Cylinders in Rational Surfaces

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Cylinder

Definition

Let S be a normal projective surface.

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Let S be a normal projective surface. A cylinder in S is a Zariski open subset in S that is isomorphic to $Z \times \mathbb{A}^1$ for some affine variety Z.

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Z is a smooth rational affine curve, i.e., $\mathbb{P}^1 - \{\text{finitely many points}\}$
S is a rational surface.

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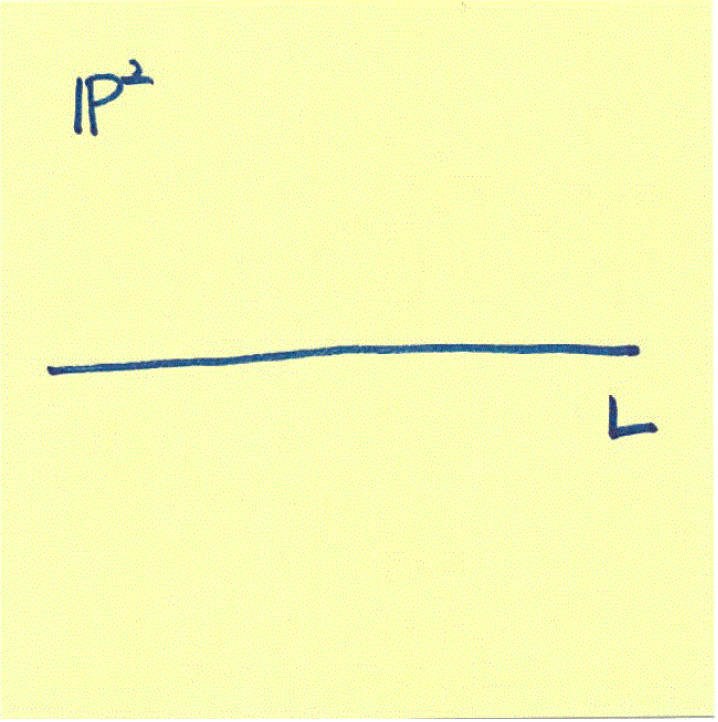
Z is a smooth rational affine curve, i.e., $\mathbb{P}^1 - \{\text{finitely many points}\}$
S is a rational surface.

We consider only the case when S is rational.

Examples



$$\mathbb{P}^2 \setminus L \cong \mathbb{A}^1 \times \mathbb{A}^1$$

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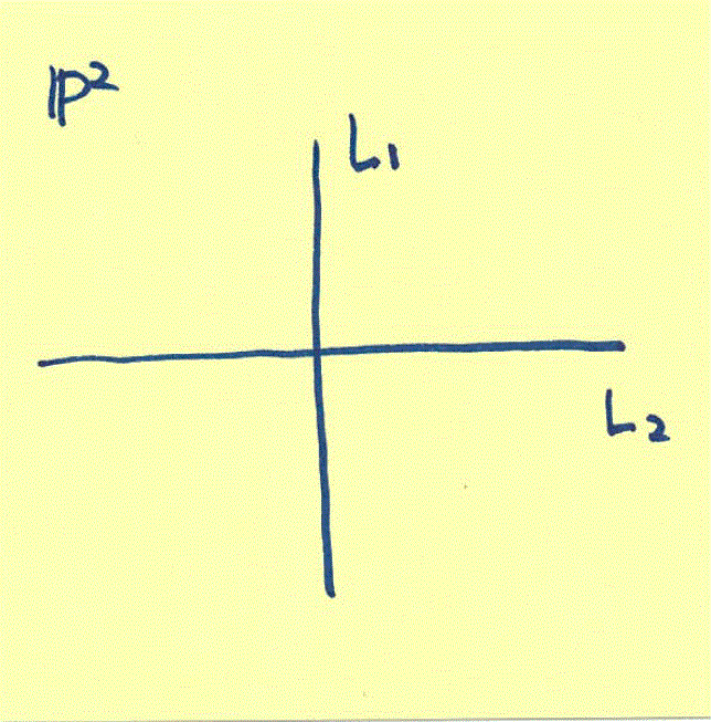

$$\mathbb{P}^2 \setminus L_1 \cong \mathbb{A}^2 \qquad \mathbb{C}^2 \setminus L_2 \cong \mathbb{A}^1 \times \mathbb{A}^1_*$$

