A characterization of complete flag manifolds

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Outline

Introduction

Fano bundles

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Later the assumption on b_4 was removed by Watanabe.

Varieties with two $\mathbb{P}^1$-bundle structures

Theorem (version with Bundles)

A Fano manifold with Picard number 2 and two $\mathbb{P}^1$-bundle structures is isomorphic to one of the following

- $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O})$
- $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$
- $\mathbb{P}_{\mathbb{P}^3}(\mathcal{N}) = \mathbb{P}_{\mathbb{Q}^3}(\mathcal{S})$ - $\mathcal{N}$ Null-correlation , $\mathcal{S}$ Spinor
- $\mathbb{P}_{\mathbb{Q}^5}(\mathcal{C}) = \mathbb{P}_{K(G_2)}(\mathcal{Q})$ - $\mathcal{C}$ Cayley, $\mathcal{Q}$ universal quotient.

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This result can be reformulated as follows:

Theorem (version with Flags)

A Fano manifold with Picard number 2 and two $\mathbb{P}^1$-bundle structures is rational homogeneous and it is isomorphic to a complete flag manifold of type $A_1 \times A_1$, A_2, B_2 or G_2.

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Theorem

X is a Fano manifold whose elementary contractions are smooth $\mathbb{P}^1$-fibrations (Flag Type manifold) if and only if X is a complete flag manifold.

Rational Homogeneous Manifolds

Lie Algebras
Root systems

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A set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Phi$ is a basis of $\mathfrak{h}^\vee$ such that the coordinates of root are integers, all ≥ 0 or all ≤ 0 .

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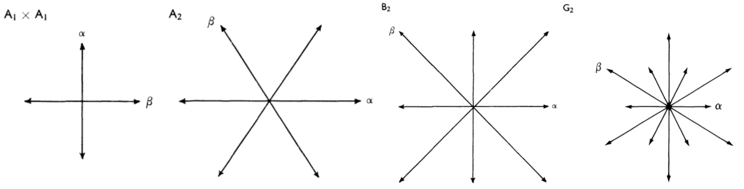
Example (n=2)



Cartan matrix

Given a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ of $\mathfrak{g}$, the Cartan matrix A of $\mathfrak{g}$ is the $n \times n$ matrix whose entries are the Cartan integers

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A and all its principal minors are positive definite and moreover

- $a_{ii} = 2$ for every i,
- $a_{ij} = 0$ iff $a_{ji} = 0$,
- if $a_{ij} \neq 0$, $i \neq j$, then $a_{ij}, a_{ji} \in \mathbb{Z}^-$ and $a_{ij}a_{ji} = 1, 2$ or 3.

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The Cartan matrices of rank 2 Lie algebras are

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

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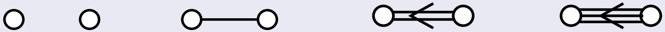

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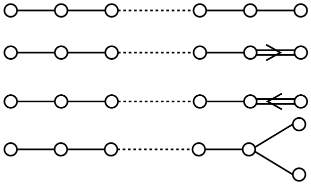

A_n	SL_{n+1}	
B_n	SO_{2n+1}	CLASSICAL
C_n	Sp_{2n}	
D_n	SO_{2n}	

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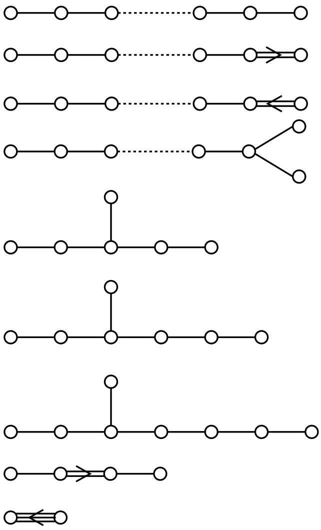

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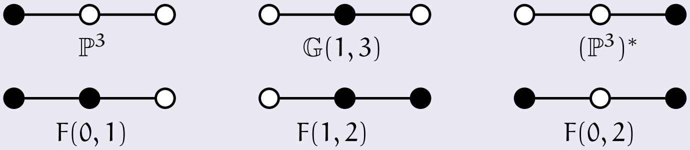

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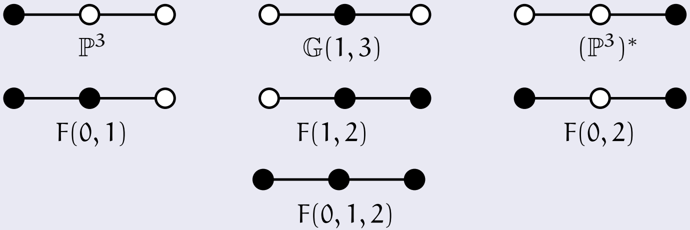

