NOETHER-LEFSCHETZ, SPACE CURVES AND MATHEMATICAL INSTANTONS

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1. INTRODUCTION

There are three areas of study of geometric entities living in the three dimensional projective space over the complex numbers which have proved to be strictly connected: the classification of space curves, surfaces and vector bundles.
Despite all the production in these areas, several key unsolved problems still challenge us.

After the work of many classical and modern authors (see [H1], [H2] for a bibliography) the problem of classifying space curves has been divided into two main parts: 1) (the so called Halphen problem) Find all the triples of integers (n, g, s) such that there exists a smooth irreducible curve $C \subset \mathbb{P}^3$ of degree n and genus g lying on a surface of degree s but not on any surface of lower degree; 2) Study the properties of the Hilbert scheme $H_{n,g}$ of smooth irreducible curves in $\mathbb{P}^3$, namely its irreducible components, dimension, singular points.

Let $G(n, s)$ be the maximum genus of a curve of degree n not contained in a surface of degree less than s. Then part 1 above falls in turn into three ranges for n and s:

$$\begin{aligned} &range\ A: \frac{1}{6}(s^2+4s+6) \leq n < \frac{1}{3}(s^2+4s+6) \\ &range\ B: \frac{1}{3}(s^2+4s+6) \leq n \leq s(s-1) \\ &range\ C: n > s(s-1). \end{aligned}$$

It is known that $G(n, s) = n(s-1) + 1 - \binom{s+2}{3}$ in range A ([BE1] for $s >> 0$, [W1]), $G(n, s) \geq G_B(n, s)$ in range B (for the definition of $G_B(n, s)$ and the proof we refer

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to [HH1]) and equality is conjectured (true in some cases [GP2], [E], [ES], [H5]), and $G(n,s) = \frac{n^2}{2s} + \frac{1}{2}n(s-4) + 1 - \epsilon$ where $\epsilon = \frac{1}{2}f(s-f-1+\frac{f}{s})$ and $n \equiv f(mod\ s)$, $0 \leq f \leq s-1$ in range C ([GP1]). To complete the solution of part 1 one has to find all (n,g,s) such that $0 \leq g \leq G(n,s)$ and there exists a smooth irreducible curve with invariants n,g,s. This appears to be a hard problem, though some results are known ([BE2], [D], [W1], [W2]).

The study of the Hilbert scheme poses even harder questions and seems to follow no particular pattern. Hartshorne in [H2] attempted to put some guiding principles so to order the large amount of results and to generate some conjectures.

(1.1) *Definitions and notation.* Let $C \subset \mathbb{P}^3$ be a curve of degree n and genus g; define $s(C) = \min\{k \geq 1 : h^0(\mathcal{I}_C(k)) \neq 0\}$, $s(n,g) = \max\{s(C) : C \in H_{n,g}\}$. C is said to be *superficially general* if $s(C) = s(n,g)$.

Such curves are candidates to be the "most general" curves. Furthermore Hartshorne's principle is that non-general behavior should depend only on the fact that the curve lies on a surface of degree $s(n,g)$. Although the situation seems to be a little bit more complicated ([W3]), one can at least hope that curves of maximal genus $G(n,s)$ will share good properties, since they should be superficially general (and they are in ranges A and C where $G(n,s)$ is known and decreasing in s). In particular since a curve C of maximal genus with n,s in range A must have $h^1(\mathcal{O}_C(s-1)) = 0$ by Clifford's theorem, we see that there is no obstruction to the vanishing of H^1 of the normal bundle of C coming from the fact that C lies on a surface of degree s. Accordingly (see also [BE2], question 4) we make the following

(1.2) Conjecture (H). *The Hilbert scheme $H_{n,G(n,s)}$ of smooth curves of maximal genus in range A is pure dimensional and every component has the expected dimension $4n$.*

This conjecture is known to be true for $s = 2$ or $s \geq 3$ and $n \leq \frac{1}{6}(s^2 + 5s + 12) + \frac{2}{s-2}$ (see Remark (2.11)).

Intrinsically related to the classification of space curves is the study of rank two vector bundles and, more recently, reflexive sheaves on $\mathbb{P}^3$. The Serre construction allows to translate problems about vector bundles to problems about curves and vice versa, therefore enriching both theories (see for example [H1],[H2], [H3], [H5], [HH1]). On the other hand the study of vector bundles on projective spaces has had itself a great development through the years. In particular the introduction of the notion of stability has given rise to varieties of moduli of vector bundles and therefore to several results and questions about them. Moreover the study of stable rank two vector bundles on $I\!P^3$ has received further input from the discovery, through the Atiyah-Penrose transformation, of their connection with the solutions of the Yang-Mills equations and therefore with modern elementary particle physics (see [AW], [H6]).

(1.3) *Definition.* Let F be a rank two vector bundle on $I\!P^3$. F is said to be an *instanton bundle* (or a *mathematical instanton*) if $c_1(F) = 0$ and $H^0(F) = H^1(F(-2)) = 0$.

While the moduli variety $M(0, c_2)$ of stable rank two vector bundles on $I\!P^3$ with $c_1 = 0$ and $c_2 > 0$ has an unpredictable behavior, for example it is generally reducible and has components of dimension larger than the expected one $8c_2 - 3$ ([BH]), the open subset $MI(0, c_2)$ of instanton vector bundles appears to have a simpler geometry: It is in fact nonsingular and irreducible for $c_2 \leq 4$ ([B1] for $c_2 = 1$, [H3] for $c_2 = 2$, [SE] for $c_2 = 3$, [B2] and [LP] for $c_2 \leq 4$) and even rational for $c_2 \leq 3$. Moreover in all those cases one has $H^2(F \otimes F^*) = 0$ for every $F \in MI(0, c_2)$ and therefore $dim MI(0, c_2) = 8c_2 - 3$ (and this is also true for the moduli of real instantons for every c_2 [OSS]). Accordingly many authors have posed the problem of finding out whether $MI(0, c_2)$ is irreducible, or at least the dimensions of its components ([OSS] problem 4.4.3, [S]). We will make the ensuing

(1.4) Conjecture (I). *$MI(0, c_2)$ is irreducible.*

(1.5) Conjecture (I′). *Every component of $MI(0, c_2)$ has dimension $8c_2 - 3$.*

