ON THE CORANK OF GAUSSIAN MAPS FOR GENERAL EMBEDDED K3 SURFACES

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ABSTRACT: Let S_g be a general prime K3 surface in $\mathbb{P}^g$ of genus $g \geq 3$ or a general double cover of $\mathbb{P}^2$ ramified along a sextic curve for $g = 2$ and $S_{i,g}$ its i-th Veronese embedding. In this article we compute the corank of the Gaussian map $\Phi_{\mathcal{O}_{S_{i,g}}(1)} : \bigwedge^2 H^0(S_{i,g}, \mathcal{O}_{S_{i,g}}(1)) \to H^0(S_{i,g}, \Omega^1_{S_{i,g}}(2))$ for $i \geq 2, g \geq 2$ and $i = 1, g \geq 17$. The main idea is to reduce the surjectivity of $\Phi_{\mathcal{O}_{S_{i,g}}(1)}$ to an application of the Kawamata-Viehweg vanishing theorem on the blow-up of $S_{i,g} \times S_{i,g}$ along the diagonal. This is seen to apply once the hyperplane divisor of the K3 surface $S_{i,g}$ can be decomposed as a sum of three suitable birationally ample divisors. We show that such a decomposition exists when $i \geq 3$ or on some K3 surfaces, constructed using the surjectivity of the period mapping, when $i = 1, g \geq 17$ or $i = 2, g \geq 7$.

1. INTRODUCTION

Let $\mathcal{H}$ be the disjoint union of the Hilbert schemes of smooth K3 surfaces in $\mathbb{P}^g$, for all $g \geq 3$. It follows by the transcendental theory that $\mathcal{H}$ has some main components $\mathcal{H}_g$ whose general element is a genus g K3 surface $S_g \subset \mathbb{P}^g$ of degree $2g-2$ with Picard group generated by its hyperplane class (the so called *prime* K3 surfaces), while all the other components $\mathcal{H}_{i,g}$ for $i \geq 2$ are obtained by re-embedding prime K3 surfaces via the i-th Veronese map.

In this paper we will compute the corank of the Gaussian map

$$\Phi_{\mathcal{O}_{S_{i,g}}(1)} : \bigwedge^2 H^0(S_{i,g}, \mathcal{O}_{S_{i,g}}(1)) \to H^0(S_{i,g}, \Omega^1_{S_{i,g}}(2))$$

on a general embedded K3 surface, that is a K3 surface $S_{i,g}$ representing a general point of $\mathcal{H}_{i,g}$ (where we set $\mathcal{H}_{1,g} = \mathcal{H}_g, S_{1,g} = S_g$ and $\mathcal{H}_2$ is the family of genus 2 K3 surfaces,

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that is double covers of $I\!P^2$ ramified along a sextic curve). This is of course equivalent to computing the corank of the Gaussian map $\Phi_{\mathcal{O}_{S_g}(i)} : \bigwedge^2 H^0(S_g, \mathcal{O}_{S_g}(i)) \to H^0(S_g, \Omega^1_{S_g}(2i))$. The main technique that we will use to study this map was suggested to us by L. Ein and it consists, as we will see in section 2, in the fact that its surjectivity follows by the Kawamata-Viehweg vanishing theorem once the hyperplane divisor of the K3 surface $S_{i,g}$ can be decomposed as a sum of three suitable birationally ample divisors (Lemmas (2.1) and (2.2)). Therefore this approach is particularly effective, and in fact it gives sharp results, when such a decomposition exists, that is on some surfaces representing points in $\mathcal{H}_g$ for high enough genus or in $\mathcal{H}_{i,g}, i \geq 2$ where the hyperplane divisor is divisible. The existence of K3 surfaces whose i-th Veronese embedding has hyperplane divisor decomposable as above will then be treated in section 3, mainly using the surjectivity of the period mapping.

Our result is as follows:

Theorem (1.1). *For $i \geq 1$ and $g \geq 2$ let $\mathcal{H}_{i,g}$ be the component of the Hilbert scheme of K3 surfaces whose general element $S_{i,g}$ is the i-th Veronese embedding of a prime genus g K3 surface. Then the corank of $\Phi_{\mathcal{O}_{S_{i,g}}(1)}$ is given by the following table:*

i	g	*corank* $\Phi_{\mathcal{O}_{S_{i,g}}(1)}$
1	≥ 17	0
2	2	3
2	≥ 3	0
3	2	8
3	≥ 3	0
≥ 4	≥ 2	0

.

Moreover for all i, g in the above table (with $i \geq 5$ if $g = 2$), the Gaussian maps $\Phi_{\mathcal{O}_{S_{i,g}}(1), \mathcal{O}_{S_{i,g}}(k)}$ are surjective for all $k \geq 2$.

Remark that it remains to compute the corank for $i = 1, g \leq 16$; on the other hand the above result for $i = 1, g \geq 17$ allows a generalization of a theorem of [CLM] (see below).

As is well-known now Gaussian maps give a new interesting approach in the study of algebraic varieties (see for example [W1], [W3], [CLM], [Z]). In particular, in the above case, the knowledge of the corank of $\Phi_{\mathcal{O}_{S_{i,g}}(1)}$ gives, upon restricting to a smooth hyperplane section $C_{i,g}$ of $S_{i,g}$, the possibility of computing the corank of the Gaussian map (or Wahl map)

$$\Phi_{\omega_{C_{i,g}}} : \bigwedge^2 H^0(C_{i,g}, \omega_{C_{i,g}}) \to H^0(C_{i,g}, \omega_{C_{i,g}}^{\otimes 3}) .$$

The latter is a particularly interesting invariant as it encodes information both on the component $\mathcal{H}_{i,g}$ of the Hilbert scheme (since it gives the dimension of the tangent space at points representing cones over $C_{i,g}$) and on the possibility of extending $C_{i,g}$ as a curve section of higher dimensional varieties (via Zak's theorem [Z], [BEL]). The main application that we have in mind is to use the knowledge of *corank* $\Phi_{\omega_{C_{i,g}}}$ to classify Fano threefolds of index greater than one, in the same vein as [CLM] where we carried out this project for prime Fano threefolds. The calculation of *corank* $\Phi_{\omega_{C_{i,g}}}$ (that gives an extension, for $g \geq 17$, of Theorem 4.1 of [CLM]) and the study of its consequences on Fano threefolds of index greater than one will appear in a forthcoming paper.

2. EIN'S APPROACH TO THE SURJECTIVITY OF GAUSSIAN MAPS

Let S be a smooth embedded K3 surface, $\mathcal{O}_S(1)$ its hyperplane bundle. In this section we will give some sufficient conditions for the surjectivity of the Gaussian map $\Phi_{\mathcal{O}_S(1)}$: $\bigwedge^2 H^0(S, \mathcal{O}_S(1)) \to H^0(S, \Omega^1_S(2))$, which is defined by $\Phi_{\mathcal{O}_S(1)}(\sigma \wedge \tau) = \sigma d\tau - \tau d\sigma$ (on any open subset where $\mathcal{O}_S(1)$ is trivial).

The main idea, that was suggested to us by L. Ein, is to blow up $S \times S$ along its diagonal Δ and then use the Kawamata-Viehweg vanishing theorem. To this end let us recall that a line bundle A on a projective variety X is said to be *nef* if $c_1(A) \cdot \Gamma \geq 0$ for every irreducible curve $\Gamma \subset X$; A is *big* if for some $m > 0$ the rational map defined by mA on X is birational. The Kawamata-Viehweg vanishing theorem asserts that $H^i(X, \omega_X \otimes A) = 0$ for $i > 0$ if X is smooth and A is big and nef.

