A characterization of complete flag manifolds

Gianluca Occhetta

with R. Muñoz, L.E. Solá Conde, K. Watanabe and J. Wiśniewski

Cortona, June 2015
1 Introduction

2 Rational Homogeneous Manifolds

3 Main result

4 Campana-Peternell Conjecture

5 An application
Flag Manifolds

Gianluca Occhetta

Introduction

Fano bundles
The problem

RH manifolds
Lie algebras
Cartan matrix
Dynkin diagrams
RH manifolds
Cone and contractions
Flag manifolds

Main result
Fibrations and reflections
Homogeneous model
Bott-Samelson varieties

CP Conjecture
Positivity of the tangent bundle
Results

An application
Homogeneity and rational curves
Comments and related results
Idea of proof
Speculations
A vector bundle $\mathcal{E}$ on a smooth complex projective variety $X$ is a Fano bundle iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.
A vector bundle $\mathcal{E}$ on a smooth complex projective variety $X$ is a Fano bundle iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on $X$ then $X$ is a Fano manifold.
A vector bundle $\mathcal{E}$ on a smooth complex projective variety $X$ is a **Fano bundle** if $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on $X$ then $X$ is a Fano manifold.

Fano bundles of rank 2 on $\mathbb{P}^m$ and $\mathbb{Q}^m$ have been classified in the ’90s (Ancona, Peternell, Sols, Szurek, Wiśniewski).
A vector bundle $\mathcal{E}$ on a smooth complex projective variety $X$ is a **Fano bundle** iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on $X$ then $X$ is a Fano manifold.

Fano bundles of rank 2 on $\mathbb{P}^m$ and $\mathbb{Q}^m$ have been classified in the ’90s (Ancona, Peternell, Sols, Szurek, Wiśniewski).

Generalization: Classification of Fano bundles of rank 2 on (Fano) manifolds with $b_2 = b_4 = 1$ (Muñoz, _____, Solá Conde, 2012).
A vector bundle $\mathcal{E}$ on a smooth complex projective variety $X$ is a Fano bundle iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on $X$ then $X$ is a Fano manifold.

Fano bundles of rank 2 on $\mathbb{P}^m$ and $\mathbb{Q}^m$ have been classified in the ’90s (Ancona, Peternell, Sols, Szurek, Wiśniewski).

Generalization: Classification of Fano bundles of rank 2 on (Fano) manifolds with $b_2 = b_4 = 1$ (Muñoz, Solá Conde, 2012).

As a special case we have the classification of Fano manifolds of Picard number two (and $b_4 = 2$) with two $\mathbb{P}^1$-bundle structures.
A vector bundle $\mathcal{E}$ on a smooth complex projective variety $X$ is a **Fano bundle** iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on $X$ then $X$ is a Fano manifold.

Fano bundles of rank 2 on $\mathbb{P}^m$ and $\mathbb{Q}^m$ have been classified in the ’90s (Ancona, Peternell, Sols, Szurek, Wiśniewski).

Generalization: Classification of Fano bundles of rank 2 on (Fano) manifolds with $b_2 = b_4 = 1$ (Muñoz, Solá Conde, 2012).

As a special case we have the classification of Fano manifolds of Picard number two (and $b_4 = 2$) with two $\mathbb{P}^1$-bundle structures.

Later the assumption on $b_4$ was removed by Watanabe.
Varieties with two $\mathbb{P}^1$-bundle structures

Theorem (version with Bundles)

A Fano manifold with Picard number 2 and two $\mathbb{P}^1$-bundle structures is isomorphic to one of the following

- $\mathbb{P}^1_1(\mathcal{O} \oplus \mathcal{O})$
- $\mathbb{P}^2_2(\mathcal{T}_{\mathbb{P}^2})$
- $\mathbb{P}^3_3(\mathcal{N}) = \mathbb{P}^5_5(\mathcal{S}) - \mathcal{N}$ Null-correlation, $\mathcal{S}$ Spinor
- $\mathbb{P}^5_5(\mathcal{C}) = \mathbb{P}^5_5(\mathcal{Q}) - \mathcal{C}$ Cayley, $\mathcal{Q}$ universal quotient.
Varieties with two $\mathbb{P}^1$-bundle structures

Theorem (version with Bundles)

A Fano manifold with Picard number 2 and two $\mathbb{P}^1$-bundle structures is isomorphic to one of the following

- $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O})$
- $\mathbb{P}_{\mathbb{P}^2}(\mathbb{T}_{\mathbb{P}^2})$
- $\mathbb{P}_{\mathbb{P}^3}(\mathcal{N}) = \mathbb{P}_{\mathbb{Q}^3}(\mathcal{S}) - \mathcal{N}$ Null-correlation, $\mathcal{S}$ Spinor
- $\mathbb{P}_{\mathbb{Q}^5}(\mathcal{C}) = \mathbb{P}_{K(G_2)}(\mathcal{Q}) - \mathcal{C}$ Cayley, $\mathcal{Q}$ universal quotient.

This result can be reformulated as follows:

Theorem (version with Flags)

A Fano manifold with Picard number 2 and two $\mathbb{P}^1$-bundle structures is rational homogeneous and it is isomorphic to a complete flag manifold of type $A_1 \times A_1$, $A_2$, $B_2$ or $G_2$. 
- Classify Fano manifolds whose elementary contractions are $\mathbb{P}^1$-bundles - or just smooth $\mathbb{P}^1$-fibrations.
A generalization

- Classify Fano manifolds whose elementary contractions are $\mathbb{P}^1$-bundles - or just smooth $\mathbb{P}^1$-fibrations.

- The vector bundle approach seems difficult to apply to this more general situation.
- Classify Fano manifolds whose elementary contractions are \( \mathbb{P}^1 \)-bundles - or just smooth \( \mathbb{P}^1 \)-fibrations.

- The vector bundle approach seems difficult to apply to this more general situation.

- Is it possible to prove the homogeneity directly, or at least recover features of the complete flags using the \( \mathbb{P}^1 \)-fibrations?
Classify Fano manifolds whose elementary contractions are $\mathbb{P}^1$-bundles - or just smooth $\mathbb{P}^1$-fibrations.

The vector bundle approach seems difficult to apply to this more general situation.

Is it possible to prove the homogeneity directly, or at least recover features of the complete flags using the $\mathbb{P}^1$-fibrations?

**Theorem**

$X$ is a Fano manifold whose elementary contractions are smooth $\mathbb{P}^1$-fibrations (Flag Type manifold) if and only if $X$ is a complete flag manifold.
Rational Homogeneous Manifolds
$G$ semisimple Lie group, $\mathfrak{g}$ Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra.
G semisimple Lie group, \( \mathfrak{g} \) Lie algebra, \( \mathfrak{h} \subset \mathfrak{g} \) Cartan subalgebra. The action of \( \mathfrak{h} \) on \( \mathfrak{g} \) defines an eigenspace decomposition, called Cartan decomposition of \( \mathfrak{g} \):

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha.
\]
G semisimple Lie group, $\mathfrak{g}$ Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra. The action of $\mathfrak{h}$ on $\mathfrak{g}$ defines an eigenspace decomposition, called Cartan decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^\vee \setminus \{0\}} \mathfrak{g}_\alpha.$$

The spaces $\mathfrak{g}_\alpha$ are defined by

$$\mathfrak{g}_\alpha = \{ g \in \mathfrak{g} \mid [h, g] = \alpha(h)g, \text{ for every } h \in \mathfrak{h} \};$$

$\alpha \neq 0$ such that $\mathfrak{g}_\alpha \neq 0$ is called a root of $\mathfrak{g}$. 
The action of $\mathfrak{h}$ on $\mathfrak{g}$ defines an eigenspace decomposition, called the Cartan decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_{\alpha}.$$

The spaces $\mathfrak{g}_{\alpha}$ are defined by

$$\mathfrak{g}_{\alpha} = \{ g \in \mathfrak{g} \mid [h, g] = \alpha(h) g, \text{ for every } h \in \mathfrak{h} \};$$

$\alpha \neq 0$ such that $\mathfrak{g}_{\alpha} \neq 0$ is called a root of $\mathfrak{g}$.

