NAKAMAYE'S THEOREM ON LOG CANONICAL PAIRS

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† *Dedicated to the memory of Fassi*

Abstract. We generalize Nakamaye's description, via intersection theory, of the augmented base locus of a big and nef divisor on a normal pair with log-canonical singularities or, more generally, on a normal variety with non-lc locus of dimension ≤ 1. We also generalize Ein-Lazarsfeld-Mustaţă-Nakamaye-Popa's description, in terms of valuations, of the subvarieties of the restricted base locus of a big divisor on a normal pair with klt singularities.

1. Introduction

Let X be a normal complex projective variety and let D be a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. The stable base locus

$$\mathbf{B}(D) = \bigcap_{E \ge 0 : E \sim_{\mathbb{Q}} D} \operatorname{Supp}(E)$$

is an important closed subset associated to D, but it is often difficult to handle. On the other hand, there are two, perhaps even more important, base loci associated to D.

One of them is the augmented base locus ([Nye], [ELMNP1, Def. 1.2])

$$\mathbf{B}_+(D) = \bigcap_{E \ge 0 : D-E \text{ ample}} \operatorname{Supp}(E)$$

where E is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Since this locus measures the failure of D to be ample, it has proved to be a key tool in several recent important results in birational geometry, such as Takayama [T], Hacon and McKernan's [HM] effective birationality of pluricanonical maps or Birkar, Cascini, Hacon and McKernan's [BCHM] finite generation of the canonical ring, just to mention a few.

One way to compute $\mathbf{B}_+(D)$ is to pick a sufficiently small ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A on X, because then one knows that $\mathbf{B}_+(D) = \mathbf{B}(D - A)$ by [ELMNP1, Prop. 1.5].

In the case when D is also nef, for every subvariety $V \subset X$ of dimension $d \ge 1$ such that $D^d \cdot V = 0$, we have that $D_{|V}$ is not big, whence $s(D-A)_{|V}$ cannot be effective for any $s \in \mathbb{N}$ and therefore $V \subseteq \mathbf{B}(D - A) = \mathbf{B}_+(D)$. Now define

$$\operatorname{Null}(D) = \bigcup_{V \subset X : D^d \cdot V = 0} V$$

so that, by what we just said,

$$\operatorname{Null}(D) \subseteq \mathbf{B}_+(D). \tag{1}$$

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A somewhat surprising result of Nakamaye [Nye, Thm. 0.3] (see also [Laz, §10.3]) asserts that, if X is *smooth* and D is big and nef, then in fact equality holds in (1).

As is well-known, in birational geometry, one must work with normal varieties with some kind of (controlled) singularities. In the light of this, it becomes apparent that it would be nice to have a generalization of Nakamaye's Theorem to normal varieties. While in positive characteristic the latter has been recently proved to hold, on any projective scheme, by Cascini, McKernan and Mustaţă [CMM, Thm. 1.1], we will show in this article a generalization to normal complex varieties with log canonical singularities. This partially answers a question in [CMM].

More precisely let us define

Definition 1.1. Let X be a normal projective variety. The **non-lc locus** of X is

$$X_{\text{nlc}} = \bigcap_{\Delta} \text{Nlc}(X, \Delta)$$

where Δ runs among all effective Weil $\mathbb{Q}$-divisors such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $\text{Nlc}(X, \Delta)$ is the locus of points $x \in X$ such that (X, Δ) is not log canonical at x.

Using Ambro's and Fujino's theory of non-lc ideal sheaves [A], [Fno] and a modification of some results of de Fernex and Hacon [dFH], we prove

Theorem 1.
Let X be a normal projective variety such that $\dim X_{\text{nlc}} \leq 1$. Let D be a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then

$$\mathbf{B}_+(D) = \text{Null}(D).$$

This easily gives the following

Corollary 1.
Let X be a normal projective variety such that $\dim \text{Sing}(X) \leq 1$ or $\dim X \leq 3$ or there exists an effective Weil $\mathbb{Q}$-divisor Δ such that (X, Δ) is log canonical.

Let D be a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then

$$\mathbf{B}_+(D) = \text{Null}(D).$$

Moreover, using a striking result of Gibney, Keel and Morrison [GKM, Thm. 0.9], we can give a very quick application to the moduli space of stable pointed curves.

Corollary 2.
Let $g \geq 1$ and let D be a big and nef $\mathbb{Q}$-divisor on $\overline{M}_{g,n}$. Then

$$\mathbf{B}_+(D) \subseteq \partial \overline{M}_{g,n}.$$

Thus, for example, one gets new compactifications of $M_{g,n}$ by taking rational maps associated to such divisors.

The second base locus associated to any pseudoeffective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D, measuring how far D is from being nef, is the restricted base locus [ELMNP1, Def. 1.12].

Definition 1.2. Let X be a normal projective variety and let D be a pseudoeffective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. The **restricted base locus** of D is

$$\mathbf{B}_-(D) = \bigcup_{A \text{ ample}} \mathbf{B}(D + A)$$

where A runs among all ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisors such that $D + A$ is a $\mathbb{Q}$-divisor.

Restricted base loci are countable unions of subvarieties by [ELMNP1, Prop. 1.19], but not always closed [Les, Thm. 1.1].

For a big $\mathbb{Q}$-divisor D on a *smooth* variety X, the subvarieties of $\mathbf{B}_-(D)$ are precisely described in [ELMNP1, Prop. 2.8] (also in positive characteristic in [M, Thm. 6.2]) in terms of asymptotic valuations.

