THE CASE $r = 2$, $n = 4$

We attach the Mathematica prints that we use in Sect. 6 and in the Appendix. Here is an itemized list to guide the reader:

(1) Out[3]–Out[14] computes the expression $f_{4,2,0}(x_1, x_2, x_3, x_4)$ that appears in the proof of Lemma A.7.
(2) Out[15]–Out[24] provides the expression of $f_{s,2,0}$ given in Lemma A.7(1).
(3) Out[30] verifies (6.3).
(4) Out[34] verifies the expression of $g_{4,s}$ appearing in (A.9).
(5) Out[36] verifies Lemma A.8(1).
(6) Out[39] verifies Lemma A.9(1).

```
(*We calculate (360/(abcd))f_{4,2,0} in variables
{a,b,c,d}. This is symmetric in {a,b,c,d}*)
```

```
In[1]:= FunctionExpand[1 - 2 * a * b * c * d * Binomial[a + b + c + d - 5, 4] + Binomial[a + b + c + d - 5, 8] -
  Binomial[a - 1, 8] - Binomial[b - 1, 8] - Binomial[c - 1, 8] - Binomial[d - 1, 8] - Binomial[
  a + a + b + c + d - 5, 8] - Binomial[b + a + b + c + d - 5, 8] - Binomial[c + a + b + c + d - 5, 8] -
  Binomial[d + a + b + c + d - 5, 8] + Binomial[a + b - 1, 8] + Binomial[a + c - 1, 8] + Binomial[a +
  d - 1, 8] + Binomial[b + c - 1, 8] + Binomial[b + d - 1, 8] + Binomial[c + d - 1, 8] + Binomial[
  a + b + a + b + c + d - 5, 8] + Binomial[a + c + a + b + c + d - 5, 8] + Binomial[a + d + a + b + c + d -
  5, 8] + Binomial[b + c + a + b + c + d - 5, 8] + Binomial[b + d + a + b + c + d - 5, 8] + Binomial[c +
  d + a + b + c + d - 5, 8] - Binomial[a + b + c - 1, 8] - Binomial[a + b + d - 1, 8] - Binomial[a +
  c + d - 1, 8] - Binomial[b + c + d - 1, 8] - Binomial[a + b + c + a + b + c + d - 5, 8] - Binomial[
  a + b + d + a + b + c + d - 5, 8] - Binomial[b + c + d + a + b + c + d - 5, 8] - Binomial[a + c + d +
  a + b + c + d - 5, 8] + Binomial[a + b + c + d - 1, 8] + Binomial[a + b + c + d + a + b + c + d - 5, 8]]
```

Out[1]=

$$1 - \frac{(-8+a)(-7+a)(-6+a)(-5+a)(-4+a)(-3+a)(-2+a)(-1+a)}{40320} - \frac{(-8+b)(-7+b)(-6+b)(\cdots 1 \cdots)\,\cdots 1 \cdots\,(-3+b)(-2+b)(-1+b)}{40320} + \cdots 39 \cdots +$$

$$\frac{\cdots 1 \cdots}{40320} - \frac{(\cdots 1 \cdots)\,\cdots 6 \cdots\,(\cdots 1 \cdots)}{40320} - \frac{(-12+a+2b+2c+2d)(-11+a+2b+2c+2d)\,\cdots 4 \cdots\,(-6+a+2b+2c+2d)(-5+a+2b+2c+2d)}{40320} +$$

$$\frac{(-6+a+b+c+d)(-5+a+b+c+d)(-4+a+b+c+d)(-3+a+b+c+d)(-11+2a+2b+2c+2d)(-9+2a+2b+2c+2d)(-7+2a+2b+2c+2d)(-5+2a+2b+2c+2d)}{2520}$$

Full expression not available (original memory size: 45.4 kB)

```
In[2]:= Expand[360 / (a * b * c * d) * %1]
```

Out[2]=

$$27861 - 24000 a + 8070 a^2 - 1200 a^3 + 66 a^4 - 24000 b + 14775 a b - 3150 a^2 b + 225 a^3 b + 8070 b^2 - 3150 a b^2 + 320 a^2 b^2 - 1200 b^3 + 225 a b^3 + 66 b^4 - 24000 c + 14775 a c - 3150 a^2 c + 225 a^3 c + 14775 b c - 5850 a b c + 600 a^2 b c - 3150 b^2 c + 600 a b^2 c + 225 b^3 c + 8070 c^2 - 3150 a c^2 + 320 a^2 c^2 - 3150 b c^2 + 600 a b c^2 + 320 b^2 c^2 - 1200 c^3 + 225 a c^3 + 225 b c^3 + 66 c^4 - 24000 d + 14775 a d - 3150 a^2 d + 225 a^3 d + 14775 b d - 5850 a b d + 600 a^2 b d - 3150 b^2 d + 600 a b^2 d + 225 b^3 d + 14775 c d - 5850 a c d + 600 a^2 c d - 5850 b c d + 1125 a b c d + 600 b^2 c d - 3150 c^2 d + 600 a c^2 d + 600 b c^2 d + 225 c^3 d + 8070 d^2 - 3150 a d^2 + 320 a^2 d^2 - 3150 b d^2 + 600 a b d^2 + 320 b^2 d^2 - 3150 c d^2 + 600 a c d^2 + 600 b c d^2 + 320 c^2 d^2 - 1200 d^3 + 225 a d^3 + 225 b d^3 + 225 c d^3 + 66 d^4$$

```
(*We calculate all the coefficients of the monomial symmetric
polynomials in {a,b,c,d} appearing in (360/(abcd))f_{4,2,0}*)
```

```
In[3]:= a1 = SeriesCoefficient[%2, {a, 0, 4}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]

Out[3]= 66

In[4]:= a2 = SeriesCoefficient[%2, {a, 0, 3}, {b, 0, 1}, {c, 0, 0}, {d, 0, 0}]

Out[4]= 225

In[5]:= a3 = SeriesCoefficient[%2, {a, 0, 2}, {b, 0, 2}, {c, 0, 0}, {d, 0, 0}]

Out[5]= 320
```

In[6]:=
```
a4 = SeriesCoefficient[%2, {a, 0, 2}, {b, 0, 1}, {c, 0, 1}, {d, 0, 0}]
```
Out[6]=
```
600
```
In[7]:=
```
a5 = SeriesCoefficient[%2, {a, 0, 1}, {b, 0, 1}, {c, 0, 1}, {d, 0, 1}]
```
Out[7]=
```
1125
```
In[8]:=
```
l6 = SeriesCoefficient[%2, {a, 0, 3}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```
Out[8]=
```
-1200
```
In[9]:=
```
l7 = SeriesCoefficient[%2, {a, 0, 2}, {b, 0, 1}, {c, 0, 0}, {d, 0, 0}]
```
Out[9]=
```
-3150
```
In[10]:=
```
l8 = SeriesCoefficient[%2, {a, 0, 1}, {b, 0, 1}, {c, 0, 1}, {d, 0, 0}]
```
Out[10]=
```
-5850
```
In[11]:=
```
l9 = SeriesCoefficient[%2, {a, 0, 2}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```
Out[11]=
```
8070
```
In[12]:=
```
l10 = SeriesCoefficient[%2, {a, 0, 1}, {b, 0, 1}, {c, 0, 0}, {d, 0, 0}]
```
Out[12]=
```
14 775
```
In[13]:=
```
l11 = SeriesCoefficient[%2, {a, 0, 1}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```
Out[13]=
```
-24 000
```
In[14]:=
```
l12 = SeriesCoefficient[%2, {a, 0, 0}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```
Out[14]=
```
27 861
```
```
(*We calculate all the coefficients of the monomial symmetric polynomials
in {x_1,...,x_s} appearing in (360/(product x_i))f_{s,2,0} using Lemma A.6*)
```
In[15]:=
```
a6 = Expand[l6 - (s - 4) * (a2)]
```
Out[15]=
```
-300 - 225 s
```
In[16]:=
```
a7 = Expand[l7 - (s - 4) * (a4)]
```
Out[16]=
```
-750 - 600 s
```
In[17]:=
```
a8 = Expand[l8 - (s - 4) * (a5)]
```
Out[17]=
```
-1350 - 1125 s
```

