ON THE EXISTENCE OF ENRIQUES-FANO THREEFOLDS OF INDEX GREATER THAN ONE

LUIS GIRALDO*

Departamento de Álgebra
Universidad Complutense de Madrid
Avenida Complutense, s/n
28040 Madrid, Spain
e-mail giraldo@eucmax.sim.ucm.es

ANGELO FELICE LOPEZ**

Dipartimento di Matematica
Università di Roma Tre
Largo San Leonardo Murialdo 1
00146 Roma, Italy
e-mail lopez@matrm3.mat.uniroma3.it

AND

ROBERTO MUÑOZ*

ESCET
Universidad Rey Juan Carlos
Campus de Móstoles - C. Tulipán, s/n
28933 Móstoles (Madrid), Spain
e-mail r.munoz@escet.urjc.es

1. INTRODUCTION

In the first half of the previous century Fano [Fa1-6] pioneered the study of higher dimensional algebraic varieties by studying projective threefolds whose curve section is a canonical curve. These varieties, and more generally Fano varieties, that is varieties with ample anticanonical bundle, have since then been studied by many authors and a classification has been achieved in dimension three ([I1-4], [MM]). In the former case the surface section is a K3 surface and it seems therefore interesting to study the threefolds whose surface section is some other surface of Kodaira dimension zero. While in the case of abelian or hyperelliptic surfaces there is no irreducible threefold (different from a cone) having them as hyperplane sections (Remark (3.12)), when the surface section is an Enriques surface a complete classification has not been achieved yet.

* Research partially supported by DGES research project, reference BFM2000-0621.
** Research partially supported by the MURST national project "Geometria Algebrica".

2000 Mathematics Subject Classification: Primary 14J30. Secondary 14J45, 14J28.

Definition (1.1). Let $X \subset I\!P^N$ be an irreducible threefold having a hyperplane section Y that is a smooth Enriques surface and such that X is not a cone over Y. X is called an Enriques-Fano threefold.

Observe that an equivalent definition is to assume that a general hyperplane section is a smooth Enriques surface. Fano himself in 1938 [Fa6] (see also the papers of Godeaux [Go1-4]) published an article in which he claimed a classification of Enriques-Fano threefolds, but his proof contains many gaps. Conte and Murre [CoMu] (see also [Co1-2]) first remarked that an Enriques-Fano threefold X must be singular (with isolated singularities), and with some assumptions on the nature of them, filled out some of these gaps. They proved that X has eight quadruple points whose tangent cone is a cone over the Veronese surface (proved also by Alexeev [Al] under the hypothesis of terminal singularities) and that X carries a birational system of K3 surfaces whose image is a Fano threefold. After the blossoming of Mori theory the problem of classifying Enriques-Fano threefolds was studied by several authors and, at least with some strong hypotheses on the singularities, a list was given by Bayle [Ba] and Sano [Sa1]. Precisely ([P]), let X be a projective threefold with only log-terminal singularities and assume that $-K_X$ is numerically equivalent to an ample Cartier divisor H and that $|H|$ contains a smooth Enriques surface. Then $n(K_X + H)$ is linearly equivalent to zero for some $n > 0$ and one can construct a cyclic covering $\widetilde{X} \to X$ with $\widetilde{X}$ a Fano threefold with Gorenstein canonical singularities. Under the additional assumption that $\widetilde{X}$ is smooth (which is equivalent to X having only terminal cyclic quotient singularities), using Mori and Mukai's classification of Fano threefolds with $B_2 \geq 2$ [MM], Bayle and Sano gave a complete list of such Enriques-Fano threefolds. Some other interesting results on Enriques-Fano threefolds are given in the works of Cheltsov, namely that such a threefold is rational [Ch2], is either a retraction of a cone or has canonical singularities [Ch1], and in this last case the genus of a curve section is at most 47 [Ch3] (in analogy with the case of Fano threefolds).

On the other hand the introduction of Gaussian maps and Zak's theorem has allowed recently to give, with very simple proofs, a classification of smooth Fano threefolds with very ample anticanonical bundle and more generally of Mukai varieties, that is projective varieties of any dimension with canonical curve section [CLM1-2]. The main theme in these papers was to compute the corank of the Wahl map $\Phi_{\omega_C} : \bigwedge^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 3})$ of a hyperplane section C of a general K3 surface. Then Zak's theorem was used to exclude the existence of Mukai varieties of dimension $n \geq \mathrm{cork}\Phi_{\omega_C} + 2$. The corank of Φ_{ω_C} was then used to calculate the dimension of the Hilbert scheme of the remaining cases and they were classified. An important role in the problem of classifying Fano threefolds is played by its index, that is the largest integer dividing the anticanonical bundle in the Picard group. One common feature of all the proofs ([I1-2], [Mu], [CLM2]) has been that the larger the index the easiest the proof. It seemed therefore reasonable to us that the same approach, applied to Enriques surfaces and Prym-canonical curves, that is curves embedded with $\omega_C \otimes \eta$, where η is a 2-torsion line bundle, should give similar results at least for index greater than one (see below). To do that it was necessary to calculate the corank of Gaussian maps of type $\Phi_{\omega_C \otimes \eta, \omega_C^r \otimes \eta^{r-1}}$. While the known results on Gaussian maps did not give an answer, we realized that any calculation of the corank of such maps, for example with the methods of [BEL], should take into consideration the Clifford index of C and ultimately the fact that, when C is hyperplane section of an Enriques surface S, if a Gaussian map as above is not surjective then S must have many trisecant lines and four-secant 2-planes. We therefore studied this problem with the usual Reider-type methods (section 2), relating the existence of many trisecant lines and four-secant 2-planes to the gonality and Clifford index of C, and then to the calculation of the cohomology of the normal bundle of S (section 3) in order to apply Zak's theorem. This led to the following very simple result on the nonexistence of Enriques-Fano threefolds of (integer) index greater than one (for an extension see Remark (3.13)).

Theorem (1.2). *Let $X \subset \mathbb{P}^N$ be an irreducible nondegenerate threefold such that X has a smooth hyperplane section Y which is the r-th Veronese embedding, for $r \geq 2$, of a linearly normal Enriques surface $S \subset \mathbb{P}^{g-1}$. Then X is a cone over Y.*

A few remarks on the index are in order. Suppose that X is a $\mathbb{Q}$-Fano n-fold, that is a normal n-dimensional variety with terminal singularities and ample anticanonical Weil divisor. Let a be the least positive integer such that $-aK_X$ is Cartier and b the largest positive integer dividing $-aK_X$ in the Picard group of X. Then b/a is called the Fano index of X. In [Sa2] Sano classified non-Gorenstein $\mathbb{Q}$-Fano n-folds with $a > 1$ and $b/a > n-2$ (for $n \geq 3$) and in particular proved that, under the same assumptions above, a and b are coprime. On an Enriques-Fano threefold X with hyperplane section H we have $-2K_X = 2H$ and if we let $H = r\Delta$ with r maximal, we get $a = 2, b = 2r$, hence, Sano's result gives also a proof of our Theorem (1.2) when X has terminal singularities. On the other hand, while we assume that $\Delta_{|H}$ is very ample and $r \geq 2$, in our theorem we have no assumptions whatsoever on the singularities (also note that by Reider's theorem [R] $\Delta_{|H}$ is automatically very ample if $H^3 \geq 10r^2$ unless $\Delta_{|H}$ is one of the linear systems listed in [CD1, Prop. 3.6.1, 3.6.2 and 3.6.3]). In the work of Bayle [Ba] and Sano [Sa1] is instead studied the case of Fano index one and cyclic quotient terminal singularities.

Besides of the application to the existence of Enriques-Fano threefolds our work on trisecant lines to Enriques surfaces has proved useful also to study the ideal of an Enriques surface. In [GLM] it was proved that any smooth linearly normal Enriques surface has homogeneous ideal generated by quadrics and cubics. Here we are able to specify when the quadrics are enough, at least scheme-theoretically.

Theorem (1.3). *Let $S \subset \mathbb{P}^{g-1}$ be a smooth linearly normal nondegenerate Enriques surface with $g \geq 11$.*

(1.4) If S does not contain a plane cubic curve then S is scheme-theoretically cut out by quadrics;

(1.5) If S contains a plane cubic curve then there are exactly two of them and the intersection of the quadrics containing S is the union of S and the two planes in which the cubic curves lie.

By [GLM, Thm. 1.1] any linearly normal Enriques-Fano threefold $X \subset \mathbb{P}^N$ is arithmetically Cohen-Macaulay (this was also one of the assumptions in [CoMu]) except when $N = 6$ and $Y = X \cap H$ is embedded with a Reye polarization. A trivial consequence of Theorem (1.3) is that, if $N \geq 11$ and the general hyperplane section through any point $p \notin X$ does not contain a plane cubic curve, then X is furthermore set-theoretically cut out by quadrics.

Finally in Remark (3.13) we give a small extension of Theorem (1.2) to other embeddings (not Veronese) and in Corollary (3.14) a consequence about Gaussian maps on Enriques surface sections.

2. TRISECANT LINES TO ENRIQUES SURFACES

This section will consist of two parts. In the first one we study trisecant lines to Enriques surfaces $S \subset \mathbb{P}^{g-1}$, essentially relating their existence to the existence of plane cubic curves on S. This will be accomplished by vector bundle techniques such as the Bogomolov-Reider method. The second part will be a description of the embedding linear systems when S contains cubic or quartic elliptic half pencils.

We denote by $\sim$ (respectively $\equiv$) the linear (respectively numerical) equivalence of divisors on S. Unless otherwise specified for the rest of the article we will denote by E (or E_1 etc.) divisors such that $|2E|$ is a genus one pencil on S, while nodal curves will be denoted by R (or R_1 etc.).

Proposition (2.1). *Let $S \subset \mathbb{P}^{g-1}$ be a smooth irreducible linearly normal Enriques surface and let $C = S \cap H$ be a smooth hyperplane section of S.*

(2.2) If S contains a plane cubic curve E then C has a trisecant line L such that $L \cap E = C \cap E$. Vice versa, if C has a trisecant line L and $g \geq 8$ then S contains a plane cubic curve E such that $L \cap E = C \cap E$;

(2.3) If C has a trisecant line, then it has a g^1_6; if $g \geq 18$ the converse holds;

(2.4) Suppose that S does not contain a plane cubic curve and that $g \geq 11$. Let $P \in C$ be a general point and let $\overline{C} \subset \mathbb{P}H^0(\mathcal{O}_C(H-P)) = \mathbb{P}^{g-3}$ be the projection of C from P. Then $\overline{C}$ has no trisecant lines.

Proof. If S contains a plane cubic curve E then H does not contain the $\mathbb{P}^2 =< E >$ span of E, hence $L = H\cap < E >$ is a trisecant line to C such that $L \cap E = C \cap E$.

