A LINEAR BOUND ON THE EULER NUMBER OF THREEFOLDS OF CALABI-YAU AND OF GENERAL TYPE

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ABSTRACT: Let $X \subset \mathbb{P}^N$ be either a threefold of Calabi-Yau or of general type (embedded with rK_X). In this article we give lower and upper bounds, linear on the degree of X and N, for the Euler number of X. As a corollary we obtain the boundedness of the region described by the Chern ratios $\frac{c_3}{c_1 c_2}, \frac{c_1^3}{c_1 c_2}$ of threefolds with ample canonical bundle and a new upper bound for the number of nodes of a complete intersection threefold.

1. INTRODUCTION

A well-known approach to the classification of algebraic varieties is to divide the problem in two different parts: first search for numerical invariants and then study the geometric properties of the varieties arising from the set of all algebraic varieties with given invariants. This approach has been successful in the study of curves (both from a projective and abstract point of view) and surfaces, especially in the case of surfaces of general type (see for example [Ha], [H], [C], [P]). For higher dimensional varieties the situation is more complicated essentially because of the enormous variety of possibilities both from a topological and birational point of view. Nevertheless the success of Mori theory and in particular the study of existence of minimal models has produced a good framework for setting the classification goals also in higher dimension. Already Mori theory indicated the importance of the study of varieties with trivial canonical bundle, but a new and unexpected event took place with the spreading in algebraic geometry of the ideas of physicists and the importance for their theories of Calabi-Yau threefolds. For physical reasons (such as the compactification of the heterotic string) it became important to study the Hodge theory of Calabi-Yau threefolds (see for example [CHSW], [COGP]) and in particular of their Euler numbers (and this carried all the consequences and beautiful predictions nowadays known

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as mirror symmetry conjectures). The similarity with K3 surfaces already suggested that some kind of finiteness result should hold for Calabi-Yau threefolds (see for example [R], [H1], [G2]) but the ideas of physics contributed both to clarify what the picture might be for the moduli space of Calabi-Yau threefolds ([CGH], [GH1], [GH2], [G1]) and, at least from a numerical point of view, what Euler numbers to expect (currently all the known examples satisfy $-960 \leq e(X) \leq 960$ [G1], [HS]).

The starting observation of our work has been that, at least from the projective point of view, there is a finiteness result: let $X \subset I\!P^N$ be a smooth linearly normal Calabi-Yau threefold, L the hyperplane bundle; by Miyaoka's positivity of $c_2(X)$ [M2], Riemann-Roch and Kodaira vanishing one easily gets $N+1 = \frac{1}{12} L \cdot c_2(X) + \frac{1}{6}\deg(X) \geq \frac{1}{6}\deg(X)$. Therefore the degree of X is bounded by $6N+6$ and, for a given N, there are only finitely many possible Hilbert polynomials, hence a finite number of families of Calabi-Yau threefolds embedded in $I\!P^N$! In particular there should be a bound on their Euler number depending only on N. Drawing on some ideas in [CKN] it is clear that a good bound on the Euler number can be given as soon as one has an effective bound on the least integer m such that $h^1(T_X(-m)) = 0$. This in turn can be estimated by applying some standard projective techniques ([E]) (that give generally better results than vanishing theorems). With these methods we found an explicit bound on the Euler number. A nice feature of the bound is that not only it works for smooth Calabi-Yau threefolds, but also for a wide class of singular ones. Our result is

Theorem (1.1). *Let X be a smooth irreducible Calabi-Yau threefold and let L be a globally generated line bundle on X such that the morphism $\varphi_L : X \to I\!PH^0(L)$ is birational. Set $N = dim|L|, d = L^3, \varepsilon_1 = \max\{0, 5-N\}, \varepsilon_2 = \max\{0, 7-N\}$ and denote by $\overline{X} = \varphi_L(X), \overline{C}$ a general curve section of $\overline{X}$ and $\rho(X)$ the Picard number of X. The following bounds hold for the Euler number of X:*

If $(*)$ $\overline{C}$ has at most ordinary singularities we have the two lower bounds

$$(1.2) \qquad e(X) \geq 2\rho(X) - \min\{2d + 60N + 82 + 6\varepsilon_1 + 4\varepsilon_2, 8d + 30N + 94 + 8\varepsilon_1 + 6\varepsilon_2\}$$

$$(1.3) \qquad e(X) \geq -\min\{2d + 58N + 80 + 6\varepsilon_1 + 4\varepsilon_2, 8d + 28N + 92 + 8\varepsilon_1 + 6\varepsilon_2\}.$$

If $(**)$ *some multiple of* L *does not contract divisors to points then*

$$e(X) \leq 2\rho(X) \leq 4d + 20N + 24. \tag{1.4}$$

If $(*)$ *and* $(**)$ *hold then*

$$e(X) \leq 22d - 24N + 68 + 2\varepsilon_1 + 2\varepsilon_2. \tag{1.5}$$

We recall that d satisfies the inequalities $3N - 7 \leq d \leq 6N + 6$, the first of which is Castelnuovo-Beauville's inequality $K_S^2 \geq 3p_g - 7$ ([Ca], [B]) applied to a general $S \in |L|$. In particular the above bounds on the Euler number can be written only in terms of N. The bound in (1.2) is better than the one in (1.3) if and only if $\rho(X) \geq N + 1$; (1.5) is better than (1.4) only for $N \leq 8$ and low values of d. As the line bundle L in the theorem is big and nef we also have $N+1 = \frac{1}{12}L \cdot c_2(X) + \frac{1}{6}L^3$ by Riemann-Roch and Kawamata-Viehweg vanishing, whence the inequalities of the theorem can also be translated into inequalities between $e(X), L \cdot c_2(X)$ and L^3 (somehow as proposed by Wilson in [W]). Moreover as $\frac{1}{2}e(X) = \rho(X) - h^1(T_X)$ we can deduce an upper bound on $h^1(T_X)$ (which improves the one given in [CK]). We do not know how sharp our bounds are. Many examples of Calabi-Yau threefolds constructed are complete intersections in weighted projective spaces and it is not a priori clear which is the smallest globally generated and birational line bundle on them. By a result of Oguiso and Peternell [OP] if $\mathcal{L}$ is an ample line bundle on a Calabi-Yau threefold then $m\mathcal{L}$ is birational for $m \geq 5$, globally generated for $m \geq 7$ and very ample for $m \geq 14$; also if $\mathcal{L}$ is ample and globally generated Gallego and Purnaprajna [GP] proved that $m\mathcal{L}$ is very ample (in fact normally generated) for $m \geq 4$, but applying these results in the case of complete intersections in weighted projective spaces gives bounds far from the actual Euler number.

In the case of threefolds of general type there has been already an attempt to study the geography of their Chern numbers, at least when the canonical bundle has some positivity properties (see [H2], [Li]). Recently Chang, Kim and Nollet [CKN] gave a quadratic bound on the Euler number of threefolds with ample canonical bundle. Applying our technique we improved this to a bound linear in K_X^3. As the functions appearing in the bound are quite long, we state here a non explicit version of this bound. The explicit version is Theorem (4.12).

