Algebraic model of local period map and Yukawa coupling.

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May 28, 2016
Joint work in progress with Marco Manetti. For detailed formulas and precise signs see arXiv:1506.05753.
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**Part 0. Introduction**

Let $X$ be a complex manifold, $\dim X = n \geq 2$. We denote by $\Theta_X$ the sheaf of holomorphic vector fields, $\Omega^i_X$ the sheaves of holomorphic differential forms,

$$i: \Theta_X \to \prod_i \text{Hom}(\Omega^i_X, \Omega^{i-1}_X), \quad i_\xi(\omega) = \xi \lrcorner \omega,$$

$$l: \Theta_X \to \prod_i \text{Hom}(\Omega^i_X, \Omega^i_X), \quad l_\xi = [\partial, i_\xi],$$

the usual contraction map and holomorphic Lie derivative respectively.
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Part 0. Introduction
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the usual contraction map and holomorphic Lie derivative respectively.

Lemma (Cartan formulas)
Every $i_\xi$ is a derivation of degree $-1$ of the graded algebra $\Omega^*_X$, and:

$$[i_\xi, i_\eta] = 0, \quad i_{[\xi, \eta]} = [i_\xi, l_\eta], \quad l_{[\xi, \eta]} = [l_\xi, l_\eta].$$
The composition of $i$ with cup product gives maps

$$i: H^p(\Theta_X) \to \text{Hom}(H^q(\Omega^i_X), H^{q+p}(\Omega^{i-1}_X))$$

Definition (Yukawa coupling)

$$\Phi: H^1(X, \Theta_X)^\wedge n \to \text{Hom}(H^0(\Omega^n_X), H^n(\Omega^0_X)),$$

$$\Phi(\xi_1, \ldots, \xi_n) = i_{\xi_1} \circ \cdots \circ i_{\xi_n}.$$
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Goal: understand $\Phi$ from the point of view of deformation theory; more precisely we want to interpret $\Phi$ as a “primary” obstruction for some geometrically meaningful deformation problem.
The model that we have in mind to imitate is the interpretation of the bracket

$$H^1(X, \Theta_X)^2 \xrightarrow{[\cdot, \cdot]} H^2(X, \Theta_X)$$

as a primary obstruction:

**Theorem (Kodaira-Spencer-Kuranishi)**

*Let $X$ be a compact complex manifold, $\mathcal{X} \to B$ its semiuniversal deformation. Then $B = q^{-1}(0)$ where

$$q: H^1(X, \Theta_X) \to H^2(X, \Theta_X),$$

is a germ of holomorphic map

$$q(\xi) = \frac{1}{2} [\xi, \xi] + \text{higher order terms}.$$*
There are formulas for the higher order terms depending on a certain Green’s operator, but in practice they are unknown. To overcome (partially) this difficulty we can work directly with the Kodaira-Spencer DG Lie algebra

$$KS_X = (A^{0,*}(\Theta_X), d = -\bar{\partial}, [-,-])$$

where $A^{p,q}$ is the space of differentiable forms of type $(p, q)$. The deformations of $X$ are controlled by $KS_X$ in the usual way, i.e., via Maurer-Cartan equation modulus gauge action.
From now on we assume $X$ compact Kähler of dimension $n \geq 2$. Under this assumption the Hodge filtration

$$F^p_X = \bigoplus_{i \geq p} A^{i,*}$$

induces a filtration in cohomology $H^*(F^p_X) \subset H^*(X, \mathbb{C})$ such that

$$\frac{H^*(F^p_X)}{H^*(F^{p+1}_X)} = H^*(\Omega^p_X)[−p] .$$
From now on we assume $X$ compact Kähler of dimension $n \geq 2$. Under this assumption the Hodge filtration

$$F_X^p = \bigoplus_{i \geq p} A_i^{i,*}$$

induces a filtration in cohomology $H^*(F_X^p) \subset H^*(X, \mathbb{C})$ such that

