Algebraic model of local period map and Yukawa coupling.

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Joint work in progress with Marco Manetti. For detailed formulas and precise signs see arXiv:1506.05753.

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Part 0. Introduction

Let X be a complex manifold, $\dim X = n \geq 2$. We denote by Θ_X the sheaf of holomorphic vector fields, Ω^i_X the sheaves of holomorphic differential forms,

$$\boldsymbol{i}\colon \Theta_X \to \prod_i \mathrm{Hom}(\Omega^i_X, \Omega^{i-1}_X), \quad \boldsymbol{i}_\xi(\omega) = \xi \lrcorner \omega,$$

$$\boldsymbol{l}\colon \Theta_X \to \prod_i \mathrm{Hom}(\Omega^i_X, \Omega^i_X), \quad \boldsymbol{l}_\xi = [\partial, \boldsymbol{i}_\xi],$$

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the usual contraction map and holomorphic Lie derivative respectively.

Lemma (Cartan formulas)

Every $\boldsymbol{i}_\xi$ is a derivation of degree -1 of the graded algebra Ω^*_X, and:

$$[\boldsymbol{i}_\xi, \boldsymbol{i}_\eta] = 0, \qquad \boldsymbol{i}_{[\xi,\eta]} = [\boldsymbol{i}_\xi, \boldsymbol{l}_\eta], \qquad \boldsymbol{l}_{[\xi,\eta]} = [\boldsymbol{l}_\xi, \boldsymbol{l}_\eta].$$

The composition of $\mathfrak{i}$ with cup product gives maps

$$\mathfrak{i}\colon H^p(\Theta_X) \to \mathrm{Hom}(H^q(\Omega^i_X), H^{q+p}(\Omega^{i-1}_X))$$

Definition (Yukawa coupling)

$$\Phi\colon H^1(X,\Theta_X)^{\wedge n} \to \mathrm{Hom}(H^0(\Omega^n_X), H^n(\Omega^0_X)),$$
$$\Phi(\xi_1,\ldots,\xi_n) = \mathfrak{i}_{\xi_1}\circ\cdots\circ\mathfrak{i}_{\xi_n}\ .$$

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Goal: understand Φ from the point of view of deformation theory; more precisely we want to interpret Φ as a "primary" obstruction for some geometrically meaningful deformation problem.

The model that we have in mind to imitate is the interpretation of the bracket

$$H^1(X,\Theta_X)^{\wedge 2} \xrightarrow{[-,-]} H^2(X,\Theta_X)$$

as a primary obstruction:

Theorem (Kodaira-Spencer-Kuranishi)

Let X be a compact complex manifold, $\mathcal{X} \to B$ its semiuniversal deformation. Then $B = q^{-1}(0)$ where

$$q\colon H^1(X,\Theta_X) \to H^2(X,\Theta_X),$$

is a germ of holomorphic map

$$q(\xi) = \frac{1}{2}[\xi,\xi] + \textit{higher order terms.}$$

There are formulas for the higher order terms depending on a certain Green's operator, but in practice they are unknown. To overcome (partially) this difficulty we can work directly with the Kodaira-Spencer DG Lie algebra

$$KS_X = (A^{0,*}(\Theta_X), d = -\overline{\partial}, [-,-])$$

where $A^{p,q}$ is the space of differentiable forms of type (p, q). The deformations of X are controlled by KS_X in the usual way, i.e., via Maurer-Cartan equation modulus gauge action.

From now on we assume X compact Kähler of dimension $n \geq 2$. Under this assumption the Hodge filtration

$$F_X^p = \oplus_{i \geq p} A^{i,*}$$

induces a filtration in cohomology $H^*(F_X^p) \subset H^*(X, \mathbb{C})$ such that

$$\frac{H^*(F_X^p)}{H^*(F_X^{p+1})} = H^*(\Omega_X^p)[-p] \, .$$

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To every small deformation $\mathcal{X} \to T$ (T contractible) there is a local period map

$$P^n \colon T \to Grass(H^*(X, \mathbb{C})), \qquad P^n(t) = H^*(F_{X_t}^n) \subset H^*(X, \mathbb{C})$$

which is holomorphic (Griffiths).