Vice versa suppose there is a zero-dimensional subscheme $Z \subset C$ of degree 3 defining the trisecant line L. We are going to apply Reider's method [R]. Of course Z is in special position with respect to $\mathcal{O}_S(1) = \mathcal{O}_S(C + K_S + K_S)$, hence by [GH, Prop. 1.33] there exists a rank two vector bundle $\mathcal{E}$ sitting in an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_Z \otimes \mathcal{O}_S(C + K_S) \to 0 \qquad (2.5)$$

and its Chern classes are $c_1(\mathcal{E}) = C + K_S, c_2(\mathcal{E}) = \deg Z = 3$. The discriminant of $\mathcal{E}$ is then $\Delta(\mathcal{E}) = c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) = 2g - 14 > 0$ and hence $\mathcal{E}$ is Bogomolov unstable [Bo], [R]. Therefore there exist two line bundles A and B on S and a zero-dimensional subscheme $Z_1 \subset S$ such that

$$0 \to A \to \mathcal{E} \to \mathcal{I}_{Z_1} \otimes B \to 0 \tag{2.6}$$

and with $c_1(\mathcal{E}) = A + B, c_2(\mathcal{E}) = A \cdot B + \deg Z_1$. Moreover $A - B$ lies in the positive cone of S, that is $(A - B) \cdot D > 0$ for every ample D and $(A - B)^2 = \Delta(\mathcal{E}) + 4\deg Z_1 > 0$. We now claim that A and B are effective, non trivial and $h^0(\mathcal{I}_Z \otimes B) > 0$. To see this notice first that $(A+B)^2 = C^2 = 2g-2$, hence $2A \cdot B = 2g-2-A^2-B^2$; $(A-B)^2 \geq \Delta(\mathcal{E}) = 2g-14$, hence $A^2 + B^2 \geq 2g - 8$. Also $0 < (A - B) \cdot C = (A - B) \cdot (A + B) = A^2 - B^2$, that is $A^2 > B^2$. If we had $A^2 \leq 0$, then $B^2 \leq -2$ and hence $2g - 8 \leq -2$, a contradiction. Therefore $A^2 \geq 2$. Note that $A \cdot C \geq 0$, otherwise $A \cdot C < 0, B \cdot C < 0$, but then both A and B are not effective and (2.6) gives $h^0(\mathcal{E}) = 0$, contradicting (2.5). Therefore $h^2(A) = 0$ and $h^0(A) \geq 2$ by the Riemann-Roch theorem. Tensoring (2.5) by $\mathcal{O}_S(-A)$ we get $h^0(\mathcal{I}_Z \otimes B) \geq h^0(\mathcal{E}(-A)) \geq 1$ by (2.6) and the claim is proved.

Now we will see that the claim implies $A \cdot B = 3, B^2 = 0$. The latter will then conclude the proof of (2.2) since $C \cdot B = 3$, hence B is a cubic of arithmetic genus one, that is a plane cubic. Moreover in this case $L =< Z >= H\cap < B >$, hence $L \cap B = C \cap B$.

Choose $B' \in |\mathcal{I}_Z \otimes B|$. Since C is not a component of B' and both C and B' contain Z we have $3 \leq B' \cdot C = B \cdot C = B \cdot (A+B) = A \cdot B + B^2$, therefore $B^2 \geq 3 - A \cdot B = \deg Z_1 \geq 0$. Now $A - B$ lies in the positive cone of the Neron-Severi group of S and B in its closure, hence the signature theorem implies that $(A - B) \cdot B > 0$ ([BPV, VIII.1]). If it were $B^2 \geq 2$, we would have $3 = c_2(\mathcal{E}) = A \cdot B + \deg Z_1 \geq A \cdot B > B^2 \geq 2$, that is $A \cdot B = 3, B^2 = 2, \deg Z_1 = 0$, but then the Hodge index theorem applied to $A - B$ and B would give the contradiction $4g - 28 = 2\Delta(\mathcal{E}) = (A - B)^2 B^2 \leq ((A - B) \cdot B)^2 = 1$. Therefore we have $B^2 = 0$ and hence $A \cdot B = 3$. This proves (2.2).

To see (2.3) suppose first that C has a trisecant line spanned by a zero-dimensional subscheme $Z \subset C$ of degree 3. Since $h^0(\mathcal{O}_C(1)(-Z)) = g - 3$ then, setting $\eta = (K_S)_{|C}$, by the Riemann-Roch theorem and Serre duality we get $h^0(\mathcal{O}_C(Z + \eta)) = 1$. Then there exists an effective divisor Z' of degree 3 such that $Z+\eta \sim Z'$, where $\sim$ denotes linear equivalence on C. Moreover, as $\eta \not\sim 0$, we get $2Z \sim 2Z', Z \neq Z'$, hence C has a g^1_6. In the other direction, setting gon(C) for the gonality of C, by [GLM, Thm. 1.4] we have that gon$(C) = 6$ and that S contains a plane cubic curve, hence C has a trisecant line.

To prove (2.4) suppose to the contrary that for the general point $P \in C$ the projection from P, $\overline{C} \subset I\!\!P^{g-3}$, has a trisecant line spanned by some zero-dimensional subscheme $\overline{Z} \subset \overline{C}$ of degree 3. This means that there is a a zero-dimensional subscheme $Z \subset C$ of degree 3 such that $h^0(\mathcal{O}_C(H-P-Z)) = g-4$. We claim that the zero-dimensional subscheme $Z_P = Z \cup \{P\}$ is in special position with respect to $\mathcal{O}_S(1)$. Notice that Z_P spans a $I\!\!P^2$, hence the map $H^0(\mathcal{O}_S(1)) \to H^0(\mathcal{O}_{Z_P}(1))$ is not surjective, and in fact $h^0(\mathcal{J}_{Z_P/S}(1)) = g-3$. Let $Z' \subset Z_P$ be a subscheme of degree 3. If $h^0(\mathcal{J}_{Z'/S}(1)) > g-3$ then, for any $Z'' \subset Z'$ of degree 2, we get $h^0(\mathcal{J}_{Z'/S}(1)) = h^0(\mathcal{J}_{Z''/S}(1)) = g-2$, that is Z' gives a trisecant line to C and then S contains a plane cubic curve by (2.2), a contradiction. Therefore we have proved that Z_P is in special position with respect to $\mathcal{O}_S(1)$. As above there exists a rank two vector bundle $\mathcal{E}$ sitting in an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{J}_{Z_P} \otimes \mathcal{O}_S(C+K_S) \to 0 \tag{2.7}$$

with Chern classes $c_1(\mathcal{E}) = C+K_S, c_2(\mathcal{E}) = \deg Z_P = 4$ and discriminant $\Delta(\mathcal{E}) = 2g-18 > 0$. Hence $\mathcal{E}$ is Bogomolov unstable and we can find two line bundles A and B on S and a zero-dimensional subscheme $Z_1 \subset S$ such that

$$0 \to A \to \mathcal{E} \to \mathcal{J}_{Z_1} \otimes B \to 0 \tag{2.8}$$

and with $c_1(\mathcal{E}) = A+B, c_2(\mathcal{E}) = A \cdot B + \deg Z_1, A-B$ lying in the positive cone of S, $(A-B)^2 = \Delta(\mathcal{E}) + 4\deg Z_1 > 0$.

Again we claim that A and B are effective, non trivial, that there exists an effective divisor $B_P \in |B|$ containing Z_P and that $A \cdot B = 4, B^2 = 0$. First we notice that $(A-B)^2 \geq 2g-18$, hence $A^2+B^2 \geq 2g-10$. Also $A^2 > B^2$, hence $A^2 \geq 2$ and $A \cdot C \geq 0$, as $h^0(\mathcal{E}) > 0$. Then $h^0(A) \geq 2$ and (2.7), (2.8) give $h^0(\mathcal{J}_{Z_P} \otimes B) \geq h^0(\mathcal{E}(-A)) \geq 1$. Now choose $B' \in |\mathcal{J}_{Z_P} \otimes B|$. Then $4 \leq B' \cdot C = B \cdot (A+B) = A \cdot B + B^2$, therefore $B^2 \geq 4 - A \cdot B = \deg Z_1 \geq 0$. As above $(A-B) \cdot B > 0$, hence it cannot be $B^2 \geq 2$, else $4 = A \cdot B + \deg Z_1 \geq A \cdot B > B^2 \geq 2$, that is $A \cdot B = 3, 4$ and $B^2 = 2$. But then the Hodge index theorem applied to $A - B$ and B gives $g \leq 10$. Therefore we have $B^2 = 0, A \cdot B = 4$. In particular $B \cdot C = B \cdot (A+B) = 4$. We now show that this implies that C has a g^1_4, contradicting as above [GLM, Thm. 1.4]. Indeed, the exact sequence

$$0 \to \mathcal{O}_S(B - C) \to \mathcal{O}_S(B) \to \mathcal{O}_C(B) \to 0$$

shows that $h^0(\mathcal{O}_C(B)) \geq h^0(\mathcal{O}_S(B))$, since $(B - C) \cdot C = -2g + 6 < 0$. Now either $h^0(\mathcal{O}_S(B)) \geq 2$ and then it cuts out a g^1_4 on C or $h^0(\mathcal{O}_S(B)) = 1$ and hence $|\mathcal{I}_{Z_P}(B)| = \{B_P\}$. Thus we have proved that there is an open subset U of C such that for every $P \in U$ and for every effective divisor Z on C of degree 3 whose projection from P gives a trisecant line, there is a well-defined effective divisor B_P on S with $Z_P = Z \cup \{P\} \subset B_P$. Let $Y \subset C^{(4)}$ be the subset of the fourth symmetric product of C described by these divisors Z_P. Define a map $\psi : Y \to \mathrm{Pic}(S)$ by $\psi(Z_P) = \mathcal{O}_S(B_P)$. Since $\mathrm{Pic}(S)$ is a countable set then there exists a line bundle $\mathcal{L} \in \mathrm{Im}\psi$ such that the set $\psi^{-1}(\mathcal{L})$ is uncountable. Therefore $\mathcal{O}_S(B_P) = \mathcal{L}$ for every element Z_P in $\psi^{-1}(\mathcal{L})$. Since $|\mathcal{O}_S(B_P)| = \{B_P\}$, this means that C is a component of B_P. But this is not possible since if $B_P = C + D$ with D effective, we have the contradiction $4 = C \cdot B \geq 2g - 2$. ∎

Let us now recall an important result about the Enriques lattice. Let B be a nef line bundle on S with $B^2 > 0$ and set

$$\Phi(B) = \inf\{B \cdot E : |2E| \textit{ is a genus one pencil}\}.$$

Then by [CD1, Cor. 2.7.1, Prop. 2.7.1 and Thm. 3.2.1] (or [Co, 2.11]) we have $\Phi(B) \leq [\sqrt{B^2}]$, where $[x]$ denotes the integer part of a real number x. In particular, if C is very ample on S, we have $3 \leq \Phi(C) \leq [\sqrt{2g-2}]$ and $g \geq 6$.

Remark (2.9). When $6 \leq g \leq 8$ any smooth irreducible Enriques surface $S \subset \mathbb{P}^{g-1}$ contains plane cubic curves, because there are always genus one pencils $|2E|$ on S with E of degree 3. On the other hand it is easy to construct examples of $S \subset \mathbb{P}^{g-1}$ containing plane cubic curves for every large g.

