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On the curves lying on a general surface containing a fixed space curve.

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ABSTRACT. - We study, up to linear equivalence, the curves lying on a general smooth surface $S \subset \mathbb{P}^3$ containing a fixed curve C. If $\deg S > \max\{3, \deg C$, degrees of the minimal generators of the ideal of $C\}$ we show that all curves on S are a linear combination of the hyperplane section H of S and C, that is $\operatorname{Pic} S \simeq \mathbb{Z}^2$ generated by $\mathcal{O}_S(1)$ and $\mathcal{O}_S(C)$. This is accomplished by proving that if $\gamma \in H^{1,1}(S)$ is the class of a curve, a general deformation S' of S, $S' \supset C$, keeps γ of type (1,1) if and only if γ is a linear combination of the classes of H and C.

KEY WORDS: *Noether-Lefschetz, Picard group.*

1. Introduction.

The study of curves lying on smooth surfaces in $\mathbb{P}^3$ is a classical and, at the same time, modern subject.

When the degree of a smooth surface S is at most three, a lot is known: a surface of degree 1 is $\mathbb{P}^2$ and the curves on it correspond to homogeneous polynomials in three variables; a smooth quadric S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and if we let x_0, x_1, y_0, y_1, be the coordinates, then every curve on S is a bihomogeneous polynomial of bidegree (a, b) (in x_0, x_1, y_0, y_1 respectively); if $d = 3$, S is isomorphic to the blowing up of $\mathbb{P}^2$ at six general points $P_1, \ldots, P_6$ and if we denote by ℓ the pull-back of a line, e_i the i^{th}-exceptional

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divisor, then the curves on S are the elements of the linear systems $a\ell - \sum_{i=1}^{6} b_i e_i$ for integers $a, b_1, \dots, b_6$, that is (essentially) correspond to plane curves with assigned singularities at $P_1, \dots, P_6$.

On the other hand no geometric description is known for surfaces of degree $d \geq 4$, except for "sufficiently general" ones: In this case a classical theorem of M. Noether and S. Lefschetz asserts that the only curves on such S are complete intersections of S with another surface. In modern language this amounts to say that the group of line bundles on S modulo isomorphism (called the Picard group of S, denoted Pic S) is $\mathbb{Z}$ generated by the hyperplane bundle $\mathcal{O}_S(1)$.

A natural question to be asked therefore, as a way of generalizing such result, is: given $C \subset \mathbb{P}^3$ and S sufficiently general containing C, what about Pic S? For example, what is the Picard group of a general surface containing a line?

The answer to this question, in the case when C is any plane curve of degree $\delta < d$ is Pic $S \cong \mathbb{Z}^2$ generated by $\mathcal{O}_S(1)$ and $\mathcal{O}_S(C)$ (see [3], [4]).

In this paper we study the above question when C is a given space curve, having in mind the goal of giving some conditions under which the same answer holds (that is Pic $S \cong \mathbb{Z}^2$). The result obtained is the following: Let $C \subset \mathbb{P}^3$ be a smooth irreducible nondegenerate curve of degree k and genus g and denote by $\alpha(C)$ the maximum degree of the polynomials in a minimal set of generators of the ideal of C. Then

THEOREM 1.1. – *If $d \geq \max\{4, \alpha(C)+1, k+1\}$ the Picard group of the general surface S of degree d containing C is generated by $\mathcal{O}_S(1)$ and $\mathcal{O}_S(C)$.*

The main idea is the use of the infinitesimal M. Noether theorem, which in turn is a remake of Lefschetz's original proof.

These basic facts are axplained in section 2. Then section 3 is devoted to see that the infinitesimal M. Noether theorem holds on the blowing up of $\mathbb{P}^3$ along the curve C. This is accomplished by a careful study of the cohomology of some sheaves on the blowing up.

It is important to note here that, while the main result of this paper (Theorem 1.1) is already contained in [4], the technical context in which the results are obtained is completely different and, I believe, of independent interest. In particular it is my opinion that the techniques used here may have more applications than the existing ones. For example one could study in a similar way surfaces containing a given curve with some multiplicity, surfaces of general type (in some cases), etc. It is with this scope in mind that I have decided to publish this paper.

2. Some basic results of Hodge theory.

The proof of the theorem is based on a deep result of Hodge theory that I wish to recall briefly here (for the proofs of all statements in this section see [2]).

Let Y be a smooth variety over the complex numbers of dimension $n+1 \geq 2$, L an ample line bundle over Y with $c_1(L) = \omega$ and $X \in |L|$ a smooth divisor.

Let $r\colon H^n(Y) \to H^n(X)$ be the restriction map and $R\colon H^{n+1}(Y-X) \to H^n(X)$ be the residue map, then there is a direct sum decomposition

$$H^n(X) = rH^n(Y) \oplus RH^{n+1}(Y-X).$$

As X varies among the smooth divisors in $|L|$ we can consider the variation of Hodge structure given by $H^n(X) = \bigoplus_{p+q=n} H^{p,q}(X)$ and it is a general fact that a global variation is completely reducible into a "fixed" component and a "variable" one, which in this case are

$$H^n_f(X) = rH^n(Y), \qquad H^n_v(X) = RH^{n+1}(Y-X).$$

Suppose now that L is such that the restriction map

$$H^0(Y, T_Y) \longrightarrow H^0(X, T_Y \otimes \mathcal{O}_X) \qquad \text{is an isomorphism}.$$

Then we can introduce an infinitesimal variation of Hodge structure as follows. Let $s \in H^0(Y, L)$ such that $X = (s)$ and denote by $T_s(|L|) = H^0(Y, L)/\mathbb{C}s \subset \mathrm{H}^0(\mathrm{X}, \mathrm{N}_{\mathrm{X/Y}})$ the tangent space to the complete linear system $|L|$ at X.