The only known component of $MI(0, c_2)$ is the one made of vector bundles which are generalizations of (a twist of) the ones associated to $c_2 + 1$ skew lines in $I\!P^3$. Let F be a general bundle in this component $\Sigma \subseteq MI(0, c_2)$. Then $H^2(F \otimes F^*) = 0$, therefore $dim\Sigma = 8c_2 - 3$ and Conjecture (I) implies Conjecture (I′). Moreover Hartshorne and Hirschowitz ([HH2]) showed that F has natural cohomology, that is at most one group $H^i(F(l))$ is different from zero for every $l \in \mathbb{Z}$. Finally, since by Riemann-Roch $\chi(F(t)) = \frac{1}{3}(t+2)[(t+1)(t+3) - 3c_2] > 0$ (for $t \geq 1$) if and only if $t > \sqrt{3c_2 + 1} - 2$, we have that

$$\min\{t : H^0(F(t)) \neq 0\} = \min\{t : t > \sqrt{3c_2 + 1} - 2\}.$$

It is then natural to suspect that this behavior will hold for general instanton bundles. Therefore we will propose two more conjectures: Let $W \subseteq MI(0,c_2)$ be an irreducible component and F be a general element of W; then

(1.6) Conjecture (N). *F has natural cohomology.*

(1.7) Conjecture (T). $\min\{t : H^0(F(t)) \neq 0\} = \min\{t : t > \sqrt{3c_2+1} - 2\}$.

Though not stated explicitly as a conjecture, the second conjecture has been already mentioned by Hartshorne ([H1] p.108, [H3] p.265). As Atiyah observed ([H3] p.265), the fact that $H^0(F(t)) \neq 0$ for $t > \sqrt{3c_2+1} - 2$ for an instanton bundle F follows from the monad representation of F. In fact Hartshorne ([H4]) proved that the same is true for any rank two reflexive sheaf with $c_1 = 0$ (and $c_1 = -1$ for $t > \sqrt{3c_2 + \frac{1}{4}} - \frac{3}{2}$). So Conjecture (T) is equivalent to say that $H^0(F(t)) = 0$ for $t \leq \sqrt{3c_2+1} - 2$.

As we saw above, Conjecture (N) implies Conjecture (T). On the other hand, if $\sqrt{3c_2+1}$ is integer, these two are equivalent (Remark (3.12)), and this particular case is of a special interest to us since it allows to attack Conjecture (I′) by means of Noether-Lefschetz theory (Theorem (1.13)).

Let $d \geq 4$ and denote by $S(d) \subset \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) = \mathbb{P}^N$ the locus of smooth surfaces of degree d in $\mathbb{P}^3$ and by $NL(d)$ the *Noether-Lefschetz locus*, that is the locus of smooth surfaces with Picard group larger than $\mathbb{Z}$. By the Noether-Lefschetz theorem $NL(d)$ is the union of countably many closed proper subvarieties of $S(d)$, called the *components* of $NL(d)$. The study of the geometry of these components has produced several interesting results, among which we will recall the ones devoted to the classification of components with given codimension c in $S(d)$. M. Green and C. Voisin characterized the components with low codimension: We have $c \geq d-3$ ([G1]), $c = d-3$ only for the component of surfaces containing a line ([G2]) and if $c \leq 2d-7$, for $d \geq 5$, then either $c = d-3$ or $c = 2d-7$ and the component is made of surfaces containing a plane conic ([V1]). On the other hand some simple Hodge-theoretic considerations show that $c \leq p_g(d) = \binom{d-1}{3}$, and that one should "in general" expect equality.

(1.8) *Definition.* A component of $NL(d)$ is called *general* if its codimension in $S(d)$ is $p_g(d)$, *special* otherwise.

C. Ciliberto, J. Harris and R. Miranda ([CHM]) showed that there are infinitely many general components and that their union is dense, in both the Zariski and the natural topology, in $S(d)$. It had been conjectured by J. Harris that there are only finitely many special components of $NL(d)$. This conjecture in turn was implied by the *Green-Ciliberto conjecture*: *For the generic element S of a special component of NL(d), some canonical divisor on S has a component whose class is not a multiple of the hyperplane class.*

The implication of course follows from the fact that the Green-Ciliberto conjecture bounds above the degree of the component C of K_S. As well-known now, both these conjectures are false, due to C. Voisin's counterexample to the Harris conjecture for $d \equiv 0$ (mod 4) ([V2]).

Even so, it seems that the Green-Ciliberto conjecture, with some restricting hypothesis, still presents some interest.

(1.9) *Definition.* Let W be a component of $Hilb\mathbb{P}^3$ and I the incidence correspondence $\{(S,C) : C \subset S\} \subset \mathbb{P}^N \times W$, together with its projections $\pi_1 : I \to \mathbb{P}^N$, $\pi_2 : I \to W$. Let $W(d) = Im\ \pi_1$ and C be a curve representing the generic point of W, $\mathcal{I}_C$ its ideal sheaf in $\mathcal{O}_{\mathbb{P}^3}$, I_C its homogeneous ideal in $\mathbb{C}[x_0, \ldots, x_3]$. Then $W(d)$ is a *regular* component of $NL(d)$ if C is smooth irreducible and satisfies

(i) $H^1(\mathcal{I}_C(d-4)) = 0$ and

(ii) I_C is generated in degree less than d.

Now let $W(d)$ be a regular component of $NL(d)$, C as above and S be a general surface of degree d containing C. We denote by C^2 the self-intersection of C in S. We want to propose the following weakening of the Green-Ciliberto conjecture:

(1.10) Conjecture (GCR). *The Green-Ciliberto conjecture holds for special regular components of NL(d) with* $C^2 \leq 0$.

While it is easily verified that there are only finitely many regular components (Remark (2.10)), the power of the above conjecture is contained in part in the fact that it reduces its study to a study of Hilbert schemes of curves in $\mathbb{P}^3$, but especially in our main results.

The first one is for negative self-intersection.

Theorem (1.11). *($d \geq 7$)*

Conjecture (H) is equivalent to Conjecture (GCR) for components of NL(d) with $C^2 < -2$.

It follows in particular that Conjecture (GCR) is true when $d \geq 7$ and $C^2 \leq -\frac{1}{6}(d^3 - 7d^2 + 12d) + \frac{2}{d-5}$ (see Remark (2.11)).

The second and third are for self-intersection zero.

Theorem (1.12). *($d \geq 7$)*
Conjecture (GCR) is true for components of NL(d) with $C^2 = 0$ *if* $d \equiv 1 \ (mod\ 3),\ d \neq 10$.