One can consider deformations of X in which the image of the period map belongs to some closed subset of the Grassmannian. For instance: consider deformations $\mathcal{X} \to (T, 0)$ such that $H(F^n_{X_t}) \subseteq H(F^1_{X_0})$ for every $t \in T$ sufficiently near to 0.

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Theorem

Let X compact Kähler of dimension n such that $H^1(\Omega^n_X) = 0$, then the above deformation problem admits a semiuniversal family $\mathcal{X} \to Y$, where $Y = q^{-1}(0)$

$$q = (q_1, q_2)\colon H^1(X, \Theta_X) \to H^2(X, \Theta_X) \times \mathrm{Hom}(H^0(\Omega^n_X), H^n(\Omega^0_X)),$$

is a germ of holomorphic map

$$q_1(\xi) = \frac{1}{2}[\xi, \xi] + \textit{higher order terms},$$

$$q_2(\xi) = \frac{1}{n!}\Phi(\xi, \dots, \xi) + \textit{higher order terms.}$$

The assumption $H^1(\Omega^n_X) = 0$ in not essential, but in general the statement is a little more tricky. We will get a better grasp at the higher order terms, which are practically unknown as they depend on certain Green's operators, after we introduce L_∞ algebras. The above theorem is not surprising and can be probably proved also by classical methods, as in Bryant and Griffiths paper "Some observations on the infinitesimal period relations..." (1982), so what's the point of having an L_∞ algebra model?

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Part 1. L_∞ algebras and deformation theory.

Recall that a DG Lie algebra structure $(L, d, [-,-])$ on a graded space $L = \oplus_{i\in\mathbb{Z}} L^i$ is the datum of a differential $d : L^i \to L^{i+1}$ and a graded antisymmetric bracket

$$x \in L^i,\ y \in L^j \ \ \to \ \ [x,y] = (-1)^{ij+1}[y,x] \in L^{i+j}$$

satisfying the Leibniz and Jacobi identities (in the graded sense)

$$d[x,y] = [dx,y] + (-1)^i [x,dy],$$

$$[x,[y,z]] = [[x,y],z] + (-1)^{ij}[y,[x,z]].$$

The set of solutions

$$\mathrm{MC}(L) = \{x \in L^1 \ \text{ s.t. } \ dx + \frac{1}{2}[x, x] = 0\}$$

of the *Maurer-Cartan equation* is stable under the *Gauge action*

$$\exp(L^0) \times \mathrm{MC}(L) \to \mathrm{MC}(L) \colon (e^a, x) \to e^a * x,$$

$$e^a * x = x + \sum_{k \geq 0} \frac{[a, -]^k}{(k+1)!}([a, x] - da)$$

We denote by $\mathrm{Def}(L)$ the set of Maurer-Cartan elements modulo Gauge equivalence.

This makes sense under suitable hypotheses ensuring convergence, in particular, it is always defined a functor of Artin rings

$$\mathrm{Def}_L\colon \mathbf{Art} \to \mathbf{Set}\colon A \to \mathrm{Def}(L \otimes \mathfrak{m}_A).$$

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Conversely, a formal moduli problem may be considered as a functor of Artin rings satisfying certain conditions, known as the Schlessinger conditions. These are always satisfied by the functor Def_L associated to a DG Lie algebra. We say that L controls a formal moduli problem $M : \mathbf{Art} \to \mathbf{Set}$ if there exist natural (in $A \in \mathbf{Art}$) isomorphisms $\mathrm{Def}_L(A) \simeq M(A)$.