Definition 1.3. ([Nma, Def. III.2.1], [ELMNP1, Lemma 3.3], [BBP, §1.3], [dFH, §2]) Let X be a normal projective variety, let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and let v be a divisorial valuation on X, that is v is a positive integer multiple of the valuation associated to a prime divisor Γ lying on a birational model $f : Y \to X$. The center of v on X is $c_X(v) = f(\Gamma)$.

If D is big, we set

$$v(\|D\|) = \inf\{v(E), E \text{ effective } \mathbb{R}\text{-Cartier } \mathbb{R}\text{-divisor on X such that } E \equiv D\};$$

if D is pseudoeffective, we pick an ample divisor A and set

$$v(\|D\|) = \lim_{\varepsilon \to 0^+} v(\|D + \varepsilon A\|).$$

If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $\kappa(D) \geq 0$ and $b \in \mathbb{N}$ is such that bD is Cartier and $|bD| \neq \emptyset$, we set (see [CD, Def. 2.14] or [ELMNP1, Def. 2.2] for the case D big)

$$v(\langle D \rangle) = \lim_{m \to +\infty} \frac{v(|mbD|)}{mb}$$

where, if g is an equation, at the generic point of $c_X(v)$, of a general element in $|mbD|$, then $v(|mbD|) = v(g)$.

Now the main content of [ELMNP1, Prop. 2.8] is that, given a discrete valuation v on a *smooth* X with center $c_X(v)$ and a big divisor D, then $c_X(v) \subseteq \mathbf{B}_-(D)$ if and only if $v(\|D\|) > 0$. Using the main result of [CD] we give a generalization to normal pairs with klt singularities.

Theorem 2.
Let X be a normal projective variety such that there exists an effective Weil $\mathbb{Q}$-divisor Δ with (X, Δ) a klt pair. Let v be a divisorial valuation on X. Then

(i) *If D is a big Cartier divisor on X we have*

$$v(\langle D \rangle) > 0 \text{ if and only if } c_X(v) \subseteq \mathbf{B}_-(D) \text{ if and only if } \limsup_{m \to +\infty} v(|mD|) = +\infty.$$

(ii) *If D is a pseudoeffective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, we have*

$$v(\|D\|) > 0 \text{ if and only if } c_X(v) \subseteq \mathbf{B}_-(D).$$

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2. Non-lc ideal sheaves

Notation and conventions 2.1. Throughout the article we work over the complex numbers. Given a variety X and a coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$, we denote by $\mathcal{Z}(\mathcal{J})$ the closed subscheme of X defined by $\mathcal{J}$. If X is a normal projective variety and Δ is a Weil $\mathbb{Q}$-divisor on X, we call (X, Δ) a **pair** if $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We refer to [KM, Def. 2.34] for the various notions of singularities of pairs.

Definition 2.2. Let X be a normal projective variety and let $\Delta = \sum_{i=1}^{s} d_i D_i$ be a Weil $\mathbb{Q}$-divisor on X, where the $D_i's$ are distinct prime divisors.

Given $a \in \mathbb{R}$ we set $\Delta^{>a} = \sum_{1 \le i \le s: d_i > a} d_i D_i$, $\Delta^+ = \Delta^{>0}$, $\Delta^- = (-\Delta)^+$ and $\Delta^{<a} = -((-\Delta)^{>-a})$. The **round up** of Δ is $\lceil \Delta \rceil = \sum_{i=1}^{s} \lceil d_i \rceil D_i$ and the **round down** is $\lfloor \Delta \rfloor = \sum_{i=1}^{s} \lfloor d_i \rfloor D_i$. We also set $\Delta^{\#} = \Delta^{<-1} + \Delta^{>-1}$.

The following is easily proved.

Remark 2.3. Let X be a normal projective variety and let Δ, Δ' be Weil $\mathbb{Q}$-divisors on X. Then

(i) $\lceil (-\Delta)^{\#} \rceil = \lceil -(\Delta^{<1}) \rceil - \lfloor \Delta^{>1} \rfloor$;
(ii) If $\Delta \le \Delta'$, then $\lceil \Delta^{\#} \rceil \le \lceil (\Delta')^{\#} \rceil$.

We recall the definition of non-lc ideal sheaves [A, Def. 4.1], [Fno, Def. 2.1].

Definition 2.4. Let (X, Δ) be a pair and let $f : Y \to X$ be a resolution of X such that $\Delta_Y := f^*(K_X + \Delta) - K_Y$ has simple normal crossing support. The **non-lc ideal sheaf associated to** (X, Δ) is

$$\mathcal{J}_{NLC}(X, \Delta) = f_* \mathcal{O}_Y(\lceil -(\Delta_Y^{<1}) \rceil - \lfloor \Delta_Y^{>1} \rfloor).$$

Remark 2.5. Non-lc ideal sheaves are well-defined by [Fno, Prop. 2.6], [A, Rmk. 4.2(iv)]. Moreover, when Δ is effective and $f : Y \to X$ is a log-resolution of (X, Δ), we have that the non-lc locus of (X, Δ) is, set-theoretically, $\mathrm{Nlc}(X, \Delta) = f(\mathrm{Supp}(\Delta_Y^{\ge 1})) = \mathcal{Z}(\mathcal{J}_{NLC}(X, \Delta))$ [Fno, Lemma 2.2].

Remark 2.6. The non-lc ideal sheaf of a pair (X, Δ) with Δ effective is an integrally closed ideal.

