

Understanding Segre invariants via higher secant varieties

Insong Choe
(with George H. Hitching)

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References (with George H. Hitching):

- ▶ Secant varieties and Hirschowitz bound on vector bundles over a curve, *Manuscr. Math.* (2010)
- ▶ Lagrangian subbundles of symplectic bundles over a curve, *Math. Proc. Camb. Phil. Soc.* (2012)
- ▶ A stratification on the moduli spaces of symplectic and orthogonal bundles over a curve, *Internat. J. Math.* (2014)
- ▶ Lagrangian subbundles of orthogonal bundles of odd rank over a curve, arXiv:1402.2816
- ▶ Non-defectivity of Grassmannian bundles over a curve, arXiv:1501.01280

Unisecants on ruled surfaces

X : smooth irreducible curve of genus $g \geq 2$ over \mathbb{C} .

S : ruled surface(scroll) over X of degree d .

([Segre](#), 1887-89)

- (1) There are unisecants of degree $\geq \frac{1}{2}(d - g)$.
- (2) If S is general and $d - g$ is odd,
there are 2^g unisecants of degree $\frac{1}{2}(d - g + 1)$.

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([Nagata](#), 1970) There are unisecants σ such that

$$\sigma^2 \leq g.$$

Note: $\sigma^2 = d - 2\deg \sigma$.

Line subbundles on rank 2 bundles

V : vector bundle over X of rank 2 and degree d .

The **Segre invariant** of V is defined by

$$s(V) := \min_{L \subset V} \{d - 2 \deg L \mid L : \text{line subbundles}\}.$$

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(Gunning, Stuhler, Lange–Narasimhan)

- (1) $s(V) \leq g$.
- (2) If $d - g$ is even/odd, $s(V) = g/g - 1$ for a general V .
- (3) If $s(V) = g$, there are 1-dimensional family of line subbundles of (maximal) degree $\frac{1}{2}(d - g)$.
- (4) A general V with $s(V) < g$ has finite number of line subbundles of maximal degree.

Generalized Segre invariant

V : principal G -bundle over X

$P \subset G$: parabolic subgroup $\rightsquigarrow V/P$: induced G/P -bundle

Generalized Segre invariant of V with respect to P is defined by

$$s_P(V) := \min_{\sigma} \{ \deg N_{\sigma} \mid \sigma : X \rightarrow V/P \}.$$

r -th Segre invariant For a vector bundle V of rank n

and degree d , and for each $r = 1, 2, \dots, n-1$,

$$s_r(V) := \min_{E \subset V} \{ rd - n \deg(E) \mid \text{rk}(E) = r \}.$$

Questions on Segre invariants

$$s_P(V) := \min_{\sigma} \{\deg N_{\sigma} \mid \sigma : X \rightarrow V/P\}.$$

$$s_r(V) := \min_{E \subset V} \{rd - n \deg(E) \mid \text{rk}(E) = r\}.$$

- ▶ For a fixed G and P , what is the (sharp) upper bound on $s_P(V)$ as V moves in the moduli of G -bundles?
- ▶ $s_P(V)$ induces a stratification on the moduli of G -bundles, called a **Segre stratification**.
Is each stratum irreducible? What is its dimension?
- ▶ A section σ of V/P is a **minimal section** if $\deg N_{\sigma} = s_P(V)$.
[A subbundle $E \subset V$ is called a **maximal subbundle** if $rd - n \deg(E) = s_r(V)$.]
How many minimal sections does a G -bundle V have, if V is general inside a fixed Segre stratum?

Upper bound

(Holla–Narasimhan 2001) For any G -bundle V ,

$$s_P(V) \leq \dim(G/P) \cdot g.$$

(Mukai–Sakai 1985) For a vector bundle V of rank n ,

$$s_r(V) \leq r(n - r)g.$$

(Hirschowitz 1986) The following bound is sharp:

$$s_r(V) \leq r(n - r)(g - 1) + k, \text{ where } 0 \leq k \leq n - 1.$$

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Observation: Consider the extensions

$$0 \rightarrow E^{(r,e)} \rightarrow V^{(n,d)} \rightarrow Q^{(n-r,d-e)} \rightarrow 0.$$

Hirschowitz bound = the smallest $rd - ne$, such that

$$\dim M(r, e) + \dim M(n-r, d-e) + \dim \mathbb{P}H^1(X, Q^* \otimes E) \geq \dim M(n, d).$$

Segre stratification (vector bundle case)