Theorem (1.6). *Let X be a smooth irreducible threefold such that there exists an integer $r \geq 1$ for which rK_X is globally generated and birational and denote by $\overline{X} = \varphi_{rK_X}(X), \overline{C}$ a general curve section of $\overline{X}$. Then there exist eight functions linear in K_X^3 of type $\alpha_i(r)K_X^3 + \beta_i(r, X), 1 \leq i \leq 8$ such that the following bounds hold for the Euler number of X:*

If $(*)$ $\overline{C}$ has at most ordinary singularities then

$$e(X) \geq \max_{1 \leq i \leq 6} \{\alpha_i(r)K_X^3 + \beta_i(r, X)\}. \tag{1.7}$$

If $(**)$ some multiple of K_X does not contract divisors to points then

$$e(X) \leq \alpha_7(r)K_X^3 + \beta_7(r, X). \tag{1.8}$$

If $(*)$ and $(**)$ hold then

$$e(X) \leq \alpha_8(r)K_X^3 + \beta_8(r, X). \tag{1.9}$$

For a smooth threefold X with ample canonical bundle results of Lee ([L1], [L2]) show that rK_X is globally generated if $r \geq 4$, separates distinct points if $r \geq 6$ and very ample if $r \geq 10$. Helmke [He] recently proved that in fact $5K_X$ separates distinct points. In particular, if K_X is ample, in Theorem (1.6) we can assume $r \leq 5$ in (1.8) and $r \leq 10$ in (1.7) and (1.9). From Lee's and Helmke's work it is reasonable to suspect that $r \leq 6$ should give at most ordinary singularities of the curve section, but no result is available yet (Fujita's conjecture predicts that $6K_X$ is very ample). We also recall that when K_X is very ample there is a simple bound $e(X) \leq 7K_X^3 - 48\chi$ due to Van de Ven (see the introduction of [H2]).

The linearity on the bound in the above theorem has a nice consequence. In fact in the case of threefolds of general type, for example with ample canonical bundle, an open question ([H2]) was whether the region described by the Chern ratios is bounded. We give an affermative answer (see also [CS]).

Corollary (1.10). *The region described by the Chern ratios $\frac{c_3}{c_1c_2}, \frac{c_1^3}{c_1c_2}$ of smooth irreducible threefolds with ample canonical bundle is bounded.*

It can be seen from the proof that the same boundedness also holds for the set of smooth irreducible threefolds X for which there exists an integer $r \geq 1$ (independently of X) such that rK_X is globally generated, birational, some multiple of K_X does not contract divisors to points and, if $\overline{X} = \varphi_{rK_X}(X)$, a general curve section of $\overline{X}$ has at most ordinary singularities.

An interesting case of a threefold of general type whose canonical bundle is birational, does not contract divisors to points but is not ample, is the small resolution of a nodal complete intersection threefold, that is the resolution obtained by replacing each node by a rational curve. By standard topological arguments (see for example [Hi]) it is clear that in such a case an upper bound on the Euler number gives an upper bound on the number of nodes. The general problem of finding the maximum number of nodes of hypersurfaces is a very classical one, going back to classical projective geometers (Severi, B. Segre, Basset and others), until more recent work of Bruce, Miyaoka, Varchenko, Chmutov and Givental (see for example [Br], [M1], [V], [Gi]). In the case of complete intersections, combining results of Kleiman [K] and Gaffney [Ga], an upper bound can be deduced (see Remark (4.27)). Applying our techniques we improve (in many cases) Kleiman-Gaffney's bound (again for an explicit formula see Theorem (4.26)):

Theorem (1.11). *Let $X \subset \mathbb{P}^{n+3}, n \geq 1$, be an irreducible complete intersection threefold of type $(d_1, \dots, d_n)$ having δ nodes and no other singularities. Then there is a function $B(d_1, \dots, d_n)$ such that*

$$\delta \leq B(d_1, \dots, d_n).$$

A similar bound can be given for smoothable nodal Calabi-Yau threefolds (Remark (4.28)).

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2. BOUNDING COHOMOLOGY OF TANGENT AND NORMAL BUNDLES

In this section we will let Y be a smooth irreducible variety of dimension n and A a globally generated line bundle on Y such that the associated map $\varphi_A : Y \to I\!P^r = I\!PH^0(A)$ is birational. We denote by N_{φ_A} the normal bundle of the map φ_A, so that we have an exact sequence

$$0 \to T_Y \to \varphi_A^* T_{I\!P^r} \to N_{\varphi_A} \to 0. \tag{2.1}$$

Our interest will focus on giving bounds on the dimensions of the cohomologies of negative twists of the tangent and normal bundles of Y. We start with the following standard fact.

Lemma (2.2). *With Y and A as above we have*

(2.3) $h^0(\varphi_A^* T_{I\!P^r} \otimes A^{-m}) = 0$ if $m \geq 2, n \geq 2$;

(2.4) $h^1(\varphi_A^* T_{I\!P^r} \otimes A^{-m}) = 0$ if either $m \geq 2$ and $n \geq 3$ or $m \geq 3$ and $n \geq 2$ or $n = m = 2, h^1(\mathcal{O}_Y) = 0$ and $g(A) > 0$;

(2.5) $h^0(N_{\varphi_A} \otimes A^{-m}) = h^1(T_Y \otimes A^{-m})$ if either $m \geq 2$ and $n \geq 3$ or $m \geq 3$ and $n \geq 2$ or $n = m = 2, h^1(\mathcal{O}_Y) = 0$ and $g(A) > 0$.

Proof: By the Euler sequence

$$0 \to A^{-m} \to H^0(A)^* \otimes A^{1-m} \to \varphi_A^* T_{I\!P^r} \otimes A^{-m} \to 0 \tag{2.6}$$

we see immediately (2.3) and also (2.4) for $n \geq 3$ by Kawamata-Viehweg vanishing theorem. In the case $n = 2$, (2.6) gives that $h^1(\varphi_A^* T_{I\!P^r} \otimes A^{-m})$ is the cokernel of the multiplication map $\mu : H^0(A) \otimes H^0(\omega_Y \otimes A^{m-1}) \to H^0(\omega_Y \otimes A^m)$. To see that μ is surjective under the hypotheses of (2.4), let $C \in |A|$ be a general curve and $V = Im\{H^0(A) \to H^0(A_{|C})\}$. We have a diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(\mathcal{O}_Y) \otimes H^0(\omega_Y \otimes A^{m-1}) & \xrightarrow{\cong} & H^0(\omega_Y \otimes A^{m-1}) \\
\downarrow & & \downarrow \\
H^0(A) \otimes H^0(\omega_Y \otimes A^{m-1}) & \xrightarrow{\mu} & H^0(\omega_Y \otimes A^m) \\
\downarrow & & \downarrow \\
V \otimes H^0(\omega_Y \otimes A^{m-1}) & \xrightarrow{\nu} & H^0(\omega_C \otimes A_{|C}^{m-1}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

and we will be done if we show that ν is surjective. Now ν is the composition of the maps $V \otimes H^0(\omega_Y \otimes A^{m-1}) \xrightarrow{\alpha} V \otimes H^0(\omega_C \otimes A_{|C}^{m-2}) \xrightarrow{\beta} H^0(\omega_C \otimes A_{|C}^{m-1})$. The map α is surjective since $H^1(\omega_Y \otimes A^{m-2}) = 0$ for $m \geq 3$ by Kawamata-Viehweg vanishing theorem and for

$m = 2$ by hypothesis. The map β is surjective by [G, Theorem 4.e.1] if $m \geq 3$ and by the base point free pencil trick if $m = 2$. (In both cases it is used that $g(C) = g(A) > 0$). As for (2.5) it is an obvious consequence of (2.3), (2.4) and the normal bundle sequence (2.1). ∎

By (2.5) it is clear that one can study the twists of either the tangent or the normal bundle. In the next Proposition we will use a standard projective result of L. Ein [E] to bound the normal bundle case.

