SURJECTIVITY OF GAUSSIAN MAPS ON CURVES IN $\mathbb{P}^r$ WITH GENERAL MODULI

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1. INTRODUCTION

Let C be a curve with general moduli of genus $g \geq 7$ and L a very ample line bundle on C; it has been known since 1975, by a result of Arbarello ([A]), that in the embedding $C \subset \mathbb{P}^r = \mathbb{P}H^0(L)$ there is no smooth algebraic surface $S \subset \mathbb{P}^{r+1}$ having C as hyperplane section, thus showing the impossibility of extending Severi's argument for the unirationality of the moduli space $\mathcal{M}_g$ of curves of genus $g \leq 10$. After Harris, Mumford, Eisenbud and Wolpert ([HM], [H], [EH], [Wo]) we know that for $g \geq 23$ $\mathcal{M}_g$ is not uniruled, thus ruling out the possibility that any (not necessarily smooth) S as above is not ruled by lines over C. On the other hand a recent theorem of F.L. Zak ([Z]; see also [Lv], [B]) gave a sufficient condition for S to be a cone over any smooth curve C in terms of its normal bundle N_C : If $h^0(N_C \otimes L^{-1}) = r + 1$ then S must be a cone.

It seems then natural to ask when Zak's condition holds for curves in $\mathbb{P}^r$ with general moduli. The relevancy of this, besides the above considerations, lies in the connection (see (1.4) below) already noted by many authors ([W3], [CM], [BEL]) between Zak's condition and the surjectivity of Gaussian maps.

(1.1) *Definitions and notation.* Let L and M be two line bundles on a smooth curve C; we define $\mu_{L,M} : H^0(L) \otimes H^0(M) \to H^0(L \otimes M)$ to be the multiplication map, $R(L, M) = Ker\mu_{L,M}$ and $\Phi_{L,M} : R(L, M) \to H^0(\omega_C \otimes L \otimes M)$ the *Gaussian map* given by $\Phi_{L,M}(\sigma \otimes \tau) = \sigma d\tau - \tau d\sigma$ (on any open subset where L and M are trivial).

We also recall (see for example [W3]) that if L is very ample, embedding C in $\mathbb{P}^r$

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with normal bundle N_C, from the Euler sequence we get

(1.2) $0 \to H^0(\Omega_{\mathbb{P}^r|C} \otimes L \otimes M) \to H^0(L) \otimes H^0(M) \stackrel{\mu_{L,M}}{\longrightarrow} H^0(L \otimes M) \to H^1(\Omega_{\mathbb{P}^r|C} \otimes L \otimes M) \to H^0(L) \otimes H^1(M) \to H^1(L \otimes M) \to 0$

hence $R(L, M) = H^0(\Omega_{\mathbb{P}^r|C} \otimes L \otimes M)$ and from the normal bundle sequence

(1.3) $0 \to H^0(N_C^* \otimes L \otimes M) \to H^0(\Omega_{\mathbb{P}^r|C} \otimes L \otimes M) \stackrel{\Phi_{L,M}}{\longrightarrow} H^0(\omega_C \otimes L \otimes M) \to H^1(N_C^* \otimes L \otimes M) \to H^1(\Omega_{\mathbb{P}^r|C} \otimes L \otimes M) \to H^1(\omega_C \otimes L \otimes M) \to 0.$

In particular from (1.2) and (1.3), setting $M = \omega_C$, and observing that $\mu_{\omega_C,L}$ is surjective unless C is rational, we have

$$h^0(N_C \otimes L^{-1}) = r + 1 + corank\ \Phi_{\omega_C,L} \tag{1.4}$$

and $\Phi_{\omega_C,L}$ is surjective if and only if $h^0(N_C \otimes L^{-1}) = r + 1$.

The importance of these maps has been brought to light by Wahl by showing the connection between the corank of Φ_{ω_C,ω_C} and the deformation theory of the cone over a canonical curve ([W1]) and proving in particular that if C lies on a K3 surface then Φ_{ω_C,ω_C} is not surjective. Ciliberto, Harris and Miranda ([CHM]) showed that Φ_{ω_C,ω_C} is surjective on a curve with general moduli of genus 10 or ≥ 12, by degenerating C to some very special stable curves. Since then there have been several results on the surjectivity of $\Phi_{\omega_C,L}$ for L of large degree. Bertram, Ein and Lazarsfeld ([BEL]) used some vector bundle techniques and the fact that C can be embedded so that it is scheme-theoretically cut out by quadrics, to prove that $\Phi_{\omega_C,L}$ is surjective on any curve C if $Cliff(C) \geq 2$ and $degL \geq 4g+1-2Cliff(C)$ or if $Cliff(C) \geq 3$ and $degL \geq 4g+1-3Cliff(C)$. It follows in particular that if C has general moduli one has surjectivity of $\Phi_{\omega_C,L}$ as soon as $g \geq 7$ and $degL \geq \frac{5}{2}g+4$ (or $\frac{5}{2}(g+1)$ for g odd), since $Cliff(C) = [\frac{g-1}{2}]$ (where [x] denotes the "integer part" of a real number x). A similar result has been obtained by Paoletti ([P]); he proved, using Voisin's idea [V] of relating the non surjectivity of Gaussian maps to the existence of non projectively normal line bundles, that $\Phi_{\omega_C,L}$ is surjective on a general C of genus $g \geq 9$ when L is general and $degL \geq \frac{3}{2}g+10$ (or $\frac{3}{2}g+\frac{9}{2}$ for g odd) or for any L with $degL \geq \frac{5}{2}g+12$ (or $\frac{5}{2}g+\frac{11}{2}$ for g odd).

We also recall the recent results of Ciliberto, Lopez and Miranda ([CLM1], [CLM2]) that have shown how Gaussian map computations can be very powerful in giving new and simple proofs of the classification of Fano threefolds and Mukai varieties.

The starting idea of this paper is that, in view of (1.4), one can study the surjectivity of Gaussian maps $\Phi_{\omega_C,L}$ by constructing smooth irreducible curves $C \subset I\!P^r$ satisfying Zak's condition; this in turn suggests using some standard projective techniques such as the ones introduced by Sernesi ([S]) and several other authors (see for example [HH]). Before stating the results let us record some notation.

(1.5) *Notation.* Let d, g, r be integers such that the Brill-Noether number $\rho(d,g,r) = g-(r+1)(g-d+r)$ is not negative. We denote by $M(d,g,r)$ the unique component of the Hilbert scheme of smooth curves of degree d, genus g in $I\!P^r$, dominating $\mathcal{M}_g$. The existence and uniqueness of this component is a basic result in Brill-Noether theory.

In section 2 we will analyze the component $M(g+r,g,r)$ of linearly normal nonspecial curves. Using the fact that such curves admit a degeneration to a curve lying on a rational normal surface scroll, we prove that they satisfy Zak's condition with a few exceptions.

Theorem (1.6). *Let $C \subset I\!P^r$ be a general linearly normal nonspecial curve of genus g, N_C its normal bundle. Then*

(i) $h^0(N_C(-1)) = r+1$ *if either* $g \geq 5$ *and* $r \geq 12$, $g \geq 9$ *and* $r \geq 9$ *or* $g \geq 3$ *and* $r \geq 14$;

(ii) $h^0(N_C(-1)) > r+1$ *if* $r = 8, 10$ *and* $g = 4$.

From this theorem one easily obtains the following.

Corollary (1.7)

(i) *If C is a curve of genus g with general moduli and L is a line bundle on C such that either*

(α) *L is general and* $degL \geq \begin{cases} g+14 & \text{for } 3 \leq g \leq 4 \\ g+12 & \text{for } 5 \leq g \leq 8 \\ g+9 & \text{for } g \geq 9 \end{cases}$

or

(β) *L is any line bundle with* $degL \geq \begin{cases} 2g+15 & \text{for } 3 \leq g \leq 4 \\ 2g+13 & \text{for } 5 \leq g \leq 8 \\ 2g+10 & \text{for } g \geq 9 \end{cases}$

then $\Phi_{\omega_C,L}$ is surjective.

(ii) *If C is any smooth curve of genus 4 and L any line bundle such that* $degL = 12, 14$ *then $\Phi_{\omega_C,L}$ is not surjective.*

(iii) *If C is any smooth nonhyperelliptic curve of genus g then there exist line bundles L*

of degree $2g-1, 2g$ *(namely* $L = \omega_C(P)$ *for any point* $P \in C, L = \omega_C(2P)$ *for a general point* $P \in C$*) such that* $\Phi_{\omega_C,L}$ *is not surjective.*

We note that (β) answers (almost completely) a question of Wahl ([W3], (2.5)). Another interesting consequence of Theorem (1.6) is that it gives a new proof of Ciliberto, Harris and Miranda's result (also proved by C. Voisin [V], R. Paoletti [P]), with the exception of $g = 10, 12$.

Corollary (1.8)

Φ_{ω_C,ω_C} *is surjective on a curve* C *with general moduli of genus* $g \geq 13$.

