ON THE PROOF OF THE GENUS BOUND FOR ENRIQUES-FANO THREEFOLDS

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Abstract. Given an Enriques surface S embedded in $\mathbb{P}^r$ with a certain linear system, we show that S is not hyperplane section of any threefold $X \subset \mathbb{P}^{r+1}$ that is not a cone over S. This special case completes the proof of the genus bound for Enriques-Fano threefolds [KLM, Thm.1.5].

1. Introduction

Given a smooth variety $Y \subset \mathbb{P}^r$, a very natural question is whether Y can be hyperplane section of a variety $X \subset \mathbb{P}^{r+1}$ that is not a cone over Y. When this does not happen $Y \subset \mathbb{P}^r$ is said to be *nonextendable*. While several classical works have addressed this question for special classes of varieties Y, in 1989 Zak [Z], [L, Thm.0.1] proved that if $\operatorname{codim} Y \geq 2$ and $h^0(N_{Y/\mathbb{P}^r}(-1)) \leq r+1$, then Y is nonextendable. The shift was then on how to compute the cohomology $h^0(N_{Y/\mathbb{P}^r}(-1))$. In the same year a result of Wahl [W, Prop.1.10] introduced, for this goal, when Y is a curve, the point of view of Gaussian maps: $h^0(N_{Y/\mathbb{P}^r}(-1)) = r + 1 + \operatorname{cork} \Phi_{H_Y,\omega_Y}$, where Φ_{H_Y,ω_Y} is the Gaussian map associated to the canonical and hyperplane bundle H_Y of Y. The pairing of these two results led to a number of articles about Gaussian maps and nonextendability of curves. In some notable cases this could also be extended to study nonextendability of surfaces, by passing to their curve section. We mention, for example, [CLM1, CLM2] where Y was a general hyperplane section of a general K3 surface.

On the other hand, until the introduction of [KLM, Thm.1.1], no general method was known when Y is a surface. This method, still based on Gaussian maps on suitable linear systems on Y, was applied in [KLM] to study nonextendability of pluricanonical embeddings of surfaces of general type and of Enriques surfaces. The study of the latter case led then in [KLM] to give a genus bound (also obtained simultaneously and independently by Prokhorov [P]) for the curve section, in analogy with the case of smooth Fano threefolds ([I1, I2, MM, CLM1, CLM2]), for Enriques-Fano threefolds, that is threefolds $X \subset \mathbb{P}^N$ having a hyperplane section Y that is a smooth Enriques surface, and such that X is not a cone over Y. Such threefolds were classically studied by Fano [F] and more recently by Conte and Murre [CM] and specially by Bayle [Ba, Thm.A] and Sano [Sa, Thm.1.1], but all of these works were not enough to get a genus bound.

The genus bound in [KLM, Thm.1.5] depends therefore on studying nonextendability of Enriques surfaces embedded in $\mathbb{P}^r$ with a very ample line bundle. Now in the course of the proof of [KLM, Thm.1.5], one explicit case of embedding line bundle (see [KLM, Prop.8.5]) was not treated, for reasons of space. In the present article we therefore complete the proof by showing nonextendability in this case.

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To be more precise let now $S \subset \mathbb{P}^r$ be an Enriques surface embedded by a very ample line bundle H of degree $H^2 = 2g-2$. Let $E > 0$ be such that $E^2 = 0$ and $E.H = \phi(H)$ (see Definition 2.3).

Suppose that H be of type (I), that is, after applying the decomposition procedure of [KLM, §6] (briefly recalled at the beginning of Section 3), we get an effective decomposition

$$H \equiv \beta E + \gamma E_1 + M_2 \text{ with } E^2 = E_1^2 = 0 \text{ and } E.E_1 = 1.$$

Then we have

Theorem 1.1. *Let H be of type* (I) *with $\beta \leq 4$, $\gamma = 2$ and $M_2 > 0$ and such that $H^2 \geq 32$ or $H^2 = 28$. Then S is nonextendable, except possibly for the following two cases, where $H^2 = 28$ and $E_2 := M_2, E_2^2 = 0$:*

(i) $H \sim 3E + 2E_1 + E_2$, $E.E_1 = E_1.E_2 = 1$, $E.E_2 = 2$,
(ii) $H \sim 4E + 2E_1 + E_2$, $E.E_1 = E.E_2 = E_1.E_2 = 1$.

The above theorem proves [KLM, Prop.8.5] and therefore completes the proof of the genus bound for Enriques-Fano threefolds [KLM, Thm.1.5]. As a suggestion to the reader, we remark that the present article and [KLM], at least in the part regarding nonextendability of Enriques surfaces, are written to be read together.

2. Basic facts on line bundles on Enriques surfaces

We first recall.

Definition 2.1. *Let L and M be line bundles on a smooth projective variety. Given $V \subseteq H^0(L)$ we denote by $\mu_{V,M} : V \otimes H^0(M) \longrightarrow H^0(L \otimes M)$ the multiplication map of sections, $\mu_{L,M}$ when $V = H^0(L)$, and by $\Phi_{L,M} : \operatorname{Ker} \mu_{L,M} \longrightarrow H^0(\Omega^1_X \otimes L \otimes M)$ the Gaussian map. This map can be defined locally by $\Phi_{L,M}(s \otimes t) = sdt - tds$* [W, 1.1].

We henceforth let S be an Enriques surface.

Definition 2.2. *We denote by $\sim$ (respectively $\equiv$) the linear (respectively numerical) equivalence of divisors (or line bundles) on S. A line bundle L on S is* **primitive** *if $L \equiv hL'$ for some line bundle L' and some integer h, implies $h = \pm 1$. An effective line bundle L on S is* **quasi-nef** [KL1] *if $L^2 \geq 0$ and $L.\Delta \geq -1$ for every Δ such that $\Delta > 0$ and $\Delta^2 = -2$.*

A **nodal** *curve on S is a smooth rational curve. A* **nodal cycle** *on S is a divisor $R > 0$ such that $(R')^2 \leq -2$ for any $0 < R' \leq R$. An* **isotropic divisor** *F on S is a divisor such that $F^2 = 0$ and $F \not\equiv 0$. An* **isotropic k-sequence** *is a set $\{f_1, \ldots, f_k\}$ of isotropic divisors such that $f_i.f_j = 1$ for $i \neq j$.*

We will often use the fact that if R is a nodal cycle, then $h^0(R) = 1$ and $h^0(R + K_S) = 0$.

Definition 2.3. *Let L be a line bundle on S with $L^2 > 0$. Following* [CD] *we define*

$$\phi(L) = \inf\{|F.L| \ : \ F \in \operatorname{Pic} S, F^2 = 0, F \not\equiv 0\}.$$

One has $\phi(L)^2 \leq L^2$ [CD, Cor.2.7.1] and, if L is nef, then there exists a genus one pencil $|2E|$ such that $E.L = \phi(L)$ [C, 2.11]. Moreover we will often use the fact that if L is nef, then it is base-point free if and only if $\phi(L) \geq 2$ [CD, Prop.3.1.6, 3.1.4 and Thm.4.4.1].

Definition 2.4. *A line bundle $L > 0$ with $L^2 \geq 0$ on S has a (nonunique) decomposition $L \equiv a_1E_1 + \ldots + a_nE_n$, where a_i are positive integers, and each E_i is primitive, effective and isotropic, cf. e.g.* [KL2, Lemma2.12]. *We will call such a decomposition an* **arithmetic genus 1 decomposition**. *An effective line bundle L on S with $L^2 \geq 0$ is said to be of* **small type** *if either $L = 0$ or if in every arithmetic genus 1 decomposition of L as above, all $a_i = 1$.*

Line bundles of small type have specific decompositions that are classified in [KLM, Lemma4.3]. We also record the following two useful results.

Lemma 2.5. *Let L be a nef and big line bundle on an Enriques surface and let F be a divisor satisfying $F.L < 2\phi(L)$ (respectively $F.L = \phi(L)$ and L is ample). Then $h^0(F) \leq 1$ and if $F > 0$ and $F^2 \geq 0$ we have $F^2 = 0$, $h^0(F) = 1$, $h^1(F) = 0$ and F is primitive and quasi-nef (resp. nef).*

Proof. If $h^0(F) \geq 2$ we can write $|F| = |M| + G$, with M the moving part and $G \geq 0$ the fixed part of $|F|$. By [CD, Prop.3.1.4] we get $F.L \geq 2\phi(L)$, a contradiction. Then $h^0(F) \leq 1$ and if $F > 0$ and $F^2 \geq 0$ it follows that $F^2 = 0$ and $h^1(F) = 0$ by Riemann-Roch. Hence F is quasi-nef and primitive by [KL1, Cor.2.5]. If $F.L = \phi(L)$, L is ample and F is not nef, by [KL2, Lemma2.4] we can write $F \sim F_0 + \Gamma$ with $F_0 > 0$, $F_0^2 = 0$ and Γ a nodal curve. But then $F_0.L < \phi(L)$. □

Lemma 2.6. *For $1 \leq i \leq 4$ let $F_i > 0$ be four isotropic divisors on S such that $F_1.F_2 = F_3.F_4 = 1$ and $F_1.F_3 = F_2.F_3 = 2$. If $F_4.(F_1 + F_2) = 4$ then $F_1.F_4 = F_2.F_4 = 2$.*

Proof. By symmetry and [KL1, Lemma2.1] we can assume, to get a contradiction, that $F_1.F_4 = 1$ and $F_2.F_4 = 3$. Then $(F_2 + F_4)^2 = 6$ and $\phi(F_2 + F_4) = 2$ whence, by [KL2, Lemma2.4], we can write $F_2 + F_4 \sim A_1 + A_2 + A_3$ with $A_i > 0$, $A_i^2 = 0$ and $A_i.A_j = 1$ for $i \neq j$. But this gives the contradiction $8 = (F_2 + F_4).(F_1 + F_2 + F_3) \geq 3\phi(F_1 + F_2 + F_3) = 9$. □

3. First reductions in the proof of Theorem 1.1

In this section we show how to use some results in [KLM] to reduce the proof of Theorem 1.1 to some explicit intersections cases (Lemma 3.1).

