DOTTORATO DI RICERCA IN MATEMATICA

CICLO XXXII

PARTIALLY AMPLE DIVISORS ON PROJECTIVE SCHEMES AND ULRICH VECTOR BUNDLES ON PROJECTIVELY COHEN-MACAULAY SURFACES IN $\mathbb{P}^4$

MARCO D'AMBRA

A.A. 2019/2020

DOCENTE RELATORE: PROF. A. F. LOPEZ

DOCENTE COORDINATORE: PROF. A. BRAIDES

Acknowledgements

I owe my deepest gratitude to my advisor Prof. Angelo Felice Lopez, for being an indispensable point of reference in these years. Without his enthusiasm, encouragement, support and continuous optimism this thesis would hardly have been completed.

Apart from my advisor, I won't forget to express the gratitude to my tutors Prof. Giuseppe Pareschi and Prof. Flaminio Flamini. Especially to the first one, who guided me wisely during my first year of PhD and helped me get started with my research.

I would like to address my sincere gratitude to Prof. Alex Küronya, for his hospitality in Frankfurt, for his availability and for sharing insightful suggestions.

I am also pleased to say thank you to Prof. Aldo Conca, Prof. Rosa Miró-Roig and Prof. Sung Rak Choi for some helpful discussion.

Heartfelt thanks go to my collegues for their support, their friendship and stimulating the discussions.

Last but not the least, my deep and sincere gratitude to my family, for supporting me spiritually throughout writing this thesis and my life in general. Expecially I would like to thank my mother, the person with the greatest indirect contribution to this work, who has taught me love for mathematics and helped me get to the end of this path. I want to thank her, my father, as well as my brother, for their patience, help and constant encouragement.

Contents

Introduction

This Thesis is naturally divided in two parts. In the first part we will study partially ample line bundles on projective schemes, while in the second part we will deal with Ulrich vector bundles on projectively Cohen-Macaulay surfaces in $\mathbb{P}^4$.

Introduction to Chapter 1

The most important notion of positivity in algebraic geometry is undoubtedly the ampleness. There are several ways to characterize the ampleness of a line bundle on a projective scheme. For example, in terms of vanishing of higher cohomologies (by the Cartan-Serre-Grothendieck's theorem), the growth rate of the dimension of higher cohomologies (by the work of de Fernex, Küronya and Lazarsfeld, see [Kür06, dFKL07]), and in terms of base loci (by the work of Ein, Lazarsfeld, Mustaţă, Nakamaye and Popa, see [ELMNP06, ELMNP09]).
By weakening these conditions we obtain various notions of 'partial ampleness', that intuitively measure how much a line bundle is far from being ample.
Chapter 1 is dedicated to the study of these notions of partial ampleness.
As can be expected, they are different from each other and they share many important properties with traditional ampleness.
The idea of comparing these various notions comes from a work of Choi (see [Choi14]). We report that some errors invalidate the main results of that article (see Remark I.3.26).

Unless otherwise specified, X will be a projective noetherian scheme of finite type and of dimension n over the complex number field and q will be a non-negative integer.
With the term variety we mean an irreducible and reduced projective noetherian scheme of finite type and of dimension n over the complex number field.
With the term subvariety we mean an irreducible and reduced closed subscheme.

To formulate the results we will get we need to recall the notions of stable base loci and augmented base loci of ($\mathbb{Q}$-)line bundles.
Let L be a $\mathbb{Q}$-line bundle on X. The stable base locus of L is

$$\mathbf{B}(L) = \bigcap_{m} \mathrm{Bs}\, |mL|$$

where m runs over all positive integers such that mL is a line bundle.
Ein, Lazarsfeld, Mustaţă, Nakamaye and Popa introduced the following approximation of the stable base locus (see [ELMNP06]). The augmented base locus of a $\mathbb{Q}$-line bundle L on X is

$$\mathbf{B}_+(L) = \bigcap_{A} \mathbf{B}(L - A)$$

where A runs over all ample $\mathbb{Q}$-line bundles. Notice that $\mathbf{B}(L) \subset \mathbf{B}_+(L)$.

Let now L be a line bundle on X. Consider the following condition.

Condition. $\dim(\mathbf{B}_+(L)) \leq q$.

We know that L is ample if and only if $\dim(\mathbf{B}_+(L)) \leq 0$, thus this condition can be seen as a first natural way to generalize the notion of ampleness. Namely, if $\dim(\mathbf{B}_+(L)) = q$, we can think about L as a line bundle that is 'q steps far from being ample'.

Another natural way to generalize the notion of ampleness of a line bundle on a projective scheme is in terms of vanishing of higher cohomologies.

Definition. *Let L be a line bundle on X. L is q-ample if for all coherent sheaves $\mathcal{F}$ we have that $H^i(X, L^m \otimes \mathcal{F}) = 0$ for all $m \gg 0$ and $i > q$.*

Observe that for $q = 0$, by Cartan-Serre-Grothendieck's theorem, we recover the notion of ampleness. Moreover, if a line bundle is q-ample, then it is also $(q+1)$-ample. Thus, if a line bundle is q-ample but not $(q-1)$-ample, we can think about it as a line bundle that is 'q steps far from being ample'.
Notice that, by a result of Küronya (see [Kür13] or Theorem I.2.27) if $\dim(\mathbf{B}_+(L)) \leq q$, then L is q-ample. Thus this notion of partial ampleness is weakest than the first one.
The first time that the notion of q-ampleness appeared in literature was in a work of Andreotti and Grauert (see [AnGr62]). A few years laters, Sommese (see [Som78]) studied the q-ampleness of semiample line bundles. More recently it has been studied, among others, by Demailly, Peternell, Schneider, Totaro, Küronya and Ottem (see [DPS96, Tot13, Kür06, Kür13, Ott12]).
Due to the work of these authors, we have a fairly exhaustive overview of the behaviour of q-ample line bundles and of the properties that they share with ample line bundles.

Even if the definition is purely cohomological, there exist in literature some interesting results that help us to interpret geometrically the notion of q-ampleness (see [Som78, Tot13, Bro12]).
One of the most important ones is due to Sommese, who gave an interesting characterization of the q-ampleness of semiample line bundles in terms of the geometry of their semiample fibration (see [Som78] or Theorem I.2.21). Namely, he proved that a semiample line bundle L is q-ample if and only if for all sufficiently large and divisible integers m the dimension of the fibers of the map $\phi_{|mL|}$ is at most q.
We will provide a generalization of this result by relaxing the hypotheses on L.

Let X be a variety, let L be a line bundle on X such that $\mathrm{k}(X, L) \geq 0$ and let $m_0 > 0$ be an integer such that $\mathbf{B}(L) = \mathrm{Bs}\,|m_0L|$. Moreover, let $\pi : \hat{X} \to X$ be the blow-up of X along the ideal sheaf $\mathcal{I} = \mathcal{I}_{\mathbf{B}(L)}$ (that is the blow-up along the base ideal $\mathfrak{b}|m_0L|$) with exceptional divisor E.
Then we have a decomposition

$$(\bullet) \qquad \pi^*(m_0L) = M + E.$$

where M is a base-point-free line bundle on $\hat{X}$.
Since M is semiample, we can characterize the partial ampleness of M using Sommese's result.
Moreover, if L is assumed to be semiample, then $M \cong m_0L$.
It is then natural to compare the partial amplenesses of the line bundles L and M.
We will prove the following result.

Theorem A (see Theorem I.2.37). *Let X be a variety, let L be a line bundle on X and such that $\mathrm{k}(X, L) \geq 0$ and let $m_0 \geq 1$ be an integer such that $\mathbf{B}(L) = \mathrm{Bs}\,|m_0L|$. Moreover, let M be the line bundle of $(\bullet)$. Then:*

(i) *If $\dim(\mathbf{B}(L)) \leq q$, then M q-ample implies L q-ample.*
(ii) *If $\dim(\mathbf{B}(L)) \leq q - 1$, then M is q-ample if and only if L is q-ample.*

We also provide examples that show that the hypotheses on $\dim(\mathbf{B}(L))$ are sharp (see Remark I.2.38 or Examples I.4.8 and I.4.9).

Every line bundle is n-ample. Moreover, Totaro showed that, if X is a variety, then a line bundle L is $(n-1)$-ample if and only if $-L$ is not pseudoeffective (see [Tot13] or Theorem I.2.23). More recently Brown proved that, if X is a smooth variety, then a big line bundle L is $(n-2)$-ample if and only if for every subvariety Z of dimension $n-1$ we have that $-L_{|Z}$ is not pseudoeffective (see [Bro12] or Theorem I.2.24).
We will generalize these two results with the following theorem, which translates the q-ampleness of a line bundle L with augmented base locus $\mathbf{B}_+(L)$ of dimension at most $q+1$ in terms of the geometry of its restriction to the subvarieties of X.

Theorem B (see Theorem I.2.41). *Let L be a line bundle on X. If $\dim(\mathbf{B}_+(L)) \leq q+1$, then the following conditions are equivalent:*

(i) *L is q-ample.*
(ii) *For all subvarieties $Z \subset X$ of dimension $q+1$ we have that $-L_{|Z}$ is not pseudoeffective.*

The pullback of an ample line bundle under a blow-up along a subvariety of codimension $e \geq 2$ is never ample. Similarly the pullback of a q-ample line bundle in never q-ample. However, the following theorem shows that we can expect some partial ampleness.

Theorem C (see Theorem I.2.46). *Let X be a variety, let $Z \subset X$ be a smooth subvariety of codimension $e \geq 1$ contained in the smooth locus of X, let $\pi : \hat{X} \to X$ be the blow-up of X along Z and let L be a line bundle on X. If L is q-ample, then π^*L is $(q+e-1)$-ample.*

A third more sophisticated way to generalize the notion of ampleness comes from the work of de Fernex, Küronya and Lazarsfeld (see [Kür06, dFKL07]).
Let X be a variety, let L be a line bundle on X and let i be an integer such that $0 \leq i \leq n$. As in the definition of volume, Küronya defined the value of the i-th asymptotic cohomological function associated to X at L to be

$$\hat{h}^i(X, L) = \limsup_{m \to \infty} \frac{h^i(X, mL)}{m^n/n!}.$$

It turns out that the value of the asymptotic cohomological functions $\hat{h}^i(X, L)$ depends only on the numerical equivalence class of a line bundle L. Moreover it can be extended uniquely to a function

$$\hat{h}^i(X, -) : \mathrm{N}^1(X)_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$$

that is continuous and homogeneous of degree n (see [Kür06] or Theorem I.3.5).
The asymptotic cohomological functions are strictly related to the notion of ampleness. Indeed, de Fernex, Küronya and Lazarsfeld proved that a line bundle L is ample if and only if there exists a neighbourhood U of $[L]$ in $\mathrm{N}^1(X)_{\mathbb{R}}$ such that

$$\hat{h}^i(X, L') = 0$$

for all $[L'] \in U$ and $i > 0$ (see [dFKL07] or Theorem I.3.9).
By weakening this condition we obtain a third interesting generalization of the notion of ampleness (see [Choi14]).

Definition. *Let X be a variety and let L be an $\mathbb{R}$-line bundle L on X. L is asymptotically q-ample if there exists a neighbourhood U of $[L]$ in $\mathrm{N}^1(X)_{\mathbb{R}}$ such that*

$$\hat{h}^i(X, L') = 0$$

for all $[L'] \in U$ and $i > q$.

If a line bundle L is q-ample, then it is asymptotically q-ample (see Proposition I.3.14). Moreover Totaro conjectured that the converse also hold (see [Tot13] or Conjecture I.3.15).
While if $q = 0$ or $q = n - 1$, the conjecture is easily true (see Theorem I.2.23), in general it seems very hard to prove. This because we have not yet properly understood the behaviour of asymptotically q-ample line bundles (for example we don't know if the asymptotic q-ampleness is preserved under pullbacks via finite morphisms or even under restriction to closed subschemes).

By a result of Küronya (see [Kür06] or Proposition I.3.8), if X is a smooth variety and L is a big line bundle on X, then $\hat{h}^i(X, L) = 0$ for all $i > \dim(\mathbf{B}(L))$. We provide the following generalization of this fact.

Proposition D (see Proposition I.3.20 (and Remark I.3.21 to see how it generalzes Küronya's result)). *Let X be a variety and let L be a line bundle on X. If $L + \epsilon A$ is asymptotically q-ample for all $\epsilon > 0$ real number and A ample line bundle, then*

$$\hat{h}^i(X, L) = 0$$

for all $i > q$.

It is interesting to study the geometry of the cones of classes of divisors in the Néron-Severi vector space $\mathrm{N}^1(X)_{\mathbb{R}}$ that satisfy the partial ampleness conditions that we have considered. To do this, denote by $\mathrm{C}^q_+(X)$ (resp. $\mathrm{Amp}^q(X)$ or $\mathrm{Amp}^q_a(X)$) the cones of classes of $\mathbb{R}$-divisors $[D] \in \mathrm{N}^1(X)_{\mathbb{R}}$ such that $\dim(\mathbf{B}_+(D)) \leq q$ (resp. $[D]$ is q-ample or $[D]$ is asymptotically q-ample).
Moreover, denote by $\mathrm{C}^q_-(X)$ (resp. $\mathrm{Alm}^q(X)$ or $\mathrm{Alm}^q_a(X)$) the cones of classes of $\mathbb{R}$-divisors $[D] \in \mathrm{N}^1(X)_{\mathbb{R}}$ such that $\dim(\mathbf{B}_+(D + \epsilon A)) \leq q$ (resp. $[D + \epsilon A]$ is q-ample or $[D + \epsilon A]$ is asymptotically q-ample) for all real numbers $\epsilon > 0$ and ample divisors A (for the definition of augmented base locus for $\mathbb{R}$-divisors see [ELMNP06] or Definition I.1.12).
It turns out that $\mathrm{C}^q_-(X)$ is the cone of classes of $\mathbb{R}$-divisors $[D] \in \mathrm{N}^1(X)_{\mathbb{R}}$ such that $\dim(\mathbf{B}_-(D)) \leq q$, where $\mathbf{B}_-(D)$ is the restricted base locus of D (that we will introduce in Section I.1) and its dimension is interpreted as in Definition I.1.31.
The following result summarizes the most important properties of these cones.

Theorem E (See Remarks I.1.49, I.2.34 and I.3.23 and Theorems I.1.50, I.2.35 and I.3.24). *Let X be a variety. Then $\mathrm{C}^q_+(X)$, $\mathrm{Amp}^q(X)$, $\mathrm{Amp}^q_a(X)$, $\mathrm{C}^q_-(X)$, $\mathrm{Alm}^q(X)$ and $\mathrm{Alm}^q_a(X)$ are full-dimensional cones. $\mathrm{C}^q_+(X)$, $\mathrm{Amp}^q(X)$, $\mathrm{Amp}^q_a(X)$ are open, while $\mathrm{C}^q_+(X)$ and $\mathrm{C}^q_-(X)$ are convex. Moreover we have inclusions*

$$\mathrm{C}^q_+(X) \subset \mathrm{C}^{q+1}_+(X), \quad \mathrm{Amp}^q(X) \subset \mathrm{Amp}^{q+1}(X), \quad \mathrm{Amp}^q_a(X) \subset \mathrm{Amp}^{q+1}_a(X),$$
$$\mathrm{C}^q_-(X) \subset \mathrm{C}^{q+1}_-(X), \quad \mathrm{Alm}^q(X) \subset \mathrm{Alm}^{q+1}(X), \quad \mathrm{Alm}^q_a(X) \subset \mathrm{Alm}^{q+1}_a(X),$$
$$\mathrm{C}^q_+(X) \subset \mathrm{Amp}^q(X) \subset \mathrm{Amp}^q_a(X), \quad \mathrm{C}^q_-(X) \subset \mathrm{Alm}^q(X) \subset \mathrm{Alm}^q_a(X)$$

and the following generalized Kleiman's theorem:

(i) $\mathrm{C}^q_+(X) = \mathrm{int}(\mathrm{C}^q_-(X))$, $\mathrm{Amp}^q(X) = \mathrm{int}(\mathrm{Alm}^q(X))$, $\mathrm{Amp}^q_a(X) = \mathrm{int}(\mathrm{Alm}^q_a(X))$.
(ii) $\mathrm{C}^q_-(X) = \overline{\mathrm{C}^q_+(X)}$, $\mathrm{Alm}^q(X) = \overline{\mathrm{Amp}^q(X)}$, $\mathrm{Alm}^q_a(X) \subset \overline{\mathrm{Amp}^q_a(X)}$.

Let X be a variety and let L be a line bundle on X. Trying to compare the various generalizations of the notion of ampleness we consider the following conditions:

(1) $\dim(\mathbf{B}_+(L)) \leq q$.
(2+) For all $A_1, \ldots, A_q$ very ample divisors and for all general $E_i \in |A_i|$ we have that $L_{|E_1 \cap \cdots \cap E_q}$ is ample.
(2−) There exists $A_1, \ldots, A_q$ very ample divisors such that for all general $E_i \in |A_i|$ we have that $L_{|E_1 \cap \cdots \cap E_q}$ is ample.
(3) L is q-ample.

(4) For all subvarieties $Z \subset X$ of dimension $> q$ we have that $-L_{|Z}$ is not pseudoeffective.
(5) For all subvarieties $Z \subset X$ of dimension $> q$ we have that $L_{|Z}$ is not numerically 0.
(6) L is asymptotically q-ample.

Conditions (1), (3) and (6) are the previous notions of partial ampleness. Conditions (2+) and (2−) come from Küronya's work (see [Kür13]) and for $q = 0$ we define them as the ampleness of L. Condition (4) comes from Totaro's characterization of $(n-1)$-ample line bundles and Theorem B. Condition (5) comes from Sommese's work (see [Som78, Remark 1.4.1]).
Note that, if $q = 0$, then Conditions (1), (2+), (2−), (3) and (6) are equivalent to the ampleness of L.
Under the additional hypotheses of Theorem A, we may consider other conditions. Assume that $\mathrm{k}(X, L) \geq 0$, let $m_0 > 0$ be an integer such that $\mathbf{B}(L) = \mathrm{Bs}\,|m_0 L|$ and let M be the line bundle of (•). We also consider the following conditions:

(1∗) $\dim(\mathbf{B}_+(M)) \leq q$.
(3∗) M is q-ample.

We will provide examples that shows that the various notions differ from each other (with the only exception of (3) and (6)).
The following theorem, which is a summary of the results of Sommese, Totaro and Küronya and the original results of the author, shows the known implications between the various conditions.

Theorem F (see Theorem I.4.4). *Let X be a variety and let L be a line bundle on X. Then we have the following implications:*

$$(1) \Rightarrow (2+) \Rightarrow (2-) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5), \quad (3) \Rightarrow (6).$$

Moreover, let $m_0 > 0$ be an integer. If

(∗) $\mathrm{k}(X, L) \geq 0$ *and* $\mathbf{B}(L) = \mathrm{Bs}\,|m_0 L|$,

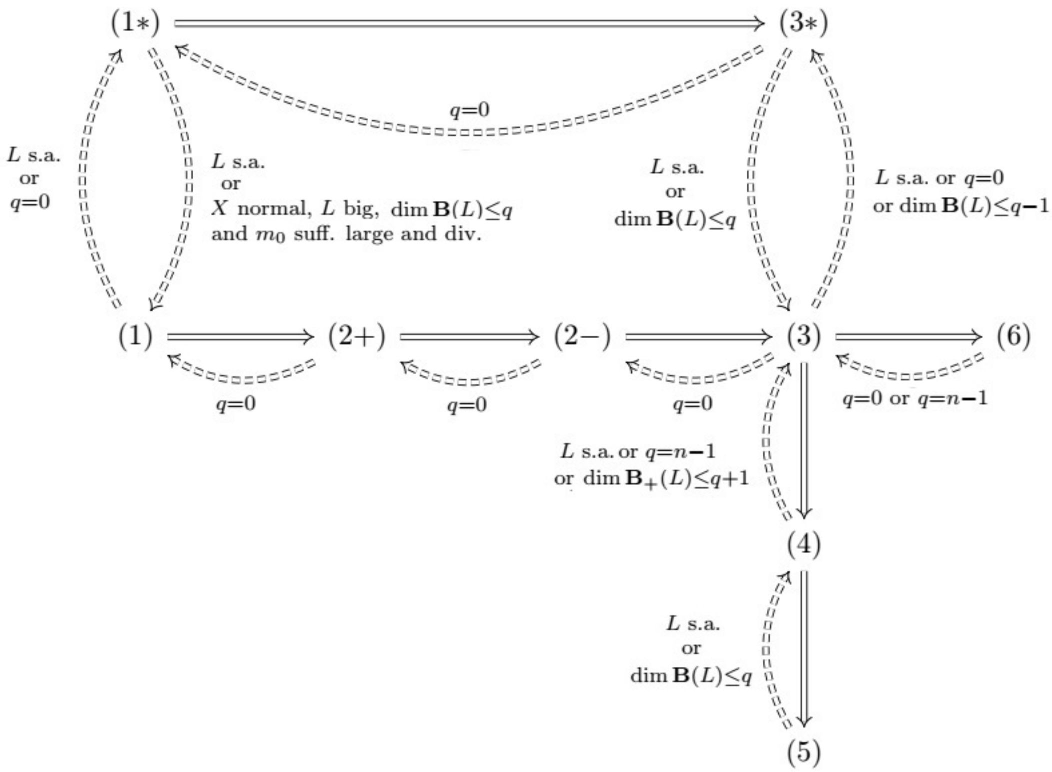
then (1∗) ⇒ (3∗).
Under additional hypotheses we have the following other implications:

(i) *Assume that* $\dim(\mathbf{B}(L)) \leq q$. *Then* (5) ⇒ (4). *Moreover, if* (∗) *is satisfied, then* (3∗) ⇒ (3). *Finally, if X is normal, L is big,* (∗) *is satisfied and m_0 is sufficiently large and divisible, then* (1∗) ⇒ (1).
(ii) *Assume that* $\dim(\mathbf{B}(L)) \leq q - 1$. *If* (∗) *is satisfied, then* (3) ⇒ (3∗).
(iii) *Assume that L is semiample. Then* (3) ⇔ (4) ⇔ (5) ⇒ (6). *Moreover, if* (∗) *is satisfied, then* (1) ⇔ (1∗) *and* (3) ⇔ (3∗).
(iv) *Assume that* $\dim(\mathbf{B}_+(L)) \leq q + 1$. *Then* (3) ⇔ (4).
(v) *Assume that* $q = 0$. *Then* (1) ⇔ (2+) ⇔ (2−) ⇔ (3) ⇔ (6). *Moreover if* (∗) *is satisfied, then* (1) ⇔ (2+) ⇔ (2−) ⇔ (3) ⇔ (6) ⇒ (1∗) ⇔ (3∗).
(vi) *Assume that* $q = n - 1$. *Then* (3) ⇔ (4) ⇔ (6).

The diagram in the next page summarizes the implications of Theorem F.

The chapter is structured as follows.
In Section I.1 we will recall the notion of stable, augmented and restricted base loci and all the properties that we will need. Moreover, we will study $\mathbb{R}$-divisors D such that $\dim(\mathbf{B}_+(D)) \leq q$, proving part of Theorem E.
In Section I.2 we will study q-ample line bundles and we will prove Theorems A, B and C and part of Theorem E.
In Section I.3 we will study asymptotically q-ample line bundles and we will prove Proposition D and the last part of Theorem E.

In Section I.4 we will compare the various notions of partial ampleness and we will prove Theorem F.



Here, to consider Conditions (1∗) and (3∗) and the related implications, we are implicitly assuming the hypothesis (∗).

Introduction to Chapter 2

Let $X \subset \mathbb{P}^n$ be a smooth complex variety and let H be the hyperplane section of X.
In algebraic geometry, one of the possible ways to describe the properties of the variety X is by looking at the behaviour of vector bundles on X. It is therefore natural to ask if there are vector bundles with many vanishing cohomology groups, such as the Ulrich vector bundles.
Recall that a vector bundle $\mathcal{E}$ on X is said to be Ulrich (with respect to H) if $h^i(X, \mathcal{E}(-pH)) = 0$ for all $0 \leq i \leq \dim(X)$ and $1 \leq p \leq \dim(X)$.
Ulrich bundles were first studied from an algebraic point of view in the 80's by Backelin, Brennan, Herzog and Ulrich among others (see [BHU87, HUB91]). The study in the algebro-geometric contest started in the new thousand with Eisenbud and Schreyer (see [ES03]). More recently they have been receiving wide attention among geometers. This is because it turned out that they have a lot of useful properties (for example they are globally generated, ACM, semistable and maximally generated, see [Bea18, CaHa12]). Moreover, their existence has many geometric consequences (such as determinantal representations and properties of the Chow form, see [Bea18]).
However it should be emphasized that, despite all the attention recently received, the basic question of whether Ulrich bundles exist or not is still widely open.

If C is a curve of degree d and genus g, then we know almost everything there is to know (see [Cos17]). Indeed, if $g = 0$, then the Ulrich bundles are those which split as a sum of line bundles of degree $d-1$. If $g=1$, then a vector bundle of rank r is Ulrich if and only if it is of the form $\bigoplus_{i=1}^{s}(\mathcal{F}_{r_i}\otimes L_i(H))$, where $s, r_i \geq 1$, $\sum r_i = r$, $\mathcal{F}_{r_i}$ is the unique indecomposable vector bundle of rank r_i, degree 0 and non-zero global sections, and L_i is a non-trivial line bundle of degree 0. Finally, if $g \geq 2$, there exist Ulrich bundles of every rank.

However, already in the case of a surface S, we still do not have an exhaustive overview on the subject. On one hand, we know that there exist Ulrich bundles with respect to mH for m sufficiently large (see [CoHu18]). On the other hand, if $m=1$ we have only partial results (the question has a positive answer for complete intersections [HUB91], K3 surfaces [Fae19, AFO17], abelian surfaces [Bea16a], bielliptic surfaces [Bea18], Enriques surfaces [Bea16b, Cas17, Cas17*, BoNu18], many regular surfaces [Cas17, Cas17*, Cas18], many irregular surfaces [Cas19, Cas19*, Lop19a, Lop19b] and many ruled surfaces [ACMR18, Bea18]).

If one wants to prove the existence of Ulrich bundles on a surface, the first case that is natural to consider is the case of codimension 1. In this context, the situation is clearer. Indeed Ulrich bundles of sufficiently large rank always exist (see [HUB91]). So the most interesting question left open is whether they exist of small rank.
Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d.
Ulrich line bundles are very rare. They exist if and only if S is linear determinantal (see [Bea00]). Thus if $1 \leq d \leq 3$ they always exist. On the other hand, if $d \geq 4$, then they don't exist on the general surface.
Even in the case of rank 2 we know a lot (note that this is the most attackable case, since the Ulrich bundles can be constructed with the well-known Serre method). For $1 \leq d \leq 4$, they always exist, while for $5 \leq d \leq 15$ they exist on the general surface. However, if $d \geq 16$ they don't exist on the general surface (see [Bea00, ES03]).
However for higher rank it is still an open problem.

In this chapter we will consider the case of a smooth irreducible non-degenerate projectively Cohen-Macaulay surface $S \subset \mathbb{P}^4$ (recall that S is projectively Cohen-Macaulay if its cone is a Cohen-Macaulay scheme). In such case we have a minimal free resolution of the form

$$(\bullet\bullet) \qquad 0 \to \bigoplus_{i=1}^{n+1}\mathcal{O}_{\mathbb{P}^4}(-m_i) \to \bigoplus_{j=1}^{n+2}\mathcal{O}_{\mathbb{P}^4}(-d_j) \to \mathcal{I}_{S/\mathbb{P}^4} \to 0$$

where $n, m_i, d_j \in \mathbb{Z}$ are integers such that $n \geq 0$, $m_i \geq 3$ for all $1 \leq i \leq n+1$ and $d_j \geq 2$ for all $1 \leq j \leq n+2$. We may also assume, without loss of generality, that $m_i \geq m_{i+1}$ for all $1 \leq i \leq n$ and $d_j \geq d_{j+1}$ for all $1 \leq j \leq n+1$.
Here the d_j's are the degrees of a minimal system of generators of the saturated graded homogeneous ideal I_S of S. Observe that, if $d_j = 1$ for some $1 \leq j \leq n+2$, then S is contained in a hyperplane. Thus the smooth irreducible projectively Cohen-Macaulay surfaces in $\mathbb{P}^4$ can be considered as a first natural generalization of the smooth irreducible surfaces in $\mathbb{P}^3$.
We denote by $S = (m_1, \dots, m_{n+1}; d_1, \dots, d_{n+2})$ a smooth irreducible non-degenerate PCM surface with minimal free resolution $(\bullet\bullet)$.
Using the machinery of the Eagon-Northcott type complexes and the results of Casnati for regular surfaces (see Theorems II.3.2 and II.3.3), we will prove the following result.

Theorem G (see Theorem II.3.6). *Let $S \subset \mathbb{P}^4$ be a smooth irreducible PCM surface in the following list:*

(1) $S = (4; 2, 2)$.

(2) $S = (3,3;2,2,2)$.
(3) $S = (4,4;3,3,2)$.
(4) $S = (4,4,4;3,3,3,3)$.
(5) $S = (5;3,2)$.
(6) $S = (5,4;3,3,3)$.
(7) $S = (5,5;4,3,3)$.
(8) $S = (5,5;4,4,2)$.
(9) $S = (5,5,5;4,4,4,3)$.
(10) $S = (5,5,5,5;4,4,4,4,4)$.

Equivalently, let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (••) such that $m_i \leq 5$ for all $1 \leq i \leq n+1$.
Then there exists a vector bundle $\mathcal{E}$ of rank 2 on S that is simple, Ulrich with respect to H and such that $c_1(\mathcal{E}) = 3H + K_S$.
Moreover, if S is not of type (2) and (10), then $\mathcal{E}$ is also μ-stable with respect to H.

In order to use Casnati's results we will first classify the smooth irreducible non-degenerate projectively Cohen-Macaulay surfaces with $p_g(S) = 0$, and those with $h^0(S, 2K_S - H) = 0$. Namely, we will prove the following result.

Proposition H (see Propositions II.2.1, II.2.2 and II.2.4). *Let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate projectively Cohen-Macaulay surface with minimal free resolution (••).*
Then the following conditions are equivalent:

(i) $p_g(S) = 0$.
(ii) $m_i \leq 4$ *for all* $1 \leq i \leq n+1$.

If they are satisfied, then S is one of the surfaces $(1), \dots, (4)$ in the above Theorem.
Moreover the following conditions are equivalent:

(iii) $h^0(S, 2K_S - H) = 0$.
(iv) $m_i \leq 5$ *for all* $1 \leq i \leq n+1$.

If they are satisfied, then S is one of the surfaces $(1), \dots, (10)$ in the above Theorem.

The chapter is structured as follows.
In Section II.1 we will recall the notion of projectively Cohen-Macaulay surfaces and some of their basic properties.
In Section II.2 we will classify the PCM surfaces with $p_g(S) = 0$ and $h^0(S, 2K_S - H) = 0$, proving Proposition H.
In Section II.3 we will prove Theorem G. We will also provide two other different proofs of (part of) the same result. The first will use the residual surface in a complete intersection of two hypersurfaces in $\mathbb{P}^4$. The second will use Brill-Noether theory.

CHAPTER 1

Partially ample divisors and base loci

Notation

Throughout Chapter 1, unless otherwise specified, q will be a non-negative integer and will be used to denote the index of 'partial ampleness'.

Throughout Sections I.1 and I.2, unless otherwise specified, X will be a projective noetherian scheme of finite type and of dimension n over the complex number field.

With the term variety we mean an irreducible and reduced projective noetherian scheme of finite type and of dimension n over the complex number field.

With the term subvariety we mean an irreducible and reduced closed subscheme.

Throughout Sections I.3 and I.4 we will assume X to be a variety.

We denote by $\mathrm{Pic}(X)$ (resp. $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathrm{Pic}(X)_{\mathbb{R}} = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$) the group of line bundles (resp. $\mathbb{Q}$-line bundles, $\mathbb{R}$-line bundles) on X.

We denote by $\mathrm{Div}(X)$ (resp. $\mathrm{Div}(X)_{\mathbb{Q}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathrm{Div}(X)_{\mathbb{R}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$) the group of Cartier (resp. $\mathbb{Q}$-Cartier, $\mathbb{R}$-Cartier) divisors on X.

With the term divisor (resp. $\mathbb{Q}$-divisor, $\mathbb{R}$-divisor) we mean a Cartier (resp. $\mathbb{Q}$-Cartier, $\mathbb{R}$-Cartier) divisor.

We denote by $\mathrm{N}^1(X)_{\mathbb{Z}}$ (resp. $\mathrm{N}^1(X)_{\mathbb{Q}} = \mathrm{N}^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{N}^1(X)_{\mathbb{R}} = \mathrm{N}^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$) the Néron-Severi group of divisors (resp. $\mathbb{Q}$-divisors, $\mathbb{R}$-divisors) on X.

We will often use additive notation for the product of $\mathbb{R}$-line bundles.

I.1. Preliminaries on base loci

NOTATION. *Throughout Section I.1, unless otherwise specified, X will be a projective noetherian scheme of finite type and of dimension n over the complex number field.*

In this section we recall the notions of stable, augmented and restricted base loci of divisors on a projective scheme and some of their basic properties. Since it is not our idea to give an exhaustive account on the argument, we present only the results that we will need later. We refer to [ELMNP06] and [Bir17] for a more complete discussion of the subject. We point out that, although in [ELMNP06] the authors work with normal varieties, the results we will cite are often valid under less restrictive hypothesis.

I.1.1. Stable, augmented and restricted base loci.

We recall the following well-known definition.

DEFINITION I.1.1. Let D be a divisor on X. The *base locus of D* is

$$\operatorname{Bs}|D| = \bigcap_{s \in H^0(X,D)} Z(s).$$

REMARK I.1.2. We consider the base locus $\operatorname{Bs}|D|$ with the reduced induced structure of closed subscheme of X.

REMARK I.1.3. Since X is projective, then the morphism $\mathcal{O}_X : \operatorname{Div}(X) \to \operatorname{Pic}(X)$ is surjective. Thus the previous definition extends to line bundles.

DEFINITION I.1.4. Let D be a $\mathbb{Q}$-divisor on X. The *stable base locus of D* is

$$\mathbf{B}(D) = \bigcap_m \operatorname{Bs}|mD|$$

where m runs over all positive integers such that mD is a divisor.

REMARK I.1.5. Unless otherwise specified, that is in Subsection I.2.2 and (part of) Section I.4, we consider the stable base locus $\mathbf{B}(D)$ with the reduced induced structure of closed subscheme of X.

REMARK I.1.6. Thanks to Proposition I.1.8 (i), the previous definition extends to $\mathbb{Q}$-line bundles via the surjective morphism $\mathcal{O}_X : \operatorname{Div}(X)_{\mathbb{Q}} \to \operatorname{Pic}(X)_{\mathbb{Q}}$.

The name 'stable base locus' is justified by the fact that, as the following lemma explains, the base loci $\operatorname{Bs}|mD|$ stabilize to $\mathbf{B}(D)$ for sufficiently large and divisible positive integers m.

LEMMA I.1.7 ([Laz04a, Proposition 2.1.21]). *Let D be a $\mathbb{Q}$-divisor on X. Then the stable base locus is the unique minimal element of the family of algebraic sets*

$$\{\operatorname{Bs}|mD| \ : \ m \geq 1, \ mD \text{ is a divisor}\}.$$

Moreover, there exists an integer $m_0 \geq 1$ such that $m_0 D$ is a divisor and

$$\mathbf{B}(D) = \operatorname{Bs}|mm_0 D|$$

for all integers $m \geq 1$.

PROPOSITION I.1.8. *Let D and D' be two $\mathbb{Q}$-divisors on X. Then:*

(i) *If $D \sim_{\mathbb{Q}} D'$, then $\mathbf{B}(D) = \mathbf{B}(D')$.*
(ii) *$\mathbf{B}(D) = \mathbf{B}(cD)$ for all $c > 0$ rational numbers.*
(iii) *$\mathbf{B}(D + D') \subset \mathbf{B}(D) \cup \mathbf{B}(D')$.*

Proof. All statements follow immediately by Lemma I.1.7. □

Definition I.1.9. Let D be an $\mathbb{R}$-divisor on X.
D is *effective* if $D = \sum_i e_i E_i$, with $e_i \geq 0$ real numbers and E_i effective divisors.
D is *big* if $D \sim_{\mathbb{R}} A + E$, where A is an ample $\mathbb{R}$-divisor and E is an effective $\mathbb{R}$-divisor.
D is *pseudoeffective* if $D + A$ is big for all ample $\mathbb{R}$-divisors A.

Remark I.1.10. It's easy to see that an $\mathbb{R}$-divisor D is big if and only if $D \sim_{\mathbb{R}} A' + E'$, where A' is an arbitrarily small ample $\mathbb{R}$-divisor and E' is an effective $\mathbb{Q}$-divisor. To prove this, suppose that D is big, that is $D \sim_{\mathbb{R}} A + E$, where A is an ample $\mathbb{R}$-divisor and E is an effective $\mathbb{R}$-divisor. If $E = 0$, then $A \sim_{\mathbb{R}} A' + E'$, with A' arbitrarily small ample $\mathbb{R}$-divisor and E' effective $\mathbb{Q}$-divisor and we conclude. Otherwise write $E = \sum_i e_i E_i$, where $e_i > 0$ are real numbers and E_i are effective divisors. Since the ampleness is an open property, then there exist sufficiently small real numbers $\delta_i > 0$ such that $e_i - \delta_i > 0$ are rationals and $A + \sum_i \delta_i E_i$ is an ample $\mathbb{R}$-divisor. Write $A + \sum_i \delta_i E_i = A' + A''$, with A' arbitrarily small ample $\mathbb{R}$-divisor and A'' ample $\mathbb{Q}$-divisor. Then $A'' + \sum_i (e_i - \delta_i) E_i \sim_{\mathbb{R}} E'$, where E' is an effective $\mathbb{Q}$-divisor and $D \sim_{\mathbb{R}} A + E \sim_{\mathbb{R}} A' + E'$.

The following lemma will be useful in the future.

Lemma I.1.11 ([Cac08, Lemma 5.3]). *Let D be an $\mathbb{R}$-divisor on X and let $||\cdot||$ be a norm on $\mathrm{N}^1(X)_{\mathbb{R}}$. Then there exists a sequence $\{A_m\}_{m\geq 1}$ of ample $\mathbb{R}$-divisors such that:*
(i) $||A_m|| \to 0$.
(ii) *$D + A_m$ is a $\mathbb{Q}$-divisor for all $m \geq 1$.*
(iii) *$A_m - A_{m+1}$ is ample for all $m \geq 1$.*

Proof. Let A be an ample $\mathbb{R}$-divisor on X. Then there exists a real number $\epsilon_A > 0$ such that $D_{\epsilon_A}([A])$, the ball of radius ϵ_A centered in $[A]$, is contained in the ample cone $\mathrm{Amp}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$. Then
$$D_{\epsilon_A}([D + A]) \cap \mathrm{N}^1(X)_{\mathbb{Q}} \neq \emptyset,$$
whence there exists a $\mathbb{Q}$-divisor D' such that $||D' - D - A|| < \epsilon_A$.
It follows that $[D' - D] \subset D_{\epsilon_A}([A]) \subset \mathrm{Amp}(X)$. Since by [Laz04a, Proposition 1.3.13] the ampleness is a numerical property, then $A' = D' - D$ is ample.
Thus we can write
$$A' = \sum_{i=1}^{s} c_i A'_i,$$
where $c_i > 0$ are real numbers and A'_i are ample divisors. For all integers $m \geq 1$ consider the divisor
$$A_m = A' - \sum_{i=1}^{s} c_{i,m} A'_i = \sum_{i=1}^{s} (c_i - c_{i,m}) A'_i,$$
where, for all i, $\{c_{i,m}\}_{m\geq 1}$ is a sequence of rational numbers such that $c_i = \lim_m c_{i,m}$ and $c_{i,m} < c_{i,m+1} < c_i$ for all $m \geq 1$.
Take $m \geq 1$ and $1 \leq i \leq s$. Since $c_{i,m} < c_i$ we get that A_m is ample, while, since $c_i = \lim_m c_{i,m}$ we have that $||A_m|| \to 0$. Moreover, since D' is a $\mathbb{Q}$-divisor and $c_{i,m}$ is rational, then
$$D + A_m = D' - A' + A_m = D' - \sum_{i=1}^{s} c_{i,m} A'_i$$
is a $\mathbb{Q}$-divisor. Finally, since $c_{i,m} < c_{i,m+1}$, we get that
$$A_m - A_{m+1} = \sum_{i=1}^{s} (c_{i,m+1} - c_{i,m}) A'_i$$

is ample. $\square$

The stable base locus turns out to be an interesting tool that gives a lot of information about linear series of divisors. However it has many pathologies that make its study complicated (e.g. it is not a numerical invariant, see [ELMNP06, Example 1.1]).
In an attempt to remedy to these problems, Ein, Lazarsfeld, Mustaţă, Nakamaye and Popa introduced the following approximation of the stable base locus.

