Supergeometria, Supervarietà di Calabi-Yau "Non-Projected" e loro Immersioni

Simone Noja

Università del Piemonte Orientale
simone.noja@uniupo.it

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Motivations and Premises

Why Supergeometry?

- Modern QFT's: bosonic/even and fermionic/odd fields
- Supersymmetry: a symmetry that relates bosonic/even and fermionic/odd fields

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- Superstring Theory: the (supposedly...) "Theory of Everything"

$$\Downarrow$$

Superstrings are complex supermanifolds ("Super Riemann Surfaces")!

Supergeometry Enters Theoretical Physics

Mathematics into Physics

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Loop Amplitudes in Superstrings - The Path Integral

Where all begins: the Path Integral

$$Z_{vac} = \int [\mathcal{D} \, \text{Fields}] \exp\left(-S_{\text{TOT}}\right)$$

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Superstring Quantization $\Longleftrightarrow$ Reduction to Supermoduli Space $\mathfrak{M}_g$

$\mathfrak{M}_g = \{$isomorphy classes of super Riemann surfaces $\mathcal{S}\Sigma_g$ of genus $g\}$

$$\dim_{\mathbb{C}} \mathfrak{M}_g = \begin{cases} 0|0 & g = 0 \\ 1|0_e \; 1|1_o & g = 1 \\ 3g-3|2g-2 & g \geq 2. \end{cases}$$

Superstring Perturbation Theory - Partition Function

Superstring Partition Function = Sum Over Topologies

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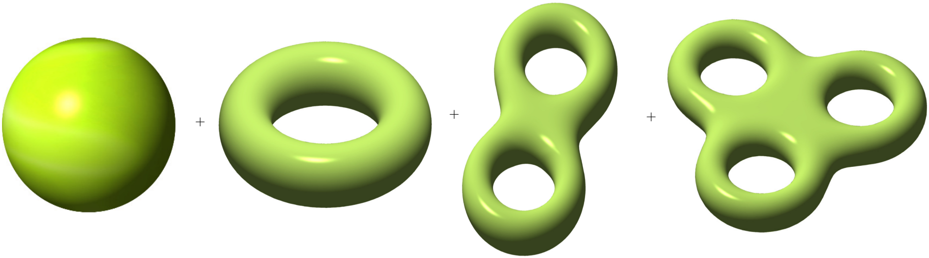

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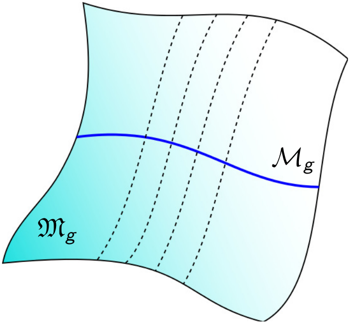

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- deal with $\mathcal{M}_g^{spin}$ instead;

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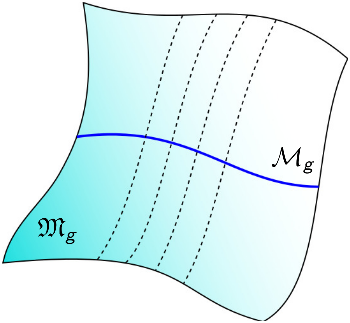

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Look for a **global holomorphic projection**

$$\pi_{hol} : \mathfrak{M}_g \to \mathcal{M}_g^{spin}$$

Supermoduli Space is Not Projected

Theorem (Donagi-Witten 2013)

For $g \geq 5$, the supermoduli space $\mathfrak{M}_g$ is **not** *projected.*
That is, there is no global holomorphic projection

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...so what?

- **The Physics:**
  1. issues in computing higher loop amplitudes: maybe divergencies?
  2. no reliable methods for higher loops amplitudes!
- **The Mathematics:**
  1. what about $g = 3$ and $g = 4$? (...and also $g = 2$)
  2. ...call for a deeper understanding of non-projected supermanifolds!