In our motivating example of deformations of the complex structure on X, the associated functor Def_X sends $A \in \mathbf{Art}$ to the set of isomorphism classes of deformations

$$\begin{array}{ccc} X & \longrightarrow & X_A \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

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of X over $\mathrm{Spec}(A)$.
Given a local Artin algebra A, together with a Maurer-Cartan element

$$\xi \in \mathrm{MC}_{KS_X}(A) = \{x \in A^{0,1}(\Theta_X) \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x,x] = 0\},$$

the corresponding deformation $X_A \to Spec(A)$ is given by the structure sheaf of flat A-modules

$$\mathcal{O}_{X_A} = \ker(\overline{\partial} + \mathfrak{l}_\xi \colon A_X^{0,0} \otimes A \to A_X^{0,1} \otimes A).$$

To better understand the connection with the theorems in the introduction we have to consider L_∞ algebras. Recall that an L_∞ algebra structure on L is the data of a differential d on L and graded antisymmetric brackets $[-,\cdots,-]: L^{\wedge n} \to L$, $n \geq 2$, of total degree $(2-n)$, satisfying "higher Jacobi identities".

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$$d[x,y]=[dx,y]+(-1)^{|x|}[x,dy],$$

$$Jac(x,y,z)=d[x,y,z]-[dx,y,z]-(-1)^{|x|}[x,dy,z]-(-1)^{|x|+|y|}[x,y,dz].$$

Under suitable hypotheses ensuring convergence, it makes sense to consider the Maurer-Cartan equation

$$dx + \sum_{n \geq 1} \frac{1}{n!}[x, \cdots, x] = 0, \qquad x \in L^1,$$

on L, in particular, it is well defined the associated functor $\mathrm{Def}_L : \mathbf{Art} \to \mathbf{Set}$ of Maurer-Cartan elements modulus an appropriate notion of gauge/homotopy equivalence.

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Even in we stick with DG Lie algebras models, it is convenient to enlarge the class of morphisms by considering the L_∞ ones. These are collections of maps $f_n : L^{\wedge n} \to M$, $n \geq 1$, of total degree $1 - n$, satisfying higher coherence relations with the brackets.

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$$df_1(x) = f_1(dx)$$

$$f_1([x,y]) - [f_1(x), f_1(y)] =$$
$$= \pm(df_2(x \wedge y) + f_2(dx) \wedge y + (-1)^{|x|} f_2(x \wedge dy)).$$

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While it is always possible to represent a morphism in the homotopy category by an L_∞ morphism $F : L \to M$, in general it is only represented by a zig-zag of DG Lie algebra morphisms

$$L \xleftarrow{\sim} N \longrightarrow M\ ,$$

As another motivation, a fundamental result, the *homotopy transfer theorem*, says that L_∞ algebra structures can be transferred along quasi-isomorphism. In particular, given a contraction of complexes

$$H \underset{\pi}{\overset{\imath}{\rightleftarrows}} L \circlearrowleft K \,,$$

and an L_∞ algebra structure on L, there are induced L_∞ structures on H, $\imath$, π: furthermore, explicit recursive formulas are available from homological perturbation theory.

Over a field of characteristic zero, it is always possible to define a contraction as above where $H = H(L)$, considered as a complex with trivial differential. The induced L_∞ algebra structure on $H(L)$ is *minimal* (which means precisely that the differential is trivial), conversely, it can be proved that a *minimal model* of L is well defined up to L_∞ isomorphism. The binary bracket on $H(L)$ is the induced one: the higher brackets $[-,\cdots,-] : H(L)^{\wedge n} \to H(L)$ are related to the higher Massey brackets on L.

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$$q(x) = \sum_{n\geq 2} \frac{1}{n!}[x,\cdots,x].$$

A DG-Lie algebra controlling Grassmannian

Let $V = F \oplus B$ a DG-vector space, $d\colon V \to V$ differential: assume $dF \subset F$ and write

$$dx = \overline{\partial}x, \qquad x \in F$$

$$dy = \overline{\partial}y + \partial y, \quad y, \overline{\partial}y \in B, \qquad \partial y \in F.$$

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Consider the DG-Lie algebra $G_{F|V} = (\oplus_i G^i_{F|V}, \delta, [-,-])$, where

$$G^i_{F|V} = \mathrm{Hom}^{i-1}(F, B), \quad \delta(\alpha) = -\overline{\partial}\alpha + (-1)^{\bar{\alpha}} \alpha \overline{\partial}$$

$$[\alpha, \beta] = \alpha \partial \beta - (-1)^{\bar{\alpha}\,\bar{\beta}} \beta \partial \alpha$$