We briefly recall here the decomposition procedure of [KLM, §6].

Let $S \subset \mathbb{P}^r$ be an Enriques surface of sectional genus g and let H be its hyperplane divisor. Let $|2E|$ be a genus one pencil such that $E.H = \phi(H)$ and, as H is not of small type by [KLM, Lemma4.3], we can define, as in [KLM, §4],

$$\alpha = \min\{k \geq 2 \quad | \quad (H - kE)^2 \geq 0 \text{ and if } (H - kE)^2 > 0 \text{ there exists } F > 0 \text{ with } F^2 = 0, F.E > 0 \text{ and } F.(H - kE) \leq \phi(H)\},$$

$L_1 = H - \alpha E$ and let $E_1 > 0$ be such that $E_1^2 = 0$ and $E_1.L_1 = \phi(L_1)$. Now repeat the procedure on L_1. Then we get a decomposition

$$H = \alpha E + \alpha_1 E_1 + \alpha_2 E_2' + \ldots + \alpha_{n-1} E_{n-1}' + L_n,$$

for some $n \geq 1, \alpha \geq 2$, $\alpha_i \geq 2$ for $1 \leq i \leq n-1$ and L_n is of small type. Removing copies of E or E_1 from L_n one gets several decompositions (see [KLM, §6]).

We say that the decomposition is of type (I) if H is not 2-divisible in $\mathrm{Num}(S)$ and we are in one of the two cases

(I-A) $n = 3, E_2' \equiv E$, or
(I-B) $n = 2$.

This allows us to write

$$H \equiv \beta E + \gamma E_1 + M_2, \text{ with } E.E_1 = 1.$$

Note that, in particular, when $\beta \leq 3$, we must be in case (I-B).

We now start the proof of Theorem 1.1.

Let H be as in Theorem 1.1. Replacing M_2 with $M_2 + K_S$, that has the same properties, we can assume

$$H \sim \beta E + 2E_1 + M_2.$$

Since by construction M_2 neither contains E nor E_1 in its arithmetic genus 1 decompositions, we have $(M_2 - E)^2 < 0$ and $(M_2 - E_1)^2 < 0$. Also $E.H \leq E_1.H$ and $E_1.L_1 \leq E.L_1$, giving

$$\frac{1}{2}M_2^2 + 1 \leq \quad E.M_2 \quad \leq E_1.M_2 + \beta - 2, \quad \text{and} \tag{1}$$

$$\frac{1}{2}M_2^2 + 1 \leq \quad E_1.M_2 \quad \leq E.M_2 + 2 - \beta + \alpha \leq E.M_2 + 2. \tag{2}$$

Also, by [KLM, Lemmas6.1 and 6.2], we have

(3) $E + E_1$ is base-component free. If $\Delta > 0$ is such that $\Delta^2 = -2$ and $\Delta.E_1 < 0,$ then Δ is a nodal curve and $E_1 \sim E + \Delta + K_S$.

Now we can give a first reduction.

Lemma 3.1. *Let H be of type* (I) *with $\beta \leq 4, \gamma = 2$ and $M_2^2 \geq 2$. Then S is nonextendable unless, possibly, we are in one of the following cases (where all the E_i's are effective and isotropic):*

(a) $M_2^2 = 2$, $M_2 \sim E_2 + E_3$, $E_2.E_3 = 1$, *and either*
  (a-i) $\beta = 2$, $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1, 2, 1, 2), (1, 2, 2, 1), (1, 1, 2, 2), (2, 2, 2, 2), (1, 2, 2, 2)$*; or*
  (a-ii) $\beta = 3$, $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (2, 2, 2, 2), (2, 2, 1, 2)$*; or*
  (a-iii) $\beta = 3, 4$, $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2)$.
(b) $M_2^2 = 4$, $M_2 \sim E_2 + E_3$, $E_2.E_3 = 2$, *and either*
  (b-i) $\beta = 2$, $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1, 2, 1, 2), (1, 2, 2, 1), (1, 2, 2, 2), (1, 2, 1, 3)$*; or*
  (b-ii) $\beta = 3$, $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1, 2, 2, 1), (1, 2, 1, 3)$.
(c) $M_2^2 = 6$, $M_2 \sim E_2 + E_3 + E_4$, $E_2.E_3 = E_2.E_4 = E_3.E_4 = 1$, *and* $\beta = 2$, $(E.E_2, E.E_3, E.E_4, E_1.E_2, E_1.E_3, E_1.E_4) = (1, 1, 2, 1, 1, 2)$.

Proof. We write $M_2 \sim E_2 + \ldots + E_{k+1}$ as in [KLM, Lemma4.3] with $k = 2$ or 3. Moreover we can assume that $1 \leq E.E_2 \leq \ldots \leq E.E_{k+1}$, whence that $E.M_2 \geq kE.E_2$.

We first consider the case $\beta = 4$.

We note that $(M_2 - 2E_2)^2 = -2$ if $M_2^2 = 2$ or 6, $(M_2 - 2E_2)^2 = -4$ if $M_2^2 = 4$ and $(M_2 - 2E_2)^2 \geq -6$ if $M_2^2 = 10$. In the latter case $E.M_2 \geq 6$ by (1), whence $E.(M_2 - 2E_2) \geq 2$. Using this and setting $B := E + E_1 + E_2$ one easily verifies that $(H - 2B)^2 = 4E.(M_2 - 2E_2) + (M_2 - 2E_2)^2 \geq 0$ and $E.(H - 2B) > 0$ (whence $H - 2B \geq 0$ by Riemann-Roch), except for the cases

$$M_2^2 = 2, 4 \text{ and } E.E_2 = E.E_3. \tag{4}$$

Moreover, except for these cases, using (1) and (2), one easily verifies that $H^2 \geq 54$, except for the case $M_2^2 = 2$ and $(E.M_2, E_1.M_2) = (3, 2)$, where $H^2 = 50$. In this case $(3B - H).H = 4 < \phi(H) = 5$, so that, if $3B - H > 0$ it must be a nodal cycle. Therefore either $h^0(3B - H) = 0$ or $h^0(3B + K_S - H) = 0$, so in any case B satisfies the conditions in [KLM, Prop.5.2] or in [KLM, Prop.5.3] and S is nonextendable.

In the remaining cases (4) we can without loss of generality assume $1 \leq E_1.E_2 \leq E_1.E_3$ and we set $B := E + E_2$. Then $(H - 2B)^2 = 8 + 4E_1.(E_3 - E_2) + (E_3 - E_2)^2 \geq 4$ and $(H - 2B).E = 2$. Using (1) and (2), one gets $H^2 \geq 64$ if $M_2^2 = 4$, $H^2 \geq 74$ if $M_2^2 = 2$ and $E.E_2 = E.E_3 = 3$, and $B.H \geq 17$ if $M_2^2 = 2$ and $E.E_2 = E.E_3 = 2$. Moreover, in the latter case, we have that again $H^2 \geq 64$ unless $E_1.M_2 = 2, 3$, which gives $E_1.E_2 = 1$ and B is nef by [KLM, Lemma6.3(c)] since $E_2.H = 11 < 2\phi(H) = 12$, whence E_2 is quasi-nef by Lemma 2.5. Therefore B satisfies the conditions in [KLM, Prop.5.2] or in [KLM, Prop.5.4] and S is nonextendable unless $M_2^2 = 2$ and $E.E_2 = E.E_3 = 1$. In the latter case, by (2) we have $2 \leq E_1.M_2 \leq 4$ and if $E_1.M_2 = 4$ then $\alpha = 4$. In this last case $L_1 \sim 2E_1 + M_2$, whence $\phi(L_1) = E_1.M_2 = 4$ and we get that $4 \leq E_i.L_1 = 2E_1.E_i + 1$ for $i = 2, 3$, so that $E_1.E_2 = E_1.E_3 = 2$. Therefore we get the cases in (a-iii) with $\beta = 4$.

We next treat the cases $\beta \leq 3$. As we know, we are in case (I-B), whence L_2 is of small type and either $L_2 \sim M_2$ or $L_2 \sim E + M_2$.

Suppose first that $L_2 \sim E + M_2$.

Then $\beta = 3$, $\alpha = 2$ and, since L_2 is of small type, by (1), we can only have $(M_2^2, E.M_2) = (2,2)$, $(2,4)$ or $(4,3)$.

If $(M_2^2, E.M_2) = (2,2)$, then $E.E_2 = E.E_3 = 1$ and by (2) we have $2 \leq E_1.M_2 \leq 3$, yielding the first two cases in (a-iii).

