}$, then Z lies on a reduced curve of degree $h-1$.*

Proof: If a zero-dimensional scheme $\Sigma \subset I\!P^r$ satisfies the Cayley-Bacharach property with respect to $|\mathcal{O}_{I\!P^r}(k)|$ we will write shortly that Σ is $CB(k)$. If $n \leq k+1$ choose hyperplanes $H_i \ni P_i$ in $I\!P^r$, $i = 1, \ldots, n-1$, $H_i \not\ni P_n$ and a hypersurface $F \not\ni P_n$ of degree $k-n+1$ ($F = \emptyset$ if $n = k+1$); then $F_k = H_1 \cup \ldots \cup H_{n-1} \cup F \supset Z - \{P_n\}$ but $F_k \not\ni P_n$.

Claim (2.6). *The lemma is true for $r = 2, 3$ and $h = 2$.*

Proof: We have $k+2 \leq n \leq 2k+1$. If $k = 1$ then $n = 3$ and any hyperplane containing P_1 and P_2 must contain P_3, so the points lie on a line; if $k \geq 2$ let α be the maximum number of points of Z on a line, so that there exists a line L and a decomposition $Z = Z_L \amalg Z'$ with $Z_L \subset L$, $Z' \cap L = \emptyset$. If $Z' \neq \emptyset$ then it is $CB(k-1)$: For every $P \in Z'$ and for every $F_{k-1} \supset Z' - \{P\}$, choose a general hyperplane $H \supset L$; then $F_{k-1} \cup H \supset Z - \{P\}$, so $P \in F_{k-1}$. But $\#Z' = n - \alpha \leq 2k-1$, hence Z' is contained on a line L' by induction on k. By definition of α we have $\#Z' = n - \alpha \leq \alpha$, hence $n - \alpha \leq \frac{n}{2} \leq k + \frac{1}{2}$, therefore $n - \alpha \leq k$. Fix a point $P \in Z'$, a general hyperplane $H_j \ni Q_j$ for every $Q_j \in Z' - \{P\}$, $j = 1, \ldots, n-\alpha-1$ and a general hypersurface $F_{k-n+\alpha}$ of degree $k-n+\alpha$ (as above $F_{k-n+\alpha} = \emptyset$ if $k = n-\alpha$); then $F_{k-n+\alpha} \cup H_1 \cup \ldots \cup H_{n-\alpha-1} \supset Z - \{P\}$ but does not contain P. This contradiction shows that Z' is empty and $Z = Z_L \subset L$. ■ (for Claim (2.6)).

Claim (2.7). *The lemma is true for $r = 2$.*

Proof: We prove it by induction on h. By Claim (2.6) we can suppose $h \geq 3$ and $n \geq (h-1)(k-h+4)$, otherwise $n \leq (h-1)(k-h+4)-1$ and by induction Z is contained in a curve of degree $h-2$. We have then $h(k-h+3)-1 \geq n \geq (h-1)(k-h+4)$ hence $k \geq 2h-3$ and either $k = 2h-3$, $n = h^2-1$ or $k \geq 2h-2$ and hence $n \geq (h-1)(k-h+4) \geq h^2$. In the second case we have $h \leq \frac{n}{h}$ and if we let $\tau(Z) = max\{i \geq 1 : H(Z,i) < n\}$, where $H(Z,i)$ is the Hilbert function of $Z \subset I\!P^2$ we have $\tau(Z) \geq k \geq h-3+\frac{n}{h}$ so by [EP], Corollaire 2, either Z is a complete intersection of curves of degree h and $\frac{n}{h}$ and $\tau(Z) = h-3+\frac{n}{h}$ or there exists an integer t such that $1 \leq t \leq h-1$ and a subset $Z' \subset Z$ of cardinality $\#Z' \geq t(k-t+3)$ contained on a curve C_t of degree t. Clearly Z cannot be a complete intersection of curves of degree h and $\frac{n}{h}$ for otherwise $h-3+\frac{n}{h} = \tau(Z) \geq k \geq h-3+\frac{n}{h}$, hence $n = h(k-h+3)$. Setting $Z = Z'_1 \amalg Z''$ where $Z'_1 \subset C_t$, $Z'' \cap C_t = \emptyset$, we see that Z'' is either empty and we are done or it is non empty and $CB(k-t)$ and $\#Z'' = n - \#Z'_1 \leq n-\#Z' \leq h(k-h+3)-1-t(k-t+3) \leq (h-t)(k-t-(h-t)+3)-1$. Notice that $h-t \geq 2$ for otherwise $t = h-1$ and $\#Z'' = n-\#Z'_1 \leq h(k-h+3)-1-(h-1)(k-h+4) = k-2h+3$ and choosing $P \in Z''$, general lines $L_j \ni Q_j$ for every $Q_j \in Z''-\{P\}$, $j = 1, \ldots, \#Z''-1$ and a curve $F \not\ni P$ of degree $k-\#Z''+2-h \geq h-1$ we see that the curve $F \cup L_1 \cup \ldots \cup L_{\#Z''-1} \cup C_t$ is of degree k and contains $Z - \{P\}$ but not P. Now the upper bound on $\#Z''$ implies, by induction, that Z'' lies on a curve of degree $h-t-1$ and hence Z lies on a curve of degree $h-1$. It remains, to finish the proof of Claim (2.7), the case $k = 2h-3$, $n = h^2-1$. We have $h-1 \leq \frac{n}{h-1}$ and $\tau(Z) \geq 2h-3 \geq h-4+\frac{n}{h-1}$, so again by [EP], either Z is a complete intersection of curves of degree $h-1$ and $\frac{n}{h-1} = h+1$ or there is a t such that $1 \leq t \leq h-2$ and a subset $Z' \subset Z$ of cardinality $\#Z' \geq t(2h-t)$ contained on a curve C_t of degree t. Again setting $Z = Z'_1 \amalg Z''$ with $Z'_1 \subset C_t$, $Z'' \cap C_t = \emptyset$, we have that Z'' is either empty and we are done or it is non empty and $CB(2h-3-t)$ and as above $\#Z'' \leq h^2-1-t(2h-t) \leq (h-t)(2h-3-t-(h-t)+3)-1$ and by induction Z'' lies on a curve of degree $h-t-1$ and hence Z lies on a curve of degree $h-1$. ■ (for Claim (2.7)).