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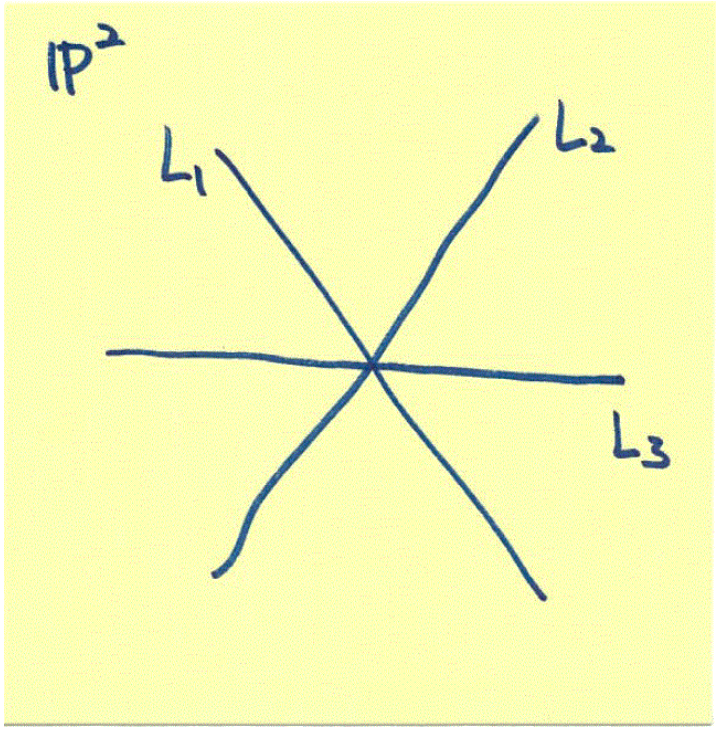

$$\mathbb{P}^2 \setminus L_1 \cong \mathbb{A}^2 \qquad \mathbb{C}^2 \setminus \{L_2 \cup L_3\} \cong \mathbb{A}^1 \times \mathbb{A}^1_{**}$$

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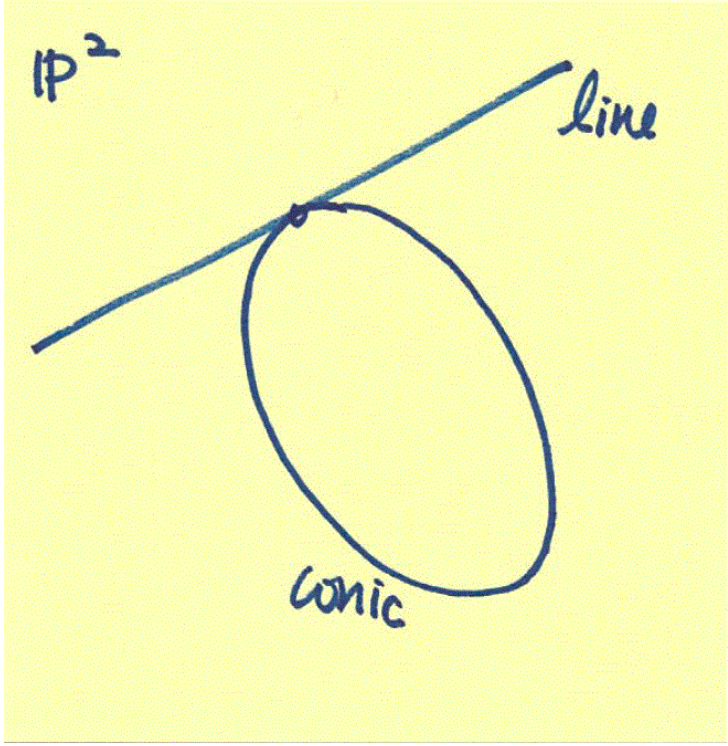

