THE MORIWAKI DIVISOR IS BIG

SALVATORE CACCIOLA, ANGELO FELICE LOPEZ AND FILIPPO VIVIANI

1. The bigness of Moriwaki's divisor

We will use the following

Criterion 1.1. *Let $g \geq 3$ and let $D \equiv a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i\delta_i$ be an $\mathbb{R}$-divisor on $\overline{M}_g$ with $a > 0$. Assume that there exists an effective $\mathbb{R}$-divisor $E \equiv \alpha\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} \beta_i\delta_i$ such that*

$$(A) \qquad \alpha > 0$$

$$(B_i) \qquad \beta_i > 0, \text{ for all } 0 \leq i \leq \lfloor g/2 \rfloor$$

and

$$(C_i) \qquad \alpha b_i < a\beta_i, \text{ for all } 0 \leq i \leq \lfloor g/2 \rfloor.$$

Then D is big.

Proof. We can choose $v \in \mathbb{R}$, $v \geq 0$ such that, for all $0 \leq i \leq \lfloor g/2 \rfloor$, we have

$$\frac{b_i}{\beta_i} \leq v < \frac{a}{\alpha}.$$

Now $D \equiv (a - v\alpha)\lambda + vE + \sum_{i=0}^{\lfloor g/2 \rfloor} (v\beta_i - b_i)\delta_i$ is big since λ is big. □

Lemma 1.2. *Let $g \geq 3$ and let $M = (8g+4)\lambda - g\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} 4i(g-i)\delta_i$ be the Moriwaki divisor on $\overline{M}_g$. Then M is big.*

Proof. We apply Criterion 1.1. As $a = 8g + 4 > 0$, $b_0 = g > 0$ and, for all $1 \leq i \leq \lfloor g/2 \rfloor$, $b_i = 4i(g - i) > 0$, we will need to verify only (A) and all (C_i)'s.

If $g + 1$ is not prime, as in [EH, Theorem 1], we can write $g + 1 = (r + 1)(s - 1)$, for some integers $s \geq 3$ and $r \geq 1$ and we can consider the Brill-Noether divisor D^r_s on $\overline{M}_g$. By [EH, Theorem 1] there exists $c > 0$ such that

$$0 \leq \frac{1}{c}D^r_s \equiv (g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i.$$

Setting $E = \frac{1}{c}D^r_s$, we have that (A) is satisfied. Also (C_0) is equivalent to $g^2 - 3g + 2 > 0$, while, for $i \geq 1$, (C_i) is equivalent to $g - 2 > 0$, so all the (C_i)'s are also satisfied.

Assume from now on that $g+1$ is prime, so that we can write $g = 2(d-1)$, for some $d \geq 3$ and we can consider the Petri divisor E^1_d on $\overline{M}_g$. By [EH, Theorem 2] there exists $c > 0$ such that

$$0 \leq \frac{1}{c} E^1_d = (6d^2 + d - 6)\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} f_i \delta_i$$

where

$$f_0 = d(d-1); \tag{1}$$

$$f_1 = (2d-3)(3d-2); \tag{2}$$

$$f_2 = 3(d-2)(4d-3). \tag{3}$$

Moreover, setting $k = d-1$ and

$$\gamma_i = (i-1)(i-2)\frac{(2k-2)!}{k!(k-1)!} - \sum_{l=1}^{\lfloor \frac{i-2}{2} \rfloor} 2(i-1-2l)\frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k-l+1)!} \tag{4}$$

by [EH, (5.3)] we have

$$f_i = -i(i-2)f_1 + \frac{i(i-1)}{2} f_2 + \frac{\gamma_i}{c} \quad \text{for all } 3 \leq i \leq d-1. \tag{5}$$

Setting $E = \frac{1}{c}E^1_d$ and recalling that $d \geq 3$, we have that (A) is satisfied.

Condition (C_0) is $(6d^2+d-6)g < (8g+4)d(d-1)$, which is equivalent to $2d^2-7d+6 > 0$, whence it is satisfied.