The (finite) set $\Phi$ of such elements is called the root system of $\mathfrak{g}$. 

G semisimple Lie group, $\mathfrak{g}$ Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra.
G semisimple Lie group, \( \mathfrak{g} \) Lie algebra, \( \mathfrak{h} \subset \mathfrak{g} \) Cartan subalgebra. The action of \( \mathfrak{h} \) on \( \mathfrak{g} \) defines an eigenspace decomposition, called Cartan decomposition of \( \mathfrak{g} \):

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^\vee \setminus \{0\}} \mathfrak{g}_\alpha.
\]

The spaces \( \mathfrak{g}_\alpha \) are defined by

\[
\mathfrak{g}_\alpha = \{ g \in \mathfrak{g} \mid [h, g] = \alpha(h)g, \text{ for every } h \in \mathfrak{h} \};
\]

\( \alpha \neq 0 \) such that \( \mathfrak{g}_\alpha \neq 0 \) is called a root of \( \mathfrak{g} \). The (finite) set \( \Phi \) of such elements is called root system of \( \mathfrak{g} \). A set of simple roots \( \Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi \) is a basis of \( \mathfrak{h}^\vee \) such that the coordinates of root are integers, all \( \geq 0 \) or all \( \leq 0 \).
\((E, \kappa)\) real vector space generated by the roots, with a symmetric bilinear positive form \(\kappa\) induced by the Killing form of \(\mathfrak{g}\).
Lie Algebras

Weyl group

$(E, \kappa)$ real vector space generated by the roots, with a symmetric bilinear positive form $\kappa$ induced by the Killing form of $\mathfrak{g}$.

The reflections with respect to the roots:

$$\sigma_\alpha(x) = x - \langle x, \alpha \rangle \alpha,$$

where $\langle x, \alpha \rangle := 2 \frac{\kappa(x, \alpha)}{\kappa(\alpha, \alpha)}$,

fix the root system and generate a finite group $W \subset \text{Gl}(E)$, called the \textbf{Weyl group} of $\mathfrak{g}$. 
(E, κ) real vector space generated by the roots, with a symmetric bilinear positive form κ induced by the Killing form of g.

The reflections with respect to the roots:

\[ \sigma_\alpha(x) = x - \langle x, \alpha \rangle \alpha, \quad \text{where} \quad \langle x, \alpha \rangle := 2\frac{\kappa(x, \alpha)}{\kappa(\alpha, \alpha)}, \]

fix the root system and generate a finite group \( W \subset \text{Gl}(E) \), called the Weyl group of g.

Example (n=2)
Given a set of simple roots \( \{\alpha_1, \ldots, \alpha_n\} \) of \( g \), the Cartan matrix \( A \) of \( g \) is the \( n \times n \) matrix whose entries are the Cartan integers

\[
\langle \alpha_i, \alpha_j \rangle = 2 \frac{\kappa(\alpha_i, \alpha_j)}{\kappa(\alpha_j, \alpha_j)}.
\]
Given a set of simple roots \( \{\alpha_1, \ldots, \alpha_n\} \) of \( g \), the Cartan matrix \( A \) of \( g \) is the \( n \times n \) matrix whose entries are the Cartan integers

\[
\langle \alpha_i, \alpha_j \rangle = 2 \frac{\kappa(\alpha_i, \alpha_j)}{\kappa(\alpha_j, \alpha_j)}.
\]

\( A \) and all its principal minors are positive definite and moreover

- \( a_{ii} = 2 \) for every \( i \),
- \( a_{ij} = 0 \) iff \( a_{ji} = 0 \),
- if \( a_{ij} \neq 0, i \neq j \), then \( a_{ij}, a_{ji} \in \mathbb{Z}^- \) and \( a_{ij}a_{ji} = 1, 2 \) or 3.
Given a set of simple roots \( \{\alpha_1, \ldots, \alpha_n\} \) of \( \mathfrak{g} \), the Cartan matrix \( A \) of \( \mathfrak{g} \) is the \( n \times n \) matrix whose entries are the Cartan integers

\[
\langle \alpha_i, \alpha_j \rangle = 2 \frac{\kappa(\alpha_i, \alpha_j)}{\kappa(\alpha_j, \alpha_j)}.
\]

\( A \) and all its principal minors are positive definite and moreover

- \( a_{ii} = 2 \) for every \( i \),
- \( a_{ij} = 0 \) iff \( a_{ji} = 0 \),
- if \( a_{ij} \neq 0, i \neq j \), then \( a_{ij}, a_{ji} \in \mathbb{Z}^- \) and \( a_{ij}a_{ji} = 1, 2 \) or \( 3 \).

**Example (n=2)**

The Cartan matrices of rank 2 Lie algebras are

\[
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix},
\begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix},
\begin{pmatrix}
2 & -1 \\
-2 & 2
\end{pmatrix},
\begin{pmatrix}
2 & -1 \\
-3 & 2
\end{pmatrix}
\]
With the matrix $A$ is associated a finite Dynkin diagram $D$, in the following way
With the matrix $A$ is associated a finite Dynkin diagram $D$, in the following way

- $D$ is a graph with $n$ nodes,
With the matrix $A$ is associated a finite Dynkin diagram $D$, in the following way

- $D$ is a graph with $n$ nodes,
- the nodes $i$ and $j$ are joined by $a_{ij}a_{ji}$ edges,
Dynkin diagrams

With the matrix $A$ is associated a finite Dynkin diagram $\mathcal{D}$, in the following way

- $\mathcal{D}$ is a graph with $n$ nodes,
- the nodes $i$ and $j$ are joined by $a_{ij}a_{ji}$ edges,
- if $|a_{ij}| > |a_{ji}|$ the edges are directed towards the node $i$. 

Example ($n=2$)

The Dynkin diagrams of rank 2 Lie algebras are

\[ \begin{align*}
\begin{array}{c}
20 \\
02 \\
\hline
\end{array}
\end{align*} \]
Dynkin diagrams

With the matrix $A$ is associated a finite Dynkin diagram $\mathcal{D}$, in the following way

- $\mathcal{D}$ is a graph with $n$ nodes,
- the nodes $i$ and $j$ are joined by $a_{ij}a_{ji}$ edges,
- if $|a_{ij}| > |a_{ji}|$ the edges are directed towards the node $i$.

Example (n=2)

The Dynkin diagrams of rank 2 Lie algebras are

\[
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}
\begin{pmatrix}
2 & -1 \\
-2 & 2
\end{pmatrix}
\begin{pmatrix}
2 & -1 \\
-3 & 2
\end{pmatrix}
\]

\[
\begin{align*}
&\quad \quad \quad \\
&\quad \quad \quad \\
&\quad \quad \quad \\
&\quad \quad \quad
\end{align*}
\]

\[
\begin{align*}
&\quad \quad \quad \\
&\quad \quad \quad \\
&\quad \quad \quad \\
&\quad \quad \quad
\end{align*}
\]
Dynkin diagrams

\begin{align*}
\text{CLASSICAL} \\
A_n & : \text{SL}_{n+1} \\
B_n & : \text{SO}_{2n+1} \\
C_n & : \text{Sp}_{2n} \\
D_n & : \text{SO}_{2n}
\end{align*}
Dynkin diagrams

CLASSICAL

A\(_n\) \(\text{SL}_{n+1}\)
B\(_n\) \(\text{SO}_{2n+1}\)
C\(_n\) \(\text{Sp}_{2n}\)
D\(_n\) \(\text{SO}_{2n}\)

EXCEPTIONAL

E\(_6\)
E\(_7\)
E\(_8\)
F\(_4\)
G\(_2\)
Subgroups $P \subset G$ s.t. $G/P$ is projective are called \textbf{parabolic}.
Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. $G/P$ is projective are called parabolic.