DEFINITION I.1.12. Let D be an $\mathbb{R}$-divisor on X. The *augmented base locus of D* is

$$\mathbf{B}_+(D) = \bigcap_A \mathbf{B}(D - A)$$

where A runs over all ample $\mathbb{R}$-divisors such that $D - A$ is a $\mathbb{Q}$-divisor.

REMARK I.1.13. If X is a variety, then

$$\mathbf{B}_+(D) = \bigcap_{D=A+E} \operatorname{Supp}(E)$$

where the intersection is taken over all decompositions $D = A + E$, where A and E are $\mathbb{R}$-divisors such that A is ample and E is effective (see [ELMNP06, Definition 1.2, Remark 1.3]).

REMARK I.1.14. We consider the augmented base locus $\mathbf{B}_+(D)$ with the reduced induced structure of closed subscheme of X.

REMARK I.1.15. Thanks to Proposition I.1.19 (i) the previous definition extends to $\mathbb{R}$-line bundles via the surjective morphism $\mathcal{O}_X : \operatorname{Div}(X)_{\mathbb{R}} \to \operatorname{Pic}(X)_{\mathbb{R}}$.

REMARK I.1.16. The name 'augmented base locus' is justified by the fact that, if D is a $\mathbb{Q}$-divisor, then for all ample $\mathbb{Q}$-divisors A we have that $\mathbf{B}(D) \subset \mathbf{B}(D-A)$. Hence we get an inclusion

$$\mathbf{B}(D) \subset \mathbf{B}_+(D).$$

The next result shows that the augmented base locus can be realized as a stable base locus.

PROPOSITION I.1.17 ([ELMNP06, Proposition 1.5, Corollary 1.6]). *Let D be an $\mathbb{R}$-divisor on X and let $||\cdot||$ be a norm on $\mathrm{N}^1(X)_{\mathbb{R}}$. Then there exists a constant $\epsilon_D > 0$ such that:*

(i) *For every ample $\mathbb{R}$-divisor A such that $||A|| < \epsilon_D$ we get*

$$\mathbf{B}_+(D) = \mathbf{B}_+(D - A).$$

(ii) *For every ample $\mathbb{R}$-divisor A such that $||A|| < \epsilon_D$ and $D - A$ is a $\mathbb{Q}$-divisor, then*

$$\mathbf{B}_+(D) = \mathbf{B}(D - A).$$

(iii) *For every $\mathbb{R}$-divisor D' such that $||D'|| < \epsilon_D$ we get*

$$\mathbf{B}_+(D - D') \subset \mathbf{B}_+(D).$$

(iv) *For every $\mathbb{R}$-divisor D' such that $||D'|| < \epsilon_D$ and $D - D'$ is a $\mathbb{Q}$-divisor, then*

$$\mathbf{B}(D - D') \subset \mathbf{B}_+(D).$$

The next result shows that the augmented base locus can be realized as a base locus.

THEOREM I.1.18 ([Bir17, (proof of) Theorem 1.4], [Lop15, Theorem 2.1]). *Let D be a $\mathbb{Q}$-divisor and let A be a very ample divisor on X. Then there exists an integer $m_0 \geq 1$ such that $m_0 D$ is a divisor and*

$$\mathbf{B}_+(D) = \mathbf{B}(mm_0D - A) = \operatorname{Bs}|mm_0D - A|$$

for all integers $m \geq 1$.

Proposition I.1.19 ([ELMNP06, Proposition 1.4, Examples 1.8 and 1.9]). *Let D and D' be two $\mathbb{R}$-divisors on X. We have:*

(i) *If D and D' are numerically equivalent, then $\mathbf{B}_+(D) = \mathbf{B}_+(D')$.*
(ii) *$\mathbf{B}_+(D) = \mathbf{B}_+(cD)$ for all $c > 0$ real numbers.*
(iii) *$\mathbf{B}_+(D + D') \subset \mathbf{B}_+(D) \cup \mathbf{B}_+(D')$.*

Proof. Let $|| \cdot ||$ be a norm on $\mathrm{N}^1(X)_{\mathbb{R}}$ and let $\epsilon_D > 0$ be the constant of Proposition I.1.17. To prove (i) it is enough to show that $\mathbf{B}_+(D') \subset \mathbf{B}_+(D)$. Since $D \equiv D'$, then $||D - D'|| = 0$, whence by Proposition I.1.17 (iii) we get that $\mathbf{B}_+(D') = \mathbf{B}_+(D - (D - D')) \subset \mathbf{B}_+(D)$ and we conclude.
To prove (ii) it is enough to show that $\mathbf{B}_+(D) \subset \mathbf{B}_+(cD)$ for all $\mathbb{R}$-divisors D and real numbers $c > 0$. Indeed we have that $\mathbf{B}_+(D) \subset \mathbf{B}_+(cD) \subset \mathbf{B}_+(\frac{1}{c}(cD)) = \mathbf{B}_+(D)$.
We first claim that $\mathbf{B}_+(D) \subset \mathbf{B}_+(cD)$ for all $\mathbb{Q}$-divisors D and rational numbers $c > 0$.
To see this let D be a $\mathbb{Q}$-divisor, let $c > 0$ be a rational number, let $\epsilon_{cD} > 0$ be the constant of Proposition I.1.17 and let A be an ample $\mathbb{Q}$-divisor such that $c||A|| < \epsilon_{cD}$.
Since $c > 0$, then cA is ample, whence by Proposition I.1.8 (ii) and Proposition I.1.17 (ii) we get that

$$\mathbf{B}_+(D) \subset \mathbf{B}(D - A) = \mathbf{B}(c(D - A)) = \mathbf{B}_+(cD)$$

and the claim follows.
We prove now that $\mathbf{B}_+(D) \subset \mathbf{B}_+(cD)$ for all $\mathbb{Q}$-divisors D and real numbers $c > 0$.
To see this let D be a $\mathbb{Q}$-divisor, let $c > 0$ be a real number, let $\epsilon_{cD} > 0$ be the constant of Proposition I.1.17 and let $c' > 0$ be a rational number such that $|c - c'|||D|| < \epsilon_{cD}$. By the result on rational numbers and Proposition I.1.17 (iii) we get that

$$\mathbf{B}_+(D) \subset \mathbf{B}_+(c'D) = \mathbf{B}_+(cD + (c' - c)D) \subset \mathbf{B}_+(cD).$$

For the general result let D be an $\mathbb{R}$-divisor, let $c > 0$ be a real number, let $\epsilon_{cD} > 0$ be the constant of Proposition I.1.17 and let A be an ample $\mathbb{R}$-divisor such that $D - A$ is a $\mathbb{Q}$-divisor and $c||A|| < \epsilon_{cD}$. By the result on $\mathbf{Q}$-divisors, Remark I.1.16 and Proposition I.1.17 (i) we get that

$$\mathbf{B}_+(D) \subset \mathbf{B}(D - A) \subset \mathbf{B}_+(D - A) \subset \mathbf{B}_+(c(D - A)) = \mathbf{B}_+(cD)$$

and we conclude (ii).
To prove (iii) let $\epsilon_{D'} > 0$ be the constant of Proposition I.1.17.
We first show the result for two $\mathbb{Q}$-divisors D and D'. Take A ample $\mathbb{Q}$-divisor such that $||A|| < \min\{2\epsilon_D, 2\epsilon_{D'}\}$. Then by Proposition I.1.8 (iii) and Proposition I.1.17 (ii) we get that

$$\mathbf{B}_+(D + D') \subset \mathbf{B}(D + D' - A) = \mathbf{B}((D - \frac{1}{2}A) + (D' - \frac{1}{2}A)) \subset$$

$$\subset \mathbf{B}(D - \frac{1}{2}A) \cup \mathbf{B}(D' - \frac{1}{2}A) = \mathbf{B}_+(D) \cup \mathbf{B}_+(D').$$

For the general result, let D and D' be two $\mathbb{R}$-divisors. As in Lemma I.1.11 there exists ample $\mathbb{R}$-divisors A and A' such that $D - A$ and $D' - A'$ are $\mathbb{Q}$-divisors and $||A|| < \epsilon_D$, $||A'|| < \epsilon_{D'}$. Then by the result for $\mathbb{Q}$-divisors and by Proposition I.1.17 (i) we get that

$$\mathbf{B}_+(D + D') \subset \mathbf{B}((D + D') - (A + A')) = \mathbf{B}((D - A) + (D' - A')) \subset$$

$$\subset \mathbf{B}_+((D - A) + (D' - A')) \subset \mathbf{B}_+(D - A) \cup \mathbf{B}_+(D' - A') = \mathbf{B}_+(D) \cup \mathbf{B}_+(D').$$

□

The following theorems give two different interpretations of the augmented base locus.

THEOREM I.1.20 (Augmented base locus and Exceptional locus; [Bir17, Theorem 1.4]). *Let X be a scheme of dimension $n \geq 1$ and let D be a nef $\mathbb{R}$-divisor on X. Then*

$$\mathbf{B}_+(D) = \bigcup_Z Z$$

where the union is taken over all subvarieties $Z \subset X$ of positive dimension such that $\mathcal{O}_X(D)_{|Z}$ is not big.

THEOREM I.1.21 (Augmented base locus and Null locus; [Nak00, Theorem 0.3]). *Let X be a variety of dimension $n \geq 1$ and let D be a nef $\mathbb{R}$-divisor on X. Then*

$$\mathbf{B}_+(D) = \bigcup_Z Z$$

where the union is taken over all subvarieties $Z \subset X$ of positive dimension such that $D^{\dim(Z)} \cdot Z = 0$.

To prove the last theorem we need the following generalization of [Laz04a, Theorem 2.2.15]. Although this is probably well-known, for completeness we present a proof.

LEMMA I.1.22. *Let X be a variety of dimension $n \geq 1$ and let D and D' be two nef $\mathbb{R}$-divisors on X such that $D^n > nD^{n-1} \cdot D'$. Then $D - D'$ is big.*

PROOF. If D and D' are $\mathbb{Q}$-divisors the result is [Laz04a, Theorem 2.2.15].
For the general case, observe first that we may assume that D and D' are ample. Indeed, if A is an ample divisor, then for all real numbers $0 < \epsilon \ll 1$ we have that

$$(D + \epsilon A)^n > n(D + \epsilon A)^{n-1} \cdot (D' + \epsilon A)$$

and

$$D - D' = (D + \epsilon A) - (D' + \epsilon A),$$

where $D + \epsilon A$ and $D' + \epsilon A$ are ample by [Laz04a, Corollary 1.4.10].
Thus we can write

$$D = \sum_{i=1}^{s} c_i A_i, \quad D' = \sum_{j=1}^{s'} c'_j A'_j,$$

where $c_i > 0$ and $c'_j > 0$ are real numbers and A_i, A'_j are ample divisors. Consider for all integers $m \geq 1$ the divisors

$$D_m = \sum_{i=1}^{s} c_{i,m} A_i, \quad D'_m = \sum_{j=1}^{s'} c'_{j,m} A'_j$$

where, for all i and j, $\{c_{i,m}\}_{m \geq 1}$ and $\{c'_{j,m}\}_{m \geq 1}$ are sequences of rational numbers such that $c_i = \lim_m c_{i,m}$, $c'_j = \lim_m c'_{j,m}$, $c_{i,m} < c_i$ and $c'_{j,m} > c'_j$ for all $m \geq 1$.
Hence

$$[D] = \lim_{m \to \infty} [D_m], \quad [D'] = \lim_{m \to \infty} [D'_m]$$

in the Néron-Severi vector space $\mathrm{N}^1(X)_{\mathbb{R}}$.
For m sufficiently large $D_m^n > nD_m^{n-1} \cdot D'_m$, whence by the result for $\mathbb{Q}$-divisors $D_m - D'_m$ is big. Moreover, since $c_{i,m} < c_i$ for all i and m and $c'_{j,m} > c'_j$ for all j and m, then $(D - D_m) - (D' - D'_m)$ is $\mathbb{R}$-linearly equivalent to an effective $\mathbb{R}$-divisor for all $m \geq 1$. It follows that

$$D - D' = (D_m - D'_m) + (D - D_m) - (D' - D'_m)$$

is big. ∎

COROLLARY I.1.23. *Let X be a variety of dimension $n \geq 1$ and let D be a nef $\mathbb{R}$-divisor on X. Then D is big if and only if $D^n > 0$.*

Proof. If $D^n > 0$, by Lemma I.1.22 with $D' = 0$, we get that D is big.
If D is big, then we can write

$$D \sim_{\mathbb{R}} \alpha A + E,$$

where $\alpha > 0$ is a real number, A is a very ample divisor and E is an effective $\mathbb{R}$-divisor.
Since D is nef, then $D^{n-1} \cdot E \geq 0$. Moreover, since A is very ample, then by induction on n we have $D^{n-1} \cdot A = (D_{|A})^{n-1} > 0$. It follows that

$$D^n = D^{n-1} \cdot (\alpha A + E) > 0.$$

□

Proof of Theorem I.1.21. Consider a subvariety $Z \subset X$ of positive dimension. Since D is nef, we obtain that $\mathcal{O}_X(D)_{|Z}$ is nef and $D^{\dim(Z)} \cdot Z \geq 0$. Then by Corollary I.1.23 we get that $\mathcal{O}_X(D)_{|Z}$ is not big if and only if $D^{\dim(Z)} \cdot Z = 0$. Thus by Theorem I.1.20 we get the desired result. □

The following result illustrates the behaviour of base loci under restriction to a closed subscheme.

Proposition I.1.24 (Base loci of restriction; [Kür06, Corollary 2.3]). *Let $Z \subset X$ be a closed subscheme and let L be an $\mathbb{R}$-line bundle on X. We have:*

(i) *If L is a $\mathbb{Q}$-line bundle, then $\mathbf{B}(L_{|Z}) \subset \mathbf{B}(L) \cap Z$.*
(ii) *$\mathbf{B}_+(L_{|Z}) \subset \mathbf{B}_+(L) \cap Z$.*

Proof. Since $\operatorname{Bs}|L_{|Z}| \subset \operatorname{Bs}|L| \cap Z$ for all line bundles L on X, then by definition

$$\mathbf{B}(L_{|Z}) = \bigcap_{m \geq 1} \operatorname{Bs}|mL_{|Z}| \subset \bigcap_{m \geq 1} \operatorname{Bs}|mL| \cap Z = \mathbf{B}(L) \cap Z.$$

This immediately extends to $\mathbb{Q}$-line bundles, thus we get (i).
Consider now an $\mathbb{R}$-line bundle L on X. Since the restriciton to Z of every ample $\mathbb{R}$-line bundle on X is ample, then by (i) we have that

$$\begin{aligned}\mathbf{B}_+(L_{|Z}) &= \bigcap_{A' \text{ ample } \mathbb{R}\text{-l.b. on } Z \,:\, L_{|Z} - A' \ \mathbb{Q}\text{-l.b.}} \mathbf{B}(L_{|Z} - A') \subset \\ &\subset \bigcap_{A \text{ ample } \mathbb{R}\text{-l.b. on } X \,:\, L - A \ \mathbb{Q}\text{-l.b.}} \mathbf{B}((L-A)_{|Z}) \subset \\ &\subset \bigcap_{A \text{ ample } \mathbb{R}\text{-l.b. on } X \,:\, L - A \ \mathbb{Q}\text{-l.b.}} \mathbf{B}(L-A) \cap Z = \mathbf{B}_+(L) \cap Z.\end{aligned}$$

Thus we get (ii) and we conclude. □

The next result is an improvement of Proposition I.1.24 that shows a relation between the base loci of a divisor and its pullback via a morphism.

Proposition I.1.25 (Base loci of pullback). *Let $f : Y \to X$ be a morphism of projective schemes and let L be an $\mathbb{R}$-line bundle on X. We have:*

(i) *If L is a $\mathbb{Q}$-line bundle, then $\mathbf{B}(f^*L) \subset f^{-1}(\mathbf{B}(L))$.*
(ii) *If f is finite, then $\mathbf{B}_+(f^*L) \subset f^{-1}(\mathbf{B}_+(L))$.*

Moreover, if $f_*\mathcal{O}_Y = \mathcal{O}_X$, then we have equalities.

PROOF. Consider a line bundle L on X. Take a point $y \in \mathrm{Bs}\,|f^*L|$, its image $x = f(y)$ and a section $s \in H^0(X, L)$. Since

$$s(x) = s(f(y)) = (f^*s)(y) = 0,$$

then $x \in \mathrm{Bs}\,|L|$, whence $y \in f^{-1}(\mathrm{Bs}\,|L|)$ and we get

$$\mathrm{Bs}\,|f^*L| \subset f^{-1}(\mathrm{Bs}\,|L|).$$

Then

$$\mathbf{B}(f^*L) = \bigcap_{m \geq 1} \mathrm{Bs}\,|f^*(mL)| \subset \bigcap_{m \geq 1} f^{-1}(\mathrm{Bs}\,|mL|) = f^{-1}(\mathbf{B}(L)).$$

This immediately extends to $\mathbb{Q}$-line bundles, thus we get (i).
Consider now an $\mathbb{R}$-line bundle L on X and assume that f is finite. Then the pullback of an ample $\mathbb{R}$-line bundle is ample, whence

$$\begin{aligned}\mathbf{B}_+(f^*L) = \bigcap_{A' \text{ ample } \mathbb{R}\text{-l.b. on } Y \,:\, f^*L - A' \ \mathbb{Q}\text{-l.b.}} \mathbf{B}(f^*L - A') \subset \\ \subset \bigcap_{A \text{ ample } \mathbb{R}\text{-l.b. on } X \,:\, L - A \ \mathbb{Q}\text{-l.b.}} \mathbf{B}(f^*(L - A)) \subset \\ \subset \bigcap_{A \text{ ample } \mathbb{R}\text{-l.b. on } X \,:\, L - A \ \mathbb{Q}\text{-l.b.}} f^{-1}(\mathbf{B}(L - A)) = f^{-1}(\mathbf{B}_+(L)).\end{aligned}$$

Thus we get (ii).
For the last statement, take a line bundle L and assume that $f_*\mathcal{O}_Y = \mathcal{O}_X$. Then we have an isomorphism

$$H^0(X, L) = H^0(X, f_*\mathcal{O}_Y \otimes L) \cong H^0(Y, f_*f^*L) \cong H^0(Y, f^*L),$$

whence

$$\mathrm{Bs}\,|f^*L| = f^{-1}(\mathrm{Bs}\,|L|).$$

Then

$$\mathbf{B}(f^*L) = \bigcap_{m \geq 1} \mathrm{Bs}\,|f^*(mL)| = \bigcap_{m \geq 1} f^{-1}(\mathrm{Bs}\,|mL|) = f^{-1}(\mathbf{B}(L)).$$

This immediately extends to $\mathbb{Q}$-line bundles, thus we get equality in (i).
Consider now an $\mathbb{R}$-line bundle L on X and assume that f is finite and $f_*\mathcal{O}_Y = \mathcal{O}_X$. Moreover, take an ample $\mathbb{R}$-line bundle A on X. Since f is finite, we get that f^*A is ample.
We apply Proposition I.1.17 (ii) to A and f^*A. We may take A sufficiently small in such a way that $L - A$ and $f^*(L - A)$ are $\mathbb{Q}$-line bundles and

$$\mathbf{B}_+(L) = \mathbf{B}(L - A),$$

$$\mathbf{B}_+(f^*L) = \mathbf{B}(f^*(L - A)).$$

Since $f_*\mathcal{O}_Y = \mathcal{O}_X$ we deduce that

$$\mathbf{B}_+(f^*L) = \mathbf{B}(f^*(L - A)) = f^{-1}(\mathbf{B}(L - A)) = f^{-1}(\mathbf{B}_+(L)).$$

Thus we get equality in (ii) and we conclude. □

In the case of a birational morphism between normal varieties, we have the following powerful improvement of Proposition I.1.25 (ii).

PROPOSITION I.1.26 ([BBP13, Proposition 2.3]). *Let $f : Y \to X$ be a birational morphism between normal varieties, let D be a big divisor on X and let F be an effective f-exceptional $\mathbb{R}$-divisor on Y. Then*

$$\mathbf{B}_+(f^*D + F) = f^{-1}(\mathbf{B}_+(D)) \cup \mathrm{Exc}(f).$$

Again with the aim of remedying the pathologies of the stable base locus, Ein, Lazarsfeld, Mustaţă, Nakamaye and Popa introduced the following other approximation.

Definition I.1.27. Let D be an $\mathbb{R}$-divisor on X. The *restricted (or diminished) base locus of* D is

$$\mathbf{B}_-(D) = \bigcup_A \mathbf{B}(D + A)$$

where A runs over all ample $\mathbb{R}$-divisors such that $D + A$ is a $\mathbb{Q}$-divisor.

Remark I.1.28. The name 'restricted base locus' is justified by the fact that, if D is a $\mathbb{Q}$-divisor, then for all ample $\mathbb{Q}$-divisors A we have that $\mathbf{B}(D + A) \subset \mathbf{B}(D)$. Hence we get an inclusion

$$\mathbf{B}_-(D) \subset \mathbf{B}(D).$$

The next proposition gives alternative ways to realize the restricted base locus.

Proposition I.1.29 ([ELMNP06, Lemma 1.14, Proposition 1.19, Remark 1.20]). *Let D be an $\mathbb{R}$-divisor on X and let $||\cdot||$ be a norm on $\mathrm{N}^1(X)_{\mathbb{R}}$. Then:*

(i)
$$\mathbf{B}_-(D) = \bigcup_A \mathbf{B}_+(D + A)$$
where A runs over all ample $\mathbb{R}$-divisors.

(ii) *Let $\{A_m\}_{m\geq 1}$ be a sequence of ample $\mathbb{R}$-divisors such that $||A_m|| \to 0$. Then*
$$\mathbf{B}_-(D) = \bigcup_{m\geq 1} \mathbf{B}_+(D + A_m).$$

(iii) *Let $\{A_m\}_{m\geq 1}$ be a sequence of ample $\mathbb{R}$-divisors such that $||A_m|| \to 0$ and $D + A_m$ is a $\mathbb{Q}$-divisor for all $m \geq 1$. Then*
$$\mathbf{B}_-(D) = \bigcup_{m\geq 1} \mathbf{B}(D + A_m).$$

Lemma I.1.30. *Let D be an $\mathbb{R}$-divisor. Then $\mathbf{B}_-(D) \subset \mathbf{B}_+(D)$.*

Proof. Let A be an ample $\mathbb{R}$-divisor. By Proposition I.1.35 (ii) we get that $\mathbf{B}_+(A) = \emptyset$. Hence by Proposition I.1.19 (iii) we get that

$$\mathbf{B}_+(D + A) \subset \mathbf{B}_+(D) \cup \mathbf{B}_+(A) = \mathbf{B}_+(D).$$

Then by Proposition I.1.29 (i) we conclude. □

Let D be an $\mathbb{R}$-divisor on X. By Proposition I.1.29 (ii) we get that $\mathbf{B}_-(D)$ is a countable union of reduced closed subschemes of X. Write $\mathbf{B}_-(D) = \bigcup_{i\in\mathbb{N}} Z_i$ where the Z_i's are closed proper subschemes of X.

Definition I.1.31. We define the *dimension of* $\mathbf{B}_-(D)$ to be

$$\dim(\mathbf{B}_-(D)) = \max\{\dim(Z_i),\ i \in \mathbb{N}\}.$$

It's easy to see that the definition does not depend on the choice of the closed proper subschemes Z_i.

Remark I.1.32. The restricted base locus is not always closed (see [Les14, Theorem 1.1]).

Proposition I.1.33 ([ELMNP06, Proposition 1.15]). *Let D and D' be two $\mathbb{R}$-divisors. We have:*

(i) *If D and D' are numerically equivalent, then $\mathbf{B}_-(D) = \mathbf{B}_-(D')$.*

(ii) $\mathbf{B}_-(D) = \mathbf{B}_-(cD)$ *for all* $c > 0$ *real numbers.*
(iii) $\mathbf{B}_-(D + D') \subset \mathbf{B}_-(D) \cup \mathbf{B}_-(D')$.

Proof. All the statements easily follow by Proposition I.1.19 and Proposition I.1.29 (i). For (i) take an ample $\mathbb{R}$-divisor A. Since $D \equiv D'$, then $D + A \equiv D' + A$, whence by Proposition I.1.19 (i) we get that $\mathbf{B}_+(D + A) = \mathbf{B}_+(D' + A)$. Thus by Proposition I.1.29 (i)

$$\mathbf{B}_-(D) = \bigcup_{A \text{ ample } \mathbb{R}\text{-divisor}} \mathbf{B}_+(D + A) = \bigcup_{A \text{ ample } \mathbb{R}\text{-divisor}} \mathbf{B}_+(D' + A) = \mathbf{B}_-(D').$$

For (ii) observe that by Proposition I.1.19 (ii) and Proposition I.1.29 (i)

$$\mathbf{B}_-(D) = \bigcup_{A \text{ ample } \mathbb{R}\text{-divisor}} \mathbf{B}_+(D + A) = \bigcup_{A \text{ ample } \mathbb{R}\text{-divisor}} \mathbf{B}_+(D + \frac{1}{c}A) =$$

$$= \bigcup_{A \text{ ample } \mathbb{R}\text{-divisor}} \mathbf{B}_+(cD + A) = \mathbf{B}_-(cD).$$

For (iii) observe that by Proposition I.1.19 (iii) and Proposition I.1.29 (i)

$$\mathbf{B}_-(D + D') = \bigcup_{A \text{ ample } \mathbb{R}\text{-divisor}} \mathbf{B}_+(D + D' + A) = \bigcup_{A \text{ ample } \mathbb{R}\text{-divisor}} \mathbf{B}_+((D + \frac{1}{2}A) + (D' + \frac{1}{2}A)) \subset$$

$$\subset \bigcup_{A \text{ ample } \mathbb{R}\text{-divisor}} \mathbf{B}_+(D + \frac{1}{2}A) \cup \bigcup_{A \text{ ample } \mathbb{R}\text{-divisor}} \mathbf{B}_+(D' + \frac{1}{2}A) = \mathbf{B}_-(D) \cup \mathbf{B}_-(D').$$

□

I.1.2. Base loci and positivity of divisors.

The following result shows that the stable (resp. augmented) base locus cannot have isolated points. The original result in [ELMNP09] is stated for smooth varieties. We provide a slightly different version of the proof, adapted to the case of normal varieties.

Proposition I.1.34 ([ELMNP09, Proposition 1.1, Remark 1.2]). *Let X be a normal variety and let D be an $\mathbb{R}$-divisor on X. We have:*

(i) *If D is a $\mathbb{Q}$-divisor, then $\mathbf{B}(D)$ has no isolated points.*
(ii) *$\mathbf{B}_+(D)$ has no isolated points.*

Proof. Let D be a $\mathbb{Q}$ divisor on X. To prove that $\mathbf{B}(D)$ has no isolated points, we assume that there exists an isolated point $x \in \mathbf{B}(D)$ and we reach a contradiction.
By Lemma I.1.7 there exists an integer $m_0 \geq 1$ such that m_0D is a divisor and $\mathrm{Bs}\,|mm_0D|$ for all $m \geq 1$. Set $Z = \mathbf{B}(D)\backslash\{x\}$. Since x is an isolated point, then Z is a closed sucbscheme of X, defining an ideal sheaf $\mathfrak{a}$. Let $\mu : \hat{X}^\nu \to X$ be the normalized blow-up of X along Z, that is the composition of the blow-up $\pi : \hat{X} \to X$ with the normalization map $\nu : \hat{X}^\nu \to \hat{X}$. We have a decomposition

$$|\mu^*(m_0D)| = |M| + E \tag{I.1.1}$$

where M and E are divisors on $\hat{X}^\nu$ such that $\mathrm{Bs}\,|M| = \mu^{-1}(x)$ and $\mathcal{O}_{\hat{X}^\nu}(-E) = \mu^{-1}(\mathfrak{a})$. Observe that, since $x \notin Z$ and X is normal, then $\mu^{-1}(x)$ is a point.
Since X is a variety, then $\hat{X}$ is a variety and π is projective and birational. Moreover $\hat{X}^\nu$ is a normal variety and ν is projective and finite, whence $\mu = \pi \circ \nu$ is projective and birational.
Since $\mathrm{Bs}\,|M| = \{\mu^{-1}(x)\}$, by Zariski-Fujita's Theorem [Laz04a, Remark 2.1.32] we get that M is semiample. It follows that there exists an integer $m_1 \geq 1$ such that m_1M is base-point-free. By (I.1.1) we get that $\mu^{-1}(x) \notin \mathrm{Bs}\,|\mu^*(m_1m_0D)|$.
Since μ is a birational projective morphism of varieties and X is normal, then by Zariski's Main Theorem [Har77, (proof of) Corollary III.11.4] we get that $\mu_*\mathcal{O}_{\hat{X}^\nu} = \mathcal{O}_X$. Hence by the proof of Proposition I.1.26 we get that

$$\mu^{-1}(\mathbf{B}(D)) = \mu^{-1}(\mathrm{Bs}\,|m_1m_0D|) = \mathrm{Bs}\,|\mu^*(m_1m_0D)|.$$

Thus $\mu^{-1}(x) \in \mathrm{Bs}\,|\mu^*(m_1m_0D)|$, whence we reach a contradiction and we conclude (i).
Let D be an $\mathbb{R}$-divisor on X. By Proposition I.1.17 (ii) we can take a sufficiently small ample $\mathbb{R}$-divisor A such that $D - A$ is a $\mathbb{Q}$-divisor and $\mathbf{B}_+(D) = \mathbf{B}(D - A)$. By (i) we have that $\mathbf{B}(D - A)$ has no isolated points, whence we get (ii) and we conclude. □

The following proposition characterizes semiampleness (resp. ampleness, nefness) in terms of emptyness the stable (resp. augmented, restricted) base locus.

Proposition I.1.35. *Let D be an $\mathbb{R}$-divisor on X. We have:*

(i) *If D is a $\mathbb{Q}$-divisor, then D is semiample if and only if $\mathbf{B}(D) = \emptyset$.*
(ii) *D is ample if and only if $\mathbf{B}_+(D) = \emptyset$.*
(iii) *D is nef if and only if $\mathbf{B}_-(D) = \emptyset$.*

Proof. For (i) let D be a $\mathbb{Q}$-divisor on X. If D is semiample, then by definition there exists an integer $m \geq 1$ such that mD is a base-point-free divisor. It follows that

$$\mathbf{B}(D) \subset \mathrm{Bs}\,|mD| = \emptyset.$$

Conversely, assume that $\mathbf{B}(D) = \emptyset$. By Lemma I.1.7 there exists an integer $m \geq 1$ such that mD is a divisor and $\mathbf{B}(D) = \mathrm{Bs}\,|mD|$. Since $\mathbf{B}(D) = \emptyset$, then mD is base-point-free, whence D is

semiample. This proves (i).
For (ii) let D be an $\mathbb{R}$-divisor on X. Assume that D is ample and take an ample $\mathbb{R}$-divisor A such that $D - A$ is an ample $\mathbb{Q}$-divisor. Then by (i)

$$\mathbf{B}_+(D) \subset \mathbf{B}(D - A) = \emptyset.$$

Conversely assume that $\mathbf{B}_+(D) = \emptyset$. By Proposition I.1.17 (ii) we can take a sufficiently small ample $\mathbb{R}$-divisor A such that $D - A$ is a $\mathbb{Q}$-divisor and $\mathbf{B}_+(D) = \mathbf{B}(D - A)$. By (i) we have that $D - A$ is semiample. Thus $D - A$ is nef, whence by [Laz04a, Corollary 1.4.10] we get that $D = (D - A) + A$ is ample. This proves (ii).
For (iii) let D be an $\mathbb{R}$-divisor on X and let $||\cdot||$ be a norm on $\mathrm{N}^1(X)_{\mathbb{R}}$. By Lemma I.1.11 there exist ample $\mathbb{R}$-divisors $\{A_m\}_{m\geq 1}$ such that $||A_m|| \to 0$, $D + A_m$ are $\mathbb{Q}$-divisors, and $A_m - A_{m+1}$ are ample for all $m \geq 1$. Then by Proposition I.1.29 (iii)

$$\mathbf{B}_-(D) = \bigcup_{m\geq 1} \mathbf{B}(D + A_m).$$

If D is nef, then by [Laz04a, Corollary 1.4.10] we get that $D + A_m$ is ample for all $m \geq 1$. It follows by (i) that $\mathbf{B}(D + A_m) = \emptyset$ for all $m \geq 1$, whence $\mathbf{B}_-(D) = \emptyset$.
Conversely assume that $\mathbf{B}_-(D) = \emptyset$. Then $\mathbf{B}(D + A_m) = \emptyset$ for all $m \geq 1$, whence by (i) we have that $D + A_m$ is semiample for all $m \geq 1$. Thus $D + A_m$ is nef for all $m \geq 1$ and, since $A_m - A_{m+1}$ is ample, by [Laz04a, Corollary 1.4.10] we have that $D + A_m = D + A_{m+1} + (A_m - A_{m+1})$ is ample for all $m \geq 1$, whence D is nef and we conclude (iii). □

Remark I.1.36. Due to Proposition I.1.35, the stable (resp. augmented or restricted) base locus is also called non-semiample (resp. non-ample or non-nef) locus.

Proposition I.1.37. *Let D be an $\mathbb{R}$-divisor on X. We have:*

(i) *If D is a $\mathbb{Q}$-divisor, then $\dim(\mathbf{B}(D)) \leq 0$ if and only if $\mathbf{B}(D) = \emptyset$.*
(ii) *$\dim(\mathbf{B}_+(D)) \leq 0$ if and only if $\mathbf{B}_+(D) = \emptyset$.*
(iii) *$\dim(\mathbf{B}_-(D)) \leq 0$ if and only if $\mathbf{B}_-(D) = \emptyset$*

Proof. To prove (i) let D be a $\mathbb{Q}$-divisor. By Proposition I.1.7 there exists an integer $m_0 \geq 0$ such that $m_0 D$ is a divisor and $\mathbf{B}(D) = \mathrm{Bs}\,|mm_0D|$ for all integers $m \geq 1$. If $\dim(\mathbf{B}(D)) \leq 0$ and $\mathbf{B}(D) \neq \emptyset$, then $\mathbf{B}(D)$ consists of a finite set of points. By Zariski-Fujita's Theorem (see [Ein00, Corollary 3] or [Laz04a, Remark 2.1.32]) we get that $m_0 D$ is semiample. It follows that there exists an integer $m_1 \geq 1$ such that $m_1 m_0 D$ is base-point-free, whence $\mathbf{B}(D) = \mathrm{Bs}\,|m_1 m_0 D| = \emptyset$. Thus we conclude (i).
To prove (ii) let D be an $\mathbb{R}$-divisor. By Proposition I.1.17 (ii) there exists an ample divisor A such that $D - A$ is a $\mathbb{Q}$-divisor and $\mathbf{B}_+(D) = \mathbf{B}(D - A)$. If $\dim(\mathbf{B}_+(D)) \leq 0$, then by (i) we get that $\mathbf{B}(D - A) = \emptyset$. This proves (ii).
To prove (iii) let D be an $\mathbb{R}$-divisor and let A be any ample $\mathbb{R}$-divisor such that $D + A$ is a $\mathbb{Q}$-divisor. Observe that, if $\dim(\mathbf{B}_-(D)) \leq 0$, then $\dim(\mathbf{B}(D + A)) \leq 0$, thus by (i) we get that $\mathbf{B}(D + A) = \emptyset$. Hence we get (iii) and we conclude. □

The following proposition characterizes effectiveness (resp. bigness, pseudoeffectiveness) in terms of dimension of the stable (resp. augmented, restricted) base locus.

Proposition I.1.38. *Let D be an $\mathbb{R}$-divisor on X. We have:*

(i) *If D is a $\mathbb{Q}$-divisor and there exists an effective $\mathbb{Q}$-divisor E such that $D \sim_{\mathbb{Q}} E$, then $\dim(\mathbf{B}(D)) < n$.*
(ii) *If D is big, then $\dim(\mathbf{B}_+(D)) < n$.*
(iii) *If D is pseudoeffective, then $\dim(\mathbf{B}_-(D)) < n$.*

Proof. For (i) let D and E be $\mathbb{Q}$-divisors on X such that E is effective and $D \sim_{\mathbb{Q}} E$. Take $m \geq 1$ such that mE is an effective divisor. Then by Proposition I.1.8 (i) we get that

$$\mathbf{B}(D) = \mathbf{B}(E) \subset \operatorname{Bs}|mE|.$$

Since mE is effective, then there exists a section $s \in H^0(X, mE)$ such that $mE = \operatorname{div}(s)$ and

$$\mathbf{B}(D) = \mathbf{B}(E) \subset \operatorname{Bs}|mE| \subset Z(s) = \operatorname{Supp}(\operatorname{div}(s)) = \operatorname{Supp}(E).$$

Hence $\dim(\mathbf{B}(D)) \leq \dim(\operatorname{Supp}(E)) < n$ and we get (i).
For (ii) let D be a big $\mathbb{R}$-divisor on X. By Remark I.1.10 we have that $D \sim_{\mathbb{R}} A + E$, with A arbitrarily small ample $\mathbb{R}$-divisor and E effective $\mathbb{Q}$-divisor. Since $D \equiv A + E$ then by Proposition I.1.19 (i) we get $\mathbf{B}_+(D) = \mathbf{B}_+(A+E)$. Moreover, take a sufficienlty small ample $\mathbb{R}$-divisor A' such that $A - A'$ is an ample $\mathbb{Q}$-divisor. Then $E' = E + A - A'$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor and by Proposition I.1.17 (ii) we get that

$$\mathbf{B}_+(D) = \mathbf{B}_+(A+E) = \mathbf{B}(A + E - A') = \mathbf{B}(E').$$

By (i) it follows that $\dim(\mathbf{B}_+(D)) = \dim(\mathbf{B}(E')) < n$, whence we get (ii).
For (iii) let D be a pseudoeffective $\mathbb{R}$-divisor and let A be an ample divisor on X. By Proposition I.1.29 (ii) we get that

$$\mathbf{B}_-(D) = \bigcup_{m \geq 1} \mathbf{B}_+(D + \frac{1}{m}A).$$

If D is pseudoeffective, then $D + \frac{1}{m}A$ is big for all $m \geq 1$. It follows by (ii) that $\dim(\mathbf{B}_+(D + \frac{1}{m}A)) < n$ for all $m \geq 1$. Hence $\dim(\mathbf{B}_-(D)) < n$ and we get (iii). □

If the scheme X is reduced and pure dimensional, then we get the following improvement of Proposition I.1.38.

Proposition I.1.39. *Let X be a reduced pure dimensional scheme and let D be an $\mathbb{R}$-divisor on X. We have:*

(i) *If D is a $\mathbb{Q}$-divisor, then there exists an effective $\mathbb{Q}$-divisor E such that $D \sim_{\mathbb{Q}} E$ if and only if* $\dim(\mathbf{B}(D)) < n$.
(ii) *D is big if and only if* $\dim(\mathbf{B}_+(D)) < n$.
(iii) *D is pseudoeffective if and only if* $\dim(\mathbf{B}_-(D)) < n$.

Proof. For (i) one implication is Proposition I.1.38 (i). Conversely, let D be a $\mathbb{Q}$ divisor such that $\dim(\mathbf{B}(D)) < n$. By Lemma I.1.7 we can take an integer $m \geq 1$ such that mD is a divisor and $\mathbf{B}(D) = \operatorname{Bs}|mD|$. Moreover, take an irreducible component X_i of X. Since $\dim(\mathbf{B}(D)) < n$ and X is pure dimensional, it follows that X_i is not contained in $\mathbf{B}(D)$. Then there exist a point $x_i \in X_i \backslash \mathbf{B}(D)$ and a section $s_i \in H^0(X, mD)$ such that $s_i(x_i) \neq 0$. It follows that $H^0(X, \mathcal{I}_{\{x_i\}/X}(mD))$ is strictly contained in $H^0(X, mD)$. Since we are dealing with vector spaces and the irreducible components of X are finitely many, then

$$\bigcup_i H^0(X, \mathcal{I}_{\{x_i\}/X}(mD)) \subsetneq H^0(X, mD),$$

where the union runs over all irreducible components of X. Hence there exists a section $s \in H^0(X, mD) \backslash \bigcup_i H^0(X, \mathcal{I}_{\{x_i\}/X}(mD))$, whence $s_{|X_i} \neq 0$ for all irreducible components X_i. Since X is reduced, it follows that $mD \sim \operatorname{div}(s)$, whence $D \sim_{\mathbb{Q}} E = \frac{1}{m}\operatorname{div}(s)$, where E is an effective $\mathbb{Q}$-divisor and we conclude.
For (ii) one implication is Proposition I.1.38 (ii). Conversely, let D be an $\mathbb{R}$-divisor such that $\dim(\mathbf{B}_+(D)) < n$. Observe that by Proposition I.1.17 (ii) there exists a sufficiently small ample $\mathbb{R}$-divisor A such that $D - A$ is a $\mathbb{Q}$-divisor and $\mathbf{B}_+(D) = \mathbf{B}(D - A)$. Since $\dim(\mathbf{B}(D - A)) < n$,

by (i) we get that there exists an effective $\mathbb{Q}$-divisor E such that $D - A \sim_{\mathbb{Q}} E$. Then $D \sim_{\mathbb{R}} A + E$ is big.
For (iii) one implication is Proposition I.1.38 (iii). Conversely, let D be an $\mathbb{R}$-divisor such that $\dim(\mathbf{B}_-(D)) < n$. By Proposition I.1.29 (i) we get that

$$\mathbf{B}_-(D) = \bigcup_A \mathbf{B}_+(D + A)$$

where A runs over all ample $\mathbb{R}$-divisors. Since $\dim(\mathbf{B}_-(D)) < n$, then $\dim(\mathbf{B}_+(D + A)) < n$ for all ample $\mathbb{R}$-divisors A. Hence by (ii) we get that $D + A$ is big for all ample $\mathbb{R}$-divisors A, whence D is pseudoeffective. □

The following example shows that we cannot remove the hypothesis that X is pure dimensional in Proposition I.1.39 (ii).