In the case of $M(n, 0)$ with respect to s_r , define

$$M(n, 0; s) := \{V \in M(n, 0) \mid s_r(V) = s\}.$$

Then,

$$M(n, 0) = \bigsqcup_{0 < k \leq k_0} M(n, 0; kn), \text{ where}$$

$$M(n, 0; kn) = \{V \in M(n, 0) \mid \deg(\text{maximal subbundle}) = -k\}.$$

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(Brambila-Paz–Lange 1998, Russo–Teixidor i Bigas 1999)

- ▶ For $0 < k \leq k_0$, $M(n, 0; kn)$ is nonempty and irreducible.
- ▶ $M(n, 0; kn) \subset \overline{M(n, 0; (k+1)n)}$ of codimension n .

Minimal sections (vector bundle case)

A subbundle $E \subset V$ of rank r is called a **maximal subbundle** of V if $s_r(V) = r \deg(V) - n \deg(E)$.

Given a stratification $M(n, 0) = \bigsqcup_{0 < k \leq k_0} M(n, 0; s_r = kn)$,

- ▶ For $0 < k < k_0$, a general $V \in M(n, 0; kn)$ has a unique maximal subbundle of rank r .
- ▶ For a general $V \in M(n, 0; k_0 n)$, the space of maximal subbundles of rank r is of dimension $k_0 n - r(n - r)(g - 1)$.
[The tangent space is given by $H^0(X, \text{Hom}(E, V/E))$.]
- ▶ When $k_0 n = r(n - r)(g - 1)$, Holla computed the number of maximal subbundles. (Oxbury: r^g for $r = 1$.)

Symplectic/orthogonal bundles of even rank

- ▶ A **symplectic(orthogonal)** bundle over X is a vector bundle V equipped with an isomorphism $\omega : V \cong V^*$, which is **anti-symmetric(symmetric)**: ${}^t\omega = -\omega$ (${}^t\omega = \omega$).
- ▶ A subbundle E of V is called **isotropic** if $\omega(E)|_E \equiv 0$, **Lagrangian** if it is isotropic and $\text{rk}(E) = \text{rk}(V)/2$.
- ▶ For a Lagrangian subbundle $E \subset V$,

$$0 \longrightarrow E \longrightarrow V \longrightarrow (E^\perp)^* \cong E^* \longrightarrow 0.$$

- ▶ The subspace $H^1(X, \text{Sym}^2 E)$ and $H^1(X, \wedge^2 E)$ parameterizes the **symplectic(orthogonal)** extensions inside $H^1(X, E \otimes E)$, with respect to which E is Lagrangian.

Segre stratification for symplectic bundles

For a symplectic bundle V of rank $2n$,

$$t(V) := \min_{E \subset V} \{-2 \deg(E) \mid E : \text{Lagrangian subbundles}\}.$$

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We get a stratification on the moduli space $MSp(2n)$:

$$MSp(2n; t) := \{[V] \in MSp(2n) \mid t(V) = t\}.$$

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Theorem (C.-Hitching)

- ▶ For any symplectic bundle V of rank $2n$, $t(V) \leq n(g - 1) + 1$.
- ▶ For each even t with $2 \leq t \leq n(g - 1) + 1$,
the stratum $MSp(2n; t)$ is nonempty and irreducible.
- ▶ A general $V \in MSp(2n)$ has 0(resp. $\frac{n+1}{2}$)–dimensional space
of maximal Lagrangian subbundles if $n(g - 1)$ is even (resp.
odd).

Segre stratification for orthogonal bundles

The moduli space has two components up to $w_2(V) \in \{0, 1\}$:

$$MO(2n) = MO(2n)^+ \bigsqcup MO(2n)^-.$$

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Theorem (C.-Hitching)

- ▶ $w_2(V) = 0/1 \iff t(V) = 0/2 \pmod{4}$.
- ▶ For any orthogonal bundle V of rank $2n$, $t(V) \leq n(g - 1) + 3$.
- ▶ For each even t with $2 \leq t \leq n(g - 1) + 3$,
the stratum $MO(2n; t) := \{[V] \in MO(2n) \mid t(V) = t\}$
is non-empty, irreducible.
- ▶ For orthogonal bundles with $t(V) \geq n(g - 1) + 2$,
maximal subbundles of rank n are not maximal Lagrangian.