Claim (2.8). *The lemma is true for $h = r = 3$.*

Proof: Since $k+2 \leq n \leq 3k-1$ we have $k \geq 2$ and if $k = 2$ then $n \leq 5$ so Z lies on a line by what we proved above. We suppose then $k \geq 3$ and $n \geq 2k+2$, otherwise again Z lies on a line. Let α be the maximum number of points of Z on a plane, H a plane containing α points and $Z = Z_H \amalg Z'$ with $Z_H \subset H$, $Z' \cap H = \emptyset$. If $Z' = \emptyset$ we have $Z \subset I\!P^2$ and we are done by Claim (2.7); otherwise Z' is $CB(k-1)$ and $\#Z' \geq k+1$. If $\#Z' \leq 2k-1$, by Claim

(2.6) we get that Z' lies on a line L' and $Z_H - Z_H \cap L'$ is $CB(k-1)$ as it can be seen by taking a general plane containing L'; but $\#(Z_H - Z_H \cap L') \leq \alpha = n - \#Z' \leq 2k-2$ hence $Z_H - Z_H \cap L'$ lies on a line and Z lies on the union of two lines. If instead $\#Z' \geq 2k$, then, by induction on k, Z' lies on a curve C_2 of degree 2 since $\#Z' = n - \alpha \leq 3k-4$ because of course $\alpha \geq 3$, and C_2 cannot be a plane conic for otherwise $Z' \subset \mathbb{P}^2$ hence $\#Z' \leq \alpha = n - \#Z'$ so $2k \leq \#Z' \leq \frac{n}{2} \leq \frac{3k-1}{2}$, a contradiction. Therefore $Z' \subset C_2 = L \cup L'$ with L, L' two skew lines and $Z_H - Z_H \cap (L \cup L')$ is $CB(k-2)$ (as usual take two general planes containing L and L') hence $\alpha \geq k$ and this is not possible since $\alpha = \#Z_H = n - \#Z' \leq k-1$. ■ (for Claim (2.8)).

Claim (2.9). *The lemma is true for $h = 4, r = 3$.*

Proof: We can suppose $3k \leq n \leq 4k-5$ and $k \geq 5$, the other cases being proved above. Again let α be the maximum number of points of Z on a plane, H a plane containing α points and $Z = Z_H \amalg Z'$ with $Z_H \subset H$, $Z' \cap H = \emptyset$. If $Z' = \emptyset$ then $Z \subset H$ and we are done; if $Z' \neq \emptyset$ then it is $CB(k-1)$ and $\#Z' \geq k+1$. If $\#Z' \leq 2k-1$, then Z' lies on a line L', $Z_H - Z_H \cap L'$ is $CB(k-1)$ and $\#(Z_H - Z_H \cap L') \leq 3k-6$, so $Z_H - Z_H \cap L'$ lies on a curve of degree 2 and Z on a curve of degree 3. So we can assume $\#Z' \geq 2k$.