$$\frac{H^*(F_X^p)}{H^*(F_X^{p+1})} = H^*(\Omega_X^p)[-p].$$

To every small deformation $\mathcal{X} \to T$ ($T$ contractible) there is a local period map

$$P^n: T \to \text{Grass}(H^*(X, \mathbb{C})), \quad P^n(t) = H^*(F_X^n_t) \subset H^*(X, \mathbb{C})$$

which is holomorphic (Griffiths).
One can consider deformations of $X$ in which the image of the period map belongs to some closed subset of the Grassmannian. For instance: consider deformations $\mathcal{X} \to (T, 0)$ such that $H(F^n_{X_t}) \subseteq H(F^1_{X_0})$ for every $t \in T$ sufficiently near to 0.
One can consider deformations of $X$ in which the image of the period map belongs to some closed subset of the Grassmannian. For instance: consider deformations $\chi \to (T, 0)$ such that $H(F^n_{\chi_t}) \subseteq H(F^1_{\chi_0})$ for every $t \in T$ sufficiently near to 0.

**Theorem**

Let $X$ compact Kähler of dimension $n$ such that $H^1(\Omega^n_X) = 0$, then the above deformation problem admits a semiuniversal family $\chi \to Y$, where $Y = q^{-1}(0)$

$$q = (q_1, q_2): H^1(X, \Theta_X) \to H^2(X, \Theta_X) \times \text{Hom}(H^0(\Omega^n_X), H^n(\Omega^0_X)),$$

is a germ of holomorphic map

$$q_1(\xi) = \frac{1}{2} [\xi, \xi] + \text{higher order terms},$$

$$q_2(\xi) = \frac{1}{n!} \Phi(\xi, \ldots, \xi) + \text{higher order terms}.$$
The assumption $H^1(\Omega^1_X) = 0$ in not essential, but in general the statement is a little more tricky. We will get a better grasp at the higher order terms, which are practically unknown as they depend on certain Green’s operators, after we introduce $L_\infty$ algebras. The above theorem is not surprising and can be probably proved also by classical methods, as in Bryant and Griffiths paper “Some observations on the infinitesimal period relations...” (1982), so what’s the point of having an $L_\infty$ algebra model?

One motivation is that it provides a systematic way to organize the large amount of algebraic data associated to the deformation problem (hidden in the higher order terms of the previous theorems), such as higher obstructions and Massey operations; another important motivation comes from derived geometry, and more precisely derived deformation theory, more on this later.
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Part 1. $L_\infty$ algebras and deformation theory.
Recall that a DG Lie algebra structure $(L, d, [-,-])$ on a graded space $L = \bigoplus_{i \in \mathbb{Z}} L^i$ is the datum of a differential $d : L^i \to L^{i+1}$ and a graded antisymmetric bracket

$$x \in L^i, \; y \in L^j \implies [x, y] = (-1)^{ij+1}[y, x] \in L^{i+j}$$

satisfying the Leibniz and Jacobi identities (in the graded sense)

$$d[x, y] = [dx, y] + (-1)^i[x, dy],$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{ij}[y, [x, z]].$$
The set of solutions

\[ \text{MC}(L) = \{ x \in L^1 \text{ s.t. } dx + \frac{1}{2}[x, x] = 0 \} \]

of the \textit{Maurer-Cartan equation} is stable under the \textit{Gauge action}

\[ \exp(L^0) \times \text{MC}(L) \rightarrow \text{MC}(L): (e^a, x) \rightarrow e^a \ast x, \]

\[ e^a \ast x = x + \sum_{k \geq 0} \frac{[a, -]^k}{(k + 1)!} ([a, x] - da) \]

We denote by Def(L) the set of Maurer-Cartan elements modulo Gauge equivalence.
This makes sense under suitable hypotheses ensuring convergence, in particular, it is always defined a functor of Artin rings

\[ \text{Def}_L : \textbf{Art} \to \textbf{Set} : A \to \text{Def}(L \otimes m_A). \]
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$$\text{Def}_L : \text{Art} \to \text{Set} : A \to \text{Def}(L \otimes \mathfrak{m}_A).$$