Given integers N, d, r, q we define some functions used below: $M_i = \max\{0, \frac{(8-2i)r+1}{2r}d + 8 - 2N\}$ for $-1 \leq i \leq 2, M_3 = \max\{0, \frac{2r+1}{2r}d + 9 - 2N\}$, $M_4 = \max\{0, \frac{d}{2r} + 2q + 7 - N\}$, $M_5 = \begin{cases} \max\{0, 6-N\} & \text{if } r \geq 2 \\ \max\{0, 7+2q-N\} & \text{if } r = 1 \end{cases}$, $M_6 = \begin{cases} 0 & \text{if } r \geq 2 \text{ or } r = 1, N \geq 5 \\ 1 & \text{if } r = 1, N = 4 \end{cases}$. Also we recall that $\varepsilon_1 = \max\{0, 5-N\}, \varepsilon_2 = \max\{0, 7-N\}$.

Proposition (2.7). *Let X be a smooth irreducible threefold and L a globally generated line bundle on X such that the map $\varphi : X \to \mathbb{P}^N = \mathbb{P}H^0(L)$ is birational and the general curve section $\overline{C}$ of its image $\overline{X} = \varphi(X)$ has at most ordinary singularities. Let $S \in |L|$ be a general surface and $C \in |Im\{H^0(L) \to H^0(L_{|S})\}|$ a general curve and denote by φ_C the map induced by the linear system $Im\{H^0(L) \to H^0(L_{|C})\}$. Assume $d = L^3 \geq 3, N \geq 4$. Then*

$$h^0(N_{\varphi_C} \otimes L_{|C}^{-m}) \leq \max\{0, h^1(L_{|C}^{m-3}) - 2N + 8\} \text{ for } m \geq 2. \tag{2.8}$$

In particular if X is a Calabi-Yau threefold we get

$$h^1(T_X \otimes L^{-m}) \leq \begin{cases} 0 & \textit{for } m \geq 6 \\ \varepsilon_1 & \textit{for } m = 5 \\ \varepsilon_2 & \textit{for } m = 4, N \geq 5 \\ \varepsilon_2 + 2 & \textit{for } m = N = 4 \\ d - 2N + 9 + 3\varepsilon_1 + 2\varepsilon_2 & \textit{for } m = 3 \\ 4d - 6N + 26 + 4\varepsilon_1 + 3\varepsilon_2 & \textit{for } m = 2 \end{cases}. \tag{2.9}$$

If X is of general type and such that there exists an integer $r \geq 1$ for which $L = rK_X$ satisfies the hypotheses above, we have,

$$h^1(T_X \otimes L^{-m}) \leq \begin{cases} 0 & \textit{for } m \geq 7 \\ \sum_{i=m}^{6} (i+1-m)M_i & \textit{for } 3 \leq m \leq 6 \end{cases}. \tag{2.10}$$

Proof: Set $W = Im\{H^0(L) \to H^0(L_{|S})\}, V = Im\{H^0(L) \to H^0(L_{|C})\}$, and denote by φ_S the map induced by W. By hypothesis it follows that S and C are smooth irreducible and φ_C, the map induced by V, is birational and unramified. If $N = 4$ we have $N_{\varphi_C} \cong \omega_C \otimes L^3_{|C}$. When $N \geq 5$ there is an exact sequence

$$0 \to \bigoplus_{j=1}^{N-4} L^{1-m}_{|C}(2P_j) \to N_{\varphi_C} \otimes L^{-m}_{|C} \to \omega_C \otimes L^{3-m}_{|C}(-2\sum_{j=1}^{N-4} P_j) \to 0$$

proved by Ein [E, Lemma 4] for $N-4$ general points $P_j \in C$ and (2.8) follows. To see (2.9) and (2.10), observe that, as S and C are general in their linear systems, we have $(N_\varphi)_{|S} = N_{\varphi_S}, (N_{\varphi_S})_{|C} = N_{\varphi_C}$ and there are two exact sequences

$$0 \to N_\varphi \otimes L^{-1} \to N_\varphi \to N_{\varphi_S} \to 0 \tag{2.11}$$

and

$$0 \to N_{\varphi_S} \otimes L^{-1}_{|S} \to N_{\varphi_S} \to N_{\varphi_C} \to 0. \tag{2.12}$$

Now suppose that X is a Calabi-Yau threefold. Then $\omega_S = L_{|S}, q(S) = 0, \omega_C = L^2_{|C}$ and $V = H^0(L_{|C})$ has dimension $N-1$. Hence

$$h^1(L^{m-3}_{|C}) = h^0(L^{5-m}_{|C}) = \begin{cases} 0 & \text{for } m \geq 6 \\ 1 & \text{for } m = 5 \\ N-1 & \text{for } m = 4 \\ g = d+1 & \text{for } m = 3 \\ 2d & \text{for } m = 2 \end{cases}.$$

By (2.8), Ein's exact sequence and the fact that in this case $d \geq 3N-7$ ([Ca], [B]), we get

$$h^0(N_{\varphi_C} \otimes L^{-m}_{|C}) \leq \begin{cases} 0 & \text{for } m \geq 6 \\ \varepsilon_1 & \text{for } m = 5 \\ \varepsilon_2 & \text{for } m = 4 \\ d-2N+9 & \text{for } m = 3 \\ 2d-2N+8 & \text{for } m = 2 \\ 3d-N+4 & \text{for } m = 1 \\ 4d+N^2-7N+12 & \text{for } m = 0 \\ (N+1)d & \text{for } m = -1 \end{cases}. \tag{2.13}$$

If $m \geq 6$ or $m = 5$ and $N \geq 5$ or $m = 4$ and $N \geq 7$, (2.12) and (2.13) imply that $H^0(N_{\varphi_S} \otimes L^{-m}_{|S}) = 0$, as C is a general element of $|L_{|S}|$. The same argument applied to

(2.11) gives $H^0(N_\varphi \otimes L^{-m}) = 0$, hence (2.9) (by (2.5)) in these cases. If $m = 5, N = 4$ or $m = 4$ and $5 \le N \le 6$ by (2.11), (2.12) we get $h^0(N_\varphi \otimes L^{-m}) \le h^0(N_{\varphi_S} \otimes L^{-m}_{|S}) \le h^0(N_{\varphi_C} \otimes L^{-m}_{|C})$ and conclude with (2.13) and (2.5). Similarly if $m = N = 4$ we get, by the above cases, $h^0(N_{\varphi_C} \otimes L^{-4}_{|C}) \le 7 - N, h^0(N_{\varphi_S} \otimes L^{-4}_{|S}) \le 8 - N$ and $h^0(N_\varphi \otimes L^{-4}) \le 9 - N$ and use again (2.5). If $m = 3$, using the same exact sequences, we deduce

$$\begin{aligned} h^1(T_X \otimes L^{-3}) = h^0(N_\varphi \otimes L^{-3}) \le h^0(N_\varphi \otimes L^{-4}) + h^0(N_{\varphi_S} \otimes L^{-3}_{|S}) \le \\ \le 3h^0(N_{\varphi_C} \otimes L^{-5}_{|C}) + 2h^0(N_{\varphi_C} \otimes L^{-4}_{|C}) + h^0(N_{\varphi_C} \otimes L^{-3}_{|C}) \le \\ \le d - 2N + 9 + 3\varepsilon_1 + 2\varepsilon_2. \end{aligned}$$

For $m = 2$, we have

$$\begin{aligned} h^1(T_X \otimes L^{-2}) = h^0(N_\varphi \otimes L^{-2}) \le h^0(N_\varphi \otimes L^{-3}) + h^0(N_{\varphi_S} \otimes L^{-2}_{|S}) \le \\ \le d - 2N + 9 + 3\varepsilon_1 + 2\varepsilon_2 + h^0(N_{\varphi_C} \otimes L^{-3}_{|C}) + h^0(N_{\varphi_C} \otimes L^{-2}_{|C}) + h^0(N_{\varphi_S} \otimes L^{-4}_{|S}) \le \\ \le 4d - 6N + 26 + 4\varepsilon_1 + 3\varepsilon_2. \end{aligned}$$