A parabolic subgroup is given by the choice of a set of simple roots, i.e. by $I \subset \Delta$, and the variety $G/P$ is denoted by marking the nodes of $I$. 
Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. $G/P$ is projective are called \textit{parabolic}.

A parabolic subgroup is given by the choice of a set of simple roots, i.e. by $I \subseteq D$, and the variety $G/P$ is denoted by marking the nodes of $I$.

$G = \text{SL}(4)$
Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. $G/P$ is projective are called parabolic.

A parabolic subgroup is given by the choice of a set of simple roots, i.e. by $I \subset D$, and the variety $G/P$ is denoted by marking the nodes of $I$.

$$G = \text{SL}(4)$$

![Diagram showing marked Dynkin diagrams representing $\mathbb{P}^3$, $G(1,3)$, and $(\mathbb{P}^3)^*$]
Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. $G/P$ is projective are called **parabolic**.

A parabolic subgroup is given by the choice of a set of simple roots, i.e. by $I \subset \Delta$, and the variety $G/P$ is denoted by marking the nodes of $I$.

Subgroups $P \subset G$ s.t. $G/P$ is projective are called **parabolic**.

A parabolic subgroup is given by the choice of a set of simple roots, i.e. by $I \subset \Delta$, and the variety $G/P$ is denoted by marking the nodes of $I$.

![Dynkin diagrams](https://example.com/dynkin_diagrams.png)
Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. $G/P$ is projective are called \textit{parabolic}.

A parabolic subgroup is given by the choice of a set of simple roots, i.e. by $I \subset D$, and the variety $G/P$ is denoted by marking the nodes of $I$.

$$G = SL(4)$$

- $P^3$
- $G(1, 3)$
- $(\mathbb{P}^3)^*$
- $F(0, 1)$
- $F(1, 2)$
- $F(0, 2)$
- $F(0, 1, 2)$
Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. $G/P$ is projective are called parabolic.

A parabolic subgroup is given by the choice of a set of simple roots, i.e. by $I \subset D$, and the variety $G/P$ is denoted by marking the nodes of $I$.

So a rational homogeneous (RH) manifold is given by a marked Dynkin diagram $(\mathcal{D}, \mathcal{I})$. 
Cone and contractions

$X$ RH given by $(\mathcal{D}, I)$.
Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

1. X is a Fano manifold;
Cone and contractions

\( X \) RH given by \((\mathcal{D}, \mathcal{I})\).

1. \( X \) is a Fano manifold;
2. The Picard number \( \rho_X \) of \( X \) is \#I;
Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

1. $X$ is a Fano manifold;
2. The Picard number $\rho_X$ of $X$ is $\# I$;
3. The cone $\text{NE}(X)$ is simplicial, and its faces correspond to proper subsets $J \subsetneq I$;
Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

1. $X$ is a Fano manifold;
2. The Picard number $\rho_X$ of $X$ is $\# I$;
3. The cone $\text{NE}(X)$ is simplicial, and its faces correspond to proper subsets $J \subsetneq I$;
4. Every contraction $\pi : X \to Y$ is of fiber type and smooth.
Cone and contractions

\( X \) RH given by \((\mathcal{D}, \mathcal{I})\).

1. \( X \) is a Fano manifold;
2. The Picard number \( \rho_X \) of \( X \) is \( \#I \);
3. The cone \( \text{NE}(X) \) is simplicial, and its faces correspond to proper subsets \( J \subsetneq I \);
4. Every contraction \( \pi : X \to Y \) is of fiber type and smooth.
5. \( Y \) is RH with marked Dynkin diagram \((\mathcal{D}, \mathcal{J})\),
Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

1. X is a Fano manifold;
2. The Picard number $\rho_X$ of X is $\#I$;
3. The cone $\text{NE}(X)$ is simplicial, and its faces correspond to proper subsets $J \subsetneq I$;
4. Every contraction $\pi : X \to Y$ is of fiber type and smooth.
5. Y is RH with marked Dynkin diagram $(\mathcal{D}, \mathcal{J})$,
6. Every fiber is RH with marked Dynkin diagram $(\mathcal{D}\setminus\mathcal{J}, \mathcal{I}\setminus\mathcal{J})$. 
Cone and contractions

\( X \) RH given by \((\mathcal{D}, I)\).

1. \( X \) is a Fano manifold;
2. The Picard number \( \rho_X \) of \( X \) is \#I;
3. The cone \( \text{NE}(X) \) is simplicial, and its faces correspond to proper subsets \( J \subsetneq I \);
4. Every contraction \( \pi : X \rightarrow Y \) is of fiber type and smooth.
5. \( Y \) is RH with marked Dynkin diagram \((\mathcal{D}, J)\),
6. Every fiber is RH with marked Dynkin diagram \((\mathcal{D} \setminus J, I \setminus J)\).

**Example**

![Diagram](image.png)
A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. The corresponding parabolic subgroup $B$ is called a Borel subgroup.
A **complete flag manifold** is a RH manifold with a diagram in which all the nodes are marked. The corresponding parabolic subgroup B is called a **Borel subgroup**.

- Every RH manifold is dominated by a complete flag manifold.
Complete flag manifolds

A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. The corresponding parabolic subgroup B is called a Borel subgroup.

- Every RH manifold is dominated by a complete flag manifold.
- $p_i : G/B \rightarrow G/P^i$ contractions corresponding to the unmarking of one node are $\mathbb{P}^1$-bundles.
A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. The corresponding parabolic subgroup B is called a Borel subgroup.

- Every RH manifold is dominated by a complete flag manifold.
- $p_i : G/B \to G/P^i$ contractions corresponding to the unmarking of one node are $\mathbb{P}^1$-bundles.
- If $\Gamma_i$ is a fiber of $p_i$, and $K_i$ the relative canonical, the intersection matrix $[-K_i \cdot \Gamma_j]$ is the Cartan matrix.
A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. The corresponding parabolic subgroup $B$ is called a Borel subgroup.

- Every RH manifold is dominated by a complete flag manifold.
- $p_i : G/B \to G/P_i$ contractions corresponding to the unmarking of one node are $\mathbb{P}^1$-bundles.
- If $\Gamma_i$ is a fiber of $p_i$, and $K_i$ the relative canonical, the intersection matrix $[-K_i \cdot \Gamma_j]$ is the Cartan matrix.

**Example ($A_n$)**

If $D = A_n$, then $G/B$ is the manifold parametrizing complete flags of linear subspaces in $\mathbb{P}^n$. 
Fano manifolds whose elementary contractions are smooth $\mathbb{P}^1$-fibrations
Relative duality

\[ \pi : M \rightarrow Y \] smooth \( \mathbb{P}^1 \)-fibration. \( \Gamma \) fiber, \( K \) relative canonical

**Lemma**

Let \( D \) be a divisor on \( M \) and set \( l := D \cdot \Gamma + 1 \). Then, \( \forall i \in \mathbb{Z} \)

\[
\begin{align*}
H^i(M, D) &\cong H^{i-1}(M, D + lK) & \text{if } l < 0 \\
H^i(M, D) &\cong \{0\} & \text{if } l = 0 \\
H^i(M, D) &\cong H^{i+1}(M, D + lK) & \text{if } l > 0
\end{align*}
\]

In particular \( \chi(M, D) = -\chi(M, D + lK) \) for any \( D \).
Idea of Proof

I - Finding a homogeneous model

Theorem

X is Flag Type manifold if and only if X is a complete flag manifold.
Theorem

\(X\) is Flag Type manifold if and only if \(X\) is a complete flag manifold.

