Sulla coomologia L^2-$\overline{\partial}$ di certe metriche Kähleriane complete

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Giornate di Geometria Algebrica e Argomenti Correlati
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- A brief reminder on the L^2-$\overline{\partial}$-cohomology

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- Applications to pinched negatively curved Kähler manifolds of finite volume

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$$0 \to \Omega_c^{p,0}(M) \stackrel{\overline{\partial}_{p,0}}{\to} \Omega_c^{p,1}(M) \stackrel{\overline{\partial}_{p,1}}{\to} \ldots \stackrel{\overline{\partial}_{p,m-1}}{\to} \Omega_c^{p,m}(M) \to 0$$

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We need to specify a closed extension

$$\overline{D}_{p,q} : \Omega_c^{p,q}(M) \to \Omega_c^{p,q+1}(M)$$

of $\overline{\partial}_{p,q} : \Omega_c^{p,q}(M) \to \Omega_c^{p,q+1}(M)$.

Maximal closure

Let $\omega \in L^2\Omega^{p,q}(M,h)$; then $\omega \in \mathcal{D}(\overline{\partial}_{p,q,\max})$ if there exists $\alpha \in L^2\Omega^{p,q+1}(M,h)$ such that

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$$\langle \omega, \overline{\partial}^t_{p,q}\phi\rangle_{L^2\Omega^{p,q}(M,h)} = \langle \alpha, \phi\rangle_{L^2\Omega^{p,q+1}(M,h)}$$

for each $\phi \in \Omega_c^{p,q+1}(M)$. We set

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$\overline{\partial}_{p,q,\max} \circ \overline{\partial}_{p,q+1,\max} = 0 \Rightarrow (\mathcal{D}(\overline{\partial}_{p,q,\max}), \overline{\partial}_{p,q,\max})$ is a complex,

$$H^{p,q}_{2,\overline{\partial},\max}(M,h)$$

denotes its cohomology (maximal L^2-$\overline{\partial}$-cohomology.)

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$\omega \in \mathcal{D}(\overline{\partial}_{p,q,\min})$ if there is a sequence $\{\phi_j\} \subset L^2\Omega^{p,q}(M,h)$ and an element α in $L^2\Omega^{p,q+1}(M,h)$ such that

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If (M,h) is complete then

$$\overline{\partial}_{p,q,\max} = \overline{\partial}_{p,q,\min} \text{ and } H^{p,q}_{2,\overline{\partial},\min}(M,h) = H^{p,q}_{2,\overline{\partial},\max}(M,h)$$

We can also consider the de Rham complex $(\Omega_c^k(M), d_k)$. Analogously to the previous case we have the maximal de Rham complex

$$(L^2\Omega^k(M,h), d_{k,\max})$$

the maximal de Rham cohomology

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- Heuristic principle: if h is a suitable metric then $H^{p,q}_{2,\overline{\partial}}(M,h)$ and $H^k_2(M,h)$ contain geometric/topological informations of $\overline{M}$, where $\overline{M}$ is a compactification of M.

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Examples:

$\overline{M}$ compact manifold with boundary; h Riemannian metric on $\overline{M}$. M interior of $\overline{M}$. Then

$$H^k_{2,\max}(M,h|_M) \cong H^k(\overline{M}),\ H^k_{2,\min}(M,h|_M) \cong H^k_c(M).$$

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X compact Thom-Mather stratified pseudomanifold; h conic metric on reg(X) (the regular part of X). Then

$$H^k_{2,\max}(\mathrm{reg}(X),h) \cong I^{\underline{m}}H^k(X,\mathbb{R})$$

and

$$H^k_{2,\min}(\mathrm{reg}(X),h) \cong I^{\overline{m}}H^k(X,\mathbb{R}).$$

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The intersection complex $I^{\underline{m}}S_\bullet(V,\mathbb{R})$ is a suitable subcomplex of $S_\bullet(V,\mathbb{R})$

$$I^{\underline{m}}S_\bullet(V,\mathbb{R}) \subset S_\bullet(V,\mathbb{R})$$

Using its homology, the middle perversity intersection homology, we can restore Poincaré duality on V:

$$I^{\underline{m}}H_k(V,\mathbb{Q}) \times I^{\underline{m}}H_{2v-k}(V,\mathbb{Q}) \to \mathbb{Q}.$$