Understanding $s_1(V)$ via secant varieties (rank 2 case)

Every vector bundle V of rank 2 and degree $d \gg 0$ fits into

$$0 \rightarrow \mathcal{O}_X \rightarrow V \rightarrow L \rightarrow 0,$$

hence $[V] \in \mathbb{P}H^1(X, L^{-1}) \cong \mathbb{P}H^0(X, K_X L)^\vee \supset X$.

(Atiyah, Lange–Narasimhan) The Segre stratification matches with the stratification given by the higher secant varieties of X :

$$s_1(V) \leq d - 2m \iff [V] \in \text{Sec}^{d-m} X.$$

For example, this reproves the Segre–Nagata's bound ($s_1(V) \leq g$).

$$s_1(V) \leq d - 2m \iff [V] \in \text{Sec}^{d-m} X:$$

Suppose $s_1(V) \leq d - 2m$. Then V has a subbundle M of degree $m (> 0)$, then we get

$$\begin{array}{ccccccc} & & \tau & & & & \tau \\ & & \uparrow & & & & \uparrow \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & V & \longrightarrow & L \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X \oplus M & \longrightarrow & M \longrightarrow 0 \end{array}$$

Hence

$$[V] \in \mathbb{P} \text{Ker} [H^1(X, L^{-1}) \rightarrow H^1(X, M^{-1}) = H^1(X, L^{-1}(\tau))]$$

for some torsion sheaf τ of degree $d - m$.

Therefore, $[V] \in \text{Sec}^{d-m} X$. And vice versa. □

Generalization

rank 2	$0 \rightarrow \mathcal{O}_X \rightarrow V \rightarrow L \rightarrow 0$
higher rank	$0 \rightarrow E^r \rightarrow V^n \rightarrow F^{n-r} \rightarrow 0$
symplectic/orthogonal	$0 \rightarrow E^n \rightarrow V^{2n} \rightarrow E^* \rightarrow 0$

rank 2	X	$\mathbb{P}H^1(X, L^*)$
higher rank	$\bigcup_{x \in X} (\mathbb{P}E_x \times \mathbb{P}F_x^*) \subset \mathbb{P}(E \otimes F^*)$	$\mathbb{P}H^1(X, E \otimes F^*)$
symplectic	$\mathbb{P}E \subset \mathbb{P}(Sym^2 E)$	$\mathbb{P}H^1(X, Sym^2 E)$
orthogonal	$Gr(2, E) \subset \mathbb{P}(\wedge^2 E)$	$\mathbb{P}H^1(X, \wedge^2 E)$

(1) Vector bundle of higher rank

For fixed E^r and F^{n-r} , the extensions

$$0 \rightarrow E \rightarrow V \rightarrow F \rightarrow 0$$

are classified by the space

$$\mathbb{P}H^1(X, \text{Hom}(F, E)) \cong \mathbb{P}H^0(\mathbb{P}\text{Hom}(F, E), \pi^*K_X \otimes \mathcal{O}_X(1))^\vee.$$

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For stable E and F with $\mu(F) - \mu(E) \gg 0$, we get a scroll

$$\mathbb{P}\text{Hom}(F, E) \subset \mathbb{P}H^1(X, \text{Hom}(F, E)),$$

which contains the Segre bundle

$$\mathcal{S}_{E,F} = \bigcup_{x \in X} (\mathbb{P}E_x \times \mathbb{P}F_x^*) \subset \mathbb{P}\text{Hom}(F, E).$$

$s_r(V)$ vs. $\text{Sec}^k(S_{E,F})$

$$S_{E,F} \subset \mathbb{P}Hom(F, E) \subset \mathbb{P}H^1(X, Hom(F, E))$$

Recall:

$$s_r(V) = \min_{H \subset V} \{rd - n \deg(H) \mid \text{rk}(H) = r\}.$$

Proposition Let $E \in M(n - r, 0)$ and $F \in M(r, d)$ with $d \gg 0$. Given $[V] \in \mathbb{P}H^1(X, Hom(F, E))$, if $[V] \in \text{Sec}^k(S_{E,F})$, then V has a subbundle H of rank r and degree $\geq d - k$, hence

$$s_r(V) \leq rd - n(d - k).$$

$$s_1(V) \leq d - 2m \iff [V] \in \text{Sec}^{d-m} X:$$

Suppose $s(V) \leq d - 2m$. Then V has a subbundle M of degree $m (> 0)$, then we get

$$\begin{array}{ccccccc} & & \tau & & & & \tau \\ & & \uparrow & & & & \uparrow \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & V & \longrightarrow & L \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X \oplus M & \longrightarrow & M \longrightarrow 0 \end{array}$$

Hence

$$[V] \in \mathbb{P} \text{Ker} [H^1(X, L^{-1}) \rightarrow H^1(X, M^{-1}) = H^1(X, L^{-1}(\tau))]$$

for some torsion sheaf τ of degree $d - m$.