As we have seen in Proposition (2.1) and in Remark (2.9) the existence of trisecant lines to an Enriques surface $S \subset \mathbb{P}^{g-1}$ depends on the existence of genus one pencils on S with low degree. In fact when such a pencil is present on S the embedding linear system $|C|$ can be decomposed as sum of suitable divisors and this fact will be useful in the next section. We will therefore proceed to find these decompositions.

Lemma (2.10). *Let $S \subset \mathbb{P}^{g-1}$ be a smooth irreducible linearly normal Enriques surface, C a smooth hyperplane section. Set $\varepsilon = 0, 1$ and suppose that $\Phi(C) = 3 + \varepsilon$ and let $|2E|$ be a genus one pencil on S such that $C \cdot E = 3 + \varepsilon$.*

(2.11) If $g = 8$ or $g \geq 9+\varepsilon$ then either $H^1(C-2E) = H^1(2C-4E) = 0$ or $C \sim \Delta+\mathcal{M}+K_S$ with $\mathcal{M}$ nef, $\mathcal{M}^2 > 0, C\cdot\mathcal{M} > 0, \Delta^2 > 0, |\Delta|$ base-component free and either $C\cdot\Delta \geq \Delta^2+5$ or $C \cdot \Delta = \Delta^2 + 4$ with Δ not hyperelliptic.

(2.12) If $\varepsilon = 1, g = 9$ or $\varepsilon = 0, 6 \leq g \leq 7$ then $C \sim E + \Delta$ with $|\Delta|$ base-point free, $\Delta^2 = 2g - 8 - 2\varepsilon, E \cdot \Delta = 3 + \varepsilon$. Moreover $H^0(\Delta + K_S - 2E) = 0$.

Proof. To see (2.11) for $g \geq 9 + \varepsilon$ set $M = C - 2E$ and notice that $M^2 = 2g - 14 - 4\varepsilon > 0, C \cdot M = 2g - 8 - 2\varepsilon > 0$ whence $h^2(M) = 0$ and $h^0(M) \geq g - 6 - 2\varepsilon \geq 2$. We first consider the case when $|M|$ is base-component free. Then it is nef, so is $2M$, hence $H^1(M) = H^1(2M) = 0$ by [CD1, Cor. 3.1.3]. Otherwise $M \sim F + \mathcal{M}$ has a nonempty base component F, $|\mathcal{M}|$ is base-component free and we will prove that $\Delta = 2E + F + K_S$ is the required divisor. Now $h^0(\mathcal{M}) = h^0(M) \geq g - 6 - 2\varepsilon$ and $h^2(\mathcal{M}) = 0$.

We first show that the case $\mathcal{M}^2 = 0$ cannot occur. In fact then by the proof of [CD1, Cor. 3.1.2] we have $\mathcal{M} \sim 2E_1$ and $M^2 = 2$, that is $\varepsilon = 1, g = 10$. Now $C \sim 2E + F + 2E_1$ and $10 = C \cdot M = C \cdot F + 2C \cdot E_1$ implies $C \cdot F = 2$. From $M^2 = 2$ we deduce $F^2 = 2 - 4F \cdot E_1$ and from $C \cdot F = 2$ we get $E \cdot F = E_1 \cdot F$. Moreover F is a conic and F^2 can only be $-2, -4, -8$, therefore it must be $E_1 \cdot F = 1$. But then $4 = C \cdot E = 1 + 2E_1 \cdot E$, a contradiction.

Therefore by [CD1, Prop. 3.1.4 and Cor. 3.1.3] we have $\mathcal{M}^2 \geq 2$ and $h^1(\mathcal{M}) = 0$. By the Riemann-Roch theorem we get $h^0(\mathcal{M}) = 1 + \frac{1}{2}\mathcal{M}^2 \geq g - 6 - 2\varepsilon$, that is $\mathcal{M}^2 \geq 2g - 14 - 4\varepsilon$. Also $C \cdot F + C \cdot \mathcal{M} = 2g - 8 - 2\varepsilon$, hence $C \cdot \mathcal{M} \leq 2g - 9 - 2\varepsilon$. Applying the Hodge index theorem to C and $\mathcal{M}$ we see that the only possible cases are: $C \cdot F = 1$ and either $\mathcal{M}^2 = 2g - 14 - 4\varepsilon$ or $\varepsilon = 1, \mathcal{M}^2 = 2g - 16$; $\varepsilon = 0, g = 9$ and $C \equiv 2\mathcal{M}$; $\varepsilon = 1, \mathcal{M}^2 = 2g - 18$ and either $2 \leq C \cdot F \leq 3$ or $C \equiv 3\mathcal{M}$. Now the cases $\varepsilon = 0, C \equiv 2\mathcal{M}$ and $\varepsilon = 1, C \equiv 3\mathcal{M}$ are excluded by $C \cdot E = 3 + \varepsilon$. When $C \cdot F = 1, \varepsilon = 1, \mathcal{M}^2 = 2g - 16$ we have that F is a line and $F^2 = -2$. But then $2g-18 = M^2 = 2g-18+2F\cdot\mathcal{M}$, that is $F\cdot\mathcal{M} = 0$, and this gives the contradiction $1 = C\cdot F = 2E\cdot F - 2$. When $C\cdot F = 1, \mathcal{M}^2 = 2g-14-4\varepsilon = M^2$ we get as above $F\cdot\mathcal{M} = 1$ and this implies $E\cdot F = 1$. In this case we easily see that Δ satisfies the properties in (2.11) by [CD1, Cor. 3.1.4]. We are then reduced to study the cases $\varepsilon = 1, \mathcal{M}^2 = 2g-18, 2 \leq C\cdot F \leq 3$. From $M^2 = 2g-18$ it follows that $F^2 = -2F\cdot\mathcal{M}$. Suppose first $C\cdot F = 2$. Then F is a conic and if $F^2 = -2$ we get $F\cdot\mathcal{M} = 1$, but this gives $2 = C\cdot F = 2E\cdot F - 1$, a contradiction. If $F = 2R$ with R a line we have $F^2 = -8$, hence $R\cdot\mathcal{M} = 2$; but this gives $1 = C\cdot R = 2E\cdot R - 2$, again impossible. When $F = R_1 + R_2$ with R_1 and R_2 disjoint lines we have $\mathcal{M}\cdot R_1 + \mathcal{M}\cdot R_2 = 2$, hence $1 = C\cdot R_1 = 2E\cdot R_1 - 2 + \mathcal{M}\cdot R_1$. Therefore $E\cdot R_1 = 1$, otherwise $E\cdot R_1 = 0, \mathcal{M}\cdot R_1 = 3$, contradicting the equality above. For the same reason we get $E\cdot R_2 = 1$. In this case we have that Δ is nef and $\Delta^2 = 4$ hence it satisfies the properties in (2.11) by [CD1, Prop. 3.1.6].

Suppose now $C\cdot F = 3$. Then F is a cubic and from $F^2 = -2F\cdot\mathcal{M}$ we deduce $F^2 = 6 - 4E\cdot F$. We deal first with the case F reduced. If F is irreducible then it is either a plane or a twisted cubic. In the first case by [CD1, Thm. 3.2.1, Prop. 3.1.2 and Prop. 3.1.4] we have that $|2F|$ is a genus one pencil, but this contradicts $\Phi(C) = 4$. If F is a twisted cubic then $F^2 = -2, E\cdot F = 2$. Therefore we have again that Δ is nef and $\Delta^2 = 6$ and it satisfies the required properties. If $F = R_1 + R_2$ with R_1 a line, R_2 an irreducible conic, we have $0 \leq R_1\cdot R_2 \leq 2$ and $F^2 = -4 + 2R_1\cdot R_2 = 6 - 4E\cdot F$, therefore $R_1\cdot R_2 = 1, F^2 = -2, E\cdot F = 2$. Hence $\mathcal{M}\cdot R_1 + \mathcal{M}\cdot R_2 = 1, E\cdot R_1 + E\cdot R_2 = 2$. Now $1 = C\cdot R_1 = 2E\cdot R_1 - 1 + \mathcal{M}\cdot R_1$, and then $E\cdot R_1 = E\cdot R_2 = 1$. When $F = R_1 + R_2 + R_3$ is union of three lines with $R_2\cdot R_3 = 1$ by $F^2 = 6 - 4E\cdot F$ we get $2E\cdot F + R_1\cdot R_2 + R_1\cdot R_3 = 5$, but $0 \leq R_1\cdot R_2 + R_1\cdot R_3 \leq 2$, hence, without loss of generality, we can assume $R_1\cdot R_2 = 1, R_1\cdot R_3 = 0$. Therefore $E\cdot F = 2, F^2 = -2, F\cdot\mathcal{M} = 1$, that is $\mathcal{M}\cdot R_1 + \mathcal{M}\cdot R_2 + \mathcal{M}\cdot R_3 = 1$. Now from $1 = C\cdot R_2 = 2E\cdot R_2 + \mathcal{M}\cdot R_2$ we get $E\cdot R_2 = 0, \mathcal{M}\cdot R_2 = 1$ and therefore $\mathcal{M}\cdot R_1 = \mathcal{M}\cdot R_3 = 0$ and by $C\cdot R_i = 1$ we get $E\cdot R_i = 1$ for $i = 1, 3$. In the last two cases for F we see that Δ is nef, $\Delta^2 = 6$ and it is as in (2.11).

If $F = R_1 + R_2 + R_3$ is union of three lines by the above case they must be disjoint, hence

$F^2 = -6, E \cdot F = \mathcal{M} \cdot F = 3$. Now $1 = C \cdot R_i = 2E \cdot R_i - 2 + \mathcal{M} \cdot R_i$ and if $E \cdot R_1 = 0$ we get $\mathcal{M} \cdot R_1 = 3$, hence $\mathcal{M} \cdot R_2 = 0$ and $2E \cdot R_2 = 3$, a contradiction. Therefore $E \cdot R_i = \mathcal{M} \cdot R_i = 1$. From $C \cdot E = 4$ we get $E \cdot \mathcal{M} = 1$, whence $\Phi(\mathcal{M}) = 1$. Now recall that $\mathcal{M}^2 = 2g - 18, C \cdot \mathcal{M} = 2g - 13$ hence using $\Phi(C) = 4$ and [CD1, Prop. 3.1.4 and Prop. 3.6.1] we contradict $C \cdot \mathcal{M} = 2g - 13$.