Proof. With notation as in Definition 2.4, set $G = \lceil -(\Delta_Y^{<1}) \rceil$ and $N = \lfloor \Delta_Y^{>1} \rfloor$, so that G and N are effective divisors without common components, G is f-exceptional and $\mathcal{J}_{NLC}(X, \Delta) = f_* \mathcal{O}_Y(G - N) = f_* \mathcal{O}_Y(-N)$ by Fujita's lemma [Fta, Lemma 2.2], [KMM, Lemma 1-3-2], [dFH, Lemma 4.5]. Therefore $\mathcal{J}_{NLC}(X, \Delta)$ is an ideal sheaf and it is integrally closed by [Laz, Prop. 9.6.11]. □

Our next goal is to prove, using techniques and results in de Fernex-Hacon [dFH], that non-lc ideal sheaves have a unique maximal element. To this end we will use some results of Fujino [Fno] and de Fernex-Hacon [dFH] that we wish to recall for the reader's convenience.

Lemma 2.7. [Fno, Lemma 2.7] *Let $g : Y' \to Y$ be a proper birational morphism between smooth varieties and let B_Y be an $\mathbb{R}$-divisor on Y having simple normal crossing support. Assume that $B_{Y'} := g^*(K_Y + B_Y) - K_{Y'}$ also has simple normal crossing support. Then*

$$g_* \mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil - \lfloor B_{Y'}^{>1} \rfloor) \cong \mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor).$$

Definition 2.8. ([dFH, Def. 3.1]) Let $f : Y \to X$ be a proper birational morphism between normal varieties and let K_Y be a canonical divisor on Y and $K_X = f_* K_Y$. For every $m \ge 1$ define $K_{m,Y/X} = K_Y - \frac{1}{m} f^{\natural}(mK_X)$, where $f^{\natural}(mK_X)$ is the divisor on Y such that $\mathcal{O}_Y(-f^{\natural}(mK_X)) = (\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y)^{\vee\vee}$.

Lemma 2.9. *Let $m \ge 1$. In (i)-(iv) below let $f : Y \to X$ be a proper birational morphism between normal varieties. Then*

(i) *If X is Gorenstein then* $K_{m,Y/X} = K_{Y/X} := K_Y + f^*(-K_X)$*;*
(ii) [dFH, Rmk 3.3] *For all* $q \geq 1$ *we have* $K_{m,Y/X} \leq K_{mq,Y/X}$*;*
(iii) [dFH, Lemma 3.5] *Assume that* mK_Y *is Cartier and* $\mathcal{O}_X(-mK_X)\cdot\mathcal{O}_Y$ *is invertible. Let* Y' *be a normal variety and let* $g : Y' \to Y$ *be a proper birational morphism. Then* $K_{m,Y'/X} = K_{m,Y'/Y} + g^*K_{m,Y/X}$*;*
(iv) [dFH, Rmk 3.9] *Let* (X,Δ) *be a pair with* Δ *effective and assume that* $m(K_X+\Delta)$ *is Cartier. Then* $K_Y + f_*^{-1}(\Delta) - f^*(K_X+\Delta) \leq K_{m,Y/X}$*;*
(v) [dFH, Thm. 5.4 and its proof] *For every* $m \geq 2$ *there exist a log-resolution* $f : Y \to X$ *of* $(X, \mathcal{O}_X(-mK_X))$ *and a Weil* $\mathbb{Q}$*-divisor* Δ_m *on* X *such that* $m\Delta_m$ *is integral,* $\lfloor \Delta_m \rfloor = 0$*,* $(\Delta_m)_Y$ *has simple normal crossing support,* f *is a log-resolution for the log-pair* $((X,\Delta_m), \mathcal{O}_X(-mK_X))$*,* $K_X + \Delta_m$ *is* $\mathbb{Q}$*-Cartier and* $K_{m,Y/X} = K_Y + f_*^{-1}(\Delta_m) - f^*(K_X+\Delta_m)$*.*

In (iv) and (v) above $f_*^{-1}(\Delta)$ is the proper transform of Δ. Note that our Δ_Y (Definition 2.4) is different from the one in [dFH, Def. 3.8].

Now we have

Proposition 2.10. *Let X be a normal projective variety. Then there exists a Weil* $\mathbb{Q}$*-divisor* Δ_0 *on* X *such that* $\lfloor \Delta_0 \rfloor = 0$*,* $K_X + \Delta_0$ *is* $\mathbb{Q}$*-Cartier and*

$$\mathcal{J}_{NLC}(X,\Delta) \subseteq \mathcal{J}_{NLC}(X,\Delta_0)$$

for every pair (X,Δ) *with* Δ *effective.*

Proof. Fix a canonical divisor K_X on X and an integer $m \geq 2$. By Lemma 2.9(v) there exist a log-resolution $f : Y \to X$ of $(X, \mathcal{O}_X(-mK_X))$ and a Weil $\mathbb{Q}$-divisor Δ_m on X with the properties in (v). In particular $K_{m,Y/X}$ is f-exceptional. Now set

$$\mathfrak{a}_m(X) = f_*\mathcal{O}_Y(\lceil (K_{m,Y/X})^{\#} \rceil).$$

As in the proof of Remark 2.6 we get that $\mathfrak{a}_m(X)$ is a coherent ideal sheaf. Let us check that its definition is independent of the choice of f. Let $f' : Y' \to X$ be another log-resolution of $(X, \mathcal{O}_X(-mK_X))$ and assume, as we may, that f' factors through f and a morphism $g : Y' \to Y$. By Lemma 2.9(iii) and (i) we have $K_{m,Y'/X} = K_{Y'/Y} + g^*K_{m,Y/X} = K_{Y'} - g^*(K_Y - K_{m,Y/X})$, whence

$$(fg)_*\mathcal{O}_{Y'}(\lceil (K_{m,Y'/X})^{\#} \rceil) = f_*(g_*\mathcal{O}_{Y'}(\lceil (K_{Y'} - g^*(K_Y - K_{m,Y/X}))^{\#} \rceil)). \tag{2}$$

