# Geometry and topology of phase tropical hypersurfaces

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## Definition

1. Let $V \subset (\mathbb{C}^*)^n$ be a complex algebraic hypersurface in the complex torus $(\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{Z}_{\geq 2}$.

## Definition

1. Let $V \subset (\mathbb{C}^*)^n$ be a complex algebraic hypersurface in the complex torus $(\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{Z}_{\geq 2}$.
2. Let $V_f$ is the zero locus of a polynomial:

$$f(z) = \sum_{\alpha \in \mathrm{supp}(f)} a_\alpha z^\alpha, \quad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n} \tag{1}$$

where $a_\alpha \in \mathbb{C}^*$ and supp(f) is a finite subset of $\mathbb{Z}^n$, called the support of the polynomial f, with convex hull, in $\mathbb{R}^n$, the Newton polytope $\Delta_f$ of f.

## Definition

1. The amoeba $\mathscr{A}_f$ of an algebraic hypersurface $V_f \subset (\mathbb{C}^*)^n$ is by definition (M. Gelfand, M.M. Kapranov and A.V. Zelevinsky, 1994) the image of $V_f$ under the map :

$$\begin{array}{rccl} \mathrm{Log} : & (\mathbb{C}^*)^n & \longrightarrow & \mathbb{R}^n \\ & (z_1, \dots, z_n) & \longmapsto & (\log|z_1|, \dots, \log|z_n|). \end{array} \tag{2}$$

2. Amoeba of $V_f$ is defined by $\mathrm{Log}(V_f) =: \mathscr{A}_f$.

## Definition

1. Let $\mathbb{K}$ be the field of the Puiseux series with real power, which is the field of the series $a(t) = \sum_{j \in A_a} \xi_j t^j$ with $\xi_j \in \mathbb{C}^*$ and $A_a \subset \mathbb{R}$ is a well-ordered set (which means that any subset has a smallest element).

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2. It is well known that the field $\mathbb{K}$ is algebraically closed and of characteristic zero, and it has a non-Archimedean valuation $\mathrm{val}(a) = -\min A_a$:

$$\begin{cases} \mathrm{val}(ab) & = & \mathrm{val}(a) + \mathrm{val}(b) \\ \mathrm{val}(a+b) & \leq & \max\{\mathrm{val}(a), \mathrm{val}(b)\}, \end{cases} \tag{3}$$

and we put $\mathrm{val}(0) = -\infty$.

## Definition

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$$f_{trop} = \max_{\alpha \in supp(f)} \{val(a_\alpha) + < \alpha, x >\},$$

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   It is called as the tropical polynomial associated to f.
3. Legendre transform of the function

$$\nu : supp(f) \to \mathbb{R}$$

   is defined by $\nu(\alpha) = -\min A_{a_\alpha}$

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### Theorem (Kapranov, 2000)

1. The tropical hypersurface $\Gamma_f$ defined by the tropical polynomial $f_{trop}$ is the subset of $\mathbb{R}^n$ image under the valuation map of the algebraic hypersurface defined by f.
2. $\Gamma_f$ is also called the non-Archimedean amoeba of the zero locus of f in $(\mathbb{K}^*)^n$.

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$$\Gamma_f = \overline{\mathrm{Val}(V_f)} = \mathrm{Val}(V_f)(\because \mathrm{Val} \text{ is surjective})$$

$$\mathrm{Val} : (\mathbb{K}^*)^n \longrightarrow \mathbb{R}^n;\ (z_1, \cdots, z_n) \mapsto (-\mathrm{val}(z_1), \cdots, -\mathrm{val}(z_n))$$

1. For $h \in \mathbb{R}_{>0}$, define the self diffeomorphism $H_h$ of $(\mathbb{C}^*)^n$ by :

$$\begin{array}{rccl} H_h : & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^n \\ & (z_1, \dots, z_n) & \longmapsto & \left( \mid z_1 \mid^h \dfrac{z_1}{\mid z_1 \mid}, \dots, \mid z_n \mid^h \dfrac{z_n}{\mid z_n \mid} \right). \end{array}$$

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5. $\lim\limits_{h \to 0} J_h = J_\infty$ is not a complex structure, now we call this as phase tropical structure.