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The Maurer-Cartan equation is

$$\alpha \overline{\partial} - \overline{\partial} \alpha + \alpha \partial \alpha = 0, \qquad \alpha \in \mathrm{Hom}^0(F, B).$$

Lemma

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Idea of proof: let $\alpha \in \mathrm{Hom}^0(F, B)$, then the graded subspace $(Id + \alpha)(F)$ is a subcomplex if and only if for every $x \in F$ we have

$$(Id - \alpha)d(Id + \alpha)x \in F$$

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$$(Id - \alpha)d(Id + \alpha)x \in F$$

The B-component of the above quantity is precisely

$$-\alpha\overline{\partial}x + \overline{\partial}\alpha(x) - \alpha\partial\alpha(x)$$

which is trivial for every $x \in F$ if and only if α is Maurer-Cartan.

Next, two Maurer-Cartan elements α, β are gauge equivalent if and only if there exists an isomorphism of complexes $\phi\colon V \to V$, homotopic to the identity such that $(Id + \alpha)(F) = \phi(Id + \beta)(F)$. In particular the maps

$$H^*((Id + \alpha)(F)) \to H^*(V), \quad H^*((Id + \beta)(F)) \to H^*(V)$$

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have the same image. Easy to see: if $(V, F) \to (W, H)$ is a morphism of pairs, such that both $V \to W$ and $F \to H$ are quasi-isomorphism then $G_{F|V}$ is quasi-isomorphic to $G_{H|W}$.

Part 2. Derived local period map.

Consider again X compact Kähler of dimension $n \geq 2$, with the notations from the introduction, our aim is to reproduce the cartesian diagram of formal pointed moduli

$$\begin{array}{ccc} Y & \longrightarrow & Grass(H^*(F^n_X)|H^*(F^1_X)) \\ \downarrow & & \downarrow \\ B & \xrightarrow{P} & Grass(H^*(F^n_X)|H^*(X;\mathbb{C})) \end{array}$$

in the category of L_∞ algebras and L_∞ morphisms.

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in the category of L_∞ algebras and L_∞ morphisms.

Theorem

X compact Kähler of dimension n. Then the local period map $t \mapsto H^*(F^n_{X_t})$ is the classical truncation of a morphism of derived deformation theories: it is represented by an L_∞ morphism $P\colon KS_X \to G_{H^*(F^n_X)|H^*(X)}$, where, with an abuse of notation, we continue to denote by $G_{H^*(F^n_X)|H^*(X)}$ a DG-Lie or L_∞-algebra controlling the local geometry of the Grassmannian at $H^*(F^n_X) \subset H^*(X)$.

Since the Grassmanian $G_{H^*(F_X^n)|H^*(X)}$ is smooth, a minimal controlling L_∞ algebra will be isomorphic to $\mathrm{Hom}(H^{n,*}(X), H^{*<n,*}(X))[-1]$ with the trivial L_∞ algebra structure, but using this minimal model it is hard to construct a model of P directly.

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$$\mathrm{Hom}(A_X^{n,*}, A_X^{*<n,*})[-1] \to \mathrm{Hom}(H^{n,*}(X), H^{*<n,*}(X))[-1]$$

induced via homotopy transfer.

Having done so, the second thechnical problem is how to replace the previous cartesian diagram. Since an L_∞ model of P is only well defined up to quasi-isomorphism, it only makes sense to consider homotopy cartesian diagrams. Then our problem becomes to find an explicit L_∞ model Yuk_X fitting into a homotopy cartesian diagram of L_∞ algebras and L_∞ morphisms

$$\begin{array}{ccc} Yuk_X & \longrightarrow & \operatorname{Hom}(H^{n,*}(X), H^{0<*<n,*}(X))[-1] \\ \downarrow & & \downarrow \\ KS_X & \xrightarrow{P} & \operatorname{Hom}(H^{n,*}(X), H^{*<n,*}(X))[-1] \end{array}$$