In[18]:=
```
a9 = Expand[l9 - (s - 4) * (a3 + a7) - Binomial[s - 4, 2] * (a4)]
```
Out[18]=

$$350 + 730\, s + 300\, s^2$$

In[19]:=
```
Expand[l10 - (s - 4) * (a4 + a8) - Binomial[s - 4, 2] * (a5)]
```
Out[19]=

$$525 + \frac{2625\, s}{2} + \frac{1125\, s^2}{2}$$

In[20]:=
```
a10 = Factor[%19]
```
Out[20]=

$$\frac{75}{2}\left(14 + 35\, s + 15\, s^2\right)$$

In[21]:=
```
Expand[l11 - (s - 4) * (a2 + a7 + a10) - Binomial[s - 4, 2] * (2 * a4 + a8) - Binomial[s - 4, 3] * (a5)]
```
Out[21]=

$$-450\, s - \frac{1275\, s^2}{2} - \frac{375\, s^3}{2}$$

In[22]:=
```
a11 = Factor[%21]
```
Out[22]=

$$-\frac{75}{2}\, s\, (1 + s)\left(12 + 5\, s\right)$$

In[23]:=
```
Expand[l12 - (s - 4) * (a1 + a6 + a9 + a11) - Binomial[s - 4, 2] * (2 * a2 +
  a3 + 2 * a7 + a10) - Binomial[s - 4, 3] * (3 * a4 + a8) - Binomial[s - 4, 4] * (a5)]
```
Out[23]=

$$-\frac{349\, s}{4} + \frac{1505\, s^2}{8} + \frac{825\, s^3}{4} + \frac{375\, s^4}{8}$$

In[24]:=
```
a12 = Factor[%23]
```
Out[24]=

$$\frac{1}{8}\, s\left(-698 + 1505\, s + 1650\, s^2 + 375\, s^3\right)$$

```
(*Defining the functions computing (1/(product x_i))f_{s,2,0}*)
```

In[25]:=
```
Fs20 = (1 / 360) * (a1 * m4 + a2 * m31 + a3 * m22 + a4 * m211 + a5 *
  m1111 + a6 * m3 + a7 * m21 + a8 * m111 + a9 * m2 + a10 * m11 + a11 * m1 + a12)
```
Out[25]=

$$\frac{1}{360}\Big(1125\, m1111 + 600\, m211 + 320\, m22 + 225\, m31 + 66\, m4 + m111\left(-1350 - 1125\, s\right) + m21\left(-750 - 600\, s\right) + m3\left(-300 - 225\, s\right) - \frac{75}{2}\, m1\, s\, (1 + s)\left(12 + 5\, s\right) + \frac{75}{2}\, m11\left(14 + 35\, s + 15\, s^2\right) + m2\left(350 + 730\, s + 300\, s^2\right) + \frac{1}{8}\, s\left(-698 + 1505\, s + 1650\, s^2 + 375\, s^3\right)\Big)$$

```
(*From now on, X is in P^{n+s}, c.i. of type (d_1,...,
d_s), E Ulrich for (X,O_X(1)), and d=deg(X)=d_1d_2...d_s*)
```

```
(*We compute the polynomial calculating (24/(rd))deg(Z) using Lemma 3.1(viii)*)
```

In[26]:=

```
Expand[(1 / 2) * ((r / 2) * (m1 - s)) ^ 2 * d - (1 / 2) * ((r / 2) * (m1 - s)) * (m1 - s - n - 1) * d + (r / 12) * (m1 - s - n - 1) ^
  2 * d + (r * d / 12) * (Binomial[n + s + 1, 2] + m1 * (m1 - s - n - 1) - m11) - (r * d / 24) * (3 * n ^ 2 + 5 * n + 2)]
```

Out[26]=

$$-\frac{1}{12} d\, m1^2\, r - \frac{d\, m11\, r}{12} + \frac{1}{8} d\, m1^2\, r^2 - \frac{d\, r\, s}{24} + \frac{1}{4} d\, m1\, r\, s - \frac{1}{4} d\, m1\, r^2\, s - \frac{1}{8} d\, r\, s^2 + \frac{1}{8} d\, r^2\, s^2$$

```
(*Specializing the above when n=4,r=2*)
```

In[27]:=

```
%26 /. {r → 2, n → 4}
```

Out[27]=

$$\frac{d\, m1^2}{3} - \frac{d\, m11}{6} - \frac{d\, s}{12} - \frac{d\, m1\, s}{2} + \frac{d\, s^2}{4}$$

In[28]:=

```
d1 = Expand[(12 / d) * %27]
```

Out[28]=

$$4\, m1^2 - 2\, m11 - s - 6\, m1\, s + 3\, s^2$$

```
(*We compute the polynomial calculating (12c_2(Z))/deg(Z) using Lemma 3.2(vii)*)
```

In[29]:=

```
FunctionExpand[Binomial[s + 5, 2] + m1 * (m1 - s - 5) - m11 - (1 / 12) * d1 + (2 * m1 - 2 * s - 5) * (m1 - s)]
```

Out[29]=

$$-m11 + m1\,(-5 + m1 - s) + (-5 + 2\, m1 - 2\, s)\,(m1 - s) + \frac{1}{2}\,(4 + s)\,(5 + s) + \frac{1}{12}\left(-4\, m1^2 + 2\, m11 + s + 6\, m1\, s - 3\, s^2\right)$$

In[30]:=

```
d2 = Expand[12 * %29]
```

Out[30]=

$$120 - 120\, m1 + 32\, m1^2 - 10\, m11 + 115\, s - 54\, m1\, s + 27\, s^2$$

```
(*We compute the polynomial calculating (144/d)c_2(Z)*)
```

In[31]:=

```
p1 = Expand[d2 * d1]
```

Out[31]=

$$480\, m1^2 - 480\, m1^3 + 128\, m1^4 - 240\, m11 + 240\, m1\, m11 - 104\, m1^2\, m11 + 20\, m11^2 - 120\, s - 600\, m1\, s + 1148\, m1^2\, s - 408\, m1^3\, s - 220\, m11\, s + 168\, m1\, m11\, s + 245\, s^2 - 996\, m1\, s^2 + 528\, m1^2\, s^2 - 84\, m11\, s^2 + 318\, s^3 - 324\, m1\, s^3 + 81\, s^4$$

```
(*We compute the polynomial calculating K_Z^2/deg(Z)*)
```

In[32]:=

```
d3 = Expand[(2 m1 - 2 s - 5) ^ 2]
```

Out[32]=

$$25 - 20\, m1 + 4\, m1^2 + 20\, s - 8\, m1\, s + 4\, s^2$$

```
(*We compute the polynomial calculating (12/d)K_Z^2*)
```

In[33]:=
```
p2 = Expand[d3 * d1]
```

Out[33]=

$100\, m1^2 - 80\, m1^3 + 16\, m1^4 - 50\, m11 + 40\, m1\, m11 - 8\, m1^2\, m11 - 25\, s - 130\, m1\, s + 196\, m1^2\, s - 56\, m1^3\, s - 40\, m11\, s + 16\, m1\, m11\, s + 55\, s^2 - 172\, m1\, s^2 + 76\, m1^2\, s^2 - 8\, m11\, s^2 + 56\, s^3 - 48\, m1\, s^3 + 12\, s^4$

```
(*We compute the polynomial calculating (144/(5d))
(K_Z^2+c_2(Z)), i.e. (1728/(5(product x_i)))g_{4,s}*)
```