Subclaim (2.10). *Claim (2.9) is true for $\alpha \geq 4$.*

Proof: We have $\#Z' \leq 4k-9$ hence Z' is contained on a curve C_3 of degree 3. First notice that Z' is not contained in a plane, so in particular C_3 is not a plane cubic, for otherwise $\#Z' \leq \alpha = n - \#Z'$ so $2k \leq \#Z' \leq \frac{n}{2} \leq \frac{4k-5}{2}$; if $C_3 = L_1 \cup L_2 \cup L_3$ union of three skew lines we see that $Z_H - Z_H \cap (L_1 \cup L_2 \cup L_3)$ is $CB(k-3)$ and $\#(Z_H - Z_H \cap (L_1 \cup L_2 \cup L_3)) \leq 2k-5$ hence $Z_H - Z_H \cap (L_1 \cup L_2 \cup L_3)$ lies on a line $L \subset H$. Write $Z = Z_H \amalg Z_1 \amalg Z_2 \amalg Z_3$ with $Z_i \subset L_i$; if one of the Z_i is empty we get that Z is contained on a curve of degree 3; otherwise $L \cap L_i = \emptyset$, $i = 1, 2, 3$ (if not we have more than α points on a plane) hence Z_H, Z_1, Z_2, Z_3 are $CB(k-3)$ and therefore $\#Z_H \geq k-1$, $\#Z_i \geq k-1$, hence $n \geq 4k-4$. If $C_3 = L \cup Q$, with L a line and Q a plane conic we write $Z = Z_H \amalg Z_L \amalg Z_Q$ with $Z_L \subset L$, $Z_Q \subset Q - Q \cap L$. Notice that $Z_L \neq \emptyset$, $Z_Q \neq \emptyset$, $\#(L \cap Q) \leq 1$ for otherwise Z' is contained on a plane. If we let H' be the plane containing Q, we see that $Z_L - H' \cap L$ and $Z_Q - L \cap Q$ are $CB(k-2)$ and non empty (or Z' is contained on a plane) hence $\#Z_L$, $\#Z_Q$ are at least k; taking a general quadric through Q and a general plane through L we get that $Z_H - (L \cup Q) \cap Z_H$ is $CB(k-3)$ and $\#Z_H \leq 4k - 5 - \#Z_L - \#Z_Q \leq 2k - 5$ hence Z_H is contained on a line $L' \subset H$. Moreover $\#Z_H \geq k - 1$, hence $\#Z_Q \leq 2k - 4$ therefore Z_Q is contained on a line and Z on a curve of degree 3. If C_3 is an irreducible twisted cubic, then there exists a quadric surface $F \supset C_3$, $F \not\supset Z_H$, else $Z_H \subset C_3$ and so is Z. In this case it must be $\alpha \geq k$ for otherwise taking a point $P \in Z_H - F$, $\alpha - 1$ general planes $H_1, \ldots, H_{\alpha-1}$ through the points $P_i \in Z_H - \{P\}$ and a general surface G of degree $k-1-\alpha$ ($G = \emptyset$ if $\alpha = k-1$), we have that $H_1 \cup \ldots \cup H_{\alpha-1} \cup F \cup G \supset Z - \{P\}$ but not P. Therefore $\#Z' \leq 3k - 5$, hence, since Z' is $CB(k-1)$, it lies on a curve of degree 2 that must be union of two skew lines $L_1 \cup L_2$. It follows that $Z_H - (L_1 \cup L_2) \cap Z_H$ is $CB(k-2)$ hence either $\alpha \leq 2k - 3$ and $Z_H - (L_1 \cup L_2) \cap Z_H$ lies on a line and Z on a cubic, or $2k - 2 \leq \alpha \leq 3k - 5$ and hence $\#Z' \leq 2k - 3$, so Z' lies on a line L_1 and $Z_H - L_1 \cap Z_H$ is $CB(k-1)$ and therefore it lies on a curve of degree 2 and Z on a cubic, or $\alpha \geq 3k - 4$ and hence $\#Z' \leq k - 1$, and this is not possible since $\#Z' \geq k + 1$ because Z' is $CB(k-1)$. ■ (for Subclaim (2.10)).

Subclaim (2.11). *Claim (2.9) is true for* $\alpha = 3$.