Conversely, a formal moduli problem may be considered as a functor of Artin rings satisfying certain conditions, known as the Schlessinger conditions. These are always satisfied by the functor $\text{Def}_L$ associated to a DG Lie algebra. We say that $L$ controls a formal moduli problem $M : \text{Art} \to \text{Set}$ if there exist natural (in $A \in \text{Art}$) isomorphisms $\text{Def}_L(A) \simeq M(A)$.
In our motivating example of deformations of the complex structure on $X$, the associated functor $\text{Def}_X$ sends $A \in \text{Art}$ to the set of isomorphism classes of deformations

$$
\begin{array}{cccc}
X & \longrightarrow & X_A \\
\downarrow & & \downarrow \\
\text{Spec}(\mathbb{C}) & \longrightarrow & \text{Spec}(A)
\end{array}
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of $X$ over $\text{Spec}(A)$. 

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\]

of $X$ over $\text{Spec}(A)$. Given a local Artin algebra $A$, together with a Maurer-Cartan element

\[
\xi \in \text{MC}_{KS_X}(A) = \{ x \in A^{0,1}(\Theta_X) \otimes m_A \mid dx + \frac{1}{2} [x, x] = 0 \},
\]

the corresponding deformation $X_A \to \text{Spec}(A)$ is given by the structure sheaf of flat $A$-modules

\[
\mathcal{O}_{X_A} = \ker(\bar{\partial} + \iota_{\xi} : A^{0,0}_X \otimes A \to A^{0,1}_X \otimes A).
\]
To better understand the connection with the theorems in the introduction we have to consider $L_\infty$ algebras. Recall that an $L_\infty$ algebra structure on $L$ is the data of a differential $d$ on $L$ and graded antisymmetric brackets $[-,\cdots,-] : L^\wedge n \to L$, $n \geq 2$, of total degree $(2 - n)$, satisfying ”higher Jacobi identities”.

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$$d[x, y] = [dx, y] + (−1)^{|x|}[x, dy],$$

$$Jac(x, y, z) = d[x, y, z]−[dx, y, z]−(−1)^{|x|}[x, dy, z]−(−1)^{|x|+|y|}[x, y, dz].$$
Under suitable hypotheses ensuring convergence, it makes sense to consider the Maurer-Cartan equation

$$dx + \sum_{n \geq 1} \frac{1}{n!} [x, \cdots, x] = 0, \quad x \in L^1,$$

on $L$, in particular, it is well defined the associated functor $\text{Def}_L : \textbf{Art} \to \textbf{Set}$ of Maurer-Cartan elements modulus an appropriate notion of gauge/homotopy equivalence.
Why to consider $L_\infty$ algebras? ...
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The correspondence between DG Lie algebras, or more in general $L_\infty$ algebras, and deformation theory, is fully realized in the derived setting. Very briefly, in derived deformation theory we deal with derived formal moduli functors, that is, functors $\tilde{M} : \text{dgArt} \to \text{Kan}$ satisfying Schlessinger-like conditions.

The underlying philosophy is that an ordinary formal moduli functor should be regarded as a classical truncation of a more fundamental derived one. There is a natural notion of weak equivalence between derived moduli functors (defined pointwise), and the corresponding homotopy category is equivalent to the one of DG Lie algebras, or more in general $L_\infty$ algebras, modulus quasi-isomorphisms (Lurie-Pridham).
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$$df_1(x) = f_1(dx)$$

$$f_1([x,y]) - [f_1(x), f_1(y)] =$$

$$= \pm (df_2(x \wedge y) + f_2(dx) \wedge y + (-1)^{|x|} f_2(x \wedge dy)).$$
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While it is always possible to represent a morphism in the homotopy category by an $L_\infty$ morphism $F : L \to M$, in general it is only represented by a zig-zag of DG Lie algebra morphisms