In section 3 we will study general linearly normal special curves $C \subset \mathbb{P}^r$ with general moduli, i.e. curves representing a general point of $M(d,g,r)$, for $d < g+r$. We will degenerate to stable curves with a rational component and a component in $M(d-1,g-1,r)$; since $\rho(d-1,g-1,r) = \rho(d,g,r)-1$ one arrives at curves with $\rho = 0$. The latter admit degenerations to curves with a rational component and a component in $M(d-r,g-r-1,r)$, until one reaches canonical curves of degree $2r$ and genus $r+1$; using [CHM] and this inductive procedure we prove the ensuing

Theorem (1.9). *Let* $C \subset \mathbb{P}^r$ *be a curve representing a general point of* $M(d,g,r)$ *for integers* d,g,r *such that* $\rho(d,g,r) \geq 0$ *and* $d < g+r$*. Then*

$$h^0(N_C(-1)) = r+1 \text{ if } \ r \geq 11 \ \text{ or } \ r = 9.$$

Equivalently if C *is a curve with general moduli and* L *is a general line bundle in* $W^r_d(C)$*, then* $\Phi_{\omega_C,L}$ *is surjective for* $r \geq 11$ *or* $r = 9$.

Finally in section 4 we will show how Bertram, Ein and Lazarsfeld's technique applied to general curves leads to a result on surjectivity of Gaussian maps $\Phi_{L,M}$.

Theorem (1.10). *Let* C *be a curve of genus* $g \geq 1$ *with general moduli,* L, M *two line bundles on* C *such that,*

$$\begin{aligned} &degL, degM \geq 2g+\sqrt{g}+1 \\ &degL + degM \geq 4g + 4\sqrt{g} + \frac{2}{\sqrt{g}+1}; \end{aligned}$$

then $\Phi_{L,M}$ *is surjective.*

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2. GAUSSIAN MAPS ON NONSPECIAL CURVES

Throughout the whole paper the technique to prove surjectivity of some Gaussian map will be to degenerate our curves to some "special" ones where surjectivity can be handled and then use the fact that the corank of Gaussian maps is semicontinuous; here we degenerate to reducible curves and use induction on the genus. With this in mind, let us record the following simple but crucial fact.

Let C and γ be two smooth irreducible curves in $I\!P^r$ meeting transversally along a divisor Δ and set $C' = C \cup \gamma$. We recall that the sheaf $\Omega_{C'}$ is not locally free, since it has torsion $\mathcal{T}$ supported at the nodes of C'. From the exact sequence (see for example [CHM, (4.1)])

$$0 \to \mathcal{T} \to \Omega_{C'} \to \Omega_C \oplus \Omega_\gamma \to 0$$

we get the following diagram

$$\begin{array}{c} 0 \to H^0(\mathcal{T} \otimes \omega_{C'}(1)) \to H^0(\Omega_{C'} \otimes \omega_{C'}(1)) \stackrel{r}{\longrightarrow} H^0(\omega_C^2(1)(\Delta)) \oplus H^0(\omega_\gamma^2(1)(\Delta)) \to 0 \\ \nwarrow \Phi_{\omega_{C'},\mathcal{O}_{C'}(1)} \qquad \nearrow \Phi \\ R(\omega_{C'}, \mathcal{O}_{C'}(1)) \end{array}$$

where by definition $\Phi = r \circ \Phi_{\omega_{C'},\mathcal{O}_{C'}(1)}$; this induces a map $\Phi_{\mathcal{T}} : Ker\ \Phi \to H^0(\mathcal{T} \otimes \omega_{C'}(1))$.

Lemma (2.1). *Let C and γ be two smooth irreducible curves in $I\!P^r$ meeting transversally along a divisor Δ; set $C' = C \cup \gamma$, $N'_{C'} = Ker\{N_{C'} \to T^1_{C'}\}$ and suppose that C' is linearly normal and*

(2.2) $h^0(N_C(-1)) = r + 1$;

(2.3) $H^0(N_{C'|\gamma}(-1)(-\Delta)) = 0$;

(2.4) $H^1(N'_{C'}) = 0$;

(2.5) the map $\Phi_{\mathcal{T}} : Ker\ \Phi \to H^0(\mathcal{T} \otimes \omega_{C'}(1))$ is surjective.

Then C' is flatly smoothable and for a general smoothing C'' of C' we have $h^0(N_{C''}(-1)) = r+1$.

Proof: By [S], Lemma (5.1), there is an exact sequence

$$0 \to N_{C'|\gamma}(-1)(-\Delta) \to N'_{C'}(-1) \to N_C(-1) \to 0$$

hence (2.2) and (2.3) imply $h^0(N'_{C'}(-1)) \leq r+1$. By the definition of $N'_{C'}$ we have

$$0 \to T_{C'}(-1) \to T_{\mathbb{P}^r|C'}(-1) \to N'_{C'}(-1) \to 0$$

hence

$$0 \to H^0(T_{\mathbb{P}^r|C'}(-1)) \to H^0(N'_{C'}(-1)) \to H^1(T_{C'}(-1)) \xrightarrow{\phi_{C'}} H^1(T_{\mathbb{P}^r|C'}(-1))$$

since clearly $H^0(T_{C'}(-1)) = 0$. On the other hand the Euler sequence of C' gives $h^0(T_{\mathbb{P}^r|C'}(-1)) \geq r+1$, hence $h^0(N'_{C'}(-1)) = h^0(T_{\mathbb{P}^r|C'}(-1)) = r+1$ and therefore $\phi_{C'}$ is injective. Applying Serre duality and the spectral sequence of local and global Ext one easily shows that $\phi^*_{C'}$ is nothing else than the map

$$\begin{aligned}\Phi : R(\omega_{C'}, \mathcal{O}_{C'}(1)) \cong H^0(\Omega_{\mathbb{P}^r|C'} \otimes \omega_{C'}(1)) \to H^1(T_{C'}(-1))^* \cong H^0(\Omega^{**}_{C'} \otimes \omega_{C'}(1)) \cong \\ \cong H^0(\omega^2_C(1)(\Delta)) \oplus H^0(\omega^2_\gamma(1)(\Delta)).\end{aligned}$$

Therefore Φ is surjective and so is $\Phi_{\omega_{C'}, \mathcal{O}_{C'}(1)}$ by the snake lemma and (2.5). From [S] we know that (2.4) implies that C' is smoothable, so a general smoothing C'' of C' has $\Phi_{\omega_{C''}, \mathcal{O}_{C''}(1)}$ surjective, by the semicontinuity of the corank of Gaussian maps, and we conclude with (1.4). ∎

One simple way to degenerate nonspecial curves is to another nonspecial curve union a chord. Before proving Theorem (1.6) we show that the last three conditions of Lemma (2.1) hold in this case.

Proposition (2.6). *Let C be a curve representing a general point of $M(g+r, g, r)$ with $g \geq 1$ (i.e. a general linearly normal nonspecial curve) and γ a general chord of C. Then $C' = C \cup \gamma$ is nonspecial linearly normal and (2.3), (2.4), (2.5) of Lemma (2.1) hold.*

Proof: The fact that C' is nonspecial linearly normal is clear by the sequence

$$0 \to \mathcal{O}_\gamma(-\Delta)(1) \to \mathcal{O}_{C'}(1) \to \mathcal{O}_C(1) \to 0$$

since $H^0(\mathcal{O}_\gamma(-\Delta)(1)) = 0$ and hence $r+1 \le h^0(\mathcal{O}_{C'}(1)) \le h^0(\mathcal{O}_C(1)) = r+1$; also $H^1(\mathcal{O}_\gamma(-\Delta)(1)) = H^1(\mathcal{O}_C(1)) = 0$ give $H^1(\mathcal{O}_{C'}(1)) = 0$. From the exact sequence

$$0 \to N_{C'_{|\gamma}}(-\Delta) \to N'_{C'} \to N_C \to 0$$

since $H^1(N_C) = 0$ (C is nonspecial) we need $H^1(N_{C'_{|\gamma}}(-\Delta)) = 0$ to show (2.4). This follows again from [S], (5.1), by the exact sequence

$$0 \to N_\gamma(-\Delta) \to N_{C'_{|\gamma}}(-\Delta) \to T^1_{C'}(-\Delta) \to 0$$

since $N_\gamma(-\Delta) \cong \mathcal{O}_{I\!P^1}(-1)^{r-1}$ and $Supp T^1_{C'} = Supp \Delta$. To prove (2.3) it is enough, by semicontinuity and the irreducibility of $M(g+r,g,r)$, to exhibit *one* linearly normal nonspecial curve $\overline{C}$ and *one* chord $\overline{\gamma}$ of $\overline{C}$ such that $H^0(N_{\overline{C}'_{|\overline{\gamma}}}(-1)(-\overline{\Delta})) = 0$, where $\overline{\Delta} = \overline{C} \cap \overline{\gamma}$, $\overline{C}' = \overline{C} \cup \overline{\gamma}$. This, for example, on a rational normal surface scroll. Let $S = I\!P_{I\!P^1}(\mathcal{O}_{I\!P^1} \oplus \mathcal{O}_{I\!P^1}(-e))$ be a rational normal surface scroll embedded in $I\!P^r$ with the line bundle $C_0 + nf$ where $r-1 = 2n-e$, C_0 is a section of the $I\!P^1$-bundle and f a fiber. We take $\overline{C}$ general in $|2C_0 + (g+1+e)f|$, $\overline{\gamma} = F \sim f$ and $e = 0,1$ depending on the parity of r. Then $\overline{C}$ is smooth, F is a chord of $\overline{C}$ and the exact sequence

$$0 \to N_{\overline{C}'/S} \to N_{\overline{C}'/I\!P^r} \to N_{S_{|\overline{C}'}} \to 0$$

restricted to F gives

$$0 \to L \to N_{\overline{C}'_{|F}} \to N_{S_{|F}} \to 0 \tag{2.7}$$

where L is a line bundle on $F \cong I\!P^1$ of degree 2: $degL = \chi(L) - 1 = \chi(N_{\overline{C}'_{|F}}) - \chi(N_{S_{|F}}) - 1 = \chi(N_{F/I\!P^r}) + \chi(T^1_{\overline{C}'}) - \chi(N_{F/I\!P^r}) + \chi(N_{F/S}) - 1 = 2$ since $N_{F/S} \cong \mathcal{O}_F$.
On the other hand, if we set $N_{S_{|F}} \cong \bigoplus_{i=1}^{r-2} \mathcal{O}_{I\!P^1}(a_i)$, from

$$0 \to N_{F/S} \to N_{F/I\!P^r} \to N_{S_{|F}} \to 0$$

twisting by $\mathcal{O}_{I\!P^1}(-2)$ we get $H^1(N_{S_{|F}}(-2)) = 0$, hence $a_i \ge 1$ for every i, and

$$\sum_{i=1}^{r-2}(a_i - 1) = h^0(N_{S_{|F}}(-2)) = h^1(N_{F/S}(-2)) = 1$$

because $N_{F/\mathbb{P}^r}(-2) \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{r-1}$.