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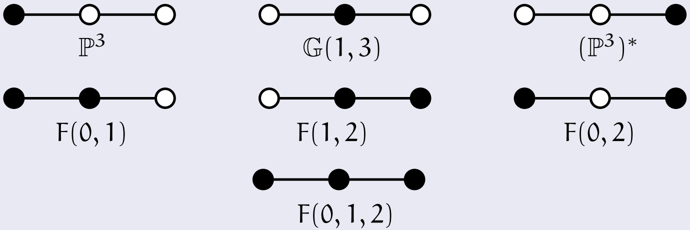

So a rational homogeneous (RH) manifold is given by a marked Dynkin diagram $(\mathcal{D}, \mathcal{I})$.

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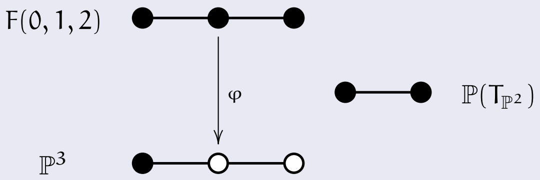
Example



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Example (A_n)

If $\mathcal{D} = A_n$, then G/B is the manifold parametrizing complete flags of linear subspaces in $\mathbb{P}^n$.

Fano manifolds whose elementary contractions are smooth $\mathbb{P}^1$-fibrations

Relative duality

$\pi : M \to Y$ smooth $\mathbb{P}^1$-fibration. Γ fiber, K relative canonical

Lemma

Let D *be a divisor on* M *and set* $l := D \cdot \Gamma + 1$. *Then,* $\forall i \in \mathbb{Z}$

$$\begin{array}{lcl} H^i(M, D) \cong & H^{i-1}(M, D + lK) & \text{if } l < 0 \\ H^i(M, D) \cong & \{0\} & \text{if } l = 0 \\ H^i(M, D) \cong & H^{i+1}(M, D + lK) & \text{if } l > 0 \end{array}$$

In particular $\chi(M, D) = -\chi(M, D + lK)$ *for any* D.

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Given $L_1, \dots, L_n$ basis of $\mathrm{Pic}(X)$,

$$\chi_X(m_1, \dots, m_n) = \chi(X, m_1 L_1 + \cdots + m_n L_n)$$

is a numerical polynomial of degree $\dim X$, so we can extend it to $\chi_X : N_1(X) \to \mathbb{R}$.

By the Lemma the affine involutions $r_i' : N^1(X) \to N^1(X)$

$$r_i'(D) := D + (D \cdot \Gamma_i + 1)K_i,$$

satisfy

$$\chi_X(D) = -\chi_X(r_i'(D)).$$

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which fixes pointwise the hyperplane $M_i := \{D \mid D \cdot \Gamma_i = 0\}$ and satisfies

$$r_i(K_i) = -K_i \qquad \chi^T(D) = -\chi^T(r_i(D));$$

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Since $K_X \cdot \Gamma_i = -2$ for every i, setting

$$\begin{array}{rcl} T(D) & := & D + K_X/2 \\ r_i & := & T^{-1} \circ r'_i \circ T \\ \chi^T & := & \chi_X \circ T \end{array}$$

we have that the map r_i is a linear involution of $N^1(X)$ given by

$$r_i(D) = D + (D \cdot \Gamma_i)K_i,$$

which fixes pointwise the hyperplane $M_i := \{D \mid D \cdot \Gamma_i = 0\}$ and satisfies

$$r_i(K_i) = -K_i \qquad \chi^T(D) = -\chi^T(r_i(D));$$

in particular χ^T vanishes on M_i for every i.

Weyl group

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Weyl group

Let $W \subset \mathrm{Gl}(N^1(X))$ be the group generated by the r_i's.

Theorem

The group W is finite and the set

$$\Phi := \{w(-K_i) \mid w \in W, \ \ i = 1, \dots, n\} \subset N^1(X),$$

is a root system, whose Weyl group is W

Idea of proof

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By the finiteness there is a scalar product $(\ , \ )$ on $N^1(X)$, which is W-invariant. In particular the r_i's are euclidean reflections.

Using that $r_i(K_i) = -K_i$ is then straightforward (but tedious) to prove that Φ is a root system with Weyl group W.