If $(M_2^2, E.M_2) = (2,4)$, then $L_2^2 = 10$ and $\phi(L_2) = 3$. As $E.E_i + 1 = L_2.E_i \geq \phi(L_2) = 3$ for $i = 2,3$, we must have $E.E_2 = E.E_3 = 2$. Now $L_1 \sim E + 2E_1 + M_2$ and $(1 + E_1.M_2)^2 = \phi(L_1)^2 \leq L_1^2 = 14 + 4E_1.M_2$ and (1) yield $E_1.M_2 = 3$ or 4. Therefore, by Lemma 2.6 and symmetry, we get the two cases in (a-ii).

If $(M_2^2, E.M_2) = (4,3)$, then $E_1.M_2 = 3$ or 4 by (2). Since $L_2^2 = 10$ and $\phi(L_2) = E.L_2 = 3$, there is by [CD, Cor.2.5.5] an isotropic effective 10-sequence $\{f_1, \dots, f_{10}\}$ such that $E = f_1$ and $3L_2 \sim f_1 + \dots + f_{10}$.

In the case $E_1.M_2 = 3$ we get $E_1.L_2 = 4$, whence we can assume, possibly after renumbering, that $E_1.f_i = 1$ for $1 \leq i \leq 8$ and $(E_1.f_9, E_1.f_{10}) = (2,2)$ or $(1,3)$. In the latter case we have $(E_1 + f_{10})^2 = 6$ and $\phi(E_1 + f_{10}) = 2$, whence we can write $E_1 + f_{10} \sim A_1 + A_2 + A_3$ for some $A_i > 0$ such that $A_i^2 = 0$, $A_i.A_j = 1$ for $i \neq j$. But $f_i.(E_1 + f_{10}) = 2$ for all $1 \leq i \leq 9$, a contradiction. Hence $(E_1.f_9, E_1.f_{10}) = (2,2)$. One easily sees that there is an isotropic divisor $f_{19} > 0$ such that $f_{19}.f_1 = f_{19}.f_9 = 2$ and $L_2 \sim f_1 + f_9 + f_{19}$. Therefore $E_1.f_{19} = 1$. Setting $E'_2 = f_9$ and $E'_3 = f_{19}$ we get the first case in (b-ii).

If $E_1.M_2 = 4$ we get $E_1.L_2 = 5$, whence we can assume, possibly after renumbering, that $E_1.f_i = 1$ for $1 \leq i \leq 5$. As above there is an isotropic divisor $f_{12} > 0$ such that $f_{12}.f_1 = f_{12}.f_2 = 2$ and $L_2 \sim f_1 + f_2 + f_{12}$. Hence $E_1.f_{12} = 3$. Setting $E'_2 = f_2$ and $E'_3 = f_{12}$ we get the second case in (b-ii).

Finally, we have left the case with $L_2 \sim M_2$, where $\beta = \alpha$. We have $L_1 \sim 2E_1 + M_2$, whence $(E_1.M_2)^2 = \phi(L_1)^2 \leq L_1^2 = 4E_1.M_2 + M_2^2$, so that (2) and [KL2, Prop.1] give $E_1.M_2 \leq 4$. In particular $M_2^2 \leq 6$ by (2).

If $\beta = \alpha = 3$, by definition of α, we have $1 + E_1.M_2 = E_1.(L_1 + E) > \phi(H) = 2 + E.M_2$, whence $E_1.M_2 = 4$, $E.M_2 = 2$ and $M_2^2 = 2$ by (1). Then $E.E_2 = E.E_3 = 1$ and, for $i = 2, 3, E_i.L_1 = 2E_i.E_1 + 1 \geq \phi(L_1) = E_1.M_2 = 4$, whence $E_1.E_2 = E_1.E_3 = 2$, that is the third case in (a-iii).

In the remaining cases we have $\beta = \alpha = 2$.

If $M_2^2 = 2$ using again $\phi(L_1)^2 \leq L_1^2$, $E_i.L_1 \geq \phi(L_1)$, (1) and (2) together with $H^2 \geq 32$ or $H^2 = 28$, we deduce the possibilities $(E.M_2, E_1.M_2) = (3,3)$, $(2,4)$, $(3,4)$ or $(4,4)$. By symmetry one easily sees that one gets the cases in (a-i).

If $M_2^2 = 4$ we similarly get $(E.M_2, E_1.M_2) = (3,3)$, $(3,4)$ or $(4,4)$. From the first two cases, using Lemma 2.6 for the second, we obtain the cases in (b-i). If $(E.M_2, E_1.M_2) = (4,4)$, we now show that H also has a decomposition of type (III) as in [KLM, §6]. It will follow that S is nonextendable by [KLM, §10]. We have $E.H = 6$, whence $(H - 3E)^2 = 8$ and $H - 3E > 0$ by [KL2, Lemma2.4]. If $\phi(H - 3E) = 1$ we can write $H - 3E \sim 4A_1 + A_2$ with $A_i > 0$, $A_i^2 = 0$ and $A_1.A_2 = 1$. Now $6 \leq H.A_1 = 3E.A_1 + 1$ gives $E.A_1 \geq 2$, whence the contradiction $6 = H.E = 4E.A_1 + E.A_2 \geq 8$. Therefore there is an $E'_1 > 0$ such that $(E'_1)^2 = 0$ and $E'_1.(H - 3E) = 2$. Since $(H - 3E - 2E'_1)^2 = 0$, by [KL2, Lemma2.4] we can write $H \sim 3E + 2E'_1 + E'_2$, with $E'_2 > 0$, $(E'_2)^2 = 0$ and $E'_1.E'_2 = 2$. From $6 \leq H.E'_1 = 3E.E'_1 + 2$ we get $E.E'_1 \geq 2$. Now from $6 = H.E = 2E.E'_1 + E.E'_2$ we see that we cannot have $E.E'_1 \geq 3$, for then $E.E'_1 = 3$, $E.E'_2 = 0$, but this gives $E'_2 \equiv qE$ for some $q \geq 1$ by [KL1, Lemma2.1], whence the contradiction $2 = E'_1.E'_2 = 3q$. Therefore $E.E'_1 = 2, E_1.E'_2 = 1$ so that E'_2 is primitive and since $E'_1.L_1 = E'_1.(H - 3E) + E'_1.E = 4 = \phi(L_1)$ we obtain a decomposition of H of type (III), as claimed.

If $M_2^2 = 6$, by (1) and (2) we get, as above, $E_1.M_2 = E.M_2 = 4$, yielding by symmetry the case in (c) in addition to the case $(E.E_2, E.E_3, E.E_4, E_1.E_2, E_1.E_3, E_1.E_4) = (1, 1, 2, 1, 2, 1)$. In the latter case we note that $\phi(H) = E.H = E_1.H = 6$ and $\phi(H - 2E_1) = \phi(2E + E_2 + E_3 + E_4) = E_3.(H - 2E_1) = 4$. Hence we can decompose H with respect to E_1 and E_3, which means that H is also of type (III) (as in [KLM, §6]) and S is nonextendable by [KLM, §10]. □

4. Conclusion of the proof of Theorem 1.1

By Lemma 3.1 we can assume that either $M_2^2 = 0$ or we are in one of the cases of that lemma. Moreover recall that H is not 2-divisible in Num(S) and we are in case (I-A) or (I-B).

4.1. **The case $M_2^2 = 0$.** We write $M_2 = E_2$ for a primitive $E_2 > 0$ with $E_2^2 = 0$.

4.1.1. $\beta = 2$. From (1) and (2) we get $1 \leq E.E_2 \leq E_1.E_2 \leq E.E_2+2$. Moreover, since $L_1 \sim 2E_1+E_2$, we get $(\phi(L_1))^2 = (E_1.E_2)^2 \leq L_1^2 = 4E_1.E_2$, whence $E_1.E_2 \leq 3$ by [KL2, Prop.1], as E_2 is primitive. Since $H^2 \geq 28$, we are left with the cases $(E.E_2, E_1.E_2) = (2, 3)$ or $(3, 3)$, so that S is nonextendable by [KLM, Lemma5.5(iii-b)].

4.1.2. $\beta = 3$. From (1) and (2) we get $1 \leq E.E_2 \leq E_1.E_2 + 1 \leq E.E_2 + \alpha$.

If $\alpha = 2$ we get $E.E_2 - 1 \leq E_1.E_2 \leq E.E_2 + 1$. Moreover, since we are in case (I-B), $L_2 \sim E + E_2$ is of small type, whence $E.E_2 \leq 3$ or $E.E_2 = 5$. Furthermore, since $L_1 \sim E + 2E_1 + E_2$, we get $(\phi(L_1))^2 = (1 + E_1.E_2)^2 \leq L_1^2 = 4 + 4E_1.E_2 + 2E.E_2$. However, in the case $(E.E_2, E_1.E_2) = (3, 4)$, we find $(L_1^2, \phi(L_1)) = (26, 5)$, which is impossible by [KL2, Prop.1]. This yields that $E.E_2 = 2, 3, 5$ if $E_1.E_2 = E.E_2 - 1$; $E.E_2 = 1, 2, 3$ if $E_1.E_2 = E.E_2$; and $E.E_2 = 1, 2$ if $E_1.E_2 = E.E_2 + 1$.