When F is non reduced of type $F = 2R_1 + R_2$ with R_1 and R_2 distinct lines, we have $0 \leq R_1 \cdot R_2 \leq 1, F^2 = -10 + 4R_1 \cdot R_2$, hence $R_1 \cdot R_2 + 2E \cdot R_1 + E \cdot R_2 = 4$. Moreover, $C \cdot R_2 = 1$ gives $2E \cdot R_2 + 2R_1 \cdot R_2 + \mathcal{M} \cdot R_2 = 3$ and $C \cdot E = 4$ implies $2E \cdot R_1 + E \cdot R_2 + E \cdot \mathcal{M} = 4$. Now either $E \cdot R_2 = 1$ and hence $R_1 \cdot R_2 = 0, \mathcal{M} \cdot R_2 = 1$, but then $2E \cdot R_1 = 3$, or $E \cdot R_2 = 0$ and hence $R_1 \cdot R_2 = 0, E \cdot R_1 = 2$, but then $E \cdot \mathcal{M} = 0$, implying the contradiction $E \equiv 0$ by the Hodge index theorem. Finally when $F = 3R$ with R a line we get $F^2 = -18, E \cdot R = 2$, but this contradicts $4 = C \cdot E = 6 + \mathcal{M} \cdot E$. This proves (2.11) with the exception of $g = 8$.

If $g = 8$ we have $\varepsilon = 0$ and we use Lemma (A.2) of the appendix to [GLM]. In the cases (A.4), (A.5) and (A.6) we have $H^1(C - 2E) = H^1(2C - 4E) = 0$. In the cases (A.7), (A.8) and (A.10) we set $\Delta = 2E + R + K_S$ (where in the cases (A.7), (A.8) $R = R_2$). In the case (A.9) we set $\Delta = 2E + R_1 + R_2$ and note that there is no genus one pencil $|2E'|$ such that $E' \cdot \Delta = 1$, therefore Δ is not hyperelliptic by [CD1, Prop. 3.1.4 and Prop. 4.5.1]. Then it is easily verified in all cases (A.4) through (A.10) that Δ satisfies the required properties.

Now to see the first part of (2.12) set $\Delta = C - E$. From $\Delta^2 = 2g - 8 - 2\varepsilon \geq 4, E \cdot \Delta = 3 + \varepsilon, C \cdot \Delta = 2g - 5 - \varepsilon > 0$ we conclude that $h^2(\Delta) = 0$ and $h^0(\Delta) \geq g - 3 - \varepsilon \geq 2$. We claim that $|\Delta|$ is base-component free. If not $\Delta \sim F + \mathcal{M}$ has a nonempty base component $F, |\mathcal{M}|$ is base-component free and $h^0(\mathcal{M}) = h^0(\Delta) \geq g - 3 - \varepsilon, h^2(\mathcal{M}) = 0$. As $\Delta^2 \geq 4$ we have $\mathcal{M}^2 > 0$ by [CD1, Cor. 3.1.2]. Therefore $h^1(\mathcal{M}) = 0$ by [CD1, Cor. 3.1.3] and $\mathcal{M}^2 \geq 2g - 8 - 2\varepsilon$ by the Riemann-Roch theorem. But also $C \cdot \mathcal{M} = C \cdot \Delta - C \cdot F \leq 2g - 6 - \varepsilon$ and this contradicts the Hodge index theorem. Now since $|\Delta|$ is base-component free to see that it has no base points by [CD1, Prop. 3.1.4, Prop. 4.5.1 and Cor. 4.5.1 of page 243] we just need to show that it cannot be $\Delta \equiv (g - 4 - \varepsilon)E_1 + E_2$ or $(g - 3 - \varepsilon)E_1 + R$ with $E_1 \cdot E_2 = E_1 \cdot R = 1$. But in both cases we get $C \cdot \Delta \geq (g - 3 - \varepsilon)(3 + \varepsilon)$, a contradiction. As for the vanishing in (2.12) observe that when $g = 6$ we have $(\Delta + K_S - 2E) \cdot \Delta = -2$ hence the vanishing holds. For $g = 7, 9$ we first prove:

(2.13) if $g = 7$ then $\Delta \equiv E_1 + E_2 + E_3$ with $E_i \cdot E_j = E \cdot E_i = 1$ for $i \neq j$ and $E_1 + E_2 - E$ is effective;

(2.14) if $g = 9$ then $\Delta \equiv E + 2E_1$ with $E \cdot E_1 = 2$.

When $g = 9$ we have $\Delta^2 = 8, C \cdot \Delta = 12$ hence $\Phi(\Delta) = 2$ and we can apply [CD1, Prop. 3.6.2]. Now $\Phi(C) = 4$ hence in all but one case we get $C \cdot \Delta > 12$, that is when $\Delta \equiv 2E_1 + E_2$ with $E_1 \cdot E_2 = 2$. From $12 = 2C \cdot E_1 + C \cdot E_2$ we deduce $C \cdot E_2 = 4$ and therefore $E \cdot E_2 = 0$, that is $E \equiv E_2$ and this gives (2.14). To prove (2.13) note that $\Delta^2 = 6, C \cdot \Delta = 9$ hence $\Phi(\Delta) = 2$ and by [CD1, Prop. 3.1.4, Lemma 4.6.1 and Thm. 4.6.3] we have that $|\Delta|$ is superelliptic hence by [CD1, Thm. 4.7.2] $\Delta \equiv E_1 + E_2 + E_3$ with $E_i \cdot E_j = 1$ for $i \neq j$ (the other two cases of [CD1, Thm. 4.7.2] give $C \cdot \Delta > 9$). Also $9 = C \cdot E_1 + C \cdot E_2 + C \cdot E_3$ and therefore $C \cdot E_i = 3, E \cdot E_i = 1$. Now by the same argument above also $E + E_1 + E_2$ is superelliptic hence by [CD1, Thm. 4.7.2] we can assume that $E_1 + E_2 - E$ is effective and (2.13) is proved.

We now proceed to prove that $H^0(\Delta + K_S - 2E) = 0$. In case (2.14) we have $(\Delta + K_S - 2E) \cdot E_1 = -2$, hence the vanishing. We finish with case (2.13). Let $\Delta_1 = E_1 + E_2 - E$; then $\Delta_1^2 = -2, C \cdot \Delta_1 = 3$. We prove first that Δ_1 is reduced. In fact if Δ_1 is nonreduced then it cannot be $\Delta_1 = 3R$ since $\Delta_1^2 = -2$, therefore $\Delta_1 = 2R_1 + R_2$ with R_1 and R_2 lines, but then we get the contradiction $R_1 \cdot R_2 = 2$. It follows that either $\Delta_1 = R$ is irreducible, or $\Delta_1 = R'_1 + R'_2$ with R'_1 a line, R'_2 an irreducible conic and $R'_1 \cdot R'_2 = 1$ or $\Delta_1 = R'_1 + R'_2 + R'_3$ with R'_1, R'_2 and R'_3 lines such that $R'_1 \cdot R'_2 = R'_1 \cdot R'_3 = 1, R'_2 \cdot R'_3 = 0$. Now let $B = \Delta + K_S - 2E$ and suppose that $H^0(B) \neq 0$. Since $B^2 = -6, C \cdot B = 3$ we are going to show that $B = R_1 + R_2 + R_3$ with R_1, R_2 and R_3 disjoint lines. In fact $C \cdot B = 3$ shows that B can have at most three irreducible components. If B is nonreduced then it cannot be $B = 3R'$ since $B^2 = -6$, therefore $B = 2R_1 + R_2$ with R_1 and R_2 two meeting lines. But then $0 = B \cdot R_2 = \Delta \cdot R_2 - 2E \cdot R_2$ and we get the contradiction $1 = C \cdot R_2 = 3E \cdot R_2$. If B is reduced then it cannot be irreducible because $B^2 = -6$, and also it cannot be $R_1 + R_2$ with R_1 a line, R_2 an irreducible conic again because $B^2 = -6$ would give $R_1 \cdot R_2 = -1$. Therefore $B = R_1 + R_2 + R_3$ and to satisfy $B^2 = -6$ we must have that R_1, R_2 and R_3 are disjoint lines. Now notice that $B \cdot \Delta_1 = -3$ hence it cannot be $\Delta_1 = R$ with R irreducible. If $\Delta_1 = R'_1 + R'_2$ as above we have $-3 = R'_1 \cdot B + R'_2 \cdot B$

and $R'_2 \cdot R_i \geq 0$, hence R'_1 must coincide with some R_i, but then $R'_1 \cdot B = -2$ and we get a contradiction. We are left with the case $\Delta_1 = R'_1 + R'_2 + R'_3$ as above. Note that $C \cdot R_i = 1$ hence $E \cdot R_i \leq 1$. Now $B \cdot \Delta_1 = -3$ shows that at least one of the lines R_i must coincide with one of the R'_j. On the other hand their intersections imply that not all of them can coincide. If only one coincidence occurs we have for example $R_3 = R'_3$ and therefore $E_3 + R'_1 + R'_2 \equiv E + R_1 + R_2$ hence $E \cdot R_1 - 2 = R_1 \cdot (E_3 + R'_1 + R'_2) \geq 0$, therefore the contradiction $E \cdot R_1 \geq 2$. The other coincidences can be excluded similarly. If two coincidences occur we have necessarily $R_2 = R'_2, R_3 = R'_3$ and therefore $E_3 + R'_1 \equiv E + R_1$ hence $E \cdot R_1 - 2 = R_1 \cdot (E_3 + R'_1) \geq 0$, again a contradiction. ∎

3. COHOMOLOGY OF THE NORMAL BUNDLE OF AN ENRIQUES SURFACE

Let $S \subset \mathbb{P}^{g-1}$ be a smooth irreducible linearly normal Enriques surface, N_S its normal bundle. As mentioned in the introduction, in order to apply Zak's theorem to study extendability of the Veronese embeddings of S, one needs to know the cohomology of the negative twists of its normal bundle. We will see that in fact the situation is particularly simple, since we always have $H^0(N_S(-2)) = 0$. This is the goal of this section. The proof will be divided in two distinct cases, depending on whether the surface contains elliptic curves of low degree or not. We start with the first case.