Also let σ be the composition map $T_Y \to T_Y \otimes \mathcal{O}_X \to N_{X/Y}$ and

$$W = \frac{T_s(|L|)}{T_s(|L|) \cap \operatorname{Im} \sigma}.$$

By the above hypothesis on L we see that $W \subset H^1(X, T_X)$ is the image of $T_s(|L|)$ under the Kodaira-Spencer mapping and this allows us to define the infinitesimal variation of Hodge structure

$V = \{H_{\mathbb{Z}}, H^{p,q}, W, \delta\}$ given by...

$H_{\mathbb{Z}} = H^n(X, \mathbb{Z}) /$ torsion

$H^{p,q} = H^{p,q}(X)$

$\delta \colon W \to \bigoplus_{p+q=n} \operatorname{Hom}(H^{p,q}, H^{p-1,q+1})$ given by the cup-product with the Kodaira-Spencer class.

Let $H^{p,q}_{i.f.}(X) = \{\psi \in H^{p,q} \colon \delta(\xi)\psi = 0 \ \forall \xi \in W\}$ = subspace of classes infinitesimally fixed under V and introduce the following definition:
L is *sufficiently ample* if

(1) $H^0(Y, T_Y) \to H^0(X, T_Y \otimes \mathcal{O}_X)$ is an isomorphism

(2) $H^0(Y, L) \otimes H^0(Y, K_Y \otimes L^q) \twoheadrightarrow H^0(Y, K_Y \otimes L^{q+1}) \quad \forall q \geq 1$

(3) $H^r(Y, \Omega^p_Y \otimes L^q) = 0 \quad \forall r > 0, \forall r > 0$.

Then

Infinitesimal M. Noether Theorem

For L sufficiently ample and any smooth $X \in |L|$ we have

$$H^{p,q}_{i.f.}(X) = H^{p,q}_f(X).$$

COROLLARY 2.1. – (Lefschetz) *Let $S \subset |L|$ be the open dense set of smooth divisors X. Then the monodromy representation*

$$\rho \colon \pi_1(S) \longrightarrow \operatorname{Aut}(H^n_v(X))$$

has no factors on which $\pi_1(S)$ acts as a finite group.

COROLLARY 2.2. – *If $X \in |L|$ is generic and $n = 2m$, then*

$$H^{m,m}(X, \mathbb{Q}) = \text{ image of } \{\mathrm{H}^{m,m}(\mathrm{Y}, \mathbb{Q}) \longrightarrow \mathrm{H}^{2m}(\mathrm{X}, \mathbb{Q})\}.$$

3. The proof of theorem 1.1.

With notation as in section 1 let $\widetilde{\mathbb{P}}^3$ be the blow-up of $\mathbb{P}^3$ along C, $E = \mathbb{P}(N^\vee_C)$ the exceptional divisor so that we have the following diagram

$$\begin{array}{ccc} E & \stackrel{j}{\hookrightarrow} & \widetilde{\mathbb{P}}^3 \\ \downarrow g & & f \downarrow \\ C & \stackrel{i}{\hookrightarrow} & \mathbb{P}^3. \end{array}$$

Also let $\widetilde{H}$ be the pull-back of a plane in $\mathbb{P}^3$ and $L = \mathcal{O}_{\widetilde{\mathbb{P}}^3}(d\widetilde{H} - E)$. The I wish to prove that, under the hypothesis on d of Theorem 1.1, L is sufficiently ample on $\widetilde{\mathbb{P}}^3$. This will imply the Infinitesimal M. Noether theorem for $Y = \widetilde{\mathbb{P}}^3$ and $X \in |L|$ and hence theorem 1.1 by using cor. 2.2.

I will start by proving that property (3) of the definition of sufficiently ample holds for L, but first I want to collect some facts about the ideal sheaf of C and of its "thickenings" that I will be using throughout this paper.

LEMMA 3.1. – *Let $\mathcal{I}_C$ be the ideal sheaf of C. Then*

(a) $$H^1\left(\frac{\mathcal{I}_C^{q-1}}{\mathcal{I}_C^q}(m)\right) = 0 \quad \forall q \geq 1, \quad \forall m \geq qd - 4;$$

(b) $$H^2\left(\mathcal{I}_C^q(m)\right) = 0 \quad \forall q \geq 1, \quad \forall m \geq qd - 4;$$

(c) $$H^1\left(\mathcal{I}_C^q(m)\right) = 0 \quad \forall q \geq 1, \quad \forall m \geq qd - 3;$$

(d) $$H^r\left(\Omega^1_{\mathbb{P}^3} \otimes \mathcal{I}_C^q(m)\right) = 0, \quad r = 1,2, \quad \forall q \geq 1, \quad \forall m \geq qd;$$

(e) $$H^1\left(\frac{\mathcal{I}_C^{q-2}}{\mathcal{I}_C^{q-1}}(qd) \otimes \Lambda^2(\mathcal{I}_C/\mathcal{I}_C^2)\right) = 0 \quad \forall q \geq 2;$$

(f) $$H^r\left(T_{\mathbb{P}^3} \otimes \mathcal{I}_C^q(m)\right) = 0, \quad r = 1,2, \quad \forall q \geq 1, \quad \forall m \geq qd - 4;$$

(g) $$H^1\left(N_C \otimes \frac{\mathcal{I}_C^{q-1}}{\mathcal{I}_C^q}(qd-4)\right) = 0 \quad \forall q \geq 1;$$

(h) $$H^1\left(T_C \otimes \frac{\mathcal{I}_C^{q-1}}{\mathcal{I}_C^q}(qd-4)\right) = 0 \quad \forall q \geq 1.$$