If $\alpha = 3$ we must have, by [KLM, (11)], that $E_1.(H - 3E) = \phi(H)$, whence $E_1.E_2 = 2 + E.E_2$. Moreover, since $L_1 \sim 2E_1 + E_2$, we get $(\phi(L_1))^2 = (E_1.E_2)^2 \leq L_1^2 = 4E_1.E_2$, whence $E_1.E_2 \leq 3$ by [KL2, Prop.1] since E_2 is primitive. Hence $E_1.E_2 = 3$ and $E.E_2 = 1$.

To summarize, using $H^2 \geq 32$ or $H^2 = 28$, we have the following cases:

$$\begin{array}{lll} E_1.E_2 = E.E_2 - 1, & E.E_2 = 2,\ 3 \text{ or } 5, & g = 15,\ 20 \text{ or } 30. \\ E_1.E_2 = E.E_2, & E.E_2 = 2 \text{ or } 3, & g = 17 \text{ or } 22. \\ E_1.E_2 = 3, & E.E_2 = 2, & g = 19. \end{array} \tag{5}$$

We will now show, in Lemmas 4.1-4.4, that S is nonextendable in the five cases of genus $g \geq 17$. The case with $g = 15$ is case (i) in Theorem 1.1.

Lemma 4.1. *In the case $(E.E_2, E_1.E_2, g) = (5, 4, 30)$ in (5), S is nonextendable.*

Proof. We have $H^2 = 58$ and $\phi(H) = E.H = E_1.H = 7$. Hence both E and E_1 are nef by Lemma 2.5. Let now $H' = H - 4E$. Then $(H')^2 = 2$ and consequently we can write $H \sim 4E + A_1 + A_2$ for $A_i > 0$ primitive with $A_i^2 = 0$ and $A_1.A_2 = 1$. Since $E.H = E.A_1 + E.A_2 = 7$ we can assume by symmetry that either (a) $(E.A_1, E.A_2) = (2, 5)$ or (b) $(E.A_1, E.A_2) = (3, 4)$. Also since $E_1.H = 7$ we have $E_1.(A_1 + A_2) = 3$, whence we have the two possibilities $(E_1.A_1, E_1.A_2) = (2, 1)$ or $(1, 2)$.

In case (b) we get $A_1.H = 13$, whence $(H - 2(E + A_1))^2 = 2$. Since $(H - 2(E + A_1)).E = 1$, we have $H - 2(E + A_1) > 0$ by Riemann-Roch, whence S is nonextendable by [KLM, Prop.5.2].

In case (a) we get $A_1.H = 9$. Now if $E_1.A_1 = 2$, we get $(H - 2(E + A_1 + E_1))^2 = 6$, and as above S is nonextendable by [KLM, Prop.5.2]. If $E_1.A_1 = 1$, then $E_1.(H - 2E) = A_1.(H - 2E) = 5$, whence $L_1 \sim H - 2E$ and $\phi(L_1) = A_1.L_1 = 5$. Therefore we can continue the decomposition with respect to A_1 instead of E_1. Since H now is of type (III) (as in [KLM, §6]), S is nonextendable by [KLM, §10] □

Claim 4.2. *Let $H \sim 3E + 2E_1 + E_2$ be as in (5) with $(E.E_2, E_1.E_2, g) = (3, 2, 20)$ (respectively $(E.E_2, E_1.E_2, g) = (3, 3, 22)$). Then there exists an isotropic effective 5-sequence $\{E, F_1, F_2, F_3, F_4\}$ (respectively an isotropic effective 4-sequence $\{E, F_1, F_2, F_3\}$ together with an isotropic divisor $F_4 > 0$ such that $E.F_4 = F_2.F_4 = F_3.F_4 = 1$ and $F_1.F_4 = 2$) such that $H \sim 2E + 2F_1 + F_2 + F_3 + F_4$ and:*

(a) *F_1 is nef and F_i is quasi-nef for $i = 2, 3, 4$;*
(b) *$|E + F_2|$ and $|F_1 + F_3|$ are without base components;*
(c) *$|E + F_1 + F_2 + F_3|$ and $|E + F_1 + F_4|$ are base-point free;*
(d) *$h^1(F_1 + F_4 - F_2) = h^2(F_1 + F_4 - F_2) = 0$.*

Proof. Since $(E + E_2)^2 = 6$ and both E and E_2 are primitive, we can write $E + E_2 \sim A_1 + A_2 + A_3$ with $A_i > 0$, $A_i^2 = 0$ and $A_i.A_j = 1$ for $i \neq j$. We easily find (possibly after renumbering) that $A_i.E = A_i.E_2 = A_1.E_1 = A_2.E_1 = 1$ for $i = 1, 2, 3$ and $A_3.E_1 = 1$ if $g = 20$ and 2 if $g = 22$. Moreover $A_i.H \leq 8 < 2\phi(H) = 10$, whence all the A_i's are quasi-nef by Lemma 2.5.

Assume now there is a nodal curve R_i with $R_i.A_i = -1$ for $(i, g) \neq (3, 22)$. Then we can as usual write $A_i \sim B_i + R_i$, with $B_i > 0$ primitive and isotropic. Since $A_i.H = 6$ we deduce that $B_i \equiv E$ or $B_i \equiv E_1$, where the latter case only occurs if $g = 20$.

If $g = 20$, then, since for $i \neq j$, we have $(E + R_i).(E + R_j) = 2 + R_i.R_j = (E_1 + R_i).(E_1 + R_j)$, we see that at most two of the A_i's can be not nef, otherwise we would get $R_i.R_j = -1$, a contradiction. Possibly after reordering the A_i's and adding K_S to two of them, we can therefore assume that A_1 is nef, and that either A_2 is nef or $A_2 \sim E + R + K_S$ for R a nodal curve with $E.R = 1$. Now E_1 is nef, by Lemma 2.5, as $E_1.H = \phi(H) = 5$, so that both $|E_1 + A_1|$ and $|E + A_2|$ are without fixed components. Setting $F_1 = E_1$, $F_2 = A_2$, $F_3 = A_1$ and $F_4 = A_3$ we therefore have the desired decomposition satisfying (a) and (b). It also follows by construction that $E + F_1 + F_2 + F_3$ and $E + F_1 + F_4$ are nef, the latter because E and F_1 are, and F_4 is either nef or $F_4 \equiv A + R'$ with $A = E$ or $A = E_1$, for R' a nodal curve with $A.R' = 1$. Therefore (c) also follows.

If $g = 22$, we similarly find that we can assume that A_1 and A_2 are nef. Moreover $A_1.L_1 = A_1.(H - 2E) = E_1.(H - 2E) = 4$, so if E_1 is not nef, we can substitute E_1 with A_1 and repeat the process. Therefore we can assume that E_1 is nef as well. Again both $|E_1 + A_1|$ and $|E + A_2|$ are without fixed components, and setting $F_1 = E_1$, $F_2 = A_2$, $F_3 = A_1$ and $F_4 = A_3$ we therefore have the desired decomposition satisfying (a) and (b). Now $E + F_1 + F_2 + F_3$ is again nef by construction. To see that $E + F_1 + F_4$ is nef, assume, to get a contradiction, that there is a nodal curve Γ with $\Gamma.(E + F_1 + F_4) < 0$. Then $\Gamma.F_4 = -1$ and $\Gamma.(E + F_1) = 0$ by (a). The ampleness of H implies $\Gamma.(F_2 + F_3) \geq 2$, whence the contradiction $(F_4 - \Gamma)^2 = 0$ and $(F_4 - \Gamma).(F_2 + F_3) \leq 0$, recalling that $F_4 - \Gamma > 0$ by [KL2, Lemma2.3]. Therefore (c) is proved. We now prove (d).

If $g = 20$ then $(F_1 + F_4 - F_2)^2 = -2$ and $(F_1 + F_4 - F_2).H = 5 = \phi(H)$, whence $h^2(F_1 + F_4 - F_2) = 0$ and if $F_1 + F_4 - F_2 > 0$ it is a nodal cycle, so that either $h^0(F_1 + F_4 - F_2) = 0$ or $h^0(F_1 + F_4 - F_2 + K_S) = 0$. Replacing F_1 with $F_1 + K_S$ if necessary, we can arrange that $h^0(F_1 + F_4 - F_2) = 0$, whence also $h^1(F_1 + F_4 - F_2) = 0$ by Riemann-Roch.

