On large theta-characteristics with prescribed vanishing

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Let C be a smooth complex projective curve of genus $g \geq 2$, and let $\mathcal{M}_g$ denote the moduli space of smooth curves of genus g.

A theta-characteristic on C is a line bundle L such that $L^{\otimes 2} \cong \omega_C$.

We are interested in studying theta-characteristics from two viewpoints at once:

1. the dimension $h^0(C, L) = \dim H^0(C, L)$ of the space of global sections;
2. the vanishing of some global section in $H^0(C, L)$.

The parity of a theta-characteristic L is the residue modulo 2 of the dimension $h^0(C, L) := \dim H^0(C, L)$ of the space of global sections. So a theta-characteristic L is said to be even (resp. odd) if $h^0(C, L)$ is.

In 1971, Mumford introduced a purely algebraic approach to theta-characteristics, and proved the following.

Theorem (Mumford - 1971)

Let $\mathcal{C} \xrightarrow{\psi} B$ be a family of smooth curves $C_b = \psi^{-1}(b)$, and let $\mathcal{L}$ be a line bundle on $\mathcal{C}$ such that the restriction $\mathcal{L}_{|C_b}$ is a theta-characteristic on C_b.
Then the function $b \longmapsto h^0\left(C_b, \mathcal{L}_{|C_b}\right)$ is constant modulo 2.

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Along these lines, Harris investigated the loci $\mathcal{M}_g^r$ in $\mathcal{M}_g$ of curves admitting a *large* theta-characteristic L, that is

$$\mathcal{M}_g^r := \left\{ [C] \in \mathcal{M}_g \;\middle|\; \begin{array}{l} \exists \text{ a theta-characteristic } L \text{ on } C \text{ such that} \\ h^0(C,L) \geq r+1 \text{ and } h^0(C,L) \equiv r+1 \pmod 2 \end{array} \right\}$$

where $r \geq 0$ is a fixed integer.

Theorem (Harris' Bound - 1982)

Either $\mathcal{M}_g^r$ is empty, or any irreducible component $\mathcal{Z} \subset \mathcal{M}_g^r$ satisfies

$$\operatorname{codim}_{\mathcal{M}_g} \mathcal{Z} \leq \frac{r(r+1)}{2}.$$

It is a classical result that any curve $[C] \in \mathcal{M}_g$ admits $2^{g-1}(2^g - 1)$ theta-characteristics with $r = 0$, and that $\mathcal{M}_g^1$ is a divisor of $\mathcal{M}_g$.

Besides, the sharpness in the cases $r = 2$ and $r = 3$ had been showed by Teixidor i Bigas.

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Moreover, Farkas achieved sharpness of Harris' bound for $r \leq 9$ and $r = 11$, by means of the following resut.

Theorem (Farkas - 2005)

Let $r \geq 2$ and $g(r)$ be integers. Assume that $\mathcal{M}^r_{g(r)}$ has an irreducible component of codimension $\frac{r(r+1)}{2}$ in $\mathcal{M}_{g(r)}$.
Then, for any $g \geq g(r)$, $\mathcal{M}^r_g$ has an irreducible component of codimension $\frac{r(r+1)}{2}$ in $\mathcal{M}_g$.

Finally, Benzo used the latter theorem to prove the sharpness for any integer r.

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On the other hand, let $\mathcal{H}_g$ be the moduli space of abelian differentials, which parameterizes isomorphism classes of pairs (C, ω) consisting of a smooth curve C of genus g endowed with a non-zero holomorphic form $\omega \in H^0(C, \omega_C)$.

Definition

A *partition of* $g-1$ is a sequence $\underline{k} = (k_1, \ldots, k_n)$ of integers such that $k_1 \geq \cdots \geq k_n > 0$ and $\displaystyle\sum_{i=1}^{n} k_i = g-1$.

Given a partition $\underline{k}$ as above, Kontsevich and Zorich approached theta-characteristics by studying connected components of the locus

$$\mathcal{H}_g(2\underline{k}) := \left\{ [C, \omega] \in \mathcal{H}_g \;\middle|\; \begin{array}{l} (\omega)_0 = 2(k_1 p_1 + \cdots + k_n p_n) \\ \text{for some } p_1, \ldots, p_n \in C \end{array} \right\}.$$

In particular, for any such a divisor $2(k_1 p_1 + \cdots + k_n p_n)$, we have that $L := \mathcal{O}_C\left(\sum_{i=1}^{n} k_i p_i\right)$ is a theta-characteristic on C.