- \(X\) Fano manifold with Picard number \(n\).
- \(\pi_i: X \to X_i\) elementary contration.
- \(K_i\) relative canonical, \(\Gamma_i\) fiber of \(\pi_i\).
I - Finding a homogeneous model

Theorem

X is Flag Type manifold if and only if X is a complete flag manifold.

- X Fano manifold with Picard number n.
- $\pi_i : X \to X_i$ elementary contraction.
- $K_i$ relative canonical, $\Gamma_i$ fiber of $\pi_i$.
- $\chi_X : \text{Pic}(X) \to \mathbb{Z}$ such that $\chi_X(L) = X(X, L)$. 
Theorem

\( X \) is Flag Type manifold if and only if \( X \) is a complete flag manifold.

- \( X \) Fano manifold with Picard number \( n \).
- \( \pi_i : X \to X_i \) elementary contraction.
- \( K_i \) relative canonical, \( \Gamma_i \) fiber of \( \pi_i \).
- \( \chi_X : \text{Pic}(X) \to \mathbb{Z} \) such that \( \chi_X(L) = \chi(X, L) \).

Given \( L_1, \ldots, L_n \) basis of \( \text{Pic}(X) \),

\[
\chi_X(m_1, \ldots, m_n) = \chi(X, m_1 L_1 + \cdots + m_n L_n)
\]

is a numerical polynomial of degree \( \dim X \), so we can extend it to \( \chi_X : N_1(X) \to \mathbb{R} \).
By the Lemma the affine involutions \( r'_i : N^1(X) \to N^1(X) \)
\[
r'_i(D) := D + (D \cdot \Gamma_i + 1)K_i,
\]
satisfy
\[
\chi_X(D) = -\chi_X(r'_i(D)).
\]
Since \( K_X \cdot \Gamma_i = -2 \) for every \( i \), setting
By the Lemma the affine involutions $r'_i : N^1(X) \to N^1(X)$

$$r'_i(D) := D + (D \cdot \Gamma_i + 1)K_i,$$

satisfy

$$\chi_X(D) = -\chi_X(r'_i(D)).$$

Since $K_X \cdot \Gamma_i = -2$ for every $i$, setting

$$T(D) := D + K_X/2$$
$$r_i := T^{-1} \circ r'_i \circ T$$
$$\chi^T := \chi_X \circ T$$
By the Lemma the affine involutions $r_i' : N^1(X) \to N^1(X)$

$$r_i'(D) := D + (D \cdot \Gamma_i + 1)K_i,$$

satisfy

$$\chi_X(D) = -\chi_X(r_i'(D)).$$

Since $K_X \cdot \Gamma_i = -2$ for every $i$, setting

$$T(D) := D + K_X/2$$

$$r_i := T^{-1} \circ r_i' \circ T$$

$$\chi^T := \chi_X \circ T$$

we have that the map $r_i$ is a linear involution of $N^1(X)$ given by

$$r_i(D) = D + (D \cdot \Gamma_i)K_i,$$
By the Lemma the affine involutions $r'_i : N^1(X) \to N^1(X)$
$$r'_i(D) := D + (D \cdot \Gamma_i + 1)K_i,$$

satisfy
$$\chi_X(D) = -\chi_X(r'_i(D)).$$

Since $K_X \cdot \Gamma_i = -2$ for every $i$, setting
$$T(D) := D + K_X/2$$
$$r_i := T^{-1} \circ r'_i \circ T$$
$$\chi^T := \chi_X \circ T$$

we have that the map $r_i$ is a linear involution of $N^1(X)$ given by
$$r_i(D) = D + (D \cdot \Gamma_i)K_i,$$

which fixes pointwise the hyperplane $M_i := \{D \mid D \cdot \Gamma_i = 0\}$ and satisfies
$$r_i(K_i) = -K_i \quad \chi^T(D) = -\chi^T(r_i(D));$$
By the Lemma the affine involutions $r_i' : N^1(X) \to N^1(X)$

$$r_i'(D) := D + (D \cdot \Gamma_i + 1)K_i,$$

satisfy

$$\chi_X(D) = -\chi_X(r_i'(D)).$$

Since $K_X \cdot \Gamma_i = -2$ for every $i$, setting

$$T(D) := D + K_X/2$$

$$r_i := T^{-1} \circ r_i' \circ T$$

$$\chi^T := \chi_X \circ T$$

we have that the map $r_i$ is a linear involution of $N^1(X)$ given by

$$r_i(D) = D + (D \cdot \Gamma_i)K_i,$$

which fixes pointwise the hyperplane $M_i := \{D \mid D \cdot \Gamma_i = 0\}$ and satisfies

$$r_i(K_i) = -K_i \quad \chi^T(D) = -\chi^T(r_i(D));$$

in particular $\chi^T$ vanishes on $M_i$ for every $i$. 
Let $W \subset \text{Gl}(N^1(X))$ be the group generated by the $r_i$’s.
Let $W \subset \text{Gl}(N^1(X))$ be the group generated by the $r_i$’s.

**Theorem**

The group $W$ is finite and the set

$$\Phi := \{w(-K_i) \mid w \in W, \ i = 1, \ldots, n\} \subset N^1(X),$$

is a root system, whose Weyl group is $W$.
For every divisor $D$ and every $w \in W$

$$\chi^T(D) = \pm \chi^T(w(D)),$$
For every divisor $D$ and every $w \in W$

$$
\chi^T(D) = \pm \chi^T(w(D)),
$$

so $\chi^T_X$ vanishes on the hyperplanes $w(M_i)$; therefore the number of these hyperplanes is bounded by the dimension of $X$. 
For every divisor $D$ and every $w \in W$

$$X^T(D) = \pm X^T(w(D)),$$

so $X^T_X$ vanishes on the hyperplanes $w(M_i)$; therefore the number of these hyperplanes is bounded by the dimension of $X$.

Then one proves that the isotropy subgroup of $M_i$ is finite (by considering the induced action on $N_1(X)$, and writing the elements of $W$ is a suitable basis).
Idea of proof

For every divisor $D$ and every $w \in W$

$$
\chi^T(D) = \pm \chi^T(w(D)),
$$

so $\chi^T_X$ vanishes on the hyperplanes $w(M_i)$; therefore the number of these hyperplanes is bounded by the dimension of $X$.

Then one proves that the isotropy subgroup of $M_i$ is finite (by considering the induced action on $N_1(X)$, and writing the elements of $W$ is a suitable basis).

By the finiteness there is a scalar product $(\ ,\ )$ on $N^1(X)$, which is $W$-invariant. In particular the $r_i$'s are euclidean reflections.
For every divisor $D$ and every $w \in W$

$$\chi^T(D) = \pm \chi^T(w(D)),$$

so $\chi_X^T$ vanishes on the hyperplanes $w(M_i)$; therefore the number of these hyperplanes is bounded by the dimension of $X$.

Then one proves that the isotropy subgroup of $M_i$ is finite (by considering the induced action on $N_1(X)$, and writing the elements of $W$ is a suitable basis).

By the finiteness there is a scalar product $(\ , \ )$ on $N^1(X)$, which is $W$-invariant. In particular the $r_i$’s are euclidean reflections.

Using that $r_i(K_i) = -K_i$ is then straightforward (but tedious) to prove that $\Phi$ is a root system with Weyl group $W$. 
Since \((\ , \ )\) is \(W\)-invariant, \((K_j, K_i) = (r_i(K_j), -K_i)\) which gives

\[-K_j \cdot \Gamma_i = 2 \frac{(K_j, K_i)}{(K_i, K_i)} = \langle K_j, K_i \rangle,
\]

so the intersection matrix \([-K_j \cdot \Gamma_i]\) is the Cartan matrix of \(\Phi\).
Since \((\ , \ )\) is \(W\)-invariant, \((K_j, K_i) = (r_i(K_j), -K_i)\) which gives

\[-K_j \cdot \Gamma_i = 2 \frac{(K_j, K_i)}{(K_i, K_i)} = \langle K_j, K_i \rangle,
\]

so the intersection matrix \([-K_j \cdot \Gamma_i]\) is the Cartan matrix of \(\Phi\).