$$L \xleftarrow{\sim} N \longrightarrow M,$$
As another motivation, a fundamental result, the homotopy transfer theorem, says that $L_\infty$ algebra structures can be transferred along quasi-isomorphism. In particular, given a contraction of complexes

$$H \xleftarrow{\iota} \xrightarrow{\pi} L \xleftarrow{K}$$

and an $L_\infty$ algebra structure on $L$, there are induced $L_\infty$ structures on $H$, $\iota$, $\pi$: furthermore, explicit recursive formulas are available from homological perturbation theory.
Over a field of characteristic zero, it is always possible to define a contraction as above where $H = H(L)$, considered as a complex with trivial differential. The induced $L_\infty$ algebra structure on $H(L)$ is \textit{minimal} (which means precisely that the differential is trivial), conversely, it can be proved that a \textit{minimal model} of $L$ is well defined up to $L_\infty$ isomorphism. The binary bracket on $H(L)$ is the induced one: the higher brackets $[-,\cdots,-]: H(L)^\wedge n \to H(L)$ are related to the higher Massey brackets on $L$. 

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$$q(x) = \sum_{n \geq 2} \frac{1}{n!} [x, ⋅ ⋅ ⋅ , x].$$
A DG-Lie algebra controlling Grassmannian

Let $V = F \oplus B$ a DG-vector space, $d: V \rightarrow V$ differential: assume $dF \subset F$ and write

$$dx = \overline{\partial} x, \quad x \in F$$

$$dy = \overline{\partial} y + \partial y, \quad y, \overline{\partial} y \in B, \quad \partial y \in F.$$
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Consider the DG-Lie algebra $G_{F|V} = (\bigoplus_i G^i_{F|V}, \delta, [-,-])$, where

$$G^i_{F|V} = \text{Hom}^{i-1}(F, B), \quad \delta(\alpha) = -\bar{\partial}\alpha + (-1)^\bar{\alpha} \alpha \bar{\partial}$$

$$[\alpha, \beta] = \alpha \partial \beta - (-1)^{\bar{\alpha} \bar{\beta}} \beta \partial \alpha$$
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$$[\alpha, \beta] = \alpha \partial \beta - (-1)^{\bar{\alpha}\bar{\beta}}\beta \partial \alpha$$

The Maurer-Cartan equation is

$$\alpha \bar{\partial} - \bar{\partial} \alpha + \alpha \partial \alpha = 0, \quad \alpha \in \text{Hom}^0(F, B).$$
Lemma

The DG-Lie algebra $G_{F|V}$ controls deformations up to ambient homotopy of the subcomplex $F$ inside $V$. 

Idea of proof: let $\alpha \in \text{Hom}_0(F, B)$, then the graded subspace $(\text{Id} + \alpha)(F)$ is a subcomplex if and only if for every $x \in F$ we have $(\text{Id} - \alpha) d (\text{Id} + \alpha) x \in F$. The $B$-component of the above quantity is precisely $- \alpha \partial x + \partial \alpha(x) - \alpha \partial \alpha(x)$ which is trivial for every $x \in F$ if and only if $\alpha$ is Maurer-Cartan.
Lemma
The DG-Lie algebra $G_{F|V}$ controls deformations up to ambient homotopy of the subcomplex $F$ inside $V$.