Therefore, $[V] \in \text{Sec}^{d-m} X$. And vice versa. □

$$[V] \in \text{Sec}^k S_{E,F} \implies s_r(V) \leq rd - n(d-k) :$$

Suppose V has a subbundle \tilde{E}^r degree $\geq d - k$ intersecting E^{n-r} trivially, then we get

$$\begin{array}{ccccccc} & \tau & \xlongequal{\quad} & \tau & & & \\ & \uparrow & & \uparrow & & & \\ 0 & \longrightarrow & E^{n-r} & \longrightarrow & V & \longrightarrow & F^r \longrightarrow 0 \\ & \parallel & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & E & \longrightarrow & E \oplus \tilde{E} & \longrightarrow & \tilde{E} \longrightarrow 0 \end{array}$$

Hence

$$[V] \in \mathbb{P} \text{Ker} \left[H^1(X, F^* \otimes E) \longrightarrow H^1(X, \tilde{E}^* \otimes E) \right]$$

for some torsion sheaf τ of degree $\leq k$.

This implies that $[V] \in \text{Sec}^k S_{E,F}$. And vice versa. □

Another proof of Hirschowitz bound (C.-Hitching 2010)

$$\mathcal{S}_{E,F} \subset \mathbb{P}Hom(F, E) \subset \mathbb{P}H^1(X, Hom(F, E)) =: \mathbb{P}^N$$

Since $\dim \mathcal{S}_{E,F} = (r - 1) + (n - r - 1) + 1 = n - 1$,

$$\dim(Sec^k(\mathcal{S}_{E,F})) = \min\{(n - 1)k + (k - 1) = nk - 1, N\}$$

provided that the higher secant varieties $\mathcal{S}_{E,F}$ are not defective.

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provided that the higher secant varieties $S_{E,F}$ are not defective.

Since $N = (n - r)(d + r(g - 1)) - 1$,

$$[V] \in Sec^m(\mathcal{S}_{E,F}) \text{ for } m = \left\lceil \frac{n - r}{n}(d + r(g - 1)) \right\rceil.$$

Hence by the previous Theorem,

$$s_r(V) \leq rd - n(d - m) \leq r(n - r)(g - 1) + (n - 1).$$

Non-defectiveness of $\text{Sec}^k(\mathcal{S}_{E,F})$

Terracini Lemma $\dim(\text{Sec}^k Z) = \dim < \mathbb{T}_{z_1} Z, \dots, \mathbb{T}_{z_k} Z >$

Hence it suffices to show that for general points $[\mu_i \otimes e_i] \in \mathcal{S}_{E,F}$,

$$\dim < \mathbb{T}_{[\mu_1 \otimes e_1]} \mathcal{S}, \dots, \mathbb{T}_{[\mu_m \otimes e_k]} \mathcal{S} > = \min\{nk - 1, \dim \mathbb{P}^N\}.$$

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Lemma For $[\mu_i \otimes e_i] \in \mathcal{S}_{E,F}|_x = \mathbb{P}F_x^* \times \mathbb{P}E_x$, consider

$$0 \longrightarrow F^* \xrightarrow{\mu_i} \widehat{F}_i^* \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

$$0 \longrightarrow E \xrightarrow{e_i} \widehat{E}_i \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

For the elementary transformation $F^* \otimes E \subset \widehat{F}_i^* \otimes \widehat{E}_i$,

$$\mathbb{T}_{[\mu_i \otimes e_i]} \mathcal{S}_{E,F} = \mathbb{P} \ker \left[H^1(X, F^* \otimes E) \longrightarrow H^1(X, \widehat{F}_i^* \otimes \widehat{E}_i) \right].$$

Therefore, for general points $[\mu_i \otimes e_i] \in \mathcal{S}_{E,F}$, $1 \leq i \leq k$,

$$\begin{aligned} & \langle \mathbb{T}_{[\mu_1 \otimes e_1]} \mathcal{S}, \dots, \mathbb{T}_{[\mu_k \otimes e_k]} \mathcal{S} \rangle \\ &= \mathbb{P} \ker \left[H^1(X, F^* \otimes E) \xrightarrow{\psi} H^1(X, \widehat{F}^* \otimes \widehat{E}) \right], \end{aligned}$$

where \widehat{F}^* (resp. \widehat{E}) are the elementary transformations of F^* (resp. E) at μ_1, \dots, μ_k (resp. e_1, \dots, e_k).