In particular the intersection matrix of \(X\) is the intersection matrix of a complete flag manifold \(G/B\), the **homogeneous model** of \(X\).
Since \((K_j, K_i) = (r_i(K_j), -K_i)\) which gives

\[-K_j \cdot \Gamma_i = 2 \frac{(K_j, K_i)}{(K_i, K_i)} = \langle K_j, K_i \rangle,
\]

so the intersection matrix \([-K_j \cdot \Gamma_i]\) is the Cartan matrix of \(\Phi\).

In particular the intersection matrix of \(X\) is the intersection matrix of a complete flag manifold \(G/B\), the **homogeneous model** of \(X\).

Define \(\psi : N^1(X) \to N^1(G/B)\), by setting \(\psi(K_i) = \overline{K}_i\).
Since \((\ , \ )\) is \(W\)-invariant, \((K_j, K_i) = (r_i(K_j), -K_i)\) which gives

\[-K_j \cdot \Gamma_i = 2 \frac{(K_j, K_i)}{(K_i, K_i)} = \langle K_j, K_i \rangle,
\]

so the intersection matrix \([-K_j \cdot \Gamma_i]\) is the Cartan matrix of \(\Phi\).

In particular the intersection matrix of \(X\) is the intersection matrix of a complete flag manifold \(G/B\), the **homogeneous model** of \(X\).

Define \(\psi : N^1(X) \to N^1(G/B)\), by setting \(\psi(K_i) = \bar{K}_i\).

**Proposition**

\[\Lambda \subset \text{Pic}(X)\] generated by the \(K_i\)'s.

- \(h^i(X, D) = h^i(G/B, \psi(D))\) for every \(D \in \Lambda, \ i \in \mathbb{Z}\).
- \(\dim X = \dim G/B;\)
Idea of Proof

II - Proving the isomorphism

- $X$ Flag Type manifold of Picard number $n$, $x \in X$ point;
- $\ell = (l_1, \ldots, l_t)$, list of indices in $\{1, \ldots, n\}$,
- $\ell[1] = (l_1, \ldots, l_{t-1})$. 

---

Flag Manifolds
Gianluca Occhetta

Introduction
- Fano bundles
- The problem
RH manifolds
- Lie algebras
- Cartan matrix
- Dynkin diagrams
- RH manifolds
- Cone and contractions
- Flag manifolds

Main result
- Fibrations and reflections
- Homogeneous model
- Bott-Samelson varieties

CP Conjecture
- Positivity of the tangent bundle
- Results

An application
- Homogeneity and rational curves
- Comments and related results
- Idea of proof
- Speculations
Idea of Proof

II - Proving the isomorphism

- X Flag Type manifold of Picard number $n$, $x \in X$ point;
- $\ell = (l_1, \ldots, l_t)$, list of indices in $\{1, \ldots, n\}$,
- $\ell[1] = (l_1, \ldots, l_{t-1})$.

The Bott-Samelson variety $Z_\ell$, with a morphism $f_\ell : Z_\ell \to X$, associated with the sequence $\ell$, is constructed in the following way:
• X Flag Type manifold of Picard number $n$, $x \in X$ point;
• $\ell = (l_1, \ldots, l_t)$, list of indices in $\{1, \ldots, n\}$,
• $\ell[1] = (l_1, \ldots, l_{t-1})$.

The Bott-Samelson variety $Z_\ell$, with a morphism $f_\ell : Z_\ell \to X$, associated with the sequence $\ell$, is constructed in the following way:

If $\ell = \emptyset$ we set $Z_\ell := \{x\}$ and $f_\ell : \{x\} \to X$ is the inclusion.
• $X$ Flag Type manifold of Picard number $n$, $x \in X$ point;
• $\ell = (l_1, \ldots, l_t)$, list of indices in \{1, \ldots, n\},
• $\ell[1] = (l_1, \ldots, l_{t-1})$.

The **Bott-Samelson variety** $Z_\ell$, with a morphism $f_\ell : Z_\ell \to X$, associated with the sequence $\ell$, is constructed in the following way:

If $\ell = \emptyset$ we set $Z_\ell := \{x\}$ and $f_\ell : \{x\} \to X$ is the inclusion.

Inductively we build $Z_\ell$ on $Z_\ell[1]$:
Idea of Proof

II - Proving the isomorphism

- $X$ Flag Type manifold of Picard number $n$, $x \in X$ point;
- $\ell = (l_1, \ldots, l_t)$, list of indices in $\{1, \ldots, n\}$,
- $\ell[1] = (l_1, \ldots, l_{t-1})$.

The **Bott-Samelson variety** $Z_\ell$, with a morphism $f_\ell : Z_\ell \to X$, associated with the sequence $\ell$, is constructed in the following way:

If $\ell = \emptyset$ we set $Z_\ell := \{x\}$ and $f_\ell : \{x\} \to X$ is the inclusion.

Inductively we build $Z_\ell$ on $Z_\ell[1]$:

\[
\begin{align*}
X & \xrightarrow{\pi_{l_t}} X_{l_t} \\
\downarrow & \downarrow \\
Z_\ell[1] & \xrightarrow{f_\ell[1]} \\
\end{align*}
\]
The Bott-Samelson variety $Z_\ell$, with a morphism $f_\ell : Z_\ell \to X$, associated with the sequence $\ell$, is constructed in the following way:

If $\ell = \emptyset$ we set $Z_\ell := \{x\}$ and $f_\ell : \{x\} \to X$ is the inclusion.

Inductively we build $Z_\ell$ on $Z_\ell[1]$:
The Bott-Samelson variety $Z_\ell$, with a morphism $f_\ell : Z_\ell \to X$, associated with the sequence $\ell$, is constructed in the following way:

If $\ell = \emptyset$ we set $Z_\ell := \{x\}$ and $f_\ell : \{x\} \to X$ is the inclusion.

Inductively we build $Z_\ell$ on $Z_{\ell[1]}$:
The image of $Z_{\ell}$ in $X$ is the set of points belonging to chains of rational curves $\Gamma_{l_1}, \Gamma_{l_2} \ldots, \Gamma_{l_t}$ starting from $x$. 
The image of \( Z_\ell \) in \( X \) is the set of points belonging to chains of rational curves \( \Gamma_1, \Gamma_2 \ldots, \Gamma_t \) starting from \( \chi \).
In the homogeneous case such loci are the Schubert varieties.
The image of $Z_\ell$ in $X$ is the set of points belonging to chains of rational curves $\Gamma_1, \Gamma_2 \ldots, \Gamma_t$ starting from $x$. In the homogeneous case such loci are the Schubert varieties.

With a list $\ell$ it is associated an element $w(\ell)$ of the Weyl group:

$$w = r_{\ell_1} \circ \cdots \circ r_{\ell_t};$$
The image of $Z_\ell$ in $X$ is the set of points belonging to chains of rational curves $\Gamma_1, \Gamma_2 \ldots, \Gamma_t$ starting from $x$.

In the homogeneous case such loci are the **Schubert varieties**.

With a list $\ell$ it is associated an element $w(\ell)$ of the Weyl group:

$$w = r_{l_1} \circ \cdots \circ r_{l_t};$$

if there is no expression of $w(\ell)$ which contains less than $t$ reflections, then $w(\ell)$ and $\ell$ are called **reduced**.
The image of $Z_\ell$ in $X$ is the set of points belonging to chains of rational curves $\Gamma_{l_1}, \Gamma_{l_2} \ldots, \Gamma_{l_t}$ starting from $x$.

