

Hilbert scheme of smooth curves

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June 30, 2015

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1. Brill-Noether theory

Definition 1.1.

$J_{d,g,r}$: Hilbert scheme whose general point corresponds to smooth irreducible and non-degenerate curves of degree d and genus g in \mathbb{P}^r .

Brill-Noether number

The geometry of $J_{d,g,r}$ is closely related with the Brill-Noether number

$$\rho(d, g, r) := g - (r+1)(g-d+r).$$

Questions

(1) When does $J_{d,g,r}$ exist?

(2) When is $J_{d,g,r}$ irreducible?

(3) Describe the general elements of an extra component of $J_{d,g,r}$.

• Brill-Noether theory

a non-degenerate map $C \rightarrow \mathbb{P}^r$ of degree d , modulo
automorphisms of \mathbb{P}^r

- \Leftrightarrow (1) a line bundle \mathcal{L} of degree d on C .
(2) an $(r+1)$ -dimensional vector space $V \subseteq H^0(\mathcal{L})$
(3) the sections $\{s \in V\}$ have no common zeroes.

- (\mathcal{L}, V) : a linear system of degree d and projective dimension r .
 $\leadsto \mathcal{G}_d^r$ for short.

• Definition

- $G_d^r(c) := \{(\mathcal{L}, V) \mid \text{degree } \mathcal{L} = d, V \subseteq H^0(\mathcal{L}), \dim V = r+1\}$
- $Q_d^r := \{(c, \mathcal{L}, V) \mid c \in M_g, (\mathcal{L}, V) \in G_d^r(c)\}$
- $W_d^r(c) := \{\mathcal{L} \mid \text{degree } \mathcal{L} = d, \dim(H^0(\mathcal{L})) \geq r+1\}$
- $W_d^r := \{(c, \mathcal{L}) \mid c \in M_g, \mathcal{L} \in W_d^r(c)\}$.
- Let g be a component of Q_d^r . Then there exists a rational map $g \dashrightarrow W_d^\alpha$ for some α .

1. Existence & non-existence theorem

- If $\rho \geq 0$, then $G_d^r(C) \neq \emptyset$ for any $C \in M_g$.
- If $\rho \geq 0$, then $\dim(G_d^r(C)) = \rho$ for general $C \in M_g$.
- If $\rho < 0$, then a general curve in M_g does not have f_d^r .

\Rightarrow (i) gonality of $C \leq \lfloor \frac{g+3}{2} \rfloor$.

(IT) If C is in general, then $\text{gon}(C) = \lfloor \frac{g+3}{2} \rfloor$.

2. The geometry of g_d^r for $r \geq 3$.

- If $\rho(d, g, r) > 0$, then there is a unique component g_0 of g_d^r which dominates M_g . (It is called "principal component" or "distinguished component".)
- $\dim(g_0) = 3g - 3 + \rho(d, g, r)$
- If g is a component of g_d^r , then $\dim(g) \geq 3g - 3 + \rho(d, g, r)$.

3. The geometry of Hilbert scheme $\mathcal{J}_{d,g,r}$ for $\rho(d,g,r) \geq 0$.

$$\begin{array}{ccc}
 \mathcal{J}'_{d,g,r} & \xrightarrow{\quad} & M_g \\
 \downarrow \text{PGL}(r+1)\text{-bundle} & & \downarrow \\
 g_d^r \xrightarrow{q_d} Q & \xrightarrow{\quad} & W_d^\alpha \\
 & \uparrow \text{G}(d,r)\text{-bundle} &
 \end{array}$$

(1) If $\rho \geq 0$, then there exists a unique irreducible component \mathcal{J}'_0 of $\mathcal{J}'_{d,g,r}$ dominating M_g .

$$\begin{aligned}
 (2) \text{ If } \rho \geq 0, \quad \dim(\mathcal{J}'_0) &= 3g-3 + \rho(d,g,r) + \dim(\text{PGL}(r+1)) \\
 &= (r+1)d - (r-3)(g-1) := \lambda_{d,g,r}
 \end{aligned}$$

4. The geometry of $J'_{d,g,r}$ for $\rho(d,g,r) < 0$.

Definition 1.2.