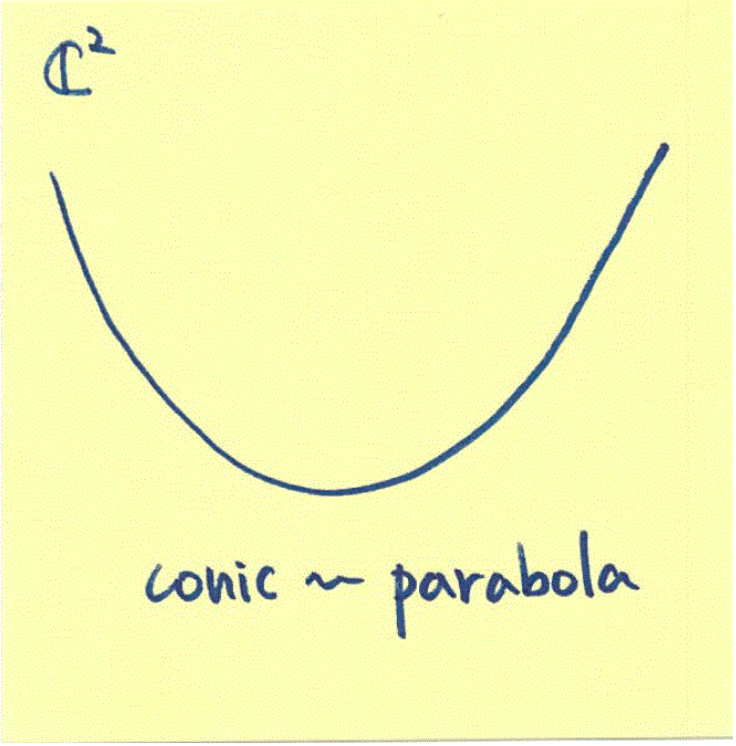

$$\mathbb{P}^2 \setminus L \cong \mathbb{A}^2 \qquad \mathbb{A}^2 \setminus C \cong \mathbb{A}^1 \times \mathbb{A}^1_*$$

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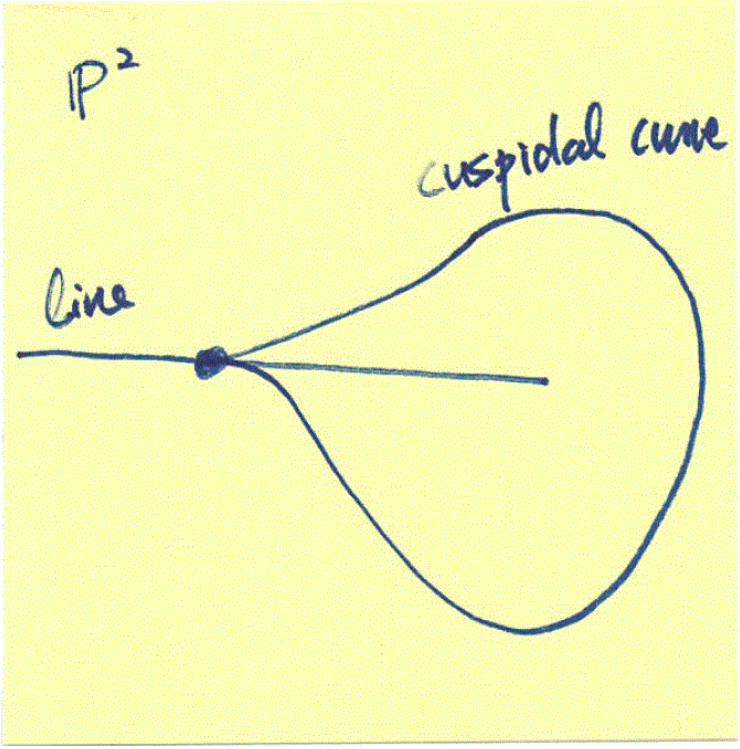