1. Hausdorff distance between two closed subsets $A, B$ of a metric space $(E, d)$ is defined by:

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}.$$

Take $E = \mathbb{R}^n \times (S^1)^n$ with the distance defined as product of the Euclidean metric on $\mathbb{R}^n$ & the flat metric on $(S^1)^n$.

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2. A phase tropical hypersurface $V_\infty \subset (\mathbb{C}^*)^n$ is the limit (with respect to the Hausdorff metric on compact sets in $(\mathbb{C}^*)^n$) of a sequence of a $J_h$-holomorphic hypersurfaces $V_h \subset (\mathbb{C}^*)^n$ when h tends to zero.

$$V_\infty = \lim_{h \to 0} V_h$$

1. Let $a = \sum_{j \in A_a} \xi_j t^j \in \mathbb{K}^*$: Puiseux series, $\xi_j \in \mathbb{C}^*$ and $A_a \subset \mathbb{R}$ is a well-ordered set with smallest element. Then we have a non-Archimedean valuation on $\mathbb{K}^*$ s.t. $\mathrm{val}(a) = -\min A_a$.
2. Complexify the valuation map as follows

$$\begin{array}{rcccl} w & : & \mathbb{K}^* & \longrightarrow & \mathbb{C}^* \\ & & a & \longmapsto & w(a) = e^{\mathrm{val}(a) + i\arg(\xi_{-\mathrm{val}(a)})} \end{array}$$

$$\begin{array}{rcccl} \mathrm{Arg} & : & \mathbb{K}^* & \longrightarrow & S^1 \\ & & a = \sum_{j \in A_a} \xi_j t^j & \longmapsto & e^{i\arg(\xi_{-\mathrm{val}(a)})} \end{array}$$

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\end{array}
$$

3. Applying this map coordinatewise, we get $W : (\mathbb{K}^*)^n \to (\mathbb{C}^*)^n$

$$
\begin{aligned}
(z_1, \cdots, z_n) \mapsto & \left( e^{\mathrm{val}(z_1) + i \arg(\xi_{-\mathrm{val}(z_1)})}, \cdots, e^{\mathrm{val}(z_n) + i \arg(\xi_{-\mathrm{val}(z_n)})} \right) \\
& = (w(z_1), \cdots, w(z_n))
\end{aligned}
$$

**Theorem (Mikhalkin, 2002)**

The set $V_\infty \subset (\mathbb{C}^*)^n$ is a phase tropical hypersurface
$\Leftrightarrow \exists$ an algebraic hypersurface $V_{\mathbb{K}} \subset (\mathbb{K}^*)^n$ over $\mathbb{K}$ such that

$$\overline{W(V_{\mathbb{K}})} = V_\infty,$$

where $\overline{W(V_{\mathbb{K}})}$ is the closure of $W(V_{\mathbb{K}})$ in $(\mathbb{C}^*)^n \approx \mathbb{R}^n \times (S^1)^n$ as a Riemannian manifold with metric defined by the standard Euclidean metric of $\mathbb{R}^n$ and the standard flat metric of the torus.

# Main theorem

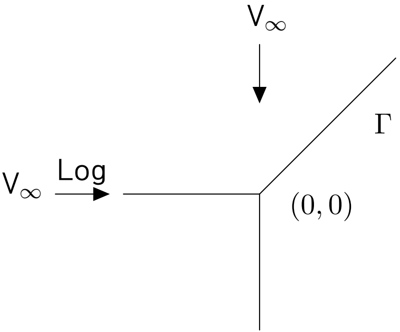
**Theorem (Y.R. Kim and M. Nisse, 2015)**

Let $V \subset (\mathbb{C}^*)^n$ be a smooth complex hypersurface and denote by $V_\infty$ the phase tropical hypersurface associated to V (i.e., the limit of $H_h(V_h)$ when h goes to zero). Then

1. For a sufficently small h we have $V_h$ is homeomorphic to $V_\infty$.
2. $V_\infty$ has a natural smooth symplectic structure.