Then necessarily $a_1 = \ldots = a_{r-3} = 1$, $a_{r-2} = 2$ and (2.7) splits because $degL = 2$. It follows that $N_{\overline{C'}_{|F}} \cong \mathcal{O}_{\mathbb{P}^1}(2)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{r-3}$ and hence (2.3) since $\mathcal{O}_F(-1)(-\Delta) \cong \mathcal{O}_{\mathbb{P}^1}(-3)$.

To see (2.5) we recall the expression of the map $\Phi_{\mathcal{T}} : Ker\ \Phi \to H^0(\mathcal{T} \otimes \omega_{C'}(1))$ in local coordinates (a simple computation similar to the one in [CHM]). Locally around a node $P \in Supp\Delta$ let C' be given by the equation $st = 0$, where $s = 0$ represents C and $t = 0$ gives γ. We have that $\omega_{C'}$ is generated by $\frac{ds}{s} - \frac{dt}{t}, \Omega_{C'}$ is generated by ds, dt with the relation $tds + sdt = 0$ and $\mathcal{T}$ by $tds - sdt$. Let $\sum\limits_i \omega_i \otimes h_i \in R(\omega_{C'}, \mathcal{O}_{C'}(1)) \subset H^0(\omega_{C'}) \otimes H^0(\mathcal{O}_{C'}(1))$ and write locally

$$\omega_i = (c_{\omega_i} + sf_{\omega_i}(s) + tg_{\omega_i}(t))\ (\frac{ds}{s} - \frac{dt}{t})$$

$$h_i = (c_{h_i} + sf_{h_i}(s) + tg_{h_i}(t))\ h,$$

where h is a local generator of $\mathcal{O}_{C'}(1)$ and the f's and g's are regular at 0. Then we get

$$\begin{aligned}
&\Phi_{\omega_{C'}, \mathcal{O}_{C'}(1)}(\sum_i \omega_i \otimes h_i) = \\
&\sum_i \{(c_{h_i} + sf_{h_i}(s) + tg_{h_i}(t))[(f_{\omega_i}(s) + sf'_{\omega_i}(s))ds + (g_{\omega_i}(t) + tg'_{\omega_i}(t))dt] - \\
&(c_{\omega_i} + sf_{\omega_i}(s) + tg_{\omega_i}(t))[(f_{h_i}(s) + sf'_{h_i}(s))ds + (g_{h_i}(t) + tg'_{h_i}(t))dt]\}(\frac{ds}{s} - \frac{dt}{t}) \otimes h = \\
&\sum_i \{[c_{h_i} f_{\omega_i}(s) - c_{\omega_i} f_{h_i}(s) + s(c_{h_i} f'_{\omega_i}(s) - c_{\omega_i} f'_{h_i}(s)) + s^2(f_{h_i}(s) f'_{\omega_i}(s) - f'_{h_i}(s) f_{\omega_i}(s))]ds \\
&+ [c_{h_i} g_{\omega_i}(t) - c_{\omega_i} g_{h_i}(t) + t(c_{h_i} g'_{\omega_i}(t) - c_{\omega_i} g'_{h_i}(t)) + t^2(g_{h_i}(t) g'_{\omega_i}(t) - g'_{h_i}(t) g_{\omega_i}(t))]dt + \\
&+ [g_{h_i}(t) f_{\omega_i}(s) - f_{h_i}(s) g_{\omega_i}(t)](tds - sdt)\}(\frac{ds}{s} - \frac{dt}{t}) \otimes h.
\end{aligned}$$

Hence if $\sum\limits_i \omega_i \otimes h_i \in Ker\ \Phi$, then

$$\Phi_{\omega_{C'}, \mathcal{O}_{C'}(1)}(\sum_i \omega_i \otimes h_i) = \sum_i [g_{h_i}(0) f_{\omega_i}(0) - f_{h_i}(0) g_{\omega_i}(0)](tds - sdt)(\frac{ds}{s} - \frac{dt}{t}) \otimes h$$

locally near P as an element of $H^0(\mathcal{T} \otimes \omega_{C'}(1))$. If we identify $H^0(\mathcal{T} \otimes \omega_{C'}(1))$ with $\mathbb{C}^\delta$, where δ is the number of nodes, we get the expression

$$\Phi_{\mathcal{T}}(\sum_i \omega_i \otimes h_i) = \sum_i [g_{h_i}(0) f_{\omega_i}(0) - f_{h_i}(0) g_{\omega_i}(0)]. \tag{2.8}$$

Now let $\overline{N}$ be a line bundle on C' and suppose are given sections $\overline{u}, \overline{v} \in H^0(\overline{N}), \overline{\sigma}_i \in H^0(\omega_{C'} \otimes \overline{N}^{-1}), \overline{\tau}_i \in H^0(\mathcal{O}_{C'}(1) \otimes \overline{N}^{-1})$. Let $\overline{w}_i = (\overline{\sigma}_i \overline{u}) \otimes (\overline{\tau}_i \overline{v}) - (\overline{\sigma}_i \overline{v}) \otimes (\overline{\tau}_i \overline{u})$ and $\overline{w} = \sum_i \overline{w}_i$. Then clearly $\overline{w} \in R(\omega_{C'}, \mathcal{O}_{C'}(1))$ and if we set $N = \overline{N}_{|C}, u = \overline{u}_{|C}, v = \overline{v}_{|C}, \sigma_i = \overline{\sigma}_{i|C}, \tau_i = \overline{\tau}_{i|C}, w_i = \overline{w}_{i|C}, w = \overline{w}_{|C}$ we notice that

$$\Phi(\overline{w}) = \Phi_{\omega_C(\Delta), \mathcal{O}_C(1)}(w) = \sum_i \Phi_{\omega_C(\Delta), \mathcal{O}_C(1)}(w_i) = 2 \sum_i \sigma_i \tau_i (udv - vdu) \tag{2.9}$$

(here we use the fact that in the case at hand we have $\delta = 2$ and γ is a line, hence $H^0(\omega_\gamma^2(1)(\Delta)) = 0$). Therefore if $\sum_i \sigma_i \otimes \tau_i \in R(\omega_C(\Delta) \otimes N^{-1}, \mathcal{O}_C(1) \otimes N^{-1})$ we get that $\overline{w} \in Ker\ \Phi$. We will construct such elements of $Ker\ \Phi$ to show (2.5). First we evaluate the expression (2.8) on these elements. Writing locally $\overline{\sigma}_i = c_{\overline{\sigma}_i} + s f_{\overline{\sigma}_i}(s) + t g_{\overline{\sigma}_i}(t)$ and so on for the other sections (we omit the local generators here for simplicity of computations), a somewhat tedious calculation gives

$$\begin{aligned}
&\Phi_{\mathcal{T}}(\overline{w}) = \sum_i [c_{\overline{v}} c_{\overline{\sigma}_i} (g_{\overline{\tau}_i}(0) f_{\overline{u}}(0) - f_{\overline{\tau}_i}(0) g_{\overline{u}}(0)) + c_{\overline{u}} c_{\overline{\tau}_i} (f_{\overline{\sigma}_i}(0) g_{\overline{v}}(0) - g_{\overline{\sigma}_i}(0) f_{\overline{v}}(0)) - \\
&c_{\overline{v}} c_{\overline{\tau}_i} (f_{\overline{\sigma}_i}(0) g_{\overline{u}}(0) - g_{\overline{\sigma}_i}(0) f_{\overline{u}}(0)) - c_{\overline{u}} c_{\overline{\sigma}_i} (g_{\overline{\tau}_i}(0) f_{\overline{v}}(0) - f_{\overline{\tau}_i}(0) g_{\overline{v}}(0)) + \\
&2 c_{\overline{\tau}_i} c_{\overline{\sigma}_i} (g_{\overline{v}}(0) f_{\overline{u}}(0) - f_{\overline{v}}(0) g_{\overline{u}}(0))] = \\
&c_{\overline{v}} f_{\overline{u}}(0) \sum_i [c_{\overline{\sigma}_i} g_{\overline{\tau}_i}(0) + g_{\overline{\sigma}_i}(0) c_{\overline{\tau}_i}] - c_{\overline{v}} g_{\overline{u}}(0) \sum_i [c_{\overline{\sigma}_i} f_{\overline{\tau}_i}(0) + f_{\overline{\sigma}_i}(0) c_{\overline{\tau}_i}] + \\
&c_{\overline{u}} g_{\overline{v}}(0) \sum_i [c_{\overline{\tau}_i} f_{\overline{\sigma}_i}(0) + f_{\overline{\tau}_i}(0) c_{\overline{\sigma}_i}] - c_{\overline{u}} f_{\overline{v}}(0) \sum_i [c_{\overline{\tau}_i} g_{\overline{\sigma}_i}(0) + g_{\overline{\tau}_i}(0) c_{\overline{\sigma}_i}] + \\
&2 (g_{\overline{v}}(0) f_{\overline{u}}(0) - f_{\overline{v}}(0) g_{\overline{u}}(0)) \sum_i c_{\overline{\tau}_i} c_{\overline{\sigma}_i}.
\end{aligned}$$