In[34]:=
```
f1 = Expand[(1 / 5) * (12 * p2 + p1)]
```

Out[34]=

$336\, m1^2 - 288\, m1^3 + 64\, m1^4 - 168\, m11 + 144\, m1\, m11 - 40\, m1^2\, m11 + 4\, m11^2 - 84\, s - 432\, m1\, s + 700\, m1^2\, s - 216\, m1^3\, s - 140\, m11\, s + 72\, m1\, m11\, s + 181\, s^2 - 612\, m1\, s^2 + 288\, m1^2\, s^2 - 36\, m11\, s^2 + 198\, s^3 - 180\, m1\, s^3 + 45\, s^4$

```
(*Linearize*)
```

In[35]:=
```
f1 /. {(m1)^2 → m2 + 2 * m11, (m1)^3 → m3 + 3 * m21 + 6 * m111, (m1)^4 → m4 + 4 * m31 +
  6 * m22 + 12 * m211 + 24 * m1111, m1 * m11 → m21 + 3 * m111, (m1)^2 * m11 → m31 + 2 * m22 +
  5 * m211 + 12 * m1111, (m11)^2 → m22 + 2 * m211 + 6 * m1111, m1 * m3 → m4 + m31, m1 *
  m21 → m31 + 2 * m22 + 2 * m211, m1 * m111 → m211 + 4 * m1111, m1 * m2 → m3 + m21, (m1)^
  2 * m2 → m4 + 2 * m31 + 2 * m22 + 2 * m211, (m2)^2 → m4 + 2 * m22, m2 * m11 → m31 + m211}
```

Out[35]=

$-168\, m11 + 336\,(2\, m11 + m2) + 144\,(3\, m111 + m21) + 4\,(6\, m1111 + 2\, m211 + m22) - 288\,(6\, m111 + 3\, m21 + m3) - 40\,(12\, m1111 + 5\, m211 + 2\, m22 + m31) + 64\,(24\, m1111 + 12\, m211 + 6\, m22 + 4\, m31 + m4) - 84\, s - 432\, m1\, s - 140\, m11\, s + 700\,(2\, m11 + m2)\, s + 72\,(3\, m111 + m21)\, s - 216\,(6\, m111 + 3\, m21 + m3)\, s + 181\, s^2 - 612\, m1\, s^2 - 36\, m11\, s^2 + 288\,(2\, m11 + m2)\, s^2 + 198\, s^3 - 180\, m1\, s^3 + 45\, s^4$

In[36]:=
```
Expand[%35]
```

Out[36]=

$504\, m11 - 1296\, m111 + 1080\, m1111 + 336\, m2 - 720\, m21 + 576\, m211 + 308\, m22 - 288\, m3 + 216\, m31 + 64\, m4 - 84\, s - 432\, m1\, s + 1260\, m11\, s - 1080\, m111\, s + 700\, m2\, s - 576\, m21\, s - 216\, m3\, s + 181\, s^2 - 612\, m1\, s^2 + 540\, m11\, s^2 + 288\, m2\, s^2 + 198\, s^3 - 180\, m1\, s^3 + 45\, s^4$

```
(* We compute (1/(product x_i))g_{4,s}*)
```

In[37]:=
```
G4s = Expand[(5 / 1728) * %36]
```

Out[37]=

$$\frac{35\, m11}{24} - \frac{15\, m111}{4} + \frac{25\, m1111}{8} + \frac{35\, m2}{36} - \frac{25\, m21}{12} + \frac{5\, m211}{3} + \frac{385\, m22}{432} - \frac{5\, m3}{6} + \frac{5\, m31}{8} + \frac{5\, m4}{27} - \frac{35\, s}{144} - \frac{5\, m1\, s}{4} + \frac{175\, m11\, s}{48} - \frac{25\, m111\, s}{8} + \frac{875\, m2\, s}{432} - \frac{5\, m21\, s}{3} - \frac{5\, m3\, s}{8} + \frac{905\, s^2}{1728} - \frac{85\, m1\, s^2}{48} + \frac{25\, m11\, s^2}{16} + \frac{5\, m2\, s^2}{6} + \frac{55\, s^3}{96} - \frac{25\, m1\, s^3}{48} + \frac{25\, s^4}{192}$$

```
(*The main relation*)
```

In[38]:= Expand[G4s - Fs20]

Out[38]=

$$\frac{m22}{432}+\frac{m4}{540}-\frac{s}{1440}-\frac{m2\,s}{432}+\frac{s^2}{864}$$

In[39]:= Factor[%38]

Out[39]=

$$\frac{10\,m22+8\,m4-3\,s-10\,m2\,s+5\,s^2}{4320}$$

THE CASE $r = 3$, $n = 4$

We attach the Mathematica prints that we use in Sect. 6 and in the Appendix. Here is an itemized list to guide the reader:

(1) Out[3]–Out[14] computes the expression $f_{4,3,0}(x_1, x_2, x_3, x_4)$ that appears in the proof of Lemma A.7.
(2) Out[15]–Out[21] provides the expression of $f_{s,3,0}$ given in Lemma A.7(2).
(3) Out[24]–Out[35] computes the expression $f_{4,3,1}(x_1, x_2, x_3, x_4)$ that appears in the proof of Lemma A.7.
(4) Out[36]–Out[42] provides the expression of $f_{s,3,1}$ given in Lemma A.7(3).
(5) Out[47] verifies the expression of δ_s appearing in (A.9).
(6) Out[49] verifies Lemma A.8(2).
(7) Out[50] verifies Lemma A.8(3).
(8) Out[59] verifies Lemma A.8(4).
(9) Out[70] verifies Lemma A.8(5).
(10) Out[72] verifies Lemma A.8(6).
(11) Out[74] verifies Lemma A.9(2).

```
(*We calculate (1920/(abcd))f_{4,3,0} in variables
{a,b,c,d}. This is symmetric in {a,b,c,d}*)
```

In[1]:=
```
FunctionExpand[1 - 3 * a * b * c * d * Binomial[3 * (a + b + c + d - 4) / 2 - 1, 4] + 2 * Binomial[3 * (a + b +
  c + d - 4) / 2 - 1, 8] - Binomial[a - 1, 8] - Binomial[b - 1, 8] - Binomial[c - 1, 8] - Binomial[d -
  1, 8] - 2 * Binomial[a - 1 + 3 * (a + b + c + d - 4) / 2, 8] - 2 * Binomial[b - 1 + 3 * (a + b + c + d - 4) / 2,
  8] - 2 * Binomial[c - 1 + 3 * (a + b + c + d - 4) / 2, 8] - 2 * Binomial[d - 1 + 3 * (a + b + c + d - 4) / 2,
  8] + Binomial[a + b - 1, 8] + Binomial[a + c - 1, 8] + Binomial[a + d - 1, 8] + Binomial[b + c - 1,
  8] + Binomial[b + d - 1, 8] + Binomial[c + d - 1, 8] + 2 * Binomial[a + b - 1 + 3 * (a + b + c + d - 4) /
  2, 8] + 2 * Binomial[a + c - 1 + 3 * (a + b + c + d - 4) / 2, 8] + 2 * Binomial[a + d - 1 + 3 * (a + b + c + d -
  4) / 2, 8] + 2 * Binomial[b + c - 1 + 3 * (a + b + c + d - 4) / 2, 8] + 2 * Binomial[b + d - 1 + 3 * (a + b +
  c + d - 4) / 2, 8] + 2 * Binomial[c + d - 1 + 3 * (a + b + c + d - 4) / 2, 8] - Binomial[a + b + c - 1, 8] -
  Binomial[a + b + d - 1, 8] - Binomial[a + c + d - 1, 8] - Binomial[b + c + d - 1, 8] - 2 * Binomial[
  a + b + c - 1 + 3 * (a + b + c + d - 4) / 2, 8] - 2 * Binomial[a + b + d - 1 + 3 * (a + b + c + d - 4) / 2, 8] - 2 *
  Binomial[b + c + d - 1 + 3 * (a + b + c + d - 4) / 2, 8] - 2 * Binomial[a + c + d - 1 + 3 * (a + b + c + d -
  4) / 2, 8] + Binomial[a + b + c + d - 1, 8] + 2 * Binomial[a + b + c + d - 1 + 3 * (a + b + c + d - 4) / 2, 8]]
```