Therefore, for general points $[\mu_i \otimes e_i] \in \mathcal{S}_{E,F}$, $1 \leq i \leq k$,

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where \widehat{F}^* (resp. \widehat{E}) are the elementary transformations of F^* (resp. E) at μ_1, \dots, μ_k (resp. e_1, \dots, e_k).

Finally, as desired,

$$\dim(\text{Sec}^k(\mathcal{S}_{E,F})) = \dim \mathbb{P}(\ker \psi) = \min\{nk - 1, \dim \mathbb{P}^N\}$$

by **Hirschowitz' lemma**: For general A and B ,

$$\mu(A \otimes B) \leq g - 1 \implies H^0(X, A \otimes B) = 0.$$



(2) Symplectic/orthogonal bundles of even rank

Recall: the subspace $H^1(X, \text{Sym}^2 E)$ ($H^1(X, \wedge^2 E)$) parameterizes the **symplectic**(**orthogonal**) extensions such that E is Lagrangian.

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For stable E with $\deg E \ll 0$,

the subscrolls $\mathbb{P}(\text{Sym}^2 E)$ and $\mathbb{P}(\wedge^2 E)$ are embedded in

$$\mathbb{P}(E \otimes E) \subset \mathbb{P}H^1(X, E \otimes E) = \mathbb{P}H^0(\mathbb{P}(E \otimes E), \pi^* K_X \otimes \mathcal{O}_X(1))^\vee.$$

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Hence we get a Veronese bundle and a Grassmannian bundle:

$$\mathbb{P}E \subset \mathbb{P}(\text{Sym}^2 E) \subset \mathbb{P}H^1(X, \text{Sym}^2 E),$$

$$Gr(2, E) \subset \mathbb{P}(\wedge^2 E) \subset \mathbb{P}H^1(X, \wedge^2 E).$$

$t(V)$ vs. $\text{Sec}^k \mathbb{P}E$ / $\text{Sec}^k Gr(2, E)$

$$\boxed{\mathbb{P}E \subset \mathbb{P}(Sym^2 E) \subset \mathbb{P}H^1(X, Sym^2 E)}$$

Proposition Given $[V] \in \mathbb{P}H^1(X, Sym^2 E)$, if $[V] \in \text{Sec}^k (\mathbb{P}E)$, then V has a Lagrangian subbundle H of degree $\geq -(\deg E + k)$, hence $t(V) \leq 2(\deg E + k)$.

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Proposition Given $[V] \in \mathbb{P}H^1(X, \wedge^2 E)$, if $[V] \in \text{Sec}^k Gr(2, E)$, then V has a Lagrangian subbundle H of degree $\geq -(\deg E + 2k)$, hence $t(V) \leq 2(\deg E + 2k)$.

- For $[e \otimes e] \in \mathbb{P}E|_x$, consider

$$0 \longrightarrow E^* \xrightarrow{e} \widehat{E} \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

$$\mathbb{T}_{[e \otimes e]} \mathbb{P}E = \mathbb{P} \ker \left[H^1(X, Sym^2 E) \longrightarrow H^1(X, Sym^2 \widehat{E}) \right].$$

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- For $[e \wedge f] \in Gr(2, E|_x)$, consider

$$0 \longrightarrow E^* \xrightarrow{e \wedge f} \widehat{E} \longrightarrow \mathbb{C}_x^{\oplus 2} \longrightarrow 0.$$

$$\mathbb{T}_{[e \wedge f]} Gr(2, E) = \mathbb{P} \ker \left[H^1(X, \wedge^2 E) \longrightarrow H^1(X, \wedge^2 \widehat{E}) \right].$$

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$$0 \longrightarrow E^* \xrightarrow{e \wedge f} \widehat{E} \longrightarrow \mathbb{C}_x^{\oplus 2} \longrightarrow 0.$$

$$\mathbb{T}_{[e \wedge f]} Gr(2, E) = \mathbb{P} \ker \left[H^1(X, \wedge^2 E) \longrightarrow H^1(X, \wedge^2 \widehat{E}) \right].$$