Proof. – Let S be a surface of degree d containing C (existing by the hypothesis $d \geq \alpha(C) + 1$) and let $H \subset S$ be the plane section, then $\forall m \in \mathbb{Z}$ we have an exact diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 \longrightarrow & \mathcal{I}_C^{q-1}(m-d) & \longrightarrow & \mathcal{I}_C^{q}(m) & \longrightarrow & \mathcal{O}_S(mH-qC) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(*) & 0 \longrightarrow & \mathcal{I}_C^{q-2}(m-d) & \longrightarrow & \mathcal{I}_C^{q-1}(m) & \longrightarrow & \mathcal{O}_S(mH-(q-1)C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 \longrightarrow & \frac{\mathcal{I}_C^{q-2}}{\mathcal{I}_C^{q-1}}(m-d) & \longrightarrow & \frac{\mathcal{I}_C^{q-1}}{\mathcal{I}_C^{q}}(m) & \longrightarrow & \mathcal{O}_C \otimes \mathcal{O}_S(mH-(q-1)C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

(a) induction on q:
$q=1$: $H^1\left(\frac{\mathcal{I}_C^0}{\mathcal{I}_C}(m)\right) = H^1(\mathcal{O}_C(m)) = 0$ since $mk > 2g-2$.
In fact by Castelnuovo's bound we have $g \leq 1/4(k^2-1)-k+1$ hence $2g-2 \leq 1/2(k^2-1)-2k < k(k-3) \leq k(d-4) \leq mk \quad (k \geq 3)$.
$q \geq 2$: from the last horizontal exact sequence of diagram (*) since $m-d \geq (q-1)d-4$ all we need to prove is
$H^1(\mathcal{O}_C \otimes \mathcal{O}_S(mH-(q-1)C)) = 0$ and this is true since
$\deg(\mathcal{O}_C \otimes \mathcal{O}_S(mH-(q-1)C)) = mk-(q-1)(2g-2-k(d-4)) > 2g-2$;
in fact $mk \geq (qd-4)k$, $2g-2-2kd+4k \leq 0$, $q \geq 2$ hence it's enough to see that $2(2g-2-2kd+4k)+kd < 0$ which is true by Castelnuovo's bound.

(b) again by induction on q.
$q=1$: $H^2(\mathcal{I}_C(m)) = H^1(\mathcal{O}_C(m)) = 0$ as above;
$q \geq 2$: this follows by the inductive hypothesis and (a) by looking at the second vertical exact sequence of (*).

(c) $q=1$: $H^1(\mathcal{I}_C(m)) = 0$ by Castelnuovo-Gruson-Peskine-Lazarsfeld theorem on the completedness of the hyperplane series since $m \geq d-3 \geq k-2$.
$q \geq 2$: modify diagram (*) by choosing a surface S containing C of degree $d-1$. Again by the second vertical exact sequence and the inductive hypothesis it will be enough to show

Claim - $H^0(\mathcal{I}_C^{q-1}(m)) \twoheadrightarrow H^0\left(\frac{\mathcal{I}_C^{q-1}}{\mathcal{I}_C^{q}}(m)\right) \quad \forall q \geq 2, \quad \forall m \geq qd-3.$

To see that we look at the global sections of (*) modified

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(\mathcal{I}_C^{q-2}(m-d+1)) & \stackrel{v}{\twoheadrightarrow} & H^0\left(\frac{\mathcal{I}_C^{q-2}}{\mathcal{I}_C^{q-1}}(m-d+1)\right) \\
\downarrow & & \downarrow \\
H^0(\mathcal{I}_C^{q-1}(m)) & \longrightarrow & H^0\left(\frac{\mathcal{I}_C^{q-1}}{\mathcal{I}_C^{q}}(m)\right) \\
\downarrow u & & \downarrow \\
H^0(\mathcal{O}_S(mH-(q-1)C)) & \stackrel{\varphi}{\twoheadrightarrow} & H^0(\mathcal{O}_C \otimes \mathcal{O}_S(mH-(q-1)C))\,.
\end{array}$$

Now u and v are surjective by induction so the claim will follow once proved that φ is surjective.

To see that I will actually prove that $H^1(\mathcal{O}_S(mH-qC)) = 0$.

Let C' be a generic curve in the base point free linear system $|\mathcal{O}_S((d-1)H-C)|$ so that
$mH - qC \sim (m-q(d-1))H + qC'$ and $m-q(d-1) \geq q-3 \geq -2$ for $q \geq 1$.
Subclaim - $H^1(\mathcal{O}_S(\alpha H + qC')) = 0$ $\forall \alpha \geq -2$ and $\forall q \geq 1$.
$q = 1$: follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(\alpha) \longrightarrow \mathcal{I}_C(\alpha+d-1) \longrightarrow \mathcal{O}_S(\alpha H + C') \longrightarrow 0$$

since $H^1(\mathcal{I}_C(\alpha+d-1)) = 0$ (as above because $\alpha+d-1 \geq d-3 \geq k-2$);
$q \geq 2$: follows by induction from the sequence

$$0 \longrightarrow \mathcal{O}_S(\alpha H + (q-1)C') \longrightarrow \mathcal{O}_S(\alpha H + qC') \longrightarrow \mathcal{O}_{C'} \otimes \mathcal{O}_S(\alpha H + qC') \longrightarrow 0$$

since deg $(\mathcal{O}_{C'} \otimes \mathcal{O}_S(\alpha H + qC')) > 2g(C') - 2$ (computation as in (a)).

This proves the subclaim, hence the claim and (c).