On the other hand, since $\sum_i \sigma_i \otimes \tau_i \in R(\omega_C(\Delta) \otimes N^{-1}, \mathcal{O}_C(1) \otimes N^{-1})$ we have the local relations

$$\sum_i c_{\overline{\tau}_i} c_{\overline{\sigma}_i} = 0$$

$$\sum_i [c_{\overline{\sigma}_i} g_{\overline{\tau}_i}(0) + g_{\overline{\sigma}_i}(0) c_{\overline{\tau}_i}] = 0$$

hence, from the above expression we get

$$\Phi_{\mathcal{T}}(\overline{w}) = [c_{\overline{u}} g_{\overline{v}}(0) - c_{\overline{v}} g_{\overline{u}}(0)] \sum_i [c_{\overline{\tau}_i} f_{\overline{\sigma}_i}(0) + f_{\overline{\tau}_i}(0) c_{\overline{\sigma}_i}]. \tag{2.10}$$

Now choose $\overline{N}$ to be any line bundle on C' such that both maps $H^0(C', \omega_{C'} \otimes \overline{N}^{-1}) \to H^0(\gamma, \omega_\gamma(\Delta) \otimes \overline{N}^{-1})$ and $H^0(C', \mathcal{O}_{C'}(1) \otimes \overline{N}^{-1}) \to H^0(\gamma, \mathcal{O}_\gamma(1) \otimes \overline{N}^{-1})$ are surjective and moreover $deg \overline{N}_{|\gamma} = 0$ and let $\epsilon \in H^0(\gamma, \omega_\gamma(1)(\Delta) \otimes \overline{N}_{|\gamma}^{-2})$ be any section vanishing with order one at $P \in \gamma$. By the hypothesis on $\overline{N}$ we have that the natural map

$$\psi : H^0(C', \omega_{C'} \otimes \overline{N}^{-1}) \otimes H^0(C', \mathcal{O}_{C'}(1) \otimes \overline{N}^{-1}) \to$$
$$\to H^0(\gamma, \omega_\gamma(\Delta) \otimes \overline{N}^{-1}) \otimes H^0(\gamma, \mathcal{O}_\gamma(1) \otimes \overline{N}^{-1}) \to H^0(\gamma, \omega_\gamma(1)(\Delta) \otimes \overline{N}_{|\gamma}^{-2})$$

is surjective, hence there exists an element

$$\sum_i \overline{\alpha}_i \otimes \overline{\beta}_i \in H^0(C', \omega_{C'} \otimes \overline{N}^{-1}) \otimes H^0(C', \mathcal{O}_{C'}(1) \otimes \overline{N}^{-1})$$

such that $\psi(\sum_i \overline{\alpha}_i \otimes \overline{\beta}_i) = \epsilon$. We can choose sections $\overline{\delta}_i \in H^0(C', \omega_{C'} \otimes \overline{N}^{-1})$ and $\overline{\epsilon}_i \in H^0(C', \mathcal{O}_{C'}(1) \otimes \overline{N}^{-1})$ such that, $\overline{\delta}_{i|\gamma} = \overline{\epsilon}_{i|\gamma} = 0$ and, setting as above $\alpha_i = \overline{\alpha}_{i|C}$, etc., $\sum_i (\alpha_i + \delta_i)(\beta_i + \epsilon_i) = 0$. Then if we define $\overline{\sigma}_i = \overline{\alpha}_i + \overline{\delta}_i, \overline{\tau}_i = \overline{\beta}_i + \overline{\epsilon}_i$ we see that the element $\sum_i \overline{\sigma}_i \otimes \overline{\tau}_i$ of $H^0(C', \omega_{C'} \otimes \overline{N}^{-1}) \otimes H^0(C', \mathcal{O}_{C'}(1) \otimes \overline{N}^{-1})$ restricts on C to an element of $R(\omega_C(\Delta) \otimes N^{-1}, \mathcal{O}_C(1) \otimes N^{-1})$ and moreover $\psi(\sum_i \overline{\sigma}_i \otimes \overline{\tau}_i) = \psi(\sum_i \overline{\alpha}_i \otimes \overline{\beta}_i) = \epsilon$. Hence, if we write local expressions for $\overline{\sigma}_i$, etc. as above, we deduce that $\psi(\sum_i \overline{\sigma}_i \otimes \overline{\tau}_i) = \sum_i \overline{\sigma}_{i|\gamma} \overline{\tau}_{i|\gamma} = \sum_i [c_{\overline{\sigma}_i} + s f_{\overline{\sigma}_i}(s)][c_{\overline{\tau}_i} + s f_{\overline{\tau}_i}(s)] = \sum_i c_{\overline{\sigma}_i} c_{\overline{\tau}_i} + s \sum_i [c_{\overline{\tau}_i} f_{\overline{\sigma}_i}(s) + f_{\overline{\tau}_i}(s) c_{\overline{\sigma}_i}] + s^2 \sum_i f_{\overline{\tau}_i}(s) f_{\overline{\sigma}_i}(s)$ and hence that the factor $\sum_i [c_{\overline{\tau}_i} f_{\overline{\sigma}_i}(0) + f_{\overline{\tau}_i}(0) c_{\overline{\sigma}_i}]$ in (2.10) is nothing other than the coefficient of the power of s in the local expression of ϵ and therefore it is different from zero by the hypothesis on ϵ. Finally we choose any two sections $\overline{u}, \overline{v} \in H^0(C', \overline{N})$ such that the local expression $c_{\overline{u}} g_{\overline{v}}(0) - c_{\overline{v}} g_{\overline{u}}(0)$ is not zero at P while it is zero at the other point $Q \in Supp \Delta$. By (2.10) we then get an element $\overline{w}_P \in Ker\ \Phi$ such that $\Phi_{\mathcal{T}}(\overline{w}_P)$ has nonzero torsion at P and zero torsion at Q. Similarly we can find an element $\overline{w}_Q \in Ker\ \Phi$ such that $\Phi_{\mathcal{T}}(\overline{w}_Q)$ has nonzero torsion at Q and zero torsion at P and hence we are done with the surjectivity of $\Phi_{\mathcal{T}}$. ■

We will now use Lemma (2.1) and Proposition (2.6) to see the first part of Theorem (1.6).

Proof of (i) of Theorem (1.6): Let g_0 be an integer such that $h^0(N_{C_0}(-1)) = r + 1$ for C_0 general in $M(g_0 + r, g_0, r)$. For $j \geq 1$ let C_j be a general deformation of $C'_j = C_{j-1} \cup \gamma_{j-1}$, where γ_{j-1} is a general chord of C_{j-1}. By induction on j assume that C_{j-1} is a general element of $M(g_0+j-1+r, g_0+j-1, r)$ satisfying $h^0(N_{C_{j-1}}(-1)) = r+1$. From Proposition (2.6) we have that C'_j satisfies (2.3), (2.4) and (2.5), hence Lemma (2.1) gives that C_j is a general element of $M(g_0 + j + r, g_0 + j, r)$ having $h^0(N_{C_j}(-1)) = r + 1$. So part (i) of Theorem (1.6) holds for $g \geq g_0$. To finish the proof we observe that for $g_0 = 3$ and $r \geq 14$, $g_0 = 5$ and $r \geq 12$ or $g_0 = 9$ and $r \geq 9$, the fact that $h^0(N_{C_0}(-1)) = r + 1$ follows from (1.4) and [W2], [BEL], [P] respectively: In fact if $g_0 = 3$, $r \geq 14$ we have $deg\mathcal{O}_{C_0}(1) = r + 3 \geq 17 = 5g_0 + 2$ and $\Phi_{\omega_{C_0},\mathcal{O}_{C_0}(1)}$ is surjective by [W2, Theorem 3.8], since C_0 has general moduli hence it is not hyperelliptic; if $g_0 = 5$, $r \geq 12$ we have $deg\mathcal{O}_{C_0}(1) \geq 17 = 21 - 2Cliff(C_0)$ since $Cliff(C_0) = 2$, hence $\Phi_{\omega_{C_0},\mathcal{O}_{C_0}(1)}$ is surjective by [BEL, Theorem 2]; if $g_0 = 9$, $r \geq 9$ the surjectivity follows from [P, Corollary 1], since $deg\mathcal{O}_{C_0}(1) \geq 18 = \frac{3(g_0-1)}{2} + 6$ and the fact that C_0 is general in $M(g_0 + r, g_0, r)$, that is C_0 is a curve with general moduli and $\mathcal{O}_{C_0}(1)$ is a general line bundle on it. ■

The reason why we have not yet discussed the non surjectivity part of Theorem (1.6) is that it is strictly connected to the non surjectivity of Φ_{ω_C,ω_C} on a curve C with general moduli of genus $g \leq 9$ or $g = 11$. We will do this next, as well as Corollary (1.8).