Proposition (3.1). *Let $S \subset \mathbb{P}^{g-1}$ be a smooth irreducible linearly normal Enriques surface, C a hyperplane section and suppose $\Phi(C) \leq 4$. Then $H^0(N_S(-2)) = 0$.*

Proof. Suppose to the contrary that $H^0(N_S(-2)) \neq 0$ and take $0 \neq \sigma \in H^0(N_S(-2))$ and $x \in S$ a general point. Let $|\Delta|$ be a linear system on S of dimension at least one, let $D \in |\Delta|$ be a general divisor containing x and suppose that D is smooth irreducible. As $\sigma_{|D}(x) = \sigma(x) \neq 0$, we deduce that $H^0(N_S(-2)_{|D}) \neq 0$. Under the hypothesis of the Proposition we are going to find such a linear system $|\Delta|$ so that $H^0(N_S(-2)_{|D}) = 0$. Consider to this end the exact sequence

$$0 \to N_{D/S}(-2) \to N_{D/\mathbb{P}^{g-1}}(-2) \to N_S(-2)_{|D} \to 0.$$

Set $< D >$ for the linear span of D. We will be done if we show

$$H^0(N_{D/<D>}(-2)) = 0 \tag{3.2}$$

and

$$\phi : H^1(N_{D/S}(-2)) \to H^1(N_{D/\mathbb{P}^{g-1}}(-2)) \text{ is injective} \tag{3.3}$$

because (3.2) is equivalent to $H^0(N_{D/\mathbb{P}^{g-1}}(-2)) = 0$. We now collect some sufficient conditions that ensure (3.2) and (3.3). To see the first one, suppose that D is linearly normal and $\dim < D >= h^0(\mathcal{O}_D(1)) - 1 \geq 2$, then the Euler sequence

$$0 \to \Omega^1_{<D>_{|D}} \otimes \omega_D(2) \to H^0(\mathcal{O}_D(1)) \otimes \omega_D(1) \to \omega_D(2) \to 0$$

shows that $H^1(\Omega^1_{<D>_{|D}} \otimes \omega_D(2)) = 0$ by the surjectivity of the standard multiplication map $H^0(\mathcal{O}_D(1)) \otimes H^0(\omega_D(1)) \to H^0(\omega_D(2))$ ([Gr, Thm. 4.e.1]). Then the normal bundle sequence

$$0 \to N^*_{D/<D>} \otimes \omega_D(2) \to \Omega^1_{<D>_{|D}} \otimes \omega_D(2) \to \omega_D^2(2) \to 0$$

implies that $h^0(N_{D/<D>}(-2)) = h^1(N^*_{D/<D>} \otimes \omega_D(2)) = \operatorname{cork}\Phi_{\mathcal{O}_D(1),\omega_D(1)}$, where the latter is the Gaussian map associated to $\mathcal{O}_D(1)$ and $\omega_D(1)$ ([W1]). Therefore (3.2) will follow from

$$H^1(C-D) = 0,\ h^0(\mathcal{O}_D(1)) \geq 3 \text{ and } \Phi_{\mathcal{O}_D(1),\omega_D(1)} \text{ is surjective} \tag{3.4}$$

because then D is linearly normal since S is. Concerning (3.3), since $N^*_{D/\mathbb{P}^{g-1}} \cong \mathcal{I}_{D/\mathbb{P}^{g-1}} \otimes \mathcal{O}_D$, we have the following commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{I}_{D/\mathbb{P}^{g-1}}(2)) \otimes H^0(\omega_D) & \longrightarrow & H^0(N^*_{D/\mathbb{P}^{g-1}}(2) \otimes \omega_D) \\ \downarrow \psi & & \downarrow \phi^* \\ H^0(\mathcal{I}_{D/S}(2)) \otimes H^0(\omega_D) & \stackrel{\chi}{\longrightarrow} & H^0(N^*_{D/S}(2) \otimes \omega_D). \end{array}$$

Now ψ is surjective since $H^1(\mathcal{I}_{S/\mathbb{P}^{g-1}}(2)) = 0$ by [GLM, Thm. 1.1] unless $g = 6$ and $\mathcal{O}_S(1)$ is a Reye polarization. If we denote by V the image of $H^0(\mathcal{O}_S(2C-D)) \to H^0(\mathcal{O}_D(2C-D))$, we see that (3.3) follows, with the above exception, if the multiplication map

$$\mu_{V,\omega_D} : V \otimes H^0(\omega_D) \to H^0(\mathcal{O}_D(2C-D) \otimes \omega_D) \text{ is surjective} \tag{3.5}$$

since then χ is surjective and so is ϕ^*.

By hypothesis we have $\Phi(C) = 3 + \varepsilon$ for $\varepsilon = 0, 1$ and let $|2E|$ be a genus one pencil on S such that $C \cdot E = 3 + \varepsilon$. Suppose first that $g = 8$ (hence $\varepsilon = 0$) or $g \geq 9 + \varepsilon$. If $H^1(C-2E) = H^1(2C-4E) = 0$, we set $\Delta = 2E$. Then for a general $D \in |2E|$ (3.4) follows by [BEL, Thm. 1]. Also (3.5) is satisfied because $V = H^0(\mathcal{O}_D(2C - D))$ and μ_{V,ω_D} is an isomorphism since D is elliptic. On the other hand if one of the two vanishings for $2E$ does not hold, we take Δ as in (2.11). Note that the general divisor $D \in |\Delta|$ is smooth irreducible by [CD1, Prop. 3.1.4 and Thm. 4.10.2] since $|\Delta|$ is base-component free and $\Delta^2 > 0$. Then $H^1(C-D) = H^1(\mathcal{M}+K_S) = 0$ by [CD1, Cor. 3.1.3], $H^2(C-D) = H^2(\mathcal{M}+K_S) = 0$ since $C \cdot \mathcal{M} > 0$. Therefore $h^0(\mathcal{O}_D(1)) = g - h^0(C - D) = C \cdot D - \frac{1}{2}D^2 \geq 5$ and (3.4) follows again by [BEL, Thm. 1]. For the same reason above we also have $H^1(2C - 2D) = 0$ hence $V = H^0(\mathcal{O}_D(2C - D))$ and μ_{V,ω_D} is surjective by the base-point free pencil trick [ArSe, Thm. 1.6], since D is not rational and $\deg \mathcal{O}_D(2C - D) \geq 2g(D) + 1$, that is $\mathcal{O}_D(2C - D)$ is very ample. Hence also (3.5) is satisfied in this case.

It remains to study the cases $g = 6, 7$ (hence $\varepsilon = 0$) or $g = 9, \varepsilon = 1$. We take Δ as in (2.12) and observe that as above the general $D \in |\Delta|$ is smooth irreducible and nonhyperelliptic (since $|\Delta|$ is base-point free by [CD1, Prop. 4.5.1, Thm. 4.5.4 and Rmk. 4.5.2]). To see (3.5) notice that $H^1(2C + K_S - D) = H^1(2E + K_S + D) = 0$ by [CD1, Cor. 3.1.3] and we have a diagram

$$\begin{array}{ccc} H^0(2C - D) \otimes H^0(D + K_S) & \xrightarrow{\gamma} & H^0(2C + K_S) \\ \downarrow & & \downarrow \pi \\ V \otimes H^0(\omega_D) & \xrightarrow{\mu_{V,\omega_D}} & H^0(\mathcal{O}_D(2C - D) \otimes \omega_D) \end{array}$$

where π is surjective. Now both $2C - D \sim 2E + D$ and $D + K_S$ are base point free by [CD1, Prop. 3.1.4 and Thm. 4.4.1] since $\Phi(D) \geq 2$. Also $h^1(2C - 2D - K_S) = h^1(2E + K_S) = 0$ and $h^2(2C - 3D) = h^0(D + K_S - 2E) = 0$ by (2.12) hence γ is surjective by Castelnuovo-Mumford and so is μ_{V,ω_D}. We now study (3.4).

When $g = 9, \varepsilon = 1$ we have that $C \cdot D = D^2 + 4$, hence (3.4) follows as above by [BEL, Thm. 1]. When $g = 7$ we have $h^0(C - D) = 1, h^0(\mathcal{O}_D(1)) = 6$ and $D \subset \mathbb{P}^5$ is a smooth irreducible nondegenerate linearly normal nonspecial curve of degree 9 and genus 4. To see (3.4) we need to show that $\Phi_{\mathcal{O}_D(1),\omega_D(1)}$ is surjective or equivalently that $H^0(N_{D/\mathbb{P}^5}(-2)) = 0$. To this end notice first that D is projectively normal by Castelnuovo's theorem hence in particular it is contained in 6 linearly independent quadrics. Moreover $\mathrm{Cliff}(D) = 1$ hence by [LS, Thm. 1.3] the intersection of the 6 quadrics containing D is the union of D and of its trisecant lines. Now observe that D cannot have infinitely many trisecant lines, otherwise their union would sweep a nondegenerate surface $X \subset \mathbb{P}^5$ intersection of 6 quadrics. Let Y be a component of X containing D, then Y is a rational normal surface scroll since its hyperplane section is a nondegenerate curve in $\mathbb{P}^4$ contained in 6 linearly independent quadrics, that is a rational normal curve. Since D has degree 9 and genus 4 and is not hyperelliptic it follows that the rulings of Y cut out on D a pencil of divisors A_λ of degree 3 such that $h^0(\mathcal{O}_D(1)(-A_\lambda)) = 4$ ([Sc, §2]). We prove that such a pencil does not exist. In fact let A be a divisor on D of degree 3 with $h^0(\mathcal{O}_D(1)(-A)) = 4$ and $h^0(A) \geq 2$. Then $h^0(\omega_D(-1)(A)) = 1$ and $\deg \omega_D(-1)(A) = 0$, that is $A \sim H - K_D$ on D. But $D \sim C - E$ on S, hence we have an exact sequence

$$0 \to \mathcal{O}_S(2E + K_S - C) \to \mathcal{O}_S(E + K_S) \to A \to 0$$

and $h^1(2E + K_S - C) = h^1(C - 2E) = 0$ by [GLM, Lemma A.2], hence $h^0(A) \leq 1$. Let now $P \in D$ be a general point and denote by $\overline{D} \subset \mathbb{P}^4$ the projection of D from P. Then $\overline{D}$ is a smooth irreducible nondegenerate linearly normal nonspecial curve of degree 8 and genus 4. As in [E, Lemma 4] or [BEL, 2.7] we have an exact sequence

$$0 \to \mathcal{O}_D(-1)(2P) \to N_{D/\mathbb{P}^5}(-2) \to N_{\overline{D}/\mathbb{P}^4}(-2)(-\overline{P}) \to 0$$

where $\overline{P} \in \overline{D}$ is the projection of P. To conclude for $g = 7$ we will therefore prove that $h^0(N_{\overline{D}/\mathbb{P}^4}(-2)(-\overline{P})) = 0$. Notice now that $h^0(\mathcal{J}_{\overline{D}/\mathbb{P}^4}(2)) = 2 + h^1(\mathcal{J}_{\overline{D}/\mathbb{P}^4}(2))$ and we cannot have three linearly independent quadrics containing $\overline{D}$ because a component Y of their intersection that contains $\overline{D}$ would be a rational normal surface scroll in $\mathbb{P}^4$, but on such surface there are no smooth irreducible curves of the given degree and genus. Therefore $h^0(\mathcal{J}_{\overline{D}/\mathbb{P}^4}(2)) = 2, h^1(\mathcal{J}_{\overline{D}/\mathbb{P}^4}(2)) = 0$ and $\mathcal{J}_{\overline{D}/\mathbb{P}^4}$ is 3-regular in the sense of Castelnuovo-Mumford. It follows that the homogeneous ideal of $\overline{D}$ is generated in degree ≤ 3. Let Q_1, Q_2 be the two quadrics containing $\overline{D}$ and let X be their complete intersection.

By the above X is irreducible and we claim that $\overline{D} \cap \mathrm{Sing}(X) = \emptyset$. To see this suppose there is a point $Q \in \overline{D} \cap \mathrm{Sing}(X)$ and let π_Q be the projection in $\mathbb{P}^3$ from Q. As $\pi_{Q|\overline{D}}$ is not $7:1$ we have that $\pi_Q(\overline{D})$ is a curve of degree 7 contained in $\pi_Q(X)$, which is therefore a surface of degree necessarily two. Hence there is a ruling L_t of $\pi_Q(X)$ intersecting $\pi_Q(\overline{D})$ in at least 4 points and $M_t =< L_t, Q >$ is a 2-plane at least 5-secant to $\overline{D}$. On the other hand $\overline{D}$ is not hyperelliptic, hence M_t is exactly 5-secant and the hyperplanes containing it cut out a g^1_3 on $\overline{D}$. But clearly this gives infinitely many g^1_3's on $\overline{D}$, a contradiction. Therefore we get an exact sequence

$$0 \to N_{\overline{D}/X}(-2)(-\overline{P}) \to N_{\overline{D}/\mathbb{P}^4}(-2)(-\overline{P}) \to \mathcal{O}_{\overline{D}}(-\overline{P})^{\oplus 2} \to 0$$

which gives the desired vanishing.

