ON THE EXISTENCE OF COMPONENTS OF THE HILBERT SCHEME WITH THE EXPECTED NUMBER OF MODULI. II

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1. INTRODUCTION

In the study of the geometry of families of algebraic curves one common tool is the comparison between the extrinsic geometry, represented by properties of their projective embeddings and of the Hilbert scheme and the intrinsic geometry, represented by abstract properties and the moduli space. This comparison is often best represented by the natural map

$$\pi : H_{n,g,r} \to \mathcal{M}_g$$

from the Hilbert scheme $H_{n,g,r}$ of smooth curves in $\mathbb{P}^r$ of degree n and genus g (or the Hurwitz scheme for $r = 1$, the Severi variety for $r = 2$) to the moduli space of smooth curves of genus g.

The behaviour of π is very strongly influenced by the nature of its fibers, the Brill-Noether varieties. Now, while $\mathcal{M}_g$ is an irreducible variety, the Hilbert scheme $H_{n,g,r}$ is in general reducible and has in fact many components for $r \geq 3$. This difference has inspired many authors in the search of what could be a good component of the Hilbert scheme. One possible answer to such a question is to apply the above principle of comparison ([Ser],

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but see also [K1], [K2] for a different point of view):

(1.1) *Definition.* Let $\rho = \rho(g, n, r) = g - (r + 1)(g - n + r)$ be the Brill-Noether number. An irreducible component W of $H_{n,g,r}$ is said to have the *expected number of moduli* if the image $\pi(W)$ has dimension $\min\{3g - 3, 3g - 3 + \rho\}$. W is said to be *regular* if $H^1(N_C) = 0$ for a general curve C in W.

When $\rho \geq 0$ the behaviour of π is simple, from the above point of view: there is only one component of the Hilbert scheme with the expected number of moduli, by Brill-Noether theory. On the other hand the case $\rho < 0$ presents a much more complicated behaviour (see [EH] for a comprehensive description). In particular it is not completely known for which ρ such that $-3g + 3 \leq \rho < 0$ there exist components with the expected number of moduli and even when there are, they can be not unique ([P]).

The existence question has a good answer for $r = 1, 2$, since both the Hurwitz scheme and the Severi variety are irreducible and have the expected number of moduli ([Seg], [Sev], [AC1], [AC2], [F], [Ser]); for $r = 3$ it is almost completely solved by Pareschi [P]. When $r \geq 4$ the best known results obtained so far are as follows: Sernesi [Ser] proved the existence of a regular component having the expected number of moduli for any r, n, g such that $-\frac{1}{r}g + \frac{r+1}{r} \leq \rho \leq 0$; Ballico-Ellia [BE] showed that there exist components for integers (n_k, g_k) such that $\rho = -\frac{7}{3}g_k$ for $r = 4$ and $\rho = -(1 + \frac{2}{r})g_k$ for $r \geq 5$ and $k \to \infty$; in our previous work [L] we extended the existence range to $-\frac{5}{4}(g - r^2) \leq \rho \leq 0$ (asymptotically in r, for a precise statement see [L, Theorem 1.2]).

The method of our previous work [L] was to attach to Sernesi's curves some general nonspecial curve at $r + 4$ general points and then apply Sernesi's smoothing techniques.

In the present article, which, as much as our previous one, was very much inspired by the reading of the beautiful paper of Sernesi, we want to improve the results mentioned above. We further extend the range of possible r, n, g such that there exists a regular component of $H_{n,g,r}$ with the expected number of moduli to $-(2 - \frac{6}{r+3})g + h(r) \leq \rho \leq 0$, by applying a nice result of J. Stevens [St] concerning canonical curves.

More precisely set

$$h(r) = \frac{4r^3 + 8r^2 - 9r + 3}{r + 3}.$$

Then we will show:

Theorem (1.2). *There exists a regular component of $H_{n,g,r}$ with the expected number of moduli for any r, n, g satisfying:*

$$-\max\left\{\frac{1}{r}g - \frac{r^2+3r+2}{r}, g-2r^2-2r-2, (2-\frac{6}{r+3})g - h(r)\right\} \le \rho(g,n,r) \le 0 \tag{1.3}$$

if $r = 5$ or $r \ge 7$;

$$-\max\left\{\frac{1}{6}g - \frac{35}{2}, g-135, \frac{30}{19}g - \frac{4369}{19}\right\} \le \rho(g,n,6) \le 0 \quad \text{if } r=6; \tag{1.4}$$

$$-\max\left\{\frac{1}{4}g - \frac{55}{4}, g-67, \frac{3}{2}g - 112\right\} \le \rho(g,n,4) \le 0 \quad \text{if } r=4. \tag{1.5}$$

It seems reasonable to expect for the existence problem that the optimal result should be $\rho_{\min}(g,r) = -3g + o(g)$. For $r = 3$ this is true [P] and is easily seen to be the best possible.