- An irreducible component f of $J'_{d,g,r}$ is said to have *expected number of moduli* if the image of $\pi: f \rightarrow M_g$ has dimension $\min\{3g-3, 3g-3+p\}$. f is said to be regular if $H^1(N_C) = 0$ for a general curve $C \in f$.

Theorem (Sernesi, Eisenbud-Harris, Ballico-Ellia, Pareschi, Lopez)

- (Sernesi '84) There exist a regular component of $J'_{d,g,r}$ having expected number of moduli if $g-d+r \geq \max\{0, 1-\rho(d,g,r)\}$.

- (Lopez '99) " if $\min\{\dots, -(2 - \frac{t}{r+3})g + h(r)\} \leq \rho(d,g,r) \leq 0$
where $h(r) = \frac{4r^3 + 8r^3 - 9r + 3}{r+3}$.

2. Severi conjecture

Severi conjecture: If $\underline{g-d+r \leq 0}$, then $f'_{d,g,r}$ is irreducible.

$$P(d,g,r) = g - (r+1)(g-r+d) \geq g$$

- Ein proved this conjecture for $r=3$ & $r=4$.
- Ein proved $f'_{d,g,r}$ is irreducible if $r \geq 6$ and $d \geq \frac{2r-2}{r+2}g + \frac{2r+3}{r+2}$.
- For $r=5$, it is still open. (H. Itoe proved that it is irr. if $d \geq \max\left\{\frac{11}{10}g+2, g+5\right\}$.)
- Severi conjecture is not true for $r \geq 6$.
(Harris (1982), Ein (1986), Mezzetti-Sacchiero (1986), Keem (1994), Kim (2001), ...)

• Ein's counter example 2.1.

Assume that $r \geq b$. Then $f'_{16r-35, 8r+6, r}$ is reducible.

Proof) (1) $p(d, g, r) = 7r^2 - 26r - 35 \geq 0$ for $r \geq b$.

(2) \exists unique component f'_0 corresponding to non-special curves.

$$\dim f'_0 = (r+1)d + (r-3)(1-g) = 8r^2 - 20.$$

(3) $M'_{g,3} := \{ c \in M_{8r+6} \mid c \text{ has } g'_3 \} \subseteq M_g$.

- Let Σ be a family of trigonal curves with $|K_c - 15g'_3|$.

- Monodromy invariant gives $|K_c - 15g'_3|$ is very ample.

$$\dim \Sigma \geq \dim (M'_{g,3}) + \dim G(r+1, 8r-24) + \dim (\text{Aut } \bar{P'})$$

$$= (16r+13) + (7r-25)(r+1) + (r+1)^2 - 1 = 8r^2 + 12 \geq 8r^2 - 20 = \dim (f'_0).$$

Keem's counter example 2.2.

Assume that $r \geq 9$. $f'_{3r-1, \frac{3r+7}{2}, r}$ is reducible with two components.

Furthermore, a general element of the exceptional component is trigonal with $|K - 2g_3'|$.

Proof) (1) $\rho(3r-1, \frac{3r+7}{2}, r) = \frac{1}{2}(r^2 - 5r - 2) \geq 0$ for $r \geq 9$.