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Moreover when $v = 2l$ we have a well defined signature:

$$I^{\underline{m}}H_{2l}(V,\mathbb{Q}) \times I^{\underline{m}}H_{2l}(V,\mathbb{Q}) \to \mathbb{Q}.$$

Question: on a compact Kähler manifold (M, h) we have the well known decomposition:

$$H^k(M) \cong \bigoplus_{p+q=m} H^{p,q}_{\overline{\partial}}(M)$$

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- Can we construct a pure Hodge decomposition on $I^{\underline{m}}H^k(V, \mathbb{Q})$?
- Can we do it using L^2-methods?

Cheeger-Goresky-MacPherson's Conjecture, 1982

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- Approach through incomplete metrics:

Let g be the Kähler metric induced on $\text{reg}(V)$, the regular part of V, by the Fubini-Study metric of $\mathbb{CP}^n$. Then:

$$d_{k,max} = d_{k,min}$$

$$H_2^k(\text{reg}(V), g) \cong I^{\underline{m}}H^k(V, \mathbb{R})$$

and the previous isomorphism induces a Hodge decomposition on $I^{\underline{m}}H^k(V, \mathbb{R})$ in terms of L^2-$\overline{\partial}$-cohomology groups.

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Advantages: Natural candidate for the Kähler metric.
Disadvantages: The metric is incomplete. The L^2-Stokes theorem is hard to prove. L^2-$\overline{\partial}$-cohomolology is not well understood.

- Approach through complete metrics:

Prove the existence of a complete Kähler metric g on $\mathrm{reg}(V)$ such that

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Advantages: The metric (if exists) is complete. The Hodge Laplacian Δ_k is essentially self-adjoint. Automatic pure Hodge structure on $I^{\underline{m}}H^k(V, \mathbb{C})$:

$$I^{\underline{m}}H^k(V, \mathbb{C}) \cong H_2^k(\text{reg}(V), g) \cong \ker(\Delta_k) \cong$$

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Moreover in both approaches the following question remains open:

Provide a geometric interpretation for $H^{p,q}_{2,\overline{\partial}}(\text{reg}(V), g)$.

To my best knowledge only results for $(0, q)$ are available. This last problem is related to the so called MacPherson's conjecture.

MacPherson's conjecture, 1983

Conjecture

Let $V \subset \mathbb{CP}^n$ *be a complex projective variety,* $\pi : \tilde{V} \longrightarrow V$ *a Hironaka resolution of* V *and let* g *be the Kähler metric on* $\text{reg}(V)$ *induced by the Fubini-Study metric.Then:*

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$$\chi_2(\text{reg}(V), g) = \chi(V, \mathcal{O}_{\tilde{V}})$$

where $\chi_2(\text{reg}(V), g)) = \sum(-1)^q \dim(H^{0,q}_{2,\overline{\partial}}(\text{reg}(V), g))$ *and* $\chi(\tilde{V}) = \sum(-1)^q \dim(H^{0,q}_{\overline{\partial}}(\tilde{V}))$.