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Let $\mathcal{M}_{g,n}$ be the moduli spaces of $(n+1)$-tuples $[C, p_1, \ldots, p_n]$, such that $[C] \in \mathcal{M}_g$ and $p_1, \ldots p_n \in C$ are distinct points.

We are interested in studying the following objects.

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For an integer $r \geq 0$ and a partition $\underline{k} = (k_1, \ldots, k_n)$ of $g-1$, we define the locus $\mathcal{G}^r_g(\underline{k}) \subset \mathcal{M}_{g,n}$ given by

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The forgetful morphism $\pi_n \colon \mathcal{M}_{g,n} \longrightarrow \mathcal{M}_g$ maps any $\mathcal{G}^r_g(\underline{k})$ to $\mathcal{M}^r_g$.

When $\underline{k} = (1, \ldots, 1)$, the locus $\mathcal{G}^r_g(\underline{k})$ dominates $\mathcal{M}^r_g$, and $\dim \mathcal{G}^r_g(\underline{k}) \geq \dim \mathcal{M}^r_g + r$ as the fibre over $[C] \in \mathcal{M}^r_g$ is described by the complete linear series $|L|$ associated to large theta-characteristics on C.

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are the loci of subcanonical points, which recently gained renewed interest.

Moreover, they shall play a crucial role in the proofs of our results.

Finally, we note that the description of the loci $\mathcal{G}^r_g(\underline{k})$ led to various applications in

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By means of Harris' bound, we prove the following.

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For any $r \geq 0$ and for any a partition $\underline{k} = (k_1, \dots, k_n)$ of $g-1$, either $\mathcal{G}^r_g(\underline{k})$ is empty, or the codimension in $\mathcal{M}_{g,n}$ of each irreducible component $\mathcal{Z}$ of $\mathcal{G}^r_g(\underline{k})$ satisfies

$$\operatorname{codim}_{\mathcal{M}_{g,n}} \mathcal{Z} \leq g - 1 + \frac{r(r-1)}{2}.$$

Accordingly, we say that an irreducible component $\mathcal{Z} \subset \mathcal{G}^r_g(\underline{k})$ has expected dimension if it satisfies equality in the latter bound, that is $\dim \mathcal{Z} = 2g - 2 + n - \frac{r(r-1)}{2}$.

When $\underline{k} = (1, \dots, 1)$, our bound agrees with Harris' one, as it gives $\dim \mathcal{Z} \geq \left(3g - 3 - \frac{r(r+1)}{2}\right) + r$.

The assertion for $\underline{k} = (g-1)$ had been proved in a joint work with Gian Pietro Pirola, and the proof of the theorem above relies on a similar argument.

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For any $r \geq 0$, let $g(r)$ be the integer defined as

$$g(r) := \begin{cases} 2 & \text{for } r = 0 \\ 3r & \text{for } 1 \leq r \leq 3 \\ \left\lfloor \frac{r^2+14r-11}{4} \right\rfloor & \text{for } r \geq 4. \end{cases}$$

Theorem 2

For any genus $g \geq g(r)$, and for any partition $\underline{k} = (k_1, \ldots, k_n)$ of $g-1$, the locus $\mathcal{G}_g^r(\underline{k})$ is non-empty, and there exists an irreducible component $\mathcal{Z} \subset \mathcal{G}_g^r(\underline{k})$ having expected dimension.

In particular, at a general point $[C, p_1, \ldots, p_n] \in \mathcal{Z}$, the large theta-characteristic $\mathcal{O}_C\left(\sum_{i=1}^n k_i p_i\right)$ possesses exactly $r+1$ independent global sections and, apart from the cases $(r, g) = (0, 2)$ and $(1, 3)$, the curve C is non-hyperelliptic.

When $\underline{k} = (1, \ldots, 1)$, is covered by the results on Harris' bound, and the value of $g(r)$ can be slightly lowered.

Our bound is meaningful as long as $g \geq \left\lfloor \frac{r^2-r+4}{4} \right\rfloor$, which is hypothetically the best value for $g(r)$ when $r >> 0$.

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Our bound is meaningful as long as $g \geq \left\lfloor \frac{r^2-r+4}{4} \right\rfloor$, which is hypothetically the best value for $g(r)$ when $r >> 0$.