Proof of Corollary (1.8) and (ii) of Theorem (1.6): For $r = 8, 10$ or $r \geq 12$ let $\Gamma \subset \mathbb{P}^r$ be a general linearly normal nonspecial curve of genus $g(\Gamma)$ such that $g(\Gamma) = 4$ if $r = 8, 10$ or $g(\Gamma) = 5$ if $r \geq 12$. Set $g = r + 1$ and $\delta = g + 1 - g(\Gamma)$; we have the following

Claim (2.11). There exist δ distinct points $P_1, \ldots, P_\delta \in \Gamma$ such that

(2.12) $dim < P_1, \ldots, P_\delta >= \delta - 2$;

(2.13) $dim < P_1, \ldots, \widehat{P_i}, \ldots, P_\delta >= \delta - 2 \;\; \forall i = 1, \ldots, \delta$.

Let us assume Claim (2.11) for now. A simple count of parameters shows that there exists a rational normal curve $\gamma \subset \mathbb{P}^{\delta-2} =< P_1, \ldots, P_\delta >$ meeting Γ transversally at $P_1, \ldots, P_\delta$; if we let $C' = \Gamma \cup \gamma$ we immediately notice that $degC' = 2g - 2$, $p_a(C') = g$ and C' is linearly normal since

$$0 \to \mathcal{O}_\gamma(-\Delta)(1) \to \mathcal{O}_{C'}(1) \to \mathcal{O}_\Gamma(1) \to 0$$

and $H^0(\mathcal{O}_\gamma(-\Delta)(1)) = 0$, $h^0(\mathcal{O}_\Gamma(1)) = r + 1 = g$.

From the exact sequence

$$0 \to N_{C'|\gamma}(-\Delta) \to N'_{C'} \to N_\Gamma \to 0$$

we see that $H^1(N'_{C'}) = 0$ and hence (2.4) follow from

$$H^1(N_{C'|\gamma}(-\Delta)) = 0. \tag{2.14}$$

Moreover (2.4) implies that C' is smoothable, so a general smoothing C of it will be a canonical curve of genus g. We will show that (2.3), (2.5) and (2.14) hold for C'. A consequence of this is that if (2.2) holds for Γ then $h^0(N_C(-1)) = g$ by Lemma (2.1) and therefore Φ_{ω_C,ω_C} is surjective by (1.4). Since Φ_{ω_C,ω_C} is not surjective for $g \le 9$ or $g = 11$ ([CHM]) this will give (ii) of Theorem (1.6). On the other hand for $g \ge 13$ we have $r \ge 12$ and $g(\Gamma) = 5$, hence (2.2) holds for Γ by part (i) of Theorem (1.6) and this will prove Corollary (1.8). It remains then to show (2.3), (2.14), (2.5) and (2.11).

To see the first two statements we degenerate again to curves lying on rational normal surface scrolls as in the proof of Proposition (2.6). With the same notation let $\overline{\Gamma}$ be general in $|2C_0+(g(\Gamma)+1+e)f|$ and $\overline{\gamma}$ general in $|C_0+(n-g(\Gamma)+1)f|$, where $r-1 = g-2 = 2n-e$, and we choose $e = 0, 1$ depending on the parity of g. Then $\overline{\Gamma}$ is a smooth linearly normal nonspecial curve of genus $g(\Gamma)$ and $\overline{\gamma}$ is a rational normal curve of degree $g-1-g(\Gamma) = \delta-2$ meeting $\overline{\Gamma}$ in $\overline{\Gamma}\cdot\overline{\gamma} = g+1-g(\Gamma) = \delta$ points. Therefore $C' = \Gamma\cup\gamma$ degenerates to $\overline{C}' = \overline{\Gamma}\cup\overline{\gamma}$ and by semicontinuity it will be enough to prove (2.3) and (2.14) for $\overline{C}'$. From the exact sequence

$$0 \to N_{\overline{C}'/S} \to N_{\overline{C}'/\mathbb{P}^r} \to N_{S_{|\overline{C}'}} \to 0$$

we get, restricting to $\overline{\gamma}$,

$$0 \to L \to N_{\overline{C}'_{|\overline{\gamma}}} \to N_{S_{|\overline{\gamma}}} \to 0 \tag{2.15}$$

and L is a line bundle on $\overline{\gamma} \cong \mathbb{P}^1$ of degree $2g+1-3g(\Gamma)$:

$deg L = \chi(L)-1 = \chi(N_{\overline{C}'_{|\overline{\gamma}}})-\chi(N_{S_{|\overline{\gamma}}})-1 = \chi(T^1_{\overline{C}'})+\chi(N_{\overline{\gamma}/S})-1 = \delta+\overline{\gamma}^2 = 2g+1-3g(\Gamma)$.

Now $deg L\otimes\mathcal{O}_{\overline{\gamma}}(-1)(-\overline{\Delta}) = 1-g(\Gamma)$ hence $H^0(L\otimes\mathcal{O}_{\overline{\gamma}}(-1)(-\overline{\Delta})) = 0$. On the other hand the exact sequence ([MS, Proposition 4])

$$0 \to \mathcal{O}_{\overline{\gamma}}(2)((4-g)f) \to N_{S_{|\overline{\gamma}}} \to \mathcal{O}_{\overline{\gamma}}(1)(2f)^{g-4} \to 0$$

gives $H^0(N_{S_{|\overline{\gamma}}}(-1)(-\overline{\Delta})) = 0$ because $H^0(\mathcal{O}_{\overline{\gamma}}(1)((4-g)f - \overline{\Delta})) = 0$ since the degree is $2-g$ and $H^0(\mathcal{O}_{\overline{\gamma}}(2f - \overline{\Delta})) = 0$ since the degree is $1 + g(\Gamma) - g < 0$. Then (2.3) for $\overline{C}'$ follows twisting (2.15) by $\mathcal{O}_{\overline{\gamma}}(-1)(-\overline{\Delta})$. Since $degL(-\overline{\Delta}) = g - 2g(\Gamma) \geq -1$ in all cases we have $H^1(L(-\overline{\Delta})) = 0$, so to see (2.14) for $\overline{C}'$ it is enough that

$$H^0(N^*_{S_{|\overline{\gamma}}}(1)) = 0 \tag{2.16}$$

because $H^0(N^*_{S_{|\overline{\gamma}}}(1))^* \cong H^1(N_{S_{|\overline{\gamma}}} \otimes \mathcal{O}_{I\!P^1}(g(\Gamma) - g - 1)) \cong H^1(N_{S_{|\overline{\gamma}}}(-\overline{\Delta}))$. From the exact sequence

$$0 \to N^*_{S_{|\overline{\gamma}}}(1) \to N^*_{\overline{\gamma}/I\!P^{g-1}}(1) \to N^*_{\overline{\gamma}/S}(1) \to 0$$

we see that (2.16) follows if we show that $\phi : H^0(N^*_{\overline{\gamma}/I\!P^{g-1}}(1)) \to H^0(N^*_{\overline{\gamma}/S}(1))$ is injective. Since $H^0(N^*_{\overline{\gamma}/I\!P^{g-1}}(1)) \cong H^0(\mathcal{I}_{\overline{\gamma}/I\!P^{g-1}}(1)) \cong H^0(\mathcal{O}_S(H - \overline{\gamma}))$ and $H^0(N^*_{\overline{\gamma}/S}(1)) \cong H^0(\mathcal{O}_{\overline{\gamma}} \otimes \mathcal{O}_S(H - \overline{\gamma}))$, the map ϕ, with these identifications, is the restriction map $H^0(\mathcal{O}_S(H - \overline{\gamma})) \to H^0(\mathcal{O}_{\overline{\gamma}} \otimes \mathcal{O}_S(H - \overline{\gamma}))$ (see [L], (2.10)$'$), hence it is injective since

$$Ker\phi \cong H^0(\mathcal{O}_S(H - 2\overline{\gamma})) \cong H^0(\mathcal{O}_S(-C_0 + (g(\Gamma) - n - 2)f)) = 0.$$

Alternatively we do not need to use (2.4) (and hence (2.14)) to show that C' is smoothable. In fact $C' = \Gamma \cup \gamma$ is a stable nodal curve, hence there is a flat family $X \to B$ such that X_b is a smooth curve of genus g and $X_0 = Y \cup R$ is abstractly isomorphic to C', i.e. Y is a smooth curve of genus $g(\Gamma)$, R a rational curve meeting Y transversally along a divisor Δ of degree δ. Then $\omega_{X/B_{|X_b}}$ embeds X_b, for $b \neq 0$, as a canonical curve, embeds Y with $\omega_{X/B_{|Y}} \cong \omega_Y(\Delta)$, hence in $I\!P^{g-1}$ as a linearly normal nonspecial curve of genus $g(\Gamma)$ and embeds R with $\omega_{X/B_{|R}} \cong \omega_R(\Delta) \cong \mathcal{O}_{I\!P^1}(\delta - 2)$, hence in $I\!P^{\delta-2}$ as a rational normal curve meeting Y in δ points. Either way the proof of Corollary (1.8) and (ii) of Theorem (1.6) will be complete as soon as we prove (2.5) and Claim (2.11). The proof of (2.5) is the same as the one given in Proposition (2.6). The only difference here is that $deg\gamma = \delta - 2$ and $\delta \geq 5$, hence $H^0(\omega^2_\gamma(1)(\Delta)) \neq 0$, but for the same choice of $\overline{N}$ the elements $\overline{w}$ still belong to $Ker\ \Phi$ (provided as above that $\sum\limits_i \sigma_i \otimes \tau_i \in R(\omega_C(\Delta) \otimes N^{-1}, \mathcal{O}_C(1) \otimes N^{-1}))$ since now as in (2.9) we have