(d) Tensoring the Euler sequence by $\mathcal{I}_C^q(m)$ we get

$$0 \longrightarrow \Omega^1_{\mathbf{P}^3} \otimes \mathcal{I}_C^q(m) \longrightarrow \mathcal{I}_C^q(m-1)^{\oplus 4} \longrightarrow \mathcal{I}_C^q(m) \longrightarrow 0$$

hence the long exact sequence ...

$$\begin{aligned}
H^0(\mathcal{I}_C^q(m-1))^{\oplus 4} &\longrightarrow H^0(\mathcal{I}_C^q(m)) \longrightarrow H^1(\Omega^1_{\mathbf{P}^3} \otimes \mathcal{I}_C^q(m)) \longrightarrow \\
&\longrightarrow H^1(\mathcal{I}_C^q(m-1))^{\oplus 4} \longrightarrow H^1(\mathcal{I}_C^q(m)) \longrightarrow \\
&\longrightarrow H^2(\Omega^1_{\mathbf{P}^3} \otimes \mathcal{I}_C^q(m)) \longrightarrow H^2(\mathcal{I}_C^q(m-1))^{\oplus 4}\,.
\end{aligned}$$

Now the first map is surjective since the ideal of C is generated in degree $\leq \alpha(C)$ and $m-1 \geq qd-1 \geq q\alpha(C)$; also
$H^1(\mathcal{I}_C^q(m-1)) = H^1(\mathcal{I}_C^q(m)) = 0$ by (c) and
$H^2(\mathcal{I}_C^q(m-1)) = 0$ by (b), so (d) follows.

(e) From the sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/\mathbf{P}^3} \longrightarrow N_{S/\mathbf{P}^3} \otimes \mathcal{O}_C \longrightarrow 0$$

we see that $\Lambda^2(\mathcal{I}_C/\mathcal{I}_C^2) \cong \mathcal{O}_C \otimes \mathcal{O}_S(-dH - C)$ so we want to prove, by induction on q,

$$H^1\left(\frac{\mathcal{I}_C^{q-2}}{\mathcal{I}_C^{q-1}}(qd) \otimes \mathcal{O}_S(-dH - C)\right) = 0 \quad \forall q \geq 2;$$

$q = 2$: $$H^1\left(\frac{\mathcal{I}_C^0}{\mathcal{I}_C}(2d) \otimes \mathcal{O}_S(-dH - C)\right) = H^1(\mathcal{O}_C \otimes \mathcal{O}_S(dH - C)) = 0$$

since deg $(\mathcal{O}_C \otimes \mathcal{O}_S(dH - C)) > 2g - 2$;
$q \geq 3$: this follows by tensoring with $\mathcal{O}_C \otimes \mathcal{O}_S(-C)$ the third horizontal exact sequence of (*) written for $q-1$ in place of q and with $m = (q-1)d$. In fact this gives

$$0 \longrightarrow \frac{\mathcal{I}_C^{q-3}}{\mathcal{I}_C^{q-2}}((q-1)d) \otimes \mathcal{O}_S(-dH - C) \longrightarrow \frac{\mathcal{I}_C^{q-2}}{\mathcal{I}_C^{q-1}}(qd) \otimes \mathcal{O}_S(-dH - C) \longrightarrow$$
$$\longrightarrow \mathcal{O}_C \otimes \mathcal{O}_S((q-1)dH - (q-1)C) \longrightarrow 0$$

so H^1 of the first sheaf is 0 by induction and so is H^1 of the third by the fact deg $\big(\mathcal{O}_C \otimes \mathcal{O}_S((q-1)dH - (q-1)C)\big) > 2g - 2$.

(f) From the dual of the Euler sequence tensored by $\mathcal{I}_C^q(m)$ we get

$$0 \longrightarrow \mathcal{I}_C^q(m) \longrightarrow \mathcal{I}_C^q(m+1)^{\oplus 4} \longrightarrow T_{\mathbf{P}^3} \otimes \mathcal{I}_C^q(m) \longrightarrow 0$$

so (f) is implied by (b) and (c).

(g) $q = 1$: as in the proof of (e) we have

$$0 \longrightarrow \mathcal{O}_C \otimes \mathcal{O}_S((d-4)H + C) \longrightarrow N_C(d-4) \longrightarrow \mathcal{O}_C(2d-4) \longrightarrow 0$$

so that $H^1(N_C(d-4)) = 0$ since both $\mathcal{O}_C(d-4)$ and $\mathcal{O}_C(2d-4)$ are non-special;

$q \geq 2$: let $m = qd - 4$ in the third horizontal exact sequence of (*) and tensor with N_C; then, by induction, it will be enough to prove that $H^1\Big(N_C \otimes \mathcal{O}_S\big((qd-4)H - (q-1)C\big)\Big) = 0$ and this follows again from the sequence of normal bundles of $C \subset S \subset \mathbb{P}^3$ tensored by $\mathcal{O}_S\big((qd-4)H - (q-1)C\big)$:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_C \otimes \mathcal{O}_S\big((qd-4)H - (q-2)C\big) \longrightarrow \\ \longrightarrow N_C \otimes \mathcal{O}_S\big((qd-4)H - (q-1)C\big) \longrightarrow \\ \longrightarrow \mathcal{O}_C \otimes \mathcal{O}_S\Big([(q+1)d-4]H - (q-1)C\Big) \longrightarrow 0 \end{aligned}$$

where both the first and the third line bundle have degree strictly greater than $2g-2$.

(h) Induction on q:
$q = 1$. $H^1(T_C(d-4)) = 0$ since deg $(T_C(d-4)) = 2 - 2g + k(d-4) > 2g-2$;
$q \geq 2$: let $m = qd - 4$ in the third horizontal exact sequence of (*) and tensor with T_C so that by induction H^1 of the first sheaf is 0 and same for the third which is $T_C \otimes \mathcal{O}_S\big((qd-4)H - (q-1)C\big)$ whose degree is $> 2g-2$.

This concludes the proof of lemma 3.1.