Proof: In this case no four points of Z lie on a plane. Take eight points on Z, choose two distinct irreducible quadrics Q_1, Q_2 through them and let C be their complete intersection; write $Z = Z_C \amalg Z'$ with $Z_C \subset C$, $Z' \cap C = \emptyset$. Suppose first that Z' is non empty; then, since the ideal of C is generated by quadrics, we see that Z' is $CB(k-2)$ and $k \leq \#Z' \leq 4k - 13$ hence Z' lies on a cubic curve C_3 which is irreducible, for otherwise, since $\alpha = 3$, we have $\#Z' \leq 6$, that is $k \leq 6$ and $\#Z' \leq 2k - 1$ so Z' lies on a line, hence $k \leq \#Z' \leq 2$. But then also Z_C is $CB(k-2)$ since the ideal of C_3 is generated by quadrics. If $\#Z_C \leq 3k - 7$ then Z_C lies on a curve of degree 2 hence $8 \leq \#Z_C \leq 4$; therefore $\#Z_C = 3k - 6, 3k - 5$ because $\#Z' \geq k$. If $\#Z_C = 3k - 5$, $\#Z' = k$ taking a general plane through two points P_1, P_2 of Z' we see that $Z' - \{P_1, P_2\}$ is $CB(k-3)$ hence $k - 1 \leq \#Z' - 2 = k - 2$ or $Z' = \{P_1, P_2\}$ and $k = 2$, both impossible. If $\#Z_C = 3k - 6$, $\#Z' = k + 1$ again $Z' - \{P_1, P_2\}$ is $CB(k-3)$ and $\#(Z' - \{P_1, P_2\}) = k - 1 \leq 2k - 5$ hence $Z' - \{P_1, P_2\}$ lies on a line and therefore $\#(Z' - \{P_1, P_2\}) \leq 2$ and hence the contradiction $k + 1 = \#Z' \leq 4$. Finally if $Z' = \emptyset$, $Z = Z_C \subset C$, then C is irreducible, for otherwise either $C = C_2 \cup C_2'$ union of two (possibly reducible) curves of degree 2, hence $n = \#Z \leq 8$ or $C = L \cup C_3$ union of a line L and an irreducible cubic C_3, hence $1 \leq Z \cap L \leq 2$ (or Z lies on a cubic) and $Z \cap L$ is $CB(k-2)$, so $k \leq 2$. Using a result of Greco ([G], Theorem 1.5) we see that Z cannot be $CB(k)$. Alternatively if C is smooth and $D = P_1 + \ldots + P_n$, $D' = P_1 + \ldots + P_{n-1}$ we have $H^0(\mathcal{I}_{D/\mathbb{P}^3}(k)) = H^0(\mathcal{I}_{D'/\mathbb{P}^3}(k))$, hence, since the surfaces of degree k cut out a complete linear series on C, $|\mathcal{O}_C(k)(-D')|$ has P_n as a base point and this is not possible since $deg\mathcal{O}_C(k)(-D') = 4k-n+1 \geq 6$. If C is singular it can have at most a double point P hence we have either $P_i \neq P$, $i = 1, \ldots, n$ or, without loss of generality, $P_n = P$, $P_i \neq P$, $i = 1, \ldots, n-1$ and as above $H^0(\mathcal{I}_{D/C}(k)) = H^0(\mathcal{I}_{D'/C}(k))$. Let $\widetilde{C}$ be the normalization of C and R_1, R_2 the points on $\widetilde{C}$ over P. Then we get that $|\mathcal{O}_{\widetilde{C}}(k\widetilde{H} - \widetilde{D}')|$ has either one or two base points and this is not the case since $\widetilde{C} \cong \mathbb{P}^1$ and $deg\mathcal{O}_{\widetilde{C}}(k\widetilde{H} - \widetilde{D}') > 0$. This proves Subclaim (2.11) and hence Claim (2.9)). ■

Before proving Theorem (1.3) let us see that the correspondences of Examples (1.2) are in fact correspondences with null trace.

Proposition (2.12). *With notation as in (1.2) Γ_{C_1,C_2}, Γ_{C_1} and $\Gamma_{T_{C_1}S}$ are correspondences with null trace of degree $d-2$ on $X \times S$, where X is $Sec(C_1, C_2)$, $Sec(C_1)$, $T_{C_1}S$ respectively provided that C_1 and C_2 do not lie in the same plane, C_1 is non degenerate, C_1 is any integral curve respectively.*