Now let Y be the blow-up of $S \times S$ along its diagonal Δ, E the exceptional divisor and for every sheaf $\mathcal{F}$ on S let us denote by $\mathcal{F}_i, i = 1, 2$, its pull-back via the map $Y \to S \times S \xrightarrow{p_i} S$ where p_i is the i-th projection. Then we have

Lemma (2.1). *Suppose there are line bundles A_1, A_2, A_3 on S such that* *(i)* $\mathcal{O}_S(1) \cong A_1 \otimes A_2 \otimes A_3$*;*

(ii) $\bigotimes_{j=1}^{3} [A_{j1} \otimes A_{j2}(-E)]$ *is big and nef on* Y*.*

Then $\Phi_{\mathcal{O}_S(1)}$ *is surjective.*

Proof: Let $\mathcal{I}_\Delta$ be the ideal sheaf of $\Delta \subset S \times S$, $L = \mathcal{O}_S(1)$ and consider the exact sequence

$$0 \to p_1^* L \otimes p_2^* L \otimes \mathcal{I}_\Delta^2 \to p_1^* L \otimes p_2^* L \otimes \mathcal{I}_\Delta \to p_1^* L \otimes p_2^* L \otimes \frac{\mathcal{I}_\Delta}{\mathcal{I}_\Delta^2} \to 0.$$

It is a standard fact that the isomorphism $H^0(\Delta, p_1^* L \otimes p_2^* L \otimes \frac{\mathcal{I}_\Delta}{\mathcal{I}_\Delta^2}) \cong H^0(S, \Omega_S^1(2))$ gives

$$Coker \Phi_L \cong Coker\{H^0(S \times S, p_1^* L \otimes p_2^* L \otimes \mathcal{I}_\Delta) \to H^0(\Delta, p_1^* L \otimes p_2^* L \otimes \frac{\mathcal{I}_\Delta}{\mathcal{I}_\Delta^2})\}$$

hence the surjectivity of Φ_L is implied by the vanishing of $H^1(S \times S, p_1^* L \otimes p_2^* L \otimes \mathcal{I}_\Delta^2) \cong H^1(Y, L_1 \otimes L_2(-2E))$. On the other hand, since $\omega_Y = \mathcal{O}_Y(E)$, we have $L_1 \otimes L_2(-2E) = \omega_Y \otimes \bigotimes_{j=1}^{3} [A_{j1} \otimes A_{j2}(-E)]$ and the required vanishing follows by (ii) from the Kawamata-Viehweg vanishing theorem. ∎

As we will see below if we have three very ample line bundles as in (i) of Lemma (2.1) then (ii) is automatically satisfied. If at least one line bundle is very ample it is still possible that (ii) holds. Some sufficient conditions that are enough for our purposes are stated in the following lemma.

Let A be a line bundle on a K3 surface S with $A^2 \geq 2$, having no base points. Recall that we have the following three cases (see [SD], [Ma]) : if $A^2 \geq 4, A \neq 2B$ with $B^2 = 2$ and the associated morphism ϕ_A is birational, then in fact it is an embedding or an isomorphism off some irreducible curves Z such that $Z^2 = -2, Z \cdot A = 0$; if $A^2 = 2$ then ϕ_A is a 2:1 morphism onto $\mathbb{P}^2$ and is finite if there are no irreducible curves Z such that $Z^2 = -2, Z \cdot A = 0$; if $A = 2B$ with $B^2 = 2$, then ϕ_A is a 2:1 morphism onto the Veronese surface in $\mathbb{P}^5$. In these three cases we have

Lemma (2.2). *Let* A_1, A_2, A_3 *be three base point free line bundles on* S *with* $A_j^2 \geq 2, j = 1, 2, 3$ *and such that* A_1 *is very ample and* A_2, A_3 *define either isomorphisms off one (possibly empty) curve* Z_2, Z_3 *respectively or 2:1 finite morphisms onto* $\mathbb{P}^2$ *or onto the Veronese surface in* $\mathbb{P}^5$*. Suppose that either*

(i) A_2 *and* A_3 *are very ample or*

(ii) $(A_1 + A_2 + A_3) \cdot Z_j \geq 3$ *whenever there is a* Z_j *and*

(iii) $(A_1 + A_2 + A_3) \cdot A_j \geq 9$ *whenever* $A_j^2 = 2$ *or* $(A_1 + A_2 + A_3) \cdot B_j \geq 9$ *whenever* $A_j = 2B_j, B_j^2 = 2$.

Then $\bigotimes_{j=1}^{3}[A_{j1} \otimes A_{j2}(-E)]$ *is big and nef on* Y.

Proof: If A_j is very ample the linear system $|A_{j1} \otimes A_{j2}(-E)|$ on Y has a sublinear system defining the morphism $Y \to \mathbb{G}\ (1, I\!PH^0(A_j)^*)$ associating to $(x, y) \in Y$ the linear span of $\phi_{A_j}(x)$ and $\phi_{A_j}(y)$ (note that this still makes sense if $(x, y) \in E$ since we can think of (x, y) as a pair with $x \in S$, $y \in I\!PT_{S_{|x}}$). Therefore $A_{j1} \otimes A_{j2}(-E)$ is nef and also big since the image of S in $I\!PH^0(A_j)^*$ is not ruled. If A_2 and/or A_3 is not very ample certainly the line bundle $\bigotimes_{j=1}^{3}[A_{j1} \otimes A_{j2}(-E)]$ is big since $A_{11} \otimes A_{12}(-E)$ is already big; moreover it can fail to be nef only on a curve contained in the indeterminacy locus of the maps $Y \to \mathbb{G}(1, I\!PH^0(A_j)^*), j = 2, 3$, that is a curve $Z \subset \{(x, y) \in Y : \phi_{A_j}(x) = \phi_{A_j}(y)\}$. If A_j is an isomorphism off one curve Z_j then $Z_j \cong I\!P^1$, $Z \subset Z_j \times Z_j$ (which we have identified with its blow-up in Y), and we have

$$\bigotimes_{j=1}^{3}[A_{j1} \otimes A_{j2}(-E)]_{|Z_j \times Z_j} = \mathcal{O}_{I\!P^1 \times I\!P^1}((A_1+A_2+A_3) \cdot Z_j, (A_1+A_2+A_3) \cdot Z_j)(-3\Delta_{I\!P^1 \times I\!P^1}) =$$

$$= \mathcal{O}_{I\!P^1 \times I\!P^1}((A_1 + A_2 + A_3) \cdot Z_j - 3, (A_1 + A_2 + A_3) \cdot Z_j - 3)$$

is nef on any curve $Z \subset Z_j \times Z_j \cong I\!P^1 \times I\!P^1$ by (ii). Suppose now that A_j gives a 2:1 finite morphism onto $I\!P^2$. If $Z \not\subset E$, letting $i : S \times S \to S \times S$ be the involution $i(x, y) = (y, x)$ we can assume that the image $\overline{Z}$ of Z in $S \times S$ is such that $i(\overline{Z}) = \overline{Z}$, because $\bigotimes_{j=1}^{3}[A_{j1} \otimes A_{j2}(-E)]$ is invariant under i. Then the first (or second) projection Z' of $\overline{Z}$ in S is $\phi_{A_j}^*(Z_1)$ for some curve $Z_1 \subset I\!P^2$. If N is a line in $I\!P^2$ and $Z_1 \sim mN$ we have

$$\bigotimes_{j=1}^{3}[A_{j1} \otimes A_{j2}(-E)] \cdot Z = 2(A_1 + A_2 + A_3) \cdot Z' - 3E \cdot Z =$$

$$= 2(A_1 + A_2 + A_3) \cdot m\phi_{A_j}^*(N) - 3E \cdot Z = 2m(A_1 + A_2 + A_3) \cdot A_j - 3E \cdot Z$$

and we will be done by (iii) if we show that $E \cdot Z = 6m$. To this end let B be the ramification divisor of ϕ_{A_j}; then B is a smooth plane sextic and $E \cdot Z = mB \cdot N = 6m$ provided that the intersection of E and Z is transverse. On the other hand there is a one to one correspondence between $E \cap Z$ and $\Delta \cap \overline{Z}$, and transversality can then be checked on $S \times S$. For our purpose it is of course enough to show that Δ and the pull-back of N on $S \times S$ intersect transversally. The latter being a local computation, we can assume that locally the double plane S is given by $z^2 = x$ and N is $y = 0$; then on $S \times S$ with coordinates y, z, y', z' the pull-back of N is defined by the equations $z' = -z, y = y' = 0$, hence it is transversal to the diagonal Δ of $S \times S$. If $Z \subset E$, since $Z \subset \{(x,y) \in Y : \phi_{A_j}(x) = \phi_{A_j}(y)\}$, then it must be the strict transform of the ramification divisor $\overline{B}$ on S of ϕ_{A_j}, hence we have that $Z \cdot E = -c_1(\mathcal{O}_{I\!PT_S}(1)) \cdot Z = -deg\ T_{\overline{B}} = 18$ and therefore $\bigotimes_{j=1}^{3}[A_{j1} \otimes A_{j2}(-E)] \cdot Z = 6(A_1 + A_2 + A_3) \cdot A_j - 3E \cdot Z \geq 0$ by (iii). If $A_j = 2B_j$ with $B_j^2 = 2$ then $\phi_{A_j} = v_2 \circ \phi_{B_j}$, where v_2 is the Veronese map, and we have $Z \subset \{(x,y) \in Y : \phi_{A_j}(x) = \phi_{A_j}(y)\} = \{(x,y) \in Y : \phi_{B_j}(x) = \phi_{B_j}(y)\}$ hence on S, $Z' = \phi_{A_j}^*(Z_1) = \phi_{B_j}^*(Z_2)$ where Z_1 is a curve on the Veronese surface in $I\!P^5$ and $Z_2 = v_2^*(Z_1) \subset I\!P^2$. Now setting $Z_2 \sim mN$ a computation as above shows that

$$\bigotimes_{j=1}^{3}[A_{j1} \otimes A_{j2}(-E)] \cdot Z = 2(A_1 + A_2 + A_3) \cdot Z' - 3E \cdot Z = 2m(A_1 + A_2 + A_3) \cdot B_j - 18m$$

and we are done by (iii) (similarly when $Z \subset E$). ■

We will now start the proof of Theorem (1.1). This will be done in two main steps: First we will compute the corank of $\Phi_{\mathcal{O}_{S_{i,g}}(1)}$ for low values of g or high values of i and then (in section 3, for $i = 1, g \geq 17$ or $i = 2, g \geq 7$) we will construct K3 surfaces in $\mathcal{H}_g$ whose i-th Veronese embedding has the hyperplane bundle decomposable as in Lemma (2.1), and hence $\Phi_{\mathcal{O}_{S_{i,g}}(1)}$ surjective.