## Example (Phase tropical line)

1. $L_{\mathbb{K}} := \{(z, w) \in (\mathbb{K}^*)^2 \mid z + w + 1 = 0\}$
2. $V_\infty = \overline{W(L_{\mathbb{K}})} \subset (\mathbb{C}^*)^2$
3. $\Gamma := \mathrm{Log}(V_\infty) := \max\{x, y, 0\}$



## Example (Phase tropical line – continued)

1. $(x, 0) \in \Gamma, \ x < 0$
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3. Let us parametrize the lines!
4. $$\begin{cases} z(t) = t^{-x} e^{i\alpha} & 0 < t < 1 \\ w(t) = -1 - t^{-x} e^{i\alpha} & \\ \alpha = \arg(z) : \text{fix}, & \beta_t = \arg(w(t)) \end{cases}$$

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5. $x < 0,\ \lim_{x\to 0} t^{-x} = 0 \ \Rightarrow\ \lim_{t\to 0} \beta_t = \pi.$
6. $\mathrm{Arg}(\mathrm{Log}^{-1}(x,0) \cap V_\infty) = \{(e^{i\alpha}, e^{i\pi}) \mid 0 \le \alpha \le 2\pi\} \simeq S^1$

## Example (Phase tropical line - continued)

1. $(0, \mathsf{y}) \in \Gamma,\ \mathsf{y} < 0$
2. $\mathsf{Arg}(\mathsf{Log}^{-1}(0, \mathsf{y}) \cap \mathsf{V}_\infty) \simeq \mathsf{S}^1$

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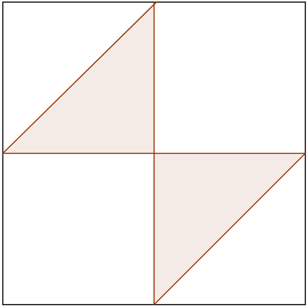
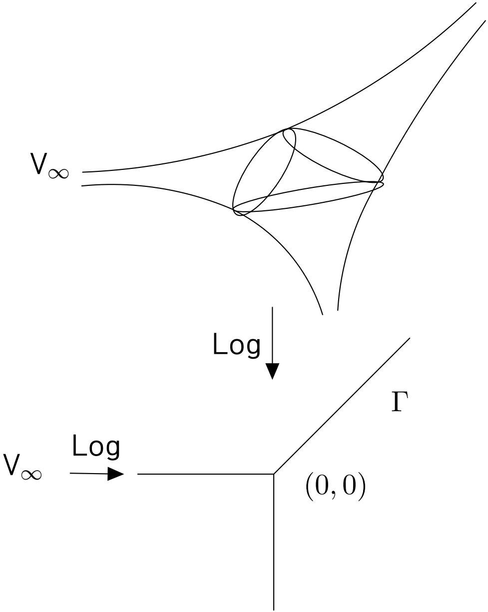


2. $\mathrm{L}_{\mathbb{K}} := \{(\mathrm{z}, \mathrm{w}) \in (\mathbb{K}^*)^2 \,|\, \mathrm{z} + \mathrm{w} + 1 = 0\} \simeq \mathrm{L}_\mathrm{h} \simeq \mathbb{CP}^1 \backslash \{3 \text{ points}\}$


V∞
Log
Γ
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## Definition (Pair-of-pants)

1. Let $\mathscr{H} \subset \mathbb{CP}^n$ be the union of $n + 2$ generic hyperplanes in $\mathbb{CP}^n$.
2. Let $\mathscr{U} \subset \mathbb{CP}^n$ be the union of their $\epsilon$-neighborhoods for every small $\epsilon > 0$.

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3. The complement $\bar{\mathscr{P}}_n = \mathbb{CP}^n \backslash \mathscr{U}$ is a manifold with corners. We call $\bar{\mathscr{P}}_n$ the n-dimensional pair-of-pants.
4. We call $\mathscr{P}_n = \mathbb{CP}^n \backslash \mathscr{H}$ the n-dimensional open pair-of-pants.

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## Example

1. $\mathscr{P}_1 = \mathbb{CP}^1 \backslash$3 generic hyperplanes in $\mathbb{CP}^1 \simeq \mathbb{CP}^1 \backslash \{3 \text{ points}\}$
2. $\bar{\mathscr{P}}_1 = \mathbb{CP}^1 \backslash \mathscr{U} \simeq$ a closed disk with 2 holes.