Out[1]=

$$1 - \frac{(-8+a)(-7+a)(-6+a)(-5+a)(-4+a)(-3+a)(-2+a)(-1+a)}{40320} - \frac{(-8+b)(-7+b)(-6+b)(-5+b)(-4+b)(-3+b)(-2+b)(-1+b)}{40320} +$$

$$\cdots 42 \cdots + \frac{(-4+a+b+c+d)\,(-28+5a+5b+5c+5d)\,(-26+5a+5b+5c+5d) \cdots 3 \cdots (-16+5a+5b+5c+5d)\,(-14+5a+5b+5c+5d)}{1032192} -$$

$$\frac{1}{8}\,a\,b\,c\,d\left(-4+\frac{3}{2}(-4+a+b+c+d)\right)\left(-3+\frac{3}{2}(-4+a+b+c+d)\right)\left(-2+\frac{3}{2}(-4+a+b+c+d)\right)\left(-1+\frac{3}{2}(-4+a+b+c+d)\right)$$

Full expression not available (original memory size: 64.8 kB)

In[2]:=
```
Expand[1920 / (a * b * c * d) * %1]
```

Out[2]=

$$681768 - 595440\,a + 199620\,a^2 - 29940\,a^3 + 1683\,a^4 - 595440\,b + 380160\,a\,b - 82740\,a^2\,b + 6060\,a^3\,b + 199620\,b^2 - 82740\,a\,b^2 + 8770\,a^2\,b^2 - 29940\,b^3 + 6060\,a\,b^3 + 1683\,b^4 - 595440\,c + 380160\,a\,c - 82740\,a^2\,c + 6060\,a^3\,c + 380160\,b\,c - 158400\,a\,b\,c + 16860\,a^2\,b\,c - 82740\,b^2\,c + 16860\,a\,b^2\,c + 6060\,b^3\,c + 199620\,c^2 - 82740\,a\,c^2 + 8770\,a^2\,c^2 - 82740\,b\,c^2 + 16860\,a\,b\,c^2 + 8770\,b^2\,c^2 - 29940\,c^3 + 6060\,a\,c^3 + 6060\,b\,c^3 + 1683\,c^4 - 595440\,d + 380160\,a\,d - 82740\,a^2\,d + 6060\,a^3\,d + 380160\,b\,d - 158400\,a\,b\,d + 16860\,a^2\,b\,d - 82740\,b^2\,d + 16860\,a\,b^2\,d + 6060\,b^3\,d + 380160\,c\,d - 158400\,a\,c\,d + 16860\,a^2\,c\,d - 158400\,b\,c\,d + 32400\,a\,b\,c\,d + 16860\,b^2\,c\,d - 82740\,c^2\,d + 16860\,a\,c^2\,d + 16860\,b\,c^2\,d + 6060\,c^3\,d + 199620\,d^2 - 82740\,a\,d^2 + 8770\,a^2\,d^2 - 82740\,b\,d^2 + 16860\,a\,b\,d^2 + 8770\,b^2\,d^2 - 82740\,c\,d^2 + 16860\,a\,c\,d^2 + 16860\,b\,c\,d^2 + 8770\,c^2\,d^2 - 29940\,d^3 + 6060\,a\,d^3 + 6060\,b\,d^3 + 6060\,c\,d^3 + 1683\,d^4$$

```
(*We calculate all the coefficients of the monomial symmetric
polynomials in {a,b,c,d} appearing in (1920/(abcd))f_{4,3,0}*)
```

In[3]:=
```
a1 = SeriesCoefficient[%2, {a, 0, 4}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```

Out[3]= 1683

In[4]:=
```
a2 = SeriesCoefficient[%2, {a, 0, 3}, {b, 0, 1}, {c, 0, 0}, {d, 0, 0}]
```

Out[4]= 6060

In[5]:=
```
a3 = SeriesCoefficient[%2, {a, 0, 2}, {b, 0, 2}, {c, 0, 0}, {d, 0, 0}]
```
Out[5]= 8770

In[6]:=
```
a4 = SeriesCoefficient[%2, {a, 0, 2}, {b, 0, 1}, {c, 0, 1}, {d, 0, 0}]
```
Out[6]= 16 860

In[7]:=
```
a5 = SeriesCoefficient[%2, {a, 0, 1}, {b, 0, 1}, {c, 0, 1}, {d, 0, 1}]
```
Out[7]= 32 400

In[8]:=
```
l6 = SeriesCoefficient[%2, {a, 0, 3}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```
Out[8]= -29 940

In[9]:=
```
l7 = SeriesCoefficient[%2, {a, 0, 2}, {b, 0, 1}, {c, 0, 0}, {d, 0, 0}]
```
Out[9]= -82 740

In[10]:=
```
l8 = SeriesCoefficient[%2, {a, 0, 1}, {b, 0, 1}, {c, 0, 1}, {d, 0, 0}]
```
Out[10]=

-158 400

In[11]:=
```
l9 = SeriesCoefficient[%2, {a, 0, 2}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```
Out[11]=

199 620

In[12]:=
```
l10 = SeriesCoefficient[%2, {a, 0, 1}, {b, 0, 1}, {c, 0, 0}, {d, 0, 0}]
```
Out[12]=

380 160

In[13]:=
```
l11 = SeriesCoefficient[%2, {a, 0, 1}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```
Out[13]=

-595 440

In[14]:=
```
l12 = SeriesCoefficient[%2, {a, 0, 0}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```
Out[14]=

681 768

```
(*We calculate all the coefficients of the monomial symmetric polynomials
in {x_1,...,x_s} appearing in (1920/(product x_i))f_{s,3,0} using Lemma A.6*)
```

In[15]:=
```
a6 = Expand[l6 - (s - 4) * (a2)]
```
Out[15]=

-5700 - 6060 s

In[16]:=
```
a7 = Expand[l7 - (s - 4) * (a4)]
```
Out[16]=

-15 300 - 16 860 s

In[17]:=
```
a8 = Expand[l8 - (s - 4) * (a5)]
```
Out[17]=

$-28\,800 - 32\,400\, s$

In[18]:=
```
a9 = Expand[l9 - (s - 4) * (a3 + a7) - Binomial[s - 4, 2] * (a4)]
```
Out[18]=

$4900 + 14\,960\, s + 8430\, s^2$

In[19]:=
```
a10 = Expand[l10 - (s - 4) * (a4 + a8) - Binomial[s - 4, 2] * (a5)]
```
Out[19]=

$8400 + 28\,140\, s + 16\,200\, s^2$

In[20]:=
```
a11 = Expand[l11 - (s - 4) * (a2 + a7 + a10) - Binomial[s - 4, 2] * (2 * a4 + a8) - Binomial[s - 4, 3] * (a5)]
```
Out[20]=

$-7500\, s - 13\,740\, s^2 - 5400\, s^3$

In[21]:=
```
a12 = Expand[l12 - (s - 4) * (a1 + a6 + a9 + a11) - Binomial[s - 4, 2] * (2 *
  a2 + a3 + 2 * a7 + a10) - Binomial[s - 4, 3] * (3 * a4 + a8) - Binomial[s - 4, 4] * (a5)]
```
Out[21]=

$-698\, s + 3305\, s^2 + 4470\, s^3 + 1350\, s^4$