$$\Phi(\overline{w}) = (\Phi_{\omega_C(\Delta),\mathcal{O}_C(1)}(w), \Phi_{\omega_\gamma(\Delta),\mathcal{O}_\gamma(1)}(\overline{w}_{|\gamma})) =$$
$$= \sum_i (\Phi_{\omega_C(\Delta),\mathcal{O}_C(1)}(w_i), \Phi_{\omega_\gamma(\Delta),\mathcal{O}_\gamma(1)}(\overline{w}_{i|\gamma})) =$$
$$(2\sum_i \sigma_i\tau_i(udv - vdu), 2\sum_i \overline{\sigma}_{i|\gamma}\overline{\tau}_{i|\gamma}(\overline{u}_{|\gamma}d\overline{v}_{|\gamma} - \overline{v}_{|\gamma}d\overline{u}_{|\gamma})) = (0,0)$$

and here $(\overline{u}_{|\gamma}d\overline{v}_{|\gamma} - \overline{v}_{|\gamma}d\overline{u}_{|\gamma}) \in H^0(\omega_\gamma \otimes \overline{N}^2_{|\gamma}) = 0$. Now proceed exactly as in the proof of Proposition (2.6) but choose $\overline{u}, \overline{v} \in H^0(C', \overline{N})$ such that the local expression $c_{\overline{u}}g_{\overline{v}}(0) - c_{\overline{v}}g_{\overline{u}}(0)$ is not zero at P while it is zero at all the other points $Q \in Supp\Delta$.

Proof of Claim (2.11): Let K_Γ be a canonical divisor and H_Γ a hyperplane section divisor on Γ. We will show that a general divisor $\Delta \in |H_\Gamma - K_\Gamma|$ will do. Now $deg\Delta = deg\Gamma - 2g(\Gamma) + 2 = r - g(\Gamma) + 2 = g + 1 - g(\Gamma) = \delta$; $dim < \Delta >= g - 1 - h^0(H_\Gamma - \Delta) = g - 1 - h^0(K_\Gamma) = \delta - 2$ and for every $P \in Supp\Delta$ we have $dim < \Delta - P >= g - 1 - h^0(H_\Gamma - \Delta + P) = g - 1 - h^0(K_\Gamma + P) = \delta - 2$. So we just need to show that Δ is a sum of δ distinct points. To this end it is of course enough to exhibit *one* nonspecial curve $\overline{\Gamma}$ having the general divisor $\overline{\Delta} \in |H_{\overline{\Gamma}} - K_{\overline{\Gamma}}|$ sum of distinct points. We take $\overline{\Gamma}$ as above general in $|2C_0 + (g(\Gamma) + 1 + e)f|$ on a rational normal surface scroll $S \subset \mathbb{P}^r$ with $e = 0, 1$. Note that by the adjunction formula

$$\mathcal{O}_{\overline{\Gamma}}(H_{\overline{\Gamma}} - K_{\overline{\Gamma}}) \cong \mathcal{O}_{\overline{\Gamma}} \otimes \mathcal{O}_S(C_0 + nf - K_S - \overline{\Gamma}) \cong \mathcal{O}_{\overline{\Gamma}} \otimes \mathcal{O}_S(C_0 + (n - g(\Gamma) + 1)f).$$

Now suppose $r \geq 12$ and $g(\Gamma) = 5$; then $n - g(\Gamma) + 1 = \frac{g-10+e}{2} > e$, so the divisor $C_0 + (n - g(\Gamma) + 1)f$ is very ample on S, hence $H_{\overline{\Gamma}} - K_{\overline{\Gamma}}$ is very ample on $\overline{\Gamma}$ and therefore $\overline{\Delta}$ is a sum of distinct points. If $r = 8, 10$ let $\overline{\gamma}$ be general in $|C_0 + (n - g(\Gamma) + 1)f|$; then $\overline{\Delta} = \overline{\Gamma} \cap \overline{\gamma}$ and $H^1(\mathcal{O}_S(\overline{\Gamma} - \overline{\gamma})) \cong H^1(\mathcal{O}_S(C_0 + (2g(\Gamma) + e - n)f)) = 0$ because $2g(\Gamma) + e - n = 2g(\Gamma) + \frac{e-g+2}{2} \geq 0$. Therefore $H^0(\mathcal{O}_S(\overline{\Gamma})) \to H^0(\mathcal{O}_{\overline{\gamma}} \otimes \mathcal{O}_S(\overline{\Gamma})) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(\delta))$ is surjective, hence $\overline{\Gamma}$ intersects $\overline{\gamma}$ in δ distinct points. ∎

(2.17) *Remark.* If $C \subset \mathbb{P}^r$ is a linearly normal nonspecial curve of genus g then $h^0(N_C(-1)) = r + 1$ implies $h^1(N_C(-1)) = r + 1 - \chi(N_C(-1)) \geq 0$, hence $(r - 5)g \geq 2r - 4$. So $h^0(N_C(-1)) = r + 1$ (and the surjectivity of $\Phi_{\omega_C, \mathcal{O}_C(1)}$) is possible only if

(2.18) $$r \geq 6 \;\text{ and }\; g \geq 2 + \frac{6}{r-5}.$$

These numerical conditions are not sufficient though, as part (ii) of Theorem (1.6) shows. Also notice that in the proof of (ii) of Theorem (1.6) we have used the non surjectivity of Φ_{ω_C,ω_C} on a general curve only in the cases $g = 9, 11$; in the other cases ($g \leq 8$) either we fall out of the numerical range (2.18) or the construction of C' is not possible since $H_\Gamma - K_\Gamma$ is not effective, as it is easily seen by degenerating Γ to a nonspecial curve on a rational normal scroll. Thus this method does not generate other examples of non surjectivity of Gaussian maps.

To end this section let us show Corollary (1.7).

Proof of Corollary (1.7) : To see (i), (α) is enough to notice that since L is general it is nonspecial, hence it embeds C as a general linearly normal nonspecial curve in $I\!P^r$, with $r = degL - g$. Therefore the surjectivity of $\Phi_{\omega_C,L}$ follows from part (i) of Theorem (1.6) and (1.4). Also in (ii) L is nonspecial, hence if $\Phi_{\omega_C,L}$ were surjective, so it would be on a general pair (C, L), violating (ii) of Theorem (1.6) (in this case we have $r = 8, 10$). Now (i), (β) follows with a standard argument (see for example [P]) : Choose a general line bundle $B \in W^1_{g+1}(C)$; then B is base point free and $\Phi_{\omega_C,L\otimes B^{-1}}$ is surjective by (i), (α). The commutative diagram

$$\begin{array}{ccc} R(\omega_C, L\otimes B^{-1})\otimes H^0(B) & \overset{\Phi_{\omega_C,L\otimes B^{-1}}\otimes id_{H^0(B)}}{\longrightarrow} & H^0(\omega_C^2\otimes L\otimes B^{-1})\otimes H^0(B) \\ \downarrow & & \downarrow\beta \\ R(\omega_C, L) & \overset{\Phi_{\omega_C,L}}{\longrightarrow} & H^0(\omega_C^2\otimes L) \end{array}$$

shows that $\Phi_{\omega_C,L}$ is surjective since β is by [Gr], Theorem (4.e.1), because $deg\omega_C^2 \otimes L \otimes B^{-2} = 2g - 6 + degL > 2g - 2$ hence $H^1(\omega_C^2 \otimes L \otimes B^{-2}) = 0$. To see (iii) suppose that C is not hyperelliptic and consider the canonical embedding $C \subset I\!P^{g-1}$ with normal bundle N_C. By (1.2) and (1.3) (the embedding line bundle is now ω_C), if L is a line bundle such that $H^1(L) = 0$ we get

$$\begin{aligned} corank\ \Phi_{\omega_C,L} &= h^1(N_C^* \otimes \omega_C \otimes L) - h^1(\Omega_{I\!P^{g-1}|C} \otimes \omega_C \otimes L) = \\ &= h^0(N_C \otimes L^{-1}) - corank\ \mu_{\omega_C,L}. \end{aligned} \tag{2.19}$$

Set now $L = \omega_C(P)$, where P is any point in C. Then

$h^0(N_C\otimes L^{-1}) = h^0(N_C\otimes\omega_C^{-1}(-P)) \geq h^0(N_C\otimes\omega_C^{-1}) - rank\ N_C = g + corank\ \Phi_{\omega_C,\omega_C} - g + 2 \geq 2$ while $corank\ \mu_{\omega_C,\omega_C(P)} = 1$ since μ_{ω_C,ω_C} is surjective. Therefore $corank\ \Phi_{\omega_C,L} \geq 1$

by (2.19). For the case of degree $2g$ consider the exact sequence ([E, proof Theorem 5])

$$0 \to \omega_C^{-3}(\sum_{i=1}^{g-3} 2P_i) \to N_C^* \otimes \omega_C \to \bigoplus_{i=1}^{g-3} \mathcal{O}_C(-2P_i) \to 0 \tag{2.20}$$

where $P_1, \ldots, P_{g-3}$ are general points of C and set $L = \omega_C(2P_1)$. Then L is base point free and defines a birational morphism, hence $\mu_{\omega_C,L}$ is surjective. Therefore by (2.19) and (2.20) we see that *corank* $\Phi_{\omega_C,L} = h^0(N_C \otimes \omega_C^{-1}(-2P_1)) = h^1(N_C^* \otimes \omega_C^2(2P_1)) \geq h^1(\omega_C) = 1$. ∎

3. GAUSSIAN MAPS ON SPECIAL CURVES

As mentioned in the introduction we will study here some suitable degenerations of linearly normal special curves in $\mathbb{P}^r$ with general moduli, that is degenerations in $M(d,g,r)$ with $d < g + r$ and $\rho(d,g,r) \geq 0$. The first case is for $\rho > 0$.