Theorem (1.13). *Suppose* $d \equiv 0, 2 \ (mod\ 6)$, $d \geq 8$ *and let* $c_2 = \frac{1}{12}(d^2 - 8d + 12)$. *Conjecture (I′) implies Conjecture (GCR) for components of NL(d) with* $C^2 = 0$ *and* $H^1(\mathcal{I}_C(\frac{d}{2} - 2)) = 0$.
Vice versa Conjecture (GCR) for components of NL(d) with $C^2 = 0$ *and* $H^1(\mathcal{I}_C(\frac{d}{2} - 2)) = 0$ *together with Conjecture (T) implies Conjecture (I′).*

Finally we remark that for $d = 8$ all the conjectures in Theorem (1.13) are true since $c_2 = 1$.

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2. REGULAR COMPONENTS OF THE NOETHER-LEFSCHETZ LOCUS

In this section we will study some properties of regular components of $NL(d)$, as defined in (1.9). The first thing to notice is that the definition makes sense, i.e. the $W(d)$'s are actually components of $NL(d)$ and not some proper subvariety of a component.

Lemma (2.1). *Let* $W \subseteq Hilb\mathbb{P}^3$ *be a component of the Hilbert scheme of smooth curves,* C *a curve representing the generic point of* W, $\mathcal{I}_C$ *its ideal sheaf in* $\mathcal{O}_{\mathbb{P}^3}$, I_C *its homogeneous ideal in* $\mathbb{C}[x_0, \ldots, x_3]$, S *a general surface of degree d containing* C *and consider the incidence correspondence*

$$I = \{(S', C') : C' \subset S'\} \subset \mathbb{P}^N \times W$$

together with its projections $\pi_1 : I \to \mathbb{P}^N$, $\pi_2 : I \to W$. *Let* $W(d) = Im\ \pi_1$ *and suppose that* C *is smooth irreducible and satisfies*

(i) $H^1(\mathcal{I}_C(d-4))=0$ *and*

(ii) I_C *is generated in degree less than d.*

Then

(2.2) PicS $\cong\mathbb{Z}^2$ *freely generated by* $\mathcal{O}_S(1)$ *and* $\mathcal{O}_S(C)$*;*

(2.3) W(d) is a component of NL(d) and

$$p_g(d)-codim_{S(d)}W(d)=h^0(\mathcal{I}_C(d-4))-h^1(\mathcal{O}_C(d))+h^1(\mathcal{I}_C(d))+dimW-4degC.$$

Proof: This lemma is proved in [CL] (Lemma 1.2) with the stronger hypothesis that $\mathcal{I}_C$ is $(d-1)$-regular in the sense of Castelnuovo-Mumford. The proof there only uses (2.2) which here follows from (ii) by Corollary II.3.8 of [L]. Therefore also (2.3) follows by the same proof since

$$\begin{aligned}
&p_g(d)-codim_{S(d)}W(d)=\\
&=h^0(\mathcal{O}_{I\!P^3}(d-4))-h^0(\mathcal{O}_{I\!P^3}(d))+dimW+h^0(\mathcal{I}_C(d))-h^1(\mathcal{O}_C(d-4))=\\
&=h^0(\mathcal{O}_{I\!P^3}(d-4))-h^1(\mathcal{O}_C(d-4))-h^0(\mathcal{O}_C(d))+h^1(\mathcal{I}_C(d))+dimW=\\
&=h^0(\mathcal{O}_{I\!P^3}(d-4))-(d-4)degC+g(C)-1-h^1(\mathcal{O}_C(d-4))-h^1(\mathcal{O}_C(d))\\
&\quad+h^1(\mathcal{I}_C(d))+dimW-4degC=\\
&=h^0(\mathcal{O}_{I\!P^3}(d-4))-h^0(\mathcal{O}_C(d-4))-h^1(\mathcal{O}_C(d))+h^1(\mathcal{I}_C(d))+dimW-4degC=\\
&=h^0(\mathcal{I}_C(d-4))-h^1(\mathcal{O}_C(d))+h^1(\mathcal{I}_C(d))+dimW-4degC. \blacksquare
\end{aligned}$$

Now that we know that regular components are as above, we can start relating them with the Green-Ciliberto conjecture. First we record two elementary observations.

Proposition (2.4). *With notation as in Lemma (2.1), let W(d) be a regular component of NL(d), and suppose that no canonical divisor on S has a component whose class is not a multiple of the hyperplane class H. Then, if* $s=s(C)$*, we have*

(iii) $d-3\le s\le d-1$;

(iv) $H^1(\mathcal{O}_C(s))=0$.

Proof: By hypothesis $C\not\subseteq K_S=(d-4)H$ hence $s\ge d-3$; of course $s\le d-1$ by (ii). Moreover let $C'\sim sH-C$ on S; then C' is effective and $C'\not\subseteq K_S$ means

$$\begin{aligned}
0=H^0(\mathcal{I}_{C'}(d-4))=H^0(\mathcal{O}_S((d-4)H-C'))=H^2(\mathcal{O}_S(C'))=\\
=H^2(\mathcal{O}_S(sH-C))\supseteq H^2(\mathcal{I}_C(s))=H^1(\mathcal{O}_C(s))
\end{aligned}$$

where the inclusion comes from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(s-d) \to \mathcal{I}_C(s) \to \mathcal{O}_S(sH-C) \to 0. \quad \blacksquare$$

Therefore conditions (iii) and (iv) are necessary for a regular component of $NL(d)$ in order to have the generic surface with canonical divisors whose only components are complete intersections of the surface. The next proposition adds another condition to give sufficiency.

Proposition (2.5). *Let W(d) be a regular component of NL(d), $S \supset C$ as above, C^2 the self-intersection and $s = s(C)$. Suppose that*

(iii) $d-3 \leq s \leq d-1$,

(iv) $H^1(\mathcal{O}_C(s)) = 0$ *and*

(v) $C^2 \leq 0$.

Then no canonical divisor on S has a component whose class is not a multiple of the hyperplane class H.

Proof: By (2.2) of Lemma (2.1) any effective divisor on S is linearly equivalent to $aH+bC$ for some $a, b \in \mathbb{Z}$. Hence we need to prove that

(2.6) $\quad H^0(\mathcal{I}_{C'}(d-4)) = 0 \ \ \forall a, b \in \mathbb{Z}$ such that $C' \sim aH + bC$ is effective and $b \neq 0$.

First we notice the following

Claim (2.7). $H^0(\mathcal{O}_S((d-4-a)H - bC)) = 0$ if $b < 0, a \geq s$.