Proof of Theorem (1.1): First of all, by the semicontinuity of the corank of Gaussian maps, in order to prove that the general surface in an irreducible family has a surjective Gaussian map, we only need to exhibit a single surface which has a surjective Gaussian map. For $i \geq 3$ and $g \geq 3$ on any smooth K3 surface representing a point in $\mathcal{H}_{i,g}$ there is an obvious decomposition of the hyperplane bundle as a tensor product of three very ample line bundles, hence satisfying (ii) of Lemma (2.1) by Lemma (2.2), and therefore giving the required surjectivity by Lemma (2.1). For $g = 2$ the values of *corank* $\Phi_{\mathcal{O}_{S_{i,2}}(1)}$ follow by a result of J. Duflot [D, Proposition 4.7]. Alternatively, for $g = 2, i \geq 5$ the surjectivity can also be proved using Lemmas (2.1) and (2.2) as follows. We have $iH = 3H + H + (i-4)H$ and the conditions of Lemma (2.2) are satisfied with $A_1 = 3H, A_2 = H, A_3 = (i-4)H$: In fact $3H$ is very ample and $(i-4)H$ is very ample for $i \geq 7$ while it defines a 2:1 finite morphism for $i = 5,6$ and we have $(A_1 + A_2 + A_3) \cdot H = 2i \geq 10$. For $i = 1, g \geq 17$ or $i = 2, g \geq 7$ there is a K3 surface T whose i-th Veronese embedding $v_i(T)$ has surjective Gaussian map by Proposition (3.1) and Lemma (2.1).

Now for $L = \mathcal{O}_{S_{i,g}}(1)$ set $R(L, L^k) = Ker\{H^0(L) \otimes H^0(L^k) \to H^0(L^{k+1})\}$ and consider the Gaussian map

$$\Phi_{L,L^k} : R(L, L^k) \to H^0(S_{i,g}, \Omega^1_{S_{i,g}} \otimes L^{k+1})$$

defined as usual by $\Phi_{L,L^k}(\sigma \otimes \tau) = \sigma d\tau - \tau d\sigma$. We recall that for $k = 1$ we have $Im\ \Phi_{L,L} = Im\ \Phi_L$, where $\Phi_L : \bigwedge^2 H^0(S_{i,g}, L) \to H^0(S_{i,g}, \Omega^1_{S_{i,g}} \otimes L^2)$. With the same notation as in Lemma (2.1) we have

$$Coker \Phi_{L,L^k} \subseteq H^1(S \times S, p_1^* L \otimes p_2^* L^k \otimes \mathcal{I}^2_\Delta) \cong H^1(Y, \omega_Y \otimes L_1 \otimes L_2^k(-3E)) = 0$$

by the Kawamata-Viehweg vanishing theorem for $i = 1$ and $g \geq 17, i = 2$ and $g \geq 7, i \geq 3$ and $g \geq 3, g = 2$ and $i \geq 5$. In fact as we have seen above in all these cases a decomposition as in (i) and (ii) of Lemma (2.1) holds for L so that $L_1 \otimes L_2(-3E) = \bigotimes_{j=1}^{3} [A_{j1} \otimes A_{j2}(-E)]$ is big and nef on Y, and therefore so is $L_1 \otimes L_2^k(-3E)$.

It remains to prove the surjectivity of Φ_{L,L^k} for $k \geq 1, i = 2, g = 3,4,5,6$. Since $S_{2,g} = v_2(S_g)$, where v_2 is the second Veronese embedding, we clearly have $corank \Phi_{L,L^k} = corank \Phi_{\mathcal{O}_{S_g}(2), \mathcal{O}_{S_g}(2k)}$. When $g \leq 5$ the K3 surface S_g is a complete intersection and the corank of the Gaussian map $\Phi_{\mathcal{O}_{S_g}(2), \mathcal{O}_{S_g}(2k)}$ can be computed by results of Wahl and S. Kumar as follows. We have a diagram

$$(2.3) \qquad \begin{array}{ccc} R(\mathcal{O}_{\mathbb{P}^g}(2), \mathcal{O}_{\mathbb{P}^g}(2k)) & \stackrel{\Phi_{\mathcal{O}_{\mathbb{P}^g}(2), \mathcal{O}_{\mathbb{P}^g}(2k)}}{\longrightarrow} & H^0(\mathbb{P}^g, \Omega^1_{\mathbb{P}^g}(2k+2)) \\ & & \downarrow \phi_g \\ \downarrow \pi_g & & H^0(S_g, \Omega^1_{\mathbb{P}^g}(2k+2)_{|S_g}) \\ & & \downarrow \psi_g \\ R(\mathcal{O}_{S_g}(2), \mathcal{O}_{S_g}(2k)) & \stackrel{\Phi_{\mathcal{O}_{S_g}(2), \mathcal{O}_{S_g}(2k)}}{\longrightarrow} & H^0(S_g, \Omega^1_{S_g}(2k+2)) \end{array}$$

and $\Phi_{\mathcal{O}_{\mathbb{P}^g}(2), \mathcal{O}_{\mathbb{P}^g}(2k)}$ is surjective by [W1, Theorem 6.4] and [K, Theorem 2.5]. Since $S_g \subset \mathbb{P}^g$ is a complete intersection surface of type (4) for $g = 3$, type $(2,3)$ for $g = 4$ and type $(2,2,2)$ for $g = 5$, we have that $N^*_{S_g/\mathbb{P}^g}(2k+2) = \mathcal{O}_{S_3}(2k-2), \mathcal{O}_{S_4}(2k) \oplus \mathcal{O}_{S_4}(2k -$

1), $\mathcal{O}_{S_5}(2k)^{\oplus 3}$ respectively and its H^1 vanishes (in fact $H^1(\mathcal{O}_{S_g}(a)) = 0$ for every integer a); therefore ψ_g is surjective since $Coker\psi_g \subseteq H^1(N^*_{S_g/\mathbb{P}^g}(2k+2)) = 0$. Now $Coker\phi_g = H^1(\Omega^1_{\mathbb{P}^g} \otimes \mathcal{I}_{S_g/\mathbb{P}^g}(2k+2))$ since $H^1(\Omega^1_{\mathbb{P}^g}(2k+2)) = 0$ by Bott vanishing. For $g = 4,5$ and the Koszul resolution of the ideal sheaf $\mathcal{I}_{S_g/\mathbb{P}^g}$ we have

$$0 \to \Omega^1_{\mathbb{P}^4}(2k-3) \to \Omega^1_{\mathbb{P}^4}(2k) \oplus \Omega^1_{\mathbb{P}^4}(2k-1) \to \Omega^1_{\mathbb{P}^4} \otimes \mathcal{I}_{S_4/\mathbb{P}^4}(2k+2) \to 0$$

and

$$0 \to \Omega^1_{\mathbb{P}^5}(2k-4) \to \Omega^1_{\mathbb{P}^5}(2k-2)^{\oplus 3} \to \Omega^1_{\mathbb{P}^5}(2k)^{\oplus 3} \to \Omega^1_{\mathbb{P}^5} \otimes \mathcal{I}_{S_5/\mathbb{P}^5}(2k+2) \to 0$$

and by Bott vanishing we see that $H^1(\Omega^1_{\mathbb{P}^g} \otimes \mathcal{I}_{S_g/\mathbb{P}^g}(2k+2)) = 0$, hence ϕ_g is surjective and so is $\Phi_{\mathcal{O}_{S_g}(2),\mathcal{O}_{S_g}(2k)}$, by diagram (2.3).
For $g = 3$ we have that *corank* $\phi_3 = h^1(\Omega^1_{\mathbb{P}^3}(2k-2)) = 0$ unless $k = 1$. In the latter case we get *corank* $\phi_3 = 1$ and $dimKer\psi_3 = h^0(\mathcal{O}_{S_3}) = 1$; hence to see the surjectivity of $\Phi_{\mathcal{O}_{S_3}(2)}$ it is enough to show that $Ker\psi_3 \not\subseteq Im\phi_3$ because then $\psi_3 \circ \phi_3$ is surjective. Now consider the diagram