## Theorem (Description of the boundary of $\partial\bar{\mathscr{P}}$, Mikhalkin, 2002)

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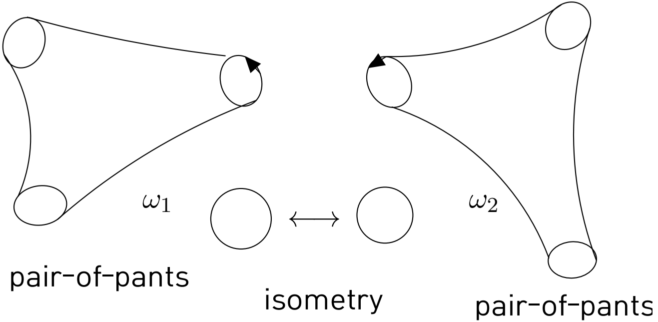
4. The closure of $\partial_j \bar{\mathscr{P}}_n$ contains $\partial_k \bar{\mathscr{P}}_n$ for all $k \leq j$.
5. The number of connected components of $\partial\bar{\mathscr{P}}_n$ is $\binom{n+2}{j+2}$.

1. $H := \{(z_1, \cdots, z_n) \in (\mathbb{K}^*)^n \mid z_1 + \cdots + z_n = 1\}$
2. $H_\infty$
3. $n = 2$ case. Think of the following picture!



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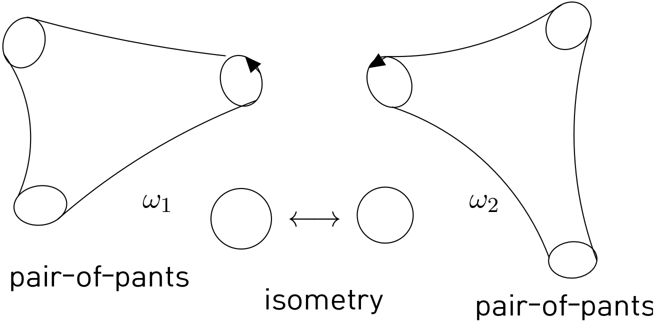
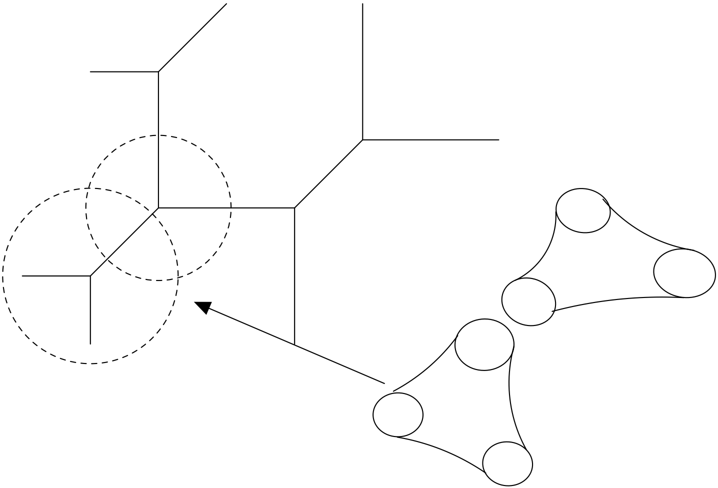



4. $\omega_1,\ \omega_2$ are symplectic forms on each pair-of-pants
5. Get a homeomorphism and a smooth symplectic structure. Topology are not changed.