Finally when $g = 6$ there are two possibilities for S. If $\mathcal{O}_S(1)$ is a Reye polarization, that is if S is contained in some quadric in $\mathbb{P}^5$, then by [CD2] (as mentioned in section 1 of [DR]), the quadric must be nonsingular and, under its identification with the Grassmann variety $\mathbb{G} = \mathbb{G}(1,3)$, S is equal to the Reye congruence of some web of quadrics. By [ArSo, 4.3] S is geometrically linked, in the complete intersection of $\mathbb{G}$ and two general cubic hypersurfaces containing S, to a smooth congruence $T \subset \mathbb{G}$ of bidegree $(2,6)$. By [K, 2.19.1 and Cor. 2.12] (note that in [K, 2.19.1] there is a misprint; in the second line one should replace $H^1(I_{X/Y}(f_i+v))$ by $H^1(I_X(f_i+v)))$ it is then enough to show that $H^0(N_{T/\mathbb{P}^5}(-2)) = 0$. This fact is probably well known, but in any case we prove it with the methods above. To this end notice that by [ArSo, 4.1] $T \cong \mathbb{P}\mathcal{E} \xrightarrow{\pi} E$ is the ruled surface with invariant -1 over an elliptic curve E and is embedded in $\mathbb{P}^5$ by the linear system $|H| = |2C_0 + \pi^*\mathcal{L}|$, where C_0 is a section and $\mathcal{L}$ a line bundle on E of degree one. Let $\Delta = C_0 + \pi^*\mathcal{L}$ and D general in $|\Delta|$. It is easily seen that D is a smooth irreducible elliptic quintic spanning a $\mathbb{P}^4$, whence (3.2) follows by [H, Prop. V.2.1]. To see (3.3) we first show that $H^1(\mathcal{J}_{T/\mathbb{P}^5}(2)) = 0$, or, equivalently, $H^1(\mathcal{J}_{T/\mathbb{G}}(2)) = 0$. By [ArSo, 4.1] if Q is the universal quotient bundle on $\mathbb{G}$, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{G}}^{\oplus 2}(-1) \to S^2Q(-1) \to \mathcal{J}_{T/\mathbb{G}}(2) \to 0$$

whence $H^1(\mathcal{J}_{T/\mathbb{G}}(2)) = 0$ (since $H^2(\mathcal{O}_{\mathbb{G}}(-1)) = H^1(S^2Q(-1)) = 0$ by [ArSo, 1.4] or Bott vanishing). Now, as above, (3.3) follows by (3.5) and the latter holds since $H^1(\mathcal{O}_T(2H -$

$2D)) = H^1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(2C_0)) = 0$. Therefore the case of a Reye polarization is concluded. Suppose instead that $g = 6$ and $\mathcal{O}_S(1)$ is not a Reye polarization. By what we proved above we are left with proving (3.4) for $D \sim C - E$ as in (2.12). We have $h^0(C-D) = 1, h^0(\mathcal{O}_D(1)) = 5$, hence D is a smooth irreducible linearly normal nonspecial curve of degree 7 and genus 3 spanning a $\mathbb{P}^4$. To finish the proof of (3.4) we prove that $H^0(N_{D/\mathbb{P}^4}(-2)) = 0$. By Castelnuovo's theorem D is projectively normal hence in particular it is contained in 3 linearly independent quadrics Q_1, Q_2, Q_3. Let X be their complete intersection and suppose first that for every genus one pencil $|2E|$ on S such that $C \cdot E = 3$ we have that X is a surface. In this case we prove directly that $h^0(N_S(-2)) = 0$. In fact as above a component Y of X containing D must be a rational normal surface scroll in $\mathbb{P}^4$ hence, taking into account the degree and genus of D and the fact that D is not hyperelliptic, it follows that the rulings of Y cut out on D a pencil of divisors A_λ of degree 3 such that $h^0(\mathcal{O}_D(1)(-A_\lambda)) = 3$ ([Sc, §2]). By the Riemann-Roch theorem we deduce $A_\lambda \sim H - K_D$ on D, hence $h^0((E + K_S)_{|D}) \geq 2$. By the exact sequence

$$0 \to \mathcal{O}_S(2E + K_S - C) \to \mathcal{O}_S(E + K_S) \to (E + K_S)_{|D} \to 0$$

and $h^1(E+K_S) = h^0(2E+K_S-C) = 0$, we see that $h^1(2E+K_S-C) = h^1(C-2E) \geq 1$. But $C\cdot(C-2E) = 4, (C-2E)^2 = -2$, hence $h^0(C-2E) = h^1(C-2E) \geq 1$ by the Riemann-Roch theorem. Notice now that it cannot be $h^0(C-2E) \geq 2$ otherwise $C - 2E$ must have a nonempty base component F by [CD1, Prop. 3.1.4] and if we set $C - 2E \sim F + \mathcal{M}$ with $|\mathcal{M}|$ base-component free, then $h^0(\mathcal{M}) = h^0(C-2E) \geq 2$ and $C \cdot \mathcal{M} = 4 - C \cdot F \leq 3$. Now either $\mathcal{M}^2 = 0$ but then $\mathcal{M} \sim 2hE_1$ and $C \cdot \mathcal{M} \geq 6$ or $\mathcal{M}^2 \geq 2$, but this contradicts the Hodge index theorem applied to C and $\mathcal{M}$. Therefore for every genus one pencil $|2E|$ on S such that $C \cdot E = 3$ we have that $h^1(2E + K_S - C) = h^0(C - 2E) = 1$. By [CoVe, Prop. 3.13] $C' = C + K_S$ is a Reye polarization. Let S' be the embedding of S in $\mathbb{P}^5$ with C'. As $h^i(T_{\mathbb{P}^5 {}_{|S}}(-2)) = h^i(T_{\mathbb{P}^5 {}_{|S'}}(-2)) = 0, i = 0, 1$ (see proof of Theorem (1.2)), then $h^0(N_{S/\mathbb{P}^5}(-2)) = h^1(T_S(-2C)) = h^1(T_{S'}(-2C')) = h^0(N_{S'/\mathbb{P}^5}(-2)) = 0$ since S' is embedded with a Reye polarization. Finally suppose that there exists a genus one pencil $|2E|$ on S such that $C \cdot E = 3$ and X is a curve. Let Z be the subscheme of X defined by $\mathcal{I}_{Z/\mathbb{P}^4}/\mathcal{I}_{X/\mathbb{P}^4} = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{O}_D, \mathcal{O}_X)$. Then by [Sw, Thm. 13, Thm. 12 and Rmk. G2] we see that D and Z are geometrically linked and that Z is a line. Of course $h^0(N_{Z/\mathbb{P}^4}(-2)) = 0$, hence the same holds for D by [K, 2.19.1 and Cor. 2.12]. ■

To deal with the case when S does not contain a plane cubic curve we first record the following simple but useful application of the techniques of [BEL].

Lemma (3.6). *Let $D \subset \mathbb{P}^s = \mathbb{P}H^0(L)$ be a smooth irreducible non degenerate linearly normal curve of genus $g(D) \geq 4$ and suppose that*

(3.7) $\deg L \geq 2g(D) + 3 - 2h^1(L) - \mathrm{Cliff}(D)$ and

(3.8) for a general point $P \in D$ the projection $\overline{D} \subset \mathbb{P}H^0(L(-P))$ has no trisecant lines.

Then $H^0(N_{D/\mathbb{P}^s}(-2)) = 0$.

Proof. Setting, as in [BEL], $R_L = N^*_{D/\mathbb{P}^s} \otimes L$, we get $h^0(N_{D/\mathbb{P}^s}(-2)) = h^1(R_L \otimes \omega_D \otimes L)$. Note that $L(-P)$ is very ample and $H^1(\omega_D \otimes L(-2P)) = 0$. We apply the exact sequence [BEL, 2.7]

$$0 \to R_{L(-P)} \otimes \omega_D \otimes L \to R_L \otimes \omega_D \otimes L \to \omega_D \otimes L(-2P) \to 0.$$

By [La, Prop. 2.4.2] and the hypotheses (3.7), (3.8) we have that $\overline{D} \subset \mathbb{P}H^0(L(-P))$ is scheme-theoretically cut out by quadrics, hence we get a surjection $L^{-2}(2P)^{\oplus a} \to N^*_{\overline{D}/\mathbb{P}^{s-1}} \cong R_{L(-P)} \otimes L^{-1}(P)$, whence a surjection $\omega_D(P)^{\oplus a} \to R_{L(-P)} \otimes \omega_D \otimes L$ and therefore $H^1(R_{L(-P)} \otimes \omega_D \otimes L) = 0$. ■

We now come to the proofs of our main results.

Proof of Theorem (1.3). By [GLM, Thm. 1.4, Rmk. 2.10 and proof of Cor. 1.10] we know that there exists a countable family $\{Z_n, n \in \mathbb{N}\}$ of zero dimensional subschemes $Z_n \subset S$ of degree two, such that if $C \subset \mathbb{P}^{g-2}$ is a hyperplane section of S not containing Z_n for every n, then it has gonality $\mathrm{gon}(C) \geq 6$ and C is not isomorphic to a smooth plane septic. In fact this implies that $\mathrm{Cliff}(C) \geq 4$: By [CoMa, Thm. 2.3] we have $\mathrm{gon}(C) - 3 \leq \mathrm{Cliff}(C) \leq \mathrm{gon}(C) - 2$, hence it cannot be $\mathrm{Cliff}(C) \leq 2$. If $\mathrm{Cliff}(C) = 3$ we have then $\mathrm{gon}(C) = 6$, hence the Clifford index of C is not computed by a g^1_k, and if we set $\mathrm{Cliffdim}(C) = \min\{\dim |A|, A \in \mathrm{Pic}(C) \text{ such that } \dim |A| \geq 1, h^1(A) \geq 2, \deg(A) - 2\dim |A| = \mathrm{Cliff}(C)\}$ ([ELMS]), then $\mathrm{Cliffdim}(C) > 1$. Let A be a line bundle on C that computes the Clifford index of C and has minimal dimension, that is such that $\mathrm{Cliff}(A) = \mathrm{Cliff}(C) = 3, s = \dim |A| = \mathrm{Cliffdim}(C) \geq 2, \deg A \leq g - 1$. Recall that A is very ample by [ELMS, Lemma

1.1]. We get $\deg(A) = 2s+3$ and by [CoMa, Thm. C] we have $2s+3 \leq 2\mathrm{Cliff}(C)+4 = 10$, that is $s \leq 3$. It cannot be $s = 3$ for then $g = 10$ by [Ma, Satz 1]. Therefore $s = 2$ and C is isomorphic to a smooth plane septic, case already excluded.