Condition (C_1) is $(6d^2+d-6)4(g-1) < (8g+4)(2d-3)(3d-2)$, which is equivalent to $2d^3 - 9d^2 + 13d - 6 > 0$, whence it is satisfied.

Condition (C_2) is $(6d^2+d-6)8(g-2) < (8g+4)3(d-2)(4d-3)$, which is equivalent to $24d^3 - 124d^2 + 203d - 102 > 0$, whence it is satisfied.

For all $i = 3, \dots, d-1$, condition (C_i) is equivalent to

$$f_i > \frac{(6d^2+d-6)4i(g-i)}{8g+4} = \frac{(6d^2+d-6)i(2d-2-i)}{4d-3}$$

and using (5) can be transformed in

$$-i(i-2)f_1 + \frac{i(i-1)}{2}f_2 + \frac{\gamma_i}{c} > \frac{(6d^2+d-6)i(2d-2-i)}{4d-3}, \; 3 \leq i \leq d-1. \tag{6}$$

To prove (6) we will show that $\gamma_i \geq 0$ for all $i = 3, \dots, d-1$ and

$$-i(i-2)f_1 + \frac{i(i-1)}{2}f_2 > \frac{(6d^2+d-6)i(2d-2-i)}{4d-3}, \; 3 \leq i \leq k. \tag{7}$$

Now (7) is equivalent to

$$i\left((4d-3)(f_2 - 2f_1) + 2(6d^2+d-6)\right) > (4d-3)(f_2 - 4f_1) + 2(6d^2+d-6)(2d-2)$$

and using (2) and (3), to

$$24d^3 - 92d^2 + 109d - 42 > (16d^2 - 47d + 30)i$$

so that, as $i \leq d-1$, we reduce it to $8d^3 - 29d^2 + 32d - 12 > 0$, whence it is satisfied.

It remains to prove that $\gamma_i \geq 0$ for all $i = 3, \dots, k = d-1$.

Note that, by (4), $\gamma_3 = \frac{2(2k-2)!}{k!(k-1)!} > 0$. Hence if we put $c_i = \frac{1}{2}(\gamma_i - \gamma_{i-1})$, we will be done if we show that $c_i \geq 0$ for all $i = 4, \dots, k$. In particular we can suppose $k \geq 4$.

To simplify the notation let

$$b_l = \frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k-l+1)!}$$

so that, by (4), we can write

$$c_i = (i-2)\frac{(2k-2)!}{k!(k-1)!} - \sum_{l=1}^{\lfloor \frac{i-2}{2} \rfloor} b_l.$$

As $c_4 = \frac{(2k-4)!}{k!(k-1)!}(2(2k-2)(2k-3)-1) \geq 0$, setting $d_i = c_i - c_{i-1}$, we are reduced to prove that $d_i \geq 0$ for all $i = 5, \dots, k$.

If i is odd, then $d_i = \frac{(2k-2)!}{k!(k-1)!} \geq 0$, so that we can assume that i is even. In particular we can put $i = 2h+2$, where $2 \leq h \leq \lfloor \frac{k-2}{2} \rfloor$, and we get

$$d_i = \frac{(2k-2)!}{k!(k-1)!} - \frac{(2h)!(2k-2-2h)!}{(h+1)!h!(k-h)!(k-h+1)!}.$$

In this way, after putting $v_h = \frac{(2h)!(2k-2-2h)!}{(h+1)!h!(k-h)!(k-h+1)!}$, we need to prove that

$$\frac{(2k-2)!}{k!(k-1)!} \geq v_h \tag{8}$$

for all $h \in \{2, \dots, \lfloor \frac{k-2}{2} \rfloor\}$ and for all $k \geq 6$.

We now claim that, for $2 \leq h \leq \lfloor \frac{k-2}{2} \rfloor$, we have $v_h \leq \max\{v_2, v_{\lfloor \frac{k-2}{2} \rfloor}\}$.