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First work by Brüning-Peyerimhoff-Schröder, Haskell, Pardon for curves and surfaces. Solved by Pardon and Stern proving a stronger result:

$$H^{0,q}_{2,\overline{\partial}_{\min}}(\mathrm{reg}(V), g) \cong H^{0,q}_{\overline{\partial}}(\tilde{V}),\ \ q = 0, ..., v$$

State of art concerning the C-G-M Conjecture

- Approach through incomplete metrics

Partial results for projective varieties with isolated singularities: Cheeger, Hsiang-Pati, Nagase, Pardon-Stern and Ohsawa.

$$H^k_{2,\max}(\text{reg}(V), h) \cong I^{\underline{m}}H^k(V, \mathbb{R})$$

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decomposition of the L^2-cohomology in term of L^2-$\overline{\partial}$-cohomology for $k \neq v-1, v, v+1$.
If $\dim(\mathrm{sing}(V)) > 0$

$$d_{0,\max} = d_{0,\min}$$

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2. $H_{2,\overline{\partial}}^{0,q}(\mathrm{reg}(V), g) \cong H_{\overline{\partial}}^{0,q}(\tilde{V})$
3. *The pure L^2-Hodge structure on $I^{\underline{m}}H^k(V, \mathbb{R})$ coincides with the structure defined by Saito. (Saper-Zucker)*

Definition (Gromov 1990)

A Kähler manifold (M, h) with fundamental form ω is said to be *d-bounded* if

$$d\eta = \omega$$

for some $\eta \in L^\infty\Omega^1(M, h) \cap \Omega^1(M)$.

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Theorem (Gromov 1990)

(M, h) Kähler, complete and d-bounded of complex dimension m. Then

$$H^{p,q}_{2,\overline{\partial}}(M, h) = 0$$

for $p + q \neq m$ and

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The notion of d-boundedness can be relaxed. It is enough $\omega = d\eta$ in the distributional sense for some $\eta \in L^\infty\Omega^1(M)$.

Theorem (Bei-Piazza 2017)

Let V be a compact and irreducible complex space of complex dimension v. Let $\pi : M \to V$ be a resolution of V with $D := \pi^{-1}(\text{sing}(V))$ a normal crossings divisor of M. Let h be a complete Hermitian metric on $\text{reg}(V)$ and let σ be the complete Hermitian metric on $M \setminus D$ defined as $\sigma := (\pi|_{M \setminus D})^* h$.

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Assume that:

- $\pi_* \mathcal{C}^{v,q}_{D,\sigma}$ *is a fine sheaf for each $q = 0, ..., v$,*
- *for each $p \in \text{sing}(V)$ there is an open neighborhood U of p and a d-bounded Kähler metric g_U on $\text{reg}(U)$ such that $h|_{\text{reg}(U)}$ and g_U are quasi-isometric.*

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Let V be a compact and irreducible complex space of complex dimension v. Let $\pi : M \to V$ be a resolution of V with $D := \pi^{-1}(\text{sing}(V))$ a normal crossings divisor of M. Let h be a complete Hermitian metric on $\text{reg}(V)$ and let σ be the complete Hermitian metric on $M \setminus D$ defined as $\sigma := (\pi|_{M\setminus D})^* h$.
Assume that:

- $\pi_* \mathcal{C}^{v,q}_{D,\sigma}$ *is a fine sheaf for each $q = 0, ..., v$,*
- *for each $p \in \text{sing}(V)$ there is an open neighborhood U of p and a d-bounded Kähler metric g_U on $\text{reg}(U)$ such that $h|_{\text{reg}(U)}$ and g_U are quasi-isometric.*

Then we have the following isomorphism for each $q = 0, ..., v$:

$$H^{v,q}_{2,\overline{\partial}}(\text{reg}(V), h) \cong H^{v,q}_{\overline{\partial}}(M)$$

$$H^{0,q}_{2,\overline{\partial}}(\text{reg}(V), h) \cong H^{0,q}_{\overline{\partial}}(M).$$

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Consider the preasheaves $C^{p,q}_{D,h}$ on M given by the assignments

$$C^{p,q}_{D,h}(U) := \{\mathcal{D}(\overline{\partial}_{p,q,\max}) \text{ on } (U \setminus U \cap D, h|_{U \setminus U \cap D})\};$$

in other words to every open subset U of M we assign the maximal domain of $\overline{\partial}_{p,q}$ over $U \setminus (U \cap D)$ with respect to the Hermitian metric $h|_{U \setminus U \cap D}$.