Superspace and Local Models

Definition (Superspace)

A superspace is a pair $(|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$, where

- $|\mathcal{M}|$ is a topological space;
- $\mathcal{O}_{\mathcal{M}}$ is a sheaf of superalgebras over $|\mathcal{M}|$ and such that the stalks $\mathcal{O}_{\mathcal{M},x}$ at every point of $|\mathcal{M}|$ are local rings.

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Definition (Local Model $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$)

Let $|\mathcal{M}|$ be a topological space and $\mathcal{E}$ a vector bundle over $|\mathcal{M}|$. Then we call $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$ the superspace such that

- $|\mathcal{M}|$ is the underlying topological space;
- $\mathcal{O}_{\mathcal{M}}$ is given by the $\mathcal{O}_{|\mathcal{M}|}$-valued sections of the exterior algebra $\bigwedge^{\bullet} \mathcal{E}^*$.

Examples of Local Models

Affine Superspaces $\mathbb{A}^{n|m}$

Affine Superspaces $\mathbb{A}^{n|m}$ are constructed as the local models $\mathfrak{S}(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n}^{\oplus m})$, where

- $\mathbb{A}^n$ is the n-dimensional affine space over the ring (or field) $\mathbb{A}$;
- $\mathcal{O}_{\mathbb{A}^n}$ is the trivial sheaf over it.

$\mathbb{R}^{n|m}$ and $\mathbb{C}^{n|m}$

- These are the most common example of superspaces in Theoretical Physics;
- They enter the definition of differentiable and complex supermanifolds respectively!

Supermanifolds

Definition (Supermanifold)

A supermanifold is a superspace $\mathcal{M}$ that is locally isomorphic to some local model $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$.
Let $\{U_i\}_{i\in I}$ be an open covering of $|\mathcal{M}|$, then $\mathcal{O}_\mathcal{M}$ is described via a collection $\{\psi_{U_i}\}_{i\in I}$ of local isomorphisms of sheaves

$$U_i \longmapsto \psi_{U_i} : \mathcal{O}_\mathcal{M}\lfloor_{U_i} \longrightarrow \bigwedge^{\bullet} \mathcal{E}^* \lfloor_{U_i}$$

where is $\bigwedge^\bullet \mathcal{E}^*$ the sheaf of sections of the exterior algebra of $\mathcal{E}$.

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Complex Supermanifolds

Are characterized by *holomorphic* local models.

- $|M|$ has a complex manifold structure.
- $\mathcal{E}$ is a holomorphic vector bundle.

Split Supermanifolds: Projective Superspaces

Split Supermanifold

If one has a **global** isomorphism $\mathcal{O}_{\mathcal{M}} \cong \bigwedge^{\bullet} \mathcal{E}^{*}$ then $\mathcal{M}$ is said **split**.

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Transition functions on an intersection $\mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta}$ of a split supermanifold:

- **even**: $z^{i}_{\alpha} = f^{i}_{\alpha\beta}(z^{1}_{\beta}, \ldots z^{n}_{\beta}) \longrightarrow$ ordinary complex manifolds;
- **odd**: $\theta^{j}_{\alpha} = \sum_{\ell=1}^{m} g^{\ell}_{\alpha\beta}(z^{1}_{\beta}, \ldots z^{n}_{\beta})\theta^{\ell}_{\beta} \longrightarrow$ vector bundle (rank $0|m$)

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A split supermanifolds can be looked at as the total space of a certain fermionic/odd vector bundle.

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Projective Superspaces $\mathbb{P}^{n|m} = \mathfrak{S}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus m})$

$$\mathcal{O}_{\mathbb{P}^{n|m}} = \bigoplus_{k \text{ even}} \bigwedge^k \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m} \oplus \bigoplus_{k \text{ odd}} \bigwedge^k \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m}$$

Projected and Non-Projected Supermanifolds

Definition (Nilpotent Sheaf $\mathcal{J}_{\mathcal{M}}$)

Given $\mathcal{M}$ we will call $\mathcal{J}_{\mathcal{M}}$ the sheaf of ideals generated by all the (nilpotent) odd sections.