For any $r \geq 0$, let $g(r)$ be the integer defined as

$$g(r) := \begin{cases} 2 & \text{for } r = 0 \\ 3r & \text{for } 1 \leq r \leq 3 \\ \left\lfloor \frac{r^2+14r-11}{4} \right\rfloor & \text{for } r \geq 4. \end{cases}$$

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For any genus $g \geq g(r)$, and for any partition $\underline{k} = (k_1, \ldots, k_n)$ of $g-1$, the locus $\mathcal{G}^r_g(\underline{k})$ is non-empty, and there exists an irreducible component $\mathcal{Z} \subset \mathcal{G}^r_g(\underline{k})$ having expected dimension.

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g=2

In this case $g-1=1$ and $\underline{k}=(1)$ and $r=0$. So we look for pairs (C,p) such that $O_C(p)^{\otimes 2}\cong\omega_C$, that is $|2p|\cong|\omega_C|\cong\mathfrak{g}^1_2$. Therefore

$$\mathcal{G}^0_2(1)=\left\{[C,p]\in\mathcal{M}_{2,1}|\ |2p|\cong\mathfrak{g}^1_2\right\}$$

which has dimension $2g-2+n-\frac{r(r-1)}{2}=3$.

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In this case $g-1=2$, $\underline{k}\in\{(1,1),(2)\}$, $r\in\{0,1\}$ and any $\mathcal{G}^r_g(\underline{k})$ has expected dimension. When C is non-hyperelliptic, its canonical model is a plane quartic, and theta-characteristics are cut out by bitangent lines. So

$$\mathcal{G}^0_3(1,1)=\left\{[C,p_1,p_2]\in\mathcal{M}_{3,2}|\,p_1,p_2 \text{ have the same tangent line}\right\},$$

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g=6 and r=2

The general point $[C] \in \mathcal{M}_6^2 \subset \mathcal{M}_6$ parameterizes a smooth curve C admitting a $\mathfrak{g}_5^2$, which maps C to a smooth plane quintic curve.

Conversely, if $C \subset \mathbb{P}^2$ is a smooth curve of degree 5, then $\omega_C \cong \mathcal{O}_C(2)$ and $L := \mathcal{O}_C(1)$ is the only theta-characteristic on C with $h^0(C, L) = 3$. In particular, $[C] \in \mathcal{M}_6^2 \subset \mathcal{M}_6$.

Therefore, given any partition $\underline{k} = (k_1, \ldots, k_n)$ of $g - 1 = 5$, we have that $[C, p_1, \ldots, p_n] \in \mathcal{G}_6^2(\underline{k})$ if and only if the divisor $k_1p_1 + \cdots + k_np_n$ is cut out on C by some line $\ell \subset \mathbb{P}^2$.

$\underline{k}$=(g-1) and hyperelliptic curves

If C is a hyperelliptic curve of genus g, then $[C, p] \in \mathcal{G}_g^r(g-1)$ if and only if $|2p| \cong \mathfrak{g}_2^1$ and $r \equiv \left\lfloor \frac{g-1}{2} \right\rfloor \pmod 2$. In particular,

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Either $\mathcal{G}^r_g(\underline{k})$ is empty, or the codimension in $\mathcal{M}_{g,n}$ of each irreducible component $\mathcal{Z}$ of $\mathcal{G}^r_g(\underline{k})$ satisfies

$$\mathrm{codim}_{\mathcal{M}_{g,n}} \mathcal{Z} \leq g - 1 + \frac{r(r-1)}{2}.$$

Proof. Let $\mathcal{S}_{g,n}$ be the moduli space of n-pointed spin curves, which consists of classes $[C, p_1, \ldots, p_n, L]$ such that $[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$ and L is a theta-characteristic on C.