```
(*We calculate (1920/(abcd))f_{4,3,1} in variables
{a,b,c,d}. This is symmetric in {a,b,c,d}*)
```

In[22]:=
```
FunctionExpand[9 - 3 * a * b * c * d * Binomial[3 * (a + b + c + d - 4) / 2 - 2, 4] + 2 * Binomial[3 * (a + b +
  c + d - 4) / 2 - 2, 8] - Binomial[a - 2, 8] - Binomial[b - 2, 8] - Binomial[c - 2, 8] - Binomial[d -
  2, 8] - 2 * Binomial[a - 2 + 3 * (a + b + c + d - 4) / 2, 8] - 2 * Binomial[b - 2 + 3 * (a + b + c + d - 4) / 2,
  8] - 2 * Binomial[c - 2 + 3 * (a + b + c + d - 4) / 2, 8] - 2 * Binomial[d - 2 + 3 * (a + b + c + d - 4) / 2,
  8] + Binomial[a + b - 2, 8] + Binomial[a + c - 2, 8] + Binomial[a + d - 2, 8] + Binomial[b + c - 2,
  8] + Binomial[b + d - 2, 8] + Binomial[c + d - 2, 8] + 2 * Binomial[a + b - 2 + 3 * (a + b + c + d - 4) /
  2, 8] + 2 * Binomial[a + c - 2 + 3 * (a + b + c + d - 4) / 2, 8] + 2 * Binomial[a + d - 2 + 3 * (a + b + c + d -
  4) / 2, 8] + 2 * Binomial[b + c - 2 + 3 * (a + b + c + d - 4) / 2, 8] + 2 * Binomial[b + d - 2 + 3 * (a + b +
  c + d - 4) / 2, 8] + 2 * Binomial[c + d - 2 + 3 * (a + b + c + d - 4) / 2, 8] - Binomial[a + b + c - 2, 8] -
  Binomial[a + b + d - 2, 8] - Binomial[a + c + d - 2, 8] - Binomial[b + c + d - 2, 8] - 2 * Binomial[
  a + b + c - 2 + 3 * (a + b + c + d - 4) / 2, 8] - 2 * Binomial[a + b + d - 2 + 3 * (a + b + c + d - 4) / 2, 8] - 2 *
  Binomial[b + c + d - 2 + 3 * (a + b + c + d - 4) / 2, 8] - 2 * Binomial[a + c + d - 2 + 3 * (a + b + c + d -
  4) / 2, 8] + Binomial[a + b + c + d - 2, 8] + 2 * Binomial[a + b + c + d - 2 + 3 * (a + b + c + d - 4) / 2, 8]]
```
Out[22]=

$$9 - \frac{(-9+a)(-8+a)(-7+a)(-6+a)(-5+a)(-4+a)(-3+a)(-2+a)}{40\,320} - \frac{(-9+b)(-8+b)(-7+b)(\cdots 1 \cdots)\,\cdots 1 \cdots\,(-4+b)(-3+b)(-2+b)}{40\,320} + \cdots 42 \cdots +$$
$$\frac{5(-6+a+b+c+d)(-4+a+b+c+d)(-28+5a+5b+5c+5d)(\cdots 1 \cdots)(\cdots 1 \cdots)(-22+5a+5b+5c+5d)(-18+5a+5b+5c+5d)(-16+5a+5b+5c+5d)}{1\,032\,192} -$$
$$\frac{1}{8}\, a\, b\, c\, d\left(-5+\frac{3}{2}(-4+a+b+c+d)\right)\left(-4+\frac{3}{2}(-4+a+b+c+d)\right)\left(-3+\frac{3}{2}(-4+a+b+c+d)\right)\left(-2+\frac{3}{2}(-4+a+b+c+d)\right)$$

Full expression not available (original memory size: 64.3 kB)

In[23]:= Expand[1920 / (a * b * c * d) * %22]

Out[23]=

865 128 - 720 720 a + 229 140 a^2 - 32 220 a^3 + 1683 a^4 - 720 720 b + 434 880 a b - 88 860 a^2 b + 6060 a^3 b + 229 140 b^2 - 88 860 a b^2 + 8770 a^2 b^2 - 32 220 b^3 + 6060 a b^3 + 1683 b^4 - 720 720 c + 434 880 a c - 88 860 a^2 c + 6060 a^3 c + 434 880 b c - 169 920 a b c + 16 860 a^2 b c - 88 860 b^2 c + 16 860 a b^2 c + 6060 b^3 c + 229 140 c^2 - 88 860 a c^2 + 8770 a^2 c^2 - 88 860 b c^2 + 16 860 a b c^2 + 8770 b^2 c^2 - 32 220 c^3 + 6060 a c^3 + 6060 b c^3 + 1683 c^4 - 720 720 d + 434 880 a d - 88 860 a^2 d + 6060 a^3 d + 434 880 b d - 169 920 a b d + 16 860 a^2 b d - 88 860 b^2 d + 16 860 a b^2 d + 6060 b^3 d + 434 880 c d - 169 920 a c d + 16 860 a^2 c d - 169 920 b c d + 32 400 a b c d + 16 860 b^2 c d - 88 860 c^2 d + 16 860 a c^2 d + 16 860 b c^2 d + 6060 c^3 d + 229 140 d^2 - 88 860 a d^2 + 8770 a^2 d^2 - 88 860 b d^2 + 16 860 a b d^2 + 8770 b^2 d^2 - 88 860 c d^2 + 16 860 a c d^2 + 16 860 b c d^2 + 8770 c^2 d^2 - 32 220 d^3 + 6060 a d^3 + 6060 b d^3 + 6060 c d^3 + 1683 d^4

(*We calculate all the coefficients of the monomial symmetric polynomials in {a,b,c,d} appearing in (1920/(abcd))f_{4,3,1}*)

In[24]:= b1 = SeriesCoefficient[%23, {a, 0, 4}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]

Out[24]=

1683

In[25]:= b2 = SeriesCoefficient[%23, {a, 0, 3}, {b, 0, 1}, {c, 0, 0}, {d, 0, 0}]

Out[25]=

6060

In[26]:= b3 = SeriesCoefficient[%23, {a, 0, 2}, {b, 0, 2}, {c, 0, 0}, {d, 0, 0}]

Out[26]=

8770

In[27]:= b4 = SeriesCoefficient[%23, {a, 0, 2}, {b, 0, 1}, {c, 0, 1}, {d, 0, 0}]

Out[27]=

16 860

In[28]:= b5 = SeriesCoefficient[%23, {a, 0, 1}, {b, 0, 1}, {c, 0, 1}, {d, 0, 1}]

Out[28]=

32 400

In[29]:= n6 = SeriesCoefficient[%23, {a, 0, 3}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]

Out[29]=

-32 220

In[30]:= n7 = SeriesCoefficient[%23, {a, 0, 2}, {b, 0, 1}, {c, 0, 0}, {d, 0, 0}]

Out[30]=

-88 860

In[31]:= n8 = SeriesCoefficient[%23, {a, 0, 1}, {b, 0, 1}, {c, 0, 1}, {d, 0, 0}]

Out[31]=

-169 920

In[32]:=
```
n9 = SeriesCoefficient[%23, {a, 0, 2}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```
Out[32]=

229 140

In[33]:=
```
n10 = SeriesCoefficient[%23, {a, 0, 1}, {b, 0, 1}, {c, 0, 0}, {d, 0, 0}]
```
Out[33]=

434 880

In[34]:=
```
n11 = SeriesCoefficient[%23, {a, 0, 1}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```
Out[34]=

-720 720

In[35]:=
```
n12 = SeriesCoefficient[%23, {a, 0, 0}, {b, 0, 0}, {c, 0, 0}, {d, 0, 0}]
```
Out[35]=

865 128