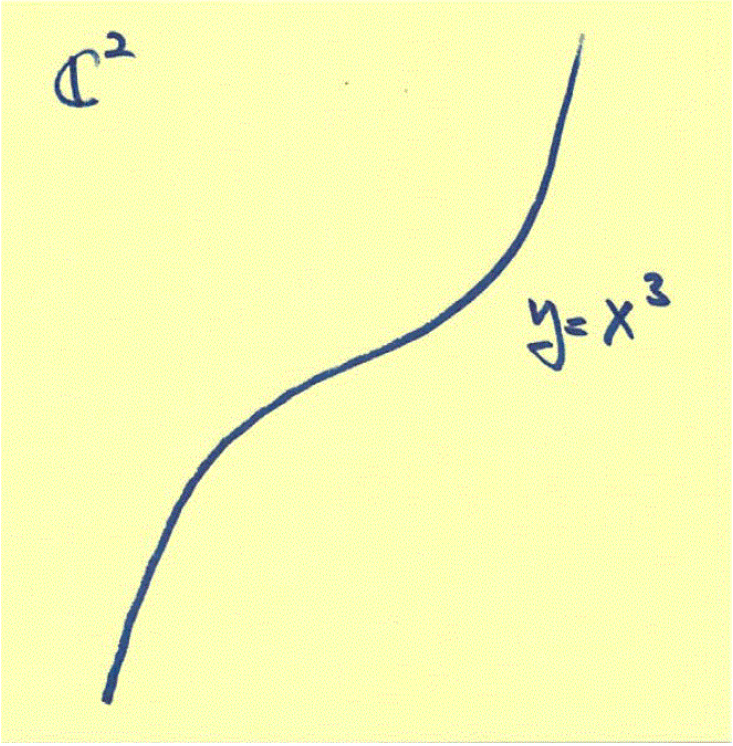

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Examples

- Suppose that S has a cylinder U.

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- A surface obtained by blowing up S at points outside U contains a cylinder.

Examples

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- A *singular* surface may not have any cylinder at all.

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- Let U be a cylinder in S, i.e., a Zariski open subset in S such that $U = \mathbb{A}^1 \times Z$ for some affine curve Z.

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where

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- π is a birational morphism resolving the indeterminacy of ψ and the singularities of S,
- ϕ is a morphism.

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- a general fiber of ϕ is $\mathbb{P}^1$.

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- The curves $C_1, \cdots, C_n$ generate the divisor class group $\mathrm{Cl}(S)$ of the surface S because $\mathrm{Cl}(U) = 0$. In particular, one has

$$n \geqslant \operatorname{rank} \mathrm{Cl}(S).$$

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- Let $E_1, \ldots, E_r$ be the π-exceptional curves, and let Γ be the section of $\bar{p}_2$, which is the complement of $\mathbb{C}^1 \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

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- Denote by $\tilde{C}_1, \ldots, \tilde{C}_n$ and $\tilde{\Gamma}$ the proper transforms of the curves $C_1, \ldots, C_n$ and Γ on the surface $\tilde{S}$, respectively.

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- All the other curves among $\tilde{C}_1, \ldots, \tilde{C}_n$ and $E_1, \ldots, E_r$ are irreducible components of some fibers of ϕ.
- We may assume either $\tilde{\Gamma} = \tilde{C}_1$ or $\tilde{\Gamma} = E_r$.

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Let $\tilde{F}$ be a general fiber of ϕ.

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Then $K_{\tilde{S}} \cdot \tilde{F} = -2$ by the adjunction formula.

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Put $F = \pi(\tilde{F})$.

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Choose arbitrary non-negative rational numbers $\lambda_1, \ldots, \lambda_n$.

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Choose arbitrary non-negative rational numbers $\lambda_1, \ldots, \lambda_n$.

$$K_{\tilde{S}} + \sum_{i=1}^{n} \lambda_i \tilde{C}_i \qquad = \pi^*\Big(K_S + \sum_{i=1}^{n} \lambda_i C_i\Big)$$

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Choose arbitrary non-negative rational numbers $\lambda_1, \ldots, \lambda_n$.

$$K_{\tilde{S}} + \sum_{i=1}^{n} \lambda_i \tilde{C}_i + \sum_{i=1}^{r} \mu_i E_i = \pi^*\Big(K_S + \sum_{i=1}^{n} \lambda_i C_i\Big)$$

for some rational numbers $\mu_1, \ldots, \mu_r$.