Now let us do the case X of general type. First we claim that

$$(2.14) \qquad h^0(N_{\varphi_C} \otimes L^{-m}_{|C}) \le \begin{cases} 0 & \text{for } m \ge 7 \\ M_m & \text{for } 2 \le m \le 6 \\ N - 4 + M_1 & \text{for } m = 1 \\ (N-4)(2q - 1 + N) + M_0 & \text{for } m = 0 \\ (N-4)(\frac{2r-1}{2r} d + 2 + \alpha) + M_{-1} & \text{for } m = -1 \end{cases}$$

where $\alpha = \begin{cases} p_g(X) & \text{if } r \ge 2 \\ N + 2q - 1 & \text{if } r = 1 \end{cases}$. To see (2.14) observe that by hypothesis we have $K_S = (r+1)K_{X|S}, K_C = (2r+1)K_{X|C}$. Hence $h^1(L^{m-3}_{|C}) = h^0((5r + 1 - mr)K_{X|C})$. This and (2.8) give (2.14) for $m \ge 6$. To do the case $m = 5$, we will first prove

$$(2.15) \qquad h^0(K_{X|C}) \le \begin{cases} N - 2 & \text{if } r \ge 2 \\ N + 2q - 1 & \text{if } r = 1 \end{cases}.$$

Of course (2.15) and (2.8) imply (2.14) for $m = 5$. To prove (2.15) for $r \ge 2$, observe that Kawamata-Viehweg vanishing gives easily $h^0(K_{X|C}) = h^0(K_{X|S}) = h^0(K_X)$. Now either $h^0(K_X) = 0$ and (2.15) is true or $h^0(K_X) \ne 0$ and hence $h^0(K_X) \le h^0((r-1)K_X) \le h^0(rK_X) - 3 = N - 2$. (The last inequality is because rK_X has no base locus). If $r = 1$, the bound (2.15) is clear just by restricting to S and C. As above, we conclude that (2.10) holds for $m \geq 5$. If $m \leq 4$, we will give a bound on $h^1(L_{|C}^{m-3})$; (2.14) and (2.10) then follow in the usual way from (2.8), (2.11), (2.12) and this bound.

For $m = 4$, we have $h^1(L_{|C}) = h^0((r+1)K_{X|_C}) = (r+1)K_X \cdot C - g + 1 + h^1((r+1)K_{X|_C})$. Now $K_X \cdot C = \frac{1}{r}d, g - 1 = \frac{2r+1}{2r}d$ and $h^1((r+1)K_{X|_C}) \leq 2q - 1 + N$, as it can be easily seen from the exact sequences

$$0 \to \mathcal{O}_S(K_{X|_S}) \to \mathcal{O}_S((r+1)K_{X|_S}) \to \mathcal{O}_C((r+1)K_{X|_C}) \to 0$$

and

$$0 \to \mathcal{O}_X((1-r)K_X) \to \mathcal{O}_X(K_X) \to \mathcal{O}_S(K_{X|_S}) \to 0.$$

When $m = 3$ we have $h^1(\mathcal{O}_C) = g = \frac{2r+1}{2r}d + 1$. The rest of the cases of (2.14) is proved in the same way as above by using Ein's exact sequence. ∎

Remark **(2.16).** From the above proof it is clear that bounds on $h^1(T_X \otimes L^{-m})$ do follow as soon as one has a good bound on $h^0(N_{\varphi_C} \otimes L_{|C}^{-m})$. As is well-known the latter cohomology is just the corank of a suitable Gaussian map. For example if X is a Calabi-Yau threefold one has $h^0(N_{\varphi_C} \otimes L_{|C}^{-m}) = \mathrm{cork}\Phi_{L_{|C}, L_{|C}^{m+1}}$, for $m \geq 2$. However the known results about Gaussian maps do not apply nicely here to give better estimates than (2.8), mainly because of the fact that the degree of $L_{|C}$ is $g+1$.

Remark **(2.17).** The above Proposition is the only place (besides of course its several applications) in the article where the hypothesis that the curve $\overline{C}$ has at most ordinary singularities is used. In fact it is possible to give similar bounds both on $h^1(T_X \otimes L^{-m})$ and on the Euler number also in the case the curve $\overline{C}$ does not have ordinary singularities. On the other hand, as these bounds get progressively worst, we will omit them.

3. THE BOUND FOR CALABI-YAU THREEFOLDS

We now let X be a smooth irreducible Calabi-Yau threefold and L a globally generated line bundle on X such that the map $\varphi : X \to \mathbb{P}^N = \mathbb{P}H^0(L)$ is birational. Again we choose $S \in |L|$ and $C \in |L_{|S}|$ general elements and denote by $\overline{X} = \varphi(X), \overline{C}$ a general curve section of $\overline{X}, N+1 = h^0(L)$ and $d = L^3$. We start with some bounds on the cohomologies of the twists of the tangent sheaf of S.

Lemma (3.1). *With notation as above we have the following bounds*

(3.2) $h^1(T_S) \leq 10N + 11$;

(3.3) $h^1(T_S \otimes L_{|S}^{-1}) \leq 11N + 10$;

(3.4) $h^1(T_S \otimes L_{|S}) \leq 2d + 10N + 12$;

(3.5) $\sum_{j=0}^{m-1} h^0(L_{|S}^j) + \sum_{j=1}^{m} h^0(\Omega_S^1 \otimes L_{|S}^j) \leq \begin{cases} 18d + 4N + 4 & \text{if } m = 3 \\ 7d + 2N + 2 & \text{if } m = 2 \end{cases}.$

If in addition $\overline{C}$ has at most ordinary singularities then also

(3.6) $\sum_{j=0}^{5} h^1(T_S \otimes L_{|S}^{-j}) \leq \min\{d + 30N + 41 + 3\varepsilon_1 + 2\varepsilon_2, 4d + 15N + 47 + 4\varepsilon_1 + 3\varepsilon_2\}.$

Proof: By [C, Theorem C], we have $h^1(T_S) \leq 10\chi(\mathcal{O}_S) + 1 = 10N + 11$. From

$$0 \to T_S \otimes L_{|S}^{-1} \to T_S \to T_{S|_C} \to 0$$

and

$$0 \to L_{|C}^{-2} \to T_{S|_C} \to L_{|C} \to 0$$

we easily deduce $h^1(T_S \otimes L_{|S}^{-1}) \leq h^1(T_S) + h^0(T_{S|_C}) \leq 11N + 10$. Similarly, we get the bounds (3.4) and (3.6) by restricting to C and using (2.4), (2.5), (2.12) and (2.13). The bound (3.5) is obtained in the same way by using Riemann-Roch. ∎

We will now prove our bounds on the Euler number of X.