(2) \exists unique component f'_0 corresponding to non-special curves.

$$\dim f'_0 = (r+1)d + (r-3)(1-g) = \frac{3}{2}r^2 + 4r + \frac{13}{2}.$$

(3) Let \mathcal{E} be a component of $f'_{3r-1, \frac{3r+7}{2}, r}$ whose general points correspond to trigonal curves with $|K - 2g_3'|$.

$$(4) \dim \mathcal{E} = (2g+1) + \dim G(r+1, g-4) + (r+1)^2 - 1 = \frac{3}{2}r^2 + 4r + \frac{13}{2} = \dim f'_0.$$

• Define 2.3.

$$A_0(g, r) := \frac{2r-4}{r+1} g + \frac{r+13}{r+1} \left(\simeq \left(2 - \frac{6}{r+1}\right) g \right)$$

$$A_1(g, r) := \frac{2r-6}{r+1} g + \frac{2r+26}{r+1} \left(\simeq \left(2 - \frac{8}{r+1}\right) g \right)$$

• Ein's counter example : $d = 16r - 35$, $g = 8r + 6$, r .

- $A_0(8r+6, r) = \frac{2r-4}{r+1}(8r+6) + \frac{r+13}{r+1} = \frac{16r^2 - 19r - 11}{r+1}$

- $0 < A_0(8r+6, r) - d = \frac{24}{r+1} < 1 \quad \text{if } r \geq 25$

• Keem's counter example : $d = 3r - 1$, $g = \frac{3r+7}{2}$, r

$$A_0\left(\frac{3r+7}{2}, r\right) = \frac{2r-4}{r+1} \cdot \frac{3r+7}{2} + \frac{r+13}{r+1} = 3r - 1 = d.$$

• Theorem (S. Kim, 2001)

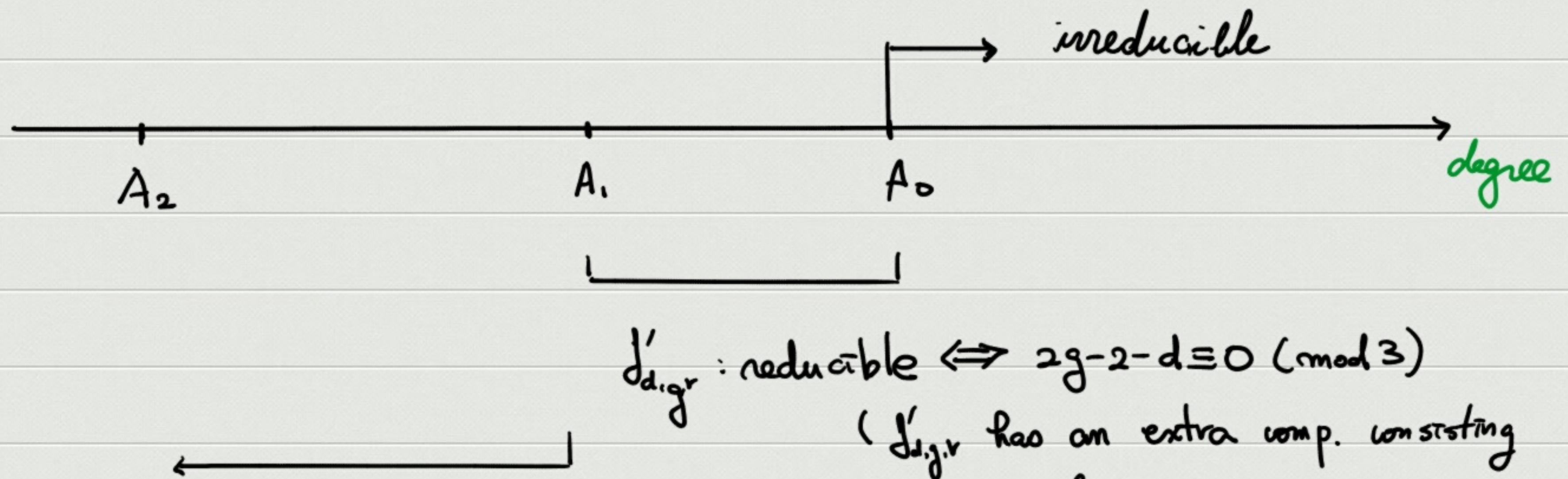
(1) If $r \geq 8$ and $d > A_0(g, r)$, then $f'_{d,g,r}$ is irreducible.