Now set $B_Y = -K_{m,Y/X}$ and $B_{Y'} = g^*(K_Y + B_Y) - K_{Y'}$ so that, using Remark 2.3(i) and Lemma 2.7, we have

$$g_*\mathcal{O}_{Y'}(\lceil (K_{Y'} - g^*(K_Y - K_{m,Y/X}))^{\#} \rceil) = g_*\mathcal{O}_{Y'}(\lceil (-B_{Y'})^{\#} \rceil) = g_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil - \lfloor B_{Y'}^{>1} \rfloor) =$$
$$= \mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor) = \mathcal{O}_Y(\lceil (-B_Y)^{\#} \rceil) = \mathcal{O}_Y(\lceil (K_{m,Y/X})^{\#} \rceil)$$

and by (2) we get

$$(fg)_*\mathcal{O}_{Y'}(\lceil (K_{m,Y'/X})^{\#} \rceil) = f_*\mathcal{O}_Y(\lceil (K_{m,Y/X})^{\#} \rceil)$$

that is $\mathfrak{a}_m(X)$ is well defined.

We now claim that the set $\{\mathfrak{a}_m(X), m \geq 2\}$ has a unique maximal element. In fact, given $m, q \geq 2$, let $f : Y \to X$ be a log-resolution of $(X, \mathcal{O}_X(-mK_X)) + \mathcal{O}_X(-mqK_X))$. By Lemma 2.9(ii) and Remark 2.3(ii) we have $\lceil (K_{m,Y/X})^{\#} \rceil \leq \lceil (K_{mq,Y/X})^{\#} \rceil$ and therefore $\mathfrak{a}_m(X) \subseteq \mathfrak{a}_{mq}(X)$. Using the ascending chain condition on ideals we conclude that $\{\mathfrak{a}_m(X), m \geq 2\}$ has a unique maximal element, which we will denote by $\mathfrak{a}_{\max}(X)$.

Next let us show that all the ideal sheaves $\mathfrak{a}_m(X)$, for $m \geq 2$ (whence in particular also $\mathfrak{a}_{\max}(X)$), are in fact non-lc ideal sheaves of a suitable pair.

Let Δ_m be as above, so that, by Remark 2.3(i) and using $\lceil(-f_*^{-1}(\Delta_m))^{\#}\rceil = 0$, we have

$$\lceil -((\Delta_m)_Y^{<1})\rceil - \lfloor(\Delta_m)_Y^{>1}\rfloor = \lceil(-(\Delta_m)_Y)^{\#}\rceil = \lceil(K_{m,Y/X} - f_*^{-1}(\Delta_m))^{\#}\rceil =$$

$$= \lceil(K_{m,Y/X})^{\#}\rceil + \lceil(-f_*^{-1}(\Delta_m))^{\#}\rceil = \lceil(K_{m,Y/X})^{\#}\rceil$$

whence

$$\mathcal{J}_{NLC}(X,\Delta_m) = f_*\mathcal{O}_Y(\lceil -((\Delta_m)_Y^{<1})\rceil - \lfloor(\Delta_m)_Y^{>1}\rfloor) = f_*\mathcal{O}_Y(\lceil(K_{m,Y/X})^{\#}\rceil) = \mathfrak{a}_m(X).$$

To finish the proof, let (X,Δ) be a pair with Δ effective and let $q \in \mathbb{N}$ be such that $q(K_X+\Delta)$ is Cartier. Let $m_0 \geq 2$ be such that $\mathfrak{a}_{\max}(X) = \mathfrak{a}_{m_0}(X) = \mathfrak{a}_{qm_0}(X)$. By what we proved above, there exists $\Delta_0 := \Delta_{qm_0}$ such that $\mathcal{J}_{NLC}(X,\Delta_0) = \mathfrak{a}_{\max}(X)$. By Lemma 2.9(iv) we have that $-\Delta_Y \leq K_Y + f_*^{-1}(\Delta) - f^*(K_X+\Delta) \leq K_{qm_0,Y/X}$, whence also, by Remark 2.3 (i) and (ii),

$$\lceil -(\Delta_Y^{<1})\rceil - \lfloor\Delta_Y^{>1}\rfloor = \lceil(-\Delta_Y)^{\#}\rceil \leq \lceil(K_{qm_0,Y/X})^{\#}\rceil$$

and therefore

$$\mathcal{J}_{NLC}(X,\Delta) = f_*\mathcal{O}_Y(\lceil -(\Delta_Y^{<1})\rceil - \lfloor\Delta_Y^{>1}\rfloor) \subseteq f_*\mathcal{O}_Y(\lceil(K_{qm_0,Y/X})^{\#}\rceil) = \mathfrak{a}_{\max}(X) = \mathcal{J}_{NLC}(X,\Delta_0).$$

□

3. Proof of Theorem 1

We record the following lemma, which is also of independent interest.