Variant of Hirschowitz' lemma For a general $F \in U_X(n, d)$, if $\mu(F \otimes F) \leq g - 1$, then $H^0(X, F \otimes F) = 0$. Therefore,

$$H^0(X, Sym^2 F) = 0 = H^0(X, \wedge^2 F).$$

Lagrangian subbundles intersecting non-trivially

Suppose two Lagrangian subbundles E_1 and E_2 intersect in H of rank > 0 . Taking quotient by H , get

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_1/H & \longrightarrow & (E_2/H)^* & \longrightarrow & \tau \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & E_1/H & \longrightarrow & H^\perp/H & \longrightarrow & (E_1/H)^* \\ & & & \uparrow & & \uparrow \\ & & E_2/H & = & E_2/H & & \end{array}$$

Lemma Let $[V] \in \mathbb{P}H^1(X, \text{Sym}^2 E_1)$. Then V admits a Lagrangian subbundle of degree $-e_2 \geq -e_1 = \deg E_1$ intersecting E_1 in H of degree $-h$ if and only if

$$[H^\perp/H] \in \text{Sec}^{(e_1+e_2-2h)} \mathbb{P}(E_1/H) \subset \mathbb{P}H^1(X, \text{Sym}^2(E_1/H)).$$

(3) Orthogonal bundles of odd rank

Given an orthogonal bundle V of rank $2n + 1$,

- ▶ An isotropic subbundle of rank n is called [Lagrangian](#).
- ▶ For Lagrangians: $0 \rightarrow E \rightarrow E^\perp \rightarrow \det V \cong \mathcal{O}_X \rightarrow 0$.
- ▶ $t(V) := \min_{E \subset V} \{-2 \deg(E) \mid E : \text{Lagrangian subbundles}\}$.

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Theorem

- ▶ $t(V) \leq (n+1)(g-1) + 3$.
- ▶ For each even t with $2 \leq t \leq (n+1)(g-1) + 3$, the strata $O(2n+1; t)$ is non-empty and irreducible.
- ▶ $O(2n; t) \subset \overline{O(2n; t+4)}$ of codimension $2n$.

Orthogonal extensions of odd rank

Let $[V] \in H^1(X, F \otimes E)$ be an extension: $0 \rightarrow E \rightarrow V \rightarrow F^* \rightarrow 0$.

Lemma The bundle V has an orthogonal structure w. r. t. which E is Lagrangian if and only if

(i) there is $[f] \in H^1(X, E)$: $0 \rightarrow E \rightarrow F \rightarrow \mathcal{O}_X \rightarrow 0$

such that

(ii) $[V] \in H^1(X, \wedge^2 F) \subset H^1(X, F \otimes F)$ and

(iii) $[V] \in q^{-1}[f]$, where

$$0 \longrightarrow H^1(X, \wedge^2 E) \longrightarrow H^1(X, \wedge^2 F) \xrightarrow{q} H^1(X, E) \longrightarrow 0.$$

$t(V)$ vs. secant varieties

$$\begin{array}{ccc} H^1(X, E) & & \\ \uparrow q & & \\ H^1(X, \wedge^2 F) & \xrightarrow{\pi} & H^1(X, \wedge^2(F/H)) \end{array}$$

An orthogonal bundle corresponding to the extension $[V] \in q^{-1}[f]$ admits another Lagrangian subbundle \tilde{E} meeting E trivially

\Updownarrow

$W = V \perp \mathcal{O}_X$ admits two Lagrangian subbundles F and \tilde{F} meeting in H of rank 1

\Updownarrow

$\pi([V])$ lies on $\text{Sec}^k \text{Gr}(2, F/H)$, $k = \frac{1}{2}(e + \tilde{e} - 2h)$.

Number of maximal subbundles vs. secant order

A general vector/symplectic/orthogonal bundle V has a finitely many maximal (Lagrangian) subbundles if

$$s_r(V) = r(n - r)(g - 1) \text{ or } t(V) = n(g - 1).$$

The number is related to the secant order:

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Example Suppose $n(g - 1)$ is even. Let $V \in MSp(2n)$ be a general symplectic bundle with a Lagrangian subbundle E , so that

$$[V] \in \mathbb{P}E \subset \mathbb{P}H^1(X, Sym^2 E).$$

Then, the number of maximal Lagrangian subbundles of V
= $1 + (\text{the secant order for } Sec^{n(g-1)} \mathbb{P}E)$.

Grazie per l'attenzione.