1. $$f(z) = \sum_{\alpha \in \mathrm{supp}(f)} a_\alpha z^\alpha, \quad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$$
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3. $\tilde{\Delta} := \mathrm{convexhull}\{(\alpha, r) \in \mathrm{supp}(f) \times \mathbb{R} \mid r \geq \min A_{a_\alpha}\}$
4. Extend the above function $\nu$:

$$\begin{array}{rcccl} \nu & : & \Delta & \longrightarrow & \mathbb{R} \\ & & \alpha & \longmapsto & \min\{r \mid (\alpha, r) \in \tilde{\Delta}\}. \end{array}$$

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5. Linearity domains of $\nu$ define a convex subdivision $\tau = \{\Delta_1, \dots, \Delta_l\}$ of $\Delta$ by taking the linear subsets of the lower boundary of $\tilde{\Delta}$.
6. $y = < x, v_i > + r_i$: eq of the hyperplane $Q_i \subset \mathbb{R}^n \times \mathbb{R}$ containing the points of coordinates $(\alpha, \nu(\alpha))$ with $\alpha \in \mathrm{Vert}(\Delta_i)$.

1. There is a duality between the subdivision $\tau$ and the subdivision of $\mathbb{R}^n$ induced by $\Gamma_f$, where each connected component of $\mathbb{R}^n \setminus \Gamma_f$ is dual to some vertex of $\tau_f$ and each k-cell of $\Gamma_f$ is dual to some $(n-k)$-cell of $\tau$.

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2. Each $(n-1)$-cell of $\Gamma_f$ is dual to some edge of $\tau$.

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5. $\mathrm{supp}A = \{\alpha_1, \ldots, \alpha_l, \alpha_{l+1}, \ldots, \alpha_r\} \subset \mathbb{Z}^n$
   $A' = \{\alpha_{l+1}, \ldots, \alpha_r\} = \mathrm{Im}(\mathrm{ord})$
6. ord is the order mapping from the set of complement components of the amoeba $\mathscr{A}$ of V to $\Delta \cap \mathbb{Z}^n$.

## Theorem (Mikael Passare and Hans Rullgård)

The spine $\Gamma$ of the amoeba $\mathscr{A}$ is given as a non-Archimedean amoeba defined by the tropical polynomial

$$f_{trop}(x) = \max_{\alpha \in A'}\{c_\alpha + < \alpha, x >\},$$

$$c_\alpha = \mathbb{R}\left(\frac{1}{(2\pi i)^n}\int_{Log^{-1}(x)} \log\left|\frac{f(z)}{z^\alpha}\right| \frac{dz_1 \wedge \ldots \wedge dz_n}{z_1 \ldots z_n}\right) \tag{4}$$

$x \in E_\alpha,\ z = (z_1, \cdots, z_n) \in (\mathbb{C}^*)^n$, $< , >$: scalar product in $\mathbb{R}^n$.

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$x \in E_\alpha$, $z = (z_1, \cdots, z_n) \in (\mathbb{C}^*)^n$, $< , >$: scalar product in $\mathbb{R}^n$.

1. The spine of $\mathscr{A}$ is defined as the set of points in $\mathbb{R}^n$ where the piecewise affine linear function $f_{trop}$ is not differentiable, or as the graph of this function where $\mathbb{R}$ is the semi-field $(\mathbb{R}; \max, +)$.

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   (i) $\nu_{|\Delta_{\mathsf{v_i}}}$ is affine linear for each $\mathsf{v_i}$,
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   - (i) $\nu_{|\Delta_{v_i}}$ is affine linear for each $v_i$,
   - (ii) if $\nu_{|U}$ is affine linear for some open set $U \subset \Delta$, then there exists $v_i$ such that $U \subset \Delta_{v_i}$.
   - (iii) $\nu(\alpha) = -c_\alpha$ for any $\alpha \in \mathrm{Im}(\mathrm{ord})$.

## Definition (Generalized s-Passare-Rullgård function)

Let $s = (s_1, \ldots, s_l) \in \mathbb{R}^l_+$ and $\nu^s_{PR} : A \longrightarrow \mathbb{R}$ be the function, called the generalized s-Passare-Rullgård function, is defined by:

$$\nu^s_{PR}(\alpha) = \begin{cases} -c_\alpha & \text{if } \alpha \in \mathrm{Im(ord)} \\ < \alpha_j, a_v > + b_v + s_j & \text{if } \alpha = \alpha_j \text{ for } j = 1, \ldots, l, \end{cases}$$

where $\alpha_j \in \Delta_v$, $\Delta_v \in \tau$ and $y =< x, a_v > + b_v$ is the equation of the hyperplane in $\mathbb{R}^n \times \mathbb{R}$ containing the points of coordinates $(\beta, -c_\beta)$ with $\beta \in \mathrm{Vert}(\Delta_v)$.