Proposition (3.1). *Let d, g, r be such that $g \geq 1$, $r+1 \leq d \leq g+r$ and $\rho(d,g,r) \geq 1$; let C be a curve representing a general point of $M(d-1, g-1, r)$ and γ a general chord of C, $C' = C \cup \gamma$. Then C' is smoothable and a general smoothing C'' of it represents a general point of $M(d,g,r)$.*

Proof: By Gieseker's result ([Gi]) we know that $H^1(T_{\mathbb{P}^r}{}_{|C}) = 0$; also if $\Delta = C \cap \gamma$ we have $H^1(T_{\mathbb{P}^r}{}_{|\gamma}(-\Delta)) = H^1(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{r-1}) = 0$ hence the exact sequence

$$0 \to T_{\mathbb{P}^r}{}_{|\gamma}(-\Delta) \to T_{\mathbb{P}^r}{}_{|C'} \to T_{\mathbb{P}^r}{}_{|C} \to 0$$

shows that $H^1(T_{\mathbb{P}^r}{}_{|C'}) = 0$. By definition of $N'_{C'}$ we also have $H^1(N'_{C'}) = 0$ hence C' is smoothable and a general smoothing C'' of it satisfies $H^1(T_{\mathbb{P}^r}{}_{|C''}) = H^1(N_{C''}) = 0$. Therefore C'' represents a point belonging to a unique component $\mathcal{H}$ of the Hilbert scheme of curves of degree d, genus g in $\mathbb{P}^r$, and $dim\mathcal{H} = h^0(N_{C''}) = \chi(N_{C''}) = (r+1)d + (r-3)(1-g)$; but $H^1(T_{\mathbb{P}^r}{}_{|C''}) = 0$ gives $dim G^r_d(C'') = \rho$, hence if $\pi : \mathcal{H} \to \mathcal{M}_g$ is the rational functorial map, then $dim\pi(\mathcal{H}) = dim\mathcal{H} - \rho - (r+1)^2 + 1 = 3g - 3$, i.e. $\mathcal{H}$ dominates $\mathcal{M}_g$, that is $\mathcal{H} = M(d,g,r)$. ∎

The second case is for $\rho = 0$.

Proposition (3.2). *Let d, g, r be such that $\rho(d,g,r) = 0$. Then there is an integer $q \geq 1$ such that $d = qr$, $g = (q-1)(r+1)$. Let C be a curve representing a general point* *in $M(d-r, g-r-1, r)$ and γ a general rational normal curve meeting C in $r+2$ points, $C' = C \cup \gamma$. Then C' is smoothable and a general smoothing C'' of it represents a general point of $M(d, g, r)$.*

Proof: Since $\rho(d, g, r) = g - (r+1)(g-d+r) = 0$ we have $rg = (r+1)(d-r)$. Hence r divides d, so $d = qr$ for some integer $q \geq 1$ and $g = (q-1)(r+1)$. As in the proof of Proposition (3.1) we just need to show that $H^1(T_{I\!P^r}{}_{|C'}) = 0$. Again $H^1(T_{I\!P^r}{}_{|C}) = 0$ and if $\Delta = C \cap \gamma$, then $H^1(T_{I\!P^r}{}_{|\gamma}(-\Delta)) = H^1(\mathcal{O}_{I\!P^1}(-1)^r) = 0$, hence we conclude with the exact sequence

$$0 \to T_{I\!P^r}{}_{|\gamma}(-\Delta) \to T_{I\!P^r}{}_{|C'} \to T_{I\!P^r}{}_{|C} \to 0\ . \ \blacksquare$$

(3.3) *Remark.* The integer q of Proposition (3.2) is nothing other than the index of speciality plus 1. In particular when $q = 1$ we get a rational normal curve, when $q = 2$ a canonical curve. The two propositions together show how to construct curves with general moduli by smoothing starting from rational normal curves.

Using the two degenerations above we will now show Zak's condition.

Proof of Theorem (1.9): In the proofs of Propositions (3.1) and (3.2) we have seen that (2.4) holds. Suppose we also prove (2.3) and (2.5) in both cases; then we can prove (2.2) by induction on d as follows. If $\rho(d, g, r) \geq 1$ by Proposition (3.1) a curve C'' representing a general point of $M(d, g, r)$ is a general smoothing of $C' = C \cup \gamma$ and C represents a general point of $M(d-1, g-1, r)$; by Lemma (2.1) if (2.2) holds for C it also holds for C''. On the other hand since $\rho(d-1, g-1, r) = \rho - 1$ and $d < g + r$ we have $d > \rho + r$ and $g > \rho$; so we can iterate the degeneration of Proposition (3.1) ρ times, reducing the proof of (2.2) to curves with degree $d - \rho > r$ and genus $g - \rho > 0$. But these curves have Brill-Noether number $\rho(d-\rho, g-\rho, r) = 0$ and degree $> r$, so it is enough to show (2.2) for curves with $\rho = 0$ and $q \geq 2$ (where q is as in Proposition (3.2)). Again by Proposition (3.2) and Lemma (2.1) we reduce to the case $q = 2$, that is to canonical curves of genus $g = r + 1$; since $r \geq 11$ or $r = 9$ we know that (2.2) holds for them by (1.4) and [CHM]. It remains to show (2.3) and (2.5). To do the first we will use the same method of [S] (Claim in the proof of Theorem (5.2)). Since γ is a rational curve we have $N_{C'}{}_{|\gamma} \cong \bigoplus_{i=1}^{r-1} \mathcal{O}_{I\!P^1}(a_i)$ and the inclusion $\eta : N_{\gamma/I\!P^r} \to N_{C'}{}_{|\gamma}$ is given by a $(r-1) \times (r-1)$ matrix; moreover η is induced by the inclusion of ideals $\mathcal{I}_{C'} \hookrightarrow \mathcal{I}_\gamma$. Let $\Delta = P_1 + \ldots + P_\delta = C \cap \gamma$ (where

$\delta = 2$ or $r+2$); since η is the identity outside $Supp\Delta$ we have $\eta = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_{r-1} \end{pmatrix}$ with $\sigma_1 \dots \sigma_{r-1} \in H^0(\gamma, \mathcal{O}_\gamma(\Delta))$. Now suppose we are in the case of Proposition (3.1) and γ is a general chord of C at P_1, P_2; then the tangent lines $T_{P_1}C$, $T_{P_2}C$ are disjoint, hence they span a $I\!P^3$ of equations $u_3 = \ldots = u_{r-1} = 0$. Let $u_i = 0$ be the equation of the span of $T_{P_i}\gamma$ and $T_{P_i}C$ in this $I\!P^3$, $i = 1,2$; then locally at P_i we have $\mathcal{I}_{\gamma,P_i} = (u_1, u_2, u_3, \ldots, u_{r-1})$ $i = 1,2$, $\mathcal{I}_{C,P_1} = (u_1, v_1, u_3, \ldots, u_{r-1})$, $\mathcal{I}_{C,P_2} = (v_2, u_2, u_3, \ldots, u_{r-1})$ so $\mathcal{I}_{C',P_1} = (u_1, v_1u_2, u_3, \ldots, u_{r-1})$ and $\mathcal{I}_{C',P_2} = (u_1v_2, u_2, u_3, \ldots, u_{r-1})$ and hence $\eta_{P_1} = \begin{pmatrix} 1 & & & & \\ & v_1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$, $\eta_{P_2} = \begin{pmatrix} v_2 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$.
Therefore we have $a_1 = a_2 = 2$, $a_3 = \ldots = a_{r-1} = 1$, hence

$$H^0(N_{C'_{|\gamma}}(-1)(-\Delta)) \cong H^0(\mathcal{O}_{I\!P^1}(-1)^2 \oplus \mathcal{O}_{I\!P^1}(-2)^{r-3}) = 0$$

that is (2.3). In the case of Proposition (3.2) where γ is a general rational normal curve meeting C in $r+2$ points $P_1, \ldots, P_{r+2}$, since $r \geq 9$ we can assume that at least four of these points $P_1, \ldots, P_4$ are general and have tangent lines in general position. Similarly to the above case we can assume that $\mathcal{I}_{\gamma,P_i} = (f_{i1}, f_{i2}, \ldots, f_{i,r-1}) \subset \mathcal{O}_{I\!P^r,P_i}, i = 1, \ldots, r+2$, and

$$\mathcal{I}_{C,P_1} = (u_1, f_{12}, \ldots, f_{1,r-1}), \quad \mathcal{I}_{C,P_2} = (f_{21}, u_2, f_{23}, \ldots, f_{2,r-1}),$$

$$\mathcal{I}_{C,P_3} = (f_{31}, f_{32}, u_3, f_{34}, \ldots, f_{3,r-1}), \quad \mathcal{I}_{C,P_4} = (f_{41}, f_{42}, f_{43}, u_4, f_{45}, \ldots, f_{4,r-1}),$$

hence

$$\mathcal{I}_{C',P_1} = (u_1f_{11}, f_{12}, \ldots, f_{1,r-1}), \quad \mathcal{I}_{C',P_2} = (f_{21}, u_2f_{22}, f_{23}, \ldots, f_{2,r-1}),$$

$$\mathcal{I}_{C',P_3} = (f_{31}, f_{32}, u_3f_{33}, f_{34}, \ldots, f_{3,r-1}), \quad \mathcal{I}_{C',P_4} = (f_{41}, f_{42}, f_{43}, u_4f_{44}, f_{45}, \ldots, f_{4,r-1})$$

and therefore

$$\eta_{P_1} = \begin{pmatrix} u_1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad \eta_{P_2} = \begin{pmatrix} 1 & & & & \\ & u_2 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix},$$

$$\eta_{P_3} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & u_3 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \quad \eta_{P_4} = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & 0 & & \\ & & 1 & & & & \\ & & & u_4 & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