What if S has a cylinder?

If $\tilde{\Gamma} = E_r$, then

$$-2 + \mu_r = \left(K_{\tilde{S}} + \sum_{i=1}^{n} \lambda_i \tilde{C}_i + \sum_{i=1}^{r} \mu_i E_i \right) \cdot \tilde{F}$$

$$= \pi^* \left(K_S + \sum_{i=1}^{n} \lambda_i C_i \right) \cdot \tilde{F} = \left(K_S + \sum_{i=1}^{n} \lambda_i C_i \right) \cdot F$$

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If $\tilde{\Gamma} = C_1$, then

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What if S has a cylinder?

If $K_S + \sum_{i=1}^{n} \lambda_i C_i$ is pseudo-effective, then

$$\left(K_S + \sum_{i=1}^{n} \lambda_i C_i\right) \cdot F \geqslant 0,$$

because $\tilde{F}$ is a general fiber of ϕ.

Therefore,

- $\mu_r \geq 2$
- $\lambda_n \geq 2$

What if S has a cylinder?

- If $K_S + \sum_{i=1}^{n} \lambda_i C_i$ is pseudo-effective, the log pair $(S, \sum_{i=1}^{n} \lambda_i C_i)$ is not log canonical.

What if S has a cylinder?

Corollary

If a rational surface with pseudo-effective canonical class has only quotient singularities then it cannot contain any cylinders.

Rational surface w/o cylinder

At this stage, many famous rational surfaces enter!

Rational surface w/o cylinder: Kollár

Let S be the hypersurface in $\mathbb{P}(w_1, w_2, w_3, w_4)$ defined by the quasi-homogeneous equation of degree d

$$x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_4 + x_4^{a_4}x_1 = 0.$$

- If $\gcd(w_1, w_2, w_3, w_4) = 1$, then S is a rational surface with 4 cyclic quotient singularities.
- If a_1, a_2, a_3, $a_4 \geq 4$, then K_S is ample.

Rational surface w/o cylinder: D. Hwang, Keum

Hwang and Keum have constructed another types of singular rational surfaces of Picard number one with ample canonical divisors.

An elliptic curve w/ CM

Let E be the the Fermat cubic curve:

$$x^3 + y^3 + z^3 = 0 \subset \mathbb{P}^2.$$

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- Its j-invariant is 0 and it is isomorphic to

$$E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$$

where $\tau = e^{\frac{2}{3}\pi}$.

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- It is the unique elliptic curve admitting an automorphism τ of order 3 such that $\tau^*(\omega) = \tau\omega$, where ω is a non-zero regular 1-form on E.

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- It is the unique elliptic curve admitting an automorphism τ of order 3 such that $\tau^*(\omega) = \tau\omega$, where ω is a non-zero regular 1-form on E.
- The automorphism σ on E has exactly three fixed points P_1, P_2, P_3, respectively, the points corresponding to 0, $\frac{2}{3} + \frac{1}{3}\tau$ and $\frac{1}{3} + \frac{2}{3}\tau$.

Rational surface w/o cylinder: cf. Campana, Oguiso, Truong, Ueno

Let S be the quotient surface

$$E \times E / \langle \mathrm{diag}(-\tau, -\tau) \rangle.$$

- $6K_S$ is linearly trivial.
- Since there is no non-zero regular 1-form on $E \times E$ invariant by $\mathrm{diag}(-\tau, -\tau)$, we obtain $h^1(S, \mathcal{O}_S) = 0$.
- The surface S is a rational log Enriques surface.

Rational surface w/o cylinder: Reid, Oguiso, Zhang

Let $\bar{S}'$ be the quotient surface

$$E \times E / \langle \mathrm{diag}(\tau, \tau^2) \rangle.$$

- The action $\mathrm{diag}(\tau, \tau^2)$ on $E \times E$ has 9 fixed points.
- These 9 fixed points become du Val singular points of type A_2 on $\bar{S}'$.