```
(*We calculate all the coefficients of the monomial symmetric polynomials
in {x_1,...,x_s} appearing in (1920/(product x_i))f_{s,3,1} using Lemma A.6*)
```

In[36]:=
```
b6 = Expand[n6 - (s - 4) * (b2)]
```
Out[36]=

$-7980 - 6060\, s$

In[37]:=
```
b7 = Expand[n7 - (s - 4) * (b4)]
```
Out[37]=

$-21\,420 - 16\,860\, s$

In[38]:=
```
b8 = Expand[n8 - (s - 4) * (b5)]
```
Out[38]=

$-40\,320 - 32\,400\, s$

In[39]:=
```
b9 = Expand[n9 - (s - 4) * (b3 + b7) - Binomial[s - 4, 2] * (b4)]
```
Out[39]=

$9940 + 21\,080\, s + 8430\, s^2$

In[40]:=
```
b10 = Expand[n10 - (s - 4) * (b4 + b8) - Binomial[s - 4, 2] * (b5)]
```
Out[40]=

$17\,040 + 39\,660\, s + 16\,200\, s^2$

In[41]:=
```
b11 = Expand[n11 - (s - 4) * (b2 + b7 + b10) - Binomial[s - 4, 2] * (2 * b4 + b8) - Binomial[s - 4, 3] * (b5)]
```
Out[41]=

$-15\,780\, s - 19\,500\, s^2 - 5400\, s^3$

In[42]:=
```
b12 = Expand[n12 - (s - 4) * (b1 + b6 + b9 + b11) - Binomial[s - 4, 2] * (2 *
  b2 + b3 + 2 * b7 + b10) - Binomial[s - 4, 3] * (3 * b4 + b8) - Binomial[s - 4, 4] * (b5)]
```
Out[42]=

$-1418\, s + 7265\, s^2 + 6390\, s^3 + 1350\, s^4$

```
(*Defining the functions computing (1/(product
x_i))f_{s,3,0} and (1/(product x_i))f_{s,3,1}*)
```

In[43]:=
```
Fs30 = (1 / 1920) * (a1 * m4 + a2 * m31 + a3 * m22 + a4 * m211 + a5 *
  m1111 + a6 * m3 + a7 * m21 + a8 * m111 + a9 * m2 + a10 * m11 + a11 * m1 + a12)
```

Out[43]=

$$\frac{1}{1920}\Big(32\,400\ m1111 + 16\,860\ m211 + 8770\ m22 + 6060\ m31 + 1683\ m4 + m111\,(-28\,800 - 32\,400\ s) + m21\,(-15\,300 - 16\,860\ s) + m3\,(-5700 - 6060\ s) - 698\ s + 3305\ s^2 + 4470\ s^3 + 1350\ s^4 + m2\,(4900 + 14\,960\ s + 8430\ s^2) + m11\,(8400 + 28\,140\ s + 16\,200\ s^2) + m1\,(-7500\ s - 13\,740\ s^2 - 5400\ s^3)\Big)$$

In[44]:=
```
Fs31 = (1 / 1920) * (b1 * m4 + b2 * m31 + b3 * m22 + b4 * m211 + b5 *
  m1111 + b6 * m3 + b7 * m21 + b8 * m111 + b9 * m2 + b10 * m11 + b11 * m1 + b12)
```

Out[44]=

$$\frac{1}{1920}\Big(32\,400\ m1111 + 16\,860\ m211 + 8770\ m22 + 6060\ m31 + 1683\ m4 + m111\,(-40\,320 - 32\,400\ s) + m21\,(-21\,420 - 16\,860\ s) + m3\,(-7980 - 6060\ s) - 1418\ s + 7265\ s^2 + 6390\ s^3 + 1350\ s^4 + m2\,(9940 + 21\,080\ s + 8430\ s^2) + m11\,(17\,040 + 39\,660\ s + 16\,200\ s^2) + m1\,(-15\,780\ s - 19\,500\ s^2 - 5400\ s^3)\Big)$$

```
(*From now on, X is in P^{n+s}, c.i. of type (d_1,...,
d_s), E Ulrich for (X,O_X(1)), and d=deg(X)=d_1d_2...d_s*)

(*We compute the polynomial calculating (24/(rd))deg(Z) using Lemma 3.1(viii)*)
```

In[45]:=
```
Expand[(1 / 2) * ((r / 2) * (m1 - s)) ^ 2 * d - (1 / 2) * ((r / 2) * (m1 - s)) * (m1 - s - n - 1) * d + (r / 12) * (m1 - s - n - 1) ^
  2 * d + (r * d / 12) * (Binomial[n + s + 1, 2] + m1 * (m1 - s - n - 1) - m11) - (r * d / 24) * (3 * n ^ 2 + 5 * n + 2)]
```

Out[45]=

$$-\frac{1}{12}\,d\,m1^2\,r - \frac{d\,m11\,r}{12} + \frac{1}{8}\,d\,m1^2\,r^2 - \frac{d\,r\,s}{24} + \frac{1}{4}\,d\,m1\,r\,s - \frac{1}{4}\,d\,m1\,r^2\,s - \frac{1}{8}\,d\,r\,s^2 + \frac{1}{8}\,d\,r^2\,s^2$$

```
(*Specializing the above when r=3,n=4*)
```

In[46]:=
```
%45 /. {r → 3, n → 4}
```

Out[46]=

$$\frac{7\,d\,m1^2}{8} - \frac{d\,m11}{4} - \frac{d\,s}{8} - \frac{3\,d\,m1\,s}{2} + \frac{3\,d\,s^2}{4}$$

In[47]:=
```
Expand[(8 / d) * %46]
```

Out[47]=

$$7\,m1^2 - 2\,m11 - s - 12\,m1\,s + 6\,s^2$$

In[48]:=
```
%47 /. {(m1) ^ 2 → m2 + 2 * m11}
```

Out[48]=

$$-2\,m11 + 7\,(2\,m11 + m2) - s - 12\,m1\,s + 6\,s^2$$

```
In[49]:= d1 = Expand[%48]
Out[49]=
12 m11 + 7 m2 - s - 12 m1 s + 6 s^2

(*Polynomial calculating (8/d)H_ZK_Z*)

In[50]:= d2 = Expand[(1 / 120) * (1920 * Fs30 - 1920 * Fs31) + d1]
Out[50]=
-60 m11 + 96 m111 - 35 m2 + 51 m21 + 19 m3 + 5 s + 57 m1 s - 96 m11 s - 51 m2 s - 27 s^2 + 48 m1 s^2 - 16 s^3

(*Polynomial calculating (32/(5d))K_Z^2 using Remark 4.4(ix)*)

In[51]:= Expand[2 * ((5 / 2) * (m1 - s) - 5) * d2]
Out[51]=
600 m11 - 300 m1 m11 - 960 m111 + 480 m1 m111 + 350 m2 - 175 m1 m2 - 510 m21 + 255 m1 m21 - 190 m3 +
 95 m1 m3 - 50 s - 545 m1 s + 285 m1^2 s + 1260 m11 s - 480 m1 m11 s - 480 m111 s + 685 m2 s - 255 m1 m2 s -
 255 m21 s - 95 m3 s + 245 s^2 - 900 m1 s^2 + 240 m1^2 s^2 + 480 m11 s^2 + 255 m2 s^2 + 295 s^3 - 320 m1 s^3 + 80 s^4