Example I.1.40. Take a disjoint union $X = X_1 \cup X_2$ of two varieties and assume that $n = \dim(X_1) > \dim(X_2) > 0$. Take now a divisor $D = (A_1, 0)$ on X, where A_1 is an ample divisor on X. Then by Proposition I.1.35 (ii)

$$\mathbf{B}_+(D) = \mathbf{B}_+(A_1) \cup \mathbf{B}_+(0) = X_2.$$

It follows that $\dim(\mathbf{B}_+(D)) < n$ but D is not big.

I.1.3. Partial ampleness via augmented base loci.

Let D be an $\mathbb{R}$-divisor on X. We look at the following condition:

$$(+) \qquad \dim(\mathbf{B}_+(D)) \leq q.$$

Remark I.1.41. By Propositions I.1.35 (ii) and I.1.37 (ii) we get that $\dim(\mathbf{B}_+(D)) \leq 0$ if and only if D is ample. Thus Condition (+) for $q = 0$ is equivalent to the ampleness of D.
Hence we obtain a first natural way to measure how much an $\mathbb{R}$-divisor is far from being ample. Namely, if $\dim(\mathbf{B}_+(D)) = q$, we can think about D as an $\mathbb{R}$-divisor that is 'q steps far from being ample'.

Remark I.1.42. If X is reduced and pure dimensional, by Proposition I.1.39 (ii) for $q = n-1$ we recover the notion of bigness.

Proposition I.1.43. *Let D and D' be two $\mathbb{R}$-divisors on X.*

(i) *Assume that D and D' are numerically equivalent. If $\dim(\mathbf{B}_+(D)) \leq q$, then $\dim(\mathbf{B}_+(D')) \leq q$.*
(ii) *If $\dim(\mathbf{B}_+(D)) \leq q$, then $\dim(\mathbf{B}_+(cD)) \leq q$ for all $c > 0$ real numbers.*
(iii) *If $\dim(\mathbf{B}_+(D)) \leq q$ and $\dim(\mathbf{B}_+(D')) \leq q$, then $\dim(\mathbf{B}_+(D+D')) \leq q$.*
(iv) *If $\dim(\mathbf{B}_+(D)) \leq q$, then there exists a neighbourhood U of $[D]$ in $\mathrm{N}^1(X)_{\mathbb{R}}$ such that $\dim(\mathbf{B}_+(D')) \leq q$ for all $[D'] \in U$.*
(v) *If $\dim(\mathbf{B}_+(D)) \leq q$ and $Z \subset X$ is a closed subscheme, then $\dim(\mathbf{B}_+(\mathcal{O}_X(D)_{|Z})) \leq q$.*

Proof. (i), (ii) and (iii) follow by Proposition I.1.19. (iv) follows by Proposition I.1.17 (iii). (v) follows by Proposition I.1.24 (ii). □

The following proposition is a characterization of Condition (+) (note the analogy with Theorems I.1.20 and I.1.21).

Proposition I.1.44 (Equivalent conditions for $\dim(\mathbf{B}_+(D)) \leq q$). *Let D be a nef $\mathbb{R}$-divisor on X. Then the following conditions are equivalent:*

(i) $\dim(\mathbf{B}_+(D)) \leq q$.
(ii) *For all subvarieties $Z \subset X$ of dimension $> q$ we have that $\mathcal{O}_X(D)_{|Z}$ is big.*
(iii) *For all subvarieties $Z \subset X$ of dimension $> q$ we have that $D^{\dim(Z)}.Z > 0$.*

Proof. For (i) ⇒ (ii) consider a subvariety $Z \subset X$ such that $\dim(Z) > q$. By Proposition I.1.43 (v) we get that $\dim(\mathbf{B}_+(\mathcal{O}_X(D)_{|Z})) \leq q$. This implies by Proposition I.1.39 (ii) that $\mathcal{O}_X(D)_{|Z}$ is big.
For (iii) ⇒ (i) we assume that $\dim(\mathbf{B}_+(D)) > q$ and we reach a contradiction. Since D is nef, by Theorem I.1.21 we get that

$$\mathbf{B}_+(D) = \bigcup_Z Z$$

where the union is taken over all subvarieties $Z \subset X$ of positive dimension such that $D^{\dim(Z)}.Z = 0$.
Let now $Z \subset \mathbf{B}_+(D)$ be an irreducible component such that $\dim(Z) > q$. Then, as in [Laz04b, Lemma 10.3.6], we get that $D^{\dim(Z)} \cdot Z = 0$. This contradicts the hypothesis, whence we reach a contradiction and we conclude.
For the last part of the assertion we prove (ii) ⇔ (iii). Consider a subvariety $Z \subset X$ such that $\dim(Z) > q$. Take $L = \mathcal{O}_X(D)$. Since L is nef, then by Corollary I.1.23 that $L_{|Z}$ is big if and only if $D^{\dim(Z)} \cdot Z = L^{\dim(Z)} \cdot Z = (L_{|Z})^{\dim(Z)} > 0$, whence we conclude. □

Let D be an $\mathbb{R}$-divisor on X. Consider the following condition:

$$(-) \qquad \dim(\mathbf{B}_-(D)) \leq q.$$

Here $\dim(\mathbf{B}_-(D))$ is interpreted as in Definition I.1.31.

Remark I.1.45. By Propositions I.1.35 (iii) and I.1.37 (iii) for $q = 0$ we recover the notion of nefness. Moreover, if X is reduced and pure dimensional, by Proposition I.1.39 (iii) for $q = n - 1$ we recover the notion of pseudoeffectiveness.

Proposition I.1.46. *Let D and D' be two $\mathbb{R}$-divisors on X.*

(i) *Assume that D and D' are numerically equivalent. If $\dim(\mathbf{B}_-(D)) \leq q$, then $\dim(\mathbf{B}_-(D')) \leq q$.*
(ii) *If $\dim(\mathbf{B}_-(D)) \leq q$, then $\dim(\mathbf{B}_-(cD)) \leq q$ for all $c > 0$ real numbers.*
(iii) *If $\dim(\mathbf{B}_-(D)) \leq q$ and $\dim(\mathbf{B}_-(D')) \leq q$, then $\dim(\mathbf{B}_-(D + D')) \leq q$.*

Proof. All statements follow by Proposition I.1.33. □

The following proposition is a characterization of Condition (−).

Proposition I.1.47 (Equivalent conditions for $\dim(\mathbf{B}_-(D)) \leq q$). *Let D be an $\mathbb{R}$-divisor and let A be an ample divisor on X. Then the following conditions are equivalent:*

(i) $\dim(\mathbf{B}_-(D)) \leq q$.
(ii) $\dim(\mathbf{B}_+(D + \epsilon A)) \leq q$ *for all real numbers $\epsilon > 0$.*

Proof. Let ϵ_1, ϵ_2 be two real numbers such that $\epsilon_2 > \epsilon_1 > 0$. By Proposition I.1.35 (ii) we get that $\mathbf{B}_+((\epsilon_2 - \epsilon_1)A) = \emptyset$. Hence by Proposition I.1.19 (iii) we get that

$$\mathbf{B}_+(D + \epsilon_2 A) = \mathbf{B}_+((D + \epsilon_1 A) + (\epsilon_2 - \epsilon_1)A) \subset$$
$$\subset \mathbf{B}_+(D + \epsilon_1 A) \cup \mathbf{B}_+((\epsilon_2 - \epsilon_1)A) = \mathbf{B}_+(D + \epsilon_1 A).$$

Since by Proposition I.1.29 (ii) we have that

$$\mathbf{B}_-(D) = \bigcup_{m \geq 1} \mathbf{B}_+(D + \frac{1}{m}A),$$

the claim easily follows. □

Since by Propositions I.1.43 (i) and I.1.46 (i) Conditions (+) and (−) are numerical properties, it is interesting to consider the cones of classes of divisors that satisfy them.

Definition I.1.48. We denote by $\mathrm{C}^q_+(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$ (resp. $\mathrm{C}^q_-(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$) the cone of classes of $\mathbb{R}$-divisors $[D] \in \mathrm{N}^1(X)_{\mathbb{R}}$ such that $\dim(\mathbf{B}_+(D)) \leq q$ (resp. $\dim(\mathbf{B}_-(D)) \leq q$).

Remark I.1.49. By Proposition I.1.43 we get that $\mathrm{C}^q_+(X)$ is an open convex cone, while by Proposition I.1.46 we get that $\mathrm{C}^q_-(X)$ is a convex cone.
By Lemma I.1.30 we get the inclusions

$$\mathrm{C}^q_+(X) \subset \mathrm{C}^q_-(X), \quad \mathrm{C}^q_+(X) \subset \mathrm{C}^{q+1}_+(X), \quad \mathrm{C}^q_-(X) \subset \mathrm{C}^{q+1}_-(X).$$

Finally, since

$$\mathrm{Amp}(X) = \mathrm{C}^0_+(X) \subset \mathrm{C}^q_+(X) \subset \mathrm{C}^q_-(X)$$

and $\mathrm{Amp}(X)$ is full-dimensional, then $\mathrm{C}^q_+(X)$ and $\mathrm{C}^q_-(X)$ are full-dimensional.

The following is a generalization of Kleiman's theorem (see [Laz04a, Theorem 1.4.23]).

Theorem I.1.50 (First generalization of Kleiman's theorem). *We have:*

(i) $\mathrm{C}^q_+(X) = \mathrm{int}(\mathrm{C}^q_-(X))$.

(ii) $\mathrm{C}^q_-(X) = \overline{\mathrm{C}^q_+(X)}$.

Proof. By Remark I.1.49 we get that $\mathrm{C}^q_+(X)$ and $\mathrm{C}^q_-(X)$ are convex full-dimensional cones, $\mathrm{C}^q_+(X)$ is open and $\mathrm{C}^q_+(X) \subset \mathrm{C}^q(X)$. Thus we have that $\mathrm{C}^q_+(X) \subset \mathrm{int}(\mathrm{C}^q_-(X))$.
Take $[D] \in \mathrm{int}(\mathrm{C}^q_-(X))$. Then there exists an open neighbourhood U_D of $[D]$ in $\mathrm{N}^1(X)_{\mathbb{R}}$ such that $[D'] \in \mathrm{C}^q_-(X)$ for all $[D'] \in U_D$. Take an ample $\mathbb{R}$-divisor A such that $[D - A] \in U_D$, whence $\dim(\mathbf{B}_-(D - A)) \leq q$. By Proposition I.1.29 (i) we get that

$$\mathbf{B}_-(D - A) = \bigcup_{A' \text{ ample } \mathbb{R}\text{-div.}} \mathbf{B}_+(D - A + A') \supset \mathbf{B}_+(D).$$

Thus $\dim(\mathbf{B}_+(D)) \leq q$ and we get $\mathrm{int}(\mathrm{C}^q_-(X)) \subset \mathrm{C}^q_+(X)$. Hence we conclude (i).
To see (ii) take $[D] \in \mathrm{C}^q_-(X)$ and an ample divisor A. Since $\dim(\mathbf{B}_-(D)) \leq q$, by Proposition I.1.47 we get that $\dim(\mathbf{B}_+(D + \epsilon A)) \leq q$ for all real numbers $\epsilon > 0$. It follows that $[D] = \lim_{\epsilon \to 0}[D + \epsilon A]$ is a limit of classes of $\mathbb{R}$-divisors in $\mathrm{C}^q_+(X)$, whence $[D] \in \overline{\mathrm{C}^q_+(X)}$. Thus $\mathrm{C}^q_-(X) \subset \overline{\mathrm{C}^q_+(X)}$.
Take now $[D] \in \overline{\mathrm{C}^q_+(X)}$. Then $[D] = \lim_{m \to \infty}[D_m]$, where $\{D_m\}_{m \geq 1}$ are $\mathbb{R}$-divisors such that $[D_m] \in \mathrm{C}^q_+(X)$ for all $m \geq 1$. Take an ample divisor A and a real number $\epsilon > 0$. Since ampleness is an open property, then for all m sufficiently large $\epsilon A + (D - D_m)$ is ample, whence by Proposition I.1.35 (ii) we get $\mathbf{B}_+(\epsilon A + (D - D_m)) = \emptyset$. It follows by Proposition I.1.19 (iii) that

$$\mathbf{B}_+(D + \epsilon A) = \mathbf{B}_+(D_m + \epsilon A + (D - D_m)) \subset \mathbf{B}_+(D_m) \cup \mathbf{B}_+(\epsilon A + (D - D_m)) = \mathbf{B}_+(D_m).$$

Thus $\dim(\mathbf{B}_+(D + \epsilon A)) \leq q$ for all $\epsilon > 0$. By Proposition I.1.47 we get that $[D] \in \mathrm{C}^q_-(X)$, whence $\overline{\mathrm{C}^q_+(X)} \subset \mathrm{C}^q_-(X)$. Hence we get (ii) and we conclude. $\square$

I.2. Partial ampleness

NOTATION. *Throughout Section I.2, unless otherwise specified, X will be a projective noetherian scheme of finite type and of dimension n over the complex number field and q will be a non-negative integer.*

I.2.1. Partially ample divisors.

We recall the following well-known characterization of ample line bundles.

THEOREM I.2.1 (Cartan-Serre-Grothendieck's theorem; [Laz04a, Theorem 1.2.6]). *Let L be a line bundle on X. Then the following conditions are equivalent:*

(i) *L is ample.*
(ii) *For all coherent sheaves $\mathcal{F}$ on X there exists an integer $m_{L,\mathcal{F}} > 0$ such that*

$$H^i(X, L^m \otimes \mathcal{F}) = 0$$

for all $m > m_{L,\mathcal{F}}$ and $i > 0$.

By weakening condition (ii) of Theorem I.2.1 we obtain a notion of 'partial ampleness', that intuitively measures how much a line bundle is far from being ample and that shares many important properties with traditional ampleness.

DEFINITION I.2.2. Let L be a line bundle on X. L is *(naively) q-ample* if for all coherent sheaves $\mathcal{F}$ on X there exists an integer $m_{L,\mathcal{F}} > 0$ such that

$$H^i(X, L^m \otimes \mathcal{F}) = 0$$

for all $m > m_{L,\mathcal{F}}$ and $i > q$.

REMARK I.2.3. By Theorem I.2.1 for $q = 0$ we recover the notion of ampleness. Moreover, if a line bundle is q-ample, then it is also $(q+1)$-ample. Finally, every line bundle is n-ample.
Thus, if a line bundle is q-ample but not $(q-1)$-ample, we can think about it as a line bundle that is 'q steps far from being ample'.

LEMMA I.2.4. *Let L be a line bundle on X. Then the following conditions are equivalent:*

(i) *L is q-ample.*
(ii) *There exists an integer $m_0 \geq 1$ such that $m_0 L$ is q-ample.*
(iii) *mL is q-ample for all integers $m \geq 1$.*

PROOF. The implications (i) $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) are obvious.
For the implication (ii) $\Rightarrow$ (i) take a coherent sheaf $\mathcal{F}$. We have to prove that there exists an integer $m_{L,\mathcal{F}} > 0$ such that

$$h^i(X, L^m \otimes \mathcal{F}) = 0$$

for all $m > m_{L,\mathcal{F}}$ and $i > q$.
Take $m \geq 1$ and write $m = km_0 + r$, with k and r integers such that $k \geq 0$, $0 \leq r \leq m_0 - 1$ and $k + r \geq 1$.
For all $0 \leq r \leq m_0 - 1$ consider the coherent sheaf $\mathcal{F}_r = L^r \otimes \mathcal{F}$. Then

$$L^m \otimes \mathcal{F} = L^{km_0+r} \otimes \mathcal{F} = (L^{m_0})^k \otimes L^r \otimes \mathcal{F} = (L^{m_0})^k \otimes \mathcal{F}_r.$$

Since by hypothesis L^{m_0} is q-ample, then for all $0 \leq r \leq m_0 - 1$ there exists an integer $m_{L,\mathcal{F},r} \geq 1$ such that

$$h^i(X, L^m \otimes \mathcal{F}) = h^i(X, (L^{m_0})^k \otimes \mathcal{F}_r) = 0$$

for all $k > m_{L,\mathcal{F},r}$ and $i > q$.
Taking $m_{L,\mathcal{F}} = m_0(\max\{m_{L,\mathcal{F},s} \,:\, 0 \leq s \leq m_0 - 1\} + 1)$ we get that, if $m > m_{L,\mathcal{F}}$, then $k > m_{L,\mathcal{F},r}$, whence we get (ii) and we conclude. $\square$

Definition I.2.5. Let L be a $\mathbb{Q}$-line bundle on X. L is *q-ample* if there exists an integer $m_0 \geq 1$ such that $m_0 L$ is a q-ample line bundle.

Remark I.2.6. It is easy to see by Lemma I.2.4 that a $\mathbb{Q}$-line bundle L is q-ample if and only if for all $m \geq 1$ such that mL is a line bundle, then mL is q-ample.

Remark I.2.7. Definition I.2.2 (resp. Definition I.2.5) extends to divisors (resp. $\mathbb{Q}$-divisors) via the morphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$.

The first time that the notion of partial ampleness appeared in literature was in a work of Andreotti and Grauert (see [AnGr62]). The authors proved that, given a compact complex manifold X, an holomorphic line bundle on X is q-ample provided that it is endowed with an hermitian metric whose curvature is a $(1,1)$-form with at least $n-q$ positive eigenvalues at every point of X. Subsequently, Sommese (see [Som78]) studied the partial ampleness of semiample line bundles, giving an interesting characterization in terms of the semiample fibration (see Theorem I.2.21). More recently a large group of authors - Demailly, Peternell, Schneider, Totaro, Küronya and Ottem among others (see [DPS96, Som78, Tot13, Kür06, Kür13, Ott12]) - studied the notion of partial ampleness in the more general setting that we present in this section.

Since it is in general complicated to deduce the vanishing of the cohomology groups of any coherent sheaf, it is useful to be able to control the partial ampleness by working with only line bundles. In order to do this, Totaro (see [Tot13]) introduced the following definition.

Definition I.2.8. Let L and A be two line bundles on X such that A is ample. L is *uniformly q-ample with respect to A* if there exists a constant $C_{L,A} > 0$ such that

$$H^i(X, mL - rA) = 0$$

for all $r > 0$, $m \geq rC_{L,A}$ and $i > q$.

The following theorem is the most important and powerful cohomological characterization of partial ampleness for line bundles.

Theorem I.2.9 (Equivalent conditions for partial ampleness; [Tot13, Theorem 7.1]). *Let A be an ample line bundle on X. Then there exists a constant $C_{X,A,q} > 0$ such that for all line bundles L on X the following conditions are equivalent:*

(i) *L is q-ample.*
(ii) *L is uniformly q-ample with respect to A.*
(iii) *There exists an integer $m_{L,A} > 0$ such that*

$$H^i(X, m_{L,A}L - rA) = 0$$

for all $1 \leq r \leq C_{X,A,q}$ and $i > q$.

Using the notion of height of a coherent sheaf, Demailly, Peternell and Schneider provided an effective control on the integer $m_{L,\mathcal{F}}$ of Definition I.2.2, as follows.
Let X be a variety, let A be an ample line bundle and let $\mathcal{F}$ be a coherent sheaf on X.
Consider a resolution of $\mathcal{F}$ by non-positive powers of A of the form

$$\cdots \to \bigoplus_{1\leq i\leq m_k} A^{-d_{k,i}} \to \cdots \to \bigoplus_{1\leq i\leq m_0} A^{-d_{0,i}} \to \mathcal{F} \to 0 \tag{I.2.2}$$

where $m_k \geq 1$ and $d_{k,i} \geq 0$ for all $k \geq 0$ and $1 \leq i \leq m_k$.
We recall that the height of $\mathcal{F}$ with respect to A is defined to be

$$\mathrm{ht}_A(\mathcal{F}) = \min\{\max_{0\leq k\leq n,\ 1\leq i\leq m_k}\{d_{k,i}\}\},$$

where the minimum is taken over the set of all resolutions of $\mathcal{F}$ of the form (I.2.2).
Demailly-Peternell-Schneider (in the smooth case) and Greb-Küronya (in the general case) proved the following result.

Proposition I.2.10 (Effective control on the constant; [DPS96, Proposition 1.2], [GrKü15, Lemma 2.31]). *Let X be a variety, let L be a line bundle on X that is uniformly q-ample with respect to some ample line bundle A, let $C_{L,A} > 0$ be the constant of Definition I.2.8 and let $\mathcal{F}$ be a coherent sheaf on X. Then*

$$H^i(X, L^m \otimes \mathcal{F}) = 0$$

for all $m \geq C_{L,A}(\mathrm{ht}_A(\mathcal{F}) + 1)$ and $i > q$.

The next proposition summarizes the first important properties that the notion of partial ampleness shares with traditional ampleness.

Proposition I.2.11 ([Tot13, Corollary 7.2], [Ott12, Proposition 2.3]). *Let L be a line bundle on X. Then:*

(i) *L is q-ample if and only if $L_{|X_{red}}$ is q-ample.*
(ii) *L is q-ample if and only if $L_{|X_i}$ is q-ample on every irreducible component X_i of X.*
(iii) *Let $f : Y \to X$ be a finite morphism of projective schemes. Then if L is q-ample we have that f^*L is q-ample. If f is surjective, then the converse also holds. In particular, if $Z \subset X$ is a closed subscheme and L is q-ample, then $L_{|Z}$ is q-ample.*

Demailly-Peternell-Schneider (in the smooth case) and Greb-Küronya (in the general case) proved that the partial ampleness is a numerical property.

Proposition I.2.12 (The partial ampleness is a numerical property; [DPS96, Proposition 1.4], [GrKü15, Theorem 2.17]). *Let X be a variety and let L and L' be two numerically equivalent line bundles on X. Then L is q-ample if and only if L' is q-ample.*

The following lemma shows that the product of a q-ample line bundle and an ample line bundle remains q-ample.

Proposition I.2.13 ([Bro12, Lemma 2.2]). *Let L and A be line bundles on X such that L is q-ample and A is ample. Then for all coherent sheaves $\mathcal{F}$ on X there exist two integers $m_{L,A,\mathcal{F}} > 0, m'_{L,A,\mathcal{F}} > 0$ such that*

$$H^i(X, L^m \otimes A^{m'} \otimes \mathcal{F}) = 0$$

for all $i > q$, $m \geq 1$, $m' \geq 1$ such that either $m > m_{L,A,\mathcal{F}}$ or $m' > m'_{L,A,\mathcal{F}}$.
As a consequence, if D and A are $\mathbb{Q}$-divisors such that D is q-ample and A is ample, then $D + A$ is q-ample.

Since by Propositions I.2.12 and I.2.13 the partial ampleness is invariant with respect to numerical equivalence and it is preserved by the product with ample line bundles, it is possible to export Definition I.2.2 to the Néron-Severi vector space $\mathrm{N}^1(X)_\mathbb{R}$ of a variety X. As in [GrKü15] we give the following definition.

Definition I.2.14. Let X be a variety and let D be an $\mathbb{R}$-divisor on X. D is *q-ample* if there exist a q-ample $\mathbb{Q}$-divisor D' and an ample $\mathbb{R}$-divisor A such that $D \equiv D' + A$.

Remark I.2.15. As in the case of line bundles, for $q = 0$ we recover the notion of ampleness.

Remark I.2.16. An $\mathbb{R}$-divisor D is q-ample if and only if

$$D \equiv cD'' + A'$$

where D'' is a q-ample divisor, $c > 0$ is a real number and A' is an ample $\mathbb{R}$-divisor. This is the original definition of Totaro (see [Tot13, Definition 8.2]).
For completeness we show the equivalence.
Assume that D is q-ample and take an integer $m_0 \geq 1$ such that $m_0 D'$ is a divisor. Set $c = \frac{1}{m_0}$, $D'' = m_0 D'$ and $A' = A$. Then $D \equiv D' + A = cD'' + A'$ and we conclude.
Conversely, assume that $D \equiv cD'' + A'$, where D'' is a q-ample divisor, $c > 0$ is a real number and A' is an ample $\mathbb{R}$-divisor. Take a norm $||\cdot||$ on $\mathrm{N}^1(X)_{\mathbb{R}}$ and a real number $\epsilon_{A'} > 0$ such that $D_{\epsilon_{A'}}([A'])$, the ball of radius $\epsilon_{A'}$ centered in $[A']$, is contained in the ample cone $\mathrm{Amp}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$.
Take a real number ϵ such that $0 \leq \epsilon ||D'|| < \epsilon_{A'}$ and $c - \epsilon > 0$ is rational and set $A = \epsilon D'' + A'$ and $D' = (c - \epsilon)D''$.
Since $||A - A'|| = ||\epsilon D''|| = \epsilon ||D''|| < \epsilon_{A'}$, then A is an ample $\mathbb{R}$-divisor. Moreover D' is a q-ample $\mathbb{Q}$-divisor, whence $D \equiv cD'' + A' = D' + A$ is q-ample.

The following proposition shows that Definitions I.2.2 and I.2.14 coincide in the case of $\mathbb{Q}$-divisors.

Proposition I.2.17. *Let X be a variety and let D be a $\mathbb{Q}$-divisor on X. Then the following conditions are equivalent:*

(i) *D is q-ample as $\mathbb{Q}$-divisor (that is as in Remark I.2.7).*
(iii) *D is q-ample as $\mathbb{R}$-divisor (that is as in Definition I.2.14).*

Proof. To see the implication (i) $\Rightarrow$ (ii), observe first that it suffices to consider the case of divisors. Let D be a divisor and let $B_1, \ldots, B_\rho$ be divisors such that their classes are a basis of $\mathrm{N}^1(X)_{\mathbb{Z}} = \mathbb{Z}^\rho$ and consider the norm $||\cdot||$ on $\mathrm{N}^1(X)_{\mathbb{R}}$ defined by

$$||\sum_{i=1}^{\rho} \lambda_i [B_i]|| = \sum_{i=1}^{\rho} |\lambda_i|$$

for all $\lambda_i \in \mathbb{R}$.
Moreover take a real number $\epsilon > 0$ and an ample $\mathbb{Q}$-divisor A such that $||A|| < \epsilon$.
We claim that, if ϵ is sufficiently small, then the $\mathbb{Q}$-divisor $D' = D - A$ is q-ample as $\mathbb{Q}$-divisor. In such a way we get a decomposition

$$D = D' + A,$$

where D' and A are $\mathbb{Q}$-divisors such that D' is q-ample as $\mathbb{Q}$-divisor and A is ample, thus we conclude.
To see this, write

$$-A \equiv \sum_{i=1}^{\rho} \lambda_i B_i, \tag{I.2.3}$$

with $\lambda_i \in \mathbb{Q}$ and take an integer $\lambda > 0$ such that λA is a divisor and $\lambda\lambda_i \in \mathbb{Z}$ for all $1 \leq i \leq \rho$.
To prove that D' is q-ample it suffices to show that $\lambda D'$ is uniformly q-ample with respect to λA. Indeed by Theorem I.2.9 this implies that $\lambda D'$ is q-ample, whence D' is q-ample.
Thus we have to show that there exists a constant $C' = C_{\lambda D', \lambda A} > 0$ such that

$$H^i(X, m\lambda D' - r\lambda A) = 0$$

for all $r > 0$, $m \geq rC'$ and $i > q$.
By (I.2.3) we get that there exists a numerically trivial divisor F such that

$$-\lambda A = \lambda \sum_{i=1}^{\rho} \lambda_i B_i + F.$$

Since D is q-ample, then by Theorem I.2.9 we get that D is uniformly q-ample with respect to λA. Let $C = C_{D,\lambda A} > 0$ be the constant of Definition I.2.8 and consider for all integers $m > 0$ and $r > 0$ the coherent sheaves

$$\mathcal{F}_{m,r} = \mathcal{O}_X(m\lambda \sum_{i=1}^{\rho} \lambda_i B_i + mF - r\lambda A).$$

Then by Proposition I.2.10 we get that

$$H^i(X, m\lambda D' - r\lambda A) = H^i(X, m\lambda D - m\lambda A - r\lambda A) = \tag{I.2.4}$$

$$= H^i(X, m\lambda D + m\lambda \sum_{i=1}^{\rho} \lambda_i B_i + mF - r\lambda A) = H^i(X, \mathcal{O}_X(D)^{m\lambda} \otimes \mathcal{F}_{m,r}) = 0$$

for all $r > 0$, $m\lambda \geq C(\mathrm{ht}_{\lambda A}(\mathcal{F}_{m,r}) + 1)$ and $i > q$.
By [GrKü15, Proposition 2.27] there exists a positive constant $M > 0$ such that $\mathrm{ht}_{\lambda A}(N) \leq M$ for all numerically trivial divisors N on X. Moreover there exists a constant $M' > 0$ such that $\mathrm{ht}_{\lambda A}(\pm B_i) < M'$ for all $1 \leq i \leq \rho$. Finally, for all $r > 0$ we have that $\mathrm{ht}_{\lambda A}(-r\lambda A) = r$.
It follows that

$$\mathrm{ht}_{\lambda A}(\mathcal{F}_{m,r}) \leq \sum_{i=1}^{\rho} \mathrm{ht}_{\lambda A}(m\lambda\lambda_i B_i) + \mathrm{ht}_{\lambda A}(mF) + \mathrm{ht}_{\lambda A}(-r\lambda A) \leq \tag{I.2.5}$$

$$\leq \sum_{i=1}^{\rho} m\lambda|\lambda_i|M' + M + r = m\lambda M'||A|| + M + r < m\lambda M'\epsilon + M + r.$$

for all $r > 0$ and $m > 0$.
Set now

$$C' = \frac{2C(M+2)}{\lambda}.$$

and fix integers $r > 0$ and $m \geq rC'$. Then

$$m \geq rC' = \frac{2C}{\lambda}(Mr + 2r) \geq \frac{2C}{\lambda}(M + r + 1),$$

whence $\frac{m\lambda}{2} \geq C(M + r + 1)$. It follows that

$$m\lambda \geq \frac{m\lambda}{2} + C(M + r + 1). \tag{I.2.6}$$

If $\epsilon < \frac{1}{2CM'}$, then by (I.2.5)

$$\mathrm{ht}_{\lambda A}(\mathcal{F}_{m,r}) < \frac{m\lambda}{2C} + M + r,$$

whence by (I.2.6)

$$C(\mathrm{ht}_{\lambda A}(\mathcal{F}_{m,r}) + 1) < C(\frac{m\lambda}{2C} + M + r + 1) = \frac{m\lambda}{2} + C(M + r + 1) \leq m\lambda.$$

It follows by (I.2.4) that

$$H^i(X, m\lambda D' - r\lambda A) = 0$$

for all $r > 0$, $m \geq rC'$ and $i > q$. Hence $\lambda D'$ is uniformly q-ample with respect to λA and we get (ii).
To see (ii) $\Rightarrow$ (i) assume that $D \equiv D' + A$, where D' is a $\mathbb{Q}$-divisor that is q-ample as $\mathbb{Q}$-divisor and A is an ample $\mathbb{R}$-divisor. Then $A' := D - D' \equiv A$ is an ample $\mathbb{Q}$-divisor. Hence by Proposition I.2.13 we get that $D = D' + A'$ is q-ample as $\mathbb{Q}$-divisor. Thus we get (i) and we conclude. $\square$

As the following result shows, the partial ampleness is an open property.

Theorem I.2.18 (Openness property of partial ampleness; [DPS96, Proposition 1.4], [GrKü15, Theorem 2.22]). *Let X be a variety. Consider the function $Q : \mathrm{N}^1(X)_{\mathbb{R}} \to \mathbb{Z}_{\geq 0}$ defined by*

$$Q([D]) := \min\{q \geq 0 \ : \ D \text{ is } q\text{-ample}\}.$$

Then $Q : \mathrm{N}^1(X)_{\mathbb{R}} \to \mathbb{Z}_{\geq 0}$ is upper-semicontinuous. As a consequence, for all q-ample $\mathbb{R}$-divisors D there exists a neighbourhood U of $[D]$ in $\mathrm{N}^1(X)_{\mathbb{R}}$ of q-ample $\mathbb{R}$-divisors.

The next result is an interesting improvement of Proposition I.2.13.

Proposition I.2.19 (Sum of partially ample divisors; [DPS96, Proposition 1.5], [Tot13, Theorem 8.3]). *Let X be a variety, let D and D' be two $\mathbb{R}$-divisors on X and let $q \geq 0, q' \geq 0$ be two integers. If D is q-ample and D' is q'-ample, then $D + D'$ is $(q + q')$-ample.*

Remark I.2.20. It is easy to see that, if $q > 0$ or $q' > 0$, in general the result is sharp, that is we cannot expect that a sum of a q-ample divisor and a q'-ample divisor is $(q + q' - 1)$-ample (see for example Theorem I.2.23). However, as a consequence of Theorem I.2.21, if L and L' are two semiample line bundles such that L is q-ample, then $L + L'$ is q-ample.

Even if the definition of partial ampleness and its main characterization Theorem I.2.9 are purely cohomological, there exist in literature some interesting results that help us to interpret geometrically the notion of partial ampleness.
First of all we recall the following theorem of Sommese, that characterizes the q-ampleness of a semiample line bundle.

Theorem I.2.21 (Characterization of partial ampleness for semiample line bundles; [Som78, Proposition 1.7], [Mat13, Theorem 1.4], [GrKü15, Theorem 2.44]). *Let X be a variety, let L be a semiample line bundle and let $m_0 > 0$ be an integer such that $m_0 L$ is base-point-free. Then the following conditions are equivalent:*

(i) *L is q-ample.*
(ii) *For all subvarieties $Z \subset X$ of dimension $> q$ we have that $L_{|Z}$ is not linearly equivalent to 0.*
(iii) *The dimension of the fibers of $\phi_{|m_0 L|}$ is $\leq q$.*
(iv) *For all subvarieties $Z \subset X$ of dimension $> q$ we have that $-L_{|Z}$ is not pseudoeffective.*

Moreover, if X is smooth, they are also equivalent to the condition

(v) *For all subvarieties $Z \subset X$ of dimension $> q$ there exists a curve $C \subset Z$ such that $L.C > 0$.*

Proof. The implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are [Som78, Proposition 1.7] or [GrKü15, Theorem 2.44].
The implication (iv) $\Rightarrow$ (ii) is obvious.
To show the implication (i) $\Rightarrow$ (iv), let $Z \subset X$ be a subvariety such that $\dim(Z) > q$. Since by hypothesis L is q-ample, then by Proposition I.2.11 (iii) we get that $L_{|Z}$ is q-ample. Since $\dim(Z) > q$, then $L_{|Z}$ is $(\dim(Z) - 1)$-ample, whence by Theorem I.2.23 we get that $-L_{|Z}$ is not pseudoeffective and we conclude.
The last statement is [Mat13, Theorem 1.4]. □

Remark I.2.22. The geometric meaning of Theorem I.2.21 is more evident in the case of normal varieties. Indeed, if X is a normal variety and L is a semiample line bundle on X, then by [Laz04a, Theorem 2.1.27] there exists an algebraic fiber space

$$\phi : X \to Y,$$

called the semiample fibration of L, such that for all integers $m \geq 1$ sufficiently large and divisible

$$\phi = \phi_{|mL|}, \quad \phi_{|mL|}(X) = Y.$$

Hence by Theorem I.2.21 we get that L is q-ample if and only if the dimension of the fibers of ϕ is $\leq q$.

The $(n-1)$-ampleness is fully understood. Indeed we have the following theorem.

THEOREM I.2.23 (Characterization of $(n-1)$-ampleness; [Tot13, Theorem 9.1]). *Let X be a variety and let D be an $\mathbb{R}$-divisor on X. Then the following conditions are equivalent:*

(i) *D is $(n-1)$-ample.*
(ii) *$-D$ is not pseudoeffective.*

PROOF. Totaro established the result for line bundles. This immediately extends to $\mathbb{Q}$-divisors. For the general case observe that if D is $(n-1)$-ample, then by Theorem I.2.18 we can take a sufficiently small ample $\mathbb{R}$-divisor A such that $D - A$ is an $(n-1)$-ample $\mathbb{Q}$-divisor. It follows that $-D + A$ is not pseudoeffective, whence $-D$ is also not pseudoeffective.
For the converse note that if $-D$ is not pseudoeffective, then for all sufficiently small ample $\mathbb{R}$-divisors A such that $D - A$ is a $\mathbb{Q}$-divisor we have that $-D + A$ is not pseudoeffective. Then by the result for $\mathbb{Q}$-divisors $D - A$ is q-ample. It follows that $D = (D - A) + A$ is q-ample. □

The $(n-2)$-ampleness of big line bundles is characterized in terms of its restrictions to the subvarieties of X of dimension $n-1$. Indeed whe have the following result of Brown.

THEOREM I.2.24 ([Bro12, Corollary 1.2]). *Let X be a smooth variety and let L be a big line bundle on X. Then the following conditions are equivalent:*

(i) *L is $(n-2)$-ample.*
(ii) *For all subvarieties $Z \subset X$ of dimension $n-1$ we have that $-L_{|Z}$ is not pseudoeffective.*

The following result establishes an interesting relation between the partial ampleness of a line bundle and its augmented base locus.

THEOREM I.2.25 (Restriction to augmented base locus; [Bro12, Theorem 1.1]). *Let L be a line bundle on X. Then L is q-ample if and only if the restriction $L_{|\mathbf{B}_+(L)}$ is q-ample.*

REMARK I.2.26. We refer to Subsections I.2.2 and I.2.3 (more precisely to Theorems I.2.37 and I.2.41) for other geometric characterizations of partial ampleness with arbitrarily q, due to the author. The first one is a generalization of Theorem I.2.21, the second one is a generalization of Theorems I.2.23 and I.2.24.

It is natural to compare the notion of partial ampleness given by Condition (+) with the usual one. The following theorem shows that the new notion is stronger than the old one.

THEOREM I.2.27 ([Kür13, Theorem B, Corollary 2.6]). *Let L be a line bundle on X. Then for every coherent sheaf $\mathcal{F}$ on X there exists an integer $m_{L,\mathcal{F}} > 0$ such that*

$$H^i(X, L^m \otimes M \otimes \mathcal{F}) = 0$$

for all nef line bundles M, $m > m_{L,\mathcal{F}}$ and $i > \dim(\mathbf{B}_+(L))$. As a consequence, if D is an $\mathbb{R}$-divisor such that $\dim(\mathbf{B}_+(D)) \leq q$, then D is q-ample.

PROOF. We only show that, if D is an $\mathbb{R}$-divisor such that $\dim(\mathbf{B}_+(D)) \leq q$, then D is q-ample (this is not explicitly contained in the original statement of Küronya). For line bundles it follows by the main statement. This immediately extends to $\mathbb{Q}$-divisors by Proposition I.1.19 (ii).
For an $\mathbb{R}$-divisor D, by Proposition I.1.17 (i) there exists an ample $\mathbb{R}$-divisor A such that $D - A$ is a $\mathbb{Q}$-divisor and $\mathbf{B}_+(D) = \mathbf{B} + (D - A)$. By the result on $\mathbb{Q}$-divisors $D - A$ is q-ample. Thus $D = (D - A) + A$ is q-ample and we conclude. □

Remark I.2.28. If $q > 0$, we cannot expect the reverse implication. Namely, in general a q-ample $\mathbb{R}$-divisor does not satisfy Condition (+).
If $q = n - 1$ this is immediate to see using Theorem I.2.23. Indeed an $(n-1)$-ample divisor is not always big, whence by Remark I.1.42 it does not satisfy Condition (+).
Moreover, Example I.4.7 shows that for $q = 1$ this is not true, even if L is a semiample line bundle. Thus we disprove a result of Choi (see [Choi14, Theorem 1.1] and Remark I.3.26).

Proposition I.2.29 (Partial ampleness in families; [Tot13, Theorem 8.1]). *Let $f : X \to T$ be a flat projective morphism of schemes over $\mathbb{Z}$ with connected fibers and let L be a $\mathbb{Q}$-line bundle on X. Suppose that $L_{t_0} = L_{|X_{t_0}}$ is q-ample for some $t_0 \in T$. Then there exists an open neighbourhood $U \subset T$ of t_0 such that $L_t = L_{|X_t}$ is q-ample for all $t \in U$.*

Proof. Totaro proved the result for line bundles. This immediately extends to $\mathbb{Q}$-line bundles by Remark I.2.6. □

Trying to generalize the notion of nefness, mimicking the partial ampleness, Lau (see [Lau19]) introduced the following definition.