If $g = 22$, then $(F_1 + F_4 - F_2)^2 = 0$ and $(F_1 + F_4 - F_2).H = 8 < 2\phi(H)$, whence (d) follows by Lemma 2.5 and [KL1, Cor.2.5]. □

Lemma 4.3. *In the cases $(E.E_2, E_1.E_2, g) = (3, 2, 20)$ or $(3, 3, 22)$ in (5), S is nonextendable.*

Proof. By Claim 4.2 we can choose $D_0 = E + F_1 + F_2 + F_3$ with $D_0^2 = 12$ and both D_0 and $H - D_0 \sim E + F_1 + F_4$ base-point free. We have $h^0(2D_0 - H) = h^0(F_2 + F_3 - F_4) \leq 1$ by Lemma 2.5, as $(F_2 + F_3 - F_4).H \leq 6 < 2\phi(H)$. Hence the map Φ_{H_D, ω_D} is surjective by [KL3, Thm.(iii)-(iv)]. To show the surjectivity of μ_{V_D, ω_D} we use Claim 4.2(b) and let $D_1 \in |E + F_2|$ and $D_2 \in |F_1 + F_3|$ be general smooth curves and apply [KLM, Lemma5.6]. Now $H - D_0 - D_1 \sim F_1 + F_4 - F_2$ whence $h^1(H - D_0 - D_1) = 0$ by Claim 4.2(d), so that $\mu_{V_{D_1}, \omega_{D_1}}$ is surjective by [KLM, (14)] since $(H - D_0).D_1 = (E + F_1 + F_4).(E + F_2) = 5$. Since $(H - D_0 - D_2).H = (E + F_4 - F_3).H \leq 7 < 2\phi(H)$

we have that $h^0(H - D_0 - D_2) \leq 1$ by Lemma 2.5 and $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by [KLM, (16)]. Therefore μ_{V_D,ω_D} is surjective whence S is nonextendable by [KLM, Prop.5.1]. □

Lemma 4.4. *If $E.E_2 = 2$ and $(E_1.E_2, g) = (2, 17)$ or $(3, 19)$ in (5), then S is nonextendable.*

Proof. We first observe that it is enough to find an isotropic divisor $F > 0$ such that $E.F = 1$, $F.H = 6$ if $g = 17$ and $F.H = 7$ if $g = 19$ and $B := E + F$ is nef. In fact the latter implies that $H \sim 2B + A$, with $A > 0$ isotropic with $E.A = 2$ and $F.A = 4$ if $g = 17$ and $F.A = 5$ if $g = 19$. As H is not 2-divisible in $\operatorname{Num} S$, A is automatically primitive and it follows that S is nonextendable by [KLM, Lemma5.5(iii-b)].

To find the desired F we first consider the case $g = 17$.

Set $Q = E + E_1 + E_2$. Then $Q^2 = 10$ and $\phi(Q) = 3$. By [CD, Cor.2.5.5] there is an isotropic effective 10-sequence $\{f_1, \ldots, f_{10}\}$ with $3Q \sim f_1 + \ldots + f_{10}$. Since $E.Q = E_1.Q = 3$, we can assume that $f_1 = E$ and $f_2 = E_1$ and then $E_2.f_i = 1$ for $i \geq 3$. We now claim that $E + f_i$ is not nef for at most one $i \in \{3, \ldots, 10\}$. Indeed, note that, for $i \geq 3$, we have $f_i.H = 6 < 2\phi(H) = 8$, whence each f_i is quasi-nef by Lemma 2.5. Now assume that $R_i.(E + f_i) < 0$ for some nodal curve R_i. Then $R_i.E = 0$ and $R_i.f_i = -1$, so that $f_i \sim \overline{f_i} + R_i$, by [KL2, Lemma2.3], with $\overline{f_i} > 0$ primitive and $\overline{f_i}^2 = 0$. Since H is ample we must have $R_i.E_j > 0$ for $j = 1$ or 2. If $R_i.E_2 > 0$ then $E_2.f_i = 1$ implies $\overline{f_i} \equiv E_2$ and $R_i.E_2 = 1$. But then we get the contradiction $E.f_i = E.(E_2 + R_i) = 2$. Therefore $R_i.E_1 > 0$, so that $\overline{f_i} \equiv E_1$ and $R_i.E_1 = 1$. Now suppose that also $E + f_j$ is not nef for $j \in \{3, \ldots, 10\} - \{i\}$. Then $R_i.R_j = (f_i - E_1).(f_j - E_1) = -1$, a contradiction. Therefore $E + f_i$ is not nef for at most one $i \in \{3, \ldots, 10\}$. Now one easily verifies that any $F \in \{f_3, \ldots, f_{10}\}$ such that $E + F$ is nef satisfies the desired numerical conditions.

We next consider the case $g = 19$.

Since $(E_1 + E_2)^2 = 6$ and $\phi(E_1 + E_2) = 2$ we can find an isotropic effective 3-sequence $\{f_3, f_4, f_5\}$ such that $E_1 + E_2 \sim f_3 + f_4 + f_5$. Since $E.(E_1 + E_2) = E_1.(E_1 + E_2) = 3$ we have $f_i.E = f_i.E_1 = 1$ for $i = 3, 4, 5$, so that we have an isotropic effective 5-sequence $\{f_1, \ldots, f_5\}$ with $f_1 = E$ and $f_2 = E_1$ such that $H \sim 3f_1 + f_2 + f_3 + f_4 + f_5$. By [CD, Cor.2.5.6] we can complete the sequence to an isotropic effective 10-sequence $\{f_1, \ldots, f_{10}\}$. Note that for $i \geq 6$ we have $f_i.H = 7 < 2\phi(H) = 8$, whence each f_i is quasi-nef by Lemma 2.5. Now the same arguments as above can be used to prove that $E + f_i$ is nef for at least one $i \in \{6, \ldots, 10\}$, whence any $F \in \{f_6, \ldots, f_{10}\}$ such that $E + F$ is nef satisfies the desired numerical conditions. □

4.1.3. $\beta = 4$. From (1) and (2) we get $1 \leq E.E_2 \leq E_1.E_2 + 2 \leq E.E_2 + \alpha$.

If $\alpha = 2$ we get $E.E_2 - 2 \leq E_1.E_2 \leq E.E_2$. Moreover, since $L_2 \sim 2E + E_2$ is not of small type, we get $(\phi(L_2))^2 = (E.E_2)^2 \leq L_2^2 = 4E.E_2$, whence $E.E_2 \leq 3$ by [KL2, Prop.1]. Therefore $(E.E_2, E_1.E_2) \in \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$. The first case is case (ii) in Theorem 1.1 and in the other cases S is nonextendable by (3) and [KLM, Lemma5.5(iii-a)].

If $\alpha = 3$ or 4 we must have $E_1.(H - \alpha E) = \phi(H)$ by [KLM, (11)], whence $E_1.E_2 = E.E_2 + \alpha - 2$. Moreover $L_1 \sim (4 - \alpha)E + 2E_1 + E_2$ and using $(\phi(L_1))^2 \leq L_1^2$, we get $E_1.E_2 \leq 4$. If equality holds then $(L_1^2, \phi(L_1)) = (26, 5)$ or $(16, 4)$, both excluded by [KL2, Prop.1], as E_2 is primitive. Therefore $(E.E_2, E_1.E_2) = (1, 2), (1, 3)$ or $(2, 3)$ and S is nonextendable by (3) and [KLM, Lemma5.5(iii-a)].

4.2. **The case $M_2^2 = 2$.** We write $M_2 = E_2 + E_3$ as in Lemma 3.1(a).

4.2.1. $\beta = 2$. By Lemma 3.1 we have left to treat the cases (a-i), that is

$$(6) \qquad (E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1, 2, 1, 2),\ (1, 2, 2, 1),\ (1, 1, 2, 2),\ (2, 2, 2, 2),\ (1, 2, 2, 2).$$

We first show that S is nonextendable in the first case of (6).

Since $E_2.H = \phi(H) = 5$ and $E_3.H = 9 < 2\phi(H)$ we have that E_2 is nef and E_3 is quasi-nef by Lemma 2.5. In particular we get that $h^1(E_2 + E_3) = h^1(E_2 + E_3 + K_S) = 0$ by [KL1, Cor.2.5] and

$h^0(E_2+E_3)=2$ by Riemann-Roch. Now $D_0:=E+E_1+E_2+E_3$ is nef by [KLM, Lemma6.3(b)] with $\phi(D_0)=3$ and $D_0^2=16$. Also $H-D_0\sim E+E_1$ is base-component free by (3) and $2D_0-H\sim E_2+E_3$. Then $h^0(2D_0-H)=2$ and $h^1(H-2D_0)=0$, so that μ_{V_D,ω_D} is surjective by [KLM, (13)] and Φ_{H_D,ω_D} is surjective by [KL3, Thm.(v)], as $\text{gon}(D)=6$ by [KL2, Cor.1], whence $\text{Cliff}(D)=4$, as D has genus 9 [ELMS, §5]. By [KLM, Prop.5.1], S is nonextendable.

We next show that S is nonextendable in the last four cases in (6).

By Lemma 2.5 and [KLM, Lemma6.3(b)] we see that E_2 and E_3 are quasi-nef and $E+E_1+E_2$ and $E+E_1+E_3$ are base-point free. Set $D_0=E+E_1+E_2$. Then $D_0^2\geq 8$, D_0 is nef, $\phi(D_0)\geq 2$ and $H-D_0\sim E+E_1+E_3$ is base-point free. Moreover $h^0(2D_0-H)=0$ as $(2D_0-H).H=(E_2-E_3).H\leq 0$, so that Φ_{H_D,ω_D} is surjective by [KL3, Thm.(iii)]. Now, in all cases except for $(E.E_2,E.E_3,E_1.E_2,E_1.E_3)=(1,2,2,2)$, we have $(H-2D_0)^2=-2$ and $(H-2D_0).H=0$, so that $h^0(H-2D_0)=h^2(H-2D_0)=0$, whence $h^1(H-2D_0)=0$ by Riemann-Roch and μ_{V_D,ω_D} is surjective by [KLM, (12)] (noting that $(H-D_0)^2=10$ in the case $(2,2,2,2)$, while $H-D_0$ is not 2-divisible in $\operatorname{Pic} S$ as either $E.(H-D_0)=3$ or $E_1.(H-D_0)=3$ in the other two cases). By [KLM, Prop.5.1], S is nonextendable in those cases.

We now prove the surjectivity of μ_{V_D,ω_D} in the case $(E.E_2,E.E_3,E_1.E_2,E_1.E_3)=(1,2,2,2)$.