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Consider a versal deformation family $\left(\mathcal{C} \xrightarrow{\phi} U, \mathcal{L} \longrightarrow \mathcal{C}, U \xrightarrow{\rho_i} \mathcal{C}\right)$ of the n-pointed curve $(C, p_1, \ldots, p_n, L)$ in $\mathcal{S}_{g,n}$. In particular,

- U is an analitic open set of dimension $3g-3+n$, endowed with a finite map $U \longrightarrow \mathcal{S}_{g,n}$;
- the fibres $C_t := \phi^{-1}(t)$ are smooth curves of genus g;
- the line bundle $\mathcal{L} \longrightarrow \mathcal{C}$ restricts to a theta-characteristics $L_t := \mathcal{L}_{|C_t}$ on each fibre;
- for $i = 1, \ldots, n$, the maps $\rho_i \colon U \longrightarrow \mathcal{C}$ are sections of ϕ with $p_{i,t} := \rho_i(t) \in C_t$;
- $(C_0, p_{1,0}, \ldots, p_{n,0}, L_0) = (C, p_1, \ldots, p_n, L)$ for some point $0 \in U$.

Then we restrict the versal deformation to the locus

$$U^r := \left\{ t \in U \,\middle|\, h^0(C_t, L_t) \geq r+1 \text{ and } h^0(C_t, L_t) \equiv r+1 \pmod 2 \right\},$$

and we consider the $(g-1)$-fold relative symmetric product $\mathcal{C}^{(g-1)} \xrightarrow{\Phi} U^r$ of the family $\mathcal{C}$, so that the fibre over each t is the $(g-1)$-fold symmetric product $C_t^{(g-1)}$ of the curve C_t.

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We define two subvarieties of $\mathcal{C}^{(g-1)}$ as

$$\mathcal{P} := \left\{ k_1 p_{1,t} + \cdots + k_n p_{n,t} \in C_t^{(g-1)} \middle| \, t \in U^r \right\},$$

which restricts to a point of the $\underline{k}$-diagonal on each fibre $C_t^{(g-1)}$, and

$$\mathcal{Y} := \left\{ q_1 + \cdots + q_{g-1} \in C_t^{(g-1)} \middle| \, t \in U^r \text{ and } \mathcal{O}_{C_t}\left(q_1 + \cdots + q_{g-1}\right) \cong L_t \right\},$$

which parameterizes effective divisors in the linear systems $|L_t|$.

If $k_1 p_{1,t} + \cdots + k_n p_{n,t} \in \mathcal{P} \cap \mathcal{Y}$, then $L_t \cong \mathcal{O}_{C_t}\left(\sum_{i=1}^n k_i p_{i,t}\right)$. Thus

$$[C_t, p_{1,t}, \ldots, p_{n,t}] \in \mathcal{G}_g^r(\underline{k}).$$

Moreover, composing the map $U \longrightarrow \mathcal{S}_{g,n}$ and the forgetful morphism $\mathcal{S}_{g,n} \longrightarrow \mathcal{M}_{g,n}$ of degree 2^{2g}, we obtain a finite map

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Therefore each irreducible component $\mathcal{Z} \subset \mathcal{G}^r_g(\underline{k})$ passing through $[C, p_1, \ldots, p_n]$ has dimension at least equal to the minimal dimension of any irreducible component of $\mathcal{P} \cap \mathcal{Y}$, that is

$$\dim \mathcal{Z} \geq \dim \mathcal{P} + \dim \mathcal{Y} - \dim \mathcal{C}^{(g-1)}.$$

Finally, we point out that

- $\dim \mathcal{P} = \dim U^r \geq \dim U - \frac{r(r+1)}{2} = 3g - 3 + n - \frac{r(r+1)}{2}$,
- $\dim \mathcal{Y} \geq \dim U^r + r$,
- $\dim \mathcal{C}^{(g-1)} = \dim U^r + g - 1$.

Thus

$$\dim \mathcal{Z} \geq 2g - 2 + n - \frac{r(r-1)}{2}$$

as claimed. $\square$

Now, given $r \geq 0$ and

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The proof consists of three main steps.

The first step is to assure that it is enoungh to prove the assertion for the locus $\mathcal{G}_g^r(g-1)$ of subcanonical points.

Proposition (Reduction to the case $\underline{k} = (g-1)$)

Assume that there exists an irreducible component $\mathcal{Z} \subset \mathcal{G}_g^r(g-1)$ having expected dimension. Then for any partition $\underline{k} = (k_1, \ldots, k_n)$ of $g-1$, there exists an irreducible component $\mathcal{Z}' \subset \mathcal{G}_g^r(\underline{k})$ having expected dimension.