Definition I.2.30. Let X be a variety and let D be an $\mathbb{R}$-divisor on X. D is *q-almost ample* if there exists an ample divisor A on X such that $D + \epsilon A$ is q-ample for all real numbers $\epsilon > 0$.

Remark I.2.31. For $q = 0$ we recover the notion of nefness.

Remark I.2.32. It's easy to see that an $\mathbb{R}$-divisor is q-almost ample if and only if there exists an ample divisor A on X such that $D + \frac{1}{m}A$ is q-ample for all integers $m \geq 1$, if and only if $D + \epsilon A$ is q-ample for all ample divisors A and real numbers $\epsilon > 0$.

It is interesting to study the shape of the cones of partially ample divisors in the Néron-Severi vector space $\mathrm{N}^1(X)_{\mathbb{R}}$.

Definition I.2.33. We denote by $\mathrm{Amp}^q(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$ (resp. $\mathrm{Alm}^q(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$) the cone of classes of q-ample (resp. q-almost ample) $\mathbb{R}$-divisors.

Remark I.2.34. By Remark I.2.16 it's easy to see that $\mathrm{Amp}^q(X)$ and $\mathrm{Alm}^q(X)$ are cones, while by Theorem I.2.18 we get that $\mathrm{Amp}^q(X)$ is open. However, by Remark I.2.20, in general $\mathrm{Amp}^q(X)$ and $\mathrm{Alm}^q(X)$ are not convex if $q > 0$.
By Remark I.2.3 and Proposition I.2.19 we get the inclusions

$$\mathrm{Amp}^q(X) \subset \mathrm{Alm}^q(X), \quad \mathrm{Amp}^q(X) \subset \mathrm{Amp}^{q+1}(X), \quad \mathrm{Alm}^q(X) \subset \mathrm{Alm}^{q+1}(X).$$

Since

$$\mathrm{Amp}(X) = \mathrm{Amp}^0(X) \subset \mathrm{Amp}^q(X) \subset \mathrm{Alm}^q(X)$$

and $\mathrm{Amp}(X)$ is full-dimensional, then $\mathrm{Amp}^q(X)$ and $\mathrm{Alm}^q(X)$ are full-dimensional.
Finally, by Theorem I.2.27 we get that $\mathrm{C}^q_+(X) \subset \mathrm{Amp}^q(X)$, while by Proposition I.1.47 and Theorem I.2.27 we get that $\mathrm{C}^q_-(X) \subset \mathrm{Alm}^q(X)$.

The following is another generalization of Kleiman's theorem.

Theorem I.2.35 (Second generalization of Kleiman's theorem). *Let X be a variety. Then:*

(i) $\mathrm{Amp}^q(X) = \mathrm{int}(\mathrm{Alm}^q(X))$.
(ii) $\mathrm{Alm}^q(X) = \overline{\mathrm{Amp}^q(X)}$.

Proof. By Remark I.2.34 we get that $\mathrm{Amp}^q(X)$ and $\mathrm{Alm}^q(X)$ are full-dimensional cones, $\mathrm{Amp}^q(X)$ is open and $\mathrm{Amp}^q(X) \subset \mathrm{Alm}^q(X)$. Thus we get the inclusion $\mathrm{Amp}^q(X) \subset \mathrm{int}(\mathrm{Alm}^q(X))$. Take now $[D] \in \mathrm{int}(\mathrm{Alm}^q(X))$. Then there exists a sufficiently small ample $\mathbb{R}$-divisor A such that $[D'] = [D - A] \in \mathrm{Alm}^q(X)$. By Remark I.2.32 we get that $[D] = [D' + A] \in \mathrm{Amp}^q(X)$. Hence we

get the inclusion $\mathrm{int}(\mathrm{Alm}^q(X)) \subset \mathrm{Amp}^q(X)$ and we conclude (i).
To see (ii) take $[D] \in \mathrm{Alm}^q(X)$. By definition there exists an ample divisor A on X such that $D + \epsilon A$ is q-ample for all real numbers $\epsilon > 0$. It follows that $[D + \epsilon A] \in \mathrm{Amp}^q(X)$, whence $[D] = \lim_{\epsilon \to 0}[D + \epsilon A] \in \overline{\mathrm{Amp}^q(X)}$. Thus we get $\mathrm{Alm}^q(X) \subset \overline{\mathrm{Amp}^q(X)}$.
Take now $[D] \in \overline{\mathrm{Amp}^q(X)}$. Then $[D] = \lim_{m\to\infty}[D_m]$, where $\{D_m\}_{m\geq 1}$ are $\mathbb{R}$-divisors such that $[D_m] \in \mathrm{Amp}^q(X)$ for all $m \geq 1$. Take an ample divisor A and a real number $\epsilon > 0$. Since ampleness is an open property, then for all m sufficiently large $\epsilon A + (D - D_m)$ is ample. By Proposition I.2.19 we get that $D + \epsilon A = D_m + \epsilon A + (D - D_m)$ is q-ample, whence $[D] \in \mathrm{Alm}^q(X)$ and we get $\overline{\mathrm{Amp}^q(X)} \subset \mathrm{Alm}^q(X)$. □

The following is a generalization of [Laz04a, Proposition 1.4.14].

PROPOSITION I.2.36 (Partial almost ampleness in families). *Let $f : X \to T$ be a flat projective morphism of schemes over $\mathbb{Z}$ with connected fibers and let L be an $\mathbb{Q}$-line bundle on X. Suppose that $L_{t_0} = L_{|X_{t_0}}$ is q-almost ample for some $t_0 \in T$. Then there exists a countable union of proper subvarieties $B \subset T$ such that $L_t = L_{|X_t}$ is q-almost ample for all $t \in T\backslash B$.*

PROOF. Let A be an ample line bundle on X and let $m \geq 1$ be an integer. Since by hypothesis L_{t_0} is q-almost ample, then by Remark I.2.32 we get that $(L + \frac{1}{m}A)_{t_0}$ is q-ample. It follows by Proposition I.2.29 that there exists an open neighbourhood $U_m \subset T$ of t_0 such that $(L + \frac{1}{m}A)_t$ is q-ample for all $t \in U_m$. Thus L_t is q-almost ample for all $t \in \bigcap_{m\geq 1} U_m$.
Set $B_m = T\backslash U_m$ for all integers $m \geq 1$ and $B = \bigcup_{m\geq 1} B_m$. Take $t \in T\backslash B$. Since $T\backslash B = \bigcap_{m\geq 1} U_m$, again by Remark I.2.32, we conclude. □

I.2.2. Partial ampleness and semiample fibration.

In this Subsection we provide a generalization of Theorem I.2.21 (see Theorem I.2.37).
Let X be a variety, let L be a $\mathbb{Q}$-line bundle on X with $\mathrm{k}(X,L) \geq 0$. By Lemma I.1.7 there exists an integer $m_0 > 0$ such that $m_0 L$ is a line bundle and $\mathbf{B}(L) = \mathrm{Bs}\,|m_0 L|$. Moreover, let $\pi : \hat{X} \to X$ be the blow-up of X along the ideal sheaf $\mathcal{I} = \mathcal{I}_{\mathbf{B}(L)}$ (that is the blow-up along the base ideal $\mathfrak{b}|m_0 L|$) with exceptional divisor E.
Then $O_{\hat{X}}(-E) = \pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{X}}(1)$ is an invertible sheaf and by [Har77, Example II.7.17.3] we have a decomposition

$$\pi^*(m_0 L) = M + E. \tag{I.2.7}$$

where M is a base-point-free line bundle on $\hat{X}$ (observe that M depends on m_0).
Since M is semiample, we can characterize the partial ampleness of M using Theorem I.2.21. Moreover, if L is assumed to be semiample, then $M \cong m_0 L$.
It is then natural to compare the partial amplenesses of L and M.
Inspired by Theorem I.2.21 we prove the following result.

THEOREM I.2.37 (First geometric characterization of partial ampleness). *Let X be a variety, let L be a $\mathbb{Q}$-line bundle on X such that $\mathrm{k}(X,L) \geq 0$ and let $m_0 \geq 1$ be an integer such that $m_0 L$ is a line bundle and $\mathbf{B}(L) = \mathrm{Bs}\,|m_0 L|$. Moreover let M be the line bundle of (I.2.7). We have:*

(i) *If $\dim(\mathbf{B}(L)) \leq q$, then M q-ample implies L q-ample.*
(ii) *If $\dim(\mathbf{B}(L)) \leq q-1$, then M is q-ample if and only if L is q-ample.*

PROOF. Let A be an ample line bundle on X such that $\hat{A} = \pi^* A - E$ is ample on $\hat{X}$ and let $C \geq 1$ be a constant. Moreover take two integers $m \geq 1$ and $t \geq 1$. We want to compare the cohomology groups $H^i(X, mm_0 L - tA)$ and $H^i(\hat{X}, mM - t\hat{A})$ with the Leray spectral sequence

$$E_1^{p,r} = H^p(X, R^{r-p}\pi_*(mM - t\hat{A})) \Rightarrow H^r(\hat{X}, mM - t\hat{A})$$

(where for spectral sequences we are using the notation of [HiSt71]).
By projection formula we get that

$$E_1^{p,r} = H^p(X, R^{r-p}\pi_*(\pi^*(mm_0 L - tA) + (t-m)E)) =$$
$$= H^p(X, \mathcal{O}_X(mm_0 L - tA) \otimes R^{r-p}\pi_* \mathcal{O}_{\hat{X}}((t-m)E)).$$

Since by [Har77, Proposition II.7.10] we get that $-E$ is π-ample, it follows by [Laz04a, Theorem 1.7.6] that for all $1 \leq t \leq C$ there exists an integer $m_t > 0$ such that

$$R^i \pi_* \mathcal{O}_{\hat{X}}((t-m)E) = 0$$

for all $i > 0$ and $m \geq m_t$. Taking $M_C = \max\{m_1, \ldots, m_C, C+1\}$ we have that

$$E_1^{p,r} = \begin{cases} 0 & \text{if } r-p \neq 0 \\ H^p(X, \mathcal{O}_X(mm_0 L - tA) \otimes \pi_* \mathcal{O}_{\hat{X}}((t-m)E)) & \text{if } r-p = 0 \end{cases} \tag{I.2.8}$$

for all $m \geq M_C$ and $1 \leq t \leq C$.
If $r-p \neq 0$ we have that $E_\infty^{p,r} = E_1^{p,r} = 0$. If $r-p=0$ consider the maps

$$E_l^{r+l+1,r+1} \to E_l^{r,r} \to E_l^{r-l-1,r-1}$$

with $l \geq 1$. Since $E_l^{r+l+1,r+1} = E_l^{r-l-1,r-1} = 0$ for all $l \geq 1$ we get that $E_\infty^{r,r} = E_1^{r,r}$.
Consider now an integer $i \geq 0$ and the filtration

$$H^i(\hat{X}, mM - t\hat{A}) = F^0 \supset F^1 \supset \cdots \supset F^k \supset F^{k+1} = 0$$

with $F^p/F^{p+1} = \mathrm{Gr}^p(H^i(\hat{X}, \pi^*(mm_0L) - t\hat{A})) = E_\infty^{p,i}$ for all $p \geq 0$.
Since $F^p/F^{p+1} = 0$ for all $p \neq i$ the filtration is

$$H^i(\hat{X}, mM - t\hat{A}) = F^0 = F^i \supset F^{i+1} = 0.$$

By the exact sequence

$$0 \to F^{i+1} \to F^i \to F^i/F^{i+1} \to 0$$

and by (I.2.8) we get that

$$H^i(\hat{X}, mM - t\hat{A}) \cong H^i(X, \mathcal{O}_X(mm_0L - tA) \otimes \pi_*\mathcal{O}_{\hat{X}}((t-m)E)) \tag{I.2.9}$$

for all $i \geq 0$, $m \geq M_C$ and $1 \leq t \leq C$.
Now we prove (i). Assume that M is q-ample and that $\dim(\mathbf{B}(L)) \leq q$. Let $C = C_{X,A,q}$ be the constant of Theorem I.2.9. To show that L is q-ample we prove that m_0L is q-ample by finding an integer $m_L > 0$ such that

$$H^i(X, m_Lm_0L - tA) = 0$$

for all $i > q$ and $1 \leq t \leq C$. We then have that L is q-ample by Lemma I.2.4.
To see this fix $i > q$, $m \geq M_C$ and $1 \leq t \leq C$. Since $m > t$, then $\pi_*\mathcal{O}_{\hat{X}}((t-m)E)$ is an ideal sheaf supported on $\mathbf{B}(L)$. Set now $Z = Z(\pi_*\mathcal{O}_{\hat{X}}((t-m)E))$ and consider the exact sequence

$$0 \to \mathcal{O}_X(mm_0L - tA) \otimes \pi_*\mathcal{O}_{\hat{X}}((t-m)E) \to \mathcal{O}_X(mm_0L - tA) \to \mathcal{O}_Z(mm_0L - tA) \to 0.$$

By (I.2.9) we get that

$$h^i(X, mm_0L - tA) \leq h^i(\hat{X}, mM - t\hat{A}) + h^i(Z, (mm_0L - tA)_{|Z}).$$

Since $\dim(\mathbf{B}(L)) \leq q$ the second term on the right is zero and we have the inequality

$$h^i(X, mm_0L - tA) \leq h^i(\hat{X}, mM - t\hat{A}).$$

Since M is q-ample we have that for all $1 \leq t \leq C$ there exists an integer $\hat{m}_t > 0$ such that

$$H^i(X, mM - t\hat{A}) = 0$$

for all $i > q$ and $m \geq \hat{m}_t$. Taking $m_L = \max\{\hat{m}_1, \ldots, \hat{m}_C, M_C\}$ we have that

$$H^i(X, m_Lm_0L - tA) = 0$$

for all $i > q$ and $1 \leq t \leq C$. This proves (i).
The proof of (ii) is similar. Assume that L is q-ample and that $\dim(\mathbf{B}(L)) \leq q-1$. Let $C = C_{\hat{X},\hat{A},q}$ be the constant of Theorem I.2.9. To show that M is q-ample we find an integer $m_M > 0$ such that

$$H^i(\hat{X}, m_MM - t\hat{A}) = 0$$

for all $i > q$ and $1 \leq t \leq C$.
Take now $i > q$, $m \geq M_C$ and $1 \leq t \leq C$. For all integers $s \geq 1$ we have an exact sequence

$$0 \to \mathcal{O}_{\hat{X}}((t+s-1-m)E) \to \mathcal{O}_{\hat{X}}((t+s-m)E) \to \mathcal{O}_E((t+s-m)E) \to 0.$$

Applying the functor π_* we get the exact sequence

$$0 \to \pi_*\mathcal{O}_{\hat{X}}((t+s-1-m)E) \xrightarrow{\varphi_s} \pi_*\mathcal{O}_{\hat{X}}((t+s-m)E) \to \mathcal{F}_s \to 0$$

where $\mathcal{F}_s = \operatorname{Coker}\varphi_s$ is a coherent sheaf supported on $\mathbf{B}(L)$. Tensoring by $\mathcal{O}_X(mm_0L - tA)$ we get the exact sequence

$$0 \to \mathcal{O}_X(mm_0L - tA) \otimes \pi_*\mathcal{O}_{\hat{X}}((t+s-1-m)E) \to \mathcal{O}_X(mm_0L - tA) \otimes \pi_*\mathcal{O}_{\hat{X}}((t+s-m)E)$$
$$\to \mathcal{O}_X(mm_0L - tA) \otimes \mathcal{F}_s \to 0.$$

By (I.2.9) we have that

$$h^i(\hat{X}, mM - t\hat{A}) = h^i(X, \mathcal{O}_X(mm_0L - tA) \otimes \pi_*\mathcal{O}_{\hat{X}}((t-m)E)) \leq$$

$$\leq h^i(X, \mathcal{O}_X(mm_0L - tA)) + \sum_{s=1}^{m-t} h^{i-1}(\mathbf{B}(L), \mathcal{O}_X(mm_0L - tA) \otimes \mathcal{F}_s).$$

Since $\dim(\mathbf{B}(L)) \leq q - 1$, the second term on the right hand side is zero. Thus we have the inequality

$$h^i(\hat{X}, mM - t\hat{A}) \leq h^i(X, mm_0L - tA).$$

Since L is q-ample, for all $1 \leq t \leq C$ there exists an integer $\hat{m}_t > 0$ such that

$$H^i(X, mm_0L - tA) = 0$$

for all $i > q$ and $m \geq \hat{m}_t$. Taking $m_M = \max\{\hat{m}_1, \ldots, \hat{m}_C, M_C\}$ we conclude. □

Remark I.2.38. Examples I.4.8 and I.4.9 show respectively that the assertions (i) and (ii) of Theorem I.2.37 are sharp. Namely, if M is q-ample and $\dim(\mathbf{B}(L)) = q + 1$, it is not always true that L is q-ample. Moreover, if L is q-ample and $\dim(\mathbf{B}(L)) = q$, it is not always true that M is q-ample.

I.2.3. Partial ampleness and restriction to subschemes.

In this Subsection we provide a generalization of Theorems I.2.23 and I.2.24, that will give a geometric characterization of the partial ampleness of a line bundle L in terms of the behaviour of its restrictions to the subvarieties of X and that will work for all integers $q \geq 0$ (see Theorem I.2.41).
Before stating the result we need some preliminary work.

Lemma I.2.39. *Let X be a variety and let L be a pseudoeffective $\mathbb{R}$-line bundle on X. Then for all very general hyperplane sections H the restriction $L_{|H}$ is pseudoeffective.*

Proof. Since L is pseudoeffective, there exists a sequence $\{D_m\}_{m\geq 1}$ of effective $\mathbb{R}$-divisors such that $[L] = \lim_{m\to\infty}[D_m]$ in $\mathrm{N}^1(X)_{\mathbb{R}}$. Since $\bigcup_{m\geq 1}\mathrm{Supp}(D_m)$ is (at most) a countable union of closed subschemes of dimension $n-1$, then we can take a hyperplane section H such that for all $m \geq 1$ and G in $\mathrm{Supp}(D_m)$ we have that $H \neq G$. It follows that $[D_{m|H}]$ is effective for all $m \geq 1$, whence $[L_{|H}] = \lim_{m\to\infty}[D_{m|H}]$ is pseudoeffective. □

With the help of Lemma I.2.39 we can prove the following result.

Proposition I.2.40. *Let L be an $\mathbb{R}$-line bundle on X. Then the following conditions are equivalent:*

(i) *For all subvarieties $Z \subset X$ of dimension $q+1$ we have that $L_{|Z}$ is q-ample.*
(ii) *For all subvarieties $Z \subset X$ of dimension $q+1$ we have that $-L_{|Z}$ is not pseudoeffective.*
(iii) *For all subvarieties $Z \subset X$ of dimension $> q$ we have that $-L_{|Z}$ is not pseudoeffective.*

Proof. To obtain (i) $\Leftrightarrow$ (ii) we observe that, since the dimension of Z is $q+1$, we can apply Theorem I.2.23.
To get (ii) $\Rightarrow$ (iii) suppose that there exists a subvariety Z of dimension $> q$ such that $-L_{|Z}$ is pseudoeffective. By hypothesis $\dim(Z) > q+1$. Taking a very general hyperplane section H on Z we have by Lemma I.2.39 that $-(L_{|Z})_{|H}$ is also pseudoeffective. If the dimension of Z is $q+2$ we get a contradiction. Otherwise we can iterate the procedure until we find a subvariety W of dimension $q+1$ such that $-L_{|W}$ is pseudoeffective. Thus we reach a contradiction and we conclude.
Finally, the implication (iii) $\Rightarrow$ (ii) is obvious. □

The rest of the section is devoted to the proof that, under the addictional hypothesis that $\dim(\mathbf{B}_+(L)) \leq q+1$, the conditions of Proposition I.2.40 are also equivalent to the q-ampleness of L.

Theorem I.2.41 (Second geometric characterization of partial ampleness). *Let L be an $\mathbb{R}$-line bundle on X such that $\dim(\mathbf{B}_+(L)) \leq q+1$. Then the following conditions are equivalent:*

(i) *L is q-ample.*
(ii) *For all subvarieties $Z \subset X$ of dimension $q+1$ we have that $L_{|Z}$ is q-ample.*
(iii) *For all subvarieties $Z \subset X$ of dimension $q+1$ we have that $-L_{|Z}$ is not pseudoeffective.*
(iv) *For all subvarieties $Z \subset X$ of dimension $> q$ we have that $-L_{|Z}$ is not pseudoeffective.*

Remark I.2.42. For $q = n-1$ we recover Theorem I.2.23, removing the hypothesis that X is a variety.
Moreover, if X is a reduced pure dimensional scheme, then by Proposition I.1.39 (ii) we have that L is big if and only if $\dim(\mathbf{B}_+(L)) \leq n-1$. Thus for $q = n-2$ we generalize Theorem I.2.24.
Finally, for $q = 0$ we get the interesting fact that an $\mathbb{R}$-line bundle L such that $\dim(\mathbf{B}_+(L)) \leq 1$ is ample if and only if it is strictly nef (that is $L.C > 0$ for all curves C in X).

The next theorem of Brown is crucial in the proof of our characterization. For completeness, we provide a slightly different version of the proof.

THEOREM I.2.43 ([Bro12, Theorem 2.1]). *Let L and L' be two line bundles on X and let $a > 0$, $b > 0$ be two integers such that $h^0(X, L') > 0$ and $A := aL - bL'$ is ample. Moreover let $s \in H^0(X, L')$ be a non-zero section of L' and let $Z(s)$ be the associated closed subscheme. If L is not q-ample, then $L_{|Z(s)}$ is not q-ample.*

PROOF. First of all we make some reductions.
We may assume that $a = 1$. Indeed by Lemma I.2.4 we get that L is q-ample if and only if aL is q-ample, while $L_{|Z(s)}$ is q-ample if and only if $aL_{|Z(s)}$ is q-ample.
We may assume that $b = 1$. To see this take the non-zero section $s^{\otimes b} \in H^0(X, bL')$. We have that $Z(s^{\otimes b})_{red} = Z(s)_{red}$. Thus if we show that $L_{|Z(s^{\otimes b})}$ is not q-ample, then, by Proposition I.2.11 (i), $L_{|Z(s)}$ is also not q-ample. From now on we assume $a = b = 1$, whence $L' = L - A$.
We may assume that L is $(q + 1)$-ample. To see this consider the integer

$$q_0 = \max\{l \geq 0 : L \text{ is not } l\text{-ample}\}$$

and observe that $q \leq q_0 < n$. By definition L is $(q_0 + 1)$-ample but not q_0-ample. If we show that $L_{|Z(s)}$ is not q_0-ample, then it is also not q-ample. Thus we can replace q with q_0.
Consider now the exact sequence

$$0 \to \mathcal{F} \to -L' \xrightarrow{\varphi} \mathcal{O}_X \to O_{Z(s)} \to 0$$

for some sheaf $\mathcal{F}$, that is non-zero if s is zero on some irreducible component of X. Let $\mathcal{G} = \operatorname{Im} \varphi$. Since X is noetherian by [Har77, Proposition II.5.7] we obtain that $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves. We have two short exact sequences

$$0 \to \mathcal{F} \to -L' \to \mathcal{G} \to 0 \tag{I.2.10}$$

and

$$0 \to \mathcal{G} \to \mathcal{O}_X \to \mathcal{O}_{Z(s)} \to 0. \tag{I.2.11}$$

We assume that $L_{|Z(s)}$ is q-ample and we reach a contradiction.
Denote by $C = C_{X,A,q}$ the constant of Theorem I.2.9. Since L is $(q+1)$-ample by Proposition I.2.13 there exists an integer $m_1 > 0$ such that

$$H^i(X, \mathcal{F}(-CA)(mL + bA)) = 0 \tag{I.2.12}$$

for all $i > q + 1$, $m > m_1$ and $b \geq 0$. Moreover, since $L_{|Z(s)}$ is q-ample, again by Proposition I.2.13 there exists an integer $m_2 > 0$ such that

$$H^i(Z(s), \mathcal{O}_{Z(s)}(-CA_{|Z(s)})((mL + bA)_{|Z(s)})) = 0 \tag{I.2.13}$$

for all $i > q$, $m > m_2$ and $b \geq 0$. Take $M_1 = \max\{m_1, m_2\}$.
Since A is ample there exists an integer $m_3 > 0$ such that

$$H^i(X, (m - r)A + kL') = 0 \tag{I.2.14}$$

for all $i > 0$, $m > m_3$, $1 \leq r \leq C$ and $0 \leq k \leq M_1$. Moreover, since L is $(q + 1)$-ample, by Theorem I.2.9 there exists an integer $m_4 > 0$ such that

$$H^i(X, mL - rA) = 0 \tag{I.2.15}$$

for all $i > q + 1$, $m > m_4$ and $1 \leq r \leq C$. Take $M_2 = \max\{m_3, m_4\}$.
Since L is not q-ample there exist an $i_0 > q$, an $m_0 > M_2$ and an integer r_0 such that $1 \leq r_0 \leq C$ and

$$H^{i_0}(X, m_0L - r_0A) \neq 0.$$

Indeed otherwise we would have that

$$H^i(X, mL - rA) = 0$$

for all $i > q$, $m > M_2$ and $1 \le r \le C$ and hence that L is q-ample by Theorem I.2.9. Since $m_0 > M_2 \ge m_4$ by (I.2.15) we have that

$$H^i(X, m_0L - r_0A) = 0$$

for all $i > q + 1$. Hence $i_0 = q + 1$ and

$$H^{q+1}(X, m_0L - r_0A) \neq 0. \tag{I.2.16}$$

Consider now the set

$$P = \{l \in \mathbb{N} \ : \ H^{q+1}(X, (m_0 - r_0)A + kL') = 0, \ \forall\ 0 \le k \le l\}.$$

Since $m_0 > M_2 \ge m_3$ by (I.2.14) we get that $l \in P$ for all $0 \le l \le M_1$. Moreover by (I.2.16) we have that $l \notin P$ for all $l \ge m_0$. Hence there is a well-defined $k_0 = \max P + 1$ such that $k_0 \le m_0$ and

$$H^{q+1}(X, (m_0 - r_0)A + (k_0 - 1)L') = 0, \quad H^{q+1}(X, (m_0 - r_0)A + k_0L') \neq 0. \tag{I.2.17}$$

Tensoring (I.2.10) by $(m_0 - r_0)A + k_0L'$ we get the exact sequence

$$0 \to \mathcal{F}((m_0 - r_0)A + k_0L') \to (m_0 - r_0)A + (k_0 - 1)L' \to \mathcal{G}((m_0 - r_0)A + k_0L') \to 0. \tag{I.2.18}$$

Since $k_0 > M_1 \ge m_1$ and $m_0 - r_0 - k_0 + C \ge 0$, by (I.2.12) we have that

$$H^{q+2}(X, \mathcal{F}((m_0 - r_0)A + k_0L')) = H^{q+2}(X, \mathcal{F}((m_0 - r_0 - k_0)A + k_0L)) =$$

$$= H^{q+2}(X, \mathcal{F}(-CA)((m_0 - r_0 - k_0 + C)A + k_0L)) = 0.$$

By (I.2.17) and (I.2.18) we get that

$$H^{q+1}(X, \mathcal{G}((m_0 - r_0)A + k_0L')) = 0. \tag{I.2.19}$$

Tensoring (I.2.11) by $(m_0 - r_0)A + k_0L'$ we get the exact sequence

$$0 \to \mathcal{G}((m_0 - r_0)A + k_0L') \to (m_0 - r_0)A + k_0L' \to ((m_0 - r_0)A + k_0L')_{|Z(s)} \to 0. \tag{I.2.20}$$

Since $k_0 > M_1 \ge m_2$ and $m_0 - r_0 - k_0 + C \ge 0$, by (I.2.13) we have that

$$H^{q+1}(Z(s), ((m_0 - r_0)A + k_0L')_{|Z(s)}) = H^{q+1}(Z(s), ((m_0 - r_0 - k_0)A + k_0L)_{|Z(s)}) =$$

$$= H^{q+1}(Z(s), \mathcal{O}_{Z(s)}(-CA_{|Z(s)})(((m_0 - r_0 - k_0 + C)A + k_0L)_{|Z(s)})) = 0.$$

It follows by (I.2.19) and (I.2.20) that

$$H^{q+1}(X, (m_0 - r_0)A + k_0L') = 0.$$

This contradicts (I.2.17), whence $L_{|Z(s)}$ is not q-ample. □

The following lemma, which will not be used later, shows that the hypothesis of the previous theorem is in fact equivalent to the bigness of L.

Lemma I.2.44. *Let L be a line bundle on X. Then the following conditions are equivalent:*

(i) *L is big, that is there exist an ample $\mathbb{Q}$-line bundle A and an effective $\mathbb{Q}$-line bundle E such that $L \sim_{\mathbb{Q}} A + E$.*

(ii) *There exist a line bundle L' and two integers $a > 0$ and $b > 0$ such that $h^0(X, L') > 0$ and $aL - bL'$ is ample.*

Proof. If L is big, then there exist an ample $\mathbb{Q}$-line bundle A and an effective $\mathbb{Q}$-line bundle E such that $L \sim_{\mathbb{Q}} A + E$. Take $k > 0$ such that kA and kE are line bundles and $kL \sim_{\mathbb{Z}} kA + kE$. Set $L' = kL - kA$. We get that $h^0(X, L') = h^0(X, kE) > 0$. Taking $a = k$ and $b = 1$ we get that $aL - bL' = kA$ is ample. This proves (i) $\Rightarrow$ (ii).
To prove (ii) $\Rightarrow$ (i) set $A = aL - bL'$ and observe that $L = \frac{1}{a}A + \frac{b}{a}L'$ is big because is a sum of an ample $\mathbb{Q}$-line bundle and of an effective $\mathbb{Q}$-line bundle. □

The following proposition is a generalization of [Bro12, Corollary 2.3] and it is the key part of the proof of Theorem I.2.41.

Proposition I.2.45. *Let X be a pure dimensional scheme and let L be an $\mathbb{R}$-line bundle on X such that $\dim(\mathbf{B}_+(L)) \le q + 1$. If L is not q-ample, then there exists a subvariety $Z \subset X$ of dimension $q + 1$ such that $L_{|Z}$ is not q-ample.*

Proof. We first prove the result for a line bundle L by induction on $n - q - 1 \ge 0$.
If $q = n - 1$ we need a variety of dimension n. Since L is not q-ample, then by Proposition I.2.11 (ii) there exists an irreducible component $X_i \subset X$ such that $L_{|X_i}$ is not q-ample. Take $Z = (X_i)_{red}$. By Proposition I.2.11 (i) we get that $L_{|Z}$ is not q-ample.
If $0 \le q \le n - 2$ consider a very ample line bundle A on X. By Theorem I.1.18 there exists an integer $m_0 \ge 1$ such that

$$\mathbf{B}_+(L) = \mathrm{Bs}\,|m_0 L - A|.$$

Denote $L' = m_0 L - A$.
By inductive hypothesis there exists a subvariety $W \subset X$ such that $\dim(W) = q + 2$ and $L_{|W}$ is not q-ample. Since $\dim(W) = q + 2 > q + 1 \ge \dim(\mathbf{B}_+(L))$, then $W \not\subset \mathbf{B}_+(L) = \mathrm{Bs}\,|L'|$, whence there exists a section $s \in H^0(X, L')$ such that $s_{|W} \ne 0$.
Then $Z(s_{|W})$ is a scheme of pure dimension $q + 1$. Since $L_{|W}$ is not q-ample and

$$A_{|W} = m_0 L_{|W} - L'_{|W}$$

is ample, by Theorem I.2.43 on W with $a = m_0$ and $b = 1$ we get that $L_{|Z(s_{|W})}$ is not q-ample. Then by Proposition I.2.11 (ii) there exists an irreducible component $W_i \subset Z(s_{|W})$ such that $L_{|W_i}$ is not q-ample. Take $Z = (W_i)_{red}$. By Proposition I.2.11 (i) we get that $L_{|Z}$ is not q-ample, whence we conclude the proof in the case of line bundles.
This immediately extends to $\mathbb{Q}$-line bundles by Remark I.2.6.
Let now L be a $\mathbb{R}$-line bundle and let A be an ample $\mathbb{R}$-line bundle such that $L + A$ is a $\mathbb{Q}$-line bunde. By Proposition I.1.35 (ii) we get that $\mathbf{B}_+(A) = \emptyset$. Hence by Proposition I.1.19 (iii) we get that

$$\mathbf{B}_+(L + A) \subset \mathbf{B}_+(L) \cup \mathbf{B}_+(A) = \mathbf{B}_+(L).$$

Thus $\dim(\mathbf{B}_+(L + A)) \le q + 1$, whence by the result for $\mathbb{Q}$-line bundles we get that there exists a subvariety $Z \subset X$ of dimension $q + 1$ such that $(L + A)_{|Z}$ is not q-ample. Since $A_{|Z}$ is ample we get that $L_{|Z}$ is not q-ample and we conclude. □

Now we are ready to give the proof of Theorem I.2.41.

Proof of Theorem I.2.41. By Proposition I.2.40 we have (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).
To get (i) $\Rightarrow$ (ii) note that, by Proposition I.2.11 (i), for all subvarieties $Z \subset X$ of dimension $q + 1$ we have that $L_{|Z}$ is q-ample.
To prove (ii) $\Rightarrow$ (i) we assume that L is not q-ample and we reach a contradiction. Since L is not q-ample, then by Proposition I.2.11 (ii) there exists an irreducible component X_i of X such that $L_{|X_i}$ is not q-ample. By Proposition I.1.24 (ii) we get that $\dim(\mathbf{B}_+(L_{|X_i})) \le \dim(\mathbf{B}_+(L) \le q + 1$, hence, by Proposition I.2.45 there exists a subvariety $Z \subset X_i \subset X$ of dimension $q + 1$ such that $L_{|Z}$ is not q-ample. This contradicts (ii), whence we conclude. □

I.2.4. Partial ampleness via blow-ups.

This section is devoted to the proof of the following theorem, that shows the behaviour of partially ample $\mathbb{Q}$-line bundles under blow-up.

THEOREM I.2.46. *Let X be a variety, let $Z \subset X$ be a smooth subvariety of codimension $e \geq 1$ contained in the smooth locus of X, let $\pi : \hat{X} \to X$ be the blow-up of X along Z and let L be a $\mathbb{Q}$-line bundle on X. If L is q-ample, then π^*L is $(q+e-1)$-ample.*

Before passing to the proof we need the following lemma. Although this is probably a well-known result, for completeness we present a proof.

LEMMA I.2.47. *Let X be a variety, let $Z \subset X$ be a smooth subvariety of codimension $e \geq 2$ contained in the smooth locus of X and let $\pi : \hat{X} \to X$ be the blow-up of X along Z, with exceptional divisor E. Then:*

(i) $\pi_*\mathcal{O}_{\hat{X}} \cong \mathcal{O}_X$.
(ii) $R^s\pi_*\mathcal{O}_{\hat{X}} = 0$ *for all* $s \geq 1$.
(iii) $\pi_*\mathcal{O}_{\hat{X}}(lE) \cong \mathcal{O}_X$ *for all* $l \geq 1$.
(iv) $R^s\pi_*\mathcal{O}_{\hat{X}}(lE) = 0$ *for all* $l \geq 1$ *and* $1 \leq s \neq e-1$.

PROOF. We write the proof only in the case $e \geq 3$ (the case $e = 2$ is very similar).
Consider the commutative diagram

$$\begin{array}{ccc} E & \overset{i}{\hookrightarrow} & \hat{X} \\ \downarrow p & & \downarrow \pi \\ Z & \overset{k}{\hookrightarrow} & X \end{array}.$$

(i) and (ii) are [Laz04a, Lemma 4.3.16] with $a = 0$, thus we have only to prove (iii) and (iv).
Since Z is a smooth subvariety of codimension e contained in the smooth locus of X, then the normal bundle $N_{Z/X}$ is a vector bundle of rank $e-1$. Hence the exceptional divisor E can be identified with the projective space bundle $\mathbb{P}(N^*_{Z/X})$ and $\mathcal{O}_E(E_{|E}) = \mathcal{O}_{\mathbb{P}(N^*_{Z/X})}(-1)$.
By [Laz04a, Appendix A] we get that

$$R^s p_*\mathcal{O}_E(lE_{|E}) = 0 \quad \forall\, l \geq 1,\ 0 \leq s \neq e-1; \tag{I.2.21}$$

$$R^{e-1} p_*\mathcal{O}_E(lE_{|E}) = 0 \quad \forall\, 1 \leq l \leq e-1. \tag{I.2.22}$$

Take now $l \geq 1$ and consider the exact sequence

$$0 \to \mathcal{O}_{\hat{X}}((l-1)E) \to \mathcal{O}_{\hat{X}}(lE) \to \mathcal{O}_E(lE) \to 0.$$

Applying the functor π_* we have a long exact sequence

$$0 \to \pi_*\mathcal{O}_{\hat{X}}((l-1)E) \to \pi_*\mathcal{O}_{\hat{X}}(lE) \to p_*\mathcal{O}_E(lE_{|E}) \to R^1\pi_*\mathcal{O}_{\hat{X}}((l-1)E) \to R^1\pi_*\mathcal{O}_{\hat{X}}(lE) \to \cdots.$$

Since by (I.2.21) we have that $p_*\mathcal{O}_E(lE_{|E}) = 0$ for all $l \geq 1$, we get that

$$\pi_*\mathcal{O}_{\hat{X}}(lE) \cong \pi_*\mathcal{O}_{\hat{X}}((l-1)E) \cong \pi_*\mathcal{O}_{\hat{X}} \quad \forall\, l \geq 1.$$

Thus by (i) we get (iii).
Since by (I.2.21) we have that $p_*\mathcal{O}_E(lE_{|E}) = R^1p_*\mathcal{O}_E(lE_{|E}) = 0$ for all $l \geq 1$, we have that

$$R^1\pi_*\mathcal{O}_{\hat{X}}(lE) \cong R^1\pi_*\mathcal{O}_{\hat{X}}((l-1)E) \cong R^1\pi_*\mathcal{O}_{\hat{X}} \quad \forall\, l \geq 1.$$

Thus by (ii) we get

$$R^1\pi_*\mathcal{O}_{\hat{X}}(lE) = 0 \quad \forall\, l \geq 1.$$

Since by (I.2.21) we have that $R^s p_* \mathcal{O}_E(lE_{|E}) = 0$ for all $l \geq 1$ and $2 \leq s \neq e-1$, we get

$$R^s \pi_* \mathcal{O}_{\hat{X}}(lE) \cong R^s \pi_* \mathcal{O}_{\hat{X}}((l-1)E) \cong R^s \pi_* \mathcal{O}_{\hat{X}} \quad \forall\, l \geq 1,\ 2 \leq s \neq e-1, e.$$

Thus by (ii) we get

$$R^s \pi_* \mathcal{O}_{\hat{X}}(lE) = 0 \quad \forall\, l \geq 1,\ 2 \leq s \neq e-1, e.$$

Hence to conclude (iv) we have only to prove that

$$R^e \pi_* \mathcal{O}_{\hat{X}}(lE) = 0 \quad \forall\, l \geq 1.$$

Since by (I.2.21) and (I.2.22) we have that $R^s p_* \mathcal{O}_E(lE_{|E}) = 0$ for all $1 \leq l \leq e-1$ and $s = e-1, e$, we get

$$R^e \pi_* \mathcal{O}_{\hat{X}}(lE) \cong R^e \pi_* \mathcal{O}_{\hat{X}}((l-1)E) \cong R^e \pi_* \mathcal{O}_{\hat{X}} \quad \forall\, 1 \leq l \leq e-1.$$

Thus by (ii) we have that

$$R^e \pi_* \mathcal{O}_{\hat{X}}(lE) = 0 \quad \forall\, 1 \leq l \leq e-1.$$

We prove now that

$$R^e \pi_* \mathcal{O}_{\hat{X}}(lE) = 0 \quad \forall\, l \geq e$$

by induction on $l \geq e$.
If $l = e$ consider the exact sequence

$$\cdots \to R^e \pi_* \mathcal{O}_{\hat{X}}((e-1)E) \to R^e \pi_* \mathcal{O}_{\hat{X}}(eE) \to R^e p_* \mathcal{O}_E(eE_{|E}) \to \cdots$$

Since $R^e \pi_* \mathcal{O}_{\hat{X}}((e-1)E) = 0$ and by (I.2.21) we have $R^e p_* \mathcal{O}_E(eE_{|E}) = 0$, we get

$$R^e \pi_* \mathcal{O}_{\hat{X}}(eE) = 0.$$

If $l \geq e+1$ consider the exact sequence

$$\cdots \to R^e \pi_* \mathcal{O}_{\hat{X}}((l-1)E) \to R^e \pi_* \mathcal{O}_{\hat{X}}(lE) \to R^e p_* \mathcal{O}_E(lE_{|E}) \to \cdots.$$

Since by induction hypothesis $R^e \pi_* \mathcal{O}_{\hat{X}}((l-1)E) = 0$ and by (I.2.21) we have $R^e p_* \mathcal{O}_E(lE_{|E}) = 0$, we get

$$R^e \pi_* \mathcal{O}_{\hat{X}}(lE) = 0 \quad \forall\, l \geq e+1.$$

Thus we obtain (iv) and we conclude. □

Proof of Theorem I.2.46. Once one has the result for line bundles, this immediately extends to $\mathbb{Q}$-line bundles. Thus we have only to deal with the case of a line bundle L.
Consider the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{i} & \hat{X} \\ \downarrow p & & \downarrow \pi \\ Z & \xrightarrow{k} & X \end{array}.$$

If $e = 1$ we have that π is an isomorphism and $\pi^* L$ is q-ample, so we may assume $e \geq 2$.
Consider an ample line bundle A on X such that $\hat{A} = \pi^* A - E$ is ample on $\hat{X}$. Moreover let $C = C_{\hat{X}, \hat{A}, q+e-1}$ be the constant of Theorem I.2.9.
To show that $\pi^* L$ is $(q+e-1)$-ample we find an integer $m_{\pi^* L, \hat{A}} > 0$ such that

$$H^i(\hat{X}, \pi^*(m_{\pi^* L, \hat{A}} L) - t\hat{A}) = 0$$

for all $i > q+e-1$ and $1 \leq t \leq C$.
Set $i > q+e-1$ and $1 \leq t \leq C$.
We want to compare the cohomology groups $H^i(X, mL - tA)$ and $H^i(\hat{X}, \pi^*(mL) - t\hat{A})$ using the Leray spectral sequence

$$E_1^{p,r} = H^p(X, R^{r-p} \pi_*(\pi^*(mL) - t\hat{A})) \Rightarrow H^r(\hat{X}, \pi^*(mL) - t\hat{A})$$

where $m \geq 1$ (for spectral sequences we are using the notation of [HiSt71]).
By projection formula

$$E_1^{p,r} = H^p(X, R^{r-p}\pi_*(\pi^*(mL - tA) + tE)) \cong H^p(X, \mathcal{O}_X(mL - tA) \otimes R^{r-p}\pi_*\mathcal{O}_{\hat{X}}(tE)).$$

Since X is a variety and Z is a smooth subvariety of codimension $e \geq 2$ contained in the smooth locus of X, then by Lemma I.2.47 we have that:

(i) $\pi_*\mathcal{O}_{\hat{X}} \cong \mathcal{O}_X$.
(ii) $R^s\pi_*\mathcal{O}_{\hat{X}} = 0$ for all $s \geq 1$.
(iii) $\pi_*\mathcal{O}_{\hat{X}}(lE) \cong \mathcal{O}_X$ for all $l \geq 1$.
(iv) $R^s\pi_*\mathcal{O}_{\hat{X}}(lE) = 0$ for all $l \geq 1$ and $1 \leq s \neq e-1$.