Proof: In all three cases they dominate X. Let us prove that they are dominant over S. To see this for Γ_{C_1,C_2} observe first that a general point P of S lies in at most finitely many secants of either C_1 or C_2 : In fact $\pi_2 : \Gamma_{C_1} \to S$ is a morphism between two varieties of the same dimension, hence there is a dense open subset of S over which the fiber is finite (or empty), that is whose points lie on finitely many secants of C_1 (and of C_2). So the projection $\pi_P : \mathbb{P}^3 - \{P\} \to \mathbb{P}^2$ is birational on C_1 and C_2. Let $d_i = degC_i, i = 1, 2$; we claim that $\#(C_1 \cap C_2) < d_1 d_2 = \#(\pi_P(C_1) \cap \pi_P(C_2))$. From this we see that there is a point $Q \in \pi_P(C_1) \cap \pi_P(C_2)$ such that $Q = \pi_P(P_1) = \pi_P(P_2)$, $P_1 \in C_1$, $P_2 \in C_2$, $P_1 \neq P_2$, so $P \in < P_1, P_2 >$, that is $\pi_2 : \Gamma_{C_1,C_2} \to S$ is dominant. To see the claim suppose first that one of the two curves, for example C_1, is non degenerate; then by [GLP] C_1 is cut out by surfaces of degree $d_1 - 1$, hence $\#(C_1 \cap C_2) \leq d_2(d_1 - 1)$. If both curves are in a plane, $C_1 \subset H_1$, $C_2 \subset H_2$, then $\#(C_1 \cap C_2) \leq min\{d_1, d_2\} < d_1 d_2$ unless $d_1 = d_2 = 1$, in which case $\#(C_1 \cap C_2) = 0$. The second correspondence Γ_{C_1} is dominating on S because C_1 is not contained in a plane and hence $\overline{\bigcup_{\substack{P,Q\in C_1\\ P\neq Q}} < P,Q >} = I\!\!P^3$ for otherwise a general projection into a plane is an isomorphism from C_1 to a plane curve $\overline{C}_1$, hence $\binom{d_1-1}{2} = p_a(\overline{C}_1) = p_a(C_1) \leq \frac{1}{4}d_1^2 - d_1 + \frac{3}{4}$ by Castelnuovo's bound, contradicting $d_1 \geq 3$. Let us now consider $\pi_2 : \Gamma_{T_{C_1}S} \to S$ where C_1 is any integral curve on S. Let $T = \{H \in (I\!\!P^3)^* : H = T_Q S \text{ for some } Q \in C_1\}$ and $I = \{(P,H) : P \in H\} \subset S \times T$. If $p_1 : I \to S$ is dominant, then through the general point $P \in S$ there is a plane $H \in T$; so $H = T_Q S$ and $< P, Q > \in T_{C_1}S$, hence π_2 is dominant. Otherwise $dimImp_1 = 1$, so $T_Q S = H$ for infinitely many $Q \in C_1$. This is impossible because by Zak's theorem on tangencies ([Z]) we have $dim\{P \in S : T_P S = H\} = 0$. Alternatively all tangent lines to C_1 lie in this fixed plane H, hence C_1 is a plane curve. Without loss of generality let us assume $S = \{F(x,y,z) = 0\}$, $C_1 = \{z = A(x,y) = 0\}$; then $F = F_1 A + F_2 z$ and $T_Q S = \{z = 0\}$ for any $Q \in C_1$, that is $F_x(Q) = F_y(Q) = 0$. But $F_x = F_{1,x}A + F_1 A_x + zF_{2,x}$, $F_y = F_{1,y}A + F_1 A_y + zF_{2,y}$, hence $F_1 A_x = F_1 A_y = 0$ along C_1. If $F_{1|C_1} = 0$ then $C_1 \cap \{F_1 = 0\} \cap \{F_2 = 0\} \neq \emptyset$ and S is singular in any point $Q \in C_1 \cap \{F_2 = 0\}$ because $F_z = F_{1,z}A + F_1 A_z + zF_{2,z} + F_2$, will vanish at Q; if instead $C_1 \not\subseteq \{F_1 = 0\}$ then $A_x = A_y = 0$ along C_1, that is C_1 is not reduced, a contradiction. The fact that these correspondences have null trace will be showed in the proof of Theorem (1.3). ■

We have now gathered enough properties of correspondences with null trace on a smooth surface $S \subset I\!\!P^3$.

Proof of Theorem (1.3): By Proposition (2.2) and Lemma (2.5) with $r = 3$, $k = d - 4$, we get $n \geq d - 2$ and if $(h-1)d + 1 \leq n \leq h(d-h-1) - 1$ for $2 \leq h \leq 4$, then $\{P_1, \ldots, P_n\} = \pi_2(\pi_1^{-1}(x))$ lies on a curve of degree $h - 1$. By Bezout's theorem this curve must have a component contained in S, hence S contains a 2-dimensional family of curves of degree at most three and this is not possible since S is smooth and of general type. If $n = d - 2$ then $\{P_1, \ldots, P_{d-2}\} = \pi_2(\pi_1^{-1}(x))$ lies on a line $L_x \subset I\!\!P^3$ for a general $x \in X$. This defines a rational map $f : X \to \mathbb{G}(1,3)$ whose image X' is a surface in the Grassmannian; let $\Gamma' = \overline{\{(L,P) : P \in L, P \neq P_i, i = 1, \ldots, d-2\}} \subset X' \times S$ together with its projections $\pi_1' : \Gamma' \to X'$, $\pi_2' : \Gamma' \to S$. We will suppose first that the two points (L,P) and (L,Q) of the general fiber $(\pi_1')^{-1}(L)$ are distinct. Observe that, by definition of Γ', π_1'