Homogeneous model

Since $(\ ,\ )$ is W-invariant, $(K_j, K_i) = (r_i(K_j), -K_i)$ which gives

$$-K_j \cdot \Gamma_i = 2\frac{(K_j, K_i)}{(K_i, K_i)} = \langle K_j, K_i \rangle,$$

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Proposition

$\Lambda \subset \mathrm{Pic}(X)$ generated by the K_i's.

- $h^i(X, D) = h^i(G/B, \psi(D))$ for every $D \in \Lambda$, $i \in \mathbb{Z}$.
- $\dim X = \dim G/B$;

Idea of Proof

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- X Flag Type manifold of Picard number n, $x \in X$ point;
- $\ell = (l_1, \dots, l_t)$, list of indices in $\{1, \dots, n\}$,
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In W there exists a unique longest element w_0, of length $\dim X$.

If ℓ_0 is a reduced list such that $w(\ell_0) = w_0$ then $f_\ell : Z_{\ell_0} \to X$ is surjective and birational.

X Flag Type manifold, G/B homogeneus model of X

Find a list ℓ_0 such that $w(\ell_0) = w_0$ and prove that

$$\mathsf{Z}_{\ell_0} \simeq \overline{\mathsf{Z}}_{\ell_0} \qquad \mathsf{f}_{\ell_0} = \overline{\mathsf{f}}_{\ell_0}$$

The idea is to show inductively that Z_{ℓ_0} depends only on the list and on the intersection matrix.

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$f_{\ell[1]}$ factors via Z_ℓ, giving a section $\sigma_{\ell[1]}$, hence an extension

$$0 \to \mathcal{O}_{Z_{\ell[1]}}(f^*_{\ell[1]}K_{l_r}) \longrightarrow \mathcal{F}_\ell \longrightarrow \mathcal{O}_{Z_{\ell[1]}} \to 0.$$

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This can be done except for G_2, (already known from the $n = 2$ case) and F_4, for which an ad hoc argument is needed.

Campana-Peternell Conjecture

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Theorem [Demailly, Peternell and Schneider (1994)]

$$T_X \text{ nef} \Rightarrow \begin{cases} X \xleftarrow{\text{étale}} X' \xrightarrow{F} A \\ A \text{ Abelian, } F \text{ Fano, } T_F \text{ nef} \end{cases}$$

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- homogeneity is checked *a posteriori*.

An application

Homogeneity via families of rational curves

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Assume that $\mathcal{M}$ is unsplit, q is smooth and that $\mathcal{M}_x := q^{-1}(x)$ is RH for every $x \in X$. Then X is RH.

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Assume $\mathcal{L}_0$ is RH. If $\mathcal{M}_x \simeq \mathcal{L}_0$ for every $x \in X$ then $X \simeq S$.

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The following are equivalent:

- $\mathcal{L}_0$ is G-homogeneous.
- P is associated to a long simple root.
- There is no arrow in the Dynkin diagram pointing towards the node corresponding to P.

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Theorem [Mok, Hong-Hwang]

If P is associated to a long simple root and $\mathcal{C}(X)_x$ is projectively equivalent to $\mathcal{C}(S)_0$, then $X \simeq S$.

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Given the smooth fibration $q : \mathcal{U} \to X$, with RH fiber F, it is possible to construct the associated flag bundle over X, whose fibers over a point are complete flag manifolds.

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Given a Borel subgroup $B \subset G$ we can define the G/B-bundle

$$\overline{\mathcal{U}} := \mathcal{U}_G \times^G G/B \to X$$

as a quotient of $\mathcal{U}_G \times G/B$ by $(x, gB) \sim (xg', g'^{-1}gB)$,

Idea of the proof

Given the smooth fibration $q : \mathcal{U} \to X$, with RH fiber F, it is possible to construct the associated flag bundle over X, whose fibers over a point are complete flag manifolds.

The fibration q is defined by a cocycle $\vartheta \in H^1(X, G)$, where G is the identity component of $\mathrm{Aut}(F)$ - here we use that X is simply connected.

The cocycle ϑ defines a principal G-bundle $\mathcal{U}_G \to X$

Given a Borel subgroup $B \subset G$ we can define the G/B-bundle

$$\overline{\mathcal{U}} := \mathcal{U}_G \times^G G/B \to X$$

as a quotient of $\mathcal{U}_G \times G/B$ by $(x, gB) \sim (xg', g'^{-1}gB)$, and we have a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{U}} \arrow[rr, "\pi"] \arrow[dr, "\overline{q}"'] & & \mathcal{U} \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

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Then X is isomorphic to a complete flag manifold.

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Remark

A similar argument has been used to conclude the proof of CP conjecture in dimension 5 by Kanemitsu (2015).

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Given a CP-manifold X, we define:

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