PROPOSITION 3.2. –

$$H^r\Big(\widetilde{\mathbb{P}}^3, \Omega^p_{\widetilde{\mathbb{P}}^3} \otimes L^q\Big) = 0 \qquad \forall r > 0, \quad \forall q > 0.$$

Proof. – I wish to use the Leray spectral sequence for the blow-up $f\colon \widetilde{\mathbb{P}}^3 \longrightarrow \mathbb{P}^3$ hence first let us prove the following

LEMMA 3.3. – *Let* $\mathcal{F} = \Omega^p_{\widetilde{\mathbb{P}}^3} \otimes L^q = \Omega^p_{\widetilde{\mathbb{P}}^3}(qd\widetilde{H} - qE)$ *for* $0 \leq p \leq 3$, $q \geq 1$, *then* $R^i f_*(\mathcal{F}) = 0$ $\forall i > 0$.

PROOF OF THE LEMMA: Let $X = \widetilde{\mathbb{P}}^3$ and $y \in \widetilde{\mathbb{P}}^3$ be a point. For each $n \geq 1$ define

$$X_n = X \mathrm{x}_{\widetilde{\mathbb{P}}^3} \operatorname{Spec}(\mathcal{O}_y / m_y^n)$$

the n^{th}-thickening of the fiber over y and let $v_n : X_n \longrightarrow X$ the natural map. If $\mathcal{F}_n = v_n^* \mathcal{F}$ and ^ denotes the completion then (theorem on formal functions)

$$R^i f_*(\mathcal{F})^{\wedge}_y \cong \varprojlim H^i(X_n, \mathcal{F}_n).$$

Now since X_n is homeomorphic to either a point or $\mathbb{P}^1$ (when $y \in \mathbb{P}^3 - C$ or $y \in C$ respectively) we see that $R^i f_*(\mathcal{F})_y = 0$ if $i \geq 2$ and any y or $i = 1$ and $y \in \mathbb{P}^3 - C$.

For $i = 1$ and $y \in C$ we need a closer look.

CLAIM - $H^1(X_n, \mathcal{F}_n) = 0$ $\forall n \geq 1$ in this case.
For $n = 1$ we have $X_1 \cong \mathbb{P}^1$ and $X_1 \cdot \widetilde{H} = 0$, $X_1 \cdot E = -1$ hence
$p = 0$: $\mathcal{F}_1 = \mathcal{O}_{\widetilde{\mathbb{P}}^3}(qd\widetilde{H} - qE) \otimes \mathcal{O}_{X_1} = \mathcal{O}_{X_1}(q) \implies H^1(X_1, \mathcal{F}_1) = 0$;
$p = 3$: $\mathcal{F}_1 = \mathcal{O}_{\widetilde{\mathbb{P}}^3}\Big((qd-4)\widetilde{H} - (q-1)E\Big) \otimes \mathcal{O}_{X_1} = \mathcal{O}_{X_1}(q-1) \implies H^1(X_1, \mathcal{F}) = 0$;
$p = 1$: $\mathcal{F}_1 = \Omega^1_{\widetilde{\mathbb{P}}^3}(qd\widetilde{H} - qE) \otimes \mathcal{O}_{X_1} = \Omega^1_{\widetilde{\mathbb{P}}^3} \otimes \mathcal{O}_{X_1}(q)$
so let's look at the exact sequence

$$0 \longrightarrow N^v_{X_1/\widetilde{\mathbb{P}}^3}(q) \longrightarrow \Omega^1_{\widetilde{\mathbb{P}}^3} \otimes \mathcal{O}_{X_1}(q) \longrightarrow \Omega^1_{X_1}(q) \longrightarrow 0\,.$$

Here $H^1\Big(\Omega^1_{X_1}(q)\Big) \cong H^1\Big(\mathcal{O}_{\mathbb{P}^1}(q-2)\Big) = 0$ and from the exact sequence of normal bundles of the triple $X_1 \subset E \subset \widetilde{\mathbb{P}}^3$ we get (after dualizing and twisting by q)

$$0 \longrightarrow \mathcal{O}_{X_1}(q+1) \longrightarrow N^v_{X_1/\widetilde{\mathbb{P}}^3}(q) \longrightarrow \mathcal{O}_{X_1}(q) \longrightarrow 0$$

hence $H^1\Big(N^v_{X_1/\widetilde{\mathbb{P}}^3}(q)\Big) = 0$ since both the sheaves on the right and on the left are non-special;

$p = 2$: $\Omega^2_{\widetilde{\mathbb{P}}^3} \cong T_{\widetilde{\mathbb{P}}^3}(-4\widetilde{H} + E)$ so that $\mathcal{F}_1 = T_{\widetilde{\mathbb{P}}^3} \otimes \mathcal{O}_{X_1}(q-1)$.
Now

$$0 \longrightarrow T_{X_1}(q-1) \longrightarrow T_{\widetilde{\mathbb{P}}^3} \otimes \mathcal{O}_{X_1}(q-1) \longrightarrow N_{X_1/\widetilde{\mathbb{P}}^3}(q-1) \longrightarrow 0$$

and again $H^1\Big(T_{X_1}(q-1)\Big) \cong H^1\Big(\mathcal{O}_{\mathbb{P}^1}(q+1)\Big) = 0$ and

$$0 \longrightarrow \mathcal{O}_{X_1}(q-1) \longrightarrow N_{X_1/\widetilde{\mathbb{P}}^3}(q-1) \longrightarrow \mathcal{O}_{X_1}(q-2) \longrightarrow 0$$

so that $H^1\Big(N_{X_1/\widetilde{\mathbb{P}}^3}(q-1)\Big) = 0$.