Furthermore, if the general point $[C, p] \in \mathcal{Z}$ consists of a non-hyperelliptic curve C such that $h^0(C, \mathcal{O}_C((g-1)p) = r+1$, then the general point $[C', p'_1, \ldots, p'_n] \in \mathcal{Z}'$ parameterizes a non-hyperelliptic curve C' such that $h^0(C', \mathcal{O}_{C'}(\sum_{i=1}^n k_i p'_i)) = r+1$.

The argument of the proof is similar to the one used to achieve the bound.

Roughly speaking, the result depends on the fact that each irreducible component of the subcanonical locus $\mathcal{G}_g^r(g-1)$ may be thought as a degeneration of any $\mathcal{G}_g^r(\underline{k})$.

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Assume that there exists an irreducible component $\mathcal{Z} \subset \mathcal{G}_g^r(g-1)$ having expected dimension. Then for any partition $\underline{k} = (k_1, \ldots, k_n)$ of $g-1$, there exists an irreducible component $\mathcal{Z}' \subset \mathcal{G}_g^r(\underline{k})$ having expected dimension.

Furthermore, if the general point $[C, p] \in \mathcal{Z}$ consists of a non-hyperelliptic curve C such that $h^0(C, \mathcal{O}_C((g-1)p) = r+1$, then the general point $[C', p'_1, \ldots, p'_n] \in \mathcal{Z}'$ parameterizes a non-hyperelliptic curve C' such that $h^0(C', \mathcal{O}_{C'}(\sum_{i=1}^n k_i p'_i)) = r+1$.

The argument of the proof is similar to the one used to achieve the bound.

Roughly speaking, the result depends on the fact that each irreducible component of the subcanonical locus $\mathcal{G}_g^r(g-1)$ may be thought as a degeneration of any $\mathcal{G}_g^r(\underline{k})$.

In the light of the Proposition, the assertion of the theorem for low values of r follows from known results on subcanonical points.

The second step is to show that for any $r \geq 2$, it suffices to prove the assertion for $g = g(r)$.

Theorem (B., Pirola - 2015)

Let $r \geq 2$ and assume that there exists an integer $g(r)$ such that $\mathcal{G}^r_{g(r)}(g(r)-1)$ admits an irreducible component $\mathcal{Z}_{g(r)}$ having expected dimension.
Then for any $g \geq g(r)$, there exists an irreducible component $\mathcal{Z}_g$ of $\mathcal{G}^r_g(g-1)$ having expected dimension, as well.
Furthermore, if the general point $[C,p] \in \mathcal{Z}_{g(r)}$ satisfies $h^0(C, O_C((g(r)-1)p)) = r+1$, then $h^0(C', O_{C'}((g-1)p')) = r+1$ for general $[C',p'] \in \mathcal{Z}_g$.

In order to prove this result, we apply Eisenbud-Harris' theory of limit linear series in the setting of Cornalba's compactification $\overline{\mathcal{S}}_g$ of the moduli space of spin curves, and we extend the notion of 'subcanonical point' in these terms.

Since $\dim \mathcal{Z}_g = 2g-1-\frac{r(r-1)}{2} < 2g-1 = \dim \mathcal{G}^{\mathrm{hyp}}_g$, the general point of $\mathcal{Z}_g$ does not consists of a hyperelliptic curve.

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Then the last result to prove is the following.

Proposition (Existence of base cases)

For any $r \geq 2$, there exists an irreducible component $\mathcal{Z}_{g(r)}$ of $\mathcal{G}^r_{g(r)}(g(r)-1)$ having expected dimension $2g(r)-1-\frac{r(r-1)}{2}$, and such that its general point $[C,p] \in \mathcal{Z}_{g(r)}$ satisfies $h^0\left(C, O_C\left(\left(g(r)-1\right)p\right)\right) = r+1$.

Idea of the Proof. Let $\mathrm{Hilb}^r_{g(r),g(r)-1}$ be the Hilbert scheme of curves $C \subset \mathbb{P}^r$ having arithmetic genus $p_a(C) = g(r)$, degree $\deg C = g(r)-1$, and at most nodal singularities.

For any $r \geq 2$, there exists an irreducible component W^r of $\mathrm{Hilb}^r_{g(r),g(r)-1}$, such that any smooth curve parameterized over W^r is a linearly normal curve $C \subset \mathbb{P}^r$, and $L := \mathcal{O}_C(1)$ is a theta-characteristic, with $h^0(C,L) \geq r+1$ and $h^0(C,L) \equiv r+1 \pmod 2$.