It follows that for all $m \geq 1$

$$E_1^{p,r} = \begin{cases} 0 & \text{if } (r-p \leq -1) \vee ((r-p \geq 1) \wedge (r-p \neq e-1)) \\ H^p(X, mL - tA) & \text{if } r-p = 0 \\ H^p(X, \mathcal{O}_X(mL - tA) \otimes R^{e-1}\pi_*\mathcal{O}_{\hat{X}}(tE)) & \text{if } r-p = e-1 \end{cases}.$$

If $(r-p \leq -1) \vee ((r-p \geq 1) \wedge (r-p \neq e-1))$ we have that $E_\infty^{p,r} = E_1^{p,r} = 0$.
If $r-p = 0$ consider the maps

$$E_l^{r+l+1,r+1} \to E_l^{r,r} \to E_l^{r-l-1,r-1}$$

with $l \geq 1$. We have that $E_l^{r+l+1,r+1} = 0$ for all $l \geq 1$ and that $E_l^{r-l-1,r-1} = 0$ for all $1 \leq l \neq e-1$. Thus $E_{e-1}^{r,r} = E_1^{r,r}$ and $E_\infty^{r,r} = E_e^{r,r}$. Looking at the maps $E_{e-1}^{r+e,r+1} \to E_{e-1}^{r,r} \to E_{e-1}^{r-e,r-1}$ and observing that $E_{e-1}^{r+e,r+1} = 0$ we obtain that $E_e^{r,r} = \mathrm{Ker}(E_{e-1}^{r,r} \to E_{e-1}^{r-e,r-1})$.
If $r-p = e-1$ consider the maps

$$E_l^{p+l+1,r+1} \to E_l^{p,r} \to E_l^{p-l-1,r-1}$$

with $l \geq 1$. We have that $E_l^{p+l+1,r+1} = 0$ for all $1 \leq l \neq e-1$ and that $E_l^{p-l-1,r-1} = 0$ for all $l \geq 1$, thus $E_{e-1}^{p,r} = E_1^{p,r}$ and $E_\infty^{p,r} = E_e^{p,r}$. Considering the maps $E_{e-1}^{p+e,r+1} \to E_{e-1}^{p,r} \to E_{e-1}^{p-e,r-1}$ we have that $E_{e-1}^{p-e,r-1} = 0$, whence $E_e^{p,r} = E_{e-1}^{p,r}/\mathrm{Im}(E_{e-1}^{p+e,r+1} \to E_{e-1}^{p,r})$.
It follows that for all $m \geq 1$

$$E_\infty^{p,r} = \begin{cases} 0 & \text{if } (r-p \leq -1) \vee ((r-p \geq 1) \wedge (r-p \neq e-1)) \\ V_r & \text{if } r-p = 0 \\ W_{p,r} & \text{if } r-p = e-1 \end{cases},$$

where V_r is contained in $E_1^{r,r}$ and $W_{p,r}$ is a quotient of $E_1^{p,r}$.
Consider now the filtration

$$H^i(\hat{X}, \pi^*(mL) - t\hat{A}) = F^0 \supset F^1 \supset \cdots \supset F^k \supset F^{k+1} = 0$$

with $F^p/F^{p+1} = \mathrm{Gr}^p(H^i(\hat{X}, \pi^*(mL) - t\hat{A})) = E_\infty^{p,i}$ for all $p \geq 0$.
Since $i > e-1$, then $F^p/F^{p+1} = 0$ for all $p \neq i, i+1-e$. Thus the filtration is

$$H^i(\hat{X}, \pi^*(mL) - t\hat{A}) = F^0 = F^{i+1-e} \supset F^{i+2-e} = F^i \supset F^{i+1} = 0.$$

By the exact sequence

$$0 \to F^{i+1} \to F^i \to F^i/F^{i+1} \to 0$$

we observe that $F^{i+2-e} = F^i = V_i$. Moreover by the exact sequence

$$0 \to F^{i+2-e} \to F^{i+1-e} \to F^{i+1-e}/F^{i+2-e} \to 0$$

we have that

$$(I.2.23)\qquad h^i(\hat{X},\pi^*(mL)-t\hat{A})\leq \dim(V_i)+\dim(W_{i+1-e,i})\leq h^i(X,mL-tA)+\dim(W_{i+1-e,i}).$$

Then we need to control the dimension of the $W_{i+1-e,i}$'s. To do this take an integer s such that $0\leq s\leq t-1$ and consider the exact sequences

$$(I.2.24)\quad 0\to\pi^*(mL-tA)+(t-s-1)E\to\pi^*(mL-tA)+(t-s)E\to(\pi^*(mL-tA)+(t-s)E)_{|E}\to 0.$$

By projection formula

$$R^j\pi_*(\pi^*(mL-tA)+(t-s)E)\cong\mathcal{O}_X(mL-tA)\otimes R^j\pi_*((t-s)E)$$

while

$$R^jp_*((\pi^*(mL-tA)+(t-s)E)_{|E})\cong R^jp_*(i^*(\pi^*(mL-tA)+(t-s)E)))\cong$$
$$\cong R^jp_*((p^*k^*(mL-tA)+(t-s)E_{|E}))\cong\mathcal{O}_Z((mL-tA)_{|Z})\otimes R^jp_*((t-s)E_{|E})).$$

Thus by (ii) and (iv) we get that

$$R^e\pi_*(\pi^*(mL-tA)+(t-s-1)E)\cong\mathcal{O}_X(mL-tA)\otimes R^e\pi_*((t-s-1)E)=0.$$

Moreover by (I.2.21) we get that

$$R^{e-2}p_*((\pi^*(mL-tA)+(t-s)E)_{|E})\cong\mathcal{O}_Z((mL-tA)_{|Z})\otimes R^{e-2}p_*((t-s)E_{|E}))=0,$$

whence by (I.2.24) get the exact sequence

$$0\to\mathcal{O}_X(mL-tA)\otimes R^{e-1}\pi_*((t-s-1)E)\to\mathcal{O}_X(mL-tA)\otimes R^{e-1}\pi_*((t-s)E)\to$$
$$\to\mathcal{O}_Z((mL-tA)_{|Z})\otimes R^{e-1}p_*((t-s)E_{|E}))\to 0.$$

It follows by (ii) that

$$(I.2.25)\qquad \dim(W_{i+1-e,i})\leq\dim(E_1^{i+1-e,i})=h^{i+1-e}(X,\mathcal{O}_X(mL-tA)\otimes R^{e-1}\pi_*(tE))\leq$$
$$\leq\sum_{s=0}^{t-1}h^{i+1-e}(Z,\mathcal{O}_Z((mL-tA)_{|Z})\otimes R^{e-1}p_*((t-s)E_{|E})).$$

By [Laz04a, Appendix A] we have that

$$(I.2.26)\qquad R^{e-1}p_*((t-s)E_{|E})=\begin{cases}0 & \text{if } s\geq t-e+1\\ \operatorname{Sym}^{t-s-e}N_{Z/X}\otimes\det N_{Z/X} & \text{if } s\leq t-e\end{cases}.$$

Since L is q-ample, there exists an integer $m_{L,A}>0$ such that

$$h^i(X,mL-tA)=0$$

for all $i>q$, $1\leq t\leq C$ and $m\geq m_{L,A}$.
If $t\leq e-1$, then by (I.2.25) and (I.2.26) we get that $\dim(W_{i+1-e,i})=0$. In particular if $e>C$, by (I.2.23) we get that

$$h^i(\hat{X},\pi^*(mL)-t\hat{A})\leq h^i(X,mL-tA)$$

for all $1\leq t\leq C$ and $i>q$. Taking $m_{\pi^*L,\hat{A}}=m_{L,A}$ we get the desired result. Thus we have only to deal with the case $e\leq C$.
If $t\geq e$, then by (I.2.25) and (I.2.26) we get that

$$\dim(W_{i+1-e,i})\leq\sum_{s=0}^{t-1}h^{i+1-e}(Z,\mathcal{O}_Z((mL-tA)_{|Z})\otimes R^{e-1}p_*((t-s)E_{|E}))\leq$$
$$\leq\sum_{s=0}^{t-e}h^{i+1-e}(Z,\mathcal{O}_Z((mL-tA)_{|Z})\otimes\operatorname{Sym}^{t-s-e}N_{Z/X}\otimes\det N_{Z/X}).$$

Moreover, by Proposition I.2.11 (iii), $L_{|Z}$ is also q-ample. Hence for all $e \leq t \leq C$ there exists an integer $m_t > 0$ such that

$$h^{i+1-e}(Z, \mathcal{O}_Z((mL - tA)_{|Z}) \otimes \operatorname{Sym}^{t-s-e} N_{Z/X} \otimes \det N_{Z/X}) = 0$$

for all $0 \leq s \leq t-e$, $i > q+e-1$ and $m \geq m_t$. It follows that

$$\dim(W_{i+1-e,i}) \leq \sum_{s=0}^{t-e} h^{i+1-e}(Z, \mathcal{O}_Z((mL - tA)_{|Z}) \otimes \operatorname{Sym}^{t-s-e} N_{Z/X} \otimes \det N_{Z/X}) = 0$$

for all $e \leq t \leq C$, $i > q+e-1$ and $m \geq m_t$.
Then taking $m_{\pi^*L,\hat{A}} = \max\{m_e, \ldots, m_C, m_{L,A}\}$ we get the desired result. □

Remark I.2.48. We remark that the previous result is sharp. Namely, we cannot expect more regularity on π^*L. Indeed, if $e = 1$, then π is an isomorphism. Hence, if L is q-ample but not $(q-1)$-ample, then $\pi^*L \cong L$ is $(q+e-1)$-ample but not $(q+e-2)$-ample.

I.3. An asymptotic version of partial ampleness

NOTATION. *Unless otherwise specified X will be a projective variety of dimension n over the complex number field and q will be a non-negative integer.*

I.3.1. Asymptotic cohomological functions.

We start by recalling the following definition (appeared for the first time in [Kür06]).

DEFINITION I.3.1. Let L be a line bundle on X and let $i = 0, \dots, n$ be an integer. The *value of the i-th asymptotic cohomological function associated to X at L* is

$$\hat{h}^i(X, L) = \limsup_{m \to \infty} \frac{h^i(X, mL)}{m^n/n!}.$$

REMARK I.3.2. For $i = 0$ we recover the notion of volume

$$\mathrm{vol}_X(L) = \limsup_{m \to \infty} \frac{h^0(X, mL)}{m^n/n!}.$$

and the lim sup is actually a lim (see [Laz04a, Section 2.2.C] and [Laz04b, Section 11.4.A] for a complete account on the argument).

REMARK I.3.3. Since $\hat{h}^i(X, mL) = m^n \hat{h}^i(X, L)$ for all integer $m \geq 1$, then the previous definition extends to $\mathbb{Q}$-line bundles. If L is a $\mathbb{Q}$-line bundle and $m_0 \geq 1$ is an integer such that $m_0 L$ is a line bundle, we set $\hat{h}^i(X, L) := \frac{1}{m_0^n} \hat{h}^i(X, m_0 L)$ for all $0 \leq i \leq n$. Observe that the definition does not depend on the integer m_0.

REMARK I.3.4. The definiton extends to $\mathbb{Q}$-divisors via the morphism $\mathcal{O}_X : \mathrm{Div}(X) \to \mathrm{Pic}(X)$.

The following theorem, that is a generalization of the well-known analogous result for the volume function (see [Laz04a, Corollary 2.2.45]), shows that the definition of asymptotic cohomological functions can be extended to $\mathbb{R}$-divisors.

THEOREM I.3.5 ([Kür06, Proposition 2.7, Theorem 5.1, Corollary 5.3]). *Let i be an integer such that $0 \leq i \leq n$. Then the value of the asymptotic cohomological function $\hat{h}^i$ depends only on the numerical equivalence class of a $\mathbb{Q}$-divisor, whence we have a function*

$$\hat{h}^i(X, -) : \mathrm{N}^1(X)_{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$$

that is continuous and homogeneous of degree n.
Moreover, it can be extended uniquely to a function

$$\hat{h}^i(X, -) : \mathrm{N}^1(X)_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$$

that is still continuous and homogeneous of degree n.

The asymptotic cohomological functions have a lot of interesting properties, that have been extensively studied by Küronya (see [Kür06]). For our purposes, we only remember the following version of Serre duality.

PROPOSITION I.3.6 (Asymptotic Serre duality; [Kür06, Corollary 2.11, Remark 2.12]). *Let D be an $\mathbb{R}$-divisor on X. Then*

$$\hat{h}^i(X, D) = \hat{h}^{n-i}(X, -D)$$

for all $0 \leq i \leq n$.

Next lemma shows a first relation between partial ampleness and asymptotic cohomological functions.

Lemma I.3.7. *Let D be an $\mathbb{R}$-divisor on X. If D is q-ample, then there exists an open neighbourhood U of $[D]$ in $\mathrm{N}^1(X)_\mathbb{R}$ such that $\hat{h}^i(X, D') = 0$ for all $[D'] \in U$ and $i > q$.*

Proof. Observe first that, if D is a q-ample divisor, then $\hat{h}^i(X, D) = 0$ for all $i > q$. This immediately extends to $\mathbb{Q}$-divisors.
Take now a q-ample $\mathbb{R}$-divisor D. By Theorem I.2.18 there exists an open neighbourhood U of $[D]$ in $\mathrm{N}^1(X)_\mathbb{R}$ such that D' is q-ample for all $[D'] \in U$. Thus we get that $\hat{h}^i(X, D') = 0$ for all $[D'] \in U \cap \mathrm{N}^1(X)_\mathbb{Q}$ and $i > q$. Since by Theorem I.3.5 the functions $\hat{h}^i(X, -)$ are continuous, we obtain that $\hat{h}^i(X, D') = 0$ for all $[D'] \in U$ and $i > q$. □

We also recall the following proposition, that relates the dimension of the stable base locus of a line bundle with the vanishing of its asymptotic cohomological functions and that we will generalize (see Proposition I.3.20).

Proposition I.3.8 ([Kür06, Proposition 2.15]). *Let X be a smooth variety and let L be a big line bundle on X. Then*

$$\hat{h}^i(X, L) = 0$$

for all $i > \dim(\mathbf{B}(L))$.

I.3.2. Asymptotic partial ampleness.

We recall the following interesting generalization of Theorem I.2.1.

THEOREM I.3.9 ([dFKL07, Corollary B]). *Let D be an $\mathbb{R}$-divisor on X. Then the following conditions are equivalent:*

(i) *D is ample.*
(ii) *There exists a neighbourhood U of $[D]$ in $\mathrm{N}^1(X)_{\mathbb{R}}$ such that*
$$\hat{h}^i(X, D') = 0$$
for all $[D'] \in U$ and $i > 0$.

Inspired by Theorem I.3.9, Choi introduced the following definition (see [Choi14]).

DEFINITION I.3.10. Let D be an $\mathbb{R}$-divisor on X. D is *asymptotically q-ample* if there exists a neighbourhood U of $[D]$ in $\mathrm{N}^1(X)_{\mathbb{R}}$ such that
$$\hat{h}^i(X, D') = 0$$
for all $[D'] \in U$ and $i > q$.

REMARK I.3.11. By Theorem I.3.9 for $q = 0$ we recover the notion of ampleness. Moreover, if an $\mathbb{R}$-divisor is asymptotically q-ample, then it is also asymptotically $(q+1)$-ample. Finally, every $\mathbb{R}$-divisor is asymptotically n-ample.
Thus we obtain another way to measure how much an $\mathbb{R}$-divisor is far from being ample.

For $q = n - 1$ we have the following extension of Theorem I.2.23.

THEOREM I.3.12. *Let D be an $\mathbb{R}$-divisor on X. Then the following conditions are equivalent:*

(i) *D is $(n-1)$-ample.*
(ii) *D is asymptotically $(n-1)$-ample.*
(iii) *$-D$ is not pseudoeffective.*

PROOF. By Theorem I.2.23 we have that (i) $\Leftrightarrow$ (iii), thus we have only to prove that (ii) $\Leftrightarrow$ (iii). To see this, observe that $-D$ is not pseudoeffective if and only if there exists a neighbourhood U of $[D]$ in $\mathrm{N}^1(X)_{\mathbb{R}}$ such that $-D'$ is not big for all $[D'] \in U$. Then by Proposition I.3.6 we have that
$$\hat{h}^n(X, D') = \hat{h}^0(X, -D') = \mathrm{vol}_X(-D').$$
Since by definition of volume $-D'$ is not big if and only if $\mathrm{vol}_X(-D') = 0$ we conclude. □

The next proposition, that is analogous to Propositon I.2.12 and Theorem I.2.18, summarizes the most important properties of asymptotic partial ampleness.

PROPOSITION I.3.13. *Let D and D' be two $\mathbb{R}$-divisors on X.*

(i) *If D is asymptotically q-ample, than cD is asymptotically q-ample for all $c > 0$ real numbers.*
(ii) *Assume that D and D' are numerically equivalent. Then if D is asymptotically q-ample, so it is D'.*
(iii) *If D is asymptotically q-ample, then there exists a neighbourhood U of $[D]$ in $\mathrm{N}^1(X)_{\mathbb{R}}$ of asymptotically q-ample $\mathbb{R}$-divisors.*

PROOF. The proof is straightforward. □

It is natural to compare the notion of asymptotic partial ampleness with the usual notion of partial ampleness. The following proposition shows that the new notion is weaker than the old one.

Proposition I.3.14. *Let D be an $\mathbb{R}$-divisor on X. If D is q-ample, then it is asymptotically q-ample.*

Proof. This is Lemma I.3.7. □

On the other hand we have the following conjecture, appeared for the first time in [Tot13].

Conjecture I.3.15 ([Tot13, Question 11.1]). *Let D be an $\mathbb{R}$-divisor on X. Then D is q-ample if and only if it is asymptotically q-ample.*

Remark I.3.16. Since we have not yet properly understood the behaviour of asymptotically q-ample divisors (for example we don't have an analogous of Propositions I.2.11 (iii) and Proposition I.2.19), the previous conjecture is widely open.
However, note that if $q = 0$ the conjecture is true by Theorem I.3.9, while if $q = n - 1$ it is true by Theorem I.3.12.

Inspired by Lau (see [Lau19]), we give the following definition.

Definition I.3.17. Let D be an $\mathbb{R}$-divisor on X. D is *asymptotically q-almost ample* if for all ample divisors A on X there exists a real number $\epsilon_{D,A} > 0$ such that $D + \epsilon A$ is asymptotically q-ample for all real numbers $0 < \epsilon < \epsilon_{D,A}$.

Remark I.3.18. For $q = 0$ we recover the notion of nefness.

Lemma I.3.19. *Let D be an $\mathbb{R}$-divisor on X. If D is asymptotically q-ample, then it is asymptotically q-almost ample.*

Proof. Let A be an ample divisor on X and let $\epsilon > 0$ be a real number. Since D is asymptotically q-ample, then there exists a real number $\epsilon_D > 0$ such that $\hat{h}^i(X, D') = 0$ for all $[D'] \in D_{\epsilon_D}([D])$ and $i > q$.
Set $\epsilon_{D,A} = \frac{\epsilon_D}{||A||}$ and take a real number ϵ_0 such that $0 < \epsilon_0 < \epsilon_{D,A}$. We show that $D + \epsilon_0 A$ is asymptotically q-ample.
To see this set $\epsilon_{D,A,\epsilon_0} = \epsilon_D - \epsilon_0||A||$ and take $[D'] \in D_{\epsilon_{D,A,\epsilon_0}}([D + \epsilon_0 A])$. We have that

$$||D - D'|| \leq ||D - (D + \epsilon_0 A)|| + ||D' - (D + \epsilon_0 A)|| < \epsilon_0||A|| + \epsilon_{D,A,\epsilon_0} = \epsilon_D,$$

whence $\hat{h}^i(X, D') = 0$ and we conclude. □

The following is a generalization of Proposition I.3.8 (see Remark I.3.21).

Proposition I.3.20. *Let D be an $\mathbb{R}$-divisor on X. If D is asymptotically q-almost ample, then*

$$\hat{h}^i(X, D) = 0$$

for all $i > q$.

Proof. Let A be an ample $\mathbb{R}$-divisor. Since D is asymptotically q-almost ample, then there exists a real number $\epsilon_{D,A} > 0$ such that $D + \epsilon A$ is asymptotically q-ample for all $0 < \epsilon < \epsilon_{D,A}$. In particular $\hat{h}^i(X, D + \epsilon A) = 0$ for all $0 < \epsilon < \epsilon_{D,A}$ and $i > q$. Since by Theorem I.3.5 the functions $\hat{h}^i(X, -)$ are continuous, we get that $\hat{h}^i(X, D) = 0$ for all $i > q$ and we conclude. □

Remark I.3.21. We generalized Proposition I.3.8 by removing the hypothesis of smoothness of X and the hypothesis of bigness of D.
Moreover, we replaced the hypothesis $\dim(\mathbf{B}(D)) \leq q$ with the weakest hypothesis that D is asymptotically q-almost ample.
To see the weakness, we need some result that we will show in the next section. If $\dim(\mathbf{B}(D)) \leq q$,

then $\dim(\mathbf{B}_-(D)) \leq q$, whence by Proposition I.1.47 there exists an ample divisor A on X such that

$$\dim(\mathbf{B}_+(D + \epsilon A)) \leq q$$

for all real numbers $\epsilon > 0$. It follows by Theorem I.2.27 that $D + \epsilon A$ is q-ample, whence by Proposition I.3.14 we get that $D + \epsilon A$ is asymptotically q-ample.

As in the case of partial ampleness, it is interesting to consider the cones of classes of asymptotically q-ample divisors in the Néron-Severi vector space $\mathrm{N}^1(X)_{\mathbb{R}}$.

Definition I.3.22. We denote by $\mathrm{Amp}^q_a(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$ (resp. $\mathrm{Alm}^q_a(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$) the cone of classes of asymptotically q-ample (resp. asymptotically q-almost ample) $\mathbb{R}$-divisors.

Remark I.3.23. By Proposition I.3.13 we get that $\mathrm{Amp}^q_a(X)$ is an open cone, while $\mathrm{Alm}^q_a(X)$ is a cone. We have the inclusions $\mathrm{Amp}^q_a(X) \subset \mathrm{Amp}^{q+1}_a(X)$ and $\mathrm{Alm}^q_a(X) \subset \mathrm{Alm}^{q+1}_a(X)$. By Lemma I.3.19 we get that $\mathrm{Amp}^q_a(X) \subset \mathrm{Alm}^q_a(X)$. By Proposition I.3.14 we have that $\mathrm{Amp}^q(X) \subset \mathrm{Amp}^q_a(X)$, whence by Remark I.2.34 we get that $\mathrm{Amp}^q_a(X)$ and $\mathrm{Alm}^q_a(X)$ are full-dimensional.
Note that, by Theorem I.3.12 and Remark I.2.34, $\mathrm{Amp}^{n-1}_a(X)$ and $\mathrm{Alm}^{n-1}_a(X)$ are not convex.

We prove now the following analogous of Theorems I.1.50 and I.2.35.

Theorem I.3.24 (Third generalization of Kleiman's theorem).

(i) $\mathrm{Amp}^q_a(X) = \mathrm{int}(\mathrm{Alm}^q_a(X))$.
(ii) $\mathrm{Alm}^q_a(X) \subset \overline{\mathrm{Amp}^q_a(X)}$.

Proof. By Remark I.3.23 we get that $\mathrm{Amp}^q_a(X)$ and $\mathrm{Alm}^q_a(X)$ are full-dimensional cones, $\mathrm{Amp}^q_a(X)$ is open and $\mathrm{Amp}^q_a(X) \subset \mathrm{Alm}^q_a(X)$. Thus we get the inclusion $\mathrm{Amp}^q_a(X) \subset \mathrm{int}(\mathrm{Alm}^q_a(X))$.
Take now $[D] \in \mathrm{int}(\mathrm{Alm}^q_a(X))$. Then there exists an open neighbourhood U of $[D]$ in $\mathrm{N}^1(X)_{\mathbb{R}}$ such that $[D'] \in \mathrm{Alm}^q_a(X)$ for all $[D'] \in U$. By Proposition I.3.20 we get that $\hat{h}^i(X, D') = 0$ for all $[D'] \in U$. Hence we get the inclusion $\mathrm{int}(\mathrm{Alm}^q_a(X)) \subset \mathrm{Amp}^q_a(X)$ and we conclude (i).
To see (ii) take $[D] \in \mathrm{Alm}^q_a(X)$ and let A be an ample divisor on X. Then there exists a real number $\epsilon_{D,A} > 0$ such that $D + \epsilon A$ is asymptotically q-ample for all $0 < \epsilon < \epsilon_{D,A}$. It follows that $[D + \epsilon A] \in \mathrm{Amp}^q_a(X)$ for all $0 < \epsilon < \epsilon_{D,A}$, whence $[D] = \lim_{\epsilon \to 0}[D + \epsilon A] \in \overline{\mathrm{Amp}^q_a(X)}$. Thus we get $\mathrm{Alm}^q_a(X) \subset \overline{\mathrm{Amp}^q_a(X)}$ and we conclude (ii). ☐

Remark I.3.25. It would be interesting to prove that, if D is an asymptotically q-ample $\mathbb{R}$-divisor and A is an ample $\mathbb{R}$-divisor, then $D + A$ is asymptotically q-ample. Indeed, as in the proof of Theorem I.2.35 (ii), we would have the equality $\mathrm{Alm}^q_a(X) = \overline{\mathrm{Amp}^q_a(X)}$ in Theorem (ii).
This is exactly the statement of [Choi14, Lemma 2.10 (ii)]. However we have doubts about the proof (see Remark I.3.26).

Remark I.3.26. We report some errors that invalidate the main results of [Choi14]:

(i) In Lemma 2.10 (i) we don't understand why the result holds for $\mathbb{R}$-divisors.
(ii) In Lemma 2.10 (ii) we don't understand why the diagonal is general enough to apply Lemma 2.10 (i).
(iii) In Corollary 2.11 there is an error in the last line of the proof.
(iv) In Proposition 3.2 there is an error in the last line of the proof.
(v) In Theorem 3.3 there is an error in the last line of the proof. Moreover the assertion is false (see Example I.4.7).

I.4. Comparing partial ampleness conditions

NOTATION. *Unless otherwise specified X will be a projective variety of dimension n over the complex number field and q will be a non-negative integer.*

Let L be a line bundle on X.

Trying to compare the various generalizations of the notion of ampleness we consider the following conditions:

(1) $\dim(\mathbf{B}_+(L)) \leq q$.
(2+) For all $A_1, \ldots, A_q$ very ample divisors and for all general $E_i \in |A_i|$ we have that $L_{|E_1 \cap \cdots \cap E_q}$ is ample.
(2−) There exists $A_1, \ldots, A_q$ very ample divisors such that for all general $E_i \in |A_i|$ we have that $L_{|E_1 \cap \cdots \cap E_q}$ is ample.
(3) L is q-ample.
(4) For all subvarieties $Z \subset X$ of dimension $> q$ we have that $-L_{|Z}$ is not pseudoeffective.
(5) For all subvarieties $Z \subset X$ of dimension $> q$ we have that $L_{|Z}$ is not numerically 0.
(6) L is asymptotically q-ample.

REMARK I.4.1. If $q = 0$, then by Remarks I.1.41, I.2.3 and I.3.11 Conditions (1), (3), (6) are equivalent to the ampleness of L. We also define Conditions (2+) and (2−) as the ampleness of L.

REMARK I.4.2. Conditions (1), (3) and (6) are the partial ampleness conditions studied in the previous Sections. Conditions (2+) and (2−) come from the work of Küronya (see [Kür13]). Condition (4) comes from Theorems I.2.23 and I.2.41. Condition (5) comes from Sommese's work (see [Som78, Remark 1.4.1]).

Under the additional hypotheses of Theorem I.2.37, we may consider other conditions. Assume that $\mathrm{k}(X, L) \geq 0$ and let $m_0 > 0$ be an integer such that $\mathbf{B}(L) = \mathrm{Bs}\,|m_0 L|$. Moreover, let $\pi : \hat{X} \to X$ be the blow-up of X along the ideal sheaf $\mathcal{I} = \mathcal{I}_{\mathbf{B}(L)}$ (that is the blow-up along the base ideal $\mathfrak{b}|m_0 L|$) with exceptional divisor E and consider the decomposition

$$\pi^*(m_0 L) = M + E, \tag{I.4.27}$$

where M is a base-point-free line bundle on $\hat{X}$ (observe that M depends on m_0).
We also consider the following conditions:

(1∗) $\dim(\mathbf{B}_+(M)) \leq q$.
(3∗) M is q-ample.

REMARK I.4.3. If $q = 0$, Conditions (1∗) and (3∗) are equivalent to the ampleness of M.

We have the following result, that summarizes the relationship between the previous conditions.

THEOREM I.4.4. *Let X be a variety and let L be a line bundle on X. Then we have the following implications:*

$$(1) \Rightarrow (2+) \Rightarrow (2-) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5), \quad (3) \Rightarrow (6).$$

Moreover, let $m_0 > 0$ be an integer. If

(∗) $\mathrm{k}(X, L) \geq 0$ *and* $\mathbf{B}(L) = \mathrm{Bs}\,|m_0 L|$,

then (1∗) ⇒ (3∗).
Under additional hypotheses we have the following other implications:

(i) *Assume that* $\dim(\mathbf{B}(L)) \leq q$. *Then* (5) ⇒ (4). *Moreover, if* (∗) *is satisfied, then* (3∗) ⇒ (3). *Finally, if X is normal, L is big,* (∗) *is satisfied and m_0 is sufficiently large and divisible, then* (1∗) ⇒ (1).
(ii) *Assume that* $\dim(\mathbf{B}(L)) \leq q - 1$. *If* (∗) *is satisfied, then* (3) ⇒ (3∗).

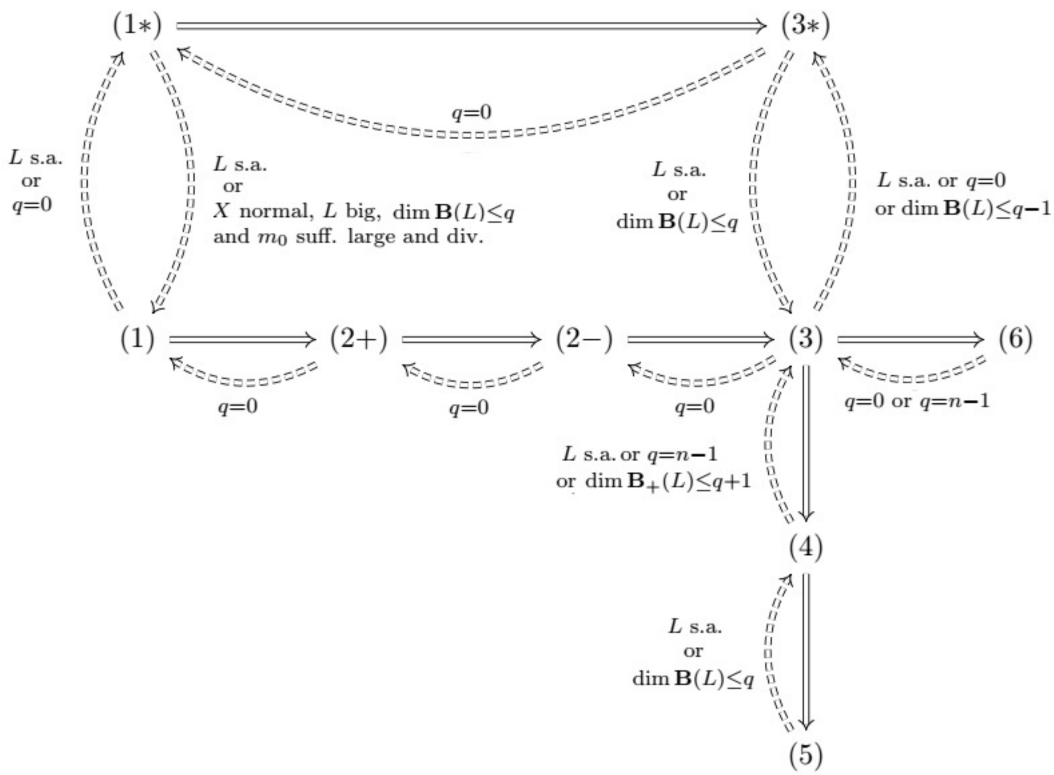
(iii) *Assume that* L *is semiample. Then* $(3) \Leftrightarrow (4) \Leftrightarrow (5) \Rightarrow (6)$. *Moreover, if* $(*)$ *is satisfied, then* $(1) \Leftrightarrow (1*)$ *and* $(3) \Leftrightarrow (3*)$.

(iv) *Assume that* $\dim(\mathbf{B}_+(L)) \leq q+1$. *Then* $(3) \Leftrightarrow (4)$.

(v) *Assume that* $q = 0$. *Then* $(1) \Leftrightarrow (2+) \Leftrightarrow (2-) \Leftrightarrow (3) \Leftrightarrow (6)$. *Moreover if* $(*)$ *is satisfied, then* $(1) \Leftrightarrow (2+) \Leftrightarrow (2-) \Leftrightarrow (3) \Leftrightarrow (6) \Rightarrow (1*) \Leftrightarrow (3*)$.

(vi) *Assume that* $q = n-1$. *Then* $(3) \Leftrightarrow (4) \Leftrightarrow (6)$.

In other words, we have the following diagram of implications:



Here, to consider Conditions $(1*)$ and $(3*)$ and the related implications, we are implicitly assuming the hypothesis $(*)$.

Before proceding to the proof, we recall the following result.

Proposition I.4.5 ([Kür13, Theorem A, Theorem 1.2]). *Let* D *be a divisor and let* $A_1, \dots, A_q$ *be very ample divisors on* X. *Suppose that for general* $E_j \in |A_j|$ *the restriction* $D_{|E_1 \cap \dots \cap E_q}$ *is ample. Then for every coherent sheaf* $\mathcal{F}$ *on* X *there exists an integer* $m_{L,A_1,\dots,A_q,\mathcal{F}} > 0$ *such that*

$$H^i(X, \mathcal{O}_X(mD + D' + \sum_{j=1}^{q} r_j A_j) \otimes \mathcal{F}) = 0$$

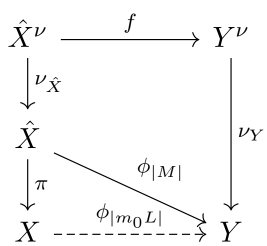
for all nef divisors D', $m > m_{L,A_1,\dots,A_q,\mathcal{F}}$, $r_j \geq 0$ *and* $i > q$. *In particular* D *is* q*-ample.*

Proof of Theorem I.4.4. To prove $(1) \Rightarrow (2+)$, take $A_1, \dots, A_q$ very ample divisors and $E_i \in |A_i|$ general. Since $\dim(\mathbf{B}_+(L)) \leq q$ and by Proposition I.1.24 (ii) the restriction to a general very ample divisor strictly decreases the dimension of the augmented base locus, then we have that $\dim(\mathbf{B}_+(L_{|E_1 \cap \dots \cap E_q})) \leq 0$. It follows by Proposition I.1.35 (ii) and Proposition I.1.37 (ii) that $L_{|E_1 \cap \dots \cap E_q}$ is ample.
The implication $(2+) \Rightarrow (2-)$ is obvious.