This completes the proof of the claim for $n = 1$.
For $n \geq 2$ since $X_{n-1} \subset X_n$ we have

$$0 \longrightarrow \mathcal{I}_{X_{n-1}, X_n} \otimes \mathcal{F} \longrightarrow \mathcal{F}_n \longrightarrow \mathcal{F}_{n-1} \longrightarrow 0$$

so by induction we need to prove $H^1(\mathcal{I}_{X_{n-1},X_n} \otimes \mathcal{F}) = 0$. Now $\forall i \geq 0$ there is an exact sequence (**)

$$0 \longrightarrow \mathcal{I}_{X_{n-i-2},X_{n-i-1}} \otimes \mathcal{F}(-(i+1)E) \longrightarrow \mathcal{I}_{X_{n-i-1},X_{n-i}} \otimes \mathcal{F}(-iE) \longrightarrow$$
$$\longrightarrow \mathcal{F}(-iE) \otimes \mathcal{O}_{X_1} \longrightarrow 0$$

and $H^1\big(\mathcal{F}(-iE) \otimes \mathcal{O}_{X_1}\big) = H^1\big(\Omega^p_{\widetilde{\mathbf{P}}^3}(qd\widetilde{H} - (q+i)E) \otimes \mathcal{O}_{X_1}\big) = 0$ by the same argument as above since $q + i \geq 1$; but for $i = n-2$

$$H^1\big(\mathcal{I}_{X_{n-i-1},X_{n-i}} \otimes \mathcal{F}(-iE)\big) = H^1\big(\mathcal{I}_{X_1,X_2} \otimes \mathcal{F}(-(n-2)E)\big) =$$
$$= H^1\big(N^v_{X_1/\widetilde{\mathbf{P}}^3} \otimes \mathcal{F}(-(n-2)E)\big) = 0$$

since we have

$$0 \longrightarrow \mathcal{O}_{X_1} \otimes \mathcal{F}(-(n-1)E) \longrightarrow N^v_{X_1/\widetilde{\mathbf{P}}^3} \otimes \mathcal{F}(-(n-2)E) \longrightarrow$$
$$\longrightarrow \mathcal{O}_{X_1} \otimes \mathcal{F}(-(n-2)E) \longrightarrow 0\,.$$

So by (**) $H^1\big(\mathcal{I}_{X_{n-i-1},X_{n-i}} \otimes \mathcal{F}(-iE)\big) = 0$ $\forall i \leq n-2$ hence for $i = 0$. ■
So Lemma 3.3 is completely proved.

Now to prove the proposition observe that by the Leray spectral sequence applied to the map $f\colon \widetilde{\mathbb{P}}^3 \longrightarrow \mathbb{P}^3$ and the lemma 3.3 we have

$$H^r\big(\widetilde{\mathbb{P}}^3, \Omega^p_{\widetilde{\mathbf{P}}^3} \otimes L^q\big) = H^r\big(\mathbb{P}^3, f_*\big(\Omega^p_{\widetilde{\mathbf{P}}^3} \otimes L^q\big)\big)$$

so we will prove that the cohomology groups on the right hand side are 0.

Again let $\mathcal{F} = \Omega^p_{\widetilde{\mathbf{P}}^3} \otimes L^q = \Omega^p_{\widetilde{\mathbf{P}}^3}(qd\widetilde{H} - qE)$ for $0 \leq p \leq 3$, $q \geq 1$, and suppose first $p \geq 1$.
If $\Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3}$ is the sheaf of relative differentials then there is an exact sequence

$$(\bullet) \qquad 0 \longrightarrow f^*\Omega^1_{\mathbf{P}^3} \longrightarrow \Omega^1_{\widetilde{\mathbf{P}}^3} \longrightarrow \Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3} \longrightarrow 0$$

so by taking Λ^p and twisting by $\mathcal{O}_{\widetilde{\mathbf{P}}^3}(qd\widetilde{H} - qE)$ we get

$$0 \longrightarrow f^*\Omega^p_{\mathbf{P}^3}(qd\widetilde{H} - qE) \longrightarrow \Omega^p_{\widetilde{\mathbf{P}}^3}(qd\widetilde{H} - qE) \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{G}$ is a sheaf supported on E.

Now, since for every $q \geq 0$ $f_*(\mathcal{O}_{\widetilde{\mathbf{P}}^3}(-qE)) = \mathcal{I}_C^q$ and the projection formula, taking f_* of the above sequence gives

$$0 \longrightarrow \Omega^p_{\mathbf{P}^3} \otimes \mathcal{I}^q_C(qd) \longrightarrow f_*\mathcal{F} \longrightarrow \mathcal{G}' \longrightarrow 0$$

where $\mathcal{G}'$ is a sheaf supported on C.

Claim - $H^3\Big(f_*(f^*\Omega^p_{\mathbf{P}^3}(\alpha\widetilde{H} - \beta E))\Big) = 0$ $\forall \alpha \geq 0$ and $\forall \beta \geq 0$ ($\alpha > 0$ if $p = 3$).

Let's prove it by induction on β.

$\beta = 0$: $H^3\Big(f_*(f^*\Omega^p_{\mathbf{P}^3}(\alpha\widetilde{H}))\Big) = H^3(\Omega^p_{\mathbf{P}^3}(\alpha)) = 0$ $\forall \alpha \geq 0$ ($\alpha > 0$ if $p = 3$).

$\beta \geq 1$: taking f_* of the sequence

$$0 \longrightarrow f^*\Omega^p_{\mathbf{P}^3}(\alpha\widetilde{H} - \beta E) \longrightarrow f^*\Omega^p_{\mathbf{P}^3}(\alpha\widetilde{H} - (\beta - 1)E) \longrightarrow$$
$$\longrightarrow f^*\Omega^p_{\mathbf{P}^3}(\alpha\widetilde{H} - (\beta - 1)E) \otimes \mathcal{O}_E \longrightarrow 0$$

we get

$$0 \longrightarrow f_*(f^*\Omega^p_{\mathbf{P}^3}(\alpha\widetilde{H} - \beta E)) \longrightarrow f_*\Big(f^*\Omega^p_{\mathbf{P}^3}(\alpha\widetilde{H} - (\beta - 1)E)\Big) \longrightarrow \mathcal{H} \longrightarrow 0$$

and here $\mathcal{H}$ is a sheaf supported on C.