Related to the question studied in this article is the study of the subvariety $\mathcal{M}^r_{g,n}$ of $\mathcal{M}_g$ parametrizing curves having a linear series of degree n and dimension at least r and of the variety $\mathcal{G}^r_n$ parametrizing pairs (curve, g^r_n). A result of Eisenbud and Harris [EH] shows that $\mathcal{G}^r_n$ has a unique component whose image in $\mathcal{M}_g$ has codimension 1 when $\rho = -1$, while for $\rho \le -2$ every component of $\mathcal{G}^r_n$ has image in $\mathcal{M}_g$ of codimension at least 2. Edidin [Ed] proved that if $g \ge 12$ and $\rho \le -3$ then every component of $\mathcal{M}^r_{g,n}$ has codimension at least 3. In a recent preprint F. Steffen [S] proved the general result that every component of $\mathcal{M}^r_{g,n}$ has codimension *at most* $-\rho$, if $\rho \le 0$. In particular it follows that $\mathcal{M}^r_{g,n}$ is irreducible of codimension 1 when $\rho = -1$, and that if $\rho = -2, \rho = -3$ and $g \ge 12$, every component of $\mathcal{M}^r_{g,n}$ has codimension 2, 3 respectively. Of course Steffen's result implies, when the Hilbert scheme $H_{n,g,r}$ is non empty, that there is a component W such that $\operatorname{codim}_{\mathcal{M}_g}\pi(W) \le -\rho$. On the other hand it is an open question (already posed in [Ser]), when $\rho < 0$, the existence of a component $V \subset H_{n,g,r}$ such that $\operatorname{codim}_{\mathcal{M}_g}\pi(V) > -\rho$ (when $\rho > 0$ examples of Harris [E], [EH] and Mezzetti-Sacchiero [MS] show the existence of components of $H_{n,g,r}$ not dominating $\mathcal{M}_g$). Also open is the existence of a component of $H_{n,g,r}, r \ge 4$ whose image in $\mathcal{M}_g$ is a point.

2. A PROPERTY OF GENERAL CANONICAL CURVES

The general strategy that we will employ to prove our main theorem is to work inductively in the following way: first construct a curve $C \subset \mathbb{P}^r$ with good properties, second attach to C some curve $\Gamma \subset \mathbb{P}^r$ at some points of C and third apply Sernesi's smoothing techniques to show that $C \cup \Gamma$ smooths to a curve with the same properties of C. The application of the smoothing technique requires some vanishing of the cohomology of the normal bundle of Γ (twisted by a suitable divisor) while the fact that the properties are kept depends additionally on the vanishing of the cohomology of the restricted tangent bundle to Γ (twisted by a suitable divisor). These vanishings, the first of which is proved by J. Stevens [St], will be dealt with in this section.

Set $\delta_{i,j}$ for the usual Kronecker symbol.

Proposition (2.1). *For $r \geq 4$ set $a(r) = r + 6 + 2\delta_{r,4} + \delta_{r,6}$ and let $P_1, \ldots, P_{a(r)}$ in $\mathbb{P}^r$ be general points. Then there exists a smooth irreducible canonical curve $\Gamma \subset \mathbb{P}^r$ of genus $r+1$, passing through $P_1, \ldots, P_{a(r)}$ and such that, if we set $D = P_1 + \ldots + P_{a(r)}$ and N_Γ for its normal bundle, we have:*

(i) $H^1(N_\Gamma(-D)) = 0$;

(ii) $H^0(T_{\mathbb{P}^r|_\Gamma}(-D)) = 0$.

Proof: As mentioned above, (i) has been proved in [St, proof of Corollary 6 and Lemma 7] for $r \geq 5$, while it is elementary for $r = 4$ as a general canonical curve is the complete intersection of three quadrics. To prove (ii) we degenerate Γ into a union of two rational normal curves. Let X and Y be two rational normal curves in $\mathbb{P}^r$ meeting transversally in $r+2$ points. Let Δ be the divisor of Y given by these points and D_1 be a divisor on Y of degree $a(r)$. Let $\Gamma' = X \cup Y$ and $D' = D_1$ as divisor on Γ'. Then we have an exact sequence

$$0 \to T_{\mathbb{P}^r|_X}(-\Delta) \to T_{\mathbb{P}^r|_{\Gamma'}}(-D') \to T_{\mathbb{P}^r|_Y}(-D_1) \to 0$$

hence it will be enough to show

$$H^0(T_{\mathbb{P}^r|_X}(-\Delta)) = H^0(T_{\mathbb{P}^r|_Y}(-D_1)) = 0 \tag{2.2}$$

since then (ii) will follow from (2.2) by semicontinuity. As is well known $T_{\mathbb{P}^r|_Y} \cong T_{\mathbb{P}^r|_X} \cong$