(2) If $r \geq 15$ and $A_1(g, r) < d \leq A_0(g, r)$, then $f'_{d,g,r}$ is reducible

if and only if $2g - 2 - d \equiv 0 \pmod{3}$.

Furthermore, if $f'_{d,g,r}$ is reducible, $f'_{d,g,r}$ has a component

whose general member corresponds to trigonal curves.



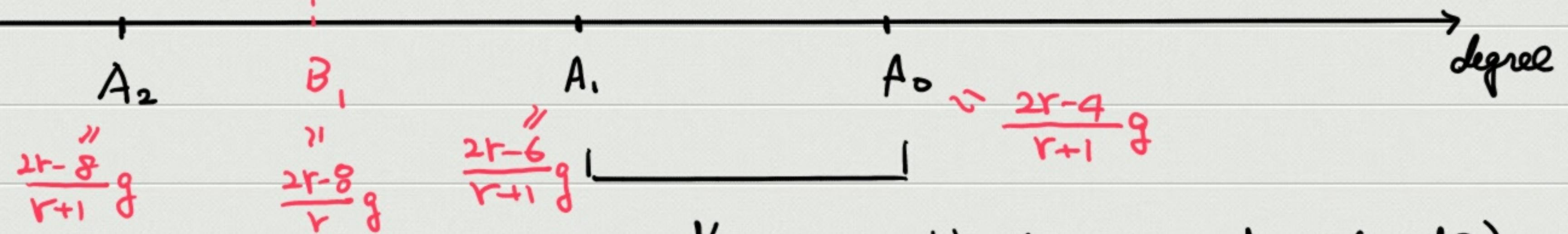
$J'_{d,g,r}$: reducible $\Leftrightarrow 2g-2-d \equiv 0 \pmod{3}$

($J'_{d,g,r}$ has an extra comp. consisting
of trigonal curves.)

$J'_{d,g,r}$ has an extra comp.
consisting 4-gonal curves

$$B_1 := \min \left\{ \left(2 - \frac{g}{r}\right)g + \left(2 + \frac{g}{r}\right), 2g - 27 \right\}$$

$J'_{dg,r}$ has an extra component
consisting of double covers of another curve



$$J'_{dg,r} : \text{reducible} \Leftrightarrow 2g - 2 - d \equiv 0 \pmod{3}$$

$J'_{dg,r}$ has an extra component
consisting 4-gonal curves.

($J'_{dg,r}$ has an extra comp. consisting
of trigonal curves.)

3. Families of double covers

Lemma 3.1.

- \mathcal{W} : irreducible closed subvariety of \mathcal{W}^ℓ , $\ell \geq 1$.
- If general element (C, \mathcal{L}) of \mathcal{W} defines a morphism $\psi_C : C \rightarrow \tilde{P} \subseteq \mathbb{P}^\ell$ of degree m and $g := \text{genus of } P$.

Then

$$\dim \mathcal{W} \leq \begin{cases} 2g - 3 - (2m-1)\nu + \frac{3e}{m} - 4\vartheta + 2m & \text{if } h^1(P, \mathcal{O}_P(1)) \geq 1 \\ 2g - 5 + 2m - 2(m-2)\nu & \text{if } h^1(P, \mathcal{O}_P(1)) = 0. \end{cases}$$

Consider

Proof) $T = \{(C, \mathcal{L}), (\tilde{P}, M) \mid (C, \mathcal{L}) \in \mathcal{W}, P = \psi_C(C), M = i^* \mathcal{O}_{\tilde{P}}(1),$

where $i : \tilde{P} \rightarrow P$ is a normalization.}

$$(C, \mathcal{L}) \in \mathcal{W} \xrightarrow{\quad P_1 \quad T} W_{\mathcal{C}_m, n}^P \xrightarrow{\quad P_2 \quad} (\tilde{P}, M)$$

• $\dim W \leq \underbrace{\dim P_1(T)}_{(\leftrightarrow)} + \underbrace{\dim (P_2^{-1}(\tilde{P}, M))}_{(\leftrightarrow)}$