Proof of the Claim: If $b = -1$, the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a-d) \to \mathcal{I}_C(a) \to \mathcal{O}_S(aH-C) \to 0$$

shows that, since $H^3(\mathcal{O}_{\mathbb{P}^3}(a-d)) = 0$ because $a-d \geq s-d \geq -3$ by (iii),

$$H^0(\mathcal{O}_S((d-4-a)H+C)) = H^2(\mathcal{O}_S(aH-C)) = H^2(\mathcal{I}_C(a)) = 0$$

as a consequence of (iv) and $a \geq s$. If $b \leq -2$, we can assume by induction on $-b$ that $H^0(\mathcal{O}_S((d-4-a)H-(b+1)C)) = 0$. Since we have

$$0 \to \mathcal{O}_S((d-4-a)H-(b+1)C) \to \mathcal{O}_S((d-4-a)H-bC) \to \mathcal{O}_C \otimes \mathcal{O}_S((d-4-a)H-bC) \to 0$$

it is enough to see that $H^0(\mathcal{O}_C \otimes \mathcal{O}_S((d-4-a)H - bC)) = 0$. This is given by (iii) and (v) since

$$deg\mathcal{O}_C \otimes \mathcal{O}_S((d-4-a)H - bC) = (d-4-a)degC - bC^2 \leq (d-4-s)degC < 0.$$

This proves Claim (2.7).

Suppose now that $C' \sim aH + bC$ is effective; if $b < 0$ we must have $a \geq s$ and hence Claim (2.7) shows that

$$H^0(\mathcal{I}_{C'}(d-4)) = H^0(\mathcal{O}_S((d-4)H - C')) = H^0(\mathcal{O}_S((d-4-a)H - bC)) = 0,$$

that is (2.6) for $b < 0$.

On the other hand if instead $b > 0$ we have

$$C' \sim aH + bC \sim (d-4-(d-4-a))H - (-b)C$$

is effective with $-b < 0$, therefore $d-4-a < s$ by Claim (2.7). So we will be done if we prove

(2.8) $\quad H^0(\mathcal{I}_{C'}(d-4)) = 0$ for $C' \sim aH + bC$ effective with $b \geq 1$ and $a > d-4-s$.

Clearly it is enough to show (2.8) for $b = 1$ since

$$H^0(\mathcal{O}_S((d-4-a)H - bC)) \subseteq H^0(\mathcal{O}_S((d-4-a)H - (b-1)C)).$$

Now the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4-a) \to \mathcal{I}_C(d-4-a) \to \mathcal{O}_S((d-4-a)H - C) \to 0$$

gives, for $b = 1$,

$$H^0(\mathcal{I}_{C'}(d-4)) = H^0(\mathcal{O}_S((d-4-a)H - C)) = 0$$

since $H^0(\mathcal{I}_C(d-4-a)) = 0$ because $d-4-a < s = s(C)$. ∎

At this point it is of particular importance the study of curves $C \subset \mathbb{P}^3$ satisfying conditions (i) through (v). In fact, with few exceptions, we will see that such curves are of a special kind.

Proposition (2.9). *Let $C \subset I\!P^3$ be a smooth irreducible curve of degree n and genus g, $s = s(C)$. If C satisfies (i), (iii) and (v) for a given $d \geq 7$ we must have $d = s + 3$, (n,s) in range A and g=G(n,s), i.e. C is a curve of maximal genus in range A, unless $\Big(n, s, h^1(\mathcal{O}_C(d-4))\Big) = (13, 4, 0), (28, 7, 0)$ or $d \equiv 0, 2$ (mod 3) and $n = \frac{1}{3}(s^2 + 4s + 6)$. In the latter case C is of maximal (conjectured) genus in range B if $h^1(\mathcal{O}_C(d-4)) = 1$.*

Vice versa a curve of maximal genus in range A or of maximal (conjectured) genus in range B and degree $n = \frac{1}{3}(s^2 + 4s + 6)$ satisfies (i), (iii), (iv) and (v) with $d = s + 3$. In the case of range A it also satisfies (ii).

Proof: By (i) and (iii) we get $h^0(\mathcal{I}_C(d-4)) = h^1(\mathcal{I}_C(d-4)) = 0$, hence

$$g = n(d-4) - \binom{d-1}{3} + 1 + u$$

where $u = h^1(\mathcal{O}_C(d-4))$. Set $s = d - 4 + \alpha$ for $\alpha = 1, 2, 3$; then by (v) we have

$$C^2 = 2g - 2 - n(d-4) = n(d-4) - 2\binom{d-1}{3} + 2u = n(s-\alpha) - 2\binom{s-\alpha+3}{3} + 2u \leq 0$$

hence

$$n \leq \frac{2}{s-\alpha}\Big[\binom{s-\alpha+3}{3} - u\Big] = \frac{1}{3}[s^2 + (6-2\alpha)s + \alpha^2 - 6\alpha + 11] + \frac{2-2u}{s-\alpha} \ .$$

If $\alpha = 2$ then $s = d - 2$ and $h^0(\mathcal{I}_C(s-1)) = 0$ gives

$$\binom{s+2}{3} - n(s-1) + g - 1 - h^1(\mathcal{O}_C(s-1)) + h^1(\mathcal{I}_C(s-1)) = 0 \ ;$$

substituting as above $g = n(s-2) - \binom{s+1}{3} + 1 + h^1(\mathcal{O}_C(s-2))$ we get

$$\binom{s+1}{2} - n = -h^1(\mathcal{I}_C(s-1)) - h^1(\mathcal{O}_C(s-2)) + h^1(\mathcal{O}_C(s-1)) \leq 0$$

so $\frac{1}{2}s(s+1) \leq n \leq \frac{1}{3}(s^2 + 2s + 3) + \frac{2-2u}{s-2} \leq \frac{1}{3}(s^2 + 2s + 3) + \frac{2}{s-2}$
therefore $s^3 - 3s^2 - 4s \leq 0$, hence $s \leq 4$ and $d \leq 6$ which is not.
Analogously if $\alpha = 3$, $s = d - 1$ and $h^0(\mathcal{I}_C(s-1)) = 0$ gives

$$g = n(s-3) - \binom{s}{3} + 1 + h^1(\mathcal{O}_C(s-3))$$

hence

$$\binom{s+2}{3} - \binom{s}{3} - 2n = -h^1(\mathcal{I}_C(s-1)) - h^1(\mathcal{O}_C(s-3)) + h^1(\mathcal{O}_C(s-1)) \leq 0$$

so $\frac{1}{2}s^2 \leq n \leq \frac{1}{3}(s^2+2) + \frac{2}{s-3}$ and again $s \leq 4$ and $d \leq 5$.

Therefore we must have $\alpha = 1$, i.e. $d = s+3$ and

$$n \leq \frac{1}{3}(s^2+4s+6) + \frac{2-2u}{s-1}$$

so either $n < \frac{1}{3}(s^2+4s+6)$ or $u \leq 1$ and $n \geq \frac{1}{3}(s^2+4s+6)$.