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The $B$-component of the above quantity is precisely

$$-\alpha \bar{\partial}x + \bar{\partial}\alpha(x) - \alpha \partial\alpha(x)$$

which is trivial for every $x \in F$ if and only if $\alpha$ is Maurer-Cartan.
Next, two Maurer-Cartan elements $\alpha, \beta$ are gauge equivalent if and only if there exists an isomorphism of complexes $\phi: V \rightarrow V$, homotopic to the identity such that $(\text{Id} + \alpha)(F) = \phi(\text{Id} + \beta)(F)$. In particular the maps

$$H^*((\text{Id} + \alpha)(F)) \rightarrow H^*(V), \quad H^*((\text{Id} + \beta)(F)) \rightarrow H^*(V)$$

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have the same image. Easy to see: if $(V, F) \to (W, H)$ is a morphism of pairs, such that both $V \to W$ and $F \to H$ are quasi-isomorphism then $G_{F|V}$ is quasi-isomorphic to $G_{H|W}$. 
Part 2. Derived local period map.
Consider again $X$ compact Kähler of dimension $n \geq 2$, with the notations from the introduction, our aim is to reproduce the cartesian diagram of formal pointed moduli

$$
\begin{align*}
Y & \longrightarrow Grass(H^*(F^n_X)|H^*(F^1_X)) \\
\downarrow & \\
B & \overset{P}{\longrightarrow} Grass(H^*(F^n_X)|H^*(X;\mathbb{C}))
\end{align*}
$$

in the category of $L_\infty$ algebras and $L_\infty$ morphisms.
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\end{align*}
$$

in the category of $L_\infty$ algebras and $L_\infty$ morphisms.

**Theorem**

$X$ compact Kähler of dimension $n$. Then the local period map $t \mapsto H^*(F^n_{X_t})$ is the classical truncation of a morphism of derived deformation theories: it is represented by an $L_\infty$ morphism $P: KS_X \to G_{H^*(F^n_X)|H^*(X)}$, where, with an abuse of notation, we continue to denote by $G_{H^*(F^n_X)|H^*(X)}$ a DG-Lie or $L_\infty$-algebra controlling the local geometry of the Grassmannian at $H^*(F^n_X) \subset H^*(X)$.
Since the Grassmanian $G_{H^*(F_X^n)|H^*(X)}$ is smooth, a minimal controlling $L_\infty$ algebra will be isomorphic to $\text{Hom}(H^n,*(X), H^*<n,*(X))[-1]$ with the trivial $L_\infty$ algebra structure, but using this minimal model it is hard to construct a model of $P$ directly.
Since the Grassmanian $G_{H^*(F^n_X)|H^*(X)}$ is smooth, a minimal controlling $L_\infty$ algebra will be isomorphic to $\text{Hom}(H^{n,*}(X), H^{*<n,*}(X))[-1]$ with the trivial $L_\infty$ algebra structure, but using this minimal model it is hard to construct a model of $P$ directly. We do it in two steps, first by considering the intermediate DG Lie algebra $G_{F^n_X|A^{*,*}_X} = \text{Hom}(A^n_{X,*}, A^{*<n,*}_X)[-1]$ controlling the local geometry of the infinite dimensional Grassmannian $\text{Grass}(F^n_X|A^{*,*}_X)$, where we find simple formulas for an $L_\infty$ model of $P$, and then by computing explicitly the $L_\infty$ morphism

$$\text{Hom}(A^n_{X,*}, A^{*<n,*}_X)[-1] \to \text{Hom}(H^{n,*}(X), H^{*<n,*}(X))[-1]$$

induced via homotopy transfer.
Having done so, the second technical problem is how to replace the previous cartesian diagram. Since an $L_\infty$ model of $P$ is only well defined up to quasi-isomorphism, it only makes sense to consider homotopy cartesian diagrams. Then our problem becomes to find an explicit $L_\infty$ model $Yuk_X$ fitting into a homotopy cartesian diagram of $L_\infty$ algebras and $L_\infty$ morphisms

\[
Yuk_X \xrightarrow{\partial} \text{Hom}(H^n,^*(X), H^0<^*<^n,^*(X))[−1] \\
\downarrow \quad \downarrow \\
KS_X \xrightarrow{P} \text{Hom}(H^n,^*(X), H^*<^n,^*(X))[−1]
\]
We want to represent the local period map by an $L_\infty$ morphism

$$P : KS_X \rightarrow G_{F^*_X|A^{*,*}_X}$$

where the second DG-Lie algebra is computed via the natural decomposition

$$A^{*,*}_X = F^*_X \oplus A^{*,<n,*}_X.$$

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$$A^{*,*}_X = F^n_X \oplus A^{*,<n,*}_X.$$