Proof of Theorem (1.1):

By Riemann-Roch, we have $\frac{1}{2}e(X) = \chi(T_X) \leq h^2(T_X) = \rho(X) = h^1(\Omega_X^1)$. Consider the exact sequences

$$0 \to \Omega_X^1 \otimes L^{-1} \to \Omega_X^1 \to \Omega_{X|_S}^1 \to 0 \tag{3.7}$$

and

$$0 \to L_{|S}^{-1} \to \Omega_{X|_S}^1 \to \Omega_S^1 \to 0. \tag{3.8}$$

By a theorem of L. Migliorini [Mi, Theorem 4.6], we have, under the hypothesis (1.4), that $H^1(\Omega_X^1 \otimes L^{-1}) = 0$. Also $H^1(L_{|S}^{-1}) = 0$ by Kawamata-Viehweg. Hence $h^1(\Omega_X^1) \leq h^1(\Omega_{X|_S}^1) \leq h^1(\Omega_S^1) = h^1(T_S \otimes L_{|S}) \leq 2d + 10N + 12$ by (3.4) and we get (1.4). To prove (1.5) we show instead that $h^1(T_S \otimes L_{|S}) \leq 11d - 12N + 34 + \varepsilon_1 + \varepsilon_2$. From the exact sequence

$$0 \to T_S \otimes L_{|S} \to \varphi_S^* T_{\mathbb{P}^{N-1}} \otimes L_{|S} \to N_{\varphi_S} \otimes L_{|S} \to 0$$

and $H^0(T_S \otimes L_{|S}) = H^1(\mathcal{O}_S) = 0$ (since $L_{|S} = \omega_S$) we deduce

$$h^1(T_S \otimes L_{|S}) \leq h^0(N_{\varphi_S} \otimes L_{|S}) + h^1(\varphi_S^* T_{I\!P^{N-1}} \otimes L_{|S}) - h^0(\varphi_S^* T_{I\!P^{N-1}} \otimes L_{|S}).$$

Now by the Euler sequence

$$0 \to \varphi_S^* \Omega^1_{I\!P^{N-1}} \to H^0(L_{|S}) \otimes L^{-1}_{|S} \to \mathcal{O}_S \to 0$$

we get $H^0(\varphi_S^* \Omega^1_{I\!P^{N-1}}) = 0, h^1(\varphi_S^* T_{I\!P^{N-1}} \otimes L_{|S}) - h^0(\varphi_S^* T_{I\!P^{N-1}} \otimes L_{|S}) = -\chi(\varphi_S^* \Omega^1_{I\!P^{N-1}}) = -N^2 - Nd + 1$. By (2.12) and (2.13) we also have $h^0(N_{\varphi_S} \otimes L_{|S}) \leq \sum_{i=-5}^{1} h^0(N_{\varphi_C} \otimes L^i_{|C}) \leq (N+11)d + N^2 - 12N + 33 + \varepsilon_1 + \varepsilon_2$ and (1.5) is proved.

To see (1.2) we use the exact sequences

$$0 \to T_X \otimes L^{-1} \to T_X \to T_{X|_S} \to 0 \tag{3.9}$$

and

$$0 \to T_S \to T_{X|_S} \to L_{|S} \to 0. \tag{3.10}$$

By (2.9) and (3.9), we deduce $h^1(T_X) \leq \sum_{i=0}^{5} h^1(T_{X|_S} \otimes L^{-i}_{|S})$. As $h^1(L^j_{|S}) = 0$ for all j by Kawamata-Viehweg, (3.10) gives $h^1(T_X) \leq \sum_{i=0}^{5} h^1(T_S \otimes L^{-i}_{|S})$ and this, together with Lemma (3.1) and $\frac{1}{2}e(X) = \rho(X) - h^1(T_X)$, implies (1.2).

Finally, to prove (1.3), we have

(3.11)

$$\chi(T_X \otimes L^{-m}) = -\frac{1}{2}(m^3+3m)d + 9m(N+1) + \frac{1}{2}e(X) \geq -h^1(T_X \otimes L^{-m}) - h^3(T_X \otimes L^{-m}).$$

For $h^3(T_X \otimes L^{-m}) = h^0(\Omega^1_X \otimes L^m)$ notice that, by (3.7) and (3.8), we deduce

$$h^0(\Omega^1_X \otimes L^m) \leq \sum_{j=1}^{m} h^0(\Omega^1_{X|_S} \otimes L^j_{|S}) \leq \sum_{j=0}^{m-1} h^0(L^j_{|S}) + \sum_{j=1}^{m} h^0(\Omega^1_S \otimes L^j_{|S}).$$

Applying (2.9) and (3.5) to (3.11), for $m = 2, 3$, we get (1.3). ■

Remark **(3.12).** In the cases $N = 4$, $N = 5$ and $\dim Sing(\overline{X}) = 0$, it follows that $\overline{X}$ is a complete intersection in $I\!P^N$ (for $N = 5$ it is a consequence of [BC]). However, as there is no further requirement on the singularities of $\overline{X}$, it seems interesting to give a bound on the Euler number even in these cases.

4. THE BOUNDS FOR THREEFOLDS OF GENERAL TYPE.

We let X be a smooth irreducible threefold with the property that there exists an integer $r \geq 1$ such that rK_X is globally generated and defines a birational morphism. We set $L = rK_X, \varphi_L : X \to I\!P^N = I\!PH^0(L), \overline{X} = \varphi_L(X), \overline{C}$ a general curve section of $\overline{X}$ and choose general sections $S \in |L|$ and $C \in |Im\{H^0(L) \to H^0(L_{|S})\}|$. Let $\chi = \chi(\mathcal{O}_X), \chi_S = \chi(\mathcal{O}_S), q = q(X), d = L^3$ and M_i as defined above Proposition (2.7). As in section 3, we have

Lemma (4.1). *With the above notation and hypotheses, the following bounds hold*

$$\chi_S \leq \frac{1}{2}(1 + \frac{2}{r} + \frac{1}{r^2})d + 3 - q; \tag{4.2}$$

(4.3) If $(r+1)K_X$ is also globally generated and birational then

$$\chi_S \leq \frac{1}{3}(1 + \frac{2}{r} + \frac{1}{r^2})d - q + \frac{10}{3};$$

(4.4) $h^2(T_S) \leq (3 + \frac{3}{r})d + N + \begin{cases} q-2 & \text{if } r \geq 2 \\ 3q & \text{if } r = 1 \end{cases}$*; if $(r+1)K_X$ is also globally generated then* $h^2(T_S) \leq (2 + \frac{4}{r} + \frac{2}{r^2})d + q + 1$*;*

(4.5) $h^1(T_S) = 10\chi_S - (2 + \frac{4}{r} + \frac{2}{r^2})d + h^2(T_S)$*;*

(4.6) $h^1(T_S \otimes L_{|S}^{-1}) \leq 10\chi_S - (2 + \frac{4}{r} + \frac{2}{r^2})d + h^2(T_S) + N + 2q - 1$*;*

(4.7) $h^1(T_S \otimes L_{|S}^{-2}) \leq 10\chi_S - (2 + \frac{4}{r} + \frac{2}{r^2})d + h^2(T_S) + N + 2q$*.*

If, in addition, $\overline{C}$ has at most ordinary singularities, we also have

(4.8) $h^1(T_S \otimes L_{|S}^{-3}) \leq \min\{10\chi_S - (2 + \frac{4}{r} + \frac{2}{r^2})d + h^2(T_S) + N + 2q, M_6 + M_5 + M_4 + M_3\}$*;*

(4.9) $h^1(T_S \otimes L_{|S}^{-4}) \leq \min\{10\chi_S - (2 + \frac{4}{r} + \frac{2}{r^2})d + h^2(T_S) + N + 2q, M_6 + M_5 + M_4\}$*;*