(+) $\dim (P_2^{-1}(\tilde{P}, M)) \leq h^*(C, N_{\tilde{P}}) = 2g - 2 - 2m(d-1)$.

where $N_{\tilde{P}}$ is a normal sheaf of \tilde{P} given by

$$C \xrightarrow{\varphi} P := \varphi(C)$$

$$\tilde{\varphi}_c \nearrow \begin{matrix} \tilde{P} \\ \downarrow i \end{matrix}$$

(*) (a) If $h^1(\Gamma, \Theta_{\Gamma}(1)) = 0$, then $\dim \mathcal{W}_{e/m, r}^l = 3r - 3 + \gamma$.

(b) If $h^1(\Gamma, \Theta_{\Gamma}(1)) > 0$, then $\dim (p_*(T)) \leq 3 \cdot \frac{e}{m} + \gamma - 4l - 1$

by the following theorem.

• Theorem (Keem-Kim, 1992)

Let $\mathcal{G} \subseteq \mathcal{G}_d^r$ be an irreducible, closed subvariety of \mathcal{G}_d^r , $r \geq 2$.

General element $(C, g_d^r) \in \mathcal{G}$ is such that g_d^r is complete, special

and birationally very ample. Then $\dim \mathcal{G} \leq 3d + g - 4r - 1$.

Lemma 3.2.

Assume that f is an irreducible component of $f'_{d,g,r}$ with $r \geq 6$ such that

$h^1(c, \mathcal{O}_c(1)) = l+1 \geq 2$ for a general $c \in f$. Let

$b :=$ the degree of the base locus of $|K_c \otimes \mathcal{O}_c(-1)|$,

$m :=$ the degree of $\Xi : c \rightarrow \mathbb{P}^l$ defined by (the moving part of) $|K_c \otimes \mathcal{O}_c(-1)|$

$d :=$ the geometric genus of $P := \Xi(c)$.

If $m \geq 2$ and $h^1(P, \mathcal{O}_P(1)) > 0$, then

$$(I). \quad d \leq \left(2 - \frac{4m}{r+3}\right)g - \frac{2m(2m-1)}{r+3}\gamma + \left(\frac{(2m-2)r + 4m^2 + 2m - 6}{r+3}\right) - \left(1 - \frac{2m}{r+3}\right)b$$

$$(II). \quad \text{dim } f \leq (r+1)d - (r-3)(d-1) - \frac{r+3}{2m}d \\ + \left(\frac{r+3}{m} - 2\right)g - (2m-1)\gamma + \left(2m + 1 + r - \frac{r+3}{m}\right) - \left(\frac{r+3}{2m} - 1\right)b.$$

Proof)

By Lemma 1., we have

$$\dim f \leq \left[\frac{3(2g-2-d-b)}{m} - 4l + 2g-2 - (2m-1)(r-1) + b \right] \\ + (r+1)(d-r) + (r+1)^2 - 1.$$

For $m=2$, $b=0$ and $\gamma := \frac{2g-2-d}{4}$, we have

$$d \leq \left(\frac{2r-g}{r} g + \frac{r+1}{r} \right) \text{ and}$$

$$\dim f \leq \lambda_{d,g,r} + \frac{r}{4} \left(\frac{2r-g}{r} g + \frac{r+1}{r} - d \right), \text{ where } \lambda_{d,g,r} := (r+1)d - (r-3)(g-1).$$

($\lambda_{d,g,r}$ is a dimension of the distinguished component
of $f'_{d,g,r}$ and a minimal dimension.)

Lemma 3.3.

Let $\underline{\pi}: C \xrightarrow{2:1} P$ be a double cover of a smooth curve P of genus ≥ 2 with $g > 6\gamma - 2$.