We denote by W^r_{sm} the non-empty open subset of W^r described by smooth curves.
The locus W^r_{sm} dominates an irreducible component of $\mathcal{M}^r_g$ having expected dimension, under the natural modular map $W^r_{\mathrm{sm}} \longrightarrow \mathcal{M}_g$.

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$$\mathcal{O}_C\left((g(r)-1)p\right) \cong \mathcal{O}_C(1) \iff \begin{array}{l} \exists \text{ a hyperplane } M \subset \mathbb{P}^r : \\ \mathrm{mult}_p(C, M) = g(r) - 1 \end{array}$$

In this case, $p \in C$ is a subcanonical point and $[C, p] \in \mathcal{G}^r_{g(r)}\left(g(r)-1\right)$.

Then we extend in this terms the notion of 'subcanonical point' to each curve parameterized by W^r.

Definition

Given $[C] \in W^r$ and a point $p \in C$, we say that p is a limit subcanonical point if there exists a hyperplane $M \subset \mathbb{P}^r$ cutting out on C a 0-dimensional scheme of length $g(r)-1$ supported at p, i.e.

$$\mathrm{mult}_p(C, M) = g(r) - 1.$$

Moreover, fixing a hyperplane $M \subset \mathbb{P}^r$, we define the locus

$$Q^r(M) := \left\{[C] \in W^r \mid \exists\, p \in M \text{ such that } \mathrm{mult}_p(C, M) = g(r) - 1\right\}.$$

of curves with a limit subcanonical point cut out by M.

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$$\dim Z_{g(r)} = 2g(r) - 2 - \frac{r(r-1)}{2} + (r+1)^2 - r.$$

Indeed, if such a component exists, the image of the modular map

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(where p is the point cut out by M on C) is an irreducible component of $\mathcal{Z}_{g(r)} \subset \mathcal{G}^r_g$ of dimension $\dim \mathcal{Z}_{g(r)} = 2g(r) - 1 - \frac{r(r-1)}{2}$.

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For any $r \geq 3$, W^r admits a divisorial component W^r_h, whose general point is a nodal reducible curve $X = C \cup E$ such that

- C is contained in a hyperplane $H \cong \mathbb{P}^{r-1}$, with $[C] \in W^{r-1}_{\mathrm{sm}}$;
- E is an elliptic normal curve of degree $h := g(r) - g(r-1)$ into a $(h-1)$-plane $H' \subset \mathbb{P}^r$;
- C and E meet transversally at h points lying the $(h-2)$-plane $H \cap H' \subset \mathbb{P}^r$.

Then $[X] \in W^r_h \cap Q^r(M)$ for some hyperplane $M \subset \mathbb{P}^r$

$$\begin{aligned} &\iff \quad \exists\, p \in X\colon\ \mathrm{mult}_p(X, M) = g(r) - 1 \\ &\iff \quad p \in C \cap E,\ \mathrm{mult}_p(C, M) = g(r-1) - 1 \text{ and } \mathrm{mult}_p(E, M) = h. \end{aligned}$$

In particular,

- $C \subset H$ has a subcanonical point at p, so that $[C] \in Q^{r-1}(M \cap H)$;
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Conversely, for any $[C] \in Z_{g(r-1)}$, we can embed $C \subset \mathbb{P}^{r-1}$ into a hyperplane $H \subset \mathbb{P}^r$, and construct curves $X = C \cup E$ such that $[X] \in W^r_h \cap Q^r(M)$.

By means of this construction, we obtain an irreducible component $Y_{g(r)}$ of $W^r_h \cap Q^r(M)$ having dimension

$$\dim Y_{g(r)} = 2g(r) - 2 - \frac{r(r-1)}{2} + (r+1)^2 - r - 1.$$

Using Ran's description of Hilbert schemes of points on nodal curves, and arguing as in the proof of Theorem 1, we deduce that each irreducible component of $Q^r(M)$ has dimension at least $\dim Y_{g(r)} + 1$.

Hence $Q^r(M) \cap W^r_{\mathrm{sm}}$ is non-empty, and it admits an irreducible component $Z_{g(r)}$ such that $Y_{g(r)} \subset \overline{Z}_{g(r)} \cap W^r_h$.

In particular, since W^r_h is a divisor in W^r, we deduce that $\dim Z_{g(r)} = \dim Y_{g(r)} + 1$, as wanted. $\square$

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