Hence

$$H^2(C, \mathcal{H}) \longrightarrow H^3\Big(f_*(f^*\Omega^p_{\mathbf{P}^3}(\alpha\widetilde{H} - \beta E))\Big) \longrightarrow$$
$$\longrightarrow H^3\Big(f_*(f^*\Omega^p_{\mathbf{P}^3}(\alpha\widetilde{H} - (\beta - 1)E))\Big)$$

so the claim follows by the inductive hypothesis.

But since $R^1f_*(\Omega^p_{\widetilde{\mathbf{P}}^3}(\alpha\widetilde{H} - \beta E))$ is supported on C we have

$$H^2\Big(R^1f_*(\Omega^p_{\widetilde{\mathbf{P}}^3}(\alpha\widetilde{H} - \beta E))\Big) = H^3\Big(f_*(\Omega^p_{\widetilde{\mathbf{P}}^3}(\alpha\widetilde{H} - \beta E))\Big) = 0$$

so the Leray spectral sequence gives $H^3\Big(\widetilde{\mathbb{P}}^3, \Omega^p_{\widetilde{\mathbf{P}}^3}(\alpha\widetilde{H} - \beta E)\Big) = 0$ $\forall \alpha \geq 0$ and $\forall \beta \geq 0$ ($\alpha > 0$ if $p = 3$).

Let us see now that $H^2\Big(\Omega^p_{\mathbf{P}^3} \otimes \mathcal{I}^q_C(qd)\Big) = 0$.

$p = 3$: $\Omega^3_{\mathbf{P}^3} \otimes \mathcal{I}^q_C(qd) = \mathcal{I}^q_C(qd - 4)$ hence its H^2 is 0 by (b) of lemma 3.1;

$p = 1$: $H^2\Big(\Omega^1_{\mathbf{P}^3} \otimes \mathcal{I}^q_C(qd)\Big) = 0$ by (d);

$p = 2$: $H^2\Big(\Omega^2_{\mathbf{P}^3} \otimes \mathcal{I}^q_C(qd)\Big) = H^2\Big(T_{\mathbf{P}^3} \otimes \mathcal{I}^q_C(qd - 4)\Big) = 0$ by (f).

So this proves the proposition for $p \geq 1$, $r = 2, 3$.

Suppose now $p = 0$.
Then $\mathcal{F} = \mathcal{O}_{\widetilde{\mathbf{P}}^3}(qd\widetilde{H} - qE)$ hence $f_*\mathcal{F} = \mathcal{I}_C^q(qd)$ so
$H^1\big(\mathcal{I}_C^q(qd)\big) = 0$ by (c), $H^2\big(\mathcal{I}_C^q(qd)\big) = 0$ by (b) and
$H^3\big(\mathcal{I}_C^q(qd)\big) = H^3(\mathcal{O}_{\mathbf{P}^3}(qd-4)\big) = 0$.
Let now $r = 1$, $p \geq 1$.
$p = 3$: $\mathcal{F} = \Omega^3_{\widetilde{\mathbf{P}}^3}(qd\widetilde{H} - qE) = \mathcal{O}_{\widetilde{\mathbf{P}}^3}((qd-4)\widetilde{H} - (q-1)E) \implies f_*\mathcal{F} = \mathcal{I}_C^{q-1}(qd-4) \implies H^1\big(\mathcal{I}_C^{q-1}(qd-4)\big) = 0$ by (c) for $q \geq 2$ and by $H^1\big(\mathcal{O}_{\mathbf{P}^3}(d-4)\big) = 0$ for $q = 1$.
$p = 1$: from the exact sequence ($\bullet$) we get

$$0 \longrightarrow \Omega^1_{\mathbf{P}^3} \otimes \mathcal{I}_C^q(qd) \longrightarrow f_*\mathcal{F} \longrightarrow f_*\Big(\Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3}(qd\widetilde{H} - qE)\Big) \longrightarrow 0$$

(since, by the projection formula for R^1f_* we have $R^1f_*\Big(f^*\Omega^1_{\mathbf{P}^3}(qd\widetilde{H} - qE)\Big) = R^1f_*\Big(\mathcal{O}_{\widetilde{\mathbf{P}}^3}(qd\widetilde{H} - qE)\Big) \otimes \Omega^1_{\mathbf{P}^3} = 0$ by lemma 3.3).
Now $H^1\big(\Omega^1_{\mathbf{P}^3} \otimes \mathcal{I}_C^q(qd)\big) = 0$ by (d) and to see that
$H^1\Big(f_*(\Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3}(qd\widetilde{H} - qE))\Big) = 0$ let us make the identification

$$E \cong \mathbb{P}(\mathcal{I}_C/\mathcal{I}_C^2) \quad \text{so that} \quad \mathcal{O}(-1) \cong \mathrm{N}_{\mathrm{E}/\widetilde{\mathbf{P}}^3} = \mathcal{O}_{\mathrm{E}} \otimes \mathcal{O}_{\widetilde{\mathbf{P}}^3}(\mathrm{E}).$$

Since $\Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3} = j_*(\Omega_{E/C})$ and by the projection formula we have

$$f_*\Big(\Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3}(qd\widetilde{H} - qE)\Big) = f_*\Big(\Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3}(-qE)\Big) \otimes \mathcal{O}_{\mathbf{P}^3}(qd) =$$
$$= g_*\Big(\Omega_{E/C}(q)\Big) \otimes \mathcal{O}_{\mathbf{P}^3}(qd).$$