By Proposition I.4.5 we get (2−) ⇒ (3).
To prove (3) ⇒ (4), consider a subvariety $Z \subset X$ of dimension $q+1$. If L is q-ample, then by Proposition I.2.11 (iii) we have that $L_{|Z}$ is q-ample, whence by Proposition I.2.40 we get the desired result.
To prove (4) ⇒ (5), consider a subvariety $Z \subset X$ of dimension $> q$. Sinche $-L_{|Z}$ is not pseudoeffective, it is not numerically 0, whence we conclude.
By Proposition I.3.14 we have that (3) ⇒ (6).
Let $m_0 > 0$ be an integer and assume (∗). Then (1∗) ⇒ (3∗) is Theorem I.2.27.
Thus we have conlcuded the implications in the general case.
To prove (i) assume that $\dim(\mathbf{B}(L)) \leq q$.
To prove (5) ⇒ (4), consider a subvariety $Z \subset X$ of dimension $> q$. Since $\dim(\mathbf{B}(L)) \leq q$ and by Proposition I.1.24 (i)

$$\mathbf{B}_{-}(L_{|Z}) \subset \mathbf{B}(L_{|Z}) \subset \mathbf{B}(L) \cap Z,$$

then $\dim(\mathbf{B}_{-}(L_{|Z})) < \dim(Z)$. It follows by Proposition I.1.39 (iii) that $L_{|Z}$ is pseudoeffective. Since by hypothesis $L_{|Z}$ is not numerically 0, we conclude that $-L_{|Z}$ is not pseudoeffective.
Let $m_0 > 0$ be an integer and assume (∗). Then the implication (3∗) ⇒ (3) is Proposition I.2.37 (i).
Let $m_0 > 0$ be an integer and assume that X is normal, L is big, (∗) holds and m_0 is sufficiently large and divisibile. We claim that (1∗) ⇒ (1).
Let Y be the Zariski-closure of the image of $X \backslash \mathbf{B}(L)$ under the map $\phi_{|m_0 L|}$ and let $\nu_Y : Y^\nu \to Y$ be its normalization. Moreover, let $\mu : \hat{X}^\nu \to X$ be the normalized blow-up of the base ideal of $|m_0 L|$ (that is the composition of π with the normalization map $\nu_{\hat{X}} : \hat{X}^\nu \to \hat{X}$).
We have a diagram



where f is the rational map induced by the morphisms $\phi_{|M|}$, $\nu_{\hat{X}}$ and ν_Y. Since L is big and m_0 is sufficiently large and divisible, then f is birational. Moreover, since $\nu_{\hat{X}}$ is finite, then by Proposition I.1.25 we get the inclusion $\mathbf{B}_+(\nu_{\hat{X}}^* M) \subset \nu_{\hat{X}}^{-1}(\mathbf{B}_+(M))$, whence

$$\dim(\mathbf{B}_+(\nu_{\hat{X}}^* M)) \leq \dim(\nu_{\hat{X}}^{-1}(\mathbf{B}_+(M))) = \dim(\mathbf{B}_+(M)). \tag{I.4.28}$$

Observe now that

$$\mu^*(m_0 L) = \nu_{\hat{X}}^* \pi^*(m_0 L) = \nu_{\hat{X}}^*(M + E) \cong \nu_{\hat{X}}^* \phi_{|M|}^* A + \nu_{\hat{X}}^* E \cong f^* \nu_Y^* A + \nu_{\hat{X}}^* E,$$

for some ample line bundle A on Y. Hence we have that

$$\nu_{\hat{X}}^* M = \mu^*(m_0 L) - \nu_{\hat{X}}^* E \cong f^* \nu_Y^* A.$$

Since A is ample and ν_Y is finite, then $\nu_Y^* A$ is ample, whence by Proposition I.1.35 (ii) we get that $\mathbf{B}_+(\nu_Y^* A) = \emptyset$. Moreover, since f is a birational map of normal varieties, then by Proposition I.1.26 we get

$$\mathbf{B}_+(\nu_{\hat{X}}^* M) = \mathbf{B}_+(f^* \nu_Y^* A) = f^{-1} \mathbf{B}_+(\nu_Y^* A) \cup \mathrm{Exc}(f) = \mathrm{Exc}(f). \tag{I.4.29}$$

We claim now that

$$\bigcup_Z \hat{Z} \subset \mathbf{B}_+(\nu_{\hat{X}}^* M), \tag{I.4.30}$$

where the union is taken over all the irreducible components Z of $\mathbf{B}_+(L)$ that are not contained in $\mathbf{B}(L)$ and $\hat{Z}$ denotes the strict transform of Z under the morphism μ.
To see this let $Z \subset \mathbf{B}_+(L)$ be an irreducible component such that $Z \not\subset \mathbf{B}(L)$. Then by [BCL14, Theorem B] we get that $\mathrm{vol}_{X/Z}(L) = 0$. Since X is normal, by Theorem I.1.34 (ii) we get that $Z \subset X$ is a positive dimensional subvariety. Since m_0 is sufficiently large and divisible, by [BCL14, Corollary 2.5] we get that $\phi_{|m_0 L|}$ contracts Z. Since ν_Y is finite, then f contracts $\hat{Z}$, whence by (I.4.29) we get that $\hat{Z} \subset \mathrm{Exc}(f) = \mathbf{B}_+(\nu_{\hat{X}}^* M)$ and we conclude.
Assume now (1$*$), that is $\dim(\mathbf{B}_+(M)) \leq q$. Hence by (I.4.28) we have that $\dim(\mathbf{B}_+(M^\nu)) \leq q$. It follows by (I.4.30) that $\dim(Z) \leq q$ for all $Z \subset \mathbf{B}_+(L)$ irreducible components such that $Z \not\subset \mathbf{B}(L)$.
Since by assumption $\dim(\mathbf{B}(L)) \leq q$ we conclude that $\dim(\mathbf{B}_+(L)) \leq q$ and we get (1).
Thus we get (i).
To prove (ii) assume that $\dim(\mathbf{B}(L)) \leq q-1$ and let $m_0 > 0$ be an integer and assume ($*$). Then by Proposition I.2.37 (ii) we have that (3) $\Rightarrow$ (3$*$).
To prove (iii) assume that L is semiample.
By Theorem I.2.21 we get (4) $\Rightarrow$ (3).
Since L is semiample, by Proposition I.1.35 we get that $\mathbf{B}(L) = \emptyset$, whence by (i) that (5) $\Rightarrow$ (4).
If m_0 is an integer and ($*$) holds, then the implications (1) $\Leftrightarrow$ (1$*$) and (3) $\Leftrightarrow$ (3$*$) are obviuos since by definition $M \cong m_0 L$.
Thus we get (iii).
To prove (iv) assume that $\dim(\mathbf{B}_+(L)) \leq q+1$. Then by Theorem I.2.41 we have that (4) $\Rightarrow$ (3).
By Remarks I.4.1 and I.4.3 to prove the statement (v) we have only to prove that, if $q = 0$, $m_0 > 0$ is an integer and ($*$) holds, then (1) $\Rightarrow$ (1$*$). This follows by Theorem I.2.37 (ii).
The statement (vi) follows by Theorems I.2.23 and I.3.12. □

I.4.1. Sharpness of some implications.

The following proposition shows that even if $q = n-1$, we cannot expect to have the implication (3) $\Rightarrow$ (2$-$)

Proposition I.4.6 ([Kür13, Proposition 1.13]). *Let L be a line bundle on X. Then the conditions:*

(i) *L is $(n-1)$-ample.*
(ii) *There exists $A_1, \ldots, A_{n-1}$ very ample divisors such that for all very general $E_i \in |A_i|$ we have that $L_{|E_1 \cap \cdots \cap E_{n-1}}$ is ample.*

are equivalent exactly if every strongly movable curve is the limit of elements of the convex cone spanned by complete intersection curves coming from very ample divisors.

The following example shows that the implication (3) $\Rightarrow$ (1) is not valid even if L is semiample or $\dim(\mathbf{B}_+(L)) \leq q+1$.

Example I.4.7. Let $\pi : X \to Y$ be the blow-up of a smooth threefold Y along a smooth curve C with exceptional divisor E, let H be a very ample divisor on Y and set $L = \pi^* H$. Since H is ample and π is birational, we have by Proposition I.1.26 that

$$\mathbf{B}_+(L) = \mathbf{B}_+(\pi^* H) = \pi^{-1}(\mathbf{B}_+(H)) \cup E = E.$$

Hence $\dim(\mathbf{B}_+(L)) = 2$. Moreover, since H is very ample, we have that L is semiample, $\mathbb{P}H^0(X, L) = \mathbb{P}H^0(Y, H)$ and the morphism $\phi_{|L|}$ factorizes through the blow-up π. Observe now that $\phi_{|H|}$ is an embedding and that the fiber of π over a point in Y can be a point or a $\mathbb{P}^1$. Hence the dimension of the fibers of the $\phi_{|L|}$ is at most 1, whence by Theorem I.2.21 we have that L is 1-ample.

The following example shows that the first assertion of Theorem I.2.37 is sharp. Namely, if $\dim(\mathbf{B}(L)) \geq q+1$, it is not always true that $(3*) \Rightarrow (3)$.

Example I.4.8. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$ containing a line C and let H be the hyperplane section. Observe that $C = \mathbf{B}(C)$ and that $C^2 = 2-d$ and $H.C = 1$. Consider now the divisor $L = H + C$. We claim that

$$C = \mathbf{B}(L) = \mathbf{B}_+(L).$$

Indeed since $L.C = 3-d$ we have that $C \subseteq \mathbf{B}(L)$. On the other hand $\mathbf{B}_+(L) \subseteq \mathbf{B}(L-H) = \mathbf{B}(C) = C$. It follows that $\dim(\mathbf{B}_+(L)) = 1$, whence by Theorem I.2.27 we conclude that L is 1-ample. However, since $L.C = 3-d < 0$, then L is not ample.
Let now $\pi : \hat{X} \to X$ be the blow-up of X along C, with exceptional divisor E. Since π is an isomorphism we have the decomposition $L = M + E = H + C$, with $M = H$ and $E = C$. Hence M is ample and $\dim(\mathbf{B}(L)) = 1$ but L is only 1-ample.

The following example shows that the second assertion of Theorem I.2.37 is sharp. Namely, if $\dim(\mathbf{B}(L)) \geq q$, it is not always true that $(3) \Rightarrow (3*)$.

Example I.4.9. Let $p : X \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ along a projective subspace $\mathbb{P}^{n-q-1}$ with exceptional divisor L. Since $\mathcal{O}_{\mathbb{P}^n}(1)$ is an ample line bundle, then $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(q+1)}$ is an ample vector bundle of rank $q+1$. It follows by [Ott12, Proposition 4.5] that $\mathbb{P}^{n-q-1}$ is ample, that is L is q-ample. Moreover we have that $\mathbf{B}(L) = L$. Let now $\pi : \hat{X} \to X$ be the blow-up of X along L with exceptional divisor E. Since π is an isomorphism we have the decomposition $L = M + E$, with $M = 0$ and $E = L$. Since $-M = 0$ is pseudoeffective, Theorem I.2.23 implies that M is not $(n-1)$-ample. On the other hand L is q-ample and $\dim(\mathbf{B}(L)) = n-1 \geq q$.

The following example shows that the implication $(2-) \Rightarrow (2+)$ is not valid even if $\dim(\mathbf{B}_+(L)) \leq q+1$.

Example I.4.10 ([Kür13, Example 1.8]). Let $\pi : \mathbb{F}_1 \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ in a point, let E be the exceptional divisor and let F be the fibre of the ruling. Then we have that $\mathrm{Div}(\mathbb{F}_1) = \mathbb{Z}E \oplus \mathbb{Z}F$, the canonical bundle is $K_{\mathbb{F}_1} = -2E - 3F$ and intersection form is $E^2 = -1$, $F^2 = 0$ and $E.F = 1$.
Consider the variety $X = \mathbb{F}_1 \times \mathbb{P}^1$, with projections $p_1 : X \to \mathbb{F}_1$ and $p_2 : X \to \mathbb{P}^1$. We have that $\mathrm{Div}(X) = \mathbb{Z}\hat{E} \oplus \mathbb{Z}\hat{F} \oplus \mathbb{Z}N$, where $\hat{E} = p_1^*(E)$, $\hat{F} = p_1^*(F)$ and $N = p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Moreover $K_X = p_1^*(K_{\mathbb{F}_1}) + p_2^*(K_{\mathbb{P}^1}) = -2\hat{E} - 3\hat{F} - 2N$.
Consider the curve $C_x = \{x\} \times \mathbb{P}^1$, with $x \in E$, and the curve $C'_y = E \times \{y\}$, with $y \in \mathbb{P}^1$.
The isomorphism $\hat{E} = E \times \mathbb{P}^1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ induces an isomorphism $\mathrm{Pic}(\hat{E}) \cong \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}^2$, under which the generators $\mathcal{O}_{\hat{E}}(C_x)$ and $\mathcal{O}_{\hat{E}}(C'_y)$ correspond to $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)$ and $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)$. Moreover $K_{\hat{E}} \sim -2C_x - 2C'_y$.
It can be easily seen that $\hat{E}.C_x = 0$, $\hat{F}.C_x = 0$, $N.C_x = 1$, $\hat{F}.C'_y = 1$ and $N.C'_y = 0$. Moreover by adjunction formula we have that $\hat{E}_{|\hat{E}} = K_{\hat{E}} - (K_X)_{|\hat{E}} = K_{\hat{E}} + (2\hat{E} + 3\hat{F} + 2N)_{|\hat{E}}$, whence

$$\hat{E}.C'_y = (\hat{E}_{|\hat{E}}).C'_y = -K_{\hat{E}}.C'_y - 3(\hat{F}_{|\hat{E}}).C'_y - 2(N_{|\hat{E}}).C'_y = 2 - 3\hat{F}.C'_y - 2N.C'_y = -1.$$

Take two integers $\lambda \geq 2, \mu \geq 2$ and consider the line bundles

$$L_\lambda = \mathcal{O}_X(\lambda\hat{E} + \hat{F} + N), \quad A_\mu = \mathcal{O}_X(\hat{E} + \mu\hat{F} + N).$$

Observe that $\lambda E + F$ is effective, while $E + \mu F$ is very ample. Then L_λ is big and A_μ is very ample. We claim that

$$\hat{E} = \mathbf{B}_-(L_\lambda) = \mathbf{B}(L_\lambda) = \mathbf{B}_+(L_\lambda). \tag{I.4.31}$$

Since $L_\lambda . C'_y = 1 - \lambda < 0$ for all $y \in \mathbb{P}^1$, we have that

$$\hat{E} = \bigcup_{y \in \mathbb{P}^1} C'_y = E \times \mathbb{P}^1 \subset \mathbf{B}_-(L_\lambda).$$

On the other hand

$$\mathbf{B}_+(L_\lambda) \subset \mathbf{B}(L_\lambda - A_\mu) = \mathbf{B}((\lambda - 1)\hat{E} + (1 - \mu)\hat{F}) \subset \operatorname{Bs}|(\lambda - 1)\hat{E} + (1 - \mu)\hat{F}| =$$
$$= \operatorname{Bs}|(\lambda - 1)E + (1 - \mu)F| \times \mathbb{P}^1 = E \times \mathbb{P}^1 = \hat{E},$$

whence we get (I.4.31).

Fix now a very general element $G_\mu \in |A_\mu|$ cutting out a smooth irreducible divisor $M_\mu := \hat{E} \cap G_\mu$ on $\hat{E}$.

We claim that $(L_\lambda)_{|G_\mu}$ is ample if and only if $\mu > \lambda$.

To see this, observe that

$$(L_\lambda)_{|\hat{E}} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}((L_\lambda).C_x, (L_\lambda).C'_y) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1 - \lambda),$$
$$(A_\mu)_{|\hat{E}} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}((A_\mu).C_x, (A_\mu).C'_y) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, \mu - 1).$$

Since C_x is the fiber of the projection $(p_1)_{|\hat{E}}$ and $(A_\mu)_{|\hat{E}}.C_x = A_\mu.C_x = \hat{E}.C_x + \mu\hat{F}.C_x + N.C_x = 1$, then $(p_1)_{|\hat{E}}$ induces an isomorphism $M_\mu \cong E \cong \mathbb{P}^1$. Moreover

$$(L_\lambda)_{|M_\mu} = \mathcal{O}_{\mathbb{P}^1}((L_\lambda)_{|\hat{E}}.M_\mu) = \mathcal{O}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1 - \lambda).\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, \mu - 1)) = \mathcal{O}_{\mathbb{P}^1}(\mu - \lambda). \tag{I.4.32}$$

By Proposition I.1.24 (ii) and (I.4.31) we get that

$$\mathbf{B}_+((L_\lambda)_{|G_\mu}) \subset \mathbf{B}_+(L_\lambda) \cap G_\mu = \hat{E} \cap G_\mu = M_\mu \subset G_\mu. \tag{I.4.33}$$

Since by Theorem I.2.25 we have that $(L_\lambda)_{|G_\mu}$ is ample if and only if $(L_\lambda)_{|\mathbf{B}_+((L_\lambda)_{|G_\mu})}$ is ample, by (I.4.32) and (I.4.33) we get that $(L_\lambda)_{|M_\mu}$ is ample if and only if $\mu > \lambda$ and the claim follows.

Hence, if $q = n - 2 = 1$, then L_λ satisfies Condition (2$-$), but does not satisfy Condition (2+).

CHAPTER 2

Ulrich vector bundles on projectively Cohen-Macaulay surfaces of codimension two

II.1. Projectively Cohen-Macaulay surfaces

Let $S \subset \mathbb{P}^4$ be a smooth projective surface over the complex number field, let H be the hyperplane section and let $\mathcal{I}_{S/\mathbb{P}^4}$ be the ideal sheaf of S.
Denote by $R = \mathbb{C}[x_0, \dots, x_4]$ the coordinate ring of $\mathbb{P}^4$ and by $I_S = \bigoplus_{l \geq 0} H^0(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l))$ the saturated graded homogeneous ideal of S.

Definition II.1.1. S is called projectively Cohen-Macaulay (or, briefly, PCM) if its cone is a Cohen-Macaulay scheme (that is all its local rings are Cohen-Macaulay).

Remark II.1.2. It can be easily seen (see [Kol13, Section 3.1] or [Mi08, Chapter 1]) that the definition is equivalent to the fact that

$$h^1(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l)) = h^2(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l)) = 0$$

for all $l \in \mathbb{Z}$ and to the fact that the homogeneous coordinate ring $R_S = R/I_S$ of S is Cohen-Macaulay.

Remark II.1.3. The PCM surfaces are often called in literature arithmetically Cohen-Macaulay (or, briefly, ACM).

Let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface.
Then there exists a minimal free resolution of graded homomorphisms of degree 0 (see [PeSz74, Section 3]) of the form

$$0 \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^4}(-m_i) \xrightarrow{\varphi} \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^4}(-d_j) \xrightarrow{\psi} \mathcal{I}_{S/\mathbb{P}^4} \to 0 \tag{II.1.34}$$

where $n, m_i, d_j \in \mathbb{Z}$ are integers such that $n \geq 0$, $m_i \geq 1$ for all $1 \leq i \leq n+1$ and $d_j \geq 2$ for all $1 \leq j \leq n+2$. Moreover we have that

$$\sum_{i=1}^{n+1} m_i = \sum_{j=1}^{n+2} d_j. \tag{II.1.35}$$

We may also assume, without loss of generality, that

$$m_i \geq m_{i+1} \quad \forall\, 1 \leq i \leq n \quad \text{and} \quad d_j \geq d_{j+1} \quad \forall\, 1 \leq j \leq n+1. \tag{II.1.36}$$

For all $1 \leq i \leq n+1$ and $1 \leq j \leq n+2$ set

$$u_{i,j} := m_i - d_j.$$

A system of generators of the ideal I_S consists of $n+2$ homogeneous polynomials $F_1, \dots, F_{n+2}$ of degrees $d_1, \dots, d_{n+2}$. Moreover, since the resolution (II.1.34) is minimal, we get that $F_1, \dots, F_{n+2}$ minimally generate I_S, that is $(F_1, \dots, \hat{F}_j, \dots, F_{n+2}) \subsetneq I_S$ for all $1 \leq j \leq n+2$.
The map φ is given by an $(n+1) \times (n+2)$ matrix $[\varphi] = (A_{i,j})$, whose entries are homogeneous polynomials.
Since the resolution (II.1.34) is minimal, then $A_{i,j}$ has degree $u_{i,j}$ if $u_{i,j} > 0$, while is zero if $u_{i,j} \leq 0$.
The map ψ is given by the $1 \times (n+2)$ matrix $[\psi] = (F_j)$.
By the Hilbert-Burch theorem we may assume that for all $1 \leq j \leq n+2$ the homogeneous generator F_j is the determinant of the matrix obtained from $[\varphi]$ by removing the j-th column.
We denote by

$$S = (m_1, \dots, m_{n+1}; d_1, \dots, d_{n+2}) \tag{II.1.37}$$

a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34).
By Grothendieck duality we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^4} \to \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^4}(d_j) \xrightarrow{\varphi^*} \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^4}(m_i) \to K_S(5) \to 0. \tag{II.1.38}$$

To simplify notation we will denote

$$F := \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^4}(d_j), \quad G := \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^4}(m_i).$$

The following easy result shows that the entries on the diagonal of the matrix $[\varphi]$ are non-zero (see also [Mi08, Lemma 1.2.20]).

Lemma II.1.4. *Let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34). Then*

$$u_{i,i} > 0 \quad \forall\ 1 \le i \le n+1.$$

Proof. We assume that there exists an integer $1 \le i_0 \le n+1$ such that $u_{i_0,i_0} \le 0$ and we reach a contradiction.
Observe that by (II.1.36) we get that $u_{i,j} \le 0$ for all $i_0 \le i \le n+1$ and $1 \le j \le i_0$. Since the resolution (II.1.34) is minimal, then $A_{i,j} = 0$ for all $i_0 \le i \le n+1$ and $1 \le j \le i_0$, whence the matrix of the map φ is

$$[\varphi] = \left(\begin{array}{ccc|ccc} A_{1,1} & \cdots & A_{1,i_0} & A_{1,i_0+1} & \cdots & A_{1,n+2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{i_0-1,1} & \cdots & A_{i_0-1,i_0} & A_{i_0-1,i_0+1} & \cdots & A_{i_0-1,n+2} \\ \hline & & & A_{i_0,i_0+1} & \cdots & A_{i_0,n+2} \\ & \mathbf{0} & & \vdots & \ddots & \vdots \\ & & & A_{n+1,i_0+1} & \cdots & A_{n+1,n+2} \end{array}\right).$$

It follows that

$$F_{n+2} = \det \left(\begin{array}{ccc|cccc} A_{1,1} & \cdots & A_{1,i_0-1} & A_{1,i_0} & A_{1,i_0+1} & \cdots & A_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{i_0-1,1} & \cdots & A_{i_0-1,i_0-1} & A_{i_0-1,i_0} & A_{i_0-1,i_0+1} & \cdots & A_{i_0-1,n+1} \\ \hline & & & 0 & A_{i_0,i_0+1} & \cdots & A_{i_0,n+1} \\ & \mathbf{0} & & \vdots & \vdots & \ddots & \vdots \\ & & & 0 & A_{n+1,i_0+1} & \cdots & A_{n+1,n+1} \end{array}\right) = 0.$$

Since the resolution (II.1.34) is minimal, we reach a contradiction and we conclude. □

The following result is an improvement of Lemma II.1.4 (based on the proof of [Sau85, Proposition 1]).

Proposition II.1.5. *Let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34) and suppose that S is not a complete intersection (that is $n \ge 1$). Then*

$$u_{i,i-1} > 0 \quad \forall\ 2 \le i \le n+1.$$

Before passing to the proof we need the following lemma.

Lemma II.1.6. *Let $n \geq 1$ be an integer and let $A = (A_{i,j})$ be an $n \times n$ matrix of homogeneous polynomials. Suppose that there exist integers $m_i, d_j \geq 1$, with $1 \leq i \leq n$ and $1 \leq j \leq n$ such that:*

(i) $m_i \geq m_{i+1}$ *for all* $1 \leq i \leq n-1$ *and* $d_j \geq d_{j+1}$ *for all* $1 \leq j \leq n-1$.
(ii) $m_i > d_i$ *for all* $1 \leq i \leq n$.
(iii) $\deg(A_{i,j}) = m_i - d_j$ *if* $m_i > d_j$, *while* $A_{i,j} = 0$ *if* $m_i \leq d_j$.

Then $\det(A)$, *if non-zero, is a homogeneous polynomial of degree* $\sum_{i=1}^{n} m_i - \sum_{j=1}^{n} d_j > 0$.

Proof. We prove the result by induction on $n \geq 1$.
If $n = 1$, then $A = (A_{1,1})$. By (ii) we get that $m_1 > d_1$, whence by (iii) we have that

$$\deg(\det(A)) = \deg(A_{1,1}) = m_1 - d_1 > 0$$

and we conclude.
If $n \geq 2$, let k be an integer such that $1 \leq k \leq n$ and let B_k be the $(n-1) \times (n-1)$ matrix obtained from A by removing the n-th row and the k-th column.
By induction, if $\det(B_k)$ is non-zero, then it is a homogeneous polynomial of degree

$$\deg(\det(B_k)) = \sum_{i=1}^{n-1} m_i - \sum_{j=1}^{k-1} d_j - \sum_{j=k+1}^{n} d_j = \Big(\sum_{i=1}^{n-1} m_i - \sum_{j=1}^{n-1} d_j\Big) + (d_k - d_n) > 0.$$

Observe now that, by the Laplace expansion of the determinant of A with respect to the n-th row, we get

$$\det(A) = \sum_{k=1}^{n} (-1)^{k+n} A_{n,k} \det(B_k).$$

Take $1 \leq k \leq n$. If $m_n \leq d_k$, then by (iii) we get $(-1)^{k+n} A_{n,k} \det(B_k) = 0$. If $m_n > d_k$, then by (iii) we get that $A_{n.k}$ is a homogeneous polynomial of degree $m_n - d_k > 0$. Moreover by induction we have that $\det(B_k)$, if non-zero, is a homogeneous polynomial of degree $\Big(\sum_{i=1}^{n-1} m_i - \sum_{j=1}^{n-1} d_j\Big) + (d_k - d_n) > 0$. It follows that $(-1)^{k+n} A_{n,k} \det(B_k)$, if non-zero, is a homogeneous polynomial of degree

$$\deg(A_{n,k}) + \deg(\det(B_k)) = (m_n - d_k) + \Big(\sum_{i=1}^{n-1} m_i - \sum_{j=1}^{n-1} d_j\Big) + (d_k - d_n) = \sum_{i=1}^{n} m_i - \sum_{j=1}^{n} d_j > 0$$

and we get the desired result. □

Proof of Proposition II.1.5. We assume that there exists an integer $2 \leq i_0 \leq n+1$ such that $u_{i_0, i_0 - 1} \leq 0$ and we reach a contradiction.
Observe that by (II.1.36) we get that $u_{i,j} \leq 0$ for all $i_0 \leq i \leq n+1$ and $1 \leq j \leq i_0 - 1$. Since the resolution (II.1.34) is minimal, then $A_{i,j} = 0$ for all $i_0 \leq i \leq n+1$ and $1 \leq j \leq i_0 - 1$, whence the matrix of the map φ is

$$[\varphi] = \left(\begin{array}{c|c} [\varphi_1] & * \\ \hline 0 & [\varphi_2] \end{array}\right) = \left(\begin{array}{ccc|ccc} A_{1,1} & \cdots & A_{1,i_0-1} & A_{1,i_0} & \cdots & A_{1,n+2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{i_0-1,1} & \cdots & A_{i_0-1,i_0-1} & A_{i_0-1,i_0} & \cdots & A_{i_0-1,n+2} \\ \hline & & & A_{i_0,i_0} & \cdots & A_{i_0,n+2} \\ & \mathbf{0} & & \vdots & \ddots & \vdots \\ & & & A_{n+1,i_0} & \cdots & A_{n+1,n+2} \end{array}\right).$$

Let $X \subset \mathbb{P}^4$ be the closed subscheme defined by the polynomial $F' = \det[\varphi_1]$ and let

$$S' = D_{n+1-i_0}(\varphi_2) = \{x \in \mathbb{P}^4 \ : \ \mathrm{rk}(\varphi_2(x)) \leq n+1-i_0\} \subset \mathbb{P}^4$$

be the top degeneracy locus of the map given by the matrix $[\varphi_2]$.
Therefore the ideal $I_{S'}$ is generated by homogeneous polynomials $F'_1, \dots, F'_{n+3-i_0}$, where for all $1 \le j \le n+3-i_0$ the polynomial F'_j is the determinant of the matrix obtained from $[\varphi_2]$ by removing the j-th column.
Note that $S' \subset S$. Indeed, if $x \in S'$, then $\mathrm{rk}(\varphi_2(x)) \le n+1-i_0$, whence by Laplace's theorem $\mathrm{rk}(\varphi(x)) \le n$. It follows that $x \in S = D_n(\varphi) = \{x \in \mathbb{P}^4 \ : \ \mathrm{rk}(\varphi(x)) \le n\}$.
Observe now that, if $j \ge i_0$, then

$$F_j = \det \left(\begin{array}{c|ccccc} & A_{1,i_0} & \cdots & \hat{A}_{1,j} & \cdots & A_{1,n+2} \\ [\varphi_1] & \vdots & \ddots & \vdots & \ddots & \vdots \\ & A_{i_0-1,i_0} & \cdots & \hat{A}_{i_0-1,j} & \cdots & A_{i_0-1,n+2} \\ \hline & A_{i_0,i_0} & \cdots & \hat{A}_{i_0,j} & \cdots & A_{i_0,n+2} \\ \mathbf{0} & \vdots & \ddots & \vdots & \ddots & \vdots \\ & A_{n+1,i_0} & \cdots & \hat{A}_{n+1,j} & \cdots & A_{n+1,n+2} \end{array}\right) = F' F'_{j-i_0+1}.$$

Since the resolution (II.1.34) is minimal, then $F_1, \dots, F_{n+2}$ is a minimal system of generators of I_S.
It follows that $F' \neq 0$ and $F'_1, \dots, F'_{n+3-i_0}$ is a minimal system of generators of $I_{S'}$.
We claim that $\dim(S') = 2$.
We first prove that S' is non-empty. To see this we assume that S' is empty and we reach a contradiction. Take a point $x \in S$. Then there exists $i_0 \le j \le n+2$ such that $F'_{j-i_0+1}(x) \neq 0$. Since $F_j(x) = 0$ we get that $F'(x) = 0$. Hence F' is a polynomial that vanishes on S, whence $\deg(F') \ge d_{n+2}$. However

$$d_{n+2} = \deg(F_{n+2}) = \deg(F') + \deg(F'_{n+3-i_0}) \ge d_{n+2},$$

from which we deduce that $\deg(F'_{n+3-i_0}) = 0$. Since

$$F'_{n+3-i_0} = \det \begin{pmatrix} A_{i_0,i_0} & \cdots & A_{i_0,n+1} \\ \vdots & \ddots & \vdots \\ A_{n+1,i_0} & \cdots & A_{n+1,n+1} \end{pmatrix}$$

by Lemma II.1.6 we get that $F'_{n+3-i_0} = 0$. This contradicts the minimality of $F'_1, \dots, F'_{n+3-i_0}$, whence S' is non-empty.
Thus the dimension of S' is at least the expected dimension, whence

$$\dim(S') \ge \dim(\mathbb{P}^4) - ((n+2-i_0) - (n+1-i_0))((n+3-i_0) - (n+1-i_0)) = 2.$$

Since $S' \subset S$ we conclude that $\dim(S') = 2$.
Moreover, since S is irreducible, we get that $S = S'$ as closed subschemes of $\mathbb{P}^4$ with the reduced induced closed subscheme structure, whence $F'_1, \dots, F'_{n+3-i_0}$ is a minimal system of generators of I_S. This contradicts the minimality of $F_1, \dots, F_{n+2}$, whence we conclude. □

Now we prove the following easy result, which will be useful in the future.

Lemma II.1.7. *Let $k \ge 1$ and $l \in \mathbb{Z}$ be integers. We have that*

$$h^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(l)) + h^0(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(l-1)) = h^0(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(l)).$$

Proof. Consider the exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^{k+1}}(l-1) \to \mathcal{O}_{\mathbb{P}^{k+1}}(l) \to \mathcal{O}_{\mathbb{P}^k}(l) \to 0.$$

Since $h^1(\mathbb{P}^{k+1}, \mathcal{O}_{\mathbb{P}^{k+1}}(l-1)) = 0$ we get the desired result. □

The following proposition gives formulas for the dimension of the cohomologies of the multiples of the hyperplane section of a PCM surface.

Proposition II.1.8. *Let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34). Moreover let $l \in \mathbb{Z}$ be an integer. We have that:*

(i)

$$h^0(S, \mathcal{O}_S(l)) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l - m_i)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l - d_j)).$$

(ii)

$$h^1(S, \mathcal{O}_S(l)) = 0.$$

In particular $q(S) = 0$ and H is non-special.

(iii)

$$h^2(S, \mathcal{O}_S(l)) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-l-5)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m_i - l - 5)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d_j - l - 5)).$$

(iv)

$$h^1(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l)) = h^2(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l)) = 0.$$

(v)

$$H^2 - H.K_S = 2\Big[4 - \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m_i - 5)) + \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d_j - 5))\Big].$$

Proof. Consider the exact sequences

$$0 \to \mathcal{I}_{S/\mathbb{P}^4}(l) \to \mathcal{O}_{\mathbb{P}^4}(l) \to \mathcal{O}_S(l) \to 0 \tag{II.1.39}$$

$$0 \to G^*(l) \to F^*(l) \to \mathcal{I}_{S/\mathbb{P}^4}(l) \to 0. \tag{II.1.40}$$

Since $h^1(\mathbb{P}^4, F^*(l)) = h^2(\mathbb{P}^4, G^*(l)) = 0$ we have by (II.1.40) that

$$h^1(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l)) = 0.$$

Analogously, since $h^2(\mathbb{P}^4, F^*(l)) = h^3(\mathbb{P}^4, G^*(l)) = 0$, we have that

$$h^2(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l)) = 0.$$

This proves (iv).
Since $h^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l)) = h^2(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l)) = 0$ we have by (II.1.39) and (iv) that

$$h^1(S, \mathcal{O}_S(l)) = h^2(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l)) = 0.$$

Thus we get (ii).
Since $h^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l)) = 0$ we have by (II.1.39) and (iv) that

$$h^0(S, \mathcal{O}_S(l)) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l)) - h^0(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l)) + h^1(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l)) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l)) - h^0(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l)).$$

Since $h^1(\mathbb{P}^4, G^*(l)) = 0$ we have by (II.1.40) that

$$h^0(\mathbb{P}^4, \mathcal{I}_{S/\mathbb{P}^4}(l)) = h^0(\mathbb{P}^4, F^*(l)) - h^0(\mathbb{P}^4, G^*(l)) = \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l - d_j)) - \sum_{i=1}^{n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l - m_i)).$$

It follows that

$$h^0(S, \mathcal{O}_S(l)) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l - m_i)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l - d_j)).$$

Hence we get (i).
Tensorizing by $\mathcal{O}_{\mathbb{P}^4}(l-5)$ the exact sequence (II.1.38) we get the exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^4}(l-5) \xrightarrow{\varphi_l} F(l-5) \to G(l-5) \to K_S(l) \to 0.$$

Defining $\mathcal{G} = \mathrm{Coker}(\varphi_0)$ we obtain the exact sequences

$$0 \to \mathcal{G}(l) \to G(l-5) \to K_S(l) \to 0 \tag{II.1.41}$$

$$0 \to \mathcal{O}_{\mathbb{P}^4}(l-5) \to F(l-5) \to \mathcal{G}(l) \to 0. \tag{II.1.42}$$

Since $h^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l-5)) = 0$ we have by (II.1.42) that

$$h^0(\mathbb{P}^4, \mathcal{G}(l)) = h^0(\mathbb{P}^4, F(l-5)) - h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l-5)) = \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d_j+l-5)) - h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l-5)).$$

Moreover, since $h^1(\mathbb{P}^4, F(l-5)) = h^2(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l-5)) = 0$, we have by (II.1.42) that

$$h^1(\mathbb{P}^4, \mathcal{G}(l)) = 0.$$

It follows by (II.1.41) that

$$h^0(S, K_S(l)) = h^0(\mathbb{P}^4, G(l-5)) - h^0(\mathbb{P}^4, \mathcal{G}(l)) =$$

$$= h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l-5)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m_i+l-5)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d_j+l-5)).$$

Hence by Serre duality we get

$$h^2(S, \mathcal{O}_S(l)) = h^0(S, K_S(-l)) =$$

$$= h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-l-5)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m_i-l-5)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d_j-l-5)).$$

Thus we get (iii).
By (i), (ii) and (iii) it follows that

$$\chi(\mathcal{O}_S(l)) = h^0(\mathcal{O}_S(l)) - h^1(\mathcal{O}_S(l)) + h^2(\mathcal{O}_S(l)) =$$

$$= h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l-m_i)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(l-d_j)) +$$

$$+ h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-l-5)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m_i-l-5)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d_j-l-5)).$$

By Riemann-Roch

$$\chi(\mathcal{O}_S(1)) = \chi(\mathcal{O}_S) + \frac{1}{2}(H^2 - H.K_S).$$

Since S is non-degenerate, then $d_j \geq 2$ for all $1 \leq j \leq n+2$. Moreover by Lemma II.1.4 we get that $m_i \geq 3$ for all $1 \leq i \leq n+1$. Thus by Lemma II.1.7 we have that

$$H^2 - H.K_S = 2(\chi(\mathcal{O}_S(1)) - \chi(\mathcal{O}_S)) =$$

$$= 2\Big[5 + \sum_{i=1}^{n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m_i-6)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d_j-6)) -$$

$$-1 - \sum_{i=1}^{n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m_i-5)) + \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d_j-5))\Big] =$$

$$= 2\Big[4 - \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m_i - 5)) + \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d_j - 5))\Big].$$

Hence we get (v) and we conclude. □

Let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34) and let $C \in |H|$ be a smooth irreducible curve. It can be proved that $C \subset \mathbb{P}^3$ is a PCM curve and its ideal sheaf $\mathcal{I}_{C/\mathbb{P}^3}$ fits into the exact sequence

(II.1.43) $$0 \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^3}(-m_i) \to \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^3}(-d_j) \to \mathcal{I}_{C/\mathbb{P}^3} \to 0.$$

The following proposition gives formulas for the dimension of the cohomologies of the multiples of the hyperplane section of C.

Proposition II.1.9. *Let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34). Moreover let $C \in |H|$ be a smooth irreducible curve and let $l \in \mathbb{Z}$ be an integer. We have that:*

(i)

$$h^0(C, \mathcal{O}_C(l)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l - m_i)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l - d_j)).$$

(ii)

$$h^1(C, \mathcal{O}_C(l)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-l - 4)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m_i - l - 4)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d_j - l - 4)).$$

(iii)

$$H^2 = 3 + \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 4)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_j - 4)).$$

(iv) *More generally*

$$H^2 = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(l - 3)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i + l - 3)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_j + l - 3)) +$$

$$+ h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-l)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-m_i - l)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d_j - l)).$$

Proof. Consider the exact sequences

(II.1.44) $$0 \to \mathcal{I}_{C/\mathbb{P}^3}(l) \to \mathcal{O}_{\mathbb{P}^3}(l) \to \mathcal{O}_C(l) \to 0.$$

Moreover set $\tilde{F} = \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^3}(d_j)$ and $\tilde{G} = \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^3}(m_i)$. By (II.1.43) we get the exact sequences

(II.1.45) $$0 \to \tilde{G}^*(l) \to \tilde{F}^*(l) \to \mathcal{I}_{C/\mathbb{P}^3}(l) \to 0.$$

Since $h^1(\mathbb{P}^3, \tilde{F}^*(l)) = h^2(\mathbb{P}^3, \tilde{G}^*(l)) = 0$ we have by (II.1.45) that

$$h^1(\mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3}(l)) = 0.$$

Since $h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) = 0$ we have by (II.1.44) that

$$h^0(C, \mathcal{O}_C(l)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) - h^0(\mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3}(l)) + h^1(\mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3}(l)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) - h^0(\mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3}(l)).$$

Since $h^1(\mathbb{P}^3, \tilde{G}^*(l)) = 0$ we have by (II.1.45) that

$$h^0(\mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3}(l)) = h^0(\mathbb{P}^3, \tilde{F}^*(l)) - h^0(\mathbb{P}^3, \tilde{G}^*(l)) = \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-d_j)) - \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-m_i)).$$

It follows that

$$h^0(C, \mathcal{O}_C(l)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-m_i)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-d_j)).$$

Hence we get (i).
By Grothendieck duality, we have the exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^3}(l-4) \xrightarrow{\varphi_l} \tilde{F}(l-4) \to \tilde{G}(l-4) \to K_C(l) \to 0.$$

Defining $\mathcal{H} = \mathrm{Coker}(\varphi_0)$ we obtain the exact sequences

(II.1.46) $$0 \to \mathcal{H}(l) \to \tilde{G}(l-4) \to K_C(l) \to 0$$

(II.1.47) $$0 \to \mathcal{O}_{\mathbb{P}^3}(l-4) \to \tilde{F}(l-4) \to \mathcal{H}(l) \to 0.$$

Since $h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-4)) = 0$ we have by (II.1.47) that

$$h^0(\mathbb{P}^3, \mathcal{H}(l)) = h^0(\mathbb{P}^3, \tilde{F}(l-4)) - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-4)).$$

Since $h^1(\mathbb{P}^3, \tilde{F}(l-4)) = h^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-4)) = 0$ we have by (II.1.47) that

$$h^1(\mathbb{P}^3, \mathcal{H}(l)) = 0.$$

It follows by (II.1.46) that

$$h^0(C, K_C(l)) = h^0(\tilde{G}(l-4)) - h^0(\mathbb{P}^3, \mathcal{H}(l)) =$$

$$= h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-4)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m_i+l-4)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d_j+l-4)).$$

Hence by Serre duality we get that

$$h^1(C, \mathcal{O}_C(l)) = h^0(C, K_C(-l)) =$$

$$= h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-l-4)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m_i-l-4)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d_j-l-4)).$$

Thus we have (ii).
By (i) and (ii) we get that

$$\chi(\mathcal{O}_C(l)) = h^0(C, \mathcal{O}_C(l)) - h^1(C, \mathcal{O}_C(l)) =$$

$$= h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-m_i)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l-d_j)) -$$

$$-h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-l-4)) - \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m_i-l-4)) + \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d_j-l-4)).$$

By Riemann-Roch

$$\chi(\mathcal{O}_C(1)) = \deg(C) + 1 - g(C) = \deg(C) + \chi(\mathcal{O}_C)$$

Since S is non-degenerate, then $d_j \geq 2$ for all $1 \leq j \leq n+2$. Moreover by Lemma II.1.4 we get that $m_i \geq 3$ for all $1 \leq i \leq n+1$. Thus by Lemma II.1.7 we have that

$$H^2 = \deg(C) = \chi(\mathcal{O}_C(1)) - \chi(\mathcal{O}_C) =$$

$$= 3 - \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m_i - 5)) + \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d_j - 5)) +$$

$$+ \sum_{i=1}^{n+1} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m_i - 4)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d_j - 4)) =$$

$$= 3 + \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 4)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_j - 4)).$$

Hence we get (iii).
To prove (iv) take $P \in |\mathcal{O}_C(1)|$ and $l \in \mathbb{Z}$. Moreover set $\hat{F} = \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^2}(d_j)$ and $\hat{G} = \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^2}(m_i)$. By Grothendieck duality we have the exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^2}(l-3) \xrightarrow{\varphi_l} \hat{F}(l-3) \to \hat{G}(l-3) \to K_P(l) \to 0. \tag{II.1.48}$$

Defining $\mathcal{F} = \mathrm{Coker}(\varphi_0)$ we obtain the exact sequences

$$0 \to \mathcal{F}(l) \to \hat{G}(l-3) \to K_P(l) \to 0 \tag{II.1.49}$$

$$0 \to \mathcal{O}_{\mathbb{P}^2}(l-3) \to \hat{F}(l-3) \to \mathcal{F}(l) \to 0. \tag{II.1.50}$$

Since $h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(l-3)) = 0$ we have by (II.1.50) that

$$h^0(\mathbb{P}^2, \mathcal{F}(l)) = h^0(\mathbb{P}^2, \hat{F}(l-3)) - h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(l-3)) = \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_j + l - 3)) - h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(l-3)).$$

Since $h^1(P, K_P(l)) = h^2(P, K_P(l)) = 0$ we have by (II.1.49) that

$$h^2(\mathbb{P}^2, \mathcal{F}(l)) = h^2(\mathbb{P}^2, \hat{G}(l-3)) = \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-m_i - l)).$$

Since $h^1(\mathbb{P}^2, \hat{F}(l-3)) = h^3(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(l-3)) = 0$ we have by (II.1.50) that

$$h^1(\mathbb{P}^2, \mathcal{F}(l)) = h^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(l-3)) - h^2(\mathbb{P}^2, \hat{F}(l-3)) + h^2(\mathbb{P}^2, \mathcal{F}(l)) =$$

$$= h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-l)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-m_i - l)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d_j - l)).$$

Since $h^1(\mathbb{P}^2, \hat{G}(l-3)) = 0$ we have by (II.1.49) that

$$H^2 = h^0(P, K_P(l)) = h^0(\mathbb{P}^2, \hat{G}(l-3)) - h^0(\mathbb{P}^2, \mathcal{F}(l)) + h^1(\mathbb{P}^2, \mathcal{F}(l)) =$$

$$= h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(l-3)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i + l - 3)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_j + l - 3)) +$$

$$+ h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-l)) + \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-m_i - l)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d_j - l)).$$

Thus we get (iv) and we conclude. □

We will now consider the residual surface in a complete intersection $X \subset \mathbb{P}^4$ of two hypersurfaces. To do this we first recall the following result.