**Theorem**

*In the above setup, an $L_\infty$ model of $P$ is given in Taylor coefficients by*

$$P_i : \bigwedge A^{0,*}(\Theta_X) \to \text{Hom}^{*-i}(F^n_X, A^{*,<n,*}_X)$$

$$P_i(\xi_1, \ldots, \xi_i) = i_{\xi_1} \circ \cdots \circ i_{\xi_i}.$$
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Using the (opposite of the) propagator $h = -\overline{\partial}^* G_{\overline{\partial}}: A^{p,q} \to A^{p,q-1}$ and the harmonic projection $\pi: A^{p,q}_\chi \to H^{p,q}(X)$ and inclusion $\iota: H^{p,q}(X) \to A^{p,q}_\chi$ we have an equivalent smaller model

$$P : KS_X \to G_{H^*(F^n_X)|H^*(X)}$$

with Taylor coefficients

$$P_i: \bigwedge^i A^0,*(\Theta_X) \to \text{Hom}^{*-i}(H^n,*(X), H^{*-n,*}(X))$$

$$P_i(\xi_1, \ldots, \xi_i) = \sum_{k=1}^i \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \pm \pi \xi_{\sigma(1)} \cdots \iota \xi_{\sigma(k)} h\xi_{\sigma(k+1)} \cdots h\xi_{\sigma(i)} \iota.$$
Part 3: $L_\infty$ extensions and homotopy pull-back

Definition
Let $I, V$ be two $L_\infty$ algebras: an extension of $V$ by $I$ is an $L_\infty$ structure on the graded vector space $I \oplus V$ such that both the inclusion $I \to I \oplus V$ and the projection $I \oplus V \to V$ are strict $L_\infty$ morphisms (strict means without Taylor components of degree $> 1$).

Theorem (Metha, Zambon, Chuang, Lazarev et al.)
There exists a natural bijection between the set of extensions of $V$ by $I$ and $L_\infty$ morphisms

$$\phi: V \to CE(I)$$

where $CE(I)$ is the Chevalley-Eilenberg DG-Lie algebra of $I$, i.e., the DG-Lie algebra of coderivations of the symmetric coalgebra $S(I[1])$. 

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\[
\begin{array}{ccc}
I & \rightarrow & I \oplus_{F\phi} W \quad \rightarrow \quad W \\
\downarrow & & \downarrow \tilde{F} \\
I & \rightarrow & I \oplus \phi V \quad \rightarrow \quad V
\end{array}
\]

It is not difficult to check that $\tilde{F}$ is a pull-back of $F$ in the category of $L_\infty$ algebras. (recall that this category does not have fibred products in general).
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\]

\[
\begin{array}{ccc}
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\downarrow F & & \downarrow F \\
V & \longrightarrow & V
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Let $G: U \rightarrow V$ be an $L_\infty$ morphism: by the above theorem plus a standard argument we can find a factorization

$G: U \xrightarrow{i} I \oplus_\phi V \rightarrow V$ where $i$ is a quasi-isomorphism and $I \oplus_\phi V$ is an extension of $V$. 

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The homotopy pull-back of $G$ along any $L_\infty$ morphism $F : W \to V$ is given by the cartesian diagram

\[
\begin{array}{ccc}
I \oplus_{\phi F} W & \longrightarrow & W \\
\downarrow \bar{F} & & \downarrow F \\
U & \xrightarrow{i} & I \oplus \phi V & \longrightarrow & V
\end{array}
\]
Particular case: $G : L \to M$ inclusion of DG-Lie algebras such that $M = L \oplus C$, $C$ abelian graded Lie subalgebra of $M$. Let $P : M \to C$ be the projection. We have a morphism (derived brackets)

$$\phi : M \to \text{Coder}(S(C), S(C)) \cong \prod_{n=0}^{\infty} \text{Hom}(C^\otimes n, C)$$

Theorem 1) (Voronov) The above $\phi$ is a morphism of DG-Lie algebras and there exists a quasi-isomorphism of $L_\infty$ algebras $L \to C[-1] \oplus \phi M$.