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Definition (Reduced Manifold $\mathcal{M}_{red}$)

Given $\mathcal{M} = (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$, we call reduced manifold $\mathcal{M}_{red}$ the ordinary manifold given as a ringed space by the pair $(|\mathcal{M}|, \mathcal{O}_{\mathcal{M}_{red}})$, where $\mathcal{O}_{\mathcal{M}_{red}}$ is defined as $\mathcal{O}_{\mathcal{M}_{red}} := \mathcal{O}_{\mathcal{M}} \, / \mathcal{J}_{\mathcal{M}} \, .$

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Definition (Structural Exact Sequence)

The sheaves $\mathcal{J}_{\mathcal{M}}$, $\mathcal{O}_{\mathcal{M}}$ and $\mathcal{O}_{\mathcal{M}_{red}}$ fit together into

$$0 \longrightarrow \mathcal{J}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}} \xrightarrow{\iota^\sharp} \mathcal{O}_{\mathcal{M}_{red}} \longrightarrow 0.$$

The maps $\iota^\sharp : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}_{red}}$ corresponds to the **inclusion** $\mathcal{M}_{red} \hookrightarrow \mathcal{M}$.

Projected and Non-Projected Supermanifolds

Does the Structural Exact Sequence split?

Does exist a morphism $\pi^\sharp : \mathcal{O}_{\mathcal{M}_{red}} \to \mathcal{O}_{\mathcal{M}}$ such that $\pi^\sharp \circ \iota^\sharp = Id_{\mathcal{O}_{\mathcal{M}}}$?

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Definition (Projected Supermanifolds)

A supermanifold that admits such a projection is said to be projected.

$$0 \longrightarrow \mathcal{J}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}_{red}} \oplus \mathcal{J}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}_{red}} \longrightarrow 0.$$

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Definition (Projected Supermanifolds)

The structure sheaf of a projected supermanifold is a sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-algebras:

ONE CAN USE ALL OF THE ORDINARY ALGEBRAIC/COMPLEX GEOMETRIC TOOLS TO STUDY PROJECTED SUPERMANIFOLDS!

$\mathcal{N} = 1$ Supermanifolds Are Projected

Definition (Fermionic Sheaf)

We call **fermionic sheaf** $\mathcal{F}_{\mathcal{M}}$ the sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules given by $\mathcal{J}_{\mathcal{M}} \,/\, \mathcal{J}^2_{\mathcal{M}}$.

Theorem (Supermanifolds of dimension $n|1$ ($\mathcal{N} = 1$))

Let $\mathcal{M} := (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$ a (complex) supermanifold of odd dimension 1. Then $\mathcal{M}$ is defined up to isomorphism by the pair $(\mathcal{M}_{red}, \mathcal{F}_{\mathcal{M}})$.

Why is it so?

- The topology is fixed by the underlying manifolds $\mathcal{M}_{red}$.
- The parity splitting is: $\mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M},0} \oplus \mathcal{O}_{\mathcal{M},1}$, then:
  1. $\mathcal{O}_{\mathcal{M},1} = \mathcal{J}_{\mathcal{M}} = \mathcal{F}_{\mathcal{M}}$;
  2. $\mathcal{O}_{\mathcal{M},0} = \mathcal{O}_{\mathcal{M}_{red}}$.

It follows that $\mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M}_{red}} \oplus \mathcal{F}_{\mathcal{M}}$.

$\mathcal{N} = 2$ Supermanifolds

$\mathcal{O}_{\mathcal{M},0}$ is an Extension of $\mathcal{O}_{\mathcal{M}_{red}}$ by $Sym^2 \mathcal{F}_{\mathcal{M}}$

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Theorem (Obstruction to Splitting)

Let $\mathcal{M}$ be a supermanifold of odd dimension 2.

- The even part of the structure sheaf $\mathcal{O}_{\mathcal{M},0}$ uniquely defines a class $\omega_{\mathcal{M}} \in H^1(\mathcal{M}_{red}, \mathcal{T}_{\mathcal{M}_{red}} \otimes Sym^2 \mathcal{F}_{\mathcal{M}})$.
- $\mathcal{M}$ is projected if and only if the obstruction class $\omega_{\mathcal{M}}$ is zero in $H^1(\mathcal{M}_{red}, \mathcal{T}_{\mathcal{M}_{red}} \otimes Sym^2 \mathcal{F}_{\mathcal{M}})$.