Note that E_1+E_2 is nef by [KLM, Lemma6.3(e)], whence base-point free, and that E_1+E_3 is quasi-nef. To see the latter, let $\Delta>0$ be such that $\Delta^2=-2$ and $\Delta.E_1+\Delta.E_3\leq -2$. As E_1 is quasi-nef by (3) and E_3 is quasi-nef we get, again by (3), that $\Delta.E_1=\Delta.E_3=-1$ and $E_1\equiv E+\Delta$, giving the contradiction $\Delta.E_3=0$. Hence E_1+E_3 is quasi-nef. To show the surjectivity of μ_{V_D,ω_D} we let $D_1=E$ and $D_2\in|E_1+E_2|$ be a general smooth curve and apply [KLM, Lemma5.6]. The map $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by [KLM, (15)] since $h^1(H-D_0-D_1)=h^1(E_1+E_3)=0$ by [KL1, Cor.2.5]. Finally, $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by [KLM, (16)], using the fact that $h^0(H-D_0-D_2)=h^0(E+E_3-E_2)\leq 1$ by Lemma 2.5, as $(E+E_3-E_2).H=7<2\phi(H)$. Therefore μ_{V_D,ω_D} is surjective and S is nonextendable by [KLM, Prop.5.1].

4.2.2. $\beta=3,4$. By Lemma 3.1 we have left to treat the cases (a-ii) and (a-iii), that is

(7) $\beta=3,\quad (E.E_2,E.E_3,E_1.E_2,E_1.E_3,E_2.E_3)=(2,2,2,2,1),$

(8) $\beta=3,\quad (E.E_2,E.E_3,E_1.E_2,E_1.E_3,E_2.E_3)=(2,2,1,2,1),$

(9) $\beta=3,4,\quad (E.E_2,E.E_3,E_1.E_2,E_1.E_3,E_2.E_3)=(1,1,2,2,1),$

(10) $\beta=3,4,\quad (E.E_2,E.E_3,E_1.E_2,E_1.E_3,E_2.E_3)=(1,1,1,2,1),$

(11) $\beta=3,4,\quad (E.E_2,E.E_3,E_1.E_2,E_1.E_3,E_2.E_3)=(1,1,1,1,1).$

Claim 4.5. *In the cases* (7)-(11) *both* E_2 *and* E_3 *are quasi-nef.*

Proof. We first prove that E_2 is quasi-nef. Assume, to get a contradiction, that there exists a $\Delta>0$ with $\Delta^2=-2$ and $\Delta.E_2\leq -2$. By [KL2, Lemma2.3] we can write $E_2\sim A+k\Delta$, for $A>0$ primitive with $A^2=0$ and $k=-\Delta.E_2=\Delta.A\geq 2$. From $E_2.E_3=1$ it follows that $\Delta.E_3\leq 0$. If $\Delta.E>0$, we get from $2\geq E.E_2=E.A+kE.\Delta$ that $E.E_2=k=2$, $E.\Delta=1$ and $E.A=0$, whence the contradiction $E\equiv A$. Hence $\Delta.E=0$ and the ampleness of H gives $\Delta.E_1\geq 2$ and the contradiction $E_1.E_2=E_1.A+kE_1.\Delta\geq 4$. Hence E_2 is quasi-nef. The same reasoning works for E_3. □

Lemma 4.6. *S is nonextendable in cases* (7)-(9) *and cases* (10)-(11) *with* $\beta=4$.

Proof. Define $D_0=2E+E_1+E_2$, which is nef by [KLM, Lemma6.3(a)] with $\phi(D_0)\geq 2$ and $D_0^2\geq 12$ in cases (7)-(9) and $D_0^2=10$ in cases (10) and (11). Also $H-D_0\sim(\beta-2)E+E_1+E_3$, whence $\phi(H-D_0)\geq 2$ and $H-D_0$ is base-point free by [KLM, Lemma6.3(b)]. We have $2D_0-H\sim(4-\beta)E+E_2-E_3$, whence $h^0(2D_0-H)\leq 1$ in the cases (7)-(9), as $(2D_0-H).H\leq\phi(H)$, and $h^0(2D_0-H)=0$ in cases (10)-(11), as $(2D_0-H).H\leq 0$. It follows from [KL3, Thm.(iii)-(iv)] that the map Φ_{H_D,ω_D} is surjective.

We next note that μ_{V_D,ω_D} is surjective by [KLM, (12)] if $h^1(H-2D_0)=h^1(E_3-(4-\beta)E-E_2)=0$.

Since $(E_3-E_2).H=0$ in cases (9) and (11) we have $h^0(E_3-E_2)=h^2(E_3-E_2)=0$, whence $h^1(E_3-E_2)=0$ by Riemann-Roch. It follows that μ_{V_D,ω_D} is surjective, whence S is nonextendable by [KLM, Prop.5.1] in cases (9) and (11) with $\beta=4$. In the remaining cases we can assume that

$$h^1(E_3-(4-\beta)E-E_2)>0. \tag{12}$$

We next show that μ_{V_D,ω_D} is surjective in case (8). For this we use (3), [KLM, Lemmas5.6 and 6.3(c)] and let $D_1\in|E+E_1|$ and $D_2\in|E+E_2|$ be general smooth members.

By Claim 4.5 and [KL1, Cor.2.5] we have that $h^1(H-D_0-D_1)=h^1(E_3)=0$, whence $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by [KLM, (14)]. Furthermore $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by [KLM, (16)], where one uses that $h^0(H-D_0-D_2)=h^0(E_1+E_3-E_2)\leq 1$ by Lemma 2.5 since $(E_1+E_3-E_2).H<2\phi(H)$. Hence μ_{V_D,ω_D} is surjective and S is nonextendable by [KLM, Prop.5.1].

Finally we treat the cases (7), (9) (with $\beta=3$) and (10) (with $\beta=4$). Since $(E_3-(4-\beta)E-E_2)^2=-2$ and $(E_3-(4-\beta)E-E_2).H=-\phi(H)$ in (7) and (9) (respectively 2 in (10)), we see that Riemann-Roch and (12) imply that $E+E_2-E_3+K_S$ is a nodal cycle in (7) and (9) and E_3-E_2 is a nodal cycle in (10). With β as above, it follows that

$$h^i(E+E_2-E_3)=0 \text{ in (7) and (9) and } h^i(E_3-E_2+K_S)=0 \text{ in (10)}, \ i=0,1,2. \tag{13}$$

We now choose a new $D_0:=(\beta-2)E+E_1+E_3$, which is nef with $\phi(D_0)\geq 2$ and with $H-D_0$ base-point free by [KLM, Lemma6.3(a)-(b)]. Then $D_0^2\geq 8$ with $h^0(2D_0-H)=h^0(E_3-E-E_2)=0$ in (7) and (9) and $D_0^2=12$ with $h^0(2D_0-H)=h^0(E_3-E_2)=1$ in (10), whence Φ_{H_D,ω_D} is surjective by [KL3, Thm.(iii)-(iv)]. Now (13) implies $h^1(H-2D_0)=0$, so that μ_{V_D,ω_D} is surjective by [KLM, (12)] and S is nonextendable by [KLM, Prop.5.1]. □

We have left the cases (10) and (11) with $\beta=3$, which we treat in Lemmas 4.7 and 4.9.

Lemma 4.7. *S is nonextendable in case* (10) *with* $\beta=3$.

Proof. Since $E_2.H=6$ one easily finds another decomposition of the same type

$$H\sim 3E+2E_2+E_1+E_3', \text{ with } E_2.E_3'=2, \tag{14}$$

and all other intersections equal to one.

We first claim that either E_1 or E_2 is nef. In fact $\phi(L_1)=E_1.L_1=E_1.(E+2E_1+E_2+E_3)=4=E_2.L_1$. By (3), if neither E_1 nor E_2 are nef, there are two nodal curves R_1 and R_2 such that $R_i.E=1$ and $E_i\equiv E+R_i$, for $i=1,2$. But then we get the absurdity $R_1.R_2=(E_1-E).(E_2-E)=-1$.

By (14) we can and will from now on assume that we have a decomposition $H\sim 3E+2E_1+E_2+E_3$ with E_2 nef.