In the homogeneous case such loci are the Schubert varieties.

With a list $\ell$ it is associated an element $w(\ell)$ of the Weyl group:

$$w = r_{l_1} \circ \cdots \circ r_{l_t};$$

if there is no expression of $w(\ell)$ which contains less than $t$ reflections, then $w(\ell)$ and $\ell$ are called reduced.

The length $\lambda(w(\ell))$ is the number of reflections appearing in a reduced expression of $w(\ell)$. 
The image of $Z_\ell$ in $X$ is the set of points belonging to chains of rational curves $\Gamma_{l_1}, \Gamma_{l_2}, \ldots, \Gamma_{l_t}$ starting from $x$.

In the homogeneous case such loci are the Schubert varieties.

With a list $\ell$ it is associated an element $w(\ell)$ of the Weyl group:

$$w = r_{l_1} \circ \cdots \circ r_{l_t};$$

if there is no expression of $w(\ell)$ which contains less than $t$ reflections, then $w(\ell)$ and $\ell$ are called reduced.

The length $\lambda(w(\ell))$ is the number of reflections appearing in a reduced expression of $w(\ell)$.

If $w(\ell)$ is reduced then $f_\ell : Z_\ell \to f_\ell(Z_\ell)$ is birational, hence

$$\dim f_\ell(Z_\ell) = \dim Z_\ell = \#(\ell) = \lambda(w(\ell)).$$
The image of $Z_\ell$ in $X$ is the set of points belonging to chains of rational curves $\Gamma_{l_1}, \Gamma_{l_2}, \ldots, \Gamma_{l_t}$ starting from $x$.

In the homogeneous case such loci are the **Schubert varieties**.

With a list $\ell$ it is associated an element $w(\ell)$ of the Weyl group:

$$w = r_{l_1} \circ \cdots \circ r_{l_t};$$

if there is no expression of $w(\ell)$ which contains less than $t$ reflections, then $w(\ell)$ and $\ell$ are called **reduced**.

The **length** $\lambda(w(\ell))$ is the number of reflections appearing in a reduced expression of $w(\ell)$.

If $w(\ell)$ is reduced then $f_\ell : Z_\ell \to f_\ell(Z_\ell)$ is birational, hence

$$\dim f_\ell(Z_\ell) = \dim Z_\ell = \#(\ell) = \lambda(w(\ell)).$$

In $W$ there exists a unique **longest element** $w_0$, of length $\dim X$.

If $\ell_0$ is a reduced list such that $w(\ell_0) = w_0$ then $f_\ell : Z_{\ell_0} \to X$ is surjective and birational.
X Flag Type manifold, $G/B$ homogeneous model of $X$

Find a list $\ell_0$ such that $w(\ell_0) = w_0$ and prove that

$$Z_{\ell_0} \simeq \overline{Z}_{\ell_0} \quad f_{\ell_0} = \overline{f}_{\ell_0}$$

The idea is to show inductively that $Z_{\ell_0}$ depends only on the list and on the intersection matrix.
X Flag Type manifold, G/B homogeneous model of X
Find a list $\ell_0$ such that $w(\ell_0) = w_0$ and prove that

$$Z_{\ell_0} \simeq \overline{Z}_{\ell_0} \quad f_{\ell_0} = \overline{f}_{\ell_0}$$

The idea is to show inductively that $Z_{\ell_0}$ depends only on the list and on the intersection matrix.
Assume that $Z_{\ell[1]} \simeq \overline{Z}_{\ell[1]}$;

$$\begin{array}{ccc}
Z_{\ell} & \xrightarrow{f_{\ell}} & X \\
\downarrow \sigma_{\ell[1]} & & \downarrow \pi_{l_r} \\
Z_{\ell[1]} & \xrightarrow{f_{\ell[1]}} & X_{l_r}
\end{array}$$
X Flag Type manifold, $G/B$ homogeneous model of $X$

Find a list $\ell_0$ such that $w(\ell_0) = w_0$ and prove that

$$Z_{\ell_0} \simeq \overline{Z}_{\ell_0}, \quad f_{\ell_0} = \overline{f}_{\ell_0}$$

The idea is to show inductively that $Z_{\ell_0}$ depends only on the list and on the intersection matrix.

Assume that $Z_{\ell[1]} \simeq \overline{Z}_{\ell[1]}$;

$$f_{\ell[1]}$$ factors via $Z_{\ell}$, giving a section $\sigma_{\ell[1]}$, hence an extension

$$0 \to \mathcal{O}_{Z_{\ell[1]}}(f_{\ell[1]}^*K_{\ell_{\tau}}) \to \mathcal{F}_{\ell} \to \mathcal{O}_{Z_{\ell[1]}} \to 0.$$
One shows easily that the following are equivalent
One shows easily that the following are equivalent

- The extension is split;
- $h^1(Z_{\ell[1]}, f_{\ell}^*(K_{L_R})) = 0$;
- the index $l_R$ does not appear in $\ell[1]$.
One shows easily that the following are equivalent:

- The extension is split;
- $h^1(Z_{\ell[1]}, f^*_\ell(K_{l_r})) = 0$;
- the index $l_r$ does not appear in $\ell[1]$.

It is enough to show that if the index $l_r$ appears in $\ell[1]$ then

$$h^1(Z_{\ell[1]}, f^*_\ell(K_{l_r})) \leq 1.$$
One shows easily that the following are equivalent

- The extension is split;
- \( h^1(Z_{\ell[1]}, f^*_\ell(K_{\ell_r})) = 0 \);
- the index \( \ell_r \) does not appear in \( \ell[1] \).

It is enough to show that if the index \( \ell_r \) appears in \( \ell[1] \) then

\[
h^1(Z_{\ell[1]}, f^*_\ell(K_{\ell_r})) \leq 1.
\]

This can be done except for \( G_2 \), (already known from the \( n = 2 \) case) and \( F_4 \), for which an ad hoc argument is needed.
Introduction

Fano bundles
The problem

RH manifolds
Lie algebras
Cartan matrix
Dynkin diagrams
RH manifolds
Cone and contractions
Flag manifolds

Main result
Fibrations and reflections
Homogeneous model
Bott-Samelson varieties

CP Conjecture

Positivity of the tangent bundle

X smooth complex projective variety.
X smooth complex projective variety.

**Theorem [Mori (1979)]**

$T_X$ ample $\iff X = \mathbb{P}^m$. 
Let $X$ be a smooth complex projective variety.

**Theorem [Mori (1979)]**

$T_X$ ample $\iff X = \mathbb{P}^m$.

- $T_X$ nef $\Rightarrow$ ??
Positivity of the tangent bundle

Let $X$ be a smooth complex projective variety.

**Theorem [Mori (1979)]**

$T_X$ ample $\iff X = \mathbb{P}^m$.

- $T_X$ nef $\Rightarrow \ ?$
- **Examples:**
  - Abelian
  - Rational

- Homogeneous manifolds

Cone and contractions
Flag manifolds

CP Conjecture

Positivity of the tangent bundle
Results

An application
Homogeneity and rational curves
Comments and related results
Idea of proof
Speculations
Positivity of the tangent bundle

$X$ smooth complex projective variety.

**Theorem [Mori (1979)]**

$T_X$ ample $\iff X = \mathbb{P}^m$.