$$\begin{array}{ccccccc}
0 & \to H^0(\Omega^1_{\mathbb{P}^3}(4)) & \to & H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(3)) & \to & H^0(\mathcal{O}_{\mathbb{P}^3}(4)) & \to & 0 \\
 & \downarrow \phi_3 & & \downarrow \cong & & \downarrow \alpha & & \\
0 & \to H^0(\Omega^1_{\mathbb{P}^3|S_3}(4)) & \to & H^0(\mathcal{O}_{S_3}(1)) \otimes H^0(\mathcal{O}_{S_3}(3)) & \to & H^0(\mathcal{O}_{S_3}(4)) & \to & 0
\end{array}$$

and let $\delta : H^0(\mathcal{I}_{S_3/\mathbb{P}^3}(4)) = Ker\alpha \to Coker\phi_3$ be the isomorphism induced by the snake lemma. Since δ is surjective and $Ker\psi_3 = H^0(N^*_{S_3/\mathbb{P}^3}(4)) = H^0(\mathcal{I}_{S_3/\mathbb{P}^3}(4))$, to show that $Ker\psi_3 \not\subseteq Im\phi_3$ is enough to see that 4δ is the composition $H^0(N^*_{S_3/\mathbb{P}^3}(4)) \to H^0(\Omega^1_{\mathbb{P}^3|S_3}(4)) \to Coker\phi_3$. To this end observe that if $F = 0$ is the equation of S_3, we have $F = \frac{1}{4}\sum_{i=0}^{3} \frac{\partial F}{\partial x_i} x_i$ hence by the snake lemma applied to the above diagram, $4\delta(F) = \sum_{i=0}^{3} x_i \otimes \frac{\partial F}{\partial x_i} + Im\phi_3$ (where we see $\sum_{i=0}^{3} x_i \otimes \frac{\partial F}{\partial x_i}$ as an element of the kernel of the multiplication map $\mu : H^0(\mathcal{O}_{S_3}(1)) \otimes H^0(\mathcal{O}_{S_3}(3)) \to H^0(\mathcal{O}_{S_3}(4))$, hence as an element of $H^0(\Omega^1_{\mathbb{P}^3|S_3}(4))$). On the other hand the map $H^0(N^*_{S_3/\mathbb{P}^3}(4)) \to H^0(\Omega^1_{\mathbb{P}^3|S_3}(4))$ takes F to $dF = \sum_{i=0}^{3} \frac{\partial F}{\partial x_i} dx_i$ which viewed as an element of the kernel of μ is in fact $\sum_{i=0}^{3} x_i \otimes \frac{\partial F}{\partial x_i}$.

Finally to see the surjectivity of $\Phi_{\mathcal{O}_{S_6}(2),\mathcal{O}_{S_6}(2k)}$ we will use the fact that S_6 is a complete intersection in a Grassmannian. We have $\mathbb{G} = \mathbb{G}(1,4) \subset \mathbb{P}^9$ in the Plücker embedding and

$S_6 = \mathbb{G} \cap H_1 \cap H_2 \cap H_3 \cap Q$ where H_i is a hyperplane and Q is a quadric hypersurface. In the diagram

$$\begin{array}{ccc} R(\mathcal{O}_{\mathbb{G}}(2), \mathcal{O}_{\mathbb{G}}(2k)) & \stackrel{\Phi_{\mathcal{O}_{\mathbb{G}}(2),\mathcal{O}_{\mathbb{G}}(2k)}}{\longrightarrow} & H^0(\mathbb{G}, \Omega^1_{\mathbb{G}}(2k+2)) \\ & & \downarrow \phi_6 \\ \downarrow \pi_6 & & H^0(S_6, \Omega^1_{\mathbb{G}}(2k+2)_{|S_6}) \\ & & \downarrow \psi_6 \\ R(\mathcal{O}_{S_6}(2), \mathcal{O}_{S_6}(2k)) & \stackrel{\Phi_{\mathcal{O}_{S_6}(2),\mathcal{O}_{S_6}(2k)}}{\longrightarrow} & H^0(S_6, \Omega^1_{S_6}(2k+2)) \end{array}$$

we have that $\Phi_{\mathcal{O}_{\mathbb{G}}(2),\mathcal{O}_{\mathbb{G}}(2k)}$ is surjective by [W1, Theorem 6.4] and [K, Theorem 2.5], ψ_6 is surjective because $Coker\psi_6 \subseteq H^1(N^*_{S_6/\mathbb{G}}(2k+2)) = H^1(\mathcal{O}_{S_6}(2k+1)^{\oplus 3} \oplus \mathcal{O}_{S_6}(2k)) = 0$. Therefore the surjectivity of $\Phi_{\mathcal{O}_{S_6}(2),\mathcal{O}_{S_6}(2k)}$ will follow by the above diagram as soon as we show that ϕ_6 is surjective. To this end note that $Coker\phi_6 \subseteq H^1(\Omega^1_{\mathbb{G}} \otimes \mathcal{I}_{S_6/\mathbb{G}}(2k+2)) = 0$: by the Koszul resolution of the ideal sheaf $\mathcal{I}_{S_6/\mathbb{G}}$ we have an exact sequence

$$0 \to \Omega^1_{\mathbb{G}}(2k-3) \to \Omega^1_{\mathbb{G}}(2k-2)^{\oplus 3} \oplus \Omega^1_{\mathbb{G}}(2k-1) \to \Omega^1_{\mathbb{G}}(2k)^{\oplus 3} \oplus \Omega^1_{\mathbb{G}}(2k-1)^{\oplus 3} \to$$

$$\to \Omega^1_{\mathbb{G}}(2k+1)^{\oplus 3} \oplus \Omega^1_{\mathbb{G}}(2k) \to \Omega^1_{\mathbb{G}} \otimes \mathcal{I}_{S_6/\mathbb{G}}(2k+2) \to 0$$

and the vanishing follows by the

Claim (2.4). *$H^p(\Omega^1_{\mathbb{G}}(q)) = 0$ for $p \geq 2, q \geq -1$ and for $p = 1, q \geq 1$.*

Proof of Claim (2.4): This is just Bott vanishing for the Grassmannian. Alternatively from the normal bundle sequence

$$0 \to N^*_{\mathbb{G}}(q) \to \Omega^1_{I\!P^9}(q)_{|\mathbb{G}} \to \Omega^1_{\mathbb{G}}(q) \to 0$$

we see that the Claim is implied by

$$H^p(\Omega^1_{I\!P^9}(q)_{|\mathbb{G}}) = 0 \text{ for } p \geq 2, q \geq -3 \text{ and for } p = 1, q \geq 1 \tag{2.5}$$

and

$$H^{p+1}(N^*_{\mathbb{G}}(q)) = 0 \text{ for } p \geq 0, q \geq -1. \tag{2.6}$$

To see (2.5) consider the Euler sequence

$$0 \to \Omega^1_{I\!P^9}(q)_{|\mathbb{G}} \to H^0(\mathcal{O}_{\mathbb{G}}(1)) \otimes \mathcal{O}_{\mathbb{G}}(q-1) \to \mathcal{O}_{\mathbb{G}}(q) \to 0.$$

We have $H^p(\mathcal{O}_{\mathbb{G}}(q-1)) = H^p(\omega_{\mathbb{G}}(4+q)) = 0$ for $p \geq 1, q \geq -3$ by Kodaira vanishing and $H^{p-1}(\mathcal{O}_{\mathbb{G}}(q)) = H^{p-1}(\omega_{\mathbb{G}}(5+q)) = 0$ for $p \geq 2, q \geq -4$ again by Kodaira vanishing, therefore (2.5) holds for $p \geq 2, q \geq -3$ and also for $p = 1, q \geq 1$ since by what we just proved we have

$$H^0(\mathcal{O}_{\mathbb{G}}(1)) \otimes H^0(\mathcal{O}_{\mathbb{G}}(q-1)) \to H^0(\mathcal{O}_{\mathbb{G}}(q)) \to H^1(\Omega^1_{I\!P^9}(q)_{|\mathbb{G}}) \to 0$$

and the above multiplication map is surjective.