Then we have

(a) $h^0(C, \underline{\pi}^* K_P) = h^0(P, K_P) = \gamma$ and

(b) $|K_C \otimes \underline{\pi}^* K_P|$ is a very ample $\mathcal{I}_{2g+2-4\gamma}^{g(g-3\gamma+2)}$ on C .

Proof) (b) Let $\mathcal{L} := K_C \otimes \underline{\pi}^* K_P$.

- $h^0(\mathcal{L}(-P)) = h^0(\mathcal{L}) - 1$ if and only if $h^0(\underline{\pi}^* K_P(P)) = h^0(\underline{\pi}^* K_P) = \gamma$.

- Suppose $\exists p \in C$ such that $h^0(\underline{\pi}^* K_P(p)) = \gamma + 1$.

- Consider $\psi_0: C \rightarrow C \subseteq \mathbb{P}^r$ where $D := |\underline{\pi}^* K_P(p)|$

- $\psi_0: C \rightarrow C' \subseteq \mathbb{P}^r$. If ψ_0 is birationally very ample, then by the

Castelnuovo genus bound, $g \leq p_a(C') \leq \binom{4}{2}(r-1) = 6r - 6$

- Assume that $\deg \psi_0 \geq 2$. On a smooth model of C' , $\hat{g}_{\frac{4r-3}{n}}$, $n := \deg \psi_0$.
- $\hat{g}_{\frac{4r-3}{n}}$ is non-special by Clifford theorem and $n=2$ or $n=3$.
- $g'(C) = \frac{4r-3}{n} - r$. $\Phi \times \psi_0: C \rightarrow \mathbb{P} \times C'$ birational.
- Contradiction to Castelnuovo - Severi inequality.

• Notation

Consider $M_g(r, m) \subseteq M_g$, $g \geq 3$, of points corresponding to curves admitting a rational map of degree $r \geq 2$ to a curves of genus γ ,

$$M_g(r, m) := \{c \in M_g \mid \exists \varphi: c \rightarrow c', c' \in M_\gamma, \deg \varphi = m\}.$$

Then $\dim M_g(r, m) = (2g-2) + (2m-3)(1-\gamma)$.

Let $\Sigma_g(r, m) \subseteq M_g(r, m)$ be an irreducible component of maximal dimension dominating M_γ .

Theorem 3.4. (S. Kim, H. Iliev, -)

Assume that $r \geq 21$, $2g-2-d \equiv 0 \pmod{4}$, $\rho(d, g, r) \geq 0$ and

$$\frac{2r-8}{r+1}g + \frac{3r+43}{r+1} < d \leq \min \left\{ \left(\frac{2r-8}{r} \right)g + \left(2 + \frac{8}{r} \right), 2g-27 \right\}. \quad \cdots (*)$$

If $\gamma := \frac{2g-2-d}{4}$ is odd, then $f_{d,g,r}$ possesses an irreducible component

$D_{d,g,r} \subseteq f'_{d,g,r}$ whose general member $C \rightarrow \mathbb{P}^r$ are

(i) $[C] \in \Sigma_g(\alpha, 2)$ is a double cover $\underline{\Phi}: C \rightarrow P$ of $P \in M_r$,

(ii) the embedding $C \hookrightarrow \mathbb{P}^r$ is given by $g_d^r \subset |K_C \otimes \underline{\Phi}^* K_P|$.

Furthermore, $\dim D_{d,g,r} = \lambda_{d,g,r} + \frac{r}{4} \left((2 - \frac{8}{r})g + (2 + \frac{8}{r}) - d \right)$.