Without loss of generality suppose $a_{r-1} \leq a_{r-2} \leq \ldots \leq a_1$; from the exact sequence

$$0 \to N_{\gamma/I\!P^r} \to N_{C'|\gamma} \to T^1_{C'} \to 0$$

we have $H^1(N_{C'|\gamma} \otimes \mathcal{O}_{I\!P^1}(-r-3)) = 0$ and $h^0(N_{C'|\gamma} \otimes \mathcal{O}_{I\!P^1}(-r-3)) = r+2$ since $SuppT^1_{C'} = Supp\Delta$ and $N_{\gamma/I\!P^r} \otimes \mathcal{O}_{I\!P^1}(-r-3) \cong \mathcal{O}_{I\!P^1}(-1)^{r-1}$. By the vanishing of the H^1 we get $a_{r-1} \geq r+2$ and the $\eta_{P_i}, i = 1, \ldots, 4$ give $a_4 \geq r+3$. Now $r+2 = h^0(N_{C'|\gamma} \otimes \mathcal{O}_{I\!P^1}(-r-3)) = \sum_{i=1}^{r-1}(a_i - r - 2)$ so $a_1 = 2r+4 - \sum_{i=2}^{r-1}(a_i - r - 2) \leq 2r+1$ and therefore $H^0(N_{C'|\gamma}(-1)(-\Delta)) \cong H^0(\bigoplus_{i=1}^{r-1} \mathcal{O}_{I\!P^1}(a_i - 2r - 2)) = 0$ and (2.3) is proved. As for (2.5) we have two cases here: $\delta = 2, r+2$. The proof is exactly the same as the one given in Proposition (2.6) in the case $\delta = 2$ or the one in the proof of Corollary (1.8) in the case $\delta = r+2$. ∎

4. GAUSSIAN MAPS $\Phi_{L,M}$

Let C be a smooth curve of genus g, L, M two line bundles on C and $A \in W^1_n(C)$ a complete base point free pencil. The use of such pencils to prove the surjectivity of Gaussian maps $\Phi_{L,M}$ has already been pointed out in [CHM], [V], [BEL], [P]. In particular in [BEL] it is remarked that an important property of A is that it is a simple covering of $I\!P^1$, i.e. all the ramification indexes are 2. From [BEL] we easily extract the following

Lemma (4.1). *Let C, L, M be as above; let $A \in W^1_n(C)$ be a complete base point free simple pencil and suppose that*

(4.2) $degL, degM \geq 2g+n$;

(4.3) $degL + degM > max\{4g - 2 + 2n + \frac{2g}{n-1}, 4g + 2n\}$.

Then $\Phi_{L,M}$ is surjective.

Proof: By [BEL] A defines a smooth irreducible curve $\Gamma \in |p_1^*A \otimes p_2^*A(-\Delta)|$ of genus $(n-1)(n+2g-3)+1-g$, where $p_i : C \times C \to C$, $i = 1, 2$ are the projections and Δ is the diagonal. By the exact sequence on $C \times C$

$$0 \to p_1^*(L \otimes A^{-1}) \otimes p_2^*(M \otimes A^{-1})(-\Delta) \to p_1^*L \otimes p_2^*M(-2\Delta) \to \mathcal{O}_\Gamma \otimes p_1^*L \otimes p_2^*M(-2\Delta) \to 0$$

one sees that the surjectivity of $\Phi_{L,M}$ follows once one proves
(4.4) $H^1(C \times C, p_1^*(L \otimes A^{-1}) \otimes p_2^*(M \otimes A^{-1})(-\Delta)) = 0$
(4.5) $H^1(\Gamma, \mathcal{O}_\Gamma \otimes p_1^*L \otimes p_2^*M(-2\Delta)) = 0$
because they give $H^1(C \times C, p_1^*L \otimes p_2^*M(-2\Delta)) = 0$ and this implies that $\Phi_{L,M}$ is surjective by the standard identification

$$\Phi_{L,M} : H^0(C \times C, p_1^*L \otimes p_2^*M(-\Delta)) \to H^0(\Delta, p_1^*L \otimes p_2^*M(-\Delta) \otimes \mathcal{O}_\Delta).$$

Since by (4.3) $deg \mathcal{O}_\Gamma \otimes p_1^*L \otimes p_2^*M(-2\Delta) = (degL + degM)(n-1) - 4n + 4 - 4g > 2g(\Gamma) - 2$ we get (4.5). On the other hand (4.2) gives $H^1(C \times C, p_1^*(L \otimes A^{-1}) \otimes p_2^*(M \otimes A^{-1})) = 0$ so $H^1(C \times C, p_1^*(L \otimes A^{-1}) \otimes p_2^*(M \otimes A^{-1})(-\Delta)) \cong Coker\mu : H^0(C, L \otimes A^{-1}) \otimes H^0(C, M \otimes A^{-1}) \to H^0(C, L \otimes M \otimes A^{-2})$. Since $L \otimes A^{-1}$ and $M \otimes A^{-1}$ are both base point free and $deg(L \otimes A^{-1}) + deg(M \otimes A^{-1}) \geq 4g + 1$ we have that μ is surjective by Green's result ([Gr], Corollary (4.e.4)). ∎

Therefore Lemma (4.1) reduces the surjectivity of $\Phi_{L,M}$ to finding suitable simple pencils of low degree. Let us denote by $H(n, w)$ the Hurwitz scheme of simple coverings of $\mathbb{P}^1$ of degree n branched at $w = 2g + 2n - 2$ points, and $\mathcal{M}^1_{g,n}$ the image of $H(n, w)$ in $\mathcal{M}_g$. If $n \leq g + 1$ the general point of $\mathcal{M}^1_{g,n}$ represents a smooth curve C_0 of genus g having a complete base point free simple pencil of degree n. Hence if L_0, M_0 are two line bundles on C_0 satisfying (4.2) and (4.3) we get that Φ_{L_0,M_0} is surjective on C_0. This in turn implies that on the general curve we also have surjectivity for *general* L and M. The crucial point here is that one can obtain the same for any L and M on a general curve using the following argument suggested by P. Pirola.

Lemma (4.6). *Suppose there exists a smooth curve C_0 of genus g and two integers l_0, $m_0 \geq 2g$ such that for any line bundles L_0, M_0 on C_0 of degrees l_0, m_0 respectively we* *have* Φ_{L_0,M_0} *surjective. Then* $\Phi_{L,M}$ *is surjective on a general curve of genus* g *for any two line bundles* L *and* M *of degrees* l_0, m_0 *respectively.*

Proof: Let $\mathcal{M}_{g,m}$ be the moduli space of curves with level m structure and $\pi : X \to \mathcal{M}_{g,m}$ the family of triples (C, L, M), $L \in Pic^{l_0}C$, $M \in Pic^{m_0}C$. Consider $U = \{(C, L, M) \in X : \Phi_{L,M}$ is surjective$\}$; then U is an open set since the corank of Gaussian maps is semicontinuous and $\pi^{-1}(C_0) \cong Pic^{l_0}(C_0) \times Pic^{m_0}(C_0) \subset U$ by hypothesis. But $\pi^{-1}(C_0)$ is compact and π is open, hence there is an open subset $V \subset \mathcal{M}_{g,m}$ containing the point $[C_0]$ such that for every $[C] \in V$ we have $\pi^{-1}([C]) \subset U$, that is a general deformation of C_0 has $\Phi_{L,M}$ surjective for any L and M of degrees l_0, m_0. ■

Proof of Theorem (1.10): Let C_0 be a curve representing a general point of $\mathcal{M}^1_{g,n}$ with $n = \lceil 1+\sqrt{g} \rceil$ (where for a real number x we let $\lceil x \rceil = min\{k \text{ integer } : k \geq x\}$). By Lemma (4.1) Φ_{L_0,M_0} is surjective for any two line bundles L_0, M_0 on C_0 of degrees l_0, m_0 satisfying (4.2) and (4.3). By Lemma (4.6) the same is true on a general C for any L and M of the same degrees. Now $l_0, m_0 \geq 2g + \lceil 1 + \sqrt{g} \rceil$ if and only if $l_0, m_0 \geq 2g + \sqrt{g} + 1$, and of course if $l_0 + m_0 \geq 4g + 4\sqrt{g} + \frac{2}{\sqrt{g}+1}$ then (4.3) holds because the function $f(x) = 2x + \frac{2g}{x-1}$ is increasing for $x \geq 1+\sqrt{g}$, hence $4g+4\sqrt{g}+\frac{2}{\sqrt{g}+1} = 4g-2+f(2+\sqrt{g}) > 4g-2+f(\lceil 1+\sqrt{g} \rceil)$.
■

Note added in proof: We recently learned of two new articles of J. Stevens [St] and G. Pareschi [Pa]. Among other results, the first author obtains a result similar to (1.7), (i), (β), while the second author improves (1.7), (i), (β) and, in some cases, (1.10).

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