Now to prove (2.6) we use Griffiths vanishing theorem ([G]). In the Plücker embedding the ideal of the Grassmannian $\mathbb{G}$ is generated by quadrics, hence $N^*_{\mathbb{G}}(2)$ is globally generated and $det(N^*_{\mathbb{G}}(2)) = det(N^*_{\mathbb{G}})(6) = \omega_{\mathbb{G}}^{-1}(-4) = \mathcal{O}_{\mathbb{G}}(1)$. Therefore setting $E = N^*_{\mathbb{G}}(2)$ we can write

$$N^*_{\mathbb{G}}(q) = E \otimes detE \otimes \mathcal{O}_{\mathbb{G}}(q+2) \otimes \omega_{\mathbb{G}}$$

with $p \geq 0, q+2 > 0$ and (2.6) follows by Griffiths vanishing theorem. ■ (for Claim (2.4))

This then concludes the proof of Theorem (1.1). ■

(2.7) *Remark.* Note that in the proof of Theorem (1.1) for $i = 2$ and $3 \leq g \leq 4$ or for $i \geq 3$ and $g \geq 3$ *we did not use* the fact that $S_{i,g}$ represents a general point of $\mathcal{H}_{i,g}$. Hence for such values of i and g the corank of $\Phi_{\mathcal{O}_{S_{i,g}}(1)}$ and of $\Phi_{\mathcal{O}_{S_{i,g}}(1),\mathcal{O}_{S_{i,g}}(k)}$ is as in Theorem (1.1) for *any smooth* $S_{i,g}$. Similarly in the case $g = 2$ we only use the fact that the ramification locus is smooth (as in [D, Proposition 4.7]) hence the computation of the corank of $\Phi_{\mathcal{O}_{S_{i,2}}(1)}$ holds just with this hypothesis.

3. CONSTRUCTION OF K3 SURFACES WITH SURJECTIVE GAUSSIAN MAP

In this section we will construct with the aid of the surjectivity of the period mapping, K3 surfaces with very ample line bundles whose multiples decompose as in (ii) of Lemma (2.1). We have

Proposition (3.1). *For every i, g such that $i = 1, g \geq 17$ or $i = 2, g \geq 7$ there exists a K3 surface T representing a point in $\mathcal{H}_g$ such that $v_i(T)$ has the hyperplane bundle decomposable as in Lemma (2.1).*

The proof of Proposition (3.1) will be in two parts. We will first construct the needed K3 surfaces using the surjectivity of the period map and then we will show how to decompose $iH, i = 1, 2$.

Let us recall that a cohomology class $h \in H^2(X, \mathbb{R})$ on a compact Kähler manifold X is a *Kähler class* if it can be represented by a Kähler form, that is by a (1,1)-form associated to a Kähler metric. In particular if h is a Kähler class and y is the class of any closed curve on X we have $h \cdot y > 0$ [BPV, Lemma I.13.1].

Proposition (3.2). *There exist smooth K3 surfaces T_{jkh} with Picard lattices Γ_{jkh} as follows:*
(i) $\Gamma_{jkh} = \mathbb{Z}D \oplus \mathbb{Z}L$ *with intersection matrix* $\begin{pmatrix} D^2 & D \cdot L \\ L \cdot D & L^2 \end{pmatrix} = \begin{pmatrix} 2h & k \\ k & 2j \end{pmatrix}$
for $j = 1, 2, k \geq j + 4, h = 2$, *or* $j = -1, k = 1, 2, h \geq 5 - 2k$ *or* $j = 1, k = 5, h = 3$;
(ii) $\Gamma_{jkh} = \mathbb{Z}D \oplus \mathbb{Z}L \oplus \mathbb{Z}R$ *with intersection matrix*

$$\begin{pmatrix} D^2 & D \cdot L & D \cdot R \\ L \cdot D & L^2 & L \cdot R \\ R \cdot D & R \cdot L & R^2 \end{pmatrix} = \begin{pmatrix} 2h & k & 2 \\ k & 4j - 2 & j \\ 2 & j & -2 \end{pmatrix}$$

for $j = 0, k = 1, 2, h \geq 5 - 2k$ *and* $j = 1, k = 4, h = 1$.
Moreover all the D's are Kähler (hence ample) classes on T_{jkh}.
Proof: Let $\Lambda = \mathbb{H}^3 \oplus E_8(-1)^2$ be the K3 lattice where $\mathbb{H}$ denotes the hyperbolic plane and E_8 the root lattice, $\Lambda_{\mathbb{C}} = \Lambda \otimes \mathbb{C}$, $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ with the quadratic form extended $\mathbb{C}$ or $\mathbb{R}$-bilinearly and let

$$(K\Omega)^0 = \{(k, [\omega]) \in \Lambda_{\mathbb{R}} \times I\!P\Lambda_{\mathbb{C}} : \omega \cdot \omega = 0, \omega \cdot \overline{\omega} > 0, k \cdot k > 0, k \cdot \omega = 0 \text{ and } k \cdot d \neq 0$$

$$\forall d \in \Lambda \text{ with } d^2 = -2, \omega \cdot d = 0\}.$$

We will show that there exists $\omega \in \Lambda_{\mathbb{C}}$ such that $(D, [\omega]) \in (K\Omega)^0$. To this end let us record the following

Claim (3.3). *The lattices Γ_{jkh} are even, nondegenerate of signature* $(1, rk\Gamma_{jkh} - 1)$ *and there is no class $d \in \Gamma_{jkh}$ such that* $d^2 = -2, D \cdot d = 0$.
Proof of the Claim: Clearly all the Γ_{jkh} are even and nondegenerate since $disc\Gamma_{jkh} =$ det (intersection matrix) $\neq 0$ (see Table (3.4)). The assertion about the signatures follows easily by looking at the signs of the principal minors of the intersection matrix. Now suppose there is $d \in \Gamma_{jkh}$ such that $d^2 = -2, D \cdot d = 0$. In case (i), since $disc\Gamma_{jkh}$ must

divide the discriminant of any rank two sublattice, we get that $4hj-k^2$ divides $disc(D,d)=-4h$ and this does not hold for the given values of j,k,h. In case (ii) set $d=\alpha D+\beta L+\gamma R$; then $d\cdot D=0$ gives $\gamma=-h\alpha-\frac{k}{2}\beta$ and from $d^2=-2$ we get

$$2h\alpha^2+(4j-2)\beta^2-2(-h\alpha-\frac{k}{2}\beta)^2+2k\alpha\beta+4\alpha(-h\alpha-\frac{k}{2}\beta)+2j\beta(-h\alpha-\frac{k}{2}\beta)=-2$$

hence

$$2h(h+1)\alpha^2+2h\beta(k+j)\alpha+(jk+\frac{k^2}{2}+2-4j)\beta^2-2=0$$

whose discriminant in α is

$$\Delta_\alpha=h[h(k+j)^2-2(h+1)(jk+\frac{k^2}{2}+2-4j)]\beta^2+4h(h+1),$$

which is easily seen to be negative for $\beta\neq 0$, while for $\beta=0$ it is not a square since $4h(h+1)$ is not a square for $h\geq 1$. ■ (for Claim (3.3))

To finish the proof of Proposition (3.2) let $\Gamma=\Gamma_{jkh}$. Since $rk\Gamma\leq 3$, by [BPV, Theorem I.2.9], Γ has a primitive embedding into the K3 lattice Λ and by Claim (3.3) the signature of $\Gamma^\perp$ is $(2,20-rk\Gamma)$. Hence there is a positive definite two dimensional space $V\subset\Gamma^\perp_{\mathbb{R}}$ such that $V^\perp\cap\Lambda=\Gamma$. Indeed let u and v be two orthogonal vectors in $\Gamma^\perp_{\mathbb{R}}$ spanning such a positive definite two dimensional space; multiplying by a real factor we can assume $u^2=v^2$, hence if we set $\omega_1=u+iv\in\Lambda_{\mathbb{C}}$ we have $\omega_1^2=0$, $\omega_1\cdot\overline{\omega_1}>0$ and $\Gamma\subseteq(\mathbb{C}\omega_1\oplus\mathbb{C}\overline{\omega_1})^\perp\cap\Lambda$. By [EEK, proof of Theorem 5.4], ω_1 can be perturbed, to a class $\omega\in\Lambda_{\mathbb{C}}$, in such a way as to preserve the first two relations and achieve equality in the third. Then we can take $V=\mathbb{R}Re\omega\oplus\mathbb{R}Im\omega$. Now let $d\in\Lambda$ be such that $d^2=-2,\omega\cdot d=0$. Then we get that $d\in V^\perp$. In fact if $d=d_1+d_2$, $d_1\in V$, $d_2\in V^\perp$, we have $d_1=aRe\omega+bIm\omega$ and $0=\omega\cdot d=\omega\cdot d_1=(Re\omega+iIm\omega)\cdot(aRe\omega+bIm\omega)=a(Re\omega)^2+ib(Im\omega)^2$ hence $a=b=0$, that is $d_1=0$. Therefore $d\in V^\perp\cap\Lambda=\Gamma$ and by Claim (3.3) we have $D\cdot d\neq 0$. Since $D^2>0$ we have that $(D,[\omega])\in(K\Omega)^0$ and hence, by the surjectivity of the refined period mapping [BPV, Theorem VIII.1.4] there exists a marked K3 surface T_{jkh} with period point $[\omega]$, Picard lattice $\Gamma=\Gamma_{jkh}$ and Kähler class D. The latter implies that D is an ample class.
■