If $u = 0$ and $n \geq \frac{1}{3}(s^2+4s+6)$ it must be either $n = \frac{1}{3}(s^2+4s+6)$ and $s \equiv 0, 2 (mod\ 3)$ since $\frac{2}{s-1} < 1$ or $s \equiv 1 (mod\ 3)$ and $\frac{2}{s-1} \geq \frac{1}{3}$, hence $s = 4, 7$ (since $s = d-3 \geq 4$), $n = 13, 28$.

If $u = 1$ and $n \geq \frac{1}{3}(s^2+4s+6)$ we must have equality.

Now in the case $n < \frac{1}{3}(s^2+4s+6)$ we have $u = h^1(\mathcal{O}_C(s-1)) = 0$ by Clifford's theorem, hence

$$g = n(s-1) - \binom{s+2}{3} + 1 = G(n,s).$$

In the case $u = 1$, $n = \frac{1}{3}(s^2+4s+6)$ we get $g = \binom{s+2}{3}$ and this is the maximal (conjectured) genus in range B for $n = degC = \frac{1}{3}(s^2+4s+6)$ ([HH1]).

Vice versa let C be a curve of maximal genus $G(n,s)$ in range A; then $h^1(\mathcal{O}_C(s-1)) = 0$ by Clifford's theorem, hence

$$h^1(\mathcal{I}_C(s-1)) = h^0(\mathcal{O}_C(s-1)) - h^0(\mathcal{O}_{\mathbb{P}^3}(s-1)) = n(s-1) - G(n,s) + 1 - \binom{s+2}{3} = 0;$$

therefore setting $d = s+3$ we get (i), (iii) and (iv). Also

$$C^2 = 2g - 2 - n(d-4) = n(s-1) - 2\binom{s+2}{3} < -2 \leq 0$$

so (v). Finally by [BE3] we have $h^1(\mathcal{I}_C(s)) = 0$ hence $\mathcal{I}_C$ is (s+1)-regular in the sense of Castelnuovo-Mumford, therefore (ii).

To conclude let C be a curve of degree $n = \frac{1}{3}(s^2+4s+6)$ and genus $g = \binom{s+2}{3}$.

For $d = s+3$ we have (iii) and $C^2 = 2g-2-n(d-4) = 0$ hence (v). From $h^0(\mathcal{I}_C(s-1)) = 0$ we get

$$\begin{aligned} h^1(\mathcal{O}_C(s-1)) &= h^0(\mathcal{O}_{I\!P^3}(s-1)) - \chi(\mathcal{O}_C(s-1)) + h^1(\mathcal{I}_C(s-1)) = \\ &= \binom{s+2}{3} - n(s-1) + \binom{s+2}{3} - 1 + h^1(\mathcal{I}_C(s-1)) = \\ &= 2g - (2g-2) - 1 + h^1(\mathcal{I}_C(s-1)) = \\ &= 1 + h^1(\mathcal{I}_C(s-1)) \ ; \end{aligned}$$

but then $\omega_C(1-s)$ is effective of degree zero, hence

$$\omega_C \cong \mathcal{O}_C(s-1) \text{ and } h^1(\mathcal{I}_C(s-1)) = 0$$

therefore (iv) and (i). ∎

We have now gathered enough information for regular components and curves in range A. Before we go into the proof of Theorem (1.11) let us see that the components we are interested in are only finitely many.

(2.10) *Remark.* There are finitely many regular components of $NL(d)$. In fact let $W(d)$ be a regular component; since I_C is generated in degree less than d we have $degC \leq (d-1)^2$ hence the projectivity of the Hilbert scheme shows that the $W(d)$'s are finitely many since the W's are.

To conclude this section we will prove our first result.

Proof of Theorem (1.11): With notation as in Lemma (2.1) let $W(d)$ be a regular special component of the Noether-Lefschetz locus such that $C^2 < -2$ and assume Conjecture (H). To see that Conjecture (GCR) holds it is enough to show that $H^0(\mathcal{I}_C(d-4)) \neq 0$, since the class of C is not a multiple of the hyperplane class of S. If $H^0(\mathcal{I}_C(d-4)) = 0$ we have $s = s(C) \geq d-3$, hence C satisfies (i), (iii) and (v). By the first part of Proposition (2.9) C must be a curve of maximal genus in range A with $s = d-3$: In fact if $(n,s) = (13,4), (28,7)$ we get $C^2 = -1, 0$ respectively; if $n = \frac{1}{3}(s^2+4s+6)$ we have

$$C^2 = 2g - 2 - n(d-4) = n(d-4) - 2\binom{d-1}{3} + 2u = -2 + 2u \geq -2$$

where $u = h^1(\mathcal{O}_C(d-4))$.

Now $h^1(\mathcal{O}_C(d-4)) = h^1(\mathcal{O}_C(s-1)) = 0$ by Clifford's theorem since we are in range A and moreover C is of maximal rank (i.e. $h^1(\mathcal{I}_C(k)) = 0\ \forall k \geq s$) by [BE3]. From (2.3) we get

$$p_g(d) - codim_{S(d)} W(d) = dim W - 4 deg C = 0$$

by the assumption. This contradicts the fact that $W(d)$ is special.

For the converse let us assume Conjecture (GCR) for components with $C^2 < -2$ and let W be a component of $H_{n,G(n,s)}$ with (n, s) in range A, C a curve representing the generic point of W and S a general surface of degree $d = s + 3$ containing C. The second part of Proposition (2.9) shows that C satisfies conditions (i) through (v) and $C^2 < -2$ (from the proof). Therefore $W(d)$ is a regular component of $NL(d)$ with $C^2 < -2$ and Proposition (2.5) shows that $W(d)$ must be a general component of $NL(d)$, i.e. of codimension $p_g(d)$, otherwise $W(d)$ would violate our assumption that Conjecture (GCR) holds. From Lemma (2.1) we get