Lemma 3.1. *Let (X,Δ) be a pair with Δ effective and let D be an effective Cartier divisor on X. Then there exists $c = c(X,\Delta,D) \in \mathbb{N}$ such that the set-theoretic equality*

$$\mathrm{Bs}\,|D| \cup \mathrm{Nlc}(X,\Delta) = \mathcal{Z}(\mathcal{J}_{NLC}(X,\Delta+E_1+\cdots+E_c))$$

holds for some $E_1,\dots,E_c \in |D|$.

Proof. Let $f : Y \to X$ be a log-resolution of (X,Δ) and of the linear series $|D|$ such that $f_*^{-1}\Delta + \mathrm{Bs}\,|f^*D| + \mathrm{Exc}(f)$ has simple normal crossing support. Write $\Delta_Y = \Delta_Y^+ - \Delta_Y^-$, where Δ_Y^+ and Δ_Y^- are effective simple normal crossing support $\mathbb{Q}$-divisors without common components. Then $\Delta_Y^- = \sum_{i=1}^s \delta_i D_i$, for some non-negative $\delta_i \in \mathbb{Q}$ and distinct prime divisors D_i's and define

$$c = \lceil\max\{\delta_i, 1 \leq i \leq s\}\rceil + 2.$$

Moreover we have that $|f^*D| = |M| + F$, where $|M|$ is base-point free and $\mathrm{Supp}(F) = \mathrm{Bs}\,|f^*D|$. By Bertini's Theorem and [Laz, Lemma 9.1.9], we can choose $M_1,\dots,M_c \in |M|$ general divisors such that, for all $j = 1,\dots,c$, M_j is smooth, every component of M_j is not a component of $\Delta_Y, M_1,\dots,M_{j-1}$ and $\Delta_Y + M_1 + \cdots + M_c + F$ has simple normal crossing support. Now, for all $j = 1,\dots,c$, $M_j + F \in |f^*D|$, so that there exists $E_j \in |D|$ such that $M_j + F = f^*E_j$. Set $E = E_1 + \cdots + E_c$ and notice that f is also a log-resolution of $(X,\Delta+E)$.

By Remark 2.5 we have $\mathrm{Nlc}(X,\Delta) = \mathcal{Z}(\mathcal{J}_{NLC}(X,\Delta)) \subseteq \mathcal{Z}(\mathcal{J}_{NLC}(X,\Delta+E))$, the latter inclusion following by Remark 2.3(i) and (ii), because E is effective. Also, for every prime divisor Γ in the support of F we get for the discrepancies

$$a(\Gamma,X,\Delta+E) = a(\Gamma,X,\Delta) - \mathrm{ord}_\Gamma(f^*E) = -\mathrm{ord}_\Gamma(\Delta_Y) - \mathrm{ord}_\Gamma(f^*E) \leq$$

$$\leq \mathrm{ord}_\Gamma(\Delta_Y^-) - \mathrm{ord}_\Gamma(f^*E) \leq \max\{\delta_i, 1 \leq i \leq s\} - \mathrm{ord}_\Gamma(M_1+\cdots+M_c+cF) \leq -2$$

whence $f(\Gamma) \subseteq \mathrm{Nlc}(X, \Delta + E)$. As $\mathrm{Bs}\,|D|$ is the union of such $f(\Gamma)$'s, using Remark 2.5, we get the inclusion $\mathrm{Bs}\,|D| \subseteq \mathrm{Nlc}(X, \Delta + E) = \mathcal{Z}(\mathcal{J}_{NLC}(X, \Delta + E))$.

On the other hand notice that $(\Delta + E)_Y = f^*(K_X + \Delta + E) - K_Y = \Delta_Y + f^*E$. Also $\Delta_Y + f^*E = \Delta_Y + M_1 + \cdots + M_c + cF$, so that

$$\mathrm{Supp}((\Delta + E)_Y^{>1}) = \mathrm{Supp}((\Delta_Y + f^*E)^{>1}) \subseteq \mathrm{Supp}(F) \cup \mathrm{Supp}(\Delta_Y^{>1})$$

whence

$$f(\mathrm{Supp}((\Delta + E)_Y^{>1})) \subseteq f(\mathrm{Supp}(F)) \cup f(\mathrm{Supp}(\Delta_Y^{>1})) = \mathrm{Bs}\,|D| \cup \mathrm{Nlc}(X, \Delta).$$

Therefore, by Remark 2.5,

$$\mathcal{Z}(\mathcal{J}_{NLC}(X, \Delta + E)) = \mathrm{Nlc}(X, \Delta + E) = f(\mathrm{Supp}((\Delta + E)_Y^{>1})) \subseteq \mathrm{Bs}\,|D| \cup \mathrm{Nlc}(X, \Delta).$$

□

Now we essentially follow the proof of Nakamaye's Theorem as in [Laz, §10.3] and [Nye, Thm. 0.3].

Proof of Theorem 1. We can assume that D is a Cartier divisor. The issue is of course to prove that $\mathbf{B}_+(D) \subseteq \mathrm{Null}(D)$, since the opposite inclusion holds on any normal projective variety, as explained in the introduction.

By Proposition 2.10 and Remark 2.5 there is an effective Weil $\mathbb{Q}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $\mathrm{Nlc}(X, \Delta) = X_{\mathrm{nlc}}$, so that $\dim \mathrm{Nlc}(X, \Delta) \leq 1$.