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The sheafification of $C^{p,q}_{D,h}$ is denoted by $\mathcal{C}^{p,q}_{D,h}$ and its sections over an open subset $U \subset M$ are:

$\mathcal{C}^{p,q}_{D,h}(U) := \{s \in L^2_{loc}\Omega^{p,q}(U \setminus U \cap D, h|_{U \setminus U \cap D})$ such that for each $p \in U$ there exists an open neighborhood W with $p \in W \subset U$ such that $s|_{W \setminus W \cap D} \in \mathcal{D}(\overline{\partial}_{p,q,\max})$ on $(W \setminus W \cap D, h|_{W \setminus W \cap D})\}$.

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Proposition

Assume that given any open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of V there exists a continuous partition of unity $\{\lambda_j\}_{j\in J}$ subordinate to $\mathcal{U}$ such that for each $j \in J$

- *$\lambda_j|_{\text{reg}(V)}$ is smooth and $\|d(\lambda_j|_{\text{reg}(V)})\|_{L^\infty\Omega^1(M,h)} < \infty$.*

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Proposition

Assume that for each $p \in \mathrm{sing}(V)$ there is an open neighborhood U, a holomorphic embedding $\phi : U \to \mathbb{C}^N$ and a positive constant c such that, with g the euclidean metric on $\mathbb{C}^N$, we have

$$(\phi|_{\mathrm{reg}(U)})^* g \leq ch|_{\mathrm{reg}(U)}.$$

Then $\pi_* \mathcal{C}^{v,q}_{D,\sigma}$ is a fine sheaf for each $q = 0, ..., v$.

Sketch of the proof

- Let $\mathcal{K}_M$ be the sheaf of holomorphic $(v, 0)$ forms on M. Let $\mathcal{K}_V := \pi_*\mathcal{K}_M$ on V. The Takegoshi vanishing theorem tells us that
$$R^k\pi_*\mathcal{K}_M = 0, \;\; k > 0$$
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- $H^{v,q}(M) \cong H^q(M,\mathcal{K}_M) \cong H^q(V,\mathcal{K}_V) \cong H^q(V,\pi_*\mathcal{C}^{v,q}_{D,\sigma}) \cong H^{v,q}_{2,\overline{\partial}}(\mathrm{reg}(V),h)$

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- $H^{v,q}(M) \cong H^q(M,\mathcal{K}_M) \cong H^q(V,\mathcal{K}_V) \cong H^q(V,\pi_*\mathcal{C}^{v,q}_{D,\sigma}) \cong H^{v,q}_{2,\overline{\partial}}(\mathrm{reg}(V),h)$
- Using the L^2-Serre duality we have: $H^{0,q}_{2,\overline{\partial}}(\mathrm{reg}(V),h) \cong H^{v,v-q}_{2,\overline{\partial}}(\mathrm{reg}(V),h) \cong H^{v,v-q}_{\overline{\partial}}(M) \cong H^{0,q}_{\overline{\partial}}(M)$.

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Let $\phi : U \to \mathbb{B}(0, c) \subset \mathbb{C}^N$ a proper holomorphic embedding. Define $\psi : \mathbb{B}(0, c) \to \mathbb{R}$ as

$$\psi := -(log(c^2 - |z|^2)).$$

Let β be the Kähler metric on $\mathbb{B}(0, c)$ whose fundamental form is $-i\partial\overline{\partial}\psi$ and let $\rho_U := (\phi|_{\mathrm{reg}(U)})^* \beta$.

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Define the following Kähler metric on reg(U):

$$\gamma_U := \rho_U + g_U.$$

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$\Rightarrow \eta \in \ker(\overline{\partial}_{v,q,\max}) \subset L^2\Omega^{v,0}(\text{reg}(U), h|_{\text{reg}(U)}) \Rightarrow \eta|_W$ is exact for any $\overline{W} \subset U \Rightarrow \{\pi_*\mathcal{C}^{p,q}_{D,\sigma}, q \geq 0\}$ is exact for $q > 0$.