$\mathcal{O}_{\mathbb{P}^1}(r+1)^{\oplus r}$, therefore (2.2) holds. ■.

3. THE CONSTRUCTION OF THE COMPONENTS

We will now pursue the strategy indicated at the beginning of section 2 and construct, using Proposition (2.1) and some smoothing techniques ([Ser], [HH]), families of smooth curves in $\mathbb{P}^r$ with good properties.

For $r \geq 4$ let $a(r) = r + 6 + 2\delta_{r,4} + \delta_{r,6}$. Given a nondegenerate curve $C \subset \mathbb{P}^r$ let us define the following properties:

Property (α) : $H^1(N_C) = 0$.

Property (β) : $h^0(T_{\mathbb{P}^r|_C}) = (r+1)^2 - 1$.

Property (γ) : there exist $a(r)$ general points $P_1, \ldots, P_{a(r)} \in \mathbb{P}^r$ and a smooth irreducible nondegenerate linearly normal canonical curve $\Gamma \subset \mathbb{P}^r$ through them, satisfying $H^1(N_\Gamma(-P_1 - \ldots - P_{a(r)})) = H^0(T_{\mathbb{P}^r|_\Gamma}(-P_1 - \ldots - P_{a(r)})) = 0$ and such that there is a deformation of C meeting Γ quasi transversally in $P_1, \ldots, P_{a(r)}$.

It is a standard fact, recalled below for completeness, that properties (α) and (β) insure the existence of components of the Hilbert scheme with the expected number of moduli. Property (γ) is instead crucial in the inductive procedure.

Lemma (3.1). *Let $C \subset \mathbb{P}^r$ be a smooth irreducible nondegenerate curve of degree n and genus g satisfying (α) and (β). Then C belongs to a unique regular component of the Hilbert scheme $H_{n,g,r}$ having the expected number of moduli.*

Proof: From general deformation theory we know that (α) implies that C belongs to a unique regular component $W \subseteq H_{n,g,r}$ and that if $\pi : W \to \mathcal{M}_g$ is the natural map, then

$$\operatorname{codim}_{\mathcal{M}_g} \pi(W) = \dim \operatorname{Coker}\{H^0(N_C) \xrightarrow{\Phi} H^1(\omega_C^{-1})\}$$

where Φ is the coboundary map associated to the exact sequence

$$0 \to \omega_C^{-1} \to T_{\mathbb{P}^r|_C} \to N_C \to 0.$$

Let $\mu_0 : H^0(\mathcal{O}_C(1)) \otimes H^0(\omega_C(-1)) \to H^0(\omega_C)$ be the Brill-Noether map of C. By (α) and the Euler sequence we get $\mathrm{Coker}\Phi = H^1(T_{\mathbb{P}^r|_C}) \cong (\mathrm{Ker}\mu_0)^*$. Now (β) implies that C is linearly normal and μ_0 is surjective (see for example [L, Lemma 4.1]). Therefore

$$\mathrm{codim}_{\mathcal{M}_g}\pi(W) = \dim \mathrm{Ker}\mu_0 = (r+1)(g-n+r) - g = -\rho(g,n,r). \quad \blacksquare$$

The starting point of our induction will be some families of curves already contained in [Ser] and [L].

Lemma (3.2). *For every $r \geq 4$ and n, g such that*

$$-\max\left\{\frac{1}{r}g - \frac{r^2+3r+2+(r+1)(5\delta_{r,4}+7\delta_{r,6})}{r}, g-2r^2-2r-2-(r+1)(5\delta_{r,4}+7\delta_{r,6})\right\} \leq$$

$$\leq \rho(g,n,r) \leq 0,$$

there exists a smooth irreducible nondegenerate curve $C \subset \mathbb{P}^r$ of degree n, genus g satisfying (α), (β) and (γ).