We want to represent the local period map by an L_∞ morphism

$$P : KS_X \to G_{F_X^n|A_X^{*,*}}$$

where the second DG-Lie algebra is computed via the natural decomposition

$$A_X^{*,*} = F_X^n \oplus A_X^{*<n,*}.$$

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Theorem

In the above setup, an L_∞ model of P is given in Taylor coefficients by

$$P_i\colon \bigwedge^i A^{0,*}(\Theta_X) \to \mathrm{Hom}^{*-i}(F^n_X, A^{*<n,*}_X)$$

$$P_i(\xi_1, \dots, \xi_i) = \boldsymbol{i}_{\xi_1} \circ \cdots \circ \boldsymbol{i}_{\xi_i} \ .$$

Similar results holds with F^p in place of F^n for every $p \leq n$ but the formulas are slightly more complicated.

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Using the (opposite of the) propagator $h = -\overline{\partial}^* G_{\overline{\partial}} \colon A^{p,q} \to A^{p,q-1}$ and the harmonic projection $\pi \colon A_X^{p,q} \to H^{p,q}(X)$ and inclusion $\imath : H^{p,q}(X) \to A_X^{p,q}$ we have an equivalent smaller model

$$P : KS_X \to G_{H^*(F_X^n)|H^*(X)}$$

with Taylor coefficients

$$P_i \colon \bigwedge^i A^{0,*}(\Theta_X) \to \mathrm{Hom}^{*-i}(H^{n,*}(X), H^{*<n,*}(X))$$

$$P_i(\xi_1, \ldots, \xi_i) = \sum_{k=1}^{i} \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \pm \pi \mathfrak{i}_{\xi_{\sigma(1)}} \cdots \mathfrak{i}_{\xi_{\sigma(k)}} h \mathfrak{l}_{\xi_{\sigma(k+1)}} \cdots h \mathfrak{l}_{\xi_{\sigma(i)}} \imath \ .$$

Part 3: L_∞ estensions and homotopy pull-back

Definition

Let I, V be two L_∞ algebras: an extension of V by I is an L_∞ structure on the graded vector space $I \oplus V$ such that both the inclusion $I \to I \oplus V$ and the projection $I \oplus V \to V$ are strict L_∞ morphisms (strict means without Taylor components of degree > 1).

Theorem (Metha, Zambon, Chuang, Lazarev et al.)

There exists a natural bijection between the set of extensions of V by I and L_∞ morphisms

$$\phi\colon V \to CE(I)$$

where $CE(I)$ is the Chevalley-Eilenberg DG-Lie algebra of I, i.e., the DG-Lie algebra of coderivations of the symmetric coalgebra $S(I[1])$.

The above theorem (plus related formulas) implies that there exists pull-backs of extensions along L_∞ morphisms.

The above theorem (plus related formulas) implies that there exists pull-backs of extensions along L_∞ morphisms. If $I \to I \oplus_\phi V \to V$ is the extension associated to an L_∞-morphism $\phi\colon V \to CE(I)$, then every L_∞ morphism $F\colon W \to V$ gives a commutative diagram

$$\begin{array}{ccccc} I & \longrightarrow & I \oplus_{F\phi} W & \longrightarrow & W \\ \| & & \downarrow \tilde{F} & & \downarrow F \\ I & \longrightarrow & I \oplus_\phi V & \longrightarrow & V \end{array}$$

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It is not difficult to check that $\tilde{F}$ is a pull-back of F in the category of L_∞ algebras. (recall that this category does not have fibred products in general).

Let $G\colon U \to V$ be an L_∞ morphism: by the above theorem plus a standard argument we can find a factorization $G\colon U \xrightarrow{i} I \oplus_\phi V \to V$ where i is a quasi-isomorphism and $I \oplus_\phi V$ is an extension of V.