Claim 4.8. *Either $h^0(E+E_3-E_2+K_S)=0$, or $h^0(E+E_2-E_3)=0$, or $h^0(E_2+E_3-E+K_S)=0$.*

Proof. Let $\Delta_1:=E+E_3-E_2+K_S$, $\Delta_2:=E+E_2-E_3$ and $\Delta_3:=E_2+E_3-E+K_S$. Assume, to get a contradiction, that $\Delta_i\geq 0$ for all $i=1,2,3$. Since $\Delta_i^2=-2$ we get that $\Delta_i>0$ for all $i=1,2,3$. We have $\Delta_2\sim 2E+K_S-\Delta_1$. Since $\Delta_1.H=6$ and $E.H=4$, we can neither have $\Delta_1\leq E$ nor $\Delta_1\leq E+K_S$. Therefore, as E and $E+K_S$ have no common components, we must have $\Delta_1=\Delta_{11}+\Delta_{12}$ with $0<\Delta_{11}\leq E$ and $0<\Delta_{12}\leq E+K_S$ and $\Delta_{11}.\Delta_{12}=0$. Moreover we have $E.\Delta_{11}=E.\Delta_{12}=0$, whence $\Delta_{1i}^2\leq 0$ for $i=1,2$. From $-2=\Delta_1^2=\Delta_{11}^2+\Delta_{12}^2$ we must have $\Delta_{1i}^2=0$ either for $i=1$ or for $i=2$. By symmetry we can assume that $\Delta_{11}^2=0$. Therefore $\Delta_{11}\equiv qE$ for some $q\geq 1$ by [KL1, Lemma2.1], but $\Delta_{11}\leq E$, whence $\Delta_{11}=E$ and $\Delta_{12}^2=-2$. Moreover $\Delta_{12}.H=2$. Now since $E+\Delta_{12}\equiv\Delta_1\equiv E+E_3-E_2$, we get $E_3\equiv E_2+\Delta_{12}$ and $E_2.\Delta_{12}=1$. Hence $\Delta_3\sim E_2+E_3-E+K_S\sim(E+E_3+K_S-\Delta_1)+E_3-E+K_S\sim 2E_3-\Delta_1\sim 2(E_2+\Delta_{12})-\Delta_1\sim 2E_2+\Delta_{12}-\Delta_{11}$, therefore

$$\Delta_{11}+\Delta_3\in|2E_2+\Delta_{12}|. \tag{15}$$

We claim that $|2E_2+\Delta_{12}| = |2E_2| + \Delta_{12}$. To see the latter observe that it certainly holds if Δ_{12} is irreducible, for then it is a nodal curve with $E_2.\Delta_{12} = 1$ (recall that $|2E_2|$ is a genus one pencil). On the other hand if Δ_{12} is reducible then, using $\Delta_{12}.H = 2$ and the ampleness of H we deduce that $\Delta_{12} = R_1 + R_2$ where R_1, R_2 are two nodal curves with $R_1.R_2 = 1$. Moreover the nefness of E_2 allows us to assume that $E_2.R_1 = 1$ and $E_2.R_2 = 0$. But then $R_2.(2E_2 + \Delta_{12}) = -1$ so that R_2 is a base-component of $|2E_2+\Delta_{12}|$ and of course R_1 is a base-component of $|2E_2+\Delta_{12}-R_2| = |2E_2+R_1|$ and the claim is proved.

Since Δ_{11} and Δ_{12} have no common components we deduce from (15) that each irreducible component of $E = \Delta_{11}$ must lie in some element of $|2E_2|$. The latter cannot hold if E is irreducible for then we would have that $2E_2 - E > 0$ and $(2E_2 - E).E_2 = -1$ would contradict the nefness of E_2. Therefore, as is well-known, we have that $E = R_1 + \ldots + R_n$ is a cycle of nodal curves and we can assume, without loss of generality, that $E_2.R_1 = 1$ and $E_2.R_i = 0$ for $2 \leq i \leq n$. As we said above, we have $2E_2 - R_1 > 0$. Now for $2 \leq i \leq n-1$ we get $R_i.(2E_2 - R_1 - \ldots - R_{i-1}) = -1$, whence $2E_2 - R_1 - \ldots - R_i > 0$. Therefore $2E_2 - R_1 - \ldots - R_{n-1} > 0$ and since $R_n.(2E_2 - R_1 - \ldots - R_{n-1}) = -2$ we deduce that $2E_2 - E > 0$, again a contradiction. □

Conclusion of the proof of Lemma 4.7. We divide the proof into the three cases of Claim 4.8.

Case A: $h^0(E + E_3 - E_2 + K_S) = 0$**.** Set $D_0 = 2E + E_1 + E_3$. Then $D_0^2 = 12$ and $\phi(D_0) = 2$. Moreover D_0 is nef by Claim 4.5 and [KLM, Lemma6.3(a)] and $H - D_0 \sim E + E_1 + E_2$ is nef since $E + E_1$ and E_2 are (the first by (3)), so that $|H - D_0|$ is base-point free, since $\phi(H - D_0) = E.(H - D_0) = 2$. We have $2D_0 - H \sim E + E_3 - E_2$ and since $(2D_0 - H).H = 6 < 2\phi(H) = 8$, we have $h^0(2D_0 - H) \leq 1$ by Lemma 2.5, so that Φ_{H_D,ω_D} is surjective by [KL3, Thm.(iii)-(iv)]. Clearly $h^0(H - 2D_0) = 0$ and we also have $h^2(H - 2D_0) = h^0(2D_0 - H + K_S) = h^0(E + E_3 - E_2 + K_S) = 0$ by assumption. Therefore $h^1(H - 2D_0) = 0$ by Riemann-Roch and μ_{V_D,ω_D} is surjective by [KLM, (12)]. Hence S is nonextendable by [KLM, Prop.5.1].

Case B: $h^0(E+E_2-E_3) = 0$**.** We set $D_0 = E+E_1+E_3$, so that $D_0^2 = 8$, $\phi(D_0) = 2$ and both D_0 and $H - D_0 \sim 2E + E_1 + E_2$ are nef by Claim 4.5 and [KLM, Lemma6.3(a)-(b)], whence base-point free. Since $2D_0 - H \sim E_3 - E - E_2$ and $(E_3 - E - E_2).H < 0$ we have $h^0(2D_0 - H) = 0$, whence Φ_{H_D,ω_D} is surjective by [KL3, Thm.(iii)]. Now by hypothesis $h^0(H - 2D_0) = 0$ and we also have $h^0(2D_0 - H + K_S) = h^0(E_3 - E - E_2 + K_S) = 0$, and by Riemann-Roch we get $h^1(H - 2D_0) = 0$ as well. Therefore μ_{V_D,ω_D} is surjective by [KLM, (12)]. Hence S is nonextendable by [KLM, Prop.5.1].

Case C: $h^0(E_2+E_3-E+K_S) = 0$**.** Set $D_0 = E+E_1+E_2+E_3$, which is nef (since $E+E_1+E_3$ is nef by Claim 4.5 and [KLM, Lemma6.3(b)] and E_2 is nef by assumption) with $D_0^2 = 14$ and $\phi(D_0) = 3$. Moreover $H - D_0 \sim 2E + E_1$ is without fixed components. We have $H - 2D_0 \sim E - E_2 - E_3$ and since $(H - 2D_0).E = -2$ we have $h^0(E - E_2 - E_3) = 0$. By hypothesis we have $h^2(E - E_2 - E_3) = 0$, whence $h^1(H - 2D_0) = 0$ by Riemann-Roch. It follows that μ_{V_D,ω_D} is surjective by [KLM, (12)]. Furthermore, since $2D_0 - H \sim E_2 + E_3 - E$ and $h^0(E_2 + E_3 - E + K_S) = 0$ we have $h^0(2D_0 - H) \leq 1$, and Φ_{H_D,ω_D} is surjective by [KL3, Thm.(iii)-(iv)]. Hence S is nonextendable by [KLM, Prop.5.1]. □

Lemma 4.9. *S is nonextendable in case* (11) *with* $\beta = 3$.

Proof. By Claim 4.5, [KLM, Lemma6.3(d)] and symmetry, and adding K_S to both E_2 and E_3 if necessary, we can assume that $|E + E_2|$ is base-component free.

Now set $D_0 = 2E+2E_1+E_3$. Then $D_0^2 = 16$ and $\phi(D_0) = 3$. Hence (3) and [KLM, Lemma6.3(b)] give that D_0 is nef and $H - D_0 \sim E + E_2$ is base-component free. We have $H - 2D_0 \sim -(2E_1 + E + E_3 - E_2)$ and we now prove that $h^0(2D_0 - H) = 2$ and $h^1(H - 2D_0) = 0$. To this end, by [KL1, Cor.2.5] and Riemann-Roch, we just need to show that $B := 2E_1 + E + E_3 - E_2$ is quasi-nef. Let $\Delta > 0$ be such that $\Delta^2 = -2$ and $\Delta.B \leq -2$. By [KL2, Lemma2.3] we can write $B \sim B_0 + k\Delta$ where $k = -\Delta.B \geq 2$, $B_0 > 0$ and $B_0^2 = B^2 = 2$. Now $2 = E.B = E.B_0 + kE.\Delta \geq 1 + 2E.\Delta$, therefore $E.\Delta = 0$. The ampleness of H implies that $E_2.\Delta \geq 2$, giving the contradiction $4 = E_2.B = E_2.B_0 + kE_2.\Delta \geq 5$. Therefore B is quasi-nef.

Let $D \in |D_0|$ be a general curve. By [KL2, Cor.1] we know that $\text{gon}(D) = 2\phi(D_0) = 6$ whence $\text{Cliff}(D) = 4$, as D has genus 9 [ELMS, §5]. Therefore the map Φ_{H_D,ω_D} is surjective by [KL3, Thm.(v)]. Also μ_{V_D,ω_D} is surjective by [KLM, (13)] and S is nonextendable by [KLM, Prop.5.1]. □

4.3. **The case** $M_2^2 = 4$**.** We write $M_2 = E_2 + E_3$ as in Lemma 3.1(b).

4.3.1. $\beta = 2$. By Lemma 3.1 we have $(E.E_2, E.E_3) = (1, 2)$ and the four cases $(E_1.E_2, E_1.E_3) = (1, 2)$, $(2, 1)$, $(2, 2)$ and $(1, 3)$. Note that in all cases $E_2.H < 2\phi(H) = 10$, whence E_2 is quasi-nef by Lemma 2.5.