Now we suppose that S does not contain a plane cubic curve and show (1.4). Since S is nondegenerate and linearly normal we have that every quadric containing a smooth hyperplane section of S lifts uniquely to a quadric containing S. Suppose that S is not scheme-theoretically cut out by quadrics at some point $P \in S$. Take a general hyperplane $H \ni P$ and let $C = S \cap H$. By [Mf, Lemma in the introduction] we have that C is not scheme-theoretically cut out by quadrics at P. On the other hand we know that $\mathrm{Cliff}(C) \geq 4$ and, as $\deg C = 2g-2$ and $h^1(\mathcal{O}_C(1)) = 0$, we have a contradiction by [La, Prop. 2.4.2], since C has no trisecant lines by (2.2).

If S does contain a plane cubic curve E then by [CD1, Thm. 3.2.1, Prop. 3.1.2 and Prop. 3.1.4] we have that $|2E|$ is a genus one pencil and let us see that the only other plane cubic on S is $E+K_S$. In fact if there is another genus one pencil $|2E_1|$ such that $C \cdot E_1 = 3$ then $E \cdot E_1 \geq 1$ but the Hodge index theorem applied to C and $E+E_1$ gives a contradiction. Now the planes spanned by the two plane cubic curves lying on S are contained in the intersection of the quadrics containing S. On the other hand let $P \in \mathbb{P}^{g-1}$ be a point not lying on S and not lying on any plane spanned by the plane cubic curves contained in S. Take a general hyperplane $H \ni P$ and let $C = S \cap H$. By (2.2) P does not lie on any trisecant line to C. By [LS, Thm. 1.3] we can find a quadric $Q' \subset H$ such that $P \notin Q'$. On the other hand there is a quadric $Q \supset S$ such that $Q' = Q \cap H$, therefore $Q \not\ni P$ and (1.5) is proved. ∎

Proof of Theorem (1.2). The goal is to show that $h^0(N_{Y/\mathbb{P}^{N-1}}(-1)) \leq N$ and apply Zak's theorem [Z], [Lv], [Bd]. As $H^1(\mathcal{O}_Y(-1)) = 0$, from the Euler sequence of $Y \subset \mathbb{P}^{N-1} = \mathbb{P}W$

$$0 \to \mathcal{O}_Y(-1) \to W^* \otimes \mathcal{O}_Y \to T_{\mathbb{P}^{N-1}}{}_{|Y}(-1) \to 0$$

we deduce that $h^0(T_{\mathbb{P}^{N-1}}{}_{|Y}(-1)) = N$. Now the normal bundle sequence

$$0 \to T_Y(-1) \to T_{\mathbb{P}^{N-1}}{}_{|Y}(-1) \to N_{Y/\mathbb{P}^{N-1}}(-1) \to 0$$

gives $h^0(N_{Y/\mathbb{P}^{N-1}}(-1)) \leq N + h^1(T_Y(-1)) = N + h^1(T_S(-r))$. We will show that

$$h^1(T_S(-r)) = 0. \tag{3.9}$$

To this end observe that the Euler sequence of $S \subset \mathbb{P}^{g-1}$

$$0 \to \mathcal{O}_S(-r) \to H^0(\mathcal{O}_S(1))^* \otimes \mathcal{O}_S(-r+1) \to T_{\mathbb{P}^{g-1}}{}_{|S}(-r) \to 0$$

implies that $h^0(T_{\mathbb{P}^{g-1}}{}_{|S}(-r)) = 0, H^1(T_{\mathbb{P}^{g-1}}{}_{|S}(-r)) = \mathrm{Coker}\mu$, where we denote by $\mu : H^0(\mathcal{O}_S(1)) \otimes H^0(\omega_S(r-1)) \to H^0(\omega_S(r))$ the multiplication map. The surjectivity of μ can be proved by restricting to a general hyperplane section C of S as follows. By the diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(\mathcal{O}_S) \otimes H^0(\omega_S(r-1)) & \xrightarrow{\cong} & H^0(\omega_S(r-1)) \\
\downarrow & & \downarrow \\
H^0(\mathcal{O}_S(1)) \otimes H^0(\omega_S(r-1)) & \xrightarrow{\mu} & H^0(\omega_S(r)) \\
\downarrow & & \downarrow \\
H^0(\mathcal{O}_C(1)) \otimes H^0(\omega_S(r-1)) & \xrightarrow{\nu} & H^0(\omega_C(r-1)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

it is enough to prove that ν is surjective. On the other hand ν is the composition of the maps $H^0(\mathcal{O}_C(1)) \otimes H^0(\omega_S(r-1)) \xrightarrow{\alpha} H^0(\mathcal{O}_C(1)) \otimes H^0(\omega_C(r-2)) \xrightarrow{\beta} H^0(\omega_C(r-1))$. The map α is surjective since $H^1(\omega_S(r-2)) = 0$, while the map β is surjective by [Gr, Thm. 4.e.1] if $r \geq 3$ and by the base-point free pencil trick if $r = 2$ [ArSe, Thm. 1.6]. Therefore also $h^1(T_{\mathbb{P}^{g-1}}{}_{|S}(-r)) = 0$, and then the normal bundle sequence of S shows that (3.9) will follow if we show

$$h^0(N_{S/\mathbb{P}^{g-1}}(-2)) = 0. \tag{3.10}$$

Let $C \subset \mathbb{P}^{g-2}$ be a general hyperplane section of S. By Proposition (3.1) we can assume $\Phi(C) \geq 5$ and hence $g \geq 14$. In particular, as in the previous proof, S does not contain a plane cubic curve. Note that since $N_{S/\mathbb{P}^{g-1}}{}_{|C} \cong N_{C/\mathbb{P}^{g-2}}$, it is enough to prove that

$$h^0(N_{C/\mathbb{P}^{g-2}}(-2)) = 0. \tag{3.11}$$

To this end we apply Lemma (3.6) to C. By (2.4) we see that (3.8) is satisfied. Since $\deg C = 2g-2$ and $h^1(\mathcal{O}_C(1)) = 0$, we deduce that (3.7) holds as soon as $\mathrm{Cliff}(C) \geq 5$. We now proceed as in the proof of Theorem (1.3). It cannot be $\mathrm{Cliff}(C) \leq 3$, otherwise $\mathrm{gon}(C) \leq 6$, contradicting [GLM, Thm. 1.4]. If $\mathrm{Cliff}(C) = 4$ we have $\mathrm{gon}(C) = 7$, again by [GLM, Thm. 1.4]. Therefore $\mathrm{Cliffdim}(C) > 1$ and we can find a very ample line bundle A on C such that $\mathrm{Cliff}(A) = \mathrm{Cliff}(C) = 4, s = \dim|A| = \mathrm{Cliffdim}(C) \geq 2, \deg A \leq g-1$. Then $\deg(A) = 2s+4$ and by [CoMa, Thm. C] we have $2s+4 \leq 2\mathrm{Cliff}(C)+4 = 12$, that is $s \leq 4$. Now, by [ELMS, §5] we have that if $s = 3, 4$, then $\mathrm{Cliff}(C) = 2s-3 = 3, 5 \neq 4$. Therefore $s = 2$ and C is isomorphic to a smooth plane octic. The latter case is excluded by [GLM, Cor. 1.10]. ■

Remark (3.12). Any polarized abelian variety (A, L) of dimension at least two satisfies $H^1(T_A \otimes L^{-i}) = 0$ for every $i \geq 1$, hence it does not lie on another variety (except from a cone) as an ample divisor with normal bundle L, by a theorem of Fujita [Fu]. The same holds for hyperelliptic surfaces as they are étale quotients with finite fibers of abelian surfaces.

Remark (3.13). As we have seen, if $S \subset \mathbb{P}^{g-1} = \mathbb{P}H^0(L)$ is a smooth irreducible Enriques surface then $H^0(N_S(-2L)) = 0$. Therefore also $H^0(N_S(-2L-L')) = 0$ for any effective line bundle L'. This means that, in many cases (see $(*)$ below), the embedding of S with $2L+L'$ is not extendable, that is it is not a hyperplane section of any threefold different from a cone. On the other hand if $g \geq 14$ and S does not contain a plane cubic curve in its embedding in $\mathbb{P}^{g-1}$, a little bit better result can be proved with the same methods above. In fact take any effective line bundle L_1 on S such that $(*)$ $L+L_1$ is very ample, $H^1(-L_1) = 0$ and the multiplication map μ_{L,K_S+L_1} is surjective. Then the embedding of S with $L+L_1$ is not extendable as soon as $H^0(N_C(-L-L_1)) = 0$, where C is a general hyperplane section of S. The latter vanishing follows, as in Lemma (3.6), if $h^0(L-L_1) \leq 1$.

As noticed in the introduction we record the following consequence of the proof of Theorem (1.2).

Corollary (3.14). *Let $S \subset \mathbb{P}^{g-1}$ be a smooth irreducible linearly normal Enriques* *surface not containing a plane cubic curve and let $C = S \cap H$ be a smooth hyperplane section of S of genus $g \geq 14$. If $g \leq 17$ or $g = 21$ assume furthermore that C is general. Then the Gaussian map $\Phi_{\mathcal{O}_C(1),\omega_C(s)}$ is surjective for any $s \geq 1$.*

Proof. By [W1, Prop. 1.10] we have $\mathrm{cork}\Phi_{\mathcal{O}_C(1),\omega_C(s)} = h^0(N_{C/\mathbb{P}^{g-2}}(-s-1))$ and the latter is zero by (the proof of) (3.11). ∎

Remark (3.15). When $s \geq 3$ the surjectivity of the above Gaussian map also follows by [W2, Thm. 2.6] while by [W1] (or [Z]) $\Phi_{\mathcal{O}_C(1),\omega_C}$ is not surjective. On the other hand $\mathcal{O}_C(1) \cong \omega_C(K_{S_{|C}})$ and we know that if (C, η) is a general pair, with C a smooth curve of genus g, η a 2-torsion line bundle on C, we have that $\Phi_{\omega_C \otimes \eta, \omega_C}$ is surjective for $g \geq 12, g \neq 13, 19$ by the result of Ciliberto-Verra [CiVe].

REFERENCES

[Al] Alexeev, V.: General elephants of $\mathbb{Q}$-Fano 3-folds. *Compositio Math.* **91**, (1994) 91-116.

[ArSe] Arbarello, E., Sernesi, E.: Petri's approach to the study of the ideal associated to a special divisor. *Invent. Math.* **49**, (1978) 99-119.

[ArSo] Arrondo, E., Sols, I.: On congruences of lines in the projective space. *Mém. Soc. Math. France* **50**, (1992).

[Ba] Bayle, L.: Classification des variétés complexes projectives de dimension trois dont une section hyperplane générale est une surface d'Enriques. *J. Reine Angew. Math.* **449**, (1994) 9-63.