In[52]:= %51 /. {(m1)^2 → m2 + 2 * m11, (m1)^3 → m3 + 3 * m21 + 6 * m111, (m1)^4 → m4 + 4 * m31 +
 6 * m22 + 12 * m211 + 24 * m1111, m1 * m11 → m21 + 3 * m111, (m1)^2 * m11 → m31 + 2 * m22 +
 5 * m211 + 12 * m1111, (m11)^2 → m22 + 2 * m211 + 6 * m1111, m1 * m3 → m4 + m31, m1 *
 m21 → m31 + 2 * m22 + 2 * m211, m1 * m111 → m211 + 4 * m1111, m1 * m2 → m3 + m21, (m1)^
 2 * m2 → m4 + 2 * m31 + 2 * m22 + 2 * m211, (m2)^2 → m4 + 2 * m22, m2 * m11 → m31 + m211}
Out[52]=
600 m11 - 960 m111 + 350 m2 - 510 m21 - 300 (3 m111 + m21) + 480 (4 m1111 + m211) - 190 m3 -
 175 (m21 + m3) + 255 (2 m211 + 2 m22 + m31) + 95 (m31 + m4) - 50 s - 545 m1 s + 1260 m11 s - 480 m111 s +
 685 m2 s + 285 (2 m11 + m2) s - 255 m21 s - 480 (3 m111 + m21) s - 95 m3 s - 255 (m21 + m3) s +
 245 s^2 - 900 m1 s^2 + 480 m11 s^2 + 255 m2 s^2 + 240 (2 m11 + m2) s^2 + 295 s^3 - 320 m1 s^3 + 80 s^4

In[53]:= d3 = Expand[%52]
Out[53]=
600 m11 - 1860 m111 + 1920 m1111 + 350 m2 - 985 m21 + 990 m211 + 510 m22 - 365 m3 +
 350 m31 + 95 m4 - 50 s - 545 m1 s + 1830 m11 s - 1920 m111 s + 970 m2 s - 990 m21 s -
 350 m3 s + 245 s^2 - 900 m1 s^2 + 960 m11 s^2 + 495 m2 s^2 + 295 s^3 - 320 m1 s^3 + 80 s^4

In[54]:= Expand[4 * ((5 / 2) * (m1 - s) - 5)^2]
Out[54]=
100 - 100 m1 + 25 m1^2 + 100 s - 50 m1 s + 25 s^2

In[55]:= %54 /. {(m1)^2 → m2 + 2 * m11}
Out[55]=
100 - 100 m1 + 25 (2 m11 + m2) + 100 s - 50 m1 s + 25 s^2
```

In[56]:=
```
Expand[%55 * d1]
```

Out[56]=
```
1200 m11 - 1200 m1 m11 + 600 m11^2 + 700 m2 - 700 m1 m2 + 650 m11 m2 + 175 m2^2 -
  100 s - 1100 m1 s + 1200 m1^2 s + 1150 m11 s - 1200 m1 m11 s + 675 m2 s - 650 m1 m2 s +
  500 s^2 - 1750 m1 s^2 + 600 m1^2 s^2 + 600 m11 s^2 + 325 m2 s^2 + 575 s^3 - 600 m1 s^3 + 150 s^4
```

In[57]:=
```
%56 /. {(m1)^2 → m2 + 2 * m11, (m1)^3 → m3 + 3 * m21 + 6 * m111, (m1)^4 → m4 + 4 * m31 +
  6 * m22 + 12 * m211 + 24 * m1111, m1 * m11 → m21 + 3 * m111, (m1)^2 * m11 → m31 + 2 * m22 +
  5 * m211 + 12 * m1111, (m11)^2 → m22 + 2 * m211 + 6 * m1111, m1 * m3 → m4 + m31, m1 *
  m21 → m31 + 2 * m22 + 2 * m211, m1 * m111 → m211 + 4 * m1111, m1 * m2 → m3 + m21, (m1)^
  2 * m2 → m4 + 2 * m31 + 2 * m22 + 2 * m211, (m2)^2 → m4 + 2 * m22, m2 * m11 → m31 + m211}
```

Out[57]=
```
1200 m11 + 700 m2 - 1200 (3 m111 + m21) + 600 (6 m1111 + 2 m211 + m22) -
  700 (m21 + m3) + 650 (m211 + m31) + 175 (2 m22 + m4) - 100 s - 1100 m1 s + 1150 m11 s +
  675 m2 s + 1200 (2 m11 + m2) s - 1200 (3 m111 + m21) s - 650 (m21 + m3) s + 500 s^2 -
  1750 m1 s^2 + 600 m11 s^2 + 325 m2 s^2 + 600 (2 m11 + m2) s^2 + 575 s^3 - 600 m1 s^3 + 150 s^4
```

In[58]:=
```
d4 = Expand[%57]
```

Out[58]=
```
1200 m11 - 3600 m111 + 3600 m1111 + 700 m2 - 1900 m21 + 1850 m211 + 950 m22 - 700 m3 +
  650 m31 + 175 m4 - 100 s - 1100 m1 s + 3550 m11 s - 3600 m111 s + 1875 m2 s - 1850 m21 s -
  650 m3 s + 500 s^2 - 1750 m1 s^2 + 1800 m11 s^2 + 925 m2 s^2 + 575 s^3 - 600 m1 s^3 + 150 s^4
```

In[59]:=
```
d5 = Expand[(1 / 5) * (4 * d3 - d4)]
```

Out[59]=
```
240 m11 - 768 m111 + 816 m1111 + 140 m2 - 408 m21 + 422 m211 + 218 m22 - 152 m3 +
  150 m31 + 41 m4 - 20 s - 216 m1 s + 754 m11 s - 816 m111 s + 401 m2 s - 422 m21 s -
  150 m3 s + 96 s^2 - 370 m1 s^2 + 408 m11 s^2 + 211 m2 s^2 + 121 s^3 - 136 m1 s^3 + 34 s^4
```