is dominant; on the other hand let us see that π_2' *is not* dominant on S: In fact we will see below that Γ' has null trace, hence if it were dominant on S it would be a correspondence of degree 2 on $X' \times S$, and by what we proved above it would follow $2 \geq d-2$. To see the trace map of Γ' we note that the diagram

$$\begin{array}{ccc} X' & \stackrel{h}{\longrightarrow} & S^{(2)} \times S^{(d-2)} \\ \downarrow g & & \swarrow h_1 \searrow h_2 \\ S^{(d)} & & S^{(2)} \qquad S^{(d-2)} \end{array}$$

induces a commutative diagram (by [Mu], Lemma 2)

$$\begin{array}{ccc} & H^{2,0}(S_0^{(d)}) & \\ \nearrow \delta_d & & \searrow g^* \\ H^{2,0}(S) & & H^{2,0}(X') \\ \searrow \delta_2 \times \delta_{d-2} & & \nearrow (h \circ h_1)^* + (h \circ h_2)^* \\ & H^{2,0}(S_0^{(2)}) \times H^{2,0}(S_0^{(d-2)}) & \end{array}$$

and hence $Trg = Tr(h \circ h_1) + Tr(h \circ h_2)$. But $h \circ h_2 = \gamma$ has null trace and also $Trg = 0$ since $g^* = 0$ because $\mathbb{G}(1,3)$ is rational; therefore $Tr(h \circ h_1) = 0$, that is Γ' has null trace. Note that this argument also shows that the correspondences (1.2) have null trace. In fact in that case Γ' does not dominate S by definition, hence it has null trace (because δ_2 factors through $H^{2,0}((\pi_2'(\Gamma'))^{(2)})$), therefore so does $\Gamma = \Gamma_{C_1,C_2}, \Gamma_{C_1}, \Gamma_{T_{C_1}S}$.

Now $\pi_1' : \Gamma' \to X'$ is $2:1$, hence Γ' is either irreducible or has two components. Let us suppose first $\Gamma' = W_1 \cup W_2$ union of two irreducible components of dimension 2 both dominating X' (otherwise we redefine Γ' removing the non dominating component); then $C_i = \pi_2'(W_i)$, $i = 1,2$ is a curve on S (if $dimC_i = 0$ then W_i does not dominate X') and the lines $L \in X'$ are of type $< P, Q >$, $P \in C_1$, $Q \in C_2$ (and therefore Γ is equivalent to the residual of the secants joining C_1 and C_2): For if $L \in X'$ is a line representing a general point, $(\pi_1')^{-1}(L) = \{(L,P),(L,Q)\}$, then $(L,P) \in W_1$ and $(L,Q) \in W_2$ because π_1' is $2:1$, and therefore $P \in C_1$, $Q \in C_2$ and $L =< P, Q >$; also X' is irreducible, hence $X' = Sec(C_1, C_2)$. If Γ' is irreducible the same argument shows that Γ is equivalent to the correspondence residual of the secants of $C_1 = \pi_2'(\Gamma')$. It remains to consider the case when no fiber of π_1' contains two distinct points. Then we can define $\Gamma_1' \subset \Gamma'$ such that $\Gamma_1' \to X'$ is $1:1$ and has null trace (a diagram like the above shows that $2Tr\gamma_1' = Tr\gamma' = 0$). So

$\pi_2' : \Gamma_1' \to S$ is not dominant, its image is an integral curve C_1 on S and Γ is equivalent to the residual of the tangents to S along C_1. ■

3. LINEAR SERIES ON CURVES ON A SMOOTH SURFACE IN $I\!P^3$

In this section we will exploit the Cayley-Bacharach property of general divisors on suitable linear series on a curve that moves on a smooth surface in $I\!P^3$. Let us record first the following basic fact.

Lemma (3.1). *Let $S \subset I\!P^3$ be a smooth surface, C an integral curve on S and g^r_n a base point free special linear series on C that is not composed with an involution if $r \geq 2$. Then the general divisor $D \in g^r_n$ satisfies the Cayley-Bacharach property with respect to $|\omega^0_C|$, where ω^0_C is the dualizing sheaf of C. Moreover if $|\mathcal{O}_C \otimes \mathcal{O}_S(C)|$ is base point free then D also satisfies the Cayley-Bacharach property with respect to $|K_S|$.*