Proof of Proposition (3.1): On the K3 surfaces T_{jkh} with Picard lattices Γ_{jkh} given in Proposition (3.2) let us define divisors H,A_1,A_2,A_3 according to the following table (in which we record also the genus $g(H)=\frac{1}{2}H^2+1$ and the discriminants):

Table (3.4)

i	j	k	h	rank	$disc\Gamma_{jkh}$	H	$g(H)$	A_1	A_2	A_3
1	1,2	$\geq j+4$	2	2	$8j-k^2$	$2D+L$	$2k+j+9$	D	D	L
1	1	5,6,7	2	2	$8-k^2$	$D+2L$	$2k+7$	D	L	L
1	1	5	3	2	-13	$D+2L$	18	D	L	L
2	-1	1	≥ 3	2	$-4h-1$	$2D-L$	$4h-2$	H	D	$D-L$
2	-1	2	≥ 1	2	$-4h-4$	$2D+L$	$4h+4$	H	D	$D+L$
2	0	1	≥ 3	3	$8h+10$	$2D-L+R$	$4h+1$	H	D	$D-L+R$
2	0	2	≥ 1	3	$8h+16$	$2D+L+R$	$4h+7$	H	$D+L$	$D+R$
2	1	5	2	2	-17	$D+L$	9	H	D	L
2	1	4	1	3	30	$D+L$	7	H	D	L

Note that the first three rows of the above table are relative to the case $i=1, g \geq 17$, while the remaining six to the case $i=2, g \geq 7$.

We will show that H is very ample and embeds T_{jkh} as a K3 surface $T \subset I\!P^g$ with the required properties. To see this first observe that $g(H)$ runs through all the integers $g \geq 17$ for $i=1$ and $g \geq 7$ for $i=2$ as Table (3.4) shows. Also notice that in all cases H is a linear combination of generators of the Picard group in which appears at least one generator with coefficient ± 1, hence H is indivisible and, by the transcendental theory of K3 surfaces, T represents a point in $\mathcal{H}_g$. To finish the proof we will see, with a case by case analysis, that H is very ample and that the decomposition of $iH, i=1,2$, given in Table (3.4) satisfies (ii) of Lemma (2.1).

Before we start we record, for the reader's convenience, the well-known theorems that we will use.

Lemma (3.5). *Let S be a smooth K3 surface and Δ an indivisible divisor on S with $\Delta^2 \geq 2$. We have*

(i) Δ is very ample if Δ is nef, $\Delta^2 \geq 4$ and there is no irreducible curve F on S such that either $F^2=0, F \cdot \Delta = 1,2$ or $F^2=-2, F \cdot \Delta = 0$;

(ii) If Δ is nef it has no base points unless there exist irreducible curves F, G and an integer $a \geq 2$ such that $\Delta \sim aF+G$, $F^2=0, G^2=-2, F \cdot G = 1$;

(iii) If $\Delta^2 = 2$ and Δ has no base points then it defines a 2:1 morphism $S \to I\!P^2$.

Moreover the morphism is finite if there is no irreducible curve F on S such that $F^2 = -2, F \cdot \Delta = 0$;

(iv) If $\Delta^2 \geq 4$, Δ has no base points and there is no irreducible curve F on S such that $F^2 = 0, F \cdot \Delta = 2$ then Δ defines a birational morphism. When the latter holds the only curves contracted by Δ are the curves F such that $F^2 = -2, F \cdot \Delta = 0$.

Proof of Lemma (3.5): By Mori's version of a theorem of Saint-Donat [Mo, Theorem 5] it follows that under the hypotheses of case (i), if Δ is not very ample then there is an irreducible curve F on S such that $\Delta \sim 2F$, hence Δ is divisible. Part (ii) follows again by [Mo, Theorem 5]; (iii) is a result of Mayer [Ma, Proposition 2] and Saint-Donat ([SD]). By a theorem of Saint-Donat ([SD]) under the hypotheses of case (iv), if Δ does not define a birational morphism then there is an irreducible curve F such that $\Delta \sim 2F$, hence Δ is divisible. The rest follows again by [SD]. ■ (for Lemma (3.5))

Claim (3.6). *For $j = -1, k = 1, h \geq 3$ we have that D is very ample and so is $D - L$ unless $h = 3$ and in that case $D - L$ defines a 2:1 finite morphism onto $\mathbb{P}^2$.*

By Claim (3.6) we see that, for $j = -1, k = 1, h \geq 3$, $H = A_1 = D + D - L$ is very ample and (ii) of Lemma (2.1) holds by Lemma (2.2) since for $h \geq 4$ A_2 and A_3 are very ample and for $h = 3$ we have $(A_1 + A_2 + A_3) \cdot A_3 = (4D - 2L) \cdot (D - L) = 14$.

Proof of Claim (3.6): By Proposition (3.2) D is a Kähler (hence ample) indivisible class; if there is an irreducible curve F such that $F^2 = 0, F \cdot D = 1, 2$ ($F \cdot D = 0$ is excluded since D is an ample class) then $disc\Gamma_{-1,1,h} = -4h - 1$ divides $disc(F, D) = -1, -4$. Hence D is very ample by (i) of Lemma (3.5). If $h \geq 4$ we have $(D - L)^2 = 2h - 4 \geq 4$ and $D - L$ is indivisible and nef since $(D - L) \cdot L = 3$ (L is effective and irreducible because $L^2 = -2$ and $D \cdot L = 1$); if there were a curve F_1 such that either $F_1^2 = 0, F_1 \cdot (D - L) = 1, 2$ or $F_1^2 = -2, F_1 \cdot (D - L) = 0$ then $-4h - 1$ would divide $disc(F_1, D - L) = -1, -4, -4h + 8$. Therefore $D - L$ is very ample by (i) of Lemma (3.5). If $h = 3$ we have $(D - L)^2 = 2$ and $D - L$ has no base points and does not contract any curve by what we saw above, hence we can apply (iii) of Lemma (3.5). ■ (for Claim (3.6))

Claim (3.7). *For $j = -1, k = 2, h \geq 1$ we have that D is very ample for $h \geq 2$, D defines a 2:1 finite morphism onto $\mathbb{P}^2$ for $h = 1$, $D + L$ defines a birational morphism that contracts only the irreducible curve L.*

By Claim (3.7) we have that, for $j=-1, k=2, h \geq 1$, $H=A_1=D+D+L$ is very ample for $h \geq 2$ and also for $h=1$ by (i) of Lemma (3.5) since in this case $H^2=14$ and if there is a curve F with $F^2=0, F\cdot H=1,2$ or $F^2=-2, F\cdot H=0$ then $disc\Gamma_{-1,2,1}=-8$ divides $disc(F,H)=-1,-4,-28$. To see (ii) of Lemma (2.1) we check again the conditions of Lemma (2.2): $(A_1+A_2+A_3)\cdot L=2H\cdot L=4$ and, for $h=1$, $(A_1+A_2+A_3)\cdot A_2=2H\cdot D=12$.

Proof of Claim (3.7): If $h \geq 2$ we have $D^2 \geq 4$ hence D is very ample since if there is a curve F such that $F^2=0, F\cdot D=1,2$ ($F \cdot D \neq 0$ since D is ample) then $disc\Gamma_{-1,2,h}=-4h-4$ divides $disc(F,D)=-1,-4$. If $h=1$ and there are irreducible curves F_1, G and an integer $a \geq 2$ such that $D \sim aF_1+G$, $F_1^2=0, G^2=-2, F_1\cdot G=1$ then $disc\Gamma_{-1,2,1}=-8$ divides $disc(F_1,G)=-1$. Therefore D has no base points and (iii) of Lemma (3.5) gives that it defines a 2:1 morphism onto $I\!\!P^2$ which is finite since D is ample. Now notice that $L^2=-2$ implies by Riemann-Roch that either L or $-L$ is effective, hence L is because D is nef and $D\cdot L=2$; also L is irreducible for if $L=L_1+L_2$ then necessarily $D\cdot L_1=D\cdot L_2=1$ (D is Kähler) hence L_1 and L_2 are irreducible and $-2=L^2=L_1^2+L_2^2+2L_1\cdot L_2$ gives that $L_1^2=L_2^2=-2, L_1\cdot L_2=1$ ($L_1 \neq L_2$ since L is indivisible) but then $disc\Gamma_{-1,2,h}=-4h-4$ divides $disc(L_1,L_2)=3$, a contradiction. Since L is irreducible and $(D+L)\cdot L=0$ we have that $D+L$ is nef and base point free because it has no base points on L as it can be seen from the exact sequence $0 \to \mathcal{O}_{T_{-1,2,h}}(D) \to \mathcal{O}_{T_{-1,2,h}}(D+L) \to \mathcal{O}_L \to 0$ and the fact that $H^1(\mathcal{O}_{T_{-1,2,h}}(D))=0$ [SD, Proposition 2.6]. Also there is no curve F_2 with $F_2^2=0, F_2\cdot(D+L)=2$ else $disc\Gamma_{-1,2,h}=-4h-4$ divides $disc(F_2,D+L)=-4$. By (iv) of Lemma (3.5) we get that $D+L$ defines a birational morphism and if F_3 is an irreducible contracted curve then $F_3\cdot(D+L)=F_3\cdot D+F_3\cdot L>0$ unless $F_3=L$. ∎ (for Claim (3.7))

Claim (3.8). *For $j=0, k=1, h \geq 3$ we have that D is very ample and $D-L+R$ defines a birational morphism that contracts only the irreducible curve R.*

By Claim (3.8) we have that, for $j=0, k=1, h \geq 3$, $H=A_1=D+D-L+R$ is very ample and (ii) of Lemma (2.1) holds by Lemma (2.2) since $(A_1+A_2+A_3)\cdot R=2H\cdot R=4$.