(4.10) $h^1(T_S \otimes L_{|S}^{-5}) \leq M_6 + M_5$*;*

(4.11) $h^1(T_S \otimes L_{|S}^{-6}) \leq M_6$*.*

Proof: As S is minimal of general type we have Noether's inequality $\chi_S \leq \frac{1}{2}(1 + \frac{2}{r} + \frac{1}{r^2})d + 3 - q$. Under the hypothesis in (4.3), $K_S = (r+1)K_{X|_S}$ is globally generated and birational, hence, by [Ca], [B], we have $p_g(S) \leq \frac{1}{3}K_S^2 + \frac{7}{3} = \frac{(r+1)^2}{3r^2}d + \frac{7}{3}$. To see the first inequality of (4.4) notice that $h^2(T_S) = h^0(\Omega_S^1((r+1)K_{X|_S}))$. Now from

$$0 \to \Omega_S^1(K_{X|_S}) \to \Omega_S^1((r+1)K_{X|_S}) \to \Omega_{S|_C}^1((r+1)K_{X|_C}) \to 0$$

and

$$0 \to \mathcal{O}_C(K_{X|_C}) \to \Omega_{S|_C}^1((r+1)K_{X|_C}) \to \mathcal{O}_C((3r+2)K_{X|_C}) \to 0$$

we deduce $h^2(T_S) \leq h^0(\Omega_S^1(K_{X|_S})) + h^0(\mathcal{O}_C(K_{X|_C})) + \frac{4r+3}{2r}d$. By

$$0 \to \Omega_S^1((1-r)K_{X|_S}) \to \Omega_S^1(K_{X|_S}) \to \Omega_{S|_C}^1(K_{X|_C}) \to 0$$

$$0 \to \mathcal{O}_C((1-r)K_{X|_C}) \to \Omega_{S|_C}^1(K_{X|_C}) \to \mathcal{O}_C((2r+2)K_{X|_C}) \to 0$$

and $h^0(\Omega_S^1((1-r)K_{X|_S})) \leq q$, we see that $h^0(\Omega_S^1(K_{X|_S})) \leq q + \frac{2r+3}{2r}d + \begin{cases} 0 & \text{if } r \geq 2 \\ 1 & \text{if r=1} \end{cases}$ and applying it above, together with (2.15), we get the first part of (4.4). The second inequality of (4.4) is in [C, Theorem C]. (4.5) is just Riemann-Roch and the fact that $h^0(T_S) = 0$ for a minimal surface of general type. All the other inequalities are obtained in the same way as Proposition (3.1). Their proof is left to the reader. ∎

We now give our result about the Euler number of X.

Theorem (4.12). *Let X be a smooth irreducible threefold such that there exists an integer $r \geq 1$ for which rK_X is globally generated and birational. Set $N = dim|rK_X|, q = q(X), \chi = \chi(\mathcal{O}_X)$, $S \in |rK_X|$ a general section and $\eta = \begin{cases} 8 & \text{if } r \geq 2 \\ 2-2q & \text{if } r = 1 \end{cases}$. Denote by $\overline{X} = \varphi_{rK_X}(X), \overline{C}$ a general curve section of $\overline{X}$. Then the following bounds hold for the Euler number of X:*

If $(*)$ $\overline{C}$ has at most ordinary singularities we have the two lower bounds

(4.13) $(r \geq 2)$

$$e(X) \geq \max\{\frac{1}{3}(10r^3+45r^2+35r+3)K_X^3 - 10N + (200r+36)\chi - 12h^2(T_S) - 2M_5 - 22q + 8,$$

$$\frac{1}{3}(8r^3+36r^2+28r+3)K_X^3 - 8N + (160r+36)\chi - 10h^2(T_S) - 4M_5 - 2M_4 - 18q + 8,$$

$$(2r^3+9r^2+7r+1)K_X^3 - 6N + (120r+36)\chi - 8h^2(T_S) - 6M_5 - 4M_4 - 2M_3 - 14q + 8\}$$

$$(r=1)\qquad e(X)\geq \max\{31K_X^3-12N+234\chi-12h^2(T_S)-4M_6-2M_5-24q,$$

$$25K_X^3-10N+194\chi-10h^2(T_S)-6M_6-4M_5-2M_4-20q,$$

$$19K_X^3-8N+154\chi-8h^2(T_S)-8M_6-6M_5-4M_4-2M_3-16q\};$$

(4.14) $e(X)\geq$

$$\max\{\frac{1}{6}(22r^3+87r^2+71r+6)K_X^3-10N+(196r+38)\chi-12h^2(T_S)-4M_6-2M_5-20q+\eta,$$

$$\frac{1}{2}(6r^3+23r^2+19r+2)K_X^3-8N+(156r+38)\chi-10h^2(T_S)-6M_6-4M_5-2M_4-16q+\eta,$$

$$\frac{1}{6}(14r^3+51r^2+43r+6)K_X^3-6N+(116r+38)\chi-8h^2(T_S)-8M_6-6M_5-4M_4-2M_3-12q+\eta\}.$$

If $(**)$ some multiple of K_X does not contract divisors to points then

(4.15)

$$e(X)\leq \frac{1}{3}(10r^3-7r+3)K_X^3+2N+(38-40r)\chi+2h^2(T_S)+\begin{cases}-4 & \textit{if } r\geq 2\\ 2h^{1,1}(X)+6q-2 & \textit{if } r=1\end{cases}.$$

If $(*)$ and $(**)$ hold then

(4.16)

$$e(X)\leq \frac{1}{6}[-(2N+46)r^3+(3N+21)r^2-(N-1)r+6]K_X^3+[38+4r(N-1)]\chi+2\sum_{i=-1}^{6}M_i+$$

$$+4(N-4)(N+q)+\begin{cases}0 & \textit{if } r\geq 2\\ 2(N-4)(2q+1)+2h^{1,1}(X)+2q & \textit{if } r=1\end{cases}.$$

Note that $\chi_S=\frac{1}{12}(2r^3+3r^2+r)K_X^3-2r\chi$ hence by Lemma (4.1) we can deduce bounds on K_X^3 in terms of χ and similarly for $h^2(T_S)$. Also applying Yau-Tsuji's inequality ([Y], [T]) $c_2\cdot K_X\geq \frac{3}{8}K_X^3$, one easily gets that $K_X^3\leq -64\chi$ (in particular $\chi<0$) and $K_X^3\leq \frac{192}{32r^3-48r^2+22r-3}(N+1)+\begin{cases}0 & \text{if } r\geq 2\\ 1-q & \text{if } r=1\end{cases}$. The bound in (4.16) is better than the one in (4.15) only in a few cases.

Proof of Theorem (4.12): By Riemann-Roch we have

$$\chi(T_X)=-\frac{d}{2r^3}-19\chi+\frac{e(X)}{2}\leq h^0(T_X)+h^2(T_X)=h^3(\Omega_X^1(K_X))+h^1(\Omega_X^1(K_X)).$$

By the hypothesis in (4.15) and Migliorini's version of Kodaira vanishing [Mi, Theorem 4.6], we have $h^3(\Omega_X^1(K_X))=0$ and $h^1(\Omega_X^1((1-r)K_X))=0$ for $r\geq 2$. By the exact sequences

$$0\to\Omega_X^1((1-r)K_X)\to\Omega_X^1(K_X)\to\Omega_X^1(K_X)_{|S}\to 0$$

and

$$0 \to \mathcal{O}_S((1-r)K_{X|_S}) \to \Omega^1_X(K_X)_{|S} \to \Omega^1_S(K_{X|_S}) \to 0$$

we deduce $h^1(\Omega^1_X(K_X)) \leq h^1(\Omega^1_X(K_X)_{|S}) + p_1$, where $p_1 = \begin{cases} 0 & \text{if } r \geq 2 \\ h^{1,1}(X) & \text{if } r = 1 \end{cases}$ and similarly $h^1(\Omega^1_X(K_X)_{|S}) \leq h^1(T_S(rK_{X|_S})) + p_2$, where $p_2 = \begin{cases} 0 & \text{if } r \geq 2 \\ q & \text{if } r = 1 \end{cases}$. (This is because $q = h^1(\mathcal{O}_X) = h^1(\mathcal{O}_S)$ and $h^1(\mathcal{O}_S((1-r)K_{X|_S})) = 0$ for $r \geq 2$, by Kawamata-Viehweg). The bound (4.15) is now a consequence of the above inequalities and

$$h^1(T_S(rK_{X|_S})) \leq (2 + \frac{3}{2r})d + h^1(T_S) + \begin{cases} N-2 & \text{if } r \geq 2 \\ N+2q-1 & \text{if } r = 1 \end{cases}. \tag{4.17}$$