Rational surface w/o cylinder: Reid, Oguiso, Zhang

Let S' be the minimal resolution of the quotient surface

$$E \times E / \langle \mathrm{diag}(\tau, \tau^2) \rangle.$$

- It is a K3 surface with 24 smooth rational curves.
- Six of them come from the six fixed curves, $\{P_i\} \times E$, $E \times \{P_i\}$ on $E \times E$. The others come from the 9 singular points of type A_2.
- Let g be the automorphism of S' induced by the action $\mathrm{diag}(\tau, 1)$ on $E \times E$. Our 24 smooth rational curves on S' are g-invariant. Among these 24 curves we can find rational tree of type D_{19}.

Rational surface w/o cylinder: Reid, Oguiso, Zhang

Let S' be the minimal resolution of the quotient surface

$$E \times E / \langle \mathrm{diag}(\tau, \tau^2) \rangle.$$

- Let $S' \to \hat{S}$ be the contraction of this tree.
- Then g acts on $\hat{S}$ and it fixes two points.
- The quotient surface $\hat{S}/\langle g \rangle$ is a rational log Enriques surface.

Rational surface w/o cylinder: Oguiso, Zhang, Wang

Rational log Enriques surfaces of ranks 19 and 18 are completely classified by Oguiso, Zhang, Wang .

Some Zoology

Some Zoology

- Elephant

Some Zoology

- Elephant
- Tiger

Some Zoology

- Elephant
- Tiger
- Cat (Tom)

Some Zoology

- Elephant
- Tiger
- Cat (Tom)
- Mouse (Jerry)

Some Zoology

Some Zoology

Definition

Let X be a projective normal variety with at most quotient singularities. A tiger on X is an effective $\mathbb{Q}$-divisor D such that

- $D \equiv -K_X$;
- (X, D) *is not log canonical.*

Rational surface w/o cylinder: Keel, McKernan

Miyanishi conjectured:

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A log del Pezzo surface S of Picard rank 1 has a finite unramified covering of $S \setminus \mathrm{Sing} S$ which contains a cylinder.

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- they have no tigers;
- their smooth loci have trivial algebraic fundamental groups π_1^{alg}.

Rational surface w/o cylinder: Cheltsov, P-, Won

Let S be a Gorenstein log del Pezzo surface of degree 1 with one of the following types of singularities

$$2\mathrm{D}_4, \quad 2\mathrm{A}_3 + 2\mathrm{A}_1, \quad 4\mathrm{A}_2.$$

- In general, if there is a cylinder, then we can construct a tiger D that does not contain the supports of any effective anticanonical divisors.

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- Let D be a tiger on S, i.e., an effective $\mathbb{Q}$-divisor in the anticanonical class of $\mathrm{Pic}(S) \otimes \mathbb{Q}$ such that (S, D) is not log canonical at some point P. Then there is an effective divisor C in $|-K_S|$ such that (S, C) is not log canonical at P and $\mathrm{Supp}(C) \subset \mathrm{Supp}(D)$.

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- The smooth loci of these surfaces are not simply connected (Miyanishi, Zhang).

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- The Gorenstein log del Pezzo surfaces of singularity types $2\mathrm{A}_3 + 2\mathrm{A}_1$ and $4\mathrm{A}_2$ are unique.
- There are infinite series of Gorenstein log del Pezzo surfaces of singularity type $2D_4$.

Polar Cylinder

- Suppose that S has a cylinder U such that

$$S \setminus U = \bigcup_{i=1}^{n} C_i.$$

- Choose arbitrary non-negative rational numbers $\lambda_1, \ldots, \lambda_n$.

$$K_{\tilde{S}} + \sum_{i=1}^{n} \lambda_i \tilde{C}_i + \sum_{i=1}^{r} \mu_i E_i = \pi^* \Big(K_S + \sum_{i=1}^{n} \lambda_i C_i \Big)$$

  for some rational numbers $\mu_1, \ldots, \mu_r$.
- If $K_S + \sum_{i=1}^{n} \lambda_i C_i$ is pseudo-effective, the log pair $(S, \sum_{i=1}^{n} \lambda_i C_i)$ is not log canonical.