$$0 = p_g(d) - codim_{S(d)} W(d) = dim W - 4 deg C$$

that is Conjecture (H) holds. ∎

(2.11) *Remark.* For $s = 2$ or $s \geq 3$ and $\frac{1}{6}(s^2 + 4s + 6) \leq n \leq \frac{1}{6}(s^2 + 5s + 12) + \frac{2}{s-2}$ Conjecture (H) is true. In fact, the case $s = 2$ being elementary, let us assume $s \geq 3$ and $n \leq \frac{1}{6}(s^2 + 5s + 12)$; we have $n \geq G(n, s) + 3$ so $H_{n,G(n,s)}$ is irreducible ([Ei]) and a general curve C of degree n and genus $G(n, s)$ is nonspecial and of maximal rank ([BE4]), hence has $s(C) = s$ and $H^1(N_C) = 0$, where N_C is the normal bundle of C. So $dim H_{n,G(n,s)} = 4n$. Otherwise $\frac{1}{6}(s^2 + 5s + 12) < n \leq \frac{1}{6}(s^2 + 5s + 12) + \frac{2}{s-2}$, therefore $n = G(n, s) + 2$ or $n = G(n, s) + 1$ and this is possible only for $(n, s) = (7, 3), (8, 3), (9, 4), (11, 5)$. In all these cases it is easy to see that $dim H_{n,G(n,s)} = 4n$ using the irreducibility of the Hilbert scheme ([KK], [K]) and some liaison techniques. The proof of Theorem (1.11) hence shows that Conjecture (GCR) is true for $d \geq 7$ and $C^2 \leq -\frac{1}{6}(d^3 - 7d^2 + 12d) + \frac{2}{d-5}$.

3. REGULAR COMPONENTS WITH $C^2 = 0$ AND MATHEMATICAL INSTANTONS

As we have seen in Proposition (2.9), and especially from what we will see later in this section, a remarkable case is the one of regular components of the Noether-Lefschetz locus with $C^2 = 0$. Let us gather first some properties of the corresponding curves.

Proposition (3.1). *Let $C \subset \mathbb{P}^3$ be a smooth irreducible curve of degree $n = \frac{1}{3}(s^2+4s+6)$ and genus $g = \binom{s+2}{3}$ where $s = s(C) \equiv 0, 2 (mod\ 3)$ and $s \geq 5$. Then $\omega_C \cong \mathcal{O}_C(s-1)$ and $H^1(\mathcal{I}_C(s-1)) = 0$, so C is zero locus of a section of a stable rank two vector bundle E on $\mathbb{P}^3$. If s is odd, i.e. $s \equiv 3, 5 (mod\ 6)$, set $F = E(-\frac{s+3}{2})$. Then the following properties hold for F:*

(3.2) $c_1(F) = 0$, $c_2(F) = \frac{1}{12}(s^2-2s-3)$;

(3.3) $\frac{s-3}{2} = \min\{t \geq 1 : H^0(F(t)) \neq 0\}$;

(3.4) F has natural cohomology ⇔ F is instanton ⇔ $H^1(\mathcal{I}_C(\frac{s-1}{2})) = 0$;

(3.5) If F is instanton then $E = F(\frac{s+3}{2})$ is (-3)-regular (in the sense of Castelnuovo-Mumford) and not (-4)-regular, $\mathcal{I}_C$ is $(s+2)$-regular and I_C is generated in degree less than $s+3$.

Proof: We have already seen in the proof of Proposition (2.9) that $\omega_C \cong \mathcal{O}_C(s-1)$ and $H^1(\mathcal{I}_C(s-1)) = 0$. Hence C is zero locus of a section of a rank two vector bundle E on $\mathbb{P}^3$ with $c_1(E) = s+3$, $c_2(E) = \frac{1}{3}(s^2+4s+6)$. The exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to E \to \mathcal{I}_C(s+3) \to 0$$

shows that

$$H^0(E(-[\frac{s+4}{2}])) = H^0(\mathcal{I}_C(s+3-[\frac{s+4}{2}])) = 0$$

since $s+3-[\frac{s+4}{2}] \leq s-1$ (where [x] denotes the integer part of a real number x). So E is stable.

If s is odd and $F = E(-\frac{s+3}{2})$ we get (3.2) and (3.3) since the exact sequence of F

$$(3.6) \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-\frac{s+3}{2}) \to F \to \mathcal{I}_C(\frac{s+3}{2}) \to 0$$

shows that $H^0(F(\frac{s-5}{2})) = H^0(\mathcal{I}_C(s-1)) = 0$ and $H^0(F(\frac{s-3}{2})) = H^0(\mathcal{I}_C(s)) \neq 0$.

If F has natural cohomology then $H^1(F(-2)) = H^2(F(-2))^*$ implies that these two must be both zero, hence F is instanton. The latter is equivalent to $H^1(F(-2)) = H^1(\mathcal{I}_C(\frac{s-1}{2})) = 0$ by (3.6).

Now suppose F instanton. To see that F has natural cohomology it is enough to prove that it has seminatural cohomology, i.e. at most one group $H^i(F(l))$ is different from zero $\forall l \geq -\frac{1}{2}c_1(F) - 2 = -2$. For such l we have $H^3(F(l)) = H^0(F(-l-4))^* = 0$ since $H^0(F) = 0$ and $H^2(F(l)) = 0$ by [BH], Lemma 4. If $-2 \leq l \leq \frac{s-5}{2}$ we have $H^0(F(l)) = 0$ by (3.3) and $h^1(F(\frac{s-5}{2})) = -\chi(F(\frac{s-5}{2})) = 0$ by Riemann-Roch. Again from [BH], Lemma 4, we conclude that $H^1(F(l)) = 0\ \forall l \geq \frac{s-5}{2}$. So (3.4) is proved.
To see (3.5) we note that

$$H^i(E(-3-i)) = H^i(F(\frac{s-3}{2} - i)) = 0$$

$\forall i > 0$, but $h^1(E(-5)) = h^1(F(\frac{s-7}{2})) = -\chi(F(\frac{s-7}{2})) \neq 0$, hence E is (-3)-regular but not (-4)-regular. Finally, by (3.6),

$$H^1(\mathcal{I}_C(s+1)) = H^1(F(\frac{s-1}{2})) = 0,$$

$$H^2(\mathcal{I}_C(s)) = H^1(\mathcal{O}_C(s)) = 0,$$

$$H^3(\mathcal{I}_C(s-1)) = H^3(\mathcal{O}_{I\!P^3}(s-1)) = 0,$$

so $\mathcal{I}_C$ is $(s+2)$-regular and I_C is generated in degree less than $s+3$ by Castelnuovo-Mumford's lemma. ∎

The Proposition (3.1) just proved, together with Proposition (2.9), gives rise to rank two instanton vector bundles on $I\!P^3$ with second Chern class c_2 as in (3.2) and satisfying (3.3). Their properties are listed below.