Proof of theorem)

Let $\mathcal{D}_{d,g,r}$ be the closure of the irreducible family of curves $C \hookrightarrow \mathbb{P}^r$

over the irreducible subset $\Sigma_{g,r}^{(r,2)}$ and embedded by $j_d^r \subseteq |K_C \otimes (\mathbb{P}^r)^*|^r|$.

$$\begin{aligned}\dim \mathcal{D}_{d,g,r} &= \dim \Sigma_{g,r}^{(r,2)} + \dim G(r+1, g-3r+3) + \dim (\text{Aut}(\mathbb{P}^r)) \\ &= (r+3)g - (3r+4)r + 3r + 1.\end{aligned}$$

Using $d = 2g + 2 - 4r$, we have

$$\dim \mathcal{D}_{d,g,r} - \lambda_{d,g,r} = \frac{r}{4} \left(\frac{2r-8}{r} g + 2 + \frac{d}{r} - d \right).$$

Notation: $B_s(r) := \frac{2r-8}{r} g + 2 + \frac{d}{r}$.

$$B_s(r_1) \leq B_s(r_2) \iff r_2 \leq r_1.$$

Claim: $\mathcal{D}_{d,g,r}$ is an irreducible component of $\mathcal{J}_{d,g,r}$.

Assume that \exists an irreducible component $E \subset \mathcal{J}_{d,g,r}$ containing $\mathcal{D}_{d,g,r}$.

Let ϕ be a morphism defined by the moving part g_E^s of $|K_E \otimes \mathcal{O}_E(-1)|$ for a general element E of E .

One can show that ϕ is a $(m:1)$ -covering of a smooth curve T , $n \geq 2$.

By Lemma 3.2 & 3.3., if $h^1(T, \mathcal{O}_T(1)) > 0$, then $m=2$ and

if $h^1(T, \mathcal{O}_T(1)) = 0$, then $m=3$ or $m=4$.

• Suppose that $m=2$. Then for a general element $\mathcal{E} \in \mathbb{P}^r$ of \mathcal{E} ,

ϕ is a double cover of a smooth curve of genus τ .

$$\therefore \dim D_{d,g,r} < \dim \mathcal{E} \leq \dim D_{d,g,r} + \frac{r+3}{4}(B_2(\tau) - B_2(\sigma)).$$

$$\therefore \tau < \gamma.$$

• Consider $\mu: \mathcal{E} \longrightarrow \overline{M}_g$.

From a general element of \mathcal{E} project to a general element of $\mu(\mathcal{E})$.

we have $\underline{\text{gon}}(\mathcal{E}) \leq 2\lceil \frac{\tau+3}{2} \rceil < 2\lceil \frac{r+3}{2} \rceil = r+3 = \underline{\text{gon}}(C)$.

for a general $[C] \in \mu(D_{d,g,r}) = \mathcal{I}_g(r,2)$ since σ is odd and $\tau < \gamma$.

This contradicts to the semi-continuity of the gonality of a curve.

- Examples of schemes $\mathbb{F}_{d,g,r}^1$ wth a component of an extra dimension.

If we take $g=10\gamma+90$, $d=16\gamma+2$, and $r=21$, then the conditions in Theorem hold.

$$(*) \quad \frac{340}{22}\gamma + \frac{106}{22} < 16\gamma+2 \leq \frac{340}{21}\gamma + \frac{50}{21}$$

$\gamma \geq 7$: inequalities hold.

For the family of double covers $\underline{\Phi}: C \rightarrow \mathbb{P}$ of genus γ ,

$\mathcal{L} = K_C \otimes \underline{\Phi}^*(K_P)$ are very ample of degree $d=16\gamma+2$ and $h^0(\mathcal{L})=\gamma\gamma+3$.

Thus we obtain an irreducible component $\mathcal{D}_{16\gamma+2, 10\gamma+90, 21}$ of $f_{16\gamma+2, 10\gamma+90, 21}$

such that

$$\dim \mathcal{D}_{16\gamma+2, 10\gamma+90, 21} - \lambda_{16\gamma+2, 10\gamma+90, 21} = \gamma+2.$$

• Examples of schemes $J'_{d,g,r}$ with an extra component of expected dimension.

Let $r \geq 9$. If g is an integer such that $z(g-1)$ is divisible by r

and $\frac{2(g-1)}{r} \geq 21$. Then

$J'_{2g+2-4r, g, \frac{2(g-1)}{r}}$ contains an extra component $D_{2g+2-4r, g, \frac{2(g-1)}{r}}$

with an expected dimension.