Proof of Claim (3.8): First notice that D is base point free, else there are irreducible curves F, G and an integer $a \geq 2$ such that $D \sim aF+G$, $F^2=0, G^2=-2, F\cdot G=1$. But then $G \neq L$ (or $D-L$ is divisible) hence $1=D\cdot L=aF\cdot L+G\cdot L$ implies $F\cdot L=0, G\cdot L=1$ and then $disc\Gamma_{0,1,h}=8h+10$ divides $disc(L,F,G)=2$. To see that D is very ample let

F_1 be an irreducible curve such that $F_1^2 = 0, F_1 \cdot D = 2$ (the case $F_1 \cdot D = 1$ is clearly impossible). Then $F_1 \neq L$ and $0 \leq L \cdot F_1 \leq 2$. In fact the map ϕ_D sends both L and F_1 to lines in $I\!P(H^0(\mathcal{O}_{T_{0,1,h}}(D))^*)$. The possible values of $disc(D, L, F_1)$ are $8, 12 - 2h, 16 - 8h$ and they are divisible by $disc\Gamma_{0,1,h} = 8h + 10$ only if $L \cdot F_1 = 1, h = 6$. Remark that in this case ϕ_D is a finite morphism of $T_{0,1,6}$ onto a rational normal scroll V of degree 6 in $I\!P^7 = I\!P(H^0(\mathcal{O}_{T_{0,1,6}}(D))^*)$, and $\phi_D(L)$ is a line directrix of V. One moment of reflection then shows that $D = 5F_1 + L + M$, where M is a rational curve such that $\phi_D(M) = \phi_D(L)$. Since $1 = L \cdot D = 3 + L \cdot M$ we have $L = M$ and $D = 5F_1 + 2L$. But then $2 = D \cdot R = 5F_1 \cdot R + 2L \cdot R = 5F_1 \cdot R$, a contradiction. Hence the very ampleness of D follows by (i) of Lemma (3.5). Note that D very ample implies that $D - L$ is nef. Indeed the general curve in $|D - L|$ is irreducible by Bertini's theorem because $|D - L|$ has no base points and is connected since $H^1(\mathcal{O}_{T_{-1,2,h}}(D - L)) = 0$ by [SD, Proposition 2.6]. Now notice that R is irreducible for if $R = R_1 + R_2$ then $2 = D \cdot R$ implies $D \cdot R_1 = D \cdot R_2 = 1$ and R_1, R_2 are irreducible; from $R^2 = -2$ we get that $R_1^2 = R_2^2 = -2, R_1 \cdot R_2 = 1$ but then $disc\Gamma_{0,1,h} = 8h + 10$ divides $disc(D, R_1, R_2) = 6h + 6$, which is impossible. Now $D - L$ nef implies that also $D - L + R$ is nef since $(D - L + R) \cdot R = 0$. Actually $|D - L + R|$ has no base points. This follows since $|D - L|$ has no base points and the exact sequence $0 \to \mathcal{O}_{T_{-1,2,h}}(D - L) \to \mathcal{O}_{T_{-1,2,h}}(D - L + R) \to \mathcal{O}_R \to 0$ and $H^1(\mathcal{O}_{T_{-1,2,h}}(D - L)) = 0$ show that there are no base points on R either. Now $(D - L + R)^2 = 2h - 2 \geq 4$ and suppose F_2 is an irreducible curve such that either $F_2^2 = -2, F_2 \cdot (D - L + R) = 0$ or $F_2^2 = 0, F_2 \cdot (D - L + R) = 2$. We will show that this is possible only when $F_2 = R$, hence (iv) of Lemma (3.5) will give that $D - L + R$ defines a birational morphism contracting only R. If $F_2 \neq R$ in the first case we have $0 = F_2 \cdot (D - L + R) = F_2 \cdot (D - L) + F_2 \cdot R$ hence $F_2 \cdot (D - L) = F_2 \cdot R = 0$ but then $disc\Gamma_{0,1,h} = 8h + 10$ divides $disc(D - L, R, F_2) = 8h - 8$. In the second case we have $2 = F_2 \cdot (D - L) + F_2 \cdot R$ hence $F_2 \cdot (D - L) = 0, 2, F_2 \cdot R = 2, 0$ (F_2 is an elliptic curve hence we cannot have $F_2 \cdot (D - L) = 1$) but then $disc\Gamma_{0,1,h} = 8h + 10$ divides $disc(D - L, R, F_2) = -8h + 16, 8$, both impossible. ∎ (for Claim (3.8))

Claim (3.9). *For $j = 0, k = 2, h \geq 1$ we have that $H = 2D + L + R$ is very ample and $D + L$ (respectively $D + R$) defines a birational morphism that contracts only the irreducible curve L (respectively R).*

By Claim (3.9) we see that, for $j=0,k=2,h\geq 1$, H is very ample and (ii) of Lemma (2.1) holds by Lemma (2.2) since $(A_1+A_2+A_3)\cdot L=(A_1+A_2+A_3)\cdot R=2H\cdot L=4$.

Proof of Claim (3.9): First we show that L and R are irreducible effective divisors. Suppose $L=L_1+L_2$ with L_1,L_2 effective. Since $D\cdot L=2$ we have $D\cdot L_1=D\cdot L_2=1$ and L_1, L_2 are irreducible; from $L^2=-2$ we get that either $L_1^2=-2,L_2^2=L_1\cdot L_2=0$ or $L_1^2=L_2^2=-2,L_1\cdot L_2=1$. But in these cases we have that $disc\Gamma_{0,2,h}=8h+16$ divides $disc(D,L_1,L_2)=2,6h+6$, both impossible. Since $D\cdot L=D\cdot R$, $L^2=R^2$ the same proof shows that R is irreducible. To see that $H=2D+L+R$ is very ample first notice that it is nef since D is and $H\cdot L=H\cdot R=2$. Let F be an irreducible curve such that either $F^2=-2,F\cdot H=0$ or $F^2=0,F\cdot H=1,2$. Then $F\neq L,R$ hence $H\cdot F=2D\cdot F+L\cdot F+R\cdot F\geq 2$, therefore necessarily $D\cdot F=1,L\cdot F=R\cdot F=0$ and $F^2=0$, but then $disc\Gamma_{0,2,h}=8h+16$ divides $disc(D,L,F)=2$. Therefore H is very ample by (i) of Lemma (3.5). Notice now that D has no base points for otherwise we have irreducible curves F_1 and G and an integer $a\geq 2$ such that $D\sim aF_1+G,F_1^2=0,G^2=-2,F_1\cdot G=1$. But then $L\neq F_1,G$ (else $D-L$ is divisible) and $2=D\cdot L=aF_1\cdot L+G\cdot L$ implies that either $F_1\cdot L=0,G\cdot L=2$ or $F_1\cdot L=1,G\cdot L=0$, but in both cases $disc(F_1,L,G)=2,4$ is not divisible by $disc\Gamma_{0,2,h}=8h+16$. D being base point free yields that $D+L$ is also base point free. This follows from $H^1(\mathcal{O}_{T_{0,2,h}}(D))=0$ and the exact sequence $0\to\mathcal{O}_{T_{0,2,h}}(D)\to\mathcal{O}_{T_{0,2,h}}(D+L)\to\mathcal{O}_L\to 0$. Since $(D+L)^2=2h+2\geq 4$ we use now (iv) of Lemma (3.5) to conclude the proof of this Claim. Let F_2 be an irreducible curve such that either $F_2^2=-2,F_2\cdot(D+L)=0$ or $F_2^2=0,F_2\cdot(D+L)=2$. We will show that this is possible only when $F_2=L$. If $F_2\neq L$ then $F_2\cdot(D+L)=F_2\cdot D+F_2\cdot L>0$ hence $F_2^2=0$, F_2 is an elliptic curve and we cannot have $F_2\cdot D=1$, therefore we have $F_2\cdot D=2,F_2\cdot L=0$ but then $disc\Gamma_{0,2,h}=8h+16$ divides $disc(D,L,F_2)=8$. Again replacing L by R we get the statement for $D+R$. ■ (for Claim (3.9))

Claim (3.10). *For $j=1,k=5,h=2$ we have that D is very ample and L defines a 2:1 finite morphism onto $\mathbb{P}^2$.*

By Claim (3.10) we obtain that, for $j=1,k=5,h=2$, $H=A_1=D+L$ is very ample and (ii) of Lemma (2.1) holds by Lemma (2.2) since $(A_1+A_2+A_3)\cdot A_3=2H\cdot L=2(D+L)\cdot L=14$.