Proof: If $r = 1$ and D is not $CB(\omega^0_C)$ there exists $P \in SuppD \cap C_{reg}$ and $\Delta \in |\omega^0_C|$ such that $\Delta \supset D - P$, $\Delta \not\ni P$. Therefore $h^0(\omega^0_C(-D+P)) > h^0(\omega^0_C(-D))$ and by Riemann-Roch (C is Gorenstein) $h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_C(D-P))$ hence P is a base point of the g^1_n. If $r \geq 2$ the assertion follows by a monodromy argument since the g^r_n defines a birational map (see for example [Ci], Proposition (1.4)). Now suppose that $|\mathcal{O}_C \otimes \mathcal{O}_S(C)|$ is base point free. Since $\omega^0_C = (\omega_S + C)_{|C}$ we have that D is $CB(\omega_S + C)$. For every $P \in SuppD$ and for every $E \in |K_S|$ such that $E \supset D - P$, take a curve $C' \subset S$, $C' \sim C$, $C' \not\ni P$; then $E + C' \supset D - P$ hence $E + C' \supset D$, therefore $P \in E$. ■

With the above lemma and Lemma (2.5) we can now give a proof of Theorems (1.4) and (1.5).

Proof of Theorem (1.4): By Lemma (3.1) the general divisor $D \in g^r_n$ is $CB(d-3)$ (the bound on n gives automatically that the g^r_n is special) hence Lemma (2.5) implies that there exists an integer h such that $h(d-h) \leq n \leq (h+1)(d-h-1) - 1$ and D lies on a curve of degree h. By Bezout's theorem we also have $n \leq hd$. ■

Proof of Theorem (1.5): By Lemma (3.1) the general divisor $D \in g^r_n$ is $CB(d-4)$ hence Lemma (2.5) implies that there exists an integer h such that $1 \leq h \leq 3$, $h(d-h-1) \leq n \leq (h+1)(d-h-2) - 1$ and D lies on a curve of degree h. Since S is smooth and of general type, it does not contain infinitely many curves of degree $h \leq 3$ hence we also have $n \leq hd$. ■

(3.2) *Remark* (i) Another application of the two Cayley-Bacharach lemmas is to complete intersection curves in $\mathbb{P}^3$. Let $C = F_a \cap F_b$ be a complete intersection curve with $5 \leq a \leq b$ and g^r_n a base point free special linear series on C that is not composed with an involution if $r \geq 2$. If $n \leq 4a + 4b - 21$ a proof similar to the above gives that the g^r_n exists if and only if there is an integer h such that $1 \leq h \leq 3$, $h(a + b - h - 1) \leq n \leq (h + 1)(a + b - h - 2) - 1$ and the surfaces F_a and F_b contain infinitely many curves of degree h that cut out the g^r_n.

(ii) Again if $C = F_a \cap F_b$ is a complete intersection curve with $2 \leq a \leq b$ then it has a unique g^3_{ab} that is base point free and not composed with an involution ([CL] for every a, b, [Ma] for $a = 3, 4$). In case $a = 3, b \geq 9$ and F_3 is smooth, this also follows from Lemmas (2.5) and (3.1). In fact the general divisor $D \in g^3_{ab}$ lies on a curve of degree 3 that is necessarily contained in F_3 (or F_3 contains a three dimensional family of curves of degree ≤ 3) and this is possible only if the curve is a plane cubic, that is D lies on a plane.

We now come to our application of Theorem (1.3) to coverings of an open subset of a smooth surface $S \subset \mathbb{P}^3$ with k-gonal curves. Before the proof let us give the following

(3.3) *Definition.* Two coverings of an open subset of a smooth surface $S \subset \mathbb{P}^3$ with k-gonal curves are *equivalent* if they give the same family of 0-cycles on S. As we will see in the proof of Corollary (1.7) this is the same as saying that the associated correspondences are equivalent.

Proof of Corollary (1.7): Let $\{C_t\}_{t \in \Delta}$ be a family of curves having a base point free g^1_k and covering an open subset of S. We will construct out of this covering a correspondence with null trace of degree k on S. For $t \in \Delta$ let us denote by H_t the Hilbert scheme of curves on S containing the point h_t representing C_t and by $H = H_\xi$ the Hilbert scheme associated to the generic point $\xi \in \Delta$. Note that $dimH \geq 1$ for if for every t we had $dimH_t = 0$, then $\bigcup_{t \in \Delta} H_t = \bigcup_{degC_t, g(C_t)} Hilb_{degC_t, g(C_t)}$ would be countable, hence an open subset of S would be covered by countably many curves. Now let $H_1 \subset H$ be a 1-dimensional family of curves containing h_ξ and $M_1 = \{(h_\lambda, P) : P \in C_\lambda, h_\lambda \in H_1\}$ the universal curve on H_1;

after a base change

$$\begin{array}{ccc} M & \to & M_1 \\ \downarrow \pi & & \downarrow p_1 \\ B & \to & H_1 \end{array}$$

we can assume to have a map $F : M \to I\!P^1 \times B$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{F} & I\!P^1 \times B \\ & \pi \searrow \quad \swarrow p_2 & \\ & B & \end{array}$$