Proof: By [L, Lemma 3.3] curves C satisfying (α) and (β) with n, g in the numerical range of the Lemma exist. To check Property (γ), we briefly recall the way these curves were constructed. Suppose one has proved that Properties (α), (β) and (γ) hold for some families of curves such that

$$(3.3) \qquad -\frac{1}{r}[g-r^2-3r-2-(r+1)(5\delta_{r,4}+7\delta_{r,6})] \leq \rho \leq 0.$$

Then curves with n, g in the next range $-g+2r^2+2r+2+(r+1)(5\delta_{r,4}+7\delta_{r,6}) \leq \rho \leq -\frac{1}{r}[g-r^2-3r-2-(r+1)(5\delta_{r,4}+7\delta_{r,6})]$ are obtained by smoothing the union of a curve C with degree and genus in range (3.3) with a rational normal curve X on a general hyperplane, such that X meets C in $r+2$ points. The fact that a general such smoothing satisfies again (γ) is trivial ([L, Sublemma 3.5]). Now for r, n, g as in (3.3) the construction is as above starting with linearly normal nonspecial curves, attaching X at $r+1$ points and smoothing. More precisely for every (r,n,g) in range (3.3), let $i = g-n+r$, $n' = n-(r-1)i$, $g' = g-ri$ and let $C_0 \subset \mathbb{P}^r$ be a smooth irreducible nondegenerate linearly normal nonspecial curve of degree n' and genus g' (note that $n' = g'+r$, $g' \geq r+2+5\delta_{r,4}+7\delta_{r,6}$). For every $j \geq 1$ let C_j be a general smoothing of $C_{j-1} \cup X$. In this way one fills up range (3.3). Therefore it is enough to see that C_1 satisfies (γ) (because $i = g-n+r \geq \frac{r+1}{r}(n-r)-n+r = \frac{n-r}{r} \geq 1$, as $n \geq 2r$). Note that at each step it is $H^1(N_{C_{j-1}\cup X}) = 0$, hence we loose no generality if we prove that C_1 satisfies (γ) by specializing C_0.

Claim (3.4). For $r \geq 4$, let $Y \subset I\!\!P^r$ be a smooth irreducible nondegenerate curve and $P_1, \ldots, P_{a(r)-1}$ be general points of Y. Then there exists a nondegenerate linearly normal nonspecial curve $C_0' \subset I\!\!P^r$ for every genus $g_0 \geq r + 2 + 5\delta_{r,4} + 7\delta_{r,6}$ meeting Y quasi-transversally at $P_1, \ldots, P_{a(r)-1}$.

Proof: Obviously we can assume $g_0 = r + 2 + 5\delta_{r,4} + 7\delta_{r,6}$. Suppose first $r \neq 4, 6$. By [L, Claim 3.7] there exists a smooth irreducible linearly normal elliptic curve $Z_1 \subset I\!\!P^r$ meeting Y quasi-transversally at $P_1, \ldots, P_{r+3}$. Choose $r+1$ general points $Q_1, \ldots, Q_{r+1}$ of Z_1. A "reducible" version of [L, Claim 3.7], whose proof is very much similar and left to the reader, shows that there exists a smooth irreducible linearly normal elliptic curve $Z_2 \subset I\!\!P^r$ passing through $Q_1, \ldots, Q_{r+1}, P_{r+4}, P_{r+5}$ and meeting Z_1 and Y quasi-transversally at these points. Now let $C_0' = Z_1 \cup Z_2$; then C_0' is a linearly normal nonspecial curve of genus $g_0 = r+2$ and meeting Y quasi-transversally at $P_1, \ldots, P_{r+5}$. When $r = 4, 6$ we proceed as above, but by attaching another elliptic curve Z_3 at $r+1$ general points of Z_2 and at the remaining points of Y. This completes the proof of Claim (3.4).

To finish the proof of Lemma (3.2) let us prove that C_1 satisfies (γ). Choose $a(r)$ general points $P_1, \ldots, P_{a(r)} \in I\!\!P^r$; choose any Γ through them as in Property (γ). Claim (3.4) gives a curve C_0' meeting Γ quasi-transversally in $P_1, \ldots, P_{a(r)-1}$. Let H be a general plane through $P_{a(r)}$ and $X \subset H$ a rational normal curve meeting C_0' in $r+1$ points and passing through $P_{a(r)}$. Then a general smoothing C_1 of $C_0' \cup X$ satisfies (γ). ∎

To extend the numerical range we now apply the smoothing procedure to curves having one component as in section 2.