Let A be an ample Cartier divisor such that $A - (K_X + \Delta)$ is ample. As in [Laz, Proof of Thm. 10.3.5]) we can choose $a, p \in \mathbb{N}$ sufficiently large such that

$$\mathbf{B}_+(D) = \mathbf{B}(aD - 2A) = \mathrm{Bs}\,|paD - 2pA|.$$

By Lemma 3.1 there exist $c \in \mathbb{N}$ and a Cartier divisor E on X such that

$$\mathbf{B}_+(D) \cup \mathrm{Nlc}(X, \Delta) = \mathcal{Z}(\mathcal{J}_{NLC}(X, \Delta + E))$$

and $E \equiv c(paD - 2pA) = qaD - 2qA$, where $q := cp \in \mathbb{N}$.

Set $Z = \mathcal{Z}(\mathcal{J}_{NLC}(X, \Delta + E))$. For $m \geq qa$, we get that

$$mD - qA - (K_X + \Delta + E) \equiv (m - qa)D + qA - (K_X + \Delta)$$

is ample, whence $H^1(X, \mathcal{J}_{NLC}(X, \Delta + E) \otimes \mathcal{O}_X(mD - qA)) = 0$, for $m \geq qa$ by [Fno, Thm. 3.2], [A, Thm. 4.4], so that the restriction map

(3) $$H^0(X, \mathcal{O}_X(mD - qA)) \to H^0(Z, \mathcal{O}_Z(mD - qA)) \text{ is surjective for } m \geq qa.$$

By contradiction let us assume that there exists an irreducible component V of $\mathbf{B}_+(D)$, such that $V \not\subseteq \mathrm{Null}(D)$. Now $V \subseteq \mathbf{B}_+(D) \subseteq \mathbf{B}(D - \frac{q}{m}A) \subseteq \mathrm{Bs}\,|mD - qA|$ for $m \in \mathbb{N}$, whence the restriction map

$$H^0(X, \mathcal{O}_X(mD - qA)) \to H^0(V, \mathcal{O}_V(mD - qA)) \text{ is zero for } m \in \mathbb{N}$$

and therefore, by (3), also

(4) $$H^0(Z, \mathcal{O}_Z(mD - qA)) \to H^0(V, \mathcal{O}_V(mD - qA)) \text{ is zero for } m \geq qa.$$

On the other hand $\dim V \geq 1$, as $\mathbf{B}_+(D)$ does not contain isolated points by [ELMNP2, Proposition 1.1](which holds on X normal). As $\dim \mathrm{Nlc}(X, \Delta) \leq 1$, this implies that V is an irreducible component of Z. Moreover, as $V \not\subseteq \mathrm{Null}(D)$, we have that $D_{|V}$ is big.

Now, by Remark 2.6, $\mathcal{J}_{NLC}(X, \Delta + E)$ is integrally closed, and exactly as in [Laz, Proof of Thm. 10.3.5] (the proof of this part holds on any normal projective variety) it follows that, for $m \gg 0$, $H^0(Z, \mathcal{O}_Z(mD - qA)) \to H^0(V, \mathcal{O}_V(mD - qA))$ is not zero, thus contradicting (4). This concludes the proof. □

Proof of Corollary 1. Note that, on any normal projective variety X, we have $X_{\mathrm{nlc}} \subseteq \mathrm{Sing}(X)$ (see for example [CD, Rmk 4.8]) and if $\dim X \leq 3$, then $\dim \mathrm{Sing}(X) \leq 1$. Then just apply Theorem 1. □

Proof of Corollary 2. By [GKM, Thm. 0.9] we know that $\mathrm{Null}(D) \subseteq \partial \overline{M}_{g,n}$. On the other hand it is well-known (see for example [BCHM, Lemma 10.1]) that $(\overline{M}_{g,n}, 0)$ is klt, whence the conclusion follows by Theorem 1. □

4. Restricted base loci on klt pairs

We first recall that, associated to a pseudoeffective divisor D, there are two more loci, one that also measures how far D is from being nef and another one that measures how far D is from being nef and abundant.

Definition 4.1. Let X be a normal projective variety and let D be a pseudoeffective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. As in [BBP, Def. 1.7], we define the **non-nef locus**

$$\mathrm{Nnef}(D) = \bigcup_{v : v(\|D\|) > 0} c_X(v)$$

where v runs among all divisorial valuations on X, $c_X(v)$ is its center and $v(\|D\|)$ is as in Definition 1.3.

Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $\kappa(D) \geq 0$. As in [CD, Def. 2.18], we define the **non nef-abundant locus**

$$\mathrm{Nna}(D) = \bigcup_{v : v(\langle D \rangle) > 0} c_X(v)$$

where again v runs among all divisorial valuations on X and $v(\langle D \rangle)$ is as in Definition 1.3.

In the sequel we will use the fact that, for D big ([ELMNP1, Lemma 3.3]) or even abundant ([Leh, Prop. 6.4]), we have $v(\|D\|) = v(\langle D \rangle)$, while in general they are different when D is only pseudoeffective ([CD, Rmk 2.16]).

We will also use (see [BFJ, page 2] and references therein)

Izumi's Theorem *Let X be a normal variety over an algebraically closed field k and let $0 \in X$ be a closed point. Let m_0 be the maximal ideal of the local ring $\mathcal{O}_{X,0}$ and set, for any $f \in \mathcal{O}_{X,0}$,* $\mathrm{ord}_0(f) = \max\{j \geq 0 : f \in m_0^j\}$. *For any divisorial valuation v of $k(X)$ centered at* 0, *there exists a constant $C = C(v) > 0$ such that*

$$C^{-1} \mathrm{ord}_0(f) \leq v(f) \leq C \,\mathrm{ord}_0(f).$$

We start by proving an analogue of [ELMNP1, Prop. 2.8] for $\mathrm{Nna}(D)$.