Proof) • Check the numerical conditions.

$$\cdot \dim D_{2g+2-4r, g, \frac{2(g-1)}{r}} = \lambda_{2g+2-4r, g, \frac{2(g-1)}{r}}.$$

4. Brill-Noether divisors

Definition 4.1

$$M_{g,d}^r = \{ c \in M_g \mid c \text{ has a } g^r_d \} \subseteq M_g$$

- If $p(d, g, r) = -1$, then $\overline{M}_{g,d}^r$ is an irreducible divisor of \overline{M}_g .
- It is called a Brill-Noether divisor and it plays a crucial role in the birational geometry of M_g for $g \geq 23$.
- Eisenbud-Harris: $M_{23,12}' \neq M_{23,17}^2 \Rightarrow$ Kodaira dim(M_{23}) ≥ 1 .
- Tocino : $M_{23,12}', M_{23,17}^2$, and $M_{23,20}^3$: mutually disjoint \Rightarrow Kodaira dim(M_{23}) ≥ 2 .
- Ballico-Fontanari : $M_{g,d}^r \neq M_{g,e}^s$ for $r \leq 2$ & s in some range.

• Theorem (S. Kim, Y. Kim, -)

Let g, r, s, d, e be positive integers with $f(d, g, r) = f(s, g, e) = -1$.

Then we have $M_{g,d}^r \neq M_{g,e}^s$ unless $e = 2g - 2 - d$.

• Let $D_{d,g,r}$ be the union of irreducible component of $f_d(g,r)$ dominating $M_{g,d}^r$.

Corollary 4.3. Let d, g, r be integers with $d > g$, $r \geq 2$, $\rho(d, g, r) = -1$.

(1) $\mathcal{I}_{d,g,r}$ is non-empty and irreducible.

(2) A general curve $C \in \mathcal{I}_{d,g,r}$ has no $(d-e)$ -secant $(r-s-1)$ -plane

for $s < r$, $\rho(e, g, s) = -1$, $e \neq 2g-2-d$.

Moreover, if $r-s=d-e-1$, then any $(d-e)$ points of C are in general position.

Example 4.4.

Consider $g=47$. $48 = (r+1)(g-d+r) = 2 \cdot 24 = 3 \cdot 16 = 4 \cdot 12 = 6 \cdot 8$

∴ We have four Brill-Noether divisors : $M_{24}^1, M_{33}^2, M_{58}^3, M_{44}^5$
" " " "
 $M_{68}^{23}, M_{59}^{15}, M_{54}^{11}, M_{48}^7$

• We have four corresponding Hilbert scheme : $\mathcal{D}_{68,47,23}, \mathcal{D}_{59,47,15}, \mathcal{D}_{54,47,11}, \mathcal{D}_{48,47,7}$.

• Let C_1 (resp. C_2) be a general member of $\mathcal{D}_{68,47,23}$ (resp. $\mathcal{D}_{59,47,15}$).

There are no 9 ($= 68 - 59$) - points 7 ($= 23 - 15 - 1$) - secant plane on C_1 .

Hence any 9 points on C_1 are in general position.

5. Questions

- (1) Are the exceptional components of $f_{d,g,r}$ reduced?
- (2) Let $C \subset \mathbb{P}^r$ be a general curve of $\mathcal{D}_{d,g,r}$, where $p(d,g,r) = -1$.
Is the embedding $C \hookrightarrow \mathbb{P}^r$ projectively normal embedding?

• Green-Lazarsfeld Theorem

For any curve $c \in M_g$, a very ample line bundle with $\deg(L) < \deg(c)$ is normally generated.

• Wang Theorem

For a general curve $c \in M_g$, a ^{general} very ample line bundle with $\deg L > g-1$ & $\text{Cliff}(L) = \text{Cliff}(c)$ is normally generated.

Grazie!

Thank you!

감사합니다!.