[Bd] Badescu, L.: Polarized varieties with no deformations of negative weights. In: *Geometry of complex projective varieties (Cetraro, 1990), Sem. Conf.* **9**, Mediterranean, Rende, 1993, 9-33.

[BEL] Bertram, A., Ein, L., Lazarsfeld, R.: Surjectivity of Gaussian maps for line bundles of large degree on curves. In: *Algebraic Geometry (Chicago, IL, 1989), Lecture Notes in Math.* **1479**, Springer, Berlin-New York: 1991, 15-25.

[Bo] Bogomolov, F.: Holomorphic tensors and vector bundles on projective varieties. *Izv. Akad. Nauk SSSR Ser. Mat.* **42**, (1978) 1227-1287, 1439.

[BPV] Barth, W., Peters, C., van de Ven, A.: Compact complex surfaces. *Ergebnisse der Mathematik und ihrer Grenzgebiete* **4**, Springer-Verlag, Berlin-New York, 1984.

[CD1] Cossec, F., Dolgachev, I.: Enriques surfaces I. *Progress in Mathematics* **76**, Birkhäuser Boston, MA, 1989.

[CD2] Cossec, F., Dolgachev, I.: Enriques surfaces II.

[Ch1] Cheltsov, I.A.: Three-dimensional algebraic varieties that have a divisor with a numerically trivial canonical class. *Russian Math. Surveys* **51**, (1996) 140-141.

[Ch2] Cheltsov, I.A.: On the rationality of non-Gorenstein $\mathbb{Q}$-Fano 3-folds with an integer Fano index. In: *Birational algebraic geometry (Baltimore, MD, 1996), Contemp. Math.*

207, Amer. Math. Soc., Providence, RI, 1997, 43-50.

[Ch3] Cheltsov, I.A.: Boundedness of Fano 3-folds of integer index. *Math. Notes* **66**, (1999) 360-365.

[CiVe] Ciliberto, C., Verra, A.: On the surjectivity of the Gaussian map for Prym-canonical line bundles on a general curve. In: *Geometry of complex projective varieties (Cetraro, 1990), Sem. Conf.* **9**, Mediterranean, Rende, 1993, 117-141.

[CLM1] Ciliberto, C., Lopez, A.F., Miranda, R.: Projective degenerations of K3 surfaces, Gaussian maps and Fano threefolds. *Invent. Math.* **114**, (1993) 641-667.

[CLM2] Ciliberto, C., Lopez, A.F., Miranda, R.: Classification of varieties with canonical curve section via Gaussian maps on canonical curves. *Amer. J. Math.* **120**, (1998), 1-21.

[Co] Cossec, F.R.: On the Picard group of Enriques surfaces. *Math. Ann.* **271**, (1985) 577-600.

[Co1] Conte, A.: Two examples of algebraic threefolds whose hyperplane sections are Enriques surfaces. In: *Algebraic geometry-open problems (Ravello, 1982), Lecture Notes in Math.* **997**, Springer, Berlin-New York, 1983, 124-130.

[Co2] Conte, A.: On the nature and the classification of Fano-Enriques threefolds. In: *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc.* **9**, Bar-Ilan Univ., Ramat Gan, 1996, 159-163.

[CoMa] Coppens, M., Martens, G.: Secant spaces and Clifford's theorem. *Compositio Math.* **78**, (1991) 193-212.

[CoMu] Conte, A., Murre, J.P.: Algebraic varieties of dimension three whose hyperplane sections are Enriques surfaces. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **12**, (1985) 43-80.

[CoVe] Conte, A., Verra, A.: Reye constructions for nodal Enriques surfaces. *Trans. Amer. Math. Soc.* **336**, (1993) 79-100.

[DR] Dolgachev, I., Reider, I.: On rank 2 vector bundles with $c_1^2 = 10$ and $c_2 = 3$ on Enriques surfaces. In: *Algebraic geometry (Chicago, IL, 1989), Lecture Notes in Math.* **1479**, Springer, Berlin, 1991, 39-49.

[E] Ein, L.: The irreducibility of the Hilbert scheme of smooth space curves. *Proceedings of Symposia in Pure Math.* **46**, (1987) 83-87.

[ELMS] Eisenbud, D., Lange, H., Martens, G., Schreyer, F.O.: The Clifford dimension of a projective curve. *Compositio Math.* **72**, (1989) 173-204.

[Fa1] Fano, G.: Sopra alcune varietà algebriche a tre dimensioni aventi tutti i generi nulli. *Atti Accad. Torino* **43**, (1908) 973-984.

[Fa2] Fano, G.: Sulle varietà algebriche a tre dimensioni aventi tutti i generi nulli. *Atti Congr. Internaz. Bologna* **IV**, (1929) 115-121.

[Fa3] Fano, G.: Su alcune varietà algebriche a tre dimensioni a curve sezioni canoniche. *Scritti Mat. offerti a L. Berzolari. Ist. Mat. R. Univ. Pavia*, Pavia, (1936) 329-349.

[Fa4] Fano, G.: Sulle varietà algebriche a tre dimensioni a curve sezioni canoniche. *Mem. Accad. d'Italia* **VIII**, (1937) 23-64.

[Fa5] Fano, G.: Nuove ricerche sulle varietà algebriche a tre dimensioni a curve-sezioni canoniche. *Pont. Acad. Sci. Comment.* **11**, (1947) 635-720.

[Fa6] Fano, G.: Sulle varietà algebriche a tre dimensioni le cui sezioni iperpiane sono superficie di genere zero e bigenere uno. *Memorie Soc. dei XL* **24**, (1938) 41-66.

[Fu] Fujita, T.: Rational retractions onto ample divisors. *Sci. Papers College Arts Sci. Univ. Tokyo* **33**, (1983) 33-39.

[GH] Griffiths, P., Harris, J.: Residues and zero-cycles on algebraic varieties. *Ann. of Math.* **108**, (1978) 461-505.

[GLM] Giraldo, L., Lopez, A.F., Muñoz, R.: On the projective normality of Enriques surfaces (with an appendix by Lopez, A.F. and Verra, A.). *Math. Ann.* **324**, (2002), 135-158.

[Go1] Godeaux, L.: Sur les variétés algébriques à trois dimensions dont les sections hyperplanes sont des surfaces de genre zéro et de bigenre un. *Bull. Acad. Belgique Cl. Sci.* **14**, (1933) 134-140.

[Go2] Godeaux, L.: Une variété algébrique à trois dimensions dont les sections hyperplanes sont des surfaces de bigenre un. *Bull. Soc. Roy. Sci. Liége* **31**, (1962) 751-756.

[Go3] Godeaux, L.: Sur les variétés algébriques à trois dimensions dont les sections hyperplanes sont des surfaces de bigenre un. *Acad. Roy. Belg. Bull. Cl. Sci.* **48**, (1962) 1251-1257.

[Go4] Godeaux, L.: Une variété algébrique à trois dimensions à sections hyperplanes de bigenre un. *Bull. Soc. Roy. Sci. Liége* **39**, (1970) 439-441.

[Gr] Green, M.: Koszul cohomology and the geometry of projective varieties. *J. Differential Geom.* **19**, (1984) 125-171.

[H] Hulek, K.: Projective geometry of elliptic curves. *Astérisque* **137** (1986).

[I1] Iskovskih, V.A.: Fano 3-folds I. *Izv. Akad. Nauk SSSR Ser. Mat.* **41**, (1977) 516-562, 717.

[I2] Iskovskih, V.A.: Fano 3-folds II. *Izv. Akad. Nauk SSSR Ser. Mat.* **42**, (1978) 506-549.

[I3] Iskovskih, V.A.: Anticanonical models of three-dimensional algebraic varieties. *Current problems in mathematics* **12**, VINITI, Moscow, 1979, 59-157, 239.

[I4] Iskovskih, V.A.: Birational automorphisms of three-dimensional algebraic varieties. *Current problems in mathematics* **12**, VINITI, Moscow, 1979, 159-236, 239.

[K] Kleppe, J.O.: Liaison of families of subschemes in $\mathbb{P}^n$. In: *Algebraic curves and projective geometry (Trento, 1988), Lecture Notes in Math.* **1389**, Springer, Berlin, 1989, 128-173.

[La] Lazarsfeld, R.: A sampling of vector bundle techniques in the study of linear series. *Lectures on Riemann surfaces (Trieste, 1987)*, World Sci. Publishing, Teaneck, NJ, 1989, 500-559.

[LS] Lange, H., Sernesi, E.: Quadrics containing a Prym-canonical curve. *J. Algebraic Geom.* **5**, (1996) 387-399.

[Lv] L'vovsky, S.: Extensions of projective varieties and deformations. I. *Michigan Math. J.* **39**, (1992) 41-51.

[Ma] Martens, G.: Über den Clifford-Index algebraischer Kurven. *J. Reine Angew. Math.* **336**, (1982) 83-90.

[Mf] Mumford, D.: Varieties defined by quadratic equations. In: *Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969)*, Edizioni Cremonese, Rome, (1970) 29-100.

[MM] Mori, S., Mukai, S.: Classification of Fano 3-folds with $B_2 \geq 2$. *Manuscripta Math.* **36**, (1981/82) 147-162.

[Mu] Murre, J.P.: Classification of Fano threefolds according to Fano and Iskovskih. In: *Algebraic threefolds (Varenna, 1981), Lecture Notes in Math.* **947**, Springer, Berlin, 1982, 35-92.

[P] Prokhorov, Y.G.: On three-dimensional varieties with hyperplane sections-Enriques surfaces. *Math. Sb.* **186**, (1995) 113-124.

[R] Reider, I.: Vector bundles of rank 2 and linear systems on algebraic surfaces. *Ann. of Math.* **127**, (1988) 309-316.

[Sa1] Sano, T.: On classification of non-Gorenstein $\mathbb{Q}$-Fano 3-folds of Fano index 1. *J. Math. Soc. Japan* **47**, (1995) 369-380.

[Sa2] Sano, T.: Classification of non-Gorenstein $\mathbb{Q}$-Fano d-folds of Fano index greater than $d-2$. *Nagoya Math. J.* **142**, (1996) 133-143.

[Sc] Schreyer, F.O.: Syzygies of canonical curves and special linear series. *Math. Ann.* **275**, (1986) 105-137.

[Sw] Schwartau, P.W.: Liaison addition and monomial ideals. *Ph.D. Thesis, Brandeis University*, Waltham, MA, (1982).

[W1] Wahl, J.: Introduction to Gaussian maps on an algebraic curve. In: *Complex Projective Geometry, Trieste-Bergen 1989, London Math. Soc. Lecture Notes Ser.* **179**, Cambridge Univ. Press, Cambridge 1992, 304-323.

[W2] Wahl, J.: Gaussian maps on algebraic curves. *J. Differential Geom.* **32**, (1990) 77-98.

[Z] Zak, F.L.: Some properties of dual varieties and their application in projective geometry. In: *Algebraic Geometry (Chicago, IL, 1989), Lecture Notes in Math.* **1479**, Springer, Berlin 1991, 273-280.