Proposition (3.7). *Let F be a rank two instanton vector bundle on $I\!P^3$ such that $c_2(F) = \frac{1}{12}(s^2 - 2s - 3)$ and $H^0(F(\frac{s-5}{2})) = 0$ for some $s \equiv 3, 5 (mod\ 6)$, $s \geq 5$. Then we have*

(3.8) $H^1(F(l)) = 0\ \forall l \geq \frac{s-5}{2}$ and $F(\frac{s+3}{2})$ is (-3)-regular;

(3.9) A general section of $H^0(F(\frac{s+3}{2}))$ has zero locus a smooth irreducible curve $C \subset I\!P^3$ of degree $\frac{1}{3}(s^2+4s+6)$, genus $\binom{s+2}{3}$ with $s(C) = s$ and $H^1(\mathcal{I}_C(\frac{s-1}{2})) = 0$;

(3.10) F has natural cohomology.

Proof: As in the proof of Proposition (3.1) we have

$$h^1(F(\frac{s-5}{2})) = -\chi(F(\frac{s-5}{2})) + h^0(F(\frac{s-5}{2})) = 0$$

by hypothesis. Hence (3.8) and (3.10) by [BH], Lemma 4. To see (3.9) notice that

$$h^0(F(\frac{s+3}{2})) - h^0(F(\frac{s+1}{2})) = \chi(F(\frac{s+3}{2})) - \chi(F(\frac{s+1}{2})) = \frac{1}{6}(s^2+19s+54) > 0$$

so C exists and by [H3], Proposition 1.4, is smooth since $F(\frac{s+3}{2})$ is generated by global sections (by Castelnuovo-Mumford's lemma and (3.8)); moreover it is connected, hence irreducible, because $H^1(F(-\frac{s+3}{2})) = 0$ (again by [BH]). Now $c_1(F(\frac{s+3}{2})) = s+3$, $c_2(F(\frac{s+3}{2})) = \frac{1}{3}(s^2+4s+6)$ hence

$$degC = c_2(F(\frac{s+3}{2})) = \frac{1}{3}(s^2+4s+6),$$

$$\omega_C \cong \mathcal{O}_C(s-1),$$

$$g(C) = \frac{1}{2}degC(s-1)+1 = \binom{s+2}{3},$$

and the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to F(\frac{s+3}{2}) \to \mathcal{I}_C(s+3) \to 0$$

gives $H^0(\mathcal{I}_C(s-1)) = H^0(F(\frac{s-5}{2})) = 0$, $h^0(\mathcal{I}_C(s)) = h^0(F(\frac{s-3}{2})) = \chi(F(\frac{s-3}{2})) \neq 0$ hence $s = s(C)$ and $H^1(\mathcal{I}_C(\frac{s-1}{2})) = H^1(F(-2)) = 0$. ■

(3.11) *Remark.* Proposition (3.7) shows in particular that smooth irreducible curves as in (3.9) exist for any $s \equiv 3, 5 (mod\ 6)$, $s \geq 5$. For if we take F to be a general bundle in the component $\Sigma \subseteq MI(0, c_2)$ of bundles associated to skew lines, with $c_2 = \frac{1}{12}(s^2-2s-3)$, then F has natural cohomology ([HH2]), hence $h^0(F(\frac{s-5}{2})) = h^1(F(\frac{s-5}{2}))$ so they are both zero. In fact such curves exist for every $s \equiv 0, 2 (mod\ 3)$ and they are the candidates for curves of maximal genus in range B ([HH1]).

(3.12) *Remark.* If F is a vector bundle with $c_1 = 0, c_2 > 0$, by Riemann-Roch we have $\chi(F(t)) = \frac{1}{3}(t+2)[(t+1)(t+3) - 3c_2]$ hence, for $t \geq 1$, $\chi(F(t)) \geq 0$ if and only if $t \geq \sqrt{3c_2+1} - 2$. So $\chi(F(t))$ can be zero only if $\sqrt{3c_2+1} \in \mathbb{Z}$, i.e. $3c_2 + 1 = a^2, a \in \mathbb{Z}$; if $s = 2a+1$ we get $c_2 = \frac{1}{12}(s^2-2s-3)$ and $s \equiv 3, 5 (mod\ 6)$. In particular if F is an instanton bundle with c_2 such that $\sqrt{3c_2+1} \in \mathbb{Z}$ and satisfies Conjecture (T),then $H^0(F(\frac{s-5}{2})) = 0$ hence F has natural cohomology by (3.10). Therefore Conjecture (T) is equivalent to Conjecture (N) in this case.

The upshot of the last two propositions is that it is equivalent to give, for some $s \equiv 3,5(mod\ 6)$, $s \geq 5$, a smooth irreducible curve $C \subset I\!P^3$ as in Proposition (3.1) with $H^1(\mathcal{I}_C(\frac{s-1}{2})) = 0$, or a rank two instanton vector bundle F on $I\!P^3$ as in Proposition (3.7). Since our interest focuses on Conjectures (I′) and (GCR), we will now study the corresponding deformations of C and F as above.

Proposition (3.13). *Let $C \subset I\!P^3$ be a smooth irreducible curve of degree $n = \frac{1}{3}(s^2 + 4s+6)$, genus $g = \binom{s+2}{3}$ with $s = s(C) \equiv 3,5(mod\ 6)$, $s \geq 5$ and $H^1(\mathcal{I}_C(\frac{s-1}{2})) = 0$ and F the corresponding instanton vector bundle with $c_2 = \frac{1}{12}(s^2-2s-3)$ and $H^0(F(\frac{s-5}{2})) = 0$. Denote by N_C the normal bundle of $C \subset I\!P^3$ and by W_C, W_F the components of $H_{n,g}$, $MI(0,c_2)$ parametrizing deformations of C and F respectively. Then*

(3.14) $H^1(N_C) \cong H^2(EndF)$;

(3.15) $dimW_C = 4n \Leftrightarrow dimW_F = 8c_2 - 3$.

Proof: From the exact sequence

$$0 \to \mathcal{O}_{I\!P^3} \to F(\frac{s+3}{2}) \to \mathcal{I}_C(s+3) \to 0$$

twisted by $F(-\frac{s+3}{2})$ we get $H^2(EndF) = H^2(F \otimes F) = H^2(\mathcal{I}_C \otimes F(\frac{s+3}{2}))$ since

$$H^2(F(-\frac{s+3}{2})) = H^1(F(\frac{s-5}{2}))^* = 0$$

by (3.8) and

$$H^3(F(-\frac{s+3}{2})) = H^0(F(\frac{s-5}{2}))^* = 0$$

by hypothesis. On the other hand, since $N_C \cong F(\frac{s+3}{2}) \otimes \mathcal{O}_C$, if we twist the sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_{I\!P^3} \to \mathcal{O}_C \to 0$$

by $F(\frac{s+3}{2})$ we obtain

$$H^1(N_C) = H^1(F(\frac{s+3}{2}) \otimes \mathcal{O}_C) = H^2(\mathcal{I}_C \otimes F(\frac{s+3}{2}))$$

since $H^1(F(\frac{s+3}{2})) = H^2(F(\frac{s+3}{2})) = 0$ by natural cohomology because $H^0(F(\frac{s+3}{2})) \neq 0$. This shows (3.14).