- $T_X$ nef $\Rightarrow$ ??
- Examples:
  - Homogeneous manifolds:
    - Abelian
    - Rational

**Theorem [Demailly, Peternell and Schneider (1994)]**

$T_X$ nef $\Rightarrow$

\[ X \xleftarrow{\text{étale}} X' \xrightarrow{F} A \]

- $A$ Abelian, $F$ Fano, $T_F$ nef
Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.
Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

**Results:**

- \( \dim X = 3 \) [Campana & Peternell(1991)]
Campana-Peternell Conjecture

Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Results:

✓ dim $X = 3$ [Campana & Peternell(1991)]
✓ dim $X = 4$ [CP (1993), Mok (2002), Hwang (2006)]
Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

**Results:**

- \( \dim X = 3 \) [Campana & Peternell (1991)]
- \( \dim X = 4 \) [CP (1993), Mok (2002), Hwang (2006)]
- \( \dim X = 5 \) and \( \rho_X > 1 \) [Watanabe (2012)]
Campana-Peternell Conjecture

Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Results:

☑ dim $X = 3$ [Campana & Peternell(1991)]
☑ dim $X = 4$ [CP (1993), Mok (2002), Hwang (2006)]
☑ dim $X = 5$ and $\rho_X > 1$ [Watanabe (2012)]
☑ $T_X$ big and 1-ample [Solá-Conde & Wiśniewski (2004)]
Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Results:

- $\dim X = 3$ [Campana & Peternell(1991)]
- $\dim X = 4$ [CP (1993), Mok (2002), Hwang (2006)]
- $\dim X = 5$ and $\rho_X > 1$ [Watanabe (2012)]
- $T_X$ big and 1-ample [Solá-Conde & Wiśniewski (2004)]

- The above results are obtained by classifying the manifolds satisfying the required properties;
Campana-Peternell Conjecture (1991)
Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Results:
- \( \dim X = 3 \) [Campana & Peternell (1991)]
- \( \dim X = 4 \) [CP (1993), Mok (2002), Hwang (2006)]
- \( \dim X = 5 \) and \( \rho_X > 1 \) [Watanabe (2012)]
- \( T_X \) big and 1-ample [Solá-Conde & Wiśniewski (2004)]

- The above results are obtained by classifying the manifolds satisfying the required properties;
- homogeneity is checked \textit{a posteriori}.
Homogeneity via families of rational curves

- $X$ Fano of Picard number one;
- $\mathcal{M}$ dominating family of rational curves of minimal degree;
- $\mathcal{U}$ universal family.

Theorem
Assume that $\mathcal{M}$ is unsplit, $q$ is smooth and that $\mathcal{M} \times := q - 1(x)$ is RH for every $x \in X$. Then $X$ is RH.

Remark
If $\mathcal{T} X$ is nef then the assumptions on $\mathcal{M}$ and $q$ hold.
Homogeneity via families of rational curves

- $X$ Fano of Picard number one;
- $M$ dominating family of rational curves of minimal degree;
- $U$ universal family.

\[ \begin{array}{c}
\text{U} \\
p \quad q \\
\text{M} \\
\rightarrow \\
\text{X} \\
\end{array} \]

Theorem: Assume that $M$ is unsplit, $q$ is smooth and that $M \times := q^{-1}(x)$ is RH for every $x \in X$. Then $X$ is RH.

Remark: If $T_X$ is nef then the assumptions on $M$ and $q$ hold.
Homogeneity via families of rational curves

- X Fano of Picard number one;
- \( \mathcal{M} \) dominating family of rational curves of minimal degree;
- \( \mathcal{U} \) universal family.

\[ \mathcal{U} \]
\[ p \]
\[ q \]
\[ \mathcal{M} \]
\[ X \]

Theorem

Assume that \( \mathcal{M} \) is unsplit, \( q \) is smooth and that \( \mathcal{M}_x := q^{-1}(x) \) is RH for every \( x \in X \). Then \( X \) is RH.
Homogeneity via families of rational curves

- $X$ Fano of Picard number one;
- $\mathcal{M}$ dominating family of rational curves of minimal degree;
- $\mathcal{U}$ universal family.

Theorem

Assume that $\mathcal{M}$ is unsplit, $q$ is smooth and that $\mathcal{M}_x := q^{-1}(x)$ is RH for every $x \in X$. Then $X$ is RH.

Remark

If $T_x$ is nef then the assumptions on $\mathcal{M}$ and $q$ hold.
Homogeneity via families of rational curves

- $X$ Fano of Picard number one;
- $\mathcal{M}$ dominating family of rational curves of minimal degree;
- $\mathcal{U}$ universal family.

\[
\begin{array}{c}
\mathcal{U} \\
p \\
\mathcal{M} \\
\ \ \ \ \ \ \ \ \\
q \\
X
\end{array}
\]

**Theorem**

Assume that $T_X$ is nef and that $\mathcal{M}_x := q^{-1}(x)$ is RH for every $x \in X$. Then $X$ is RH.

**Remark**

If $T_X$ is nef then the assumptions on $\mathcal{M}$ and $q$ hold.
Recognizing homogeneous spaces

- $X$ Fano of Picard number one, $T_X$ nef;
- $S = G/P$ RH space of Picard number one;
- $\mathcal{M}$, $\mathcal{L}$ minimal dominating families of rational curves;
Recognizing homogeneous spaces

- $X$ Fano of Picard number one, $T_X$ nef;
- $S = G/P$ RH space of Picard number one;
- $\mathcal{M}$, $\mathcal{L}$ minimal dominating families of rational curves;

**Corollary**

Assume $\mathcal{L}_0$ is RH. If $\mathcal{M}_x \simeq \mathcal{L}_0$ for every $x \in X$ then $X \simeq S$. 
Recognizing homogeneous spaces

- $X$ Fano of Picard number one, $T_X$ nef;
- $S = G/P$ RH space of Picard number one;
- $\mathcal{M}$, $\mathcal{L}$ minimal dominating families of rational curves;

**Corollary**

Assume $\mathcal{L}_0$ is RH. If $\mathcal{M}_x \cong \mathcal{L}_0$ for every $x \in X$ then $X \cong S$.

The following are equivalent:

- $\mathcal{L}_0$ is $G$-homogeneous.
- $P$ is associated to a long simple root.
- There is no arrow in the Dynkin diagram pointing towards the node corresponding to $P$. 
Recognizing homogeneous spaces

- $X$ Fano of Picard number one;
- $S = G/P$ RH space of Picard number one;
- $\mathcal{M}, \mathcal{L}$ minimal dominating families of rational curves;
- $C_0(S)$ VMRT of $S$;
- $C_x(X)$ VMRT of $X$ at a general point;
Recognizing homogeneous spaces

- $X$ Fano of Picard number one;
- $S = G/P$ RH space of Picard number one;
- $\mathcal{M}$, $\mathcal{L}$ minimal dominating families of rational curves;
- $C_0(S)$ VMRT of $S$;
- $C_x(X)$ VMRT of $X$ at a general point;

**Theorem [Mok, Hong-Hwang]**

If $P$ is associated to a long simple root and $C(X)_x$ is projectively equivalent to $C(S)_0$, then $X \sim S$. 
Idea of the proof

Given the smooth fibration $q: \mathcal{U} \to X$, with RH fiber $F$, it is possible to construct the associated flag bundle over $X$, whose fibers over a point are complete flag manifolds.
Idea of the proof

Given the smooth fibration $q : \mathcal{U} \to X$, with RH fiber $F$, it is possible to construct the associated flag bundle over $X$, whose fibers over a point are complete flag manifolds.

The fibration $q$ is defined by a cocycle $\vartheta \in H^1(X, G)$, where $G$ is the identity component of $\text{Aut}(F)$ - here we use that $X$ is simply connected.
Idea of the proof

Given the smooth fibration $q : \mathcal{U} \to X$, with RH fiber $F$, it is possible to construct the associated flag bundle over $X$, whose fibers over a point are complete flag manifolds.

The fibration $q$ is defined by a cocycle $\vartheta \in H^1(X, G)$, where $G$ is the identity component of Aut$(F)$ - here we use that $X$ is simply connected.