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The homotopy pull-back of G along any L_∞ morphism $F\colon W \to V$ is given by the cartesian diagram

$$\begin{array}{ccccc} & & I \oplus_{\phi F} W & \longrightarrow & W \\ & & \big\downarrow{\scriptstyle \tilde{F}} & & \big\downarrow{\scriptstyle F} \\ U & \xrightarrow{\ i\ } & I \oplus_\phi V & \longrightarrow & V \end{array}$$

Particular case: $G\colon L \to M$ inclusion of DG-Lie algebras such that $M = L \oplus C$, C abelian graded Lie subalgebra of M. Let $P\colon M \to C$ be the projection. We have a morphism (derived brackets)

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Theorem

1) (Voronov) The above ϕ is a morphism of DG-Lie algebras and there exists a quasi-isomorphism of L_∞ algebras $L \to C[-1] \oplus_\phi M$.
2) Therefore for every L_∞ morphism $F\colon W \to M$, the extension $C[-1] \oplus_{\phi F} W$ is the homotopy fiber product $W \times^h_M L$.

Part 4: Models for the "Yukawa algebra"

X compact Kähler of dimension $n \geq 2$. The morphism of Grassmannians $Grass(H^*(F^1_X)) \to Grass(H^*(F^0_X))$ is induced by the incusion of DG-Lie algebras

$$G\colon \operatorname{Hom}^*(A_X^{n,*}, A_X^{0<*<n,*})[-1] \to \operatorname{Hom}^*(A_X^{n,*}, A_X^{*<n,*})[-1]$$

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$$C := \operatorname{Hom}^*(A_X^{n,*}, A_X^{0,*})[-1] \subset \operatorname{Hom}^*(A_X^{n,*}, A_X^{*<n,*})[-1]$$

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is an abelian graded Lie subalgebra, and an algebraic complement of the image of G. We can apply Voronov's construction and then we get a quasi-isomorphism

$$\begin{array}{c}\operatorname{Hom}^*(A_X^{n,*}, A_X^{0<*<n,*})[-1] \\ \downarrow \\ \operatorname{Hom}^*(A_X^{n,*}, A_X^{0,*})[-2] \oplus_\phi \operatorname{Hom}^*(A_X^{n,*}, A_X^{*<n,*})[-1]\end{array}$$

We are now ready to give a first explicit model for Yuk_X, namely:

$$\mathrm{Yuk}_X = \mathrm{Hom}^*(A_X^{n,*}, A_X^{0,*})[-2] \oplus_{\phi P} KS_X$$

the "classifying" map ϕP annihilates many higher brackets and the L_∞ structure takes a simple form: the only non trivial brackets of the L_∞ structure on Yuk_X are (up to sign)

$$[\xi_1, \xi_2], \quad \xi_1, \xi_2 \in KS_X$$

$$[\xi, a] \mapsto \pm a \circ \boldsymbol{\mathfrak{l}}_\xi, \quad \xi \in KS_X, a \in \mathrm{Hom}^*(A_X^{n,*}, A_X^{0,*})[-2]$$

$$[\xi_1, \cdots, \xi_n] = \pm \boldsymbol{\mathfrak{i}}_{\xi_1} \cdots \boldsymbol{\mathfrak{i}}_{\xi_n}, \quad n = \dim X$$

A second model is obtained using the minimal models of the Grassmannians and the previous considerations, namely, we find the model

$$\mathrm{Yuk}_X = \mathrm{Hom}^*(H^{n,*}(X), H^{0,*}(X))[-2] \oplus_{\phi P} KS_X,$$

where the non trivial brackets are $[\xi_1, \xi_2]$, $\xi_k \in KS_X$, and for $i \geq n = \dim X$

$$[\xi_1, \ldots, \xi_i] = \sum_{\sigma \in S(n,1,\ldots,1)} \pm \pi \mathfrak{i}_{\xi_{\sigma(1)}} \cdots \mathfrak{i}_{\xi_{\sigma(n)}} h\mathfrak{l}_{\xi_{\sigma(n+1)}} \cdots h\mathfrak{l}_{\xi_{\sigma(i)}} \imath.$$

Using this model we recover the theorem from the introduction.

Thanks for the attention.