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Theorem (Supermanifolds of dimension $n|2$)

Let $\mathcal{M} := (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$ be a complex supermanifold of dimension $n|2$. Then $\mathcal{M}$ is defined up to isomorphism by the triple $(\mathcal{M}_{red}, \mathcal{F}_{\mathcal{M}}, \omega_{\mathcal{M}})$ where $\omega_{\mathcal{M}} \in H^1(\mathcal{M}_{red}, \mathcal{T}_{\mathcal{M}_{red}} \otimes Sym^2\mathcal{F}_{\mathcal{M}})$.

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Even Transition Functions

The even transition functions gets "corrected" by $\omega_{\mathcal{M}}$!

In an intersection $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ we have:

$$z^i_\alpha(\underline{z}_\beta, \underline{\theta}_\beta) = z^i_\alpha(\underline{z}_\beta) + \omega_{\alpha\beta}(\underline{z}_\beta, \underline{\theta}_\beta)(z^i_\alpha) \qquad i = 1, \ldots, n,$$

where

- $\omega_{\alpha\beta}$ is a representative of $\omega_{\mathcal{M}}$;
- the theta's can only appear through their product $\theta_{1\beta}\theta_{2\beta}$ in $\omega_{\alpha\beta}$: indeed $\omega_{\alpha\beta}$ takes values into $Sym^2 \mathcal{F}_{\mathcal{M}}$

$\mathcal{N} = 2$ Supermanifolds over $\mathbb{P}^n$

How to evaluate $H^1(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n} \otimes Sym^2 \mathcal{F}_{\mathcal{M}})$?

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$Sym^2 \mathcal{F}_{\mathcal{M}} \cong \mathcal{O}_{\mathbb{P}^n}(k)$

$\mathcal{F}_{\mathcal{M}}$ is a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules of rank $0|2 \Rightarrow$ $Sym^2 \mathcal{F}_{\mathcal{M}}$ is a line bundle on $\mathbb{P}^n$ $\Rightarrow$ $Sym^2 \mathcal{F}_{\mathcal{M}} \cong \mathcal{O}_{\mathbb{P}^n}(k)$ for some $k \in \mathbb{Z}$.

It follows that

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How to evaluate $H^1(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}(k))$?

We use the twisted Euler sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k+1)^{\oplus n+1} \longrightarrow \mathcal{T}_{\mathbb{P}^n}(k) \longrightarrow 0.$$

Non-Projected $\mathcal{N} = 2$ Supermanifolds over $\mathbb{P}^n$

$\mathcal{M}_{red} = \mathbb{P}^1$

- $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2+k)) \neq 0 \iff k \leq -4.$
- $\mathcal{M}$ non-projected $\iff \mathcal{F}_{\mathcal{M}} = \mathcal{O}_{\mathbb{P}^1}(\ell_1) \oplus \mathcal{O}_{\mathbb{P}^1}(\ell_2)$ such that $\ell_1 + \ell_2 \leq -4.$

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$\mathcal{M}_{red} = \mathbb{P}^2$

- $H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(k)) \neq 0 \iff k = -3.$ In particular

$$H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-3)) \cong H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \cong H^2(\mathbb{P}^2, K_{\mathbb{P}^2}) \cong \mathbb{C}.$$

Non-Projected $\mathcal{N} = 2$ Supermanifolds over $\mathbb{P}^n$

$\mathcal{M}_{red} = \mathbb{P}^1$

- $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2+k)) \neq 0 \iff k \leq -4.$
- $\mathcal{M}$ non-projected $\iff \mathcal{F}_{\mathcal{M}} = \mathcal{O}_{\mathbb{P}^1}(\ell_1) \oplus \mathcal{O}_{\mathbb{P}^1}(\ell_2)$ such that $\ell_1 + \ell_2 \leq -4.$

$\mathcal{M}_{red} = \mathbb{P}^2$

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$\mathcal{M}_{red} = \mathbb{P}^n$ for $n \geq 3$

- $H^1(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}(k)) = 0 \ \forall k \in \mathbb{Z}.$
- All of the supermanifolds $\mathcal{N} = 2$ over $\mathbb{P}^n$ with $n \geq 3$ are projected!.