The cocycle $\vartheta$ defines a principal $G$-bundle $\mathcal{U}_G \to X$
Idea of the proof

Given the smooth fibration $q : U \to X$, with RH fiber $F$, it is possible to construct the associated flag bundle over $X$, whose fibers over a point are complete flag manifolds.

The fibration $q$ is defined by a cocycle $\vartheta \in H^1(X, G)$, where $G$ is the identity component of $\text{Aut}(F)$ - here we use that $X$ is simply connected.

The cocycle $\vartheta$ defines a principal $G$-bundle $U_G \to X$.

Given a Borel subgroup $B \subset G$ we can define the $G/B$-bundle

$$\overline{U} := U_G \times^G G/B \to X$$

as a quotient of $U_G \times G/B$ by $(x, gB) \sim (xg', g'^{-1}gB)$,
Idea of the proof

Given the smooth fibration \( q : \mathcal{U} \to X \), with RH fiber \( F \), it is possible to construct the associated flag bundle over \( X \), whose fibers over a point are complete flag manifolds.

The fibration \( q \) is defined by a cocycle \( \vartheta \in H^1(X, G) \), where \( G \) is the identity component of \( \text{Aut}(F) \) - here we use that \( X \) is simply connected.

The cocycle \( \vartheta \) defines a principal \( G \)-bundle \( \mathcal{U}_G \to X \)

Given a Borel subgroup \( B \subset G \) we can define the \( G/B \)-bundle

\[
\overline{\mathcal{U}} := \mathcal{U}_G \times^G G/B \to X
\]

as a quotient of \( \mathcal{U}_G \times G/B \) by \( (x, gB) \sim (xg', g'^{-1}gB) \), and we have a commutative diagram

\[
\begin{array}{ccc}
\overline{\mathcal{U}} & \xrightarrow{\pi} & \mathcal{U} \\
\downarrow{\overline{q}} & & \downarrow{q} \\
X & & X
\end{array}
\]
The flag bundle $\overline{U}$ has Picard number $\rho(G/B) + 1$, and has $\rho(G/B)$ contractions (over $X$) which are smooth $\mathbb{P}^1$-fibrations.
The flag bundle $\overline{U}$ has Picard number $\rho(G/B) + 1$, and has $\rho(G/B)$ contractions (over $X$) which are smooth $\mathbb{P}^1$-fibrations.
The flag bundle $\overline{U}$ has Picard number $\rho(G/B) + 1$, and has $\rho(G/B)$ contractions (over $X$) which are smooth $\mathbb{P}^1$-fibrations.

$$
\begin{array}{c}
\overline{U} \\
\downarrow \pi \\
U \\
\downarrow q \\
X
\end{array}
$$

Idea: show that the $\mathbb{P}^1$-fibration $p : U \to M$
The flag bundle $\overline{U}$ has Picard number $\rho(G/B) + 1$, and has $\rho(G/B)$ contractions (over $X$) which are smooth $\mathbb{P}^1$-fibrations.

Idea: show that the $\mathbb{P}^1$-fibration $p : U \to M$
The flag bundle $\overline{U}$ has Picard number $\rho(G/B) + 1$, and has $\rho(G/B)$ contractions (over $X$) which are smooth $\mathbb{P}^1$-fibrations.

Idea: show that the $\mathbb{P}^1$-fibration $p : \mathcal{U} \to \mathcal{M}$ can be lifted to $\overline{U}$. 

Idea of the proof
The flag bundle $\overline{U}$ has Picard number $\rho(G/B) + 1$, and has $\rho(G/B)$ contractions (over $X$) which are smooth $\mathbb{P}^1$-fibrations.

\[
\begin{array}{ccc}
\overline{M} & \xleftarrow{\overline{p}} & \overline{U} \\
\downarrow & & \downarrow \pi \\
M & \xleftarrow{p} & U \\
\downarrow & & \downarrow q \\
X
\end{array}
\]

Idea: show that the $\mathbb{P}^1$-fibration $p : U \to M$ can be lifted to $\overline{U}$. 
So $\mathcal{U}$ has a number of $\mathbb{P}^1$-fibrations equal to its Picard number.
So $\overline{U}$ has a number of $\mathbb{P}^1$-fibrations equal to its Picard number. A priori it is not a Fano manifold; however we can prove a slightly stronger version of the main theorem.
So $\overline{U}$ has a number of $\mathbb{P}^1$-fibrations equal to its Picard number.

A priori it is not a Fano manifold; however we can prove a slightly stronger version of the main theorem

**Theorem**

Let $X$ be a smooth projective variety of Picard number $n$, with $n$ elementary contractions which are smooth $\mathbb{P}^1$-fibrations. Then $X$ is isomorphic to a complete flag manifold.
So $\overline{U}$ has a number of $\mathbb{P}^1$-fibrations equal to its Picard number. A priori it is not a Fano manifold; however we can prove a slightly stronger version of the main theorem

**Theorem**

Let $X$ be a smooth projective variety of Picard number $n$, with $n$ elementary contractions which are smooth $\mathbb{P}^1$-fibrations. Then $X$ is isomorphic to a complete flag manifold.

and get that $\overline{U}$ is a complete flag manifold;
So $\overline{U}$ has a number of $\mathbb{P}^1$-fibrations equal to its Picard number.

A priori it is not a Fano manifold; however we can prove a slightly stronger version of the main theorem

**Theorem**

Let $X$ be a smooth projective variety of Picard number $n$, with $n$ elementary contractions which are smooth $\mathbb{P}^1$-fibrations. Then $X$ is isomorphic to a complete flag manifold.

and get that $\overline{U}$ is a complete flag manifold; hence $X$, being the image of a contraction of $U$ is homogeneous.
So $\mathcal{U}$ has a number of $\mathbb{P}^1$-fibrations equal to its Picard number.

A priori it is not a Fano manifold; however we can prove a slightly stronger version of the main theorem

**Theorem**

Let $X$ be a smooth projective variety of Picard number $n$, with $n$ elementary contractions which are smooth $\mathbb{P}^1$-fibrations. Then $X$ is isomorphic to a complete flag manifold.

and get that $\mathcal{U}$ is a complete flag manifold; hence $X$, being the image of a contraction of $\mathcal{U}$ is homogeneous.

**Remark**

A similar argument has been used to conclude the proof of CP conjecture in dimension 5 by Kanemitsu (2015).
Given a CP-manifold $X$, we define:

$$\tau(X) := \sum_R (\ell(R) - 2)$$

where the sum is taken over the extremal rays of $\overline{NE}(X)$. 
Given a CP-manifold $X$, we define:

$$\tau(X) := \sum_{R} (\ell(R) - 2)$$

where the sum is taken over the extremal rays of $\overline{NE}(X)$.

In particular $\tau(X) = 0$ if and only if $X$ is a Flag Type manifold.
Given a CP-manifold $X$, we define:

$$\tau(X) := \sum_{R} (\ell(R) - 2)$$

where the sum is taken over the extremal rays of $\overline{NE}(X)$.

In particular $\tau(X) = 0$ if and only if $X$ is a Flag Type manifold.

CP conjecture will then follow from:

**Conjecture**

Given a CP-manifold satisfying $\tau(X) > 0$, there exists a contraction $f : X' \to X$ from a CP-manifold $X'$ satisfying $\tau(X') < \tau(X)$. 
Given a CP-manifold $X$, we define:

$$\tau(X) := \sum_{R} (\ell(R) - 2)$$

where the sum is taken over the extremal rays of $\overline{\text{NE}}(X)$.

In particular $\tau(X) = 0$ if and only if $X$ is a Flag Type manifold.

CP conjecture will then follow from:

Conjecture

Given a CP-manifold satisfying $\tau(X) > 0$, there exists a contraction $f : X' \to X$ from a CP-manifold $X'$ satisfying $\tau(X') < \tau(X)$. 