1. Assume that we have a hypersurface $V \subset (\mathbb{C}^*)^n$ defined by the polynomial $f(z) = \sum_{\alpha \in A} a_\alpha z^\alpha$ with $a_\alpha \in \mathbb{C}^*$, A is a finite subset of $\mathbb{Z}^n$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n}$.
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2. Denote by $\Delta$ the convex hull of A in $\mathbb{R}^n$ which is the Newton polytope of f.
3. Consider now the family of hypersurfaces $V_{f_{(t;\, s)}} \subset (\mathbb{C}^*)^n$ defined by the following family of polynomials :

$$f_{(t;\, s)}(z) = \sum_{\alpha \in A} \xi_\alpha t^{\nu^s_{PR}(\alpha)} z^\alpha, \tag{5}$$

with $\xi_\alpha = a_\alpha e^{\nu^s_{PR}(\alpha)}$.

1. $\nu$ be the piecewise affine linear map.
2. $\tilde{\Delta}$ be the extended polyhedron of $\Delta$ associated to $\nu$, that is the convex hull of the set $\{(\alpha, \mathsf{u}) \in \Delta \times \mathbb{R} \,|\, \mathsf{u} \geq \nu(\alpha)\}$.

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3. For any $\Delta_{\mathsf{v}_\mathsf{i}} \in \tau$, let $\lambda(\mathsf{x}) =< \mathsf{x}, \mathsf{a}_{\mathsf{v}_\mathsf{i}} > + \mathsf{b}_{\mathsf{v}_\mathsf{i}}$ be the affine linear map defined on $\Delta$ such that $\lambda_{|\Delta_{\mathsf{v}_\mathsf{i}}} = \nu_{|\Delta_{\mathsf{v}_\mathsf{i}}}$ where $<,>$ is the scalar product in $\mathbb{R}^\mathsf{n}$, $\mathsf{a}_{\mathsf{v}_\mathsf{i}} = (\mathsf{a}_{\mathsf{v}_\mathsf{i},1}, \ldots, \mathsf{a}_{\mathsf{v}_\mathsf{i},\mathsf{n}}) \in \mathbb{R}^\mathsf{n}$ (which is the coordinates of the vertex of the spine $\Gamma$, dual to $\Delta_{\mathsf{v}_\mathsf{i}}$), and $\mathsf{b}_{\mathsf{v}_\mathsf{i}}$ is a real number.

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4. $\mathsf{s} \in \mathbb{R}^{\mathsf{l}}_{+}$ as above and put $\nu' = \nu^{(\mathsf{s})}_{\mathsf{PR}} - \lambda$ and we define the family of polynomials $\{\mathsf{f}'_{(\mathsf{t};\,\mathsf{s})}\}_{\mathsf{t} \in (0, \frac{1}{\mathsf{e}}]}$ by:

$$\mathsf{f}'_{\mathsf{t}}(\mathsf{z}) = \sum_{\alpha \in \mathsf{A}} \xi_\alpha \mathsf{t}^{\nu'(\alpha)} \mathsf{z}^\alpha$$

where $\xi_\alpha \in \mathbb{C}$.

1.

$$
\begin{aligned}
f'_t(z) &= t^{-b_v} \sum_{\alpha \in A} \xi_\alpha t^{\nu^{(s)}_{PR}(\alpha)} (z_1 t^{-a_{v_i,1}})^{\alpha_1} \dots (z_n t^{-a_{v_i,n}})^{\alpha_n} \\
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2. $f_{(t;s)}$ is the polynomial defined in (5)

3.

$$\begin{array}{rccl} \Phi_{\Delta_{v_i},t} \quad : & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^n : \text{diffeomorphism} \\ & (z_1, \dots, z_n) & \longmapsto & (z_1 t^{a_{v_i,1}}, \dots, z_n t^{a_{v_i,n}}). \end{array}$$

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4. This means that the polynomials $f'_{(t;\,s)}$ and $f_{(t;\,s)} \circ \Phi^{-1}_{\Delta_{v_i},t}$ define the same hypersurface.