2) Therefore for every $L_\infty$ morphism $F : W \to M$, the extension $C[-1] \oplus \phi F W$ is the homotopy fiber product $W \times_h M L$. 

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\phi(m)(1) = Pm, \quad \phi(m)(c_1, \ldots, c_n) = P[..[[m, c_1], c_2], \ldots], c_n]
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Part 4: Models for the “Yukawa algebra”

$X$ compact Kähler of dimension $n \geq 2$. The morphism of Grassmannians $\text{Grass}(H^*(F^1_X)) \rightarrow \text{Grass}(H^*(F^0_X))$ is induced by the inclusion of DG-Lie algebras

$$G : \text{Hom}^*(A^n_X, A^{0<\ast<\ast}_{<n}X)[−1] \rightarrow \text{Hom}^*(A^n_X, A^{\ast<n}_{<n}X)[−1]$$
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Notice that

$$C := \text{Hom}^*(A^n_X, A^0_X)[-1] \subset \text{Hom}^*(A^n_X, A^*_{<n^n})[-1]$$

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$$G : \text{Hom}^*(A_{X}^{n,*}, A_X^{0\ll \ll n,*})[-1] \to \text{Hom}^*(A_{X}^{n,*}, A_X^{\ll n,*})[-1]$$

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is an abelian graded Lie subalgebra, and an algebraic complement of the image of $G$. We can apply Voronov’s construction and then we get a quasi-isomorphism

$$\text{Hom}^*(A_{X}^{n,*}, A_X^{0\ll \ll n,*})[-1] \downarrow \text{Hom}^*(A_{X}^{n,*}, A_X^{0,*})[-2] \oplus \phi \text{Hom}^*(A_{X}^{n,*}, A_X^{\ll n,*})[-1]$$
We are now ready to give a first explicit model for \( \text{Yuk}_X \), namely:

\[
\text{Yuk}_X = \text{Hom}^*(A_X^{n,*}, A_X^{0,*})[-2] \oplus_{\phi P} KS_X
\]

the “classifying” map \( \phi P \) annihilates many higher brackets and the \( L_\infty \) structure takes a simple form: the only non trivial brackets of the \( L_\infty \) structure on \( \text{Yuk}_X \) are (up to sign)

\[
[\xi_1, \xi_2], \quad \xi_1, \xi_2 \in KS_X
\]

\[
[\xi, a] \mapsto \pm a \circ [\xi], \quad \xi \in KS_X, a \in \text{Hom}^*(A_X^{n,*}, A_X^{0,*})[-2]
\]

\[
[\xi_1, \cdots, \xi_n] = \pm i_{\xi_1} \cdots i_{\xi_n}, \quad n = \text{dim } X
\]
A second model is obtained using the minimal models of the Grassmannians and the previous considerations, namely, we find the model

\[ \text{Yuk}_X = \text{Hom}^*(H^n,*(X), H^0,*(X))[-2] \oplus_{\phi_P} KS_X, \]

where the non trivial brackets are \([\xi_1, \xi_2], \xi_k \in KS_X\), and for \(i \geq n = \text{dim} X\)

\[ [\xi_1, \ldots, \xi_i] = \sum_{\sigma \in S(n,1,\ldots,1)} \pm \pi i_{\xi_{\sigma(1)}} \cdots i_{\xi_{\sigma(n)}} h_{\xi_{\sigma(n+1)}} \cdots h_{\xi_{\sigma(i)}}. \]

Using this model we recover the theorem from the introduction.
Thanks for the attention.