Non-Projected $\mathcal{N} = 2$ Supermanifolds over $\mathbb{P}^1$

Theorem ($\mathbb{P}^1_\omega(m, n)$)

Every non-projected $\mathcal{N} = 2$ supermanifold over $\mathbb{P}^1$ is characterised up to isomorphism by a triple $(\mathbb{P}^1, \mathcal{F}_\mathcal{M}, \omega)$ where $\mathcal{F}_\mathcal{M}$ is a rank $0|2$ locally-free sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules such that

$$\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n)$$

with $m + n = -\ell$, $\ell \geq 4$ and ω is a non-zero cohomology class in $H^1(\mathcal{O}_{\mathbb{P}^1}(2 - \ell)) \cong \mathbb{C}^{\ell-3}$.
The even transition function of the supermanifold reads:

$$z = \frac{1}{w} + \sum_{j=1}^{\ell-3} \lambda_j \frac{\psi_1 \psi_2}{w^{2+j}},$$

where $\lambda_i \in \mathbb{C}$ for $i = 1, \ldots, \ell - 3$.

Non-Projected $\mathcal{N} = 2$ Supermanifolds over $\mathbb{P}^2$

Theorem ($\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$)

Let $\mathcal{M}$ be a supermanifold over $\mathbb{P}^2$ having odd dimension equal to 2. Then $\mathcal{M}$ is non-projected if and only if it arises from a triple $(\mathbb{P}^2, \mathcal{F}_\mathcal{M}, \omega)$ where $\mathcal{F}_\mathcal{M}$ is a rank $0|2$ locally free sheaf of $\mathcal{O}_{\mathbb{P}^2}$-modules such that $Sym^2\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ and ω is a non-zero cohomology class $\omega \in H^2(\mathcal{O}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$.

One can write the transition functions for an element of the family $\mathbb{P}^2_\omega(\mathcal{F})$ from coordinates on $\mathcal{U}_0$ to coordinates on $\mathcal{U}_1$ as follows

$$z_{10} = \frac{1}{z_{11}}, \qquad z_{20} = \frac{z_{21}}{z_{11}} + \lambda \frac{\theta_{11}\theta_{21}}{(z_{11})^2}$$

$$\begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix} = M \begin{pmatrix} \theta_{11} \\ \theta_{21} \end{pmatrix}$$

where $\lambda \in \mathbb{C}$ is a representative of the class $\omega \in H^1(\mathcal{T}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$ and M is a 2×2 matrix with coefficients in $\mathbb{C}[z_{11}, z_{11}^{-1}, z_{21}]$ such that $\det M = 1/z_{11}^3$. Similar transformations hold between the other pairs of open sets.

Properties of $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$

Theorem ($\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ is Calabi-Yau)

Regardless how one chooses $\mathcal{F}_\mathcal{M}$, if it is such that $Sym^2\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, then $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ is a Calabi-Yau supermanifold, that is

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Definition (CY Manifold)

A CY manifold is a Kähler manifold with trivial canonical sheaf $K_M = \bigwedge^{top} T^*_M$.

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...this definition is meaningless in supergeometry:

- there is no top-form in supergeometry: $(d\theta)^n := d\theta \wedge \ldots \wedge d\theta \neq 0 \quad \forall n \in \mathbb{N}$
- the De Rham complex is not bounded from above on a supermanifold!

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- $\mathcal{B}er(\mathcal{M}) \rightsquigarrow [dx_1 \ldots dx_n | d\theta_1 \ldots d\theta_m] = \mathrm{ber}(\mathcal{S}Jac\,\Phi)[dx'_1 \ldots dx'_n | d\theta'_1 \ldots d\theta'_m].$

Sigma Models and Embeddings

...what to do now?

The idea is to look at these “strange” non-projected geometries inside “friendly” varieties, such as projective superspaces $\mathbb{P}^{n|m}$.