If $(E_1.E_2, E_1.E_3) = (1, 2)$ we claim that either $E + E_2$ or $E_1 + E_2$ is nef. Indeed if there is a nodal curve Γ such that $\Gamma.(E + E_2) < 0$ then $\Gamma.E_2 = -1$ and $\Gamma.E = 0$. By [KLM, Lemma6.3(a)] we have $\Gamma.E_1 > 0$, so that $E_2 \equiv E_1 + \Gamma$ and $E_1 + E_2 \equiv 2E_1 + \Gamma$ is nef. By symmetry the same arguments work if there is a nodal curve Γ such that $\Gamma.(E_1 + E_2) < 0$ and the claim is proved.

By symmetry between E and E_1 we can now assume that $E+E_2$ is nef. Setting $A := H-2E-2E_2$ we have $A^2 = 0$. As $E.A = 3$ and $E_2.A = 4$ we have that $A > 0$ is primitive and S is nonextendable by [KLM, Lemma5.5(iii-b)].

If $(E_1.E_2, E_1.E_3) = (2, 1)$ one easily sees that $H \sim 2(E_1 + E_2) + A$, with $A^2 = 0$, $E_1.A = 1$ and $E_2.A = 4$. Then $A > 0$ is primitive, $E_1 + E_2$ is nef by [KLM, Lemma6.3(e)] and S is nonextendable by [KLM, Lemma5.5(ii)].

If $(E_1.E_2, E_1.E_3) = (1, 3)$ we have $(E_1 + E_3)^2 = 6$ and we can write $E_1 + E_3 \sim A_1 + A_2 + A_3$ with $A_i > 0$, $A_i^2 = 0$ and $A_i.A_j = 1$ for $i \neq j$. Then $E.A_i = E_1.A_i = E_2.A_i = E_3.A_i = 1$ and $A_i.H = 6$.

We now claim that either A_i is nef or $A_i \equiv E + \Gamma_i$ for a nodal curve Γ_i with $\Gamma_i.E = 1$. In particular, at least two of the A_i's are nef. If there is a nodal curve Γ with $\Gamma.A_i < 0$, then since $A_i.L_1 = 4 = \phi(L_1)$ we must have $\Gamma.L_1 \leq 0$, whence $\Gamma.E > 0$ by the ampleness of H and the first statement immediately follows. If two of the A_i's are not nef, say $A_1 \equiv E + \Gamma_1$ and $A_2 \equiv E + \Gamma_2$ then $1 = A_1.A_2 = (E+\Gamma_1).(E+\Gamma_2) = 2+\Gamma_1.\Gamma_2$ yields the contradiction $\Gamma_1.\Gamma_2 = -1$ and the claim is proved.

We can therefore assume that A_1 and A_2 are nef. Let $A = H - 2A_1 - 2A_2$. Then $A^2 = 0$ and $E.A = 1$, whence $A > 0$ is primitive. As $A_1.A = A_2.A = 4$ and $\phi(H) = 5$, we have that S is nonextendable by [KLM, Lemma5.5(iii-b)].

If $(E_1.E_2, E_1.E_3) = (2, 2)$, note first that $E_1 + E_2$ is nef by [KLM, Lemma6.3(e)]. Set $A := H - 2E_1 - 2E_2$. Then $A^2 = 0$ and $A.E = 1$, so that $A > 0$ is primitive. As $(E_1 + E_2).A = 6$, we have that S is nonextendable by [KLM, Lemma5.5(ii)].

4.3.2. $\beta = 3$. By Lemma 3.1 we have $(E.E_2, E.E_3) = (1, 2)$ and $(E_1.E_2, E_1.E_3) = (1, 3)$ or $(2, 1)$.

We first show that E_i is quasi-nef for $i = 2, 3$. We have $H.E_2 \leq 9 < 2\phi(H) = 10$, whence E_2 is quasi-nef by Lemma 2.5. Now let $\Delta > 0$ be such that $\Delta^2 = -2$ and $\Delta.E_3 \leq -2$. By [KL2, Lemma2.3] we can write $E_3 \sim A + k\Delta$, for $A > 0$ primitive with $A^2 = 0$, $k = -\Delta.E_3 = \Delta.A \geq 2$. If $\Delta.E > 0$, from $E.E_3 = E.A + k\Delta.E$ we get that $k = 2$, $\Delta.E = 1$ and $E.A = 0$, whence the contradiction $E \equiv A$. Hence $\Delta.E = 0$. We get the same contradiction if $\Delta.E_2 > 0$. Hence, by the ampleness of H we must have $\Delta.E_1 \geq 2$, but this gives the contradiction $E_1.E_3 = E_1.A+k\Delta.E_1 \geq 4$. Hence also E_3 is quasi-nef.

We now treat the case $(E_1.E_2, E_1.E_3) = (1, 3)$.

Let $D_0 = 2E + E_1 + E_2$. Then $D_0^2 = 10, \phi(D_0) = 2$ and D_0 and $H - D_0 \sim E + E_1 + E_3$ are base-point free by [KLM, Lemma6.3 (a)-(b)]. Moreover $2D_0 - H \sim E + E_2 - E_3$, and since $(2D_0 - H).E = -1$, we have $h^0(2D_0 - H) = 0$ and it follows from [KL3, Thm.(iii)] that the map Φ_{H_D,ω_D} is surjective.

After possibly adding K_S to both E_2 and E_3, we can assume, by (3) and [KLM, Lemma6.3(c)], that the general members of both $|E+E_1|$ and $|E+E_2|$ are smooth irreducible curves. Let $D_1 \in |E+E_1|$ and $D_2 \in |E+E_2|$ be two such curves. By [KL1, Cor.2.5] we have $h^1(H-D_0-D_1)=h^1(E_3)=0$, whence $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by [KLM, (14)].

We now claim that $h^0(E_1+E_3-E_2) \leq 2$. Indeed, assume that $h^0(E_1+E_3-E_2) \geq 3$. Then $|E_1+E_3-E_2| = |M|+G$, with G the base-component and $|M|$ base-component free with $h^0(M) \geq 3$. If $M^2=0$, then $M \sim lP$, for an elliptic pencil P and an integer $l \geq 2$. But then $14=(E_1+E_3-E_2).H=(lP+G).H \geq lP.H \geq 4\phi(H)=20$, a contradiction. Hence $M^2 \geq 4$, but since $M.H \leq (E_1+E_3-E_2).H=14$, this contradicts the Hodge index theorem.

Therefore we have shown that $h^0(E_1+E_3-E_2) \leq 2$ and $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by [KLM, (16)]. By [KLM, Lemma5.6], μ_{V_D,ω_D} is surjective and by [KLM, Prop.5.1], S is nonextendable.

Next we treat the case $(E_1.E_2.E_1.E_3)=(2,1)$.

Let $D_0=2E+E_1+E_3$. Then $D_0^2=14$, $\phi(D_0)=3$ and D_0 and $H-D_0 \sim E+E_1+E_2$ are base-point free by [KLM, Lemma6.3(a)-(b)]. Moreover $2D_0-H \sim E+E_3-E_2$, and since $E+E_3$ is nef by [KLM, Lemma6.3(c)] and $(2D_0-H).(E+E_3)=(E+E_3-E_2).(E+E_3)=1$, we get that $h^0(2D_0-H) \leq 1$. It follows from [KL3, Thm.(iii)-(iv)] that the map Φ_{H_D,ω_D} is surjective. Let $D_1 \in |E+E_1|$ and $D_2 \in |E+E_3|$ be two general members. By [KL1, Cor.2.5] we have that $h^1(H-D_0-D_1)=h^1(E_2)=0$, whence $\mu_{V_{D_1},\omega_{D_1}}=\mu_{\mathcal{O}_{D_1}(H-D_0),\omega_{D_1}}$. Since ω_{D_1} is a base-point free pencil we get that $\mu_{\mathcal{O}_{D_1(H-D_0)},\omega_{D_1}}$ is surjective by the base-point free pencil trick because $\deg(\mathcal{O}_{D_1}(H-D_0-D_1+K_S))=3$, whence $h^1(\mathcal{O}_{D_1}(H-D_0-D_1+K_S))=0$. We have $(E_1+E_2-E_3).H=5=\phi(H)$, whence $h^0(E_1+E_2-E_3) \leq 1$ and $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by [KLM, (16)]. By [KLM, Lemma5.6], μ_{V_D,ω_D} is surjective and, by [KLM, Prop.5.1], S is nonextendable.

4.4. **The case** $M_2^2=6$**.** By Lemma 3.1 we have $\beta=2$ and $M_2=E_2+E_3+E_4$ as in that lemma. We note that E_1, E_2 and E_3 are nef by Lemma 2.5 and E_4 is quasi-nef by the same lemma.

By the ampleness of H it follows that $D_0 := E+E_1+E_2+E_3+E_4$ is nef with $D_0^2=24$, $\phi(D_0)=4$ and $H-D_0 \sim E+E_1$ is base-component free. Since $H-2D_0 \sim -(E_2+E_3+E_4)$ we have $h^1(H-2D_0)=0$ by [KL1, Cor.2.5] and $h^0(2D_0-H)=4$ by Riemann-Roch. Then μ_{V_D,ω_D} is surjective by [KLM, (13)] and so is Φ_{H_D,ω_D} by [KL3, Thm.(v)], since $\text{gon}(D)=8$ by [KL2, Cor.1], whence Cliff $D=6$ by [ELMS, §5], as $g(D)=13$. Hence S is nonextendable by [KLM, Prop.5.1].

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