In fact, for all $h \geq 3$, we can write $v_h - v_{h-1} = C_{k,h}N_{k,h}$, where

$$C_{k,h} = \frac{2(2h-2)!(2k-2h-2)!}{h!(h-1)!(k-h+1)!(k-h)!(h+1)(k-h+2)(k-h+1)} \geq 0$$

for all $3 \leq h \leq \lfloor \frac{k-2}{2} \rfloor, k \geq 6$, and

$$\begin{aligned} N_{k,h} &= (2h-1)(k-h+2)(k-h+1) - (k-h)(2k-2h-1)(h+1) \\ &= -3k^2 + 13kh - 2k - 10h^2 + 6h - 2. \end{aligned}$$

In particular $v_h \leq v_{h-1}$ if and only if $N_{k,h} \leq 0$, if and only if $h \leq k_1 := \frac{13k+6-\sqrt{49k^2+76k-44}}{20}$ or $h \geq k_2 := \frac{13k+6+\sqrt{49k^2+76k-44}}{20}$. Thus the claim follows by noticing that, for all $k \geq 6$, we have $k_2 > \lfloor \frac{k-2}{2} \rfloor$.

Thanks to the claim it suffices to prove that (8) holds for $h = 2$ and $h = \lfloor \frac{k-2}{2} \rfloor$. Since $v_2 = \frac{2(2k-6)!}{(k-2)!(k-1)!}$, we have that (8) holds for $h = 2$.

Suppose $h = \lfloor \frac{k-2}{2} \rfloor$. If k is even, then $k = 2h+2$, so that (8) is equivalent to

$$\frac{(4h+2)!}{(2h+2)!(2h+1)!} \geq \frac{(2h)!(2h+2)!}{(h+1)!h!(h+2)!(h+3)!}$$

which in turn is verified if and only if

$$a_h := \frac{(4h+2)!h!(h+1)!(h+2)!(h+3)!}{((2h+2)!)^2(2h+1)!(2h)!} \geq 1$$

for all $h \geq 2$. But $a_2 = \frac{10!3!2!}{(6!)^2} \geq 1$, and, for all $h \geq 3$, we have

$$a_h - a_{h-1} = S_h(T_h - 1)$$

where

$$S_h = \frac{(4h-2)!(h-1)!h!(h+1)!(h+2)!}{((2h)!)^2(2h-1)!(2h-2)!} \geq 0$$

for all $h \geq 2$, and

$$T_h = \frac{(4h+1)(4h-1)(h+2)(h+3)}{(2h+1)^2(2h+1)(2h+2)}.$$

An easy computation gives that $T_h \geq 1$ if and only if $56h^3 + 91h^2 + h - 4 \geq 0$, which, in particular, is true for all $h \geq 2$. Thus, for all $h \geq 2$, $a_h \geq a_2 \geq 1$.

If k is odd, then $k = 2h + 3$, and (8) is verified if and only if

$$a'_h := \frac{(4h+4)!h!(h+1)!(h+3)!(h+4)!}{(2h+3)!(2h+2)!(2h)!(2h+4)!} \geq 1$$

for all $h \geq 2$. Again $a'_2 = \frac{11\cdot 10\cdot 9}{7} \geq 1$, and

$$a'_h - a'_{h-1} = S'_h(T'_h - 1)$$

where

$$S'_h = \frac{(4h)!(h-1)!h!(h+2)!(h+3)!}{(2h+1)!(2h)!(2h-2)(2h+2)!} \geq 0$$

for all $h \geq 2$, and

$$T'_h = \frac{(4h+3)(4h+1)(h+3)(h+4)}{2(2h+3)^2(2h-1)(h+2)}$$

so that $T'_h \geq 1$ if and only if $56h^3 + 215h^2 + 207h + 72 \geq 0$, which, in particular, is true for all $h \geq 2$, and we conclude as before. □

References

[EH] D. Eisenbud, J. Harris. *The Kodaira dimension of the moduli space of curves of genus* ≥ 23. Invent. Math. **90**, (1987) 359-387.