commutes and for every $b \in B$ the induced map $F_b : M_b = \pi^{-1}(b) \to I\!P^1$ is the given g^1_k. Let $X = I\!P^1 \times B$ and $\Gamma = \{(x, Q) \in X \times S : x = (z, b), Q \in F_b^{-1}(z)\}$; then $\pi_1 : \Gamma \to X$ is dominant of degree k and also $\pi_2 : \Gamma \to S$ is dominant since for a general $P \in S$ there is a $b \in B$ such that $M_b \ni P$ and a point $h_\lambda \in H_1$ such that $C_\lambda \ni P$, so if m is the corresponding point in M we have $F(m) = (z, b)$ and hence $F_b(P) = z$, that is $P = \pi_2((z, b), P), ((z, b), P) \in \Gamma$. Moreover since X is ruled Γ has null trace, therefore by Theorem (1.3) we have $d - 2 \leq k \leq d$, $2d - 6 \leq k \leq 2d$ or $3d - 12 \leq k \leq 3d$ and if $k = d - 2$ then Γ must be equivalent to one of the correspondences (1.2). If Γ is equivalent to the residual Γ_{C_1,C_2} of the secants joining two curves C_1 and C_2 on S then $Sec(C_1, C_2)$ is birational to X, hence is covered by rational curves. We claim that in this case either C_1 or C_2 is a rational curve. First notice that $Sec(C_i) \not\supseteq Sec(C_1, C_2)$, $i = 1, 2$. For if $Sec(C_1, C_2) \subset Sec(C_1)$ then they are equal since the latter is irreducible; now if C_1 is contained in a plane N_1 then $C_2 \not\subset N_1$ (or Γ_{C_1,C_2} does not dominate S), hence there exists a line $L \subset N_1$ such that $L \cap C_2 = \emptyset$, so $L \in Sec(C_1)$ but $L \notin Sec(C_1, C_2)$; if C_1 is not a plane curve let P, Q be two general points of C_1 and $R \in C_2$ such that $< P, Q > \ni R$. Now if P_1 is another point in C_1 there is a point $Q_1 \in < P_1, R > \cap C_1$ since $< P_1, R > \in Sec(C_1, C_2) = Sec(C_1)$, hence the lines $< P, P_1 >$ and $< Q, Q_1 >$ lie in the plane spanned by $< P, R >$ and $< P_1, R >$; when P_1 approaches P along C_1, Q_1 approaches Q, hence the tangent lines to C_1 at P and Q meet and therefore C_1 is a plane curve (because P, Q are general points of C_1). Therefore $Sec(C_i) \not\supseteq Sec(C_1, C_2)$, $i = 1, 2$, in all cases and the natural rational map $\phi : C_1 \times C_2 \to Sec(C_1, C_2)$ given by $\phi(P, Q) =< P, Q >$ is generically one to one, hence birational. Since $Sec(C_1, C_2)$ is covered by rational curves there is a rational curve $C \subset C_1 \times C_2$ and if C is not dominant over C_1, then its image is a point $P \in C_1$ hence $C \subset \{P\} \times C_2$, i.e. C_2 is rational. In either case the covering is equivalent to the ones of (1.6) given by a rational curve. If instead Γ is equivalent to the residual Γ_{C_1} of the secants to a curve $C_1 \subset S$ then C_1 is non degenerate and again $Sec(C_1)$ is covered by rational curves. By the trisecant lemma the rational map $\psi : C_1^{(2)} \to Sec(C_1)$ defined by $\psi(P+Q) =< P, Q >$ is birational, hence also $C_1^{(2)}$ is covered by rational curves and therefore $g(C_1) \leq 1$: In fact if $g(C_1) \geq 2$ then Abel's map $C_1^{(2)} \to J(C_1)$ into the Jacobian of C_1 is birational; but this map contracts all the rational curves in $C_1^{(2)}$ and this is impossible since they cover it. Again in this case the covering is equivalent to one of (1.6). Finally if Γ is equivalent to $\Gamma_{T_{C_1}S}$ and $C_Q = \{\text{lines through } Q \text{ in } T_QS\}$ then $T_{C_1}S = \bigcup_{Q \in C_1} C_Q$ and $S = \bigcup_{Q \in C_1} \pi_2(\pi_1^{-1}(C_Q)) = \bigcup_{Q \in C_1} T_QS \cap S$, i.e. the covering is equivalent to the one given by tangent plane sections. ∎

Proof of Corollary (1.8): From [X], Theorem 1, it follows that every curve on a general S has geometric genus $g \geq 2$, hence we conclude with Corollary (1.7). ∎

Note added in proof: A remarkable generalization of Mumford's technique has been recently introduced by C. Voisin in [V]. Among other results she proves that every curve on a general S of degree $d \geq 6$ has geometric genus $g \geq 2$.

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