Theorem 4.2. *Let X be a normal projective variety, let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $\kappa(D) \geq 0$ and let v be a divisorial valuation on X. Then*

$$c_X(v) \subseteq \mathrm{Nna}(D) \text{ if and only if } v(\langle D \rangle) > 0.$$

Proof. We can assume that D is Cartier and effective. By definition of $\mathrm{Nna}(D)$, we just need to prove that if $c_X(v) \subseteq \mathrm{Nna}(D)$, then $v(\langle D \rangle) > 0$.

We first prove the theorem when X is smooth. For any $p \in \mathbb{N}$ let $b(|pD|)$ be the base ideal of $|pD|$, $\mathcal{J}(X, \|pD\|))$ the asymptotic multiplier ideal and denote by b_p and j_p the corresponding images in R_v, the DVR associated to v. As in [ELMNP1, §2], we get

$$v(\langle D \rangle) = \lim_{p \to +\infty} \frac{v(b_p)}{p} \geq \lim_{p \to +\infty} \frac{v(j_p)}{p} = \sup_{p \in \mathbb{N}} \{\frac{v(j_p)}{p}\}. \tag{5}$$

By [CD, Cor. 5.2] we have the set-theoretic equality

$$\mathrm{Nna}(D) = \bigcup_{p\in\mathbb{N}} \mathcal{Z}(\mathcal{J}(X, \|pD\|))$$

whence there exists $p_0 \in \mathbb{N}$ such that $c_X(v) \subseteq \mathcal{Z}(\mathcal{J}(X, \|p_0 D\|))$, so that $v(j_{p_0}) > 0$ and (5) gives that $v(\langle D\rangle) > 0$.

We now prove the theorem for a divisorial valuation ν on X such that $c_X(\nu) = \{x\}$ is a closed point.

As $c_X(\nu) \subseteq \mathrm{Nna}(D)$, there exists a divisorial valuation v_0 on X such that $v_0(\langle D\rangle) > 0$ and $x \in c_X(v_0)$. Let E_0 be a prime divisor over X such that $v_0 = k \operatorname{ord}_{E_0}$ for some $k \in \mathbb{N}$. We can assume that there is a birational morphism $\mu : Y \to X$ from a smooth variety Y such that $E_0 \subset Y$. As $\mu(E_0) = c_X(\operatorname{ord}_{E_0}) = c_X(v_0)$, there is a point $y \in E_0$ such that $\mu(y) = x$. Let $\pi : Y' \to Y$ be the blow-up of Y on y with exceptional divisor E_y. For any $m \in \mathbb{N}$ and $G \in |mD|$ we have

$$\operatorname{ord}_{E_y}(G) = \operatorname{ord}_{E_y}(\pi^*(\mu^*G)) = \operatorname{ord}_y(\mu^*G) \geq \operatorname{ord}_{E_0}(\mu^*G) = \operatorname{ord}_{E_0}(G)$$

therefore $\operatorname{ord}_{E_y}(\langle D\rangle) \geq \operatorname{ord}_{E_0}(\langle D\rangle) = \frac{1}{k} v_0(\langle D\rangle) > 0$. Since $c_X(\operatorname{ord}_{E_y}) = \{x\}$, by Izumi's Theorem applied twice, there exist $C > 0, C' > 0$ such that for all $m \in \mathbb{N}$ and $G \in |mD|$ we have $\operatorname{ord}_{E_y}(G) \leq C' \operatorname{ord}_x(G) \leq C\nu(G)$. Hence $\nu(\langle D\rangle) \geq \frac{1}{C} \operatorname{ord}_{E_y}(\langle D\rangle) > 0$.

Finally let v be any divisorial valuation on X with $c_X(v) \subseteq \mathrm{Nna}(D)$. As above there is a birational morphism $f : Z \to X$ from a smooth variety Z and a prime divisor $E \subset Z$ such that $v = h \operatorname{ord}_E$ for some $h \in \mathbb{N}$. For every closed point $z \in E$ we have that $\nu := \operatorname{ord}_z$ is a divisorial valuation with $c_X(\nu) \subseteq c_X(\operatorname{ord}_E) \subseteq \mathrm{Nna}(D)$ and $c_X(\nu)$ is a closed point. Thus, by what we proved above, we have that $\operatorname{ord}_z(\langle f^*(D)\rangle) = \operatorname{ord}_z(\langle D\rangle) > 0$ for all $z \in E$, so that $E \subseteq \mathrm{Nna}(f^*(D))$. As Z is smooth, we get $v(\langle D\rangle) = h \operatorname{ord}_E(\langle D\rangle) = h \operatorname{ord}_E(\langle f^*(D)\rangle) > 0$. □

We next prove an analogous result for $\mathrm{Nnef}(D)$.