Finally, for every $U \subset M$, we have

$$\ker(\overline{\partial}_{v,0,\max}) \text{ on } L^2\Omega^{v,0}(U, \sigma|_{U\setminus(U\cap D)}) = \mathcal{K}_M(U).$$

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In conclusion we proved that $\{\pi_* \mathcal{C}^{p,q}_{D,\sigma}, q \geq 0\}$ is a fine resolutions of $\mathcal{K}_V$.

Let $V \subset M$ be an analytic subvariety of a compact complex manifold M and let ω be the fundamental $(1,1)$-form of a hermitian metric on M.

Applications to Saper type Kähler metrics

Let $V \subset M$ be an analytic subvariety of a compact complex manifold M and let ω be the fundamental $(1,1)$-form of a hermitian metric on M.

Let $\pi : \tilde{M} \to M$ be a holomorphic map of a compact complex manifold $\tilde{M}$ to M whose exceptional set E is a divisor with normal crossings in $\tilde{M}$ and such that the restriction

$$\pi|_{\tilde{M}\setminus E} : \tilde{M} \setminus E \longrightarrow M \setminus \mathrm{sing}(V)$$

is a biholomorphism.

Example: $\pi : \tilde{M} \to M$ resolution of singularities

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Let L_E be the line bundle on $\tilde{M}$ associated to E and let τ be a hermitian metric on L_E.

Let $s : \tilde{M} \to L_E$ be a global holomorphic section whose associated divisor (s) equals E (in particular s vanishes exactly on E) and let $\|s\|_\tau$ be the norm of s with respect to τ.

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$$l\pi^*\omega - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log\|s\|_h^2)^2$$

for l a positive integer, will be called a Saper-type metric, distinguished with respect to the map π.

The corresponding metric on $M \setminus \text{sing } V \cong \tilde{M} \setminus E$ and its restriction to $V \setminus \text{sing } V$ are also called Saper-type metric.

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Concerning the existence Grant Melles and Milman proved the following theorem.

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Let V be an analytic subvariety of a compact Kähler manifold M and let ω be the Kähler $(1,1)$-form of a Kähler metric on M.

There exists a C^∞ function F on M, vanishing only on $\mathrm{sing}(V)$, such that the $(1,1)$-form

$$\omega_S = \omega - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log F)^2$$

is the Kähler form of a complete Saper-type metric on $M \setminus \mathrm{sing}(V)$ and hence on $V \setminus \mathrm{sing}(V)$.

In the general case (no assumptions on sing(V)) we have the following result for the L^2-$\overline{\partial}$-cohomology of a Saper type metric:

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Let $\pi : \widetilde{V} \to V$ be a resolution of V. Finally let g_S be a Saper-type metric on $\text{reg}(V)$ as constructed by Grant Melles-Milman.

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Then the following isomorphisms hold:

$$H^{v,q}_{2,\overline{\partial}}(\mathrm{reg}(V), g_S) \cong H^{v,q}_{\overline{\partial}}(\widetilde{V})$$

and

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for every $q = v, ..., n$.

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- Theorem B.-Piazza

Applications to negatively curved Kähler manifolds

Theorem (Siu-Yau, 1982)

(M, h) complete Kähler manifold of finite volume and $-a^2 \leq sec_h \leq -b^2 < 0$. Then there exists a complex projective variety $V \subset \mathbb{CP}^N$ with only isolated singularities and a biholomorphism $\phi : M \to \mathrm{reg}(V)$.

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Theorem

Let (M, h) be as in the previous theorem. Let V be the Siu-Yau compactification and let $\pi : N \to V$ be a resolution of V. Then

$$H^{m,q}_{2,\overline{\partial}}(M, h) \cong H^{m,q}_{\overline{\partial}}(N)$$

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Grazie per l'attenzione!