Lemma (3.5). *Let $C \subset I\!\!P^r$ be a curve satisfying (α), (β) and (γ) and let $\Gamma \subset I\!\!P^r$ be a general linearly normal canonical curve meeting a general deformation $\overline{C}$ of C quasi-transversally at $a(r)$ general points. Then $C' = \overline{C} \cup \Gamma$ is flatly smoothable and its general smoothings satisfy (α), (β) and (γ).*

Proof: To show that C' is flatly smoothable, it is enough to prove [Ser, Proposition 1.6] that

$$H^1(N'_{C'}) = 0 \tag{3.6}$$

where $N'_{C'} = \mathrm{Ker}\{N_{C'} \to T^1_{C'}\}$, $T^1_{C'}$ being the first cotangent sheaf of C'.
Let $\{P_1, \ldots, P_{a(r)}\} = \overline{C} \cap \Gamma$ and $D = P_1 + \ldots + P_{a(r)}$ on Γ. From the exact sequences ([Ser], Lemma 5.1)

$$0 \to N_{C'} \otimes \mathcal{O}_\Gamma(-D) \to N'_{C'} \to N_{\overline{C}} \to 0$$

$$0 \to N_\Gamma(-D) \to N_{C'} \otimes \mathcal{O}_\Gamma(-D) \to T^1_{C'} \to 0$$

we get (3.6) since $H^1(N_\Gamma(-D)) = 0$ by Proposition 2.1 (i), $H^1(T^1_{C'}) = 0$ because $T^1_{C'}$ is supported on $\overline{C} \cap \Gamma$ and $H^1(N_{\overline{C}}) = 0$ since C satisfies (α). The definition of $N'_{C'}$ and (3.6) then imply $H^1(N_{C'}) = 0$, hence (α) holds for a general smoothing of C' by semicontinuity.
To see (β), consider the exact sequence

$$0 \to T_{\mathbb{P}^r|_\Gamma}(-D) \to T_{\mathbb{P}^r|_{C'}} \to T_{\mathbb{P}^r|_{\overline{C}}} \to 0.$$

By Proposition 2.1 (ii) and (β) for C, we get

$$h^0(T_{\mathbb{P}^r|_{C'}}) \leq h^0(T_{\mathbb{P}^r|_{\overline{C}}}) \leq h^0(T_{\mathbb{P}^r|_C}) = (r+1)^2 - 1$$

hence (β) holds for any general smoothing of C' (again by using semicontinuity and the inequality $h^0(T_{\mathbb{P}^r|_Z}) \geq (r+1)^2 - 1$ that holds for any smooth curve $Z \subset \mathbb{P}^r$ [L, Lemma 4.1]). Moreover C' satisfies (γ) (just take another Γ through $a(r)$ general points of $\overline{C}$), hence so does a general smoothing of C'. ∎

We now have all the ingredients to prove our main result.

Proof of Theorem (1.2). Set

$$h(r) = \frac{4r^3 + 8r^2 - 9r + 3}{r+3} + \frac{433}{7}\delta_{r,4} + \frac{6134}{57}\delta_{r,6}.$$

By Lemma (3.1) it is enough to show, for every $r \geq 4$, n, g in ranges (1.3), (1.4), (1.5), the existence of smooth irreducible nondegenerate curves $C \subset \mathbb{P}^r$ of degree n and genus g satisfying (α) and (β). By Lemma (3.2) we need to show the existence only in the case

$$-(2 - \frac{6}{r+3} + \frac{5}{14}\delta_{r,4} + \frac{14}{57}\delta_{r,6})g + h(r) \leq \rho \leq -g + 2r^2 + 2r + 2 + (r+1)(5\delta_{r,4} + 7\delta_{r,6}). \tag{3.7}$$

Now in Lemma (3.5) we have $\deg C' = \deg C + 2r, p_a(C') = g(C) + r + a(r)$, hence, given (n, g) in ranges (1.3), (1.4), (1.5) and an integer $i \geq 0$, set $n' = n - 2ri, g' = g - (r + a(r))i$. By Lemmas (3.5) and (3.2), a curve of degree n, genus g as in (3.7), satisfying (α) and (β) will exist as soon as we can find $i \geq 0$ such that

(3.8)

$$-\max\left\{\frac{1}{r}g' - \frac{r^2+3r+2+(r+1)(5\delta_{r,4}+7\delta_{r,6})}{r}, g' - 2r^2 - 2r - 2 - (r+1)(5\delta_{r,4}+7\delta_{r,6})\right\} \leq$$

$$\leq \rho(g', n', r) \leq 0.$$

It easily seen that $i = \left[\frac{rg-(r+1)(n-r)}{4r+8\delta_{r,4}+6\delta_{r,6}}\right]$, where [x] denotes the integer part of a rational number x, satisfies (3.8). ∎

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