Theorem 4.3. *Let X be a normal projective variety, let D be a pseudoeffective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and let v be a divisorial valuation on X. Then*

$$c_X(v) \subseteq \mathrm{Nnef}(D) \textit{ if and only if } v(\|D\|) > 0.$$

Proof. Again we need to prove that $v(\|D\|) > 0$ if $c_X(v) \subseteq \mathrm{Nnef}(D)$. By [CD, Lemmas 2.13 and 2.12], there exists a sequence of ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisors $\{A_m\}_{m\in\mathbb{N}}$ such that $\|A_m\| \to 0$, $D + A_m$ is a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor for all $m \in \mathbb{N}$ and

$$\mathrm{Nnef}(D) = \bigcup_{m\in\mathbb{N}} \mathrm{Nnef}(D + A_m).$$

Then there is $m_0 \in \mathbb{N}$ such that $c_X(v) \subseteq \mathrm{Nnef}(D + A_{m_0})$. As $D + A_{m_0}$ is big, we have $\mathrm{Nnef}(D + A_{m_0}) = \mathrm{Nna}(D + A_{m_0})$, whence $v(\|D + A_{m_0}\|) = v(\langle D + A_{m_0}\rangle) > 0$ by Theorem 4.2. Therefore $0 < v(\|D + A_{m_0}\|) \leq v(\|D\|) + v(\|A_{m_0}\|) = v(\|D\|)$. □

Remark 4.4. Note that, given a normal projective variety X, Theorems 4.2 and 4.3 can be rewritten as follows (where x is a closed point).

If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $\kappa(D) \geq 0$, then

$$\mathrm{Nna}(D) = \bigcup_{x\in X} \{x \mid \{x\} = c_X(v) \text{ for some divisorial valuation } v \text{ with } v(\langle D\rangle) > 0\}.$$

If D is a pseudoeffective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, then

$$\mathrm{Nnef}(D) = \bigcup_{x \in X} \{x \mid \{x\} = c_X(v) \text{ for some divisorial valuation } v \text{ with } v(\|D\|) > 0\}.$$

Next we will prove Theorem 2. We will use a singular version (see for example [CD, Def. 2.2]) of standard asymptotic multiplier ideal sheaves [Laz, Ch. 11].

Proof of Theorem 2. In both cases we have that $\mathrm{Nnef}(D) = \mathbf{B}_-(D)$ by [CD, Thm. 1.2], whence also $\mathrm{Nna}(D) = \mathbf{B}_-(D)$ in case (i). Then (ii) follows by Theorem 4.3 and the first equivalence in (i) by Theorem 4.2. To complete the proof of (i) we need to show that if $\limsup_{m \to +\infty} v(|mD|) = +\infty$ then $v(\langle D \rangle) > 0$, the reverse implication being obvious. We will proceed similarly to [ELMNP1, Proof of Prop. 2.8] and [CD, Proof of Lemma 4.1]. If $v(\langle D \rangle) = 0$, by what we just proved, we have that $c_X(v) \not\subseteq \mathbf{B}_-(D)$ and, by [CD, Cor. 5.2], we have the set-theoretic equality

$$\mathbf{B}_-(D) = \bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|))$$

where $\mathcal{J}((X, \Delta); \|pD\|)$ is as in [CD, Def. 2.2]. Therefore $c_X(v) \not\subseteq \mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|))$ for any $p \in \mathbb{N}$. Let H be a very ample Cartier divisor such that $H - (K_X + \Delta)$ is ample and let $n = \dim X$. By Nadel's vanishing theorem [Laz, Thm. 9.4.17], we deduce that $\mathcal{J}((X, \Delta); \|pD\|) \otimes \mathcal{O}_X((n+1)H + pD)$ is 0-regular in the sense of Castelnuovo-Mumford, whence globally generated, for every $p \in \mathbb{N}$, and therefore $c_X(v) \not\subseteq \mathrm{Bs}\,|(n+1)H + pD|$. On the other hand, as D is big, there is $m_0 \in \mathbb{N}$ such that $m_0 D \sim (n+1)H + E$ for some effective Cartier divisor E. Hence, for any $m \geq m_0$, we get $v(|mD|) = v(|(m-m_0)D + (n+1)H + E|) \leq v(|(m - m_0)D + (n+1)H|) + v(|E|) = v(|E|)$ and the theorem follows. □

We end the section with an observation on the behavior of these loci under birational maps.

Corollary 4.5. *Let $f : Y \to X$ be a projective birational morphism between normal projective varieties. Then:*

(i) *For every $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X such that $\kappa(D) \geq 0$, we have*

$$\mathrm{Nna}(f^*(D)) = f^{-1}(\mathrm{Nna}(D));$$

(ii) *For every pseudoeffective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, we have*

$$\mathrm{Nnef}(f^*(D)) = f^{-1}(\mathrm{Nnef}(D));$$

(iii) *If there exist effective Weil $\mathbb{Q}$-divisors Δ_X on X and Δ_Y on Y such that (X, Δ_X) and (Y, Δ_Y) are klt pairs, then, for every pseudoeffective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, we have*

$$\mathbf{B}_-(f^*(D)) = f^{-1}(\mathbf{B}_-(D))$$

Proof. To see (i), for every closed point $y \in Y$, let v_y be a divisorial valuation such that $c_Y(v_y) = \{y\}$. Then, by Theorem 4.2, we have,

$$y \in f^{-1}(\mathrm{Nna}(D)) \Leftrightarrow \{f(y)\} = c_X(v_y) \subseteq \mathrm{Nna}(D) \Leftrightarrow$$

$$\Leftrightarrow v_y(\langle f^*(D) \rangle) = v_y(\langle D \rangle) > 0 \Leftrightarrow \{y\} = c_Y(v_y) \subseteq \mathrm{Nna}(f^*(D)).$$

Now (ii) can be proved exactly in the same way by using Theorem 4.3, while (iii) follows from (ii) and [CD, Thm. 1.2]. □

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