To see (4.17) we consider the exact sequences

$$0 \to T_S \to T_S(rK_{X|_S}) \to T_{S|_C}(rK_{X|_C}) \to 0$$

and

$$0 \to T_C(rK_{X|_C}) \to T_{S|_C}(rK_{X|_C}) \to \mathcal{O}_C(2rK_{X|_C}) \to 0.$$

We get $h^1(T_S(rK_{X|_S})) \leq h^1(T_S) + h^0(K_{X|_C}) + h^0((3r+2)K_{X|_C})$ and $h^0((3r+2)K_{X|_C}) = (3r+2)K_X \cdot C - g + 1 = \frac{4r+3}{2r}d$ by Riemann-Roch. Combining this with (2.15), we have (4.17). Hence (4.15) is proved. To prove (4.16) we give instead the upper bound

$$h^1(T_S(rK_{X|_S})) \leq (N - 8r + 3)\frac{d}{2r} - (N-1)\chi_S + \sum_{i=-1}^{6} M_i + 2(N-4)(N+q) +$$

$$+ \begin{cases} 0 & \text{if } r \geq 2 \\ (N-4)(2q+1) & \text{if } r = 1 \end{cases}.$$

The latter is proved much in the same way as we did for (1.5) and we just outline the proof, leaving the easy details to the reader. Twisting by $\mathcal{O}_S(rK_{X|_S})$ the pull-back by φ_S of the normal bundle sequence and of the Euler sequence we get

$$h^1(T_S(rK_{X|_S})) \leq h^0(N_{\varphi_S}(rK_{X|_S})) - \chi(\varphi_S^*\Omega^1_{\mathbb{P}^{N-1}}(K_{X|_S})).$$

Now $\chi(\varphi_S^*\Omega^1_{\mathbb{P}^{N-1}}(K_{X|_S})) = -(N-1)\chi_S - (2N(r-1)+1)\frac{d}{2r}$. Applying (2.12), (2.14) and the easy inequality $p_g(X) \leq \begin{cases} N-2 & \text{if } r \geq 2 \\ N & \text{if } r = 1 \end{cases}$ (this is as in (2.15)), we deduce $h^0(N_{\varphi_S} \otimes L_{|S}) \leq \sum_{i=-1}^{6} h^0(N_{\varphi_C} \otimes L^{-i}_{|C}) \leq \sum_{i=-1}^{6} M_i + (N-4)(2q + 2N + \frac{2r-1}{2r}d + \begin{cases} 0 & \text{if } r \geq 2 \\ 2q+1 & \text{if } r = 1 \end{cases})$, and (4.16) is proved.

To get the lower bound in (4.13), we recall that by [CKN, Lemma 1] and (2.10) we have

$$h^1(T_X) \leq \sum_{i=0}^{6} h^1(T_S \otimes L_{|S}^{-i}) + q + h^1(L_{|S}) \tag{4.18}$$

and

$$h^3(T_X) \leq q + h^2(T_S) + \begin{cases} 0 & \text{if } r \geq 2 \\ 1 & \text{if } r = 1 \end{cases}. \tag{4.19}$$

As $h^1(L) = h^1(\mathcal{O}_X(rK_X)) = \begin{cases} 0 & \text{if } r \geq 2 \\ h^2(\mathcal{O}_X) & \text{if } r = 1 \end{cases}$ from the exact sequence

$$0 \to \mathcal{O}_X \to L \to L_{|S} \to 0$$

we deduce

$$h^1(L_{|S}) \leq \begin{cases} h^2(\mathcal{O}_X) & \text{if } r \geq 2 \\ 2h^2(\mathcal{O}_X) & \text{if } r = 1 \end{cases}. \tag{4.20}$$

Note that $h^2(\mathcal{O}_X) = \chi + p_g + q - 1$. Applying these together with (4.18), (4.19), (4.20) and Lemma (4.1) in

$$-\frac{d}{2r^3} - 19\chi + \frac{e(X)}{2} = \chi(T_X) \geq -h^1(T_X) - h^3(T_X)$$

we get (4.13).

Finally to prove (4.14), we use the inequality

$$-\frac{d}{4r^3}(2m^3r^3 + 5m^2r^2 + 5mr + 2) - (19 + 18mr)\chi + \frac{e(X)}{2} = \tag{4.21}$$

$$= \chi(T_X \otimes L^{-m}) \geq -h^1(T_X \otimes L^{-m}) - h^3(T_X \otimes L^{-m})$$

for $m = 3, 4, 5$. An upper bound on $h^1(T_X \otimes L^{-m})$ is given in (2.10). As for $h^3(T_X \otimes L^{-m}) = h^0(\Omega^1_X(K_X) \otimes L^m)$ we proceed exactly as in the proof of (1.3) by restricting to S and C. As most of the calculations are the same, we will limit ourselves to give the bounds obtained, leaving the easy proof to the reader. We get

$$h^3(T_X \otimes L^{-5}) \leq 18d + 42(g-1) - 2q + 4h^1(L_{|C}) + 6h^2(T_S) + 4p_g(S) + 6 + h^2(L_{|S}) + p \tag{4.22}$$

$$h^3(T_X \otimes L^{-4}) \leq 6d + 24(g-1) - q + 3h^1(L_{|C}) + 5h^2(T_S) + 3p_g(S) + 4 + h^2(L_{|S}) + p \tag{4.23}$$

(4.24) $h^3(T_X \otimes L^{-3}) \leq d + 11(g-1) + 2h^1(L_{|C}) + 4h^2(T_S) + 2p_g(S) + 2 + h^2(L_{|S}) + p$

where $p = \begin{cases} 0 & \text{if } r \geq 2 \\ 1 & \text{if } r = 1 \end{cases}$. Applying the easy bounds

$$h^1(L_{|C}) \leq \frac{d}{2r} + N + 2q - 1, \quad h^2(L_{|S}) = h^0(K_{X|_S}) \leq \begin{cases} N-2 & \text{if } r \geq 2 \\ N+q & \text{if } r = 1 \end{cases} \tag{4.25}$$

together with (2.10), (4.22), (4.23) and (4.24) to (4.21), we get (4.14). ∎

Rewriting the bounds in terms of Chern numbers leads to the boundedness of the Chern ratios.

Proof of Corollary (1.10): Let $r \geq 2$ be such that $L = rK_X$ is very ample ($r \leq 10$ by Lee's results). By Riemann-Roch we get $N + 1 = -\frac{1}{24}(2r-1)c_1c_2 - \frac{1}{12}(2r^3 - 3r^2 + r)c_1^3$. By (4.4) we have $h^2(T_S) \leq -\frac{1}{24}(2r-1)c_1c_2 - \frac{1}{12}(38r^3 + 33r^2 + r)c_1^3 + q - 3$ and $q = h^1(\mathcal{O}_S) \leq h^1(\mathcal{O}_C) = g = -\frac{1}{2}(2r^3 + r^2)c_1^3 + 1$. Now the first bound in (4.13) gives an inequality of type $c_3 \geq Ac_1^3 + Bc_1c_2$ while the bound in (4.15) leads to $c_3 \leq -Cc_1^3 - Dc_1c_2 - 10$, where A, B, C, D are polynomials in r positive for some large explicit r. As $c_1c_2 = 24\chi < 0$ and $\frac{c_1^3}{c_1c_2} \leq \frac{8}{3}$ ([Y], [T]), it is clear that the region described by the Chern ratios $\frac{c_3}{c_1c_2}, \frac{c_1^3}{c_1c_2}$ is bounded. ∎

We now give the upper bound on the number of nodes of a complete intersection threefold of type $(d_1, \dots, d_n)$ in $\mathbb{P}^{n+3}$. As the bound depends on Chern numbers it is more convenient and clear to express it in terms of Newton functions. For $i = 1, \dots, n$ let $a_i = d_i - 1$ and set, for $1 \leq j \leq 3$, $s_j = s_j(d_1, \dots, d_n) = \sum_{i=1}^{n} a_i^j$.