5. $V_{f'_{(t;\,s)}} = V_{f_{(t;\,s)}\circ\Phi^{-1}_{\Delta_{v_i},t}} = \Phi_{\Delta_{v_i},t}(V_{f_{(t;\,s)}})$

1. Let $U(v_i)$ be a small ball in $\mathbb{R}^n$ with center the vertex of $\Gamma_{(t;\,s)}$ dual to $\Delta_{v_i}$ where $\Gamma_{(t;\,s)}$ is the spine of the amoeba $\mathscr{A}_{H_t(V_{f_{(t;\,s)}})}$ where $H_t$ denotes the self diffeomorphism of $(\mathbb{C}^*)^n$ defined by $H_h$ with $h = -\frac{1}{\log t}$ and $\mathrm{Log}_t = \mathrm{Log} \circ H_t$.

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2. Let $f_{(t;\,s)}^{\Delta_{v_i}}$ be the truncation of $f_{(t;\,s)}$ to $\Delta_{v_i}$, and $V_{\infty,\,\Delta_{v_i}}$ be the complex tropical hypersurface with tropical coefficients of index $\alpha \in \Delta_{v_i}$ (i.e., $V_{\infty,\,\Delta_{v_i}} = \lim_{t \to 0} H_t(V_{f_{(t;\,s)}^{\Delta_{v_i}}})$).

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3. Using Kapranov's theorem, we obtain the following Theorem called a tropical localization by Mikhalkin.

**Theorem (Tropical localization, Y.R. Kim and M. Nisse, 2015)**

Let $s \in \mathbb{R}^l_+$, then for any $\varepsilon > 0$ there exists $t_0$ such that if $t \leq t_0$ then the image under

$$\Phi_{\Delta_{v_i}, t} \circ H_t^{-1} \text{ of } H_t(V_{f_{(t;\, s)}}) \cap \mathrm{Log}^{-1}(U(v_i))$$

is contained in the $\varepsilon$-neighborhood of the image under

$$\Phi_{\Delta_{v_i}, t} \circ H_t^{-1}$$

of the complex tropical hypersurface

$$V_{\infty, \Delta_{v_i}},$$

with respect to the product metric in $(\mathbb{C}^*)^n \approx \mathbb{R}^n \times (S^1)^n$.

# Idea of proof:

1. By decomposition of $f'_t$, we obtain:

$$f'_t(z) = t^{-b_v} \sum_{\alpha \in \Delta_v \cap A} \xi_\alpha t^{\nu(\alpha) - <\alpha, a_v>} z^\alpha + \sum_{\alpha \in A \setminus \Delta_v} \xi_\alpha t^{\nu(\alpha) - <\alpha, a_v> - b_v} z^\alpha \tag{6}$$

2. 

$$\begin{array}{ccc} (\mathbb{C}^*)^n & \xrightarrow{\Phi_{\Delta_v, t}} & (\mathbb{C}^*)^n \\ \Big\downarrow \mathrm{Log}_t & & \Big\downarrow \mathrm{Log}_t \\ \mathbb{R}^n & \xrightarrow{\phi_{\Delta_v}} & \mathbb{R}^n \end{array} \tag{7}$$

3. More words.

**Theorem (Y.R. Kim and M. Nisse, 2015)**

A phase tropical hyperplane $\mathscr{H} \subset (\mathbb{C}^*)^n$ is homeomorphic to the complex projective space $\mathbb{CP}^{n-1}$ minus a tubular neighborhood of the union $\mathscr{H}$ of $n+1$ generic hyperplanes in $\mathbb{CP}^{n-1}$.

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**Theorem (Y.R. Kim and M. Nisse, 2015)**

Let $V \subset (\mathbb{C}^*)^n$ be a smooth complex hypersurface and denote by $V_\infty$ the phase tropical hypersurface associated to $V$ (i.e., the limit of $H_h(V_h)$ when h goes to zero). Then

1. For a sufficently small h we have $V_h$ is homeomorphic to $V_\infty$.
2. $V_\infty$ has a natural smooth symplectic structure.

# Grazie!

Grazie! 감사합니다!

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