Now, since C is smooth irreducible, we have

$$dimW_F = dimW_C - h^0(F(\frac{s+3}{2})) + dim\ \{\ \text{isomorphisms}\ \xi : \omega_C \cong \mathcal{O}_C(s-1)\ \} =$$
$$= dimW_C - \chi(F(\frac{s+3}{2})) + 1 =$$
$$= dimW_C - \frac{1}{3}(s+7)(2s+6) + 1 =$$
$$= dimW_C - 4n + 8c_2 - 3.\ \blacksquare$$

Let us give now the proof of our two results for regular components with $C^2 = 0$.

Proof of Theorems (1.12) and (1.13): With notation as in Lemma (2.1) let $W(d)$ be a regular special component of $NL(d)$ with $C^2 = 0$. Then either $H^0(\mathcal{I}_C(d-4)) \neq 0$ and Conjecture (GCR) holds or $H^0(\mathcal{I}_C(d-4)) = 0$. In the latter case we have $s \geq d-3$, so C satisfies (i), (iii) and (v). By the first part of Proposition (2.9) it must be $d = s+3$ and

$$\text{either}\ \ d \equiv 0, 2\ (mod\ 3)\ \text{and}\ n = degC = \frac{1}{3}(s^2 + 4s + 6)$$
$$\text{or}\ \ d = 10:$$

In fact if C is of maximal genus in range A then $C^2 < -2$ and if $n = degC = 13$, $s = 4$, $h^1(\mathcal{O}_C(d-4)) = 0$ then $C^2 = -1$.

This proves Theorem (1.12).

To see Theorem (1.13) we also assume that $d \equiv 0, 2 (mod\ 6)$ and $H^1(\mathcal{I}_C(\frac{d}{2}-2)) = 0$. As in the proof of Proposition (2.9) we have

$$g = n(d-4) - \binom{d-1}{3} + 1 + u = n(s-1) - \binom{s+2}{3} + 1 + u$$

where $u = h^1(\mathcal{O}_C(d-4)) = h^1(\mathcal{O}_C(s-1))$. But

$$0 = C^2 = 2g - 2 - n(d-4) = n(s-1) - 2\binom{s+2}{3} + 2u = 2u - 2$$

hence $u = h^1(\mathcal{O}_C(s-1)) = 1$ and $g = \binom{s+2}{3}$. Also $s = d-3 \equiv 3, 5 (mod\ 6)$, $s \geq 5$ and $H^1(\mathcal{I}_C(\frac{s-1}{2})) = H^1(\mathcal{I}_C(\frac{d}{2}-2)) = 0$; therefore $\mathcal{I}_C$ is $(s+2)$-regular by (3.5).

Now if Conjecture (I′) is true for $c_2 = \frac{1}{12}(d^2 - 8d + 12) = \frac{1}{12}(s^2 - 2s - 3)$, by Proposition (3.13) we get $dimW = 4n$.

Hence Lemma (2.1) gives $p_g(d) - codim_{S(d)}W(d) = 0$ because $\mathcal{I}_C$ is $(s+2)$-regular and therefore $H^1(\mathcal{O}_C(s+3)) = H^1(\mathcal{I}_C(s+3)) = 0$.

Since we assumed that $W(d)$ is special, we see that the case $H^0(\mathcal{I}_C(d-4)) = 0$ cannot happen and hence that Conjecture (GCR) holds.

Vice versa let us assume now Conjecture (GCR) for components of $NL(d)$ with $C^2 = 0$ and $H^1(\mathcal{I}_C(\frac{d}{2}-2)) = 0$ plus Conjecture (T). Let W_1 be a component of $MI(0, c_2)$, with $c_2 = \frac{1}{12}(d^2 - 8d + 12)$ and F a general bundle in W_1. Set $s = d-3$, so that $s \equiv 3, 5 (mod\ 6)$, $s \geq 5$ and $c_2 = \frac{1}{12}(s^2 - 2s - 3)$; also $\sqrt{3c_2+1} - 2 = \frac{s-5}{2}$, hence $H^0(F(\frac{s-5}{2})) = 0$ from Conjecture (T). By Proposition (3.7) we get a component W of the Hilbert scheme, whose generic curve C is as in (3.9). Therefore C satisfies (i) and (ii) by Proposition (3.1) and $C^2 = 0$, $H^1(\mathcal{I}_C(\frac{d}{2}-2)) = 0$.

From Lemma (2.1) we get

(3.16) $$p_g(d) - codim_{S(d)}W(d) =$$
$$= h^0(\mathcal{I}_C(s-1)) - h^1(\mathcal{O}_C(s+3)) + h^1(\mathcal{I}_C(s+3)) + dimW - 4n =$$
$$= dimW - 4n.$$

But $W(d)$ cannot be special, otherwise we would have a regular component with $C^2 = 0$ and $H^1(\mathcal{I}_C(\frac{d}{2}-2)) = 0$ that violates Conjecture (GCR) by Proposition (2.5). So, by (3.16),

$$dimW - 4n = p_g(d) - codim_{S(d)}W(d) = 0$$

and therefore $dimW_1 = 8c_2 - 3$ because of (3.15). ■

(3.17) *Remark.* As a corollary of the proof of Theorem (1.13) we get that every component of the Hilbert scheme $H_{n,g}$ of curves C with $n = \frac{1}{3}(s^2+4s+6)$, $g = \binom{s+2}{3}$, $s = s(C) \equiv 3, 5 (mod\ 6)$, $s \geq 5$ and $H^1(\mathcal{I}_C(\frac{s-1}{2})) = 0$ has the "expected" dimension $4n$ if and only if Conjecture (GCR) holds for such components. In particular both assertions above hold in the case of curves associated to vector bundles in the component $\Sigma \subseteq MI(0, c_2)$. In fact for these bundles one has $H^2(EndF) = 0$, hence $H^1(N_C) = 0$ by (3.14).

(3.18) *Remark.* According to Walter's table ([W3], Theorem 3.1 and following remarks) it is likely that $H^1(N_C) = 0$ for C of degree $\frac{1}{3}(s^2+4s+6)$ and genus $\binom{s+2}{3}$ for $s \leq 8$.

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