Theorem (4.26). *Let $X \subset \mathbb{P}^{n+3}, n \geq 1$, be an irreducible complete intersection threefold of type $(d_1, \dots, d_n)$ having δ nodes and no other singularities. Then $\delta \leq B(d_1, \dots, d_n)$ where*

$$B(d_1, \dots, d_n) = \frac{1}{48} d_1 d_2 \cdots d_n \Big[13s_1^3 + 5s_1s_2 + 16s_3 - 128s_1^2 + 548s_1 + 50s_2 - 720 \Big] + n^2 + 7n +$$

$$+12 - \sum_{j=1}^{n} \binom{d_j + n}{n} + \sum_{j=1}^{n} \sum_{h=1}^{n} (-1)^{h+1} \sum_{1 \leq i_1 < \dots < i_h \leq n} h^0(\mathcal{O}_{\mathbb{P}^n}(d_j - d_{i_1} - \dots - d_{i_h})).$$

Remark **(4.27).** The above bound is worst than Varchenko's bound for hypersurfaces. In the case of complete intersection threefolds, though not explicitly stated in the literature, there is already an upper bound on the number of nodes δ. In fact generalizing Teissier's formula Kleiman [K] proved that the class c of $X \subset I\!P^n$ satisfies

$$c = d_1 \cdots d_n(\frac{s_3}{3} + \frac{s_1 s_2}{2} + \frac{s_1^3}{6}) - \sum_{x \in Sing(X)} b(x)$$

where $b(x)$ denotes the Buchsbaum-Rim multiplicity of the Jacobian map. But Gaffney [Ga] proved that $b(x)$ is the sum of the Milnor numbers of X and of a general hyperplane section at x. Hence for a node we have $b(x) = 2$ and Kleiman's formula gives

$$\delta \leq d_1 \cdots d_n(\frac{s_3}{6} + \frac{s_1 s_2}{4} + \frac{s_1^3}{12}).$$

Our bound improves the above in many cases.

Proof of Theorem (4.26): Let $X_t \subset I\!P^n$ be a smoothing of X (that is $X_t \subset I\!P^n$ is a smooth complete intersection threefold of type $(d_1, \ldots, d_n)$) and $\widetilde{X}$ a small resolution of X. Let L be the line bundle on $\widetilde{X}$ that defines the birational map to X and note that this map is an isomorphism away from the curves contracted to the nodes. In particular a general $S \in |L|$ is just (isomorphic to) a smooth complete intersection surface of type $(d_1, \ldots, d_n)$ in $I\!P^{n+2}$. Set $k = s_1 - 4, d = d_1 d_2 \cdots d_n$. We have $K_{\widetilde{X}} = kL$, $L^3 = d$ and the Chern classes of X_t are $c_1 = (4 - s_1)H, c_2 = \frac{1}{2}(s_1^2 + s_2 - 6s_1 + 12)H^2, c_3 = e(X_t) = -\frac{1}{6}(s_1^3 + 3s_1 s_2 + 2s_3 - 6s_1^2 - 6s_2 + 18s_1 - 24)d$ where H is the hyperplane divisor. It follows that $\chi = \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\widetilde{X}}) = \chi(\mathcal{O}_{X_t}) = -\frac{1}{24}kH \cdot c_2(X_t) = \frac{d}{48}(4 - s_1)(s_1^2 + s_2 - 6s_1 + 12)$ and, by Riemann-Roch, for every integer l we have $\chi(\mathcal{O}_{X_t}(l)) = h^0(\mathcal{O}_{X_t}(l)) - h^3(\mathcal{O}_{X_t}(l)) = \frac{d}{48}(4 - s_1 + 2l)(s_1^2 + s_2 - 6s_1 + 12 + 4l^2 + 16l - 4ls_1)$. By a standard topological fact (see [Hi]) it follows that $e(\widetilde{X}) = e(X_t) + 2\delta$ hence a bound on δ follows from a bound on $e(\widetilde{X})$. As $\chi(T_{\widetilde{X}}) = -\frac{1}{2}K_{\widetilde{X}}^3 - 19\chi + \frac{e(\widetilde{X})}{2}$ again we need to bound $h^0(T_{\widetilde{X}}) + h^2(T_{\widetilde{X}})$. First notice that $h^0(T_{\widetilde{X}}) = h^3(\Omega^1_{\widetilde{X}}(K_{\widetilde{X}})) = 0$ and similarly $h^1(\Omega^1_{\widetilde{X}}(-L)) = 0$ both by Migliorini's result [Mi, Theorem 4.6] (note that $\widetilde{X}$ is in general only a compact complex Moishezon threefold, as it may not be projective [Hi], but the theorem of Migliorini does apply to this case to the line bundle L (note that we need [D])). Finally $h^2(T_{\widetilde{X}}) = h^1(\Omega^1_{\widetilde{X}}(kL))$ which we bound by restricting to S. By the exact sequences

$$0 \to \Omega^1_{\widetilde{X}}((i-1)L) \to \Omega^1_{\widetilde{X}}(iL) \to \Omega^1_{\widetilde{X}}(iL)_{|S} \to 0$$

and

$$0 \to \mathcal{O}_S(i-1) \to \Omega^1_{\widetilde{X}}(iL)_{|S} \to \Omega^1_S(i) \to 0$$

we deduce $h^1(\Omega^1_{\widetilde{X}}(kL)) \leq \sum_{i=0}^{k} h^1(\Omega^1_S(i))$. But now the Euler and normal bundle sequences of S give $h^1(\Omega^1_S) = 1 + \sum_{j=1}^{n} h^0(\mathcal{O}_S(k+1+d_j)) - nh^0(\mathcal{O}_S(k+2)) + h^0(\mathcal{O}_S(k+1))$ and, for $i \geq 1$, $h^1(\Omega^1_S(i)) = \sum_{j=1}^{n} h^0(\mathcal{O}_S(k+1+d_j-i)) - nh^0(\mathcal{O}_S(k+2-i)) + h^0(\mathcal{O}_S(k+1-i))$. Notice that $\sum_{i=0}^{l} h^0(\mathcal{O}_S(i)) = h^0(\mathcal{O}_{X_t}(l))$ (these numbers depend only on $d_1, \ldots, d_n$) and the latter, for $l \geq k+1$, can be calculated by the above formula for $\chi(\mathcal{O}_{X_t}(l))$ and the fact that $h^3(\mathcal{O}_{X_t}(l)) = 0$. Similarly $\sum_{i=0}^{k} h^0(\mathcal{O}_S(k+1+d_j-i)) = h^0(\mathcal{O}_{X_t}(k+1+d_j)) - h^0(\mathcal{O}_{X_t}(d_j))$. Calculating the latter by the Koszul resolution of the ideal sheaf of X_t we get the function $B(d_1, \ldots, d_n)$ and the proof is complete. ∎

Remark **(4.28).** It is clear that the methods of the above proof also allow to give a bound on the number of nodes of a smoothable nodal Calabi-Yau threefold (for example when it is nodal and $\mathbb{Q}$-factorial [N]).

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