Proof of Claim (3.10): In this case the K3 surface $T_{1,5,2}$ with Picard lattice $\Gamma_{1,5,2}$ can be taken to be a smooth quartic surface in $\mathbb{P}^3$ with Picard group generated by the hyperplane section D and a smooth irreducible genus 2 quintic curve L ($T_{1,5,2}$ exists by [Mo]). Moreover there is no irreducible curve F such that $F^2 = -2, F \cdot L = 0$ else $disc\Gamma_{1,5,2} = -17$ divides $disc(L, F) = -4$. Therefore L defines a 2:1 finite morphism. ∎ (for Claim (3.10))

Claim (3.11). *For $j = 1, k = 4, h = 1$ we have that $H = D + L$ is very ample and both D and L define 2:1 finite morphisms onto $\mathbb{P}^2$.*

By Claim (3.11) we see that, for $j = 1, k = 4, h = 1$, (ii) of Lemma (2.1) holds by Lemma (2.2) since $(A_1 + A_2 + A_3) \cdot A_2 = (A_1 + A_2 + A_3) \cdot A_3 = 2H \cdot D = 2H \cdot L = 12$.

Proof of Claim (3.11): First of all notice that D is base point free otherwise there are irreducible curves F, G and an integer $a \geq 2$ such that $D \sim aF + G$ but then $2 = D^2 = aF \cdot D + G \cdot D \geq 3$. Now let us see that R is irreducible. Suppose $R = R_1 + R_2$ with R_1, R_2 effective; since $D \cdot R = 2$ we have $D \cdot R_1 = D \cdot R_2 = 1$ hence R_1, R_2 are irreducible and $R_1^2 = R_2^2 = -2$ because D has no base points. But then we have that $disc\Gamma_{1,4,1} = 30$ divides $disc(D, R_1, R_2) = 12$, a contradiction. Also L can be assumed irreducible. To see this observe first that R is not a fixed component of the linear system $|L|$. Indeed $L - R$ is effective because $(L - R)^2 = -2, (L - R) \cdot D = 2$ and connected since if we had $L - R = B_1 + B_2$ with B_1, B_2 effective then $D \cdot B_1 = D \cdot B_2 = 1$ hence B_1, B_2 are irreducible and distinct (because $L - R$ is indivisible), therefore $B_1 \cdot B_2 \geq 0$ and if $B_1 \cdot B_2 = 0$, since $B_1^2 = B_2^2 = -2$, we get that $disc\Gamma_{1,4,1} = 30$ divides $disc(D, B_1, B_2) = 12$, a contradiction. Since $h^0(\mathcal{O}_{T_{1,4,1}}(L)) \geq 3, h^0(\mathcal{O}_{T_{1,4,1}}(L - R)) = 1$ we deduce that R is not a fixed component of $|L|$. Let $L' = L_1 + L_2$ be a divisor of $|L|$ not containing R, with L_1, L_2 effective. We have $R \cdot L_i \geq 0, i = 1, 2$, hence $0 \leq R \cdot L_i \leq 1$ and $1 \leq D \cdot L_i \leq 3$. If $D \cdot L_1 = 1$ then $disc\Gamma_{1,4,1} = 30$ divides $disc(D, L_1, R) = 18, 20$ which is impossible. If $D \cdot L_1 = 3$ then $D \cdot L_2 = 1$ and we conclude as above replacing L_1 with L_2. Hence $D \cdot L_1 = D \cdot L_2 = 2$ and they are both irreducible. The linear system cut out on L_1 by $|D|$ has dimension 2 otherwise $D - L_1$ is effective and $D \cdot (D - L_1) = 0$ hence $D = L_1$ and $-1 = R \cdot (L - D) = R \cdot L_2 \geq 0$, a contradiction. This implies that L_1 is rational and therefore $L_1^2 = -2$ and the same holds for L_2. Therefore we can assume $R \cdot L_1 = 0$ and this is impossible because $disc\Gamma_{1,4,1} = 30$ does not divide $disc(D, L_1, R) = 24$. This shows that L is irreducible. Since $L^2 = 2$ we have that L is nef and so is $H = D + L$. Let F be an irreducible curve such that either $F^2 = -2, F \cdot H = 0$ or $F^2 = 0, F \cdot H = 1, 2$. Then $F \neq L$ hence $H \cdot F = D \cdot F + L \cdot F \geq 1$ and therefore we have $F^2 = 0$ and either $D \cdot F = 1, L \cdot F = 0$ or $D \cdot F = L \cdot F = 1$ or $D \cdot F = 2, L \cdot F = 0$, but then $disc\Gamma_{1,4,1} = 30$ divides $disc(D, F, L) = -2, 4, -8$, a contradiction. Therefore H is very ample by (i) of Lemma (3.5). Since D has no base points, by (ii) and (iii) of Lemma (3.5) we have that D defines a 2:1 finite morphism onto $I\!P^2$ because D is ample. Similarly for L (which is irreducible, hence base point free) there is no irreducible curve F_2 such that $F_2 \cdot L = 0, F_2^2 = -2$ else setting $x = R \cdot F_2$ we get that $disc\Gamma_{1,4,1} = 30$ divides $disc(R, L, F_2) = -2x^2 + 10$, but then $2x^2 \equiv 4$ $(mod\ 6)$ and this is not possible for an integer x. ■ (for Claim (3.11))

Claim (3.12). *For $j = 1, 2, k \geq j + 4, h = 2, 3$ we have that D is very ample and so is L, except for $j = 1$, and in that case L defines a 2:1 finite morphism onto $I\!P^2$.*

From Claim (3.12) we see that, for $j = 1, 2, k \geq j + 4, h = 2, 3$, $2D + L$ and $D + 2L$ are very ample and both satisfy (ii) of Lemma (2.1) by Lemma (2.2) since $(A_1 + A_2 + A_3) \cdot L = (2D+L) \cdot L = 2k+2j \geq 4j+8 \geq 12$ or $(A_1+A_2+A_3) \cdot L = (D+2L) \cdot L = k+4j \geq 5j+4 \geq 9$.

Proof of Claim (3.12): If there is a curve F such that $F^2 = 0, F \cdot D = 1, 2$ ($F \cdot D = 0$ is excluded since D is an ample class) then $disc\Gamma_{jkh} = 8j - k^2$ divides $disc(F, D) = -1, -4$, but $k^2 - 8j \geq 13$ does not divide 1 nor 4. Hence D is very ample by (i) of Lemma (3.5). When $h = 2$ it is shown in [Mo] that L can be assumed to be irreducible base point free. Hence, by (iii) of Lemma (3.5), it defines a 2:1 finite morphism onto $I\!P^2$ for $j = 1$ since there is no irreducible curve F_1 such that $F_1 \cdot L = 0, F_1^2 = -2$ else $disc\Gamma_{1,k,2} = 8 - k^2 \leq -25$ divides $disc(L, F_1) = -4$. For $j = 2$ if there were a curve F_2 such that either $F_2^2 = 0, F_2 \cdot L = 1, 2$ or $F_2^2 = -2, F_2 \cdot L = 0$ then $disc(F_2, L) = -1, -4, -8$ would be divisible by $disc\Gamma_{2,k,2} = 16 - k^2$. But $k^2 - 16 \geq 20$, therefore L is very ample by (i) of Lemma (3.5). For $h = 3$ it suffices to show that L is irreducible: In fact it is then nef, base point free and defines a 2:1 finite morphism onto $I\!P^2$ again by (iii) of Lemma (3.5) since if there were a curve F_3 such that $F_3^2 = -2, F_3 \cdot L = 0$ then $disc(F_3, L) = -4$ would be divisible by $disc\Gamma_{1,5,3} = -13$. To show that L can be assumed to be irreducible notice that $D - L$ is effective by Riemann-Roch since $(D - L)^2 = -2, (D - L) \cdot D = 1$ and in fact it is a line L' in the embedding given by D. Now a general element $L \in |D - L'|$ is smooth since $D - L'$ is base point free and connected (as in the proof of Claim (3.8)). ∎ (for Claim (3.12))

The proof of Proposition (3.1) is now complete. ∎

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