```
(*Polynomial calculating (64/d)c_2(Z) using Lemma 3.2(viii)*)
```

In[60]:=
```
p = (3 / 2) * (m1 - s)
```

Out[60]=

$$\frac{3\,(m1 - s)}{2}$$

In[61]:=
```
q = m1 - s - 5
```

Out[61]=
```
-5 + m1 - s
```

In[62]:=
```
Expand[(q + 2 * p) * d2]
```

Out[62]=
```
300 m11 - 240 m1 m11 - 480 m111 + 384 m1 m111 + 175 m2 - 140 m1 m2 - 255 m21 + 204 m1 m21 - 95 m3 +
  76 m1 m3 - 25 s - 265 m1 s + 228 m1^2 s + 720 m11 s - 384 m1 m11 s - 384 m111 s + 395 m2 s - 204 m1 m2 s -
  204 m21 s - 76 m3 s + 115 s^2 - 576 m1 s^2 + 192 m1^2 s^2 + 384 m11 s^2 + 204 m2 s^2 + 188 s^3 - 256 m1 s^3 + 64 s^4
```

In[63]:=
```
%62 /. {(m1)^2 → m2 + 2 * m11, (m1)^3 → m3 + 3 * m21 + 6 * m111, (m1)^4 → m4 + 4 * m31 +
  6 * m22 + 12 * m211 + 24 * m1111, m1 * m11 → m21 + 3 * m111, (m1)^2 * m11 → m31 + 2 * m22 +
  5 * m211 + 12 * m1111, (m11)^2 → m22 + 2 * m211 + 6 * m1111, m1 * m3 → m4 + m31, m1 *
  m21 → m31 + 2 * m22 + 2 * m211, m1 * m111 → m211 + 4 * m1111, m1 * m2 → m3 + m21, (m1)^
  2 * m2 → m4 + 2 * m31 + 2 * m22 + 2 * m211, (m2)^2 → m4 + 2 * m22, m2 * m11 → m31 + m211}
```

Out[63]=

$$300\,m11 - 480\,m111 + 175\,m2 - 255\,m21 - 240\,(3\,m111 + m21) + 384\,(4\,m1111 + m211) - 95\,m3 - 140\,(m21 + m3) + 204\,(2\,m211 + 2\,m22 + m31) + 76\,(m31 + m4) - 25\,s - 265\,m1\,s + 720\,m11\,s - 384\,m111\,s + 395\,m2\,s + 228\,(2\,m11 + m2)\,s - 204\,m21\,s - 384\,(3\,m111 + m21)\,s - 76\,m3\,s - 204\,(m21 + m3)\,s + 115\,s^2 - 576\,m1\,s^2 + 384\,m11\,s^2 + 204\,m2\,s^2 + 192\,(2\,m11 + m2)\,s^2 + 188\,s^3 - 256\,m1\,s^3 + 64\,s^4$$

In[64]:=
```
d6 = Expand[%63]
```

Out[64]=

$$300\,m11 - 1200\,m111 + 1536\,m1111 + 175\,m2 - 635\,m21 + 792\,m211 + 408\,m22 - 235\,m3 + 280\,m31 + 76\,m4 - 25\,s - 265\,m1\,s + 1176\,m11\,s - 1536\,m111\,s + 623\,m2\,s - 792\,m21\,s - 280\,m3\,s + 115\,s^2 - 576\,m1\,s^2 + 768\,m11\,s^2 + 396\,m2\,s^2 + 188\,s^3 - 256\,m1\,s^3 + 64\,s^4$$

In[65]:=
```
Expand[8 * (Binomial[s + 5, 2] + m1 * q - m11 - (1 / 8) * d1 - p^2 - q^2 - 2 * q * p)]
```

Out[65]=

$$-120 + 160\,m1 - 42\,m1^2 - 20\,m11 - 7\,m2 - 163\,s + 104\,m1\,s - 52\,s^2$$

In[66]:=
```
%65 /. {(m1)^2 → m2 + 2 * m11}
```

Out[66]=

$$-120 + 160\,m1 - 20\,m11 - 7\,m2 - 42\,(2\,m11 + m2) - 163\,s + 104\,m1\,s - 52\,s^2$$

In[67]:=
```
Expand[%66 * d1]
```

Out[67]=

$$-1440\,m11 + 1920\,m1\,m11 - 1248\,m11^2 - 840\,m2 + 1120\,m1\,m2 - 1316\,m11\,m2 - 343\,m2^2 + 120\,s + 1280\,m1\,s - 1920\,m1^2\,s - 1852\,m11\,s + 2496\,m1\,m11\,s - 1092\,m2\,s + 1316\,m1\,m2\,s - 557\,s^2 + 2812\,m1\,s^2 - 1248\,m1^2\,s^2 - 1248\,m11\,s^2 - 658\,m2\,s^2 - 926\,s^3 + 1248\,m1\,s^3 - 312\,s^4$$

In[68]:=
```
%67 /. {(m1)^2 → m2 + 2 * m11, (m1)^3 → m3 + 3 * m21 + 6 * m111, (m1)^4 → m4 + 4 * m31 +
  6 * m22 + 12 * m211 + 24 * m1111, m1 * m11 → m21 + 3 * m111, (m1)^2 * m11 → m31 + 2 * m22 +
  5 * m211 + 12 * m1111, (m11)^2 → m22 + 2 * m211 + 6 * m1111, m1 * m3 → m4 + m31, m1 *
  m21 → m31 + 2 * m22 + 2 * m211, m1 * m111 → m211 + 4 * m1111, m1 * m2 → m3 + m21, (m1)^
  2 * m2 → m4 + 2 * m31 + 2 * m22 + 2 * m211, (m2)^2 → m4 + 2 * m22, m2 * m11 → m31 + m211}
```

Out[68]=

$$-1440\,m11 - 840\,m2 + 1920\,(3\,m111 + m21) - 1248\,(6\,m1111 + 2\,m211 + m22) + 1120\,(m21 + m3) - 1316\,(m211 + m31) - 343\,(2\,m22 + m4) + 120\,s + 1280\,m1\,s - 1852\,m11\,s - 1092\,m2\,s - 1920\,(2\,m11 + m2)\,s + 2496\,(3\,m111 + m21)\,s + 1316\,(m21 + m3)\,s - 557\,s^2 + 2812\,m1\,s^2 - 1248\,m11\,s^2 - 658\,m2\,s^2 - 1248\,(2\,m11 + m2)\,s^2 - 926\,s^3 + 1248\,m1\,s^3 - 312\,s^4$$

In[69]:=
```
d7 = Expand[%68]
```

Out[69]=

$$-1440\, m11 + 5760\, m111 - 7488\, m1111 - 840\, m2 + 3040\, m21 - 3812\, m211 - 1934\, m22 + 1120\, m3 - 1316\, m31 - 343\, m4 + 120\, s + 1280\, m1\, s - 5692\, m11\, s + 7488\, m111\, s - 3012\, m2\, s + 3812\, m21\, s + 1316\, m3\, s - 557\, s^2 + 2812\, m1\, s^2 - 3744\, m11\, s^2 - 1906\, m2\, s^2 - 926\, s^3 + 1248\, m1\, s^3 - 312\, s^4$$

In[70]:=
```
d8 = Expand[8 * d6 + d7]
```

Out[70]=

$$960\, m11 - 3840\, m111 + 4800\, m1111 + 560\, m2 - 2040\, m21 + 2524\, m211 + 1330\, m22 - 760\, m3 + 924\, m31 + 265\, m4 - 80\, s - 840\, m1\, s + 3716\, m11\, s - 4800\, m111\, s + 1972\, m2\, s - 2524\, m21\, s - 924\, m3\, s + 363\, s^2 - 1796\, m1\, s^2 + 2400\, m11\, s^2 + 1262\, m2\, s^2 + 578\, s^3 - 800\, m1\, s^3 + 200\, s^4$$

```
(*Polynomial calculating (768/(12d))(K_Z^2+c_2(Z)) i.e., (768/(product x_i))\chi_s'*)
```

In[71]:=
```
Chisprime = Expand[(1 / 12) * ((5 / 32) * d5 + (1 / 64) * d8)]
```

Out[71]=

$$\frac{35\, m11}{8} - 15\, m111 + \frac{135\, m1111}{8} + \frac{245\, m2}{96} - \frac{255\, m21}{32} + \frac{281\, m211}{32} + \frac{585\, m22}{128} - \frac{95\, m3}{32} + \frac{101\, m31}{32} + \frac{225\, m4}{256} - \frac{35\, s}{96} - \frac{125\, m1\, s}{32} + \frac{469\, m11\, s}{32} - \frac{135\, m111\, s}{8} + \frac{997\, m2\, s}{128} - \frac{281\, m21\, s}{32} - \frac{101\, m3\, s}{32} + \frac{441\, s^2}{256} - \frac{229\, m1\, s^2}{32} + \frac{135\, m11\, s^2}{16} + \frac{281\, m2\, s^2}{64} + \frac{149\, s^3}{64} - \frac{45\, m1\, s^3}{16} + \frac{45\, s^4}{64}$$

In[72]:=
```
Expand[768 * Chisprime]
```

Out[72]=

$$3360\, m11 - 11\,520\, m111 + 12\,960\, m1111 + 1960\, m2 - 6120\, m21 + 6744\, m211 + 3510\, m22 - 2280\, m3 + 2424\, m31 + 675\, m4 - 280\, s - 3000\, m1\, s + 11\,256\, m11\, s - 12\,960\, m111\, s + 5982\, m2\, s - 6744\, m21\, s - 2424\, m3\, s + 1323\, s^2 - 5496\, m1\, s^2 + 6480\, m11\, s^2 + 3372\, m2\, s^2 + 1788\, s^3 - 2160\, m1\, s^3 + 540\, s^4$$

```
(*Main relation*)
```

In[73]:=
```
Expand[Chisprime - Fs30]
```

Out[73]=

$$\frac{m22}{384} + \frac{3\, m4}{1280} - \frac{s}{960} - \frac{m2\, s}{384} + \frac{s^2}{768}$$

In[74]:=

$$\mathrm{Factor}\left[\frac{m22}{384} + \frac{3\, m4}{1280} - \frac{s}{960} - \frac{m2\, s}{384} + \frac{s^2}{768}\right]$$

Out[74]=

$$\frac{10\, m22 + 9\, m4 - 4\, s - 10\, m2\, s + 5\, s^2}{3840}$$