Embeddings of Supermanifolds

...in other words, we are looking for an **embedding** of supermanifolds

$$\varphi : \mathcal{M} \hookrightarrow \mathbb{P}^{n|m}$$

Theorem (“Very Ample” Line Bundles and Embedding)

If $\mathcal{E}$ is a certain globally-generated sheaf of $\mathcal{O}_{\mathcal{M}}$-modules of rank $1|0$, having $n+1|m$ global sections $\{s_0, \ldots, s_n | \xi_1, \ldots, \xi_m\}$, then there exists a morphism $\phi_{\mathcal{E}} : \mathcal{M} \to \mathbb{P}^{n|m}$ such that $\mathcal{E} = \phi^*_{\mathcal{E}}(\mathcal{O}_{\mathbb{P}^{n|m}}(1))$ and such that $s_i = \phi^*_{\mathcal{E}}(X_i)$ and $\xi_j = \phi^*_{\mathcal{E}}(\Theta_j)$ for $i = 0, \ldots, n$ and $j = 1, \ldots, m$.

Embeddings

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The idea is to look at these “strange” non-projected geometries inside more regular and friendly varieties, such as projective superspaces $\mathbb{P}^{n|m}$.

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Theorem (“Embedding for Projected Supermanifolds)

Any projected supermanifold whose reduced manifold $\mathcal{M}_{red}$ is projective, i.e. $\exists\, \varphi_{red} : \mathcal{M}_{red} \to \mathbb{P}^n$, is **super-projective**, *i.e. $\exists\, \varphi : \mathcal{M} \to \mathbb{P}^{n|m}$.*

“Proof”

Let $\pi : \mathcal{M} \to \mathcal{M}_{red}$ be the projection and $\mathcal{L}_{red}$ a very ample line bundle on $\mathcal{M}_{red}$. Then $\pi^* \mathcal{L}_{red}$ is very ample on $\mathcal{M}$.

Obstructions to Embed a Supermanifolds

...Issues and Obstructions

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Theorem

Let $\mathcal{M}$ be a complex supermanifold and let $\varphi_{red} : \mathcal{M}_{red} \hookrightarrow \mathbb{P}^n$ an embedding of its reduced manifold. Then the obstructions to extending φ_{red} to an embedding $\varphi : \mathcal{M} \hookrightarrow \mathbb{P}^{n|m}$ are elements of $H^2(\mathcal{M}, Sym^{2k}\mathcal{F}_{\mathcal{M}}) \neq 0$ for $k = 1, \ldots, \operatorname{rank}\mathcal{F}_{\mathcal{M}}/2$

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Any Super Curve is Super Projective

...indeed $H^2(\mathcal{M}, \mathcal{G}) = 0$ for any coherent sheaf $\mathcal{G}$ on a (compact) curve, simply because $\dim \mathcal{M}_{red} = 1 < 2$!

Embedding of a Super Curve

The Supermanifold $\mathbb{P}^1_\omega(2,2)$

Let us consider the easiest example of **non-projected** supermanifold over $\mathbb{P}^1$: it is characterized by transition functions

$$z = \frac{1}{w} + \frac{\psi_1\psi_2}{w^3}, \qquad \theta_1 = \frac{\psi_1}{w^2}, \qquad \theta_2 = \frac{\psi_2}{w^2}.$$

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It has a very ample line bundle that admits an embedding $\varphi : \mathbb{P}^1_\omega(2,2) \hookrightarrow \mathbb{P}^{2|2}$, whose image in $\mathbb{P}^{2|2}$ is given by the equation

$$X_0^2 + X_1^2 + X_2^2 + \Theta_1 \Theta_2 = 0.$$

where X_i, Θ_j are the homogeneous coordinates of $\mathbb{P}^{2|2}$.

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...so what?

Non-projected supermanifolds are ubiquitous in complex supergeometry!

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The non-projected supermanifold $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ cannot be embedded in any projective superspace $\mathbb{P}^{n|m}$, regardless how one chooses $\mathcal{F}_\mathcal{M}$!

Indeed one finds that:

1. it has trivial Picard group $H^1(\mathbb{P}^2, \mathcal{O}^*_{\mathcal{M},0}) \cong 0$;
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...$\mathbb{P}^{n|m}$ is not special!