Proposition II.1.10 ([PeSz74, (statement and proof of) Proposition 4.1]). *Let V be a closed subscheme without embedded components, that is a generically complete intersection of codimension 2 in $\mathbb{P}^l$, with $l \geq 2$, defined by a graded ideal I_V in $\mathbb{C}[x_0, \dots, x_l]$.*
Let U_1 be the open set of points in V where V is a locally complete intersection in $\mathbb{P}^l$ and let U_2 be the open set of regular points in V.
Then there exist homogeneous equations $\alpha_1, \alpha_2 \in I_V$ defining two hypersurfaces X_1 and X_2 on $\mathbb{P}^l$ which intersect properly and such that $V \cup V' = X_1 \cap X_2$, where

(i) *V and V' have no common components.*
(ii) *$V \cap V' \cap U_1$ is a locally complete intersection.*
(iii) *V' is a complete intersection at all points of codimension ≤ 3 of $V' \cap U_1$.*
(iv) *$V' \backslash (V \cap V')$ is smooth.*
(v) *$V \cap V' \cap U_2$ is smooth in codimension 2 and $\operatorname{codim}(V \cap V' \cap (U_1 \backslash U_2), (U_1 \backslash U_2)) \geq 1$.*
(vi) *V' is smooth at all points of $V' \cap U_2$ of codimension ≤ 3 in V'.*

Moreover one can take the elements α_1, α_2 of degree d_1, d_2 provided that

$$d_1 \geq \inf\{s : I_s\mathbb{C}[x_0, \dots, x_l] \text{ defines } X\}$$
$$d_2 \geq \inf\{t : I_t\mathbb{C}[x_0, \dots, x_l]/\alpha_1 \text{ defines } X \text{ in } \mathbb{C}[x_0, \dots, x_l]/\alpha_1\}.$$

Let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34) that is not a complete intersection (that is $n \geq 1$). Moreover, let X_1 and X_2 be two hypersurfaces in $\mathbb{P}^4$ defined by two minimal homogeneous generators of I_S of degree d_1 and d_2, respectively. Then we have that $X = S \cup S' = X_1 \cap X_2$, where S' is the residual scheme (see [PeSz74, Proposition 1.2]), that is the scheme defined by the ideal sheaf $\mathcal{I}_{S'/\mathbb{P}^4}$ such that

$$\mathcal{I}_{S'/\mathbb{P}^4}/\mathcal{I}_{X/\mathbb{P}^4} = \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{O}_{\mathrm{S}}, \mathcal{O}_{\mathrm{X}}).$$

Denote by $\Gamma = S \cap S'$ the scheme-theoretic intersection.

Proposition II.1.11. *In the previous setting, $\Gamma \sim (d_1 + d_2 - 5)H - K_S$ is a smooth irreducible curve of degree*

$$H.\Gamma = (d_1 + d_2 - 5)H^2 - H.K_S$$

and genus

$$g(\Gamma) = 1 + \frac{1}{2}(d_1 + d_2 - 5)H.\Gamma.$$

Proof. By Proposition II.1.10 we have that Γ is smooth.
Moreover, by [PeSz74, Remark 1.5] we get that

$$\mathcal{I}_{S'/\mathbb{P}^4}/\mathcal{I}_{X/\mathbb{P}^4} \cong \mathcal{O}_S(K_S + (5 - e)H)$$

where $e = d_1 + d_2$ is the sum of the degrees of the equations defining X.
By the exact sequence

$$0 \to \mathcal{I}_{X/\mathbb{P}^4} \to \mathcal{I}_{S'/\mathbb{P}^4} \to \mathcal{I}_{\Gamma/S} \to 0$$

it follows that

$$\mathcal{O}_S(K_S + (5 - d_1 - d_2)H) \cong \mathcal{I}_{S'/\mathbb{P}^4}/\mathcal{I}_{X/\mathbb{P}^4} \cong \mathcal{I}_{\Gamma/S} = \mathcal{O}_S(-\Gamma),$$

whence $\Gamma \sim (d_1 + d_2 - 5)H - K_S$.
Since by adjunction formula

$$K_\Gamma = (K_S + \Gamma)_{|\Gamma}$$

we have the exact sequence

$$0 \to K_S \to K_S + \Gamma \to K_\Gamma \to 0.$$

Since $h^1(S, K_S + \Gamma) = h^1(S, (d_1 + d_2 - 5)H) = 0$, by Serre duality we have that

$$h^0(\Gamma, \mathcal{O}_\Gamma) = h^1(\Gamma, K_\Gamma) \leq h^2(S, K_S) = h^0(S, \mathcal{O}_S) = 1.$$

It follows that Γ is connected, whence irreducible.
Moreover, the degree of Γ is

$$H.\Gamma = ((d_1 + d_2 - 5)H - K_S).H = (d_1 + d_2 - 5)H^2 - H.K_S$$

and, since $2g(\Gamma) - 2 = \deg(K_\Gamma) = (d_1 + d_2 - 5)H.\Gamma$, the genus of Γ is

$$g(\Gamma) = 1 + \frac{1}{2}(d_1 + d_2 - 5)H.\Gamma.$$

□

II.2. Projectively Cohen-Macaulay surfaces with $p_g(S)=0$ and $h^0(S,2K_S-H)=0$

Since we are going to prove the existence of Ulrich vector bundles on some classes of PCM surfaces using Casnati's results (see Theorems II.3.2 and II.3.3), we are interested in the classification of PCM surfaces with $p_g(S)=0$ and $h^0(S,2K_S-H)=0$.

The following result classifies smooth irreducible non-degenerate PCM surfaces with $p_g(S)=0$.

PROPOSITION II.2.1. *Let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34). Then the following conditions are equivalent:*

(i) $p_g(S)=0$.
(ii) $m_1 \le 4$.

Moreover, if they are satisfied, then S is one of the surfaces in the following list:

(1) $S=(4;2,2)$, *the complete intersection of two quadrics.*
(2) $S=(3,3;2,2,2)$, *the Hirzebruch surface $S=B_{p_1}(\mathbb{P}^2)$ embedded by the linear system $|2\hat{H}-E_1|$.*
(3) $S=(4,4;3,3,2)$, *the Castelnuovo surface $S=B_{p_1,\dots,p_8}(\mathbb{P}^2)$ embedded by the linear system $|4\hat{H}-2E_1-\sum_{\alpha=2}^{8}E_\alpha|$.*
(4) $S=(4,4,4;3,3,3,3)$, *the Bordiga surface $S=B_{p_1,\dots,p_{10}}(\mathbb{P}^2)$ embedded by the linear system $|4\hat{H}-\sum_{\alpha=1}^{10}E_\alpha|$.*

Here we are using notation (II.1.37). Moreover $B_{p_1,\dots,p_s}(\mathbb{P}^2)$ is the blow-up of $\mathbb{P}^2$ at some points $p_1,\dots,p_s\in\mathbb{P}^2$, $\hat{H}$ is the strict transform of a line in $\mathbb{P}^2$ and the E_α's are the exceptional divisors.

PROOF. By Proposition II.1.8 (iii) we have that

$$p_g(S)=\sum_{i=1}^{n+1}h^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(m_i-5))-\sum_{j=1}^{n+2}h^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(d_j-5)).$$

If $m_1\le 4$, then by (II.1.36) we get that $m_i\le 4$ for all $1\le i\le n+1$. Thus

$$p_g(S)=-\sum_{j=1}^{n+2}h^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(d_j-5))\le 0.$$

This forces $p_g(S)=0$ and we get the implication (ii) $\Rightarrow$ (i).
To prove the implication (i) $\Rightarrow$ (ii) assume that $p_g(S)=0$ and take a smooth irreducible curve $C\in|H|$.
Since by adjunction formula

$$K_C=(K_S+C)_{|C}$$

we get the exact sequence

$$0\to K_S-H\to K_S\to K_C-H_{|C}\to 0.$$

Observe that by Proposition II.1.8 (ii) and Serre duality

$$h^1(S,K_S-H)=h^1(S,H)=0.$$

Thus we get that

$$p_g(S)=h^0(S,K_S)=h^0(S,K_S-H)+h^0(C,K_C-H_{|C}).$$

Since $p_g(S)=0$, it follows that $h^0(S,K_S-H)=h^0(C,K_C-H_{|C})=0$.
Moreover, by Lemma II.1.7 and Proposition II.1.9 (ii) with $l=1$ and $l=2$

$$h^0(C,K_C-H_{|C})=\sum_{i=1}^{n+1}h^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(m_i-5))-\sum_{j=1}^{n+2}h^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(d_j-5))=$$

$$= \sum_{i=1}^{n+1} \Big[h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m_i-6)) + h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i-5))\Big] - \sum_{j=1}^{n+2} \Big[h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d_j-6)) + h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_j-5))\Big] =$$

$$= h^0(C, K_C - 2H_{|C}) + \Big[\sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 5)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_j - 5))\Big].$$

Since $h^0(C, K_C - H_{|C}) = 0$, it follows that $h^0(C, K_C - 2H_{|C}) = 0$, whence

$$\Big[\sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 5)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_j - 5))\Big] = 0. \tag{II.2.51}$$

Hence by Proposition II.1.9 (iv) with $l = -2$ we get that

$$H^2 = 6 + \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 5)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_j - 5)) + \tag{II.2.52}$$

$$+ \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-m_i + 2)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d_j + 2)) =$$

$$= 6 + \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-m_i + 2)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d_j + 2)).$$

Consider first the case $d_{n+2} \leq 4$.
We get that $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_{n+2} - 5)) = 0$, whence by (II.2.51) we have that

$$\sum_{i=1}^{n+1} \Big[h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 5)) - h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_i - 5))\Big] = 0.$$

Since by Lemma II.1.4 we have that $m_i > d_i$ for all $1 \leq i \leq n + 1$, then

$$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 5)) - h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_i - 5)) \geq 0.$$

Thus $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 5)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_i - 5))$ for all $1 \leq i \leq n + 1$.
This forces $m_1 \leq 4$ and we conclude.
Thus we have only to deal with the case $d_{n+2} \geq 5$.
Then by (II.1.36) we get that $d_j \geq 5$ for all $1 \leq j \leq n + 2$. Moreover by Lemma II.1.4 we get that $m_i \geq 6$ for all $1 \leq i \leq n + 1$. It follows by (II.2.52) that $\deg(C) = 6$.
Since C is a smooth PCM curve with minimal free resolution

$$0 \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^3}(-m_i) \to \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^3}(-d_j) \to \mathcal{I}_{C/\mathbb{P}^3} \to 0 \tag{II.2.53}$$

and $d_j \geq 5$ for all $1 \leq j \leq n + 2$, it follows that C is not contained in a surface of degree 4.
However such a curve does not exist.
To see this consider the exact sequence

$$0 \to \mathcal{I}_{C/\mathbb{P}^3}(4) \to \mathcal{O}_{\mathbb{P}^3}(4) \to \mathcal{O}_C(4) \to 0.$$

Since C is not contained in a surface of degree 4, then $h^0(\mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3}(4)) = 0$. Moreover by (II.2.53) we get that $h^1(\mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3}(4)) = 0$. Hence

$$h^0(C, \mathcal{O}_C(4)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) = 35.$$

Since $\deg(C)$ is even, by Castelnuovo's theorem [Har77, Theorem IV.6.4] we get

$$g(C) \leq \frac{1}{4}\deg(C)^2 - \deg(C) + 1 = 4.$$

Thus $\deg(\mathcal{O}_C(4)) = 24 > 6 \geq 2g(C) - 2$, whence $h^1(C, \mathcal{O}_C(4)) = 0$. Hence by Riemann-Roch we have

$$35 = h^0(C, \mathcal{O}_C(4)) = h^0(C, \mathcal{O}_C(4)) - h^1(C, \mathcal{O}_C(4)) = \chi(\mathcal{O}_C(4)) = \deg(\mathcal{O}_C(4)) + 1 - g(C) = 25 - g(C).$$

Since $g(C) \geq 0$ we reach a contradiction and we get the implication (i) $\Rightarrow$ (ii).

Take now a smooth irreducible non-degenerate PCM surface S with minimal free resolution (II.1.34) and such that $p_g(S) = 0$.
To get the complete list of possible S's we will classify the collections of integers

$$(n, \{m_i\}_{i=1}^{n+1}, \{d_j\}_{j=1}^{n+2})$$

such that $n \geq 0$, $d_j \geq 2$ for all $1 \leq j \leq n+2$, $m_1 \leq 4$, and that satisfy (II.1.35), (II.1.36) and the conditions of Lemma II.1.4 and Proposition II.1.5.
Since $m_1 \leq 4$, then by (II.1.36) we have that $m_i \leq 4$ for all $1 \leq i \leq n+1$.
Recall that by Lemma II.1.4 we have that

$$u_{i,i} > 0 \quad \forall\ 1 \leq i \leq n+1. \tag{II.2.54}$$

Thus by (II.1.36) we get that $d_j \leq 3$ for all $1 \leq j \leq n+2$.
Moreover, since S is non-degenerate, then $d_j \geq 2$ for all $1 \leq j \leq n+2$. Also by (II.2.54) we get that $m_i \geq 3$ for all $1 \leq i \leq n+1$.
It follows that $3 \leq m_i \leq 4$ for all $1 \leq i \leq n+1$ and $2 \leq d_j \leq 3$ for all $1 \leq j \leq n+2$.
Set

$$t = |\{j : d_j = 3\}|, \quad q = |\{j : d_j = 2\}|,$$
$$k = |\{i : m_i = 4\}|, \quad l = |\{i : m_i = 3\}|.$$

If S is a complete intersection, that is $n = 0$, we get that $m_1 = d_1 + d_2$. Since $m_1 \leq 4$ and $d_1, d_2 \geq 2$ the only possibility is

$$(1): \quad S = (4; 2, 2).$$

Thus to complete the list we have only to deal with the case $n \geq 1$.
By Proposition II.1.5 we have that

$$u_{i,i-1} > 0 \quad \forall\ 2 \leq i \leq n+1. \tag{II.2.55}$$

Consider first the case $t = 0$.
Then $d_j = 2$ for all $1 \leq j \leq n+2$, whence by (II.1.35) we get

$$4k + 3(n+1-k) = 2(n+2),$$

that is $n + k = 1$. Since $n \geq 1$ the only possible case is $n = 1$, $k = 0$, $l = n+1-k = 2$ and $q = n+2 = 3$, that is

$$(2): \quad S = (3, 3; 2, 2, 2).$$

Consider now the case $1 \leq t \leq n$.
Then by (II.2.55) we get that $m_i \geq m_{t+1} > d_t = 3$ for all $1 \leq i \leq t+1$, whence $k \geq t+1$. By (II.1.35) we get

$$4k + 3(n+1-k) = 3t + 2(n+2-t),$$

that is $n + k = t + 1$. Since $n \geq 1$ we get that $t + 1 = n + k \geq 1 + (t+1) = t+2$ and we reach a contradiction.
Thus we have only to deal with the case $n+1 \leq t \leq n+2$.

Then by (II.2.54) we get that $m_i \geq m_{n+1} > d_{n+1} = 3$ for all $1 \leq i \leq n+1$, whence $k = n+1$. By (II.1.35) we get

$$4(n+1) = 3t + 2(n+2-t),$$

that is $2n = t$. Since $n \geq 1$ and $t \leq n+2$ we get that $1 \leq n \leq 2$.
If $n = 1$, then $k = 2$, $t = 2$ and $q = n+2-t = 1$, thus we get

$$(3): \quad S = (4,4;3,3,2).$$

If $n = 2$, then $k = 3$, $t = 4$ and $q = n+2-t = 0$, thus we get

$$(4): \quad S = (4,4,4;3,3,3,3).$$

□

The following result classifies smooth irreducible non-degenerate PCM surfaces with $h^0(S, K_S - H) = 0$.

Proposition II.2.2. *Let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34). Then the following conditions are equivalent:*

(i) $h^0(S, K_S - H) = 0$.
(ii) $m_1 \leq 5$.

Moreover, if they are satisfied, then S is one of the surfaces of Proposition II.2.1 or one of the surfaces in the following list:

(5) $S = (5;3,2)$.
(6) $S = (5,4;3,3,3)$.
(7) $S = (5,5;4,3,3)$.
(8) $S = (5,5;4,4,2)$.
(9) $S = (5,5,5;4,4,4,3)$.
(10) $S = (5,5,5,5;4,4,4,4,4)$.

Proof. By Serre duality and Proposition II.1.8 (iii) we have that

$$h^0(S, K_S - H) = h^2(S, H) = \sum_{i=1}^{n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m_i - 6)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d_j - 6)).$$

If $m_1 \leq 5$, then by (II.1.36) we get that $m_i \leq 5$ for all $1 \leq i \leq n+1$. Thus

$$h^0(S, K_S - H) = -\sum_{j=1}^{n+2} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d_j - 6)) \leq 0.$$

This forces $h^0(S, K_S - H) = 0$ and we get the implication (ii) ⇒ (i).
To prove the implication (i) ⇒ (ii) assume that $h^0(S, K_S - H) = 0$ and take a smooth irreducible curve $C \in |H|$.
As in the proof of Proposition II.2.1 we have that

$$h^0(S, K_S - H) = h^0(S, K_S - 2H) + h^0(C, K_C - 2H_{|C}).$$

Moreover, by Lemma II.1.7 and Proposition II.1.9 (ii) with $l = 2$ and $l = 3$

$$h^0(C, K_C - 2H_{|C}) = h^0(C, K_C - 3H_{|C}) + \Big[\sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 6)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_j - 6))\Big].$$

Since $h^0(S, K_S - H) = 0$, it follows that $h^0(C, K_C - 2H_{|C}) = 0$, whence $h^0(C, K_C - 3H_{|C}) = 0$ and

$$\sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 6)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_j - 6)) = 0. \tag{II.2.56}$$

By Proposition II.1.9 (iv) with $l = -3$ we get that

$$H^2 = 10 + \sum_{i=1}^{n+1} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-m_i + 3)) - \sum_{j=1}^{n+2} h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d_j + 3)). \tag{II.2.57}$$

Consider first the case $d_{n+2} \leq 5$.
We get that $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_{n+2} - 6)) = 0$, whence by (II.2.56) we have that

$$\sum_{i=1}^{n+1} \Big[h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 6)) - h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_i - 6)) \Big] = 0.$$

Since by Lemma II.1.4 we have $m_i > d_i$ for all $1 \leq i \leq n+1$, then

$$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 6)) - h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_i - 6)) \geq 0.$$

Thus $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m_i - 6)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_i - 6))$ for all $1 \leq i \leq n+1$.
This forces $m_1 \leq 5$ and we conclude.
Thus we have only to deal with the case $d_{n+2} \geq 6$.
Then by (II.1.36) we get $d_j \geq 6$ for all $1 \leq j \leq n+2$. Moreover by Lemma II.1.4 we get that $m_i \geq 7$ for all $1 \leq i \leq n+1$. It follows by (II.2.57) that $\deg(C) = 10$.
Since C is a smooth PCM curve with minimal free resolution

$$0 \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^3}(-m_i) \to \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^3}(-d_j) \to \mathcal{I}_{C/\mathbb{P}^3} \to 0 \tag{II.2.58}$$

and $d_j \geq 6$ for all $1 \leq j \leq n+2$, it follows that C is not contained in a surface of degree 5.
However such a curve does not exist.
To see this consider the exact sequence

$$0 \to \mathcal{I}_{C/\mathbb{P}^3}(5) \to \mathcal{O}_{\mathbb{P}^3}(5) \to \mathcal{O}_C(5) \to 0.$$

Since C is not contained in a surface of degree 5, then $h^0(\mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3}(5)) = 0$. Moreover by (II.2.58) we get that $h^1(\mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3}(5)) = 0$. Hence

$$h^0(C, \mathcal{O}_C(5)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5)) = 56.$$

Since $\deg(C)$ is even, by Castelnuovo's theorem [Har77, Theorem IV.6.4] we get

$$g(C) \leq \frac{1}{4}\deg(C)^2 - \deg(C) + 1 = 16.$$

Thus $\deg(\mathcal{O}_C(5)) = 50 > 30 \geq 2g(C) - 2$, whence $h^1(C, \mathcal{O}_C(5)) = 0$. Hence by Riemann-Roch we have

$$56 = h^0(C, \mathcal{O}_C(5)) = h^0(C, \mathcal{O}_C(5)) - h^1(C, \mathcal{O}_C(5)) = \chi(\mathcal{O}_C(5)) = \deg(\mathcal{O}_C(5)) + 1 - g(C) = 51 - g(C).$$

Since $g(C) \geq 0$ we reach a contradiction and we get the implication (i) $\Rightarrow$ (ii).

Take now a smooth irreducible non-degenerate PCM surface S with minimal free resolution (II.1.34) and such that $h^0(S, K_S - H) = 0$.
To get the complete list of possible S's we will classify the collections of integers

$$(n, \{m_i\}_{i=1}^{n+1}, \{d_j\}_{j=1}^{n+2})$$

such that $n \geq 0$, $d_j \geq 2$ for all $1 \leq j \leq n+2$, $m_1 \leq 5$, and that satisfy (II.1.35), (II.1.36) and the conditions of Lemma II.1.4 and Proposition II.1.5.
We will then verify that, for every collection obtained, there exists a smooth irreducible non-degenerate PCM surface S with a corresponding minimal free resolution.
Since $m_1 \leq 5$, then by (II.1.36) we have that $m_i \leq 5$ for all $1 \leq i \leq n+1$.
Recall that by Lemma II.1.4 we have that

$$u_{i,i} > 0 \quad \forall\, 1 \leq i \leq n+1. \tag{II.2.59}$$

Thus by (II.1.36) we get that $d_j \leq 4$ for all $1 \leq j \leq n+2$.
Moreover, since S is non-degenerate, then $d_j \geq 2$ for all $1 \leq j \leq n+2$. Also by (II.2.59) we get that $m_i \geq 3$ for all $1 \leq i \leq n+1$.
It follows that $3 \leq m_i \leq 5$ for all $1 \leq i \leq n+1$ and $2 \leq d_j \leq 4$ for all $1 \leq j \leq n+2$.
Set

$$s = |\{j : d_j = 4\}|, \quad t = |\{j : d_j = 3\}|, \quad q = |\{j : d_j = 2\}|,$$
$$r = |\{i : m_i = 5\}|, \quad k = |\{i : m_i = 4\}|, \quad l = |\{i : m_i = 3\}|.$$

If $m_1 \leq 4$, then S is one of the surfaces of Proposition II.2.1, thus we may assume $r \geq 1$.
If S is a complete intersection, that is $n = 0$, we get that $m_1 = d_1 + d_2$. Since $r \geq 1$, then $m_1 = 5$.
Moreover $d_1 \geq d_2 \geq 2$, whence the only possibility is

$$(5): \quad (5; 3, 2).$$

Thus to complete the list we have only to deal with the case $n \geq 1$.
By Proposition II.1.5 we have that

$$u_{i,i-1} > 0 \quad \forall\, 2 \leq i \leq n+1. \tag{II.2.60}$$

We distinguish the following cases for the value of s:

(A) $s = 0$.
(B) $1 \leq s \leq n$.
(C) $n+1 \leq s \leq n+2$.

Consider first the case (A), that is $s = 0$.
Then we get that $d_j \leq 3$ for all $1 \leq j \leq n+2$.
If $t = 0$, then $d_j = 2$ for all $1 \leq j \leq n+2$ and by (II.1.35) we get

$$5r + 4k + 3(n+1-r-k) = 2(n+2),$$

that is $n + 2r + k = 1$. Since $r \geq 1$ we get a contradiction.
If $1 \leq t \leq n$, then by (II.2.60) we get that $m_i \geq m_{t+1} > d_t = 3$ for all $1 \leq i \leq t+1$, whence $r + k \geq t + 1$.
By (II.1.35) we get that

$$5r + 4k + 3(n+1-r-k) = 3t + 2(n+2-t),$$

that is $n + 2r + k = t + 1$. Since $t + 1 = n + r + (r + k) \geq n + 1 + (t+1) = n + t + 2$ we get a contradiction.
If $n+1 \leq t \leq n+2$, then by (II.2.59) we get that $m_i \geq m_{n+1} > d_{n+1} = 3$ for all $1 \leq i \leq n+1$, whence $r + k = n + 1$.
Hence by (II.1.35) we get that

$$5r + 4(n+1-r) = 3t + 2(n+2-t),$$

that is $2n + r = t$.
If $t = n+1$, then $n + r = 1$. Since $n \geq 1$ and $r \geq 1$ we get a contradiction.

If $t = n+2$, then $n+r = 2$. Since $n \geq 1$ and $r \geq 1$ the only possibility is $n = 1$, $r = 1$, $k = n+1-r = 1$, $t = n+2 = 3$ and $q = n+2-t = 0$, that is

$$(6): \quad (5,4;3,3,3).$$

Thus we have concluded case (A).
Consider now the case (B), that is $1 \leq s \leq n$.
Then by (II.2.60) we get that $m_i \geq m_{s+1} > d_s = 4$ for all $1 \leq i \leq s+1$, whence $r \geq s+1$.
If $0 \leq t \leq n-s$, then by (II.2.60) we get that $m_i \geq m_{s+t+1} > d_{s+t} \geq 3$ for all $1 \leq i \leq s+t+1$, whence $r+k \geq s+t+1$.
By (II.1.35) we get

$$5r + 4k + 3(n+1-r-k) = 4s + 3t + 2(n+2-s-t),$$

that is $n + 2r + k = 2s + t + 1$.
Since $2s+t+1 = n+r+(r+k) \geq n+(s+1)+(s+t+1) = n+2s+t+2$ we get a contradiction.
If $n+1-s \leq t \leq n+2-s$, then by (II.2.59) we get that $m_i \geq m_{n+1} > d_{n+1} = 3$ for all $1 \leq i \leq n+1$, whence $r+k = n+1$.
By (II.1.35) we get

$$5r + 4(n+1-r) = 4s + 3t + 2(n+2-s-t),$$

that is $2n + r = 2s + t$.
If $t = n-s+1$, then $r+n = s+1$. Since $r \geq s+1$ and $n \geq 1$ we get a contradiction.
If $t = n-s+2$, then $r+n = s+2$. Since $r \geq s+1$ and $n \geq 1$ the only possibility is $n = 1$ and $r = s+1$. Moreover, since $s \geq 1$ we get that $r = s+1 \geq 2$, whence the only possibility is $r = 2$, $k = n+1-r = 0$, $s = r-1 = 1$, $t = n-s+2 = 2$ and $q = n+2-s-t = 0$, that is

$$(7): \quad (5,5;4,3,3).$$

Consider now the case (C), that is $n+1 \leq s \leq n+2$.
Then (II.2.59) we get that $m_i \geq m_{n+1} > d_{n+1} = 4$ for all $1 \leq i \leq n+1$, whence $r = n+1$.
Hence by (II.1.35) we get

$$5(n+1) = 4s + 3t + 2(n+2-s-t),$$

that is $3n + 1 = 2s + t$.
If $s = n+1$, then $0 \leq t \leq 1$ and $n = t+1$.
If $t = 0$, then $n = 1$, $r = n+1 = 2$, $s = n+1 = 2$ and $q = n+2-s-t = 1$, that is

$$(8): \quad (5,5;4,4,2).$$

If $t = 1$, then $n = 2$, $r = n+1 = 3$, $s = n+1 = 3$ and $q = n+2-s-t = 0$, that is

$$(9): \quad (5,5,5;4,4,4,3).$$

If $s = n+2$, then $t = 0$ and $n = 3$. Thus $r = n+1 = 4$, $s = n+2 = 5$ and $q = n+2-s-t = 0$, that is

$$(10): \quad (5,5,5,5;4,4,4,4,4).$$

To conclude the proof, observe that every collection of integers $(m_1, \ldots, m_{n+1}; d_1, \ldots, d_{n+2})$ found satisfies the relation

$$u_{i,j} > 0 \quad \forall\, 1 \leq i \leq n+1, \ 1 \leq j \leq n+2.$$

Thus by [PeSz74, Theorem 6.2] we get that there exists a smooth irreducible PCM surface with a corresponding minimal free resolution. □

To classify the smooth irreducible non-degenerate PCM surfaces with $h^0(S, 2K_S - H) = 0$ we need the following proposition, that uses the machinery of the Eagon-Northcott type complexes.

Proposition II.2.3. *Let $S \subset \mathbb{P}^4$ a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34). Then there exists a resolution of the form*

$$0 \to \bigwedge^2 F \to F \otimes G \to S^2 G \to 10H + 2K_S \to 0.$$

Proof. Consider the Eagon-Northcott type complex (EN_2) associated to the map $\varphi^* : F \to G$, that is the complex

$$0 \to \bigwedge^2 F \to F \otimes G \to S^2 G \to 0$$

(we refer to [Laz04a, Appendix B] for a complete account).
Since S is generated by the minors of $[\varphi]$, then by definition it is the top degeneracy locus of the map φ

$$S = D_n(\varphi) = \{x \in \mathbb{P}^4 \ : \ \mathrm{rk}(\varphi(x)) \leq n\} = D_n(\varphi^*).$$

Since S has codimension 2, then by [Laz04a, Theorem B.2.2 (iii)] we have that the complex (EN_2) is acyclic.
Observe that by (II.1.38) we have that $\mathrm{Coker}(\varphi^*) = 5H + K_S$. Hence by [KlMi17, Proposition 2.2 (ii)] we get that (EN_2) is a resolution of $S^2\,\mathrm{Coker}(\varphi^*) = 10H + 2K_S$, whence we conclude. □

With the help of Proposition II.2.3, we can give the following improvement of Proposition II.2.2.

Proposition II.2.4. *Let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34). Then the following conditions are equivalent:*

(i) $h^0(S, K_S - H) = 0$.
(ii) $h^0(S, 2K_S - H) = 0$.
(iii) $m_1 \leq 5$.

Proof. By Proposition II.2.2 we have the equivalence (i) ⇔ (iii).
Moreover the implication (ii) ⇒ (iii) is easy. Indeed, if $p_g(S) = 0$, then by Proposition II.2.1 we have that $m_1 \leq 4$, while if $p_g(S) > 0$, then $h^0(S, K_S - H) \leq h^0(S, 2K_S - H) = 0$ and by Proposition II.2.2 we get that $m_1 \leq 5$.
To show the implication (iii) ⇒ (ii) we use Proposition II.2.3. We obtain two short exact sequences

$$0 \to \mathcal{F} \to S^2 G(-11) \xrightarrow{\varphi} 2K_S - H \to 0 \tag{II.2.61}$$

$$0 \to \bigwedge^2 F(-11) \to F \otimes G(-11) \to \mathcal{F} \to 0, \tag{II.2.62}$$

where $\mathcal{F} = \mathrm{Ker}(\varphi)$. Observe that

$$F \otimes G = \bigoplus_{1 \leq i \leq n+1, 1 \leq j \leq n+2} \mathcal{O}_{\mathbb{P}^4}(m_i + d_j),$$

$$\bigwedge^2 F = \bigoplus_{1 \leq j_1 < j_2 \leq n+2} \mathcal{O}_{\mathbb{P}^4}(d_{j_1} + d_{j_2}),$$

$$S^2 G = \bigoplus_{1 \leq i_1 \leq i_2 \leq n+1} \mathcal{O}_{\mathbb{P}^4}(m_{i_1} + m_{i_2}).$$

Since $m_1 \leq 5$, then $m_{i_1} + m_{i_2} - 11 \leq -1$ for all $1 \leq i_1 \leq i_2 \leq n+1$. Hence

$$h^0(\mathbb{P}^4, S^2 G(-11)) = \sum_{1 \leq i_1 \leq i_2 \leq n+1} h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m_{i_1} + m_{i_2} - 11)) = 0.$$

By (II.2.62), since $h^1(\mathbb{P}^4, F \otimes G(-11)) = h^2(\mathbb{P}^4, \bigwedge^2 F(-11)) = 0$, we get $h^1(\mathbb{P}^4, \mathcal{F}) = 0$. Hence, by (II.2.61) we deduce that $h^0(S, 2K_S - H) = 0$.

□

II.3. Ulrich vector bundles on projectively Cohen-Macaulay surfaces

Let S be smooth projective surface, let H be a very ample line bundle and let $\mathcal{E}$ be a vector bundle on S.

Definition II.3.1. $\mathcal{E}$ is Ulrich with respect to H if

$$h^i(S, \mathcal{E}(-pH)) = 0 \ \ \forall\, 0 \leq i \leq 2,\ 1 \leq p \leq 2.$$

We recall the following results, that guarantee the existence of Ulrich vector bundles for some classes of surfaces.

Theorem II.3.2 ([Cas17, Theorem 1.1, Theorem 1.2]). *Let S be a smooth surface with $q(S) = 0$ and $p_g(S) = 0$, and let H be a very ample non-special line bundle on S.*
Then there exists a vector bundle $\mathcal{E}$ of rank 2 that is Ulrich with respect to H and such that $c_1(\mathcal{E}) = 3H + K_S$. Moreover, if H does not embed S as a rational normal scroll and $(S, H) \ncong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, then $\mathcal{E}$ is also stable with respect to H.

Theorem II.3.3 ([Cas18, Theorem 1.1] and private communication). *Let S be a smooth surface with $q(S) = 0$ and $p_g(S) \geq 1$, and let H be a very ample non-special line bundle on S such that $h^0(S, 2K_S - H) = 0$ and $H^2 - H.K_S \geq -4$.*
Then there exists a vector bundle $\mathcal{E}$ of rank 2 on S that is simple, Ulrich with respect to H and such that $c_1(\mathcal{E}) = 3H + K_S$. Moreover, if $H^2 - H.K_S \geq 1$ and the complement S_0 of the union of smooth rational curves is dense in S, then $\mathcal{E}$ is also stable with respect to H.

Moreover we have the following theorems, that show the existence of Ulrich bundles on some classes of PCM surfaces in $\mathbb{P}^4$.

Theorem II.3.4 ([HUB91, Theorem 2.5]). *Let $S \subset \mathbb{P}^4$ be a smooth complete intersection surface, that is a smooth PCM surface of type $S = (d_1 + d_2; d_1, d_2)$, with $d_2 \geq d_1 \geq 1$.*
Then S has an Ulrich vector bundle with respect to H.

Theorem II.3.5 ([MiPo13, Theorem in the introduction]). *Let $S \subset \mathbb{P}^4$ be a general linear standard determinantal surface, that is a general PCM surface of type $S = (\underbrace{d+1, \dots, d+1}_{d}; \underbrace{d, \cdots, d}_{d+1})$, with $d \geq 1$.*
Then S supports rank 1 and 2 indecomposable Ulrich vector bundles with respect to H.

Using Theorems II.3.2 and II.3.3 we can prove the following result.

Theorem II.3.6. *Let $S \subset \mathbb{P}^4$ be a smooth irreducible PCM surface in the following list (that is, one of the surfaces of Propositions II.2.1, II.2.2 and II.2.4):*

(1) $S = (4; 2, 2)$.
(2) $S = (3, 3; 2, 2, 2)$.
(3) $S = (4, 4; 3, 3, 2)$.
(4) $S = (4, 4, 4; 3, 3, 3, 3)$.
(5) $S = (5; 3, 2)$.
(6) $S = (5, 4; 3, 3, 3)$.
(7) $S = (5, 5; 4, 3, 3)$.
(8) $S = (5, 5; 4, 4, 2)$.
(9) $S = (5, 5, 5; 4, 4, 4, 3)$.
(10) $S = (5, 5, 5, 5; 4, 4, 4, 4, 4)$.

Equivalently, let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34) such that $m_i \leq 5$ for all $1 \leq i \leq n + 1$.

Then there exists a vector bundle $\mathcal{E}$ of rank 2 on S that is simple, Ulrich with respect to H and such that $c_1(\mathcal{E}) = 3H + K_S$.
Moreover, if S is not of type (2) and (10), then $\mathcal{E}$ is also μ-stable with respect to H.

PROOF. Observe first that by Proposition II.1.8 (ii) we get that $q(S) = 0$ and H is non-special.

Take S of type $(1), (2), (3)$ or (4). Then by Proposition II.2.1 we are considering the smooth irreducible non-degenerate PCM surfaces with

$$p_g(S) = 0.$$

Hence by Theorem II.3.2 there exists a vector bundle $\mathcal{E}$ of rank 2 that is Ulrich with respect to H and such that $c_1(\mathcal{E}) = 3H + K_S$.
Take S of type (1), (3) and (4). Since H does not embed S as a rational normal scroll and $(S, H) \ncong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, then again by Theorem II.3.2 we get that the Ulrich bundle $\mathcal{E}$ is stable with respect to H. By [CaHa12, Theorem 2.9] we get that $\mathcal{E}$ is also μ-stable with respect to H.
Take S of type (2). We don't know if $\mathcal{E}$ is simple. The existence of a vector bundle with all the required properties follows by [ACMR18, Proposition 3.1].

Take S of type $(5), (6), (7), (8), (9)$ or (10). Then by Propositions II.2.1, II.2.2 and II.2.4 we are considering the smooth irreducible non-degenerate PCM surfaces with

$$p_g(S) \geq 1, \quad h^0(S, 2K_S - H) = 0.$$

Set $r = |\{i : m_i = 5\}|$. By Proposition II.1.8 (v) we have that

$$H^2 - H.K_S = 2(4 - r).$$

Since $r \leq 4$ we have that $H^2 - H.K_S \geq 0$. It follows by Theorem II.3.3 that there exists a vector bundle $\mathcal{E}$ of rank 2 on S that is simple, Ulrich with respect to H and such that $c_1(\mathcal{E}) = 3H + K_S$.
Take S of type (5), (6), (7), (8) and (9). Observe that $H^2 - H.K_S > 0$.
Moreover, since $p_g(S) \geq 1$, then the complement S_0 of the union of smooth rational curves is not dense in S. Thus by Theorem II.3.3 we get that $\mathcal{E}$ is also stable. Again by [CaHa12, Theorem 2.9] we get that $\mathcal{E}$ is also μ-stable with respect to H and we conclude. □

REMARK II.3.7. We know a birational description of the surfaces of type (1), (2), (3), (4) and (5). In particular, the surfaces of type (1), (2), (3), (4) may be realized as blow-ups of $\mathbb{P}^2$ at some points (see Proposition II.2.1), while the surface of type (5) is a $K3$.
On the other hand, we don't know a birational description of the surfaces of type (6), (7), (8), (9) and (10). Note that the surface S of type (6) is not minimal. Indeed by Propositions II.1.8 and II.1.9 and by genus-degree formula it can be easily seen that $p_g(S) > 0$ and $K_S^2 < 0$ (we refer to the proof of Proposition II.3.15 for more details). Thus S is not minimal, since it is not ruled nor with nef canonical bundle.

REMARK II.3.8. One may ask if the surfaces of the previous theorem admit Ulrich line bundles. On the one hand, by Theorem II.3.5 the general surface of type (2), (4) and (10) admits Ulrich line bundles with respect to H. On the other hand, by Lefschetz's theorem the Picard group of the general surface S of type (5) is $\text{Pic}(S) \cong \mathbb{Z}[H]$, whence S has no Ulrich line bundles with respect to H.

II.3.1. A first alternative proof of part of Theorem G.

In this subsection we describe a method that we tried to use to prove the existence of Ulrich vector bundles for smooth PCM surfaces, altough unfortunatley we succed only to reprove part of Theorem II.3.6 (see Proposition II.3.13).

First of all we recall the following well-known definition.
Let X be smooth a variety, let M be a line bundle on X and let $Z \subset X$ be a 0-dimensional reduced subscheme.

DEFINITION II.3.9. Z has the Cayley-Bacharach property with respect to $|M|$ if for all $Z' \subset Z$ subscheme such that $\text{length}(Z') = \text{length}(Z) - 1$ and for all $s \in H^0(X, M)$ we get that $s_{|Z'} = 0$ implies $s_{|Z} = 0$.