Polar Cylinder

Definition

Let H be a $\mathbb{R}$-divisor on S. An H-polar cylinder in S is an Zariski open subset U of S such that

(C) $U = \mathbb{A}^1 \times Z$ *for some affine curve Z, i.e., U is a cylinder in S,*

(P) *there is an effective $\mathbb{R}$-divisor D on S with $D \equiv H$ and $U = S \setminus \mathrm{Supp}(D)$.*

$$H \equiv \sum_{i=1}^{n} \lambda_i C_i$$

for some *positive* real numbers $\lambda_1, \ldots, \lambda_n$.

Polar Cylinder

Let $\mathrm{Amp}(S)$ be the ample cone of S. Denote by $\mathrm{Amp}^c(S)$ the set

$$\{H \in \mathrm{Amp}(S) : \text{ there is an } H\text{-polar cylinder on } S\}.$$

- The set $\mathrm{Amp}^c(S)$ can be empty.
- $\mathrm{Amp}^c(S) \neq \varnothing$ if S is smooth.

Finer Obstruction for $(-K)$-cylinder

- Let S be a Gorenstein del Pezzo surface.
- Suppose that S has a $(-K_S)$-polar cylinder, i.e., there is an effective $\mathbb{Q}$-divisor D on S with $D \sim_{\mathbb{Q}} -K_S$ and $S \setminus \mathrm{Supp}(D)$ is a cylinder.
- The divisor D is a tiger on S.
- There is a tiger D' such that
  - (S, D') is not log canonical at a point P;
  - there is a divisor $T \in |-K_S|$ such that (S, T) is not log canonical at P and $\mathrm{Supp}(T) \not\subset \mathrm{Supp}(D')$.

$(-K)$-Polar Cylinder

Theorem (KPZ; CPW)

Let S_d be a Gorenstein del Pezzo surface of degree $d \leqslant 3$ satisfying the following singularity condition:

- *If $d = 3$, S_d is smooth;*
- *If $d = 2$, S_d allows only ordinary double points;*
- *If $d = 1$, S_d allows types A_1, A_2, A_2, or D_4.*

Let D be a tiger on S_d such that the log pair (S_d, D) is not log canonical at a point P. Then there exists a divisor T in the anticanonical linear system $|-K_{S_d}|$ such that

- *the log pair (S_d, T) is not log canonical at the point P;*
- $\mathrm{Supp}(T) \subset \mathrm{Supp}(D)$.

$(-K)$-Polar Cylinder

Theorem

$-K_{S_d} \in \mathrm{Amp}^c(S_d)$ *if and only if one of the following conditions holds:*

- $d \geqslant 4$,
- $d = 3$ *and* S_d *is singular,*
- $d = 2$ *and* S_d *has a singular point that is not a singular point of type* $\mathbb{A}_1$,
- $d = 1$ *and* S_d *has a singular point that is not a singular point of types* A_1, A_2, A_3, *or* D_4.

$(-K)$-Polar Cylinder

Theorem

Let S_d be a smooth del Pezzo surface of degree d.

- *For $4 \leqslant d \leqslant 9$, one has $\mathrm{Amp}^c(S_d) = \mathrm{Amp}(S_d)$.*
- *For $d = 3$, the set $\mathrm{Amp}^c(S_3)$ is the cone $\mathrm{Amp}(S_3)$ without the ray generated by $-K_{S_3}$.*

Polar Cylinders on smooth del Pezzo surfaces of low degrees

Some partial results on $\mathrm{Amp}^c(S)$ in the case when S is a smooth del Pezzo surface of degree ≤ 2.

Polar Cylinders on smooth del Pezzo surfaces of low degrees

Sir. Peter Swinnerton-Dyer

Polar Cylinders on smooth del Pezzo surfaces of low degrees

"If your research adviser gives you a problem involving del Pezzo surfaces of degree 2 and 1,

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Polar Cylinders on smooth del Pezzo surfaces of low degrees

"If your research adviser gives you a problem involving del Pezzo surfaces of degree 2 and 1, it means he really hates you."

Sir. Peter Swinnerton-Dyer