- $\mathbb{P}^{n|m}$ is not a privileged ambient for complex algebraic supergeometry!

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- $\mathbb{P}^{n|m}$ is not a privileged ambient for complex algebraic supergeometry!
- ...is there any suitable embedding space though?

Embedding in Super Grassmannians

Theorem (Existence of Embedding)

Let $\mathcal{M}$ be a non-projected supermanifold of the family $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ and $\mathcal{T}_\mathcal{M}$ its tangent sheaf. Let $V = H^0(Sym^k \mathcal{T}_\mathcal{M})$. Then, for any $k \gg 0$ the evaluation map $V \otimes \mathcal{O}_\mathcal{M} \to Sym^k \mathcal{T}_\mathcal{M}$ induces an embedding

$$\Phi_k : \mathcal{M} \longrightarrow \mathbb{G}(2k|2k, V).$$

Super Grassmannians

Definition (Super Grassmannians)

A super Grassmannian $\mathbb{G}(a|b; V^{n|m})$ is a universal parameter space for $a|b$-dimensional linear subspaces of a given $n|m$-dimensional space $V^{n|m}$

Properties of Super Grassmannians

- Super Grassmannians are in general **non-projected**;
- Super Grassmannians are in general **non-projective**.

$\mathbb{G}(1|1; \mathbb{C}^{2|2})$

- $\mathbb{G}(1|1; \mathbb{C}^{2|2})$ is non-projected:

$$H^1(\mathcal{T}_{\mathbb{P}^1_0 \times \mathbb{P}^1_1} \otimes Sym^2 \mathcal{F}_G) \cong \mathbb{C} \oplus \mathbb{C}$$

- $\mathbb{G}(1|1; \mathbb{C}^{2|2})$ is non-projective: $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\ell, -\ell)$ lifts to $\mathbb{G}(1|1; \mathbb{C}^{2|2})$ but it has no cohomology!

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...Super Grassmannians as universal embedding spaces?

Conjecture

Let $\mathcal{M}$ be a smooth complex supermanifold and let $\mathcal{M}_{red}$ be projective.

Then $\mathcal{M}$ can be embedded in some super Grassmannians.

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...some open problems:

- The Theorem is not effective!
  1. it does not identify the target super Grassmannian;
  2. k depends heavily on the choice of $\mathcal{F}_\mathcal{M}$;
  3. $\Rightarrow$ calculate a uniform k and dim V.
- Extend the Theorem to **all** (non-projected) supermanifolds $dim(\mathcal{M}_{red}) \geq 2$!
  1. $\mathcal{T}_{\mathcal{M}_{red}}$ is not ample for $\mathcal{M}_{red} \neq \mathbb{P}^n$;
  2. $\Rightarrow$ identify an ample locally-free sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules on $\mathcal{M}_{red}$ and extend it to a locally-free sheaf on $\mathcal{M}$.

Two Explicit Examples

Two Homogeneous Fermionic Sheaves

- **Decomposable:** $\mathcal{F}_{\mathcal{M}} = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$;
- **Non-Decomposable:** $\mathcal{F}_{\mathcal{M}} = \Omega^1_{\mathbb{P}^2}$.

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Theorem (Embedding using $\mathcal{T}_{\mathcal{M}}$)

- **Decomposable:** $i : \mathbb{P}^2_\omega(\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)) \longhookrightarrow \mathbb{G}(2|2; \mathbb{C}^{12|12})$.
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$\mathcal{F}_{\mathcal{M}} = \Omega^1_{\mathbb{P}^2}$: a Minimal Embedding

$$i_M : \mathbb{P}^2_\omega(\Omega^1_{\mathbb{P}^2}) \hookrightarrow \mathbb{G}(1|1; \mathbb{C}^{3|3}).$$

$$i_M(\mathcal{M})\lfloor_{\mathcal{Z}_0} = \left(\begin{array}{ccc||ccc} 1 & z_{10} & z_{20} & 0 & \theta_{10} & \theta_{20} \\ \hline 0 & -\theta_{10} & -\theta_{20} & 1 & z_{10} & z_{20} \end{array} \right)$$

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