As the following result explains, the Cayley-Bacharach property turns out to be useful to construct vector bundles of rank 2 on a surface.

PROPOSITION II.3.10 ([HuLe10, Theorem 5.1.1]). *Let S be a smooth surface, let $Z \subset S$ be a 0-dimensional reduced subscheme and let M be a line bundle on S. Then the following conditions are equivalent:*

(i) *Z has the Cayley-Bacharach property with respect to $|M|$.*
(ii) *There exists an extension*
$$0 \to \mathcal{O}_S \to \mathcal{F} \to \mathcal{I}_Z(M - K_S) \to 0$$
with $\mathcal{F}$ a vector bundle of rank 2.

Moreover $c_1(\mathcal{F}) = M - K_S$ and $c_2(\mathcal{F}) = \text{length}(Z)$.

To deduce the Cayley-Bacharach property of Proposition II.3.10, one can use the following result.

LEMMA II.3.11. *Let C be a smooth and irreducible curve, let L and M be two line bundles on C such that L is base-point-free and let $Z \in |L|$ be a general divisor. If $h^0(C, K_C + L - M) > 0$, then Z has the Cayley-Bacharach property with respect to $|M|$.*

PROOF. Z has the Cayley-Bacharach property with respect to $|M|$ if and only if
$$h^0(C, M - L) = h^0(C, M - (L - p)) \;\; \forall\, p \in \text{Supp}(Z).$$
By Serre duality this is equivalent to
$$h^1(C, K_C + L - M) = h^1(C, K_C + L - M - p) \;\; \forall\, p \in \text{Supp}(Z).$$
Observe now that by Riemann-Roch
$$h^0(C, K_C + L - M) = h^1(C, K_C + L - M) + \deg(K_C + L - M) + 1 - g(C)$$
and
$$h^0(C, K_C + L - M - p) = h^1(C, K_C + L - M - p) + \deg(K_C + L - M - p) + 1 - g(C) =$$
$$= h^1(C, K_C + L - M - p) + \deg(K_C + L - M) - g(C).$$
It follows that Z has the Cayley-Bacharach property with respect to $|M|$ if and only if
$$h^0(C, K_C + L - M) = h^0(C, K_C + L - M - p) + 1 \;\; \forall\, p \in \text{Supp}(Z).$$
Since $h^0(C, K_C + L - M) > 0$, it follows that $\text{Bs}\,|K_C + L - M|$ is 0-dimensional. Since Z is general and L is base-point-free, it follows that $Z \cap \text{Bs}\,|K_C + L - M| = \emptyset$. Hence for all $p \in Z$ we have that
$$h^0(C, K_C + L - M) = h^0(C, K_C + L - M - p) + 1$$

and we conclude. □

Once one has constructed a vector bundle on S, he may try to prove that it is an Ulrich vector bundle by using the following result.

Proposition II.3.12 ([Cas17, Propositon 2.1]). *Let S be a smooth surface, let H be a very ample line bundle and let $\mathcal{E}$ be a vector bundle of rank 2 on S. Then the following conditions are equivalent:*

(i) *$\mathcal{E}$ is an Ulrich vector bundle with respect to H.*

(ii)

$$h^0(S, \mathcal{E}(-H)) = 0, \quad h^0(S, \mathcal{E}^*(2H + K_S)) = 0,$$

$$c_1(\mathcal{E}).H = 3H^2 + H.K_S, \quad c_2(\mathcal{E}) = \frac{1}{2}(c_1(\mathcal{E})^2 - c_1(\mathcal{E}).K_S) - 2(H^2 - \chi(\mathcal{O}_S)).$$

Proposition II.3.13. *Let $S \subset \mathbb{P}^4$ be a smooth irreducible PCM surface of type $S = (4, 4; 3, 3, 2)$ or $S = (4, 4, 4; 3, 3, 3, 3)$.*
Then there exists an Ulrich vector bundle $\mathcal{E}$ of rank 2 with respect to H with $c_1(\mathcal{E}) = 3H + K_S$.

Proof. Set $k = |\{i : m_i = 4\}|$.
By Propositions II.1.8 (i), (ii), (iii), (v) and II.1.9 (iii) we have that

$$H^2 = 3 + k, \quad H.K_S = H^2 - (H^2 - H.K_S) = -5 + k, \quad \chi(\mathcal{O}_S) = 1.$$

Let now $\Gamma = S \cap S'$ be the scheme-theoretic intersection of S with the residual scheme S' in the complete intersection $X = S \cup S' = X_1 \cap X_2$, where X_1 and X_2 are two hypersurfaces in $\mathbb{P}^4$ defined by two minimal homogeneous generators of I_S of degree d_1 and d_2, respectively. By Proposition II.1.11 we have that Γ is a smooth irreducible curve and

$$\Gamma \sim (d_1 + d_2 - 5)H - K_S = H - K_S,$$

$$H.\Gamma = H^2 - H.K_S = 8,$$

$$g(\Gamma) = 1 + \frac{1}{2}(d_1 + d_2 - 5)H.\Gamma = 5.$$

By adjunction formula we have that

$$K_\Gamma = (K_S + \Gamma)_{|\Gamma} = H_{|\Gamma}.$$

Let L be a general line bundle on Γ of degree $6 = g(\Gamma) + 1$. Then we have that L is effective, base-point-free and non-special.
Take $Z \in |L|$ general. Since

$$h^0(\Gamma, K_\Gamma + L - H_{|\Gamma}) = h^0(\Gamma, L) > 0$$

by Lemma II.3.11 it follows that Z has the Cayley-Bacharach property with respect to $|H_{|\Gamma}|$.
Hence it also has the Cayley-Bacharach property with respect to $|H|$, whence by Proposition II.3.10 there exists a rank 2 vector bundle $\mathcal{F}$ on S fitting into an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{F} \to \mathcal{I}_{Z/S}(H - K_S) \to 0 \tag{II.3.63}$$

and such that

$$c_1(\mathcal{F}) = H - K_S, \quad c_2(\mathcal{F}) = \text{length}(Z) = \deg(L) = 6.$$

Set now $\mathcal{E} = \mathcal{F}(H + K_S)$. We will show, using Proposition II.3.12, that $\mathcal{E}$ is an Ulrich vector bundle with respect to H.
Observe that

$$c_1(\mathcal{E}) = c_1(\mathcal{F}) + 2(H + K_S) = 3H + K_S.$$

Moreover

$$c_2(\mathcal{E}) = c_2(\mathcal{F}) + c_1(\mathcal{F})(H+K_S) + (H+K_S)^2 = \deg(L) + (H-K_S)(H+K_S) + (H+K_S)^2 =$$
$$= \deg(L) + 2H^2 + 2H.K_S = 2+4k$$

and

$$\frac{1}{2}(c_1(\mathcal{E})^2 - c_1(\mathcal{E}).K_S) - 2(H^2 - \chi(\mathcal{O}_S)) = \frac{1}{2}(3H+K_S)3H - 2(H^2-\chi(\mathcal{O}_S)) =$$
$$= \frac{5}{2}H^2 + \frac{3}{2}H.K_S + 2\chi(\mathcal{O}_S) = 2+4k.$$

Finally

$$\mathcal{E}^* \cong \mathcal{E}(-c_1(\mathcal{E})) = \mathcal{E}(-3H-K_S),$$

whence we get that $h^0(S, \mathcal{E}^*(2H+K_S)) = h^0(S, \mathcal{E}(-H))$.
Thus to apply Proposition II.3.12 we have only to show that

$$h^0(S, \mathcal{E}(-H)) = 0.$$

Twisting by K_S the exact sequence (II.3.63) we obtain the exact sequece

$$0 \to K_S \to \mathcal{E}(-H) \to \mathcal{I}_{Z/S}(H) \to 0.$$

Since by Proposition II.2.1 we have that $h^0(S, K_S) = 0$, the results follows if we show that

$$h^0(S, \mathcal{I}_{Z/S}(H)) = 0.$$

To do this consider the exact sequence

$$0 \to \mathcal{I}_{\Gamma/S}(H) \to \mathcal{I}_{Z/S}(H) \to \mathcal{I}_{Z/\Gamma}(H) \to 0.$$

Since $h^0(S, \mathcal{I}_{\Gamma/S}(H)) = h^0(S, H-\Gamma) = h^0(S, K_S) = 0$, it is sufficient to show that $h^0(\Gamma, \mathcal{I}_{Z/\Gamma}(H)) = 0$.
Since L is non-special, by Serre duality we have that

$$h^0(\Gamma, \mathcal{I}_{Z/\Gamma}(H)) = h^0(\Gamma, H_{|\Gamma} - L) = h^1(\Gamma, K_\Gamma - H_{|\Gamma} + L) = h^1(\Gamma, L) = 0$$

and we conclude. □

II.3.2. A second alternative proof of part of Theorem G.

In this section, we provide another proof of part of Theorem II.3.6, that uses Brill-Noether theory instead of the exactness of the Eagon-Northcott type complex of Proposition II.2.3.
We start by recalling the following result.

LEMMA II.3.14 ([KnLo07, Lemma 3.1]). *Let S be a smooth surface with $-K_S \geq 0$ and $h^1(S, \mathcal{O}_S) = 0$, let C be a smooth irreducible curve of genus g on S and let $|A|$ be a base-point-free g^1_k on C, with $k \geq 1$.*
Suppose that

$$2g - 2 - K_S.C - 4k \geq \max\{0, 3 - 4\chi(\mathcal{O}_S)\}.$$

Then there exist two line bundles L and M on S and a 0-dimensional subscheme $Z \subset S$ such that the following conditions hold:

(i) $C \sim M + L$.
(ii) $k = M.L + \mathrm{length}(Z) \geq M.L \geq L^2 \geq 0$.
(iii) *There exists an effective divisor D on C of degree $M.L + L^2 - k \geq 0$ such that $A \cong L_{|C}(-D)$.*
(iv) *If $L^2 = 0$, then $M.L = k$ and $A \cong L_{|C}$.*
(v) *L is nef, base-component-free and nontrivial.*

We can then prove the following statement.

PROPOSITION II.3.15. *Let $S \subset \mathbb{P}^4$ be a smooth irreducible PCM surface in the following list (that is, one of the surfaces of Propositions II.2.1, II.2.2 and II.2.4 not of type (10)):*

(1) $S = (4; 2, 2)$.
(2) $S = (3, 3; 2, 2, 2)$.
(3) $S = (4, 4; 3, 3, 2)$.
(4) $S = (4, 4, 4; 3, 3, 3, 3)$.
(5) $S = (5; 3, 2)$.
(6) $S = (5, 4; 3, 3, 3)$.
(7) $S = (5, 5; 4, 3, 3)$.
(8) $S = (5, 5; 4, 4, 2)$.
(9) $S = (5, 5, 5; 4, 4, 4, 3)$.

Equivalently, let $S \subset \mathbb{P}^4$ be a smooth irreducible non-degenerate PCM surface with minimal free resolution (II.1.34) such that $m_i \leq 5$ for all $1 \leq i \leq n+1$ and S is not of type (10) of Theorem II.3.6.
Then there exists a vector bundle $\mathcal{E}$ of rank 2 on S that is simple, Ulrich with respect to H and such that $c_1(\mathcal{E}) = 3H + K_S$.
Moreover, if S is not of type (2), then $\mathcal{E}$ is also μ-stable with respect to H.

PROOF. If S is of type $(1), (2), (3)$ or (4), then the proof is the same as the one of Theorem II.3.6 (observe that it doesn't use Proposition II.2.3).
If S is of type $(5), (6), (7), (8)$ or (9), we would like to apply Theorem II.3.3.
Observe that by (II.1.36) we get that $m_i \leq 5$ for all $1 \leq i \leq n+1$. Moreover by Lemma II.1.4 and (II.1.36) we get that $d_j \leq 4$ for all $1 \leq j \leq n+2$.
Set

$$r = |\{i : m_i = 5\}|, \quad k = |\{i : m_i = 4\}|, \quad s = |\{j : d_j = 4\}|.$$

By Proposition II.1.8 (i), (ii), (iii) (with $l = 0$) and (v) we get

$$\chi(\mathcal{O}_S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) = 1 + r,$$

$$H^2 - H.K_S = 8 - 2r.$$

Moreover by Proposition II.1.9 (iii) we get

$$H^2 = 3 + 3r + (k - s),$$

$$H.K_S = H^2 - (H^2 - H.K_S) = 5r + (k - s) - 5.$$

By the genus-degree formula for a smooth projective surface in $\mathbb{P}^4$ (see [Har77, Example A.4.1.3]) we get that

$$K_S^2 = \frac{1}{2}(H^2(H^2 - 10) - 5H.K_S + 12\chi(\mathcal{O}_S)) = \frac{1}{2}(16 - 25r - 9(k - s) + 9r^2 + (k - s)^2 + 6r(k - s)).$$

By Proposition II.1.8 (ii) we have that $q(S) = 0$ and H is non-special. Moreover by Proposition II.1.8 (iii) with $l = 0$ we get that $p_g(S) \geq 1$. Finally, since $r \leq 3$, then $H^2 - H.K_S > 0$. Hence we have only to check that $h^0(S, 2K_S - H) = 0$ to conclude.

Take S of type (5), (6) or (8). Since $H.(2K_S - H) = 2H.K_S - H^2 < 0$, then $h^0(S, 2K_S - H) = 0$ and we conclude.

Take S of type (7). Then $H^2 = 8$, $H.K_S = 4$ and $K_S^2 = 0$.
We claim that this implies that $h^0(S, 2K_S - H) = 0$.
Indeed, if $h^0(S, 2K_S - H) > 0$, then there exists an effective divisor $D \sim 2K_S - H$. However $H.D = H.(2K_S - H) = 0$, whence we have that $D \sim 0$. Hence $2K_S \sim H$ and we obtain $4K_S^2 = H^2 = 8$, reaching a contradiction.

Take S of type (9). Then $H^2 = 9$, $H.K_S = 7$ and $K_S^2 = 2$.
We claim that this implies that $h^0(S, 2K_S - H) = 0$.
Observe that

$$K_S.(2K_S - H) = 2K_S^2 - H.K_S = -3 < 0.$$

Hence if K_S is nef we conclude.
So we have only to consider the case where K_S is not nef. We assume that $h^0(S, 2K_S - H) > 0$ and we reach a contradiction.
Since by Proposition II.1.8 we have that $h^0(S, K_S) = h^2(S, \mathcal{O}_S) = 3$, then there exists an effective divisor $D \in |K_S|$. Since by assumption K_S is not nef, then there is an irreducible curve B in S with $D.B < 0$.
We claim that the curve B is a line.
Writing $D = \sum a_\alpha D_\alpha$, with $a_\alpha > 0$ and D_α irreducible divisors, we see that there exists an α such that $B.D_\alpha < 0$. It follows that $B = D_\alpha$ and $B^2 \leq -1$. Since $B^2 \leq -1$ and $B.K_S \leq -1$ we have that $B^2 + B.K_S \leq -2$. Since B is irreducible, by the genus formula we obtain that

$$p_a(B) = 1 + \frac{1}{2}(B^2 + B.K_S) = 0,$$

whence B is smooth, rational and $B^2 = B.K_S = -1$.
Since by assumption $h^0(S, 2K_S - H) > 0$, there exists an effective divisor $D' \in |2K_S - H|$. Since B is an irreducible curve and $B.D' < 0$, then B is contained in the support of D', whence $h^0(S, 2K_S - H - B) > 0$. Analogously, since $h^0(S, 2K_S - H - B) > 0$ and $B.(2K_S - H - B) = -1 - \deg(B) < 0$, we get that $h^0(S, 2K_S - H - 2B) > 0$. Finally, since $h^0(S, 2K_S - H - 2B) > 0$ and $B.(2K_S - H - 2B) = -\deg(B) < 0$ we obtain that $h^0(S, 2K_S - H - 3B) > 0$. Since $H.(2K_S - H - 3B) = 5 - 3\deg(B) \geq 0$, it follows $\deg(B) = 1$.
Consider now an irreducible smooth curve $C \in |H|$.

C has degree 9 and by genus formula

$$g(C) = 1 + \frac{1}{2}(H^2 + H.K_S) = 9.$$

We claim that C has a base-point-free complete g^1_k, with $2 \leq k \leq 5$ (that is a base-point-free complete linear system of dimension 1 and degree k).
To do this consider the exact sequence

$$0 \to H \to H + B \to (H+B)_{|B} \to 0.$$

By Proposition II.1.8 we get that $h^1(S, H) = 0$. Moreover, since B is a line with $B^2 = -1$ we get that $h^1(B, (H+B)_{|B}) = 0$. It follows that $h^1(S, H+B) = 0$.
Consider the exact sequence

$$0 \to K_S - B - H \to K_S - B \to (K_S - B)_{|C} \to 0.$$

Since

$$H.(K_S - B - H) = H.K_S - H.B - H^2 = -3 < 0$$

we get that $h^0(S, K_S - B - H) = 0$. Moreover by Serre duality $h^1(S, K_S - B - H) = h^1(S, B+H) = 0$.
Finally, since $B.K_S = -1$, we get that B is a base component of $|K_S|$, whence $h^0(S, K_S - B) = h^0(S, K_S) = h^2(S, \mathcal{O}_S) = 3$. It follows that $h^0(C, (K_S - B)_{|C}) = 3$.
Since $\deg((K_S - B)_{|C}) = H.K_S - H.B = 6$, then the linear series $|(K_S - B)_{|C}|$ is a complete g^2_6 on C. Moreover, if $p \in C$ is a general point, then $|(K_S - B)_{|C} - p|$ is a complete g^1_5 on C.
Set now $k = \mathrm{gon}(C)$. Then C has a base-point-free complete g^1_k. Since C is not a $\mathbb{P}^1$ we observe that $2 \leq k \leq 5$.
We want to rule out the existence of such base-point-free complete g^1_k on C.
Write now $S \cup S' = F \cap F'$, where F and F' are two hypersurfaces of degree 3 and 4 in $\mathbb{P}^4$. Intersecting with a hyperplane in $\mathbb{P}^4$ we have $C \cup C' = G \cap G'$, where G and G' are two surfaces of degree 3 and 4 in $\mathbb{P}^3$.
Since G is an irreducible surface of degree 3, it is the anticanonical image of the blow-up $p : \tilde{G} \to \mathbb{P}^2$ at 6 possibly infinitely near points. Denote by $\tilde{H}$ the strict transform of a line in $\mathbb{P}^2$ and by E_α the total inverse image of the blown-up points. We have that $H_{\tilde{G}} = 3\tilde{H} - \sum E_\alpha$ and $K_{\tilde{G}} = -H_{\tilde{G}}$.
Moreover denote by $\varphi : \tilde{G} \to \mathbb{P}^3$ the morphism given by the linear series $|H_{\tilde{G}}|$. Then $\varphi(\tilde{G}) = G$.
Since C is a smooth irreducible curve of degree 9 and genus 9 on G, then its strict transform $\tilde{C}$ is a smooth irreducible curve of genus 9 on $\tilde{G}$ such that $H_{\tilde{G}}.\tilde{C} = 9$.
Indeed, since the morphism $\varphi : \tilde{G} \to G$ is birational and contracts a finite number of curves, also the restriction morphism $\varphi_{|\tilde{C}} : \tilde{C} \to C$ is birational. It follows that there exists a subscheme W of C such that $\varphi_{|\tilde{C}}$ is a blow-up of C along W. Since C is smooth, then W is a Cartier divisor, whence $\varphi_{|\tilde{C}}$ is an isomorphism.
Moreover, since C has a complete base-point-free g^1_k, with $2 \leq k \leq 5$, then $\tilde{C}$ also has a base-point-free complete g^1_k. Let $|A|$ be any base-point-free complete g^1_k on $\tilde{C}$.
Since

$$2g(\tilde{C}) - 2 - K_{\tilde{G}}.\tilde{C} - 4k = 25 - 4k \geq 0 = \max\{0, 3 - 4\chi(\tilde{G})\},$$

then by Lemma II.3.14 there exist two line bundles L and M on $\tilde{G}$ and a 0-dimensional subscheme $Z \subset \tilde{G}$ such that:

(i) $\tilde{C} \sim M + L$.
(ii) $k = M.L + \mathrm{length}(Z) \geq M.L \geq L^2 \geq 0$.
(iii) There exists an effective divisor D on $\tilde{C}$ of degree $M.L + L^2 - k \geq 0$ such that $A \cong L_{|\tilde{C}}(-D)$.
(iv) If $L^2 = 0$, then $M.L = k$ and $A \cong L_{|\tilde{C}}$.

(v) L is nef, base-component-free and nontrivial.

Observe that, since by genus formula

$$9 = g(\tilde{C}) = 1 + \frac{1}{2}(\tilde{C}^2 + \tilde{C}.K_{\tilde{G}}) = 1 + \frac{1}{2}(\tilde{C}^2 - 9)$$

we have that $\tilde{C}^2 = 25$.

By the Hodge index theorem, since $\tilde{C}^2 > 0$, then

$$25L^2 = \tilde{C}^2.L^2 \leq (\tilde{C}.L)^2 = (M.L + L^2)^2 = (M.L)^2 + 2(M.L)L^2 + (L^2)^2,$$

that is

$$(M.L)^2 + 2L^2(M.L) + (L^2)^2 - 25L^2 \geq 0.$$

By (ii) it turns out that the only possibilities are:

(1) $k = 5$, $L^2 = 1$ and $M.L = 5$.
(2) $k = 5$, $L^2 = 1$ and $M.L = 4$.
(3) $k = 4$, $L^2 = 1$ and $M.L = 4$.
(4) $2 \leq k \leq 5$, $L^2 = 0$ and $M.L = k$.

In all cases we will reach a contradiction.

In case (1) we first show that

$$h^0(\tilde{G}, L) \leq 3, \quad h^2(\tilde{G}, L) = 0.$$

Since L is nef and $L.(K_{\tilde{G}} - L) = L.K_{\tilde{G}} - L^2 = -L.H_{\tilde{G}} - 1 < 0$, it follows that $h^2(\tilde{G}, L) = h^0(\tilde{G}, K_{\tilde{G}} - L) = 0$.

Take now the exact sequence

$$0 \to A \to A(D) \to A(D)_{|D} \to 0.$$

Since by (iii) we have that $A = L_{|\tilde{C}}(-D)$ we get that

$$6 - \deg(D) = L.\tilde{C} - \deg(D) = \deg(L_{|\tilde{C}}(-D)) = \deg(A) = 5,$$

whence $\deg(D) = 1$. Since D is effective, it follows that D is a point, whence

$$h^0(\tilde{C}, A(D)) \leq h^0(\tilde{C}, A) + h^0(D, A(D)_{|D}) = h^0(\tilde{C}, A) + 1 = 3.$$

Since L is nef and $L.(L - \tilde{C}) = L.(-M) = -5 < 0$ we get that $h^0(\tilde{G}, -M) = 0$, whence

$$h^0(\tilde{G}, L) \leq h^0(\tilde{C}, L_{|\tilde{C}}) = h^0(\tilde{C}, A(D)) \leq 3.$$

Now we claim that

$$L.(-K_{\tilde{G}}) = 2.$$

Since $(-K_{\tilde{G}})^2 = 3 > 0$, then by the Hodge index theorem we get that

$$3 = (-K_{\tilde{G}})^2 L^2 \leq (L.(-K_{\tilde{G}}))^2.$$

Moreover, since L is nef we have that $L.(-K_{\tilde{G}}) \geq 0$. It follows that $L.(-K_{\tilde{G}}) \geq 2$.

Observe that by Riemann-Roch, since $h^2(\tilde{G}, L) = 0$ we get that

$$h^0(\tilde{G}, L) - 1 - \frac{1}{2}(L^2 + L.(-K_{\tilde{G}})) = h^0(\tilde{G}, L) + h^2(\tilde{G}, L) - \chi(L) = h^1(\tilde{G}, L) \geq 0,$$

whence

$$h^0(\tilde{G}, L) \geq \frac{1}{2}(3 + L.(-K_{\tilde{G}})).$$

Since $h^0(\tilde{G}, L) \leq 3$ we get that $L.(-K_{\tilde{G}}) \leq 3$.
To show that $L.(-K_{\tilde{G}}) \leq 2$ observe that, since $9 = \tilde{C}.(-K_{\tilde{G}}) = L.(-K_{\tilde{G}}) + M.(-K_{\tilde{G}})$ and $2 \leq L.(-K_{\tilde{G}}) \leq 3$, then

$$6 \leq M.(-K_{\tilde{G}}) \leq 7.$$

On the other hand by the Hodge index theorem, since $M^2 = \tilde{C}^2 - L^2 - 2M.L = 14 > 0$ we have that

$$42 = M^2(-K_{\tilde{G}})^2 \leq (M.(-K_{\tilde{G}}))^2.$$

It follows that $M.(-K_{\tilde{G}}) = 7$, whence $L.(-K_{\tilde{G}}) = 2$.
Since L is a nef line bundle on $\tilde{G}$ we have that $L \sim a\tilde{H} - \sum b_\alpha E_\alpha$, with $a, b_\alpha \geq 0$ for all α. Moreover we have that

$$L^2 = a^2 - \sum b_\alpha^2 = 1, \quad L.(-K_{\tilde{G}}) = 3a - \sum b_\alpha = 2.$$

By Cauchy-Schwartz inequality $(\sum b_\alpha)^2 \leq 6 \sum b_\alpha^2$ we obtain that

$$(3a-2)^2 \leq 6(a^2-1).$$

The only solution is $a = 2$. Since $\sum b_\alpha^2 = a^2 - 1 = 3$ and $b_\alpha \geq 0$ for all α, the only possibility is that the b_α's are three $0's$ and three $1's$. Hence $\sum b_\alpha = 3$ and $L.(-K_{\tilde{G}}) = 3a - \sum b_\alpha = 3$.
Thus we get a contradiction and the case (1) is done.
Cases (2) and (3) are easily excluded. Indeed, in both cases, we have that

$$\tilde{C}^2.L^2 = (\tilde{C}.L)^2 = (M.L + L^2)^2 = 25,$$

whence

$$\tilde{C} = L^2\tilde{C} \equiv (L.\tilde{C})L = (M.L + L^2)L = 5L.$$

Hence we reach a contradiction because $9 = -K_{\tilde{G}}.\tilde{C} = 5(-K_{\tilde{G}}.L)$.
Thus we have only to consider the case (4).
Note that by (iv) we have that $A \cong L_{|\tilde{C}}$.
We first claim that

$$L.(-K_{\tilde{G}}) \geq 1.$$

Since L is nef we have that $L.(-K_{\tilde{G}}) \geq 0$. Moreover, if $L.(-K_{\tilde{G}}) = 0$, since $(-K_{\tilde{G}})^2 = 3 > 0$ by the Hodge index theorem

$$0 = (-K_{\tilde{G}})^2 L^2 = (L.(-K_{\tilde{G}}))^2.$$

It follows that $3L \equiv 0$, that is impossible.
We claim now that

$$h^0(\tilde{G}, L) \leq 2, \quad h^2(\tilde{G}, L) = 0.$$

Since L is nef and $L.(K_{\tilde{G}} - L) = L.K_{\tilde{G}} - L^2 = L.K_{\tilde{G}} < 0$, it follows that $h^2(\tilde{G}, L) = h^0(\tilde{G}, K_{\tilde{G}} - L) = 0$.
Moreover, since $L.(L - \tilde{C}) = L.(-M) = -k < 0$ we get that $h^0(\tilde{G}, L - \tilde{C}) = 0$, whence

$$h^0(\tilde{G}, L) \leq h^0(\tilde{C}, L_{|\tilde{C}}) = h^0(\tilde{C}, A) = 2.$$

We claim now that

$$L.(-K_{\tilde{G}}) = 2.$$

To see this observe that by Riemann-Roch

$$\chi(L) = \chi(\mathcal{O}_{\tilde{G}}) + \frac{1}{2}(L^2 + L.(-K_{\tilde{G}})) = \chi(\mathcal{O}_{\tilde{G}}) + \frac{1}{2}L.(-K_{\tilde{G}}),$$

whence $L.(-K_{\tilde{G}})$ is even. Moreover, again by Riemann-Roch, we have that

$$h^0(\tilde{G}, L) - 1 - \frac{1}{2}(L^2 - L.K_{\tilde{G}}) = h^0(\tilde{G}, L) + h^2(\tilde{G}, L) - \chi(L) = h^1(\tilde{G}, L) \geq 0,$$

whence

$$h^0(\tilde{G}, L) \geq 1 + \frac{1}{2} L.(-K_{\tilde{G}}).$$

Since $h^0(\tilde{G}, L) \leq 2$ we get that $L.(-K_{\tilde{G}}) \leq 2$. It follows that $L.(-K_{\tilde{G}}) = 2$.
Since $9 = \tilde{C}.(-K_{\tilde{G}}) = L.(-K_{\tilde{G}}) + M.(-K_{\tilde{G}})$ and $L.(-K_{\tilde{G}}) = 2$ we get that

$$M.(-K_{\tilde{G}}) = 7.$$

By the Hodge index theorem, since $M^2 = \tilde{C}^2 - L^2 - 2M.L = 25 - 2k > 0$ we have that

$$3(25 - 2k) = M^2(-K_{\tilde{G}})^2 \leq (M.(-K_{\tilde{G}}))^2 \leq 49,$$

whence $k = 5$.
Since L is a nef line bundle on $\tilde{G}$ we have that $L \sim a\tilde{H} - \sum b_\alpha E_\alpha$, with $a, b_\alpha \geq 0$ for all α. Moreover we have that

$$L^2 = a^2 - \sum b_\alpha^2 = 0, \quad L.(-K_{\tilde{G}}) = 3a - \sum b_\alpha = 2.$$

Since there are only finitely many 7-ples of integers $(a, b_1, \ldots, b_6)$ that satisfy these conditions, it follows that $\tilde{C}$ has a finite number of base-point-free complete g^1_5's. This leads to a contradiction. Indeed, if $\tilde{C}$ has a finite number of base-point-free complete g^1_5's, then also C has a finite number of base-point-free complete g^1_5's. However we have shown that $|(K_S - B)_{|C} - p|$ is a complete g^1_5 for every $p \in C$ general point and, since $5 = \text{gon}(C)$, it is also base-point-free. Observe now that $(K_S - B)_{|C} - p \not\sim (K_S - B)_{|C} - q$ for all $p \neq q \in C$ general points. Indeed, if $(K_S - B)_{|C} - p \sim (K_S - B)_{|C} - q$, then $p \sim q$. Since $g(C) > 0$, that implies $p = q$. Hence we get a contradiction and we conclude.
The proof of the μ-stability is the same as the one of Theorem II.3.6. □

Bibliography

[AnGr62] A. Andreotti, H. Grauert. *Théorème de finitude pour la cohomologie des espaces complexes.* Bull. Soc. Math. France **90** (1962), 193-259. ii, 20

[ACMR18] M. Aprodu, L. Costa, R.M. Miró-Roig. *Ulrich bundles on ruled surfaces.* J. Pure Appl. Algebra **222** (2018), no. 1, 131-138. vii, 73

[AFO17] M. Aprodu, G. Farkas, A. Ortega. *Minimal resolutions, Chow forms and Ulrich vector bundles on $K3$ surfaces.* J. Reine Angew. Math. **730** (2017), 225-249. vii

[Bea00] A. Beauville. *Determinantal hypersurfaces.* Michigan Math. J. **48** (2000), 39-64. vii

[Bea16a] A. Beauville. *Ulrich bundles on abelian surfaces.* Proc. Amer. Math. Soc. **144** (2016), no. 11, 4609-4611. vii

[Bea16b] A. Beauville. *Ulrich bundles on surfaces with $p_g = q = 0$.* Preprint arXiv:1607.00895 [math.AG]. vii

[Bea18] A. Beauville. *An introduction to Ulrich bundles.* Eur. J. Math. **4** (2018), no. 1, 26-36. vi, vii

[Bir17] C. Birkar. *The augmented base locus of real divisors over arbitrary fields.* Math. Ann. **368** (2017), no. 3-4, 905-921. 3, 5, 7

[BoNu18] L. Borisov, H. Nuer. *Ulrich bundles on Enriques surfaces.* Int. Math. Res. Not. IMRN **2018**, no. 13, 4171-4189. vii

[BBP13] S. Boucksom, A. Broustet, G. Pacienza. *Uniruledness of stable base loci of adjoint linear systems via Mori theory.* Math. Z. **275** (2013), no. 1-2, 499-507. 9

[BCL14] S. Boucksom, S. Cacciola, A.F. Lopez. *Augmented base loci and restricted volumes on normal varieties.* Math. Z. **278** (2014), no. 3-4, 979-985. 48

[BHU87] J.P. Brennan, J. Herzog, B. Ulrich. *Maximally generated Cohen-Macaulay modules.* Mathematica Scandinavica, Vol. 61 (1987), pp. 181-203 vi

[Bro12] M.V. Brown. *Big q-ample line bundles.* Compos. Math. **148** (2012), no. 3, 790-798. ii, iii, 21, 25, 32, 34

[Cac08] S. Cacciola. *Asymptotic invariants of line bundles, semiampleness and finite generation.* Undergraduate Thesis, 2008, http://ricerca.mat.uniroma3.it/users/lopez/Tesi-Cacciola.pdf. 4

[CaHa12] M. Casanellas, R. Hartshorne. *Stable Ulrich bundles.* With an appendix by F. Geiss, F.-O. Schreyer. Internat. J. Math. **23** (2012), no. 8, 1250083, 50 pp. vi, 73

[Cas17] G. Casnati. *Special Ulrich bundles on non-special surfaces with $p_g = q = 0$.* Internat. J. Math. **28** (2017), no. 8, 1750061, 18 pp. vii, 72, 75

[Cas17*] G. Casnati. *Erratum: 'Special Ulrich bundles on non-special surfaces with $p_g = q = 0$'.* Internat. J. Math. **29** (2018), no. 5, 1892001, 3 pp. vii

[Cas19] G. Casnati. *Ulrich bundles on non-special surfaces with $p_g = 0$ and $q = 1$.* Rev. Math. Complut. **32** (2019). vii

[Cas19*] G. Casnati. *Corrigendum to Ulrich bundles on non-special surfaces with $p_g = 0$ and $q = 1$.* Preprint (2018). vii

[Cas18] G. Casnati. *Special Ulrich bundles on regular surfaces with non-negative Kodaira dimension.* Preprint arXiv:1809.08565 [math.AG]. vii, 72

[Choi14] S.R. Choi. *On partially ample divisors.* Math. Nachr. 287 (2014), no. 14-15, 1632-1641. i, iii, 26, 42, 44

[CoHu18] I. Coskun, J. Huizenga. *Brill-Noether theorems and globally generated vector bundles on Hirzebruch surfaces.* Nagoya Math. J. (2018). vii

[Cos17] E. Coskun. *A Survey of Ulrich bundles.* Proceedings of Discussion Meeting on Algebraic and Analytic Geometry Related to Bundles Held in Mumbai, March 2015, edited by Anilatmaja Aryasomayajula, Indranil Biswas, Archana S. Morye and A.J. Parameswaran, 85-106. vi

[dFKL07] T. de Fernex, A. Küronya, R. Lazarsfeld. *Higher cohomology of divisors on a projective variety.* Math. Ann. **337** (2007), no. 2, 443-455. i, iii, 42

[DPS96] J.-P. Demailly, T. Peternell, M. Schneider. *Holomorphic line bundles with partially vanishing cohomology.* Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, pp. 165-198. ii, 20, 21, 24

[Ein00] L. Ein. *Linear systems with removable base loci.* 13
[ELMNP06] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, M. Popa. *Asymptotic invariants of base loci.* Ann. Inst. Fourier (Grenoble) **56** (2006), no. 6, 1701-1734. i, iv, 3, 5, 6, 10
[ELMNP09] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, M. Popa. *Restricted volumes and base loci of linear series.* Amer. J. Math. **131** (2009), no. 3, 607-651. i, 12
[ES03] D. Eisenbud, F.-O. Schreyer. *Resultants and Chow forms via exterior syzigies.* With an appendix by J. Weyman. J. Amer. Math. Soc. **16** (2003), no. 3), 537-579. vi, vii
[Fae19] D. Faenzi. *Ulrich bundles on $K3$ surfaces.* Alg. Number Th. **13** (2019), 1443-1454. vii
[GrKü15] D. Greb, A. Küronya. *Partial positivity: geometry and cohomology of q-ample line bundles.* Recent advances in algebraic geometry, 207-239, London Math. Soc. Lecture Note Ser., 417, Cambridge Univ. Press, Cambridge, 2015. 21, 23, 24
[GrHa] P. Griffiths, J. Harris, *Residues and zero-cycles on algebraic varieties.* Annals of Mathematics, 108 (1978), 461-450
[Har77] R. Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. 12, 28, 32, 65, 67, 78
[HiSt71] P.J. Hilton, U. Stammbach. *A Course in Homological Algebra.* Graduate Texts in Mathematics, Vol. 4. Springer-Verlag, New York-Berlin, 1971. 28, 37
[HUB91] J. Herzog, B. Ulrich, J. Backelin. *Linear maximal Cohen-Macaulay modules over strict complete intersections.* J. Pure Appl. Algebra **71** (1991), no. 2-3, 187 - 202. vi, vii, 72
[HuLe10] D. Huybrechts, M. Lehn. *The geometry of moduli spaces of sheaves.* Cambridge Mathematical Library. Cambridge University press, Cambridge, 2010. 74
[KlMi17] J.O. Kleppe, R.M. Miró-Roig. *The Representation Type of Determinantal Varieties.* Algebr Represent Theor (2017) 20:1029–1059 70
[KnLo07] A.L. Knutsen, A.F. Lopez. *Surjectivity of Gaussian maps for curves on Enriques surfaces.* Adv. Geom. **7** (2007), no. 2, 215-247. 77
[Kol13] J. Kollár. *Singularities of the minimal model program.* Cambridge Tracts in Mathematics. Cambridge University press, Cambridge, 2013. 52
[Kür06] A. Küronya. *Asymptotic cohomological functions on projective varieties.* Amer. J. Math. **128** (2006), no. 6, 1475-1519. i, ii, iii, iv, 8, 20, 40, 41
[Kür13] A. Küronya. *Positivity on subvarieties and vanishing of higher cohomology.* Ann. Inst. Fourier (Grenoble) **63** (2013) no. 5, 1717-1737. ii, v, 20, 25, 45, 46, 48, 49
[Lau19] C.C. Lau. *On nef subvarieties.* Adv. Math. **353** (2019), 396-430. 26, 43
[Laz04a] R. Lazarsfeld. *Positivity in algebraic geometry I.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge **48**, Springer-Verlag, Berlin, 2004. 3, 4, 7, 12, 13, 17, 19, 24, 27, 28, 35, 38, 40, 70
[Laz04b] R. Lazarsfeld. *Positivity in algebraic geometry II.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge **49**, Springer-Verlag, Berlin, 2004. 16, 40
[Les14] J. Lesieutre. *The diminuished base locus is not always closed.* Composition Math. **150** (2014) 1729-1741. 10
[Lop15] A.F. Lopez. *Augmented base loci and restricted volumes on normal varieties II: the case of real divisors.* Mathematical Proceedings of the Cambridge Philosophical Society **159** (2015), no- 3, 517-527. 5
[Lop19a] A.F. Lopez. *On the existence of Ulrich vector bundles on some irregular surfaces.* Preprint arXiv.1812.10195. vii
[Lop19b] A.F. Lopez. *On the existence of Ulrich vector bundles on some surfaces with maximal albanese dimension.* Eur. J. Math. **5** (2019), no. 3, 958-963. vii
[Mat13] S. Matsumura. *Asymptotic cohomology vanishing and a converse to the Andreotti-Grauert theorem on a surface.* Ann. Inst. Fourier (Grenoble) **63** (2013) no. 6, 2199-2221. 24
[Mi08] R.M. Miró-Roig. *Determinantal ideals.* Progress in Mathematics, vol. 264, Birkhäuser, 2008. 52, 53
[MiPo13] R.M. Miró-Roig, J. Pons-Llopis. *Reprsentation type of rational ACM surfaces $X \subset \mathbb{P}^4$.* Algebr. Represent. Theor. **16** (2013), 1135–1157. 72
[Nak00] M. Nakamaye. *Stable base loci of linear series.* Math. Ann. **318** (2000), 837–847. 7
[Ott12] J.C. Ottem. *Ample subvarieties and q-ample divisors.* Adv. Math. **229** (2012), no. 5, 2868-2887. ii, 20, 21, 49
[PeSz74] C. Peskine, L. Szpiro. *Liaison des variétés algébriques I.* Invent. math. **26** (1974), 271-302. 52, 61, 69
[Sau85] T. Sauer. *Smoothing projectively Cohen-Macaulay space curves.* Math. Ann. **272** (1985), 83-90. 53
[Som78] A.J. Sommese. *Submanifolds of Abelian varieties.* Math. Ann. **233** (1978), no. 3, 229-256. ii, v, 20, 24, 45

[Tot13] B. Totaro. *Line bundles with partially vanishing cohomology.* J. Eur. Math. Soc. **15** (2013), no. 3, 731-754. ii, iii, iv, 20, 21, 22, 24, 25, 26, 43