Let $\mathcal{E} = \mathcal{I}_C/\mathcal{I}_C^2$, then the Euler sequence for the vector bundle $\mathcal{E}$ is

$$0 \longrightarrow \Omega_{E/C} \longrightarrow (g^*\mathcal{E})(-1) \longrightarrow \mathcal{O}_E \longrightarrow 0$$

so that $\Omega_{E/C} = g^*(\Lambda^2\mathcal{E})(-2)$ and by the projection formula

$$g_*\Big(\Omega_{E/C}(q)\Big) = g_*\Big[g^*(\Lambda^2\mathcal{E}) \otimes \mathcal{O}_E(q-2)\Big] = \Lambda^2\mathcal{E} \otimes g_*\Big(\mathcal{O}_E(q-2)\Big).$$

But

$$g_*\Big(\mathcal{O}_E(q-2)\Big) = \begin{cases} 0 & \text{if } q = 1 \\ \mathcal{I}_C^{q-2}/\mathcal{I}_C^{q-1} & \text{if } q \geq 2 \end{cases}$$

so we have (for $q \geq 2$)

$$f_*\left(\Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3}(qd\widetilde{H} - qE)\right) = \frac{\mathcal{I}_C^{q-2}}{\mathcal{I}_C^{q-1}}(qd) \otimes \Lambda^2\left(\mathcal{I}_C/\mathcal{I}_C^2\right)$$

hence $H^1\left(f_*\left(\Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3}(qd\widetilde{H} - qE)\right)\right) = 0$ by (e).

Let now $p = 2$.

Let F the sheaf defined by the exact sequence

$$0 \longrightarrow N_{E/\widetilde{\mathbf{P}}^3} \longrightarrow g^* N_{C/\mathbf{P}^3} \longrightarrow F \longrightarrow 0$$

then there is an exact sequence

$$(\bullet\bullet) \qquad 0 \longrightarrow T_{\widetilde{\mathbf{P}}^3} \longrightarrow f^* T_{\mathbf{P}^3} \longrightarrow j_* F \longrightarrow 0$$

where the second map is given by the composition

$$f^* T_{\mathbf{P}^3} \longrightarrow f^* T_{\mathbf{P}^3} \otimes \mathcal{O}_E = g^*\left(T_{\mathbf{P}^3} \otimes \mathcal{O}_C\right) \longrightarrow g^* N_{C/\mathbf{P}^3} \longrightarrow F\,.$$

Now tensor $(\bullet\bullet)$ by $\mathcal{O}_{\widetilde{\mathbf{P}}^3}((qd-4)\widetilde{H} - (q-1)E)$ to get

$$0 \longrightarrow T_{\widetilde{\mathbf{P}}^3}((qd-4)\widetilde{H} - (q-1)E) \longrightarrow f^* T_{\mathbf{P}^3}((qd-4)\widetilde{H} - (q-1)E) \longrightarrow$$

$$\longrightarrow j_* F((qd-4)\widetilde{H} - (q-1)E) \longrightarrow 0\,.$$

Recall that $\Omega^2_{\widetilde{\mathbf{P}}^3} \cong T_{\widetilde{\mathbf{P}}^3}(-4\widetilde{H} + E) \implies \mathcal{F} = T_{\widetilde{\mathbf{P}}^3}((qd-4)\widetilde{H} - (q-1)E)$ so taking f_* of the above sequence, by lemma 3.3, gives

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow T_{\mathbf{P}^3} \otimes \mathcal{I}_C^{q-1}(qd-4) \stackrel{\varphi}{\longrightarrow} f_*\left(j_* F((qd-4)\widetilde{H} - (q-1)E)\right) \longrightarrow 0\,.$$

Since $H^1\left(T_{\mathbf{P}^3} \otimes \mathcal{I}_C^{q-1}(qd-4)\right) = 0$ (for $q \geq 2$ by (f) and for $q = 1$ since $T_{\mathbf{P}^3} \otimes \mathcal{I}_C^{q-1}(qd-4) = \Omega^2_{\mathbf{P}^3}$ (d)), we need to see that φ is surjective in global sections.

So according to the previous description, φ is given by taking f_* of

$$f^* T_{\mathbf{P}^3} \otimes \mathcal{O}_{\widetilde{\mathbf{P}}^3}((qd-4)\widetilde{H} - (q-1)E) \longrightarrow$$

$$\longrightarrow g^*(T_{\mathbf{P}^3} \otimes \mathcal{O}_C) \otimes \mathcal{O}_{\widetilde{\mathbf{P}}^3}((qd-4)\widetilde{H} - (q-1)E) \longrightarrow$$

$$\longrightarrow g^* N_{C/\mathbf{P}^3} \otimes \mathcal{O}_{\widetilde{\mathbf{P}}^3}((qd-4)\widetilde{H} - (q-1)E) \longrightarrow$$

$$\longrightarrow F \otimes \mathcal{O}_{\widetilde{\mathbf{P}}^3}((qd-4)\widetilde{H} - (q-1)E)$$

i.e. by the composition of

$$T_{\mathbf{P}^3} \otimes \mathcal{I}_C^{q-1}(qd-4) \xrightarrow{\varphi_1} g_*\Big[g^*\Big(T_{\mathbf{P}^3} \otimes \mathcal{O}_C(qd-4)\Big) \otimes \mathcal{O}_E(q-1)\Big] =$$

$$= T_{\mathbf{P}^3} \otimes \frac{\mathcal{I}_C^{q-1}}{\mathcal{I}_C^q}(qd-4)$$

$$T_{\mathbf{P}^3} \otimes \frac{\mathcal{I}_C^{q-1}}{\mathcal{I}_C^q}(qd-4) \xrightarrow{\varphi_2} g_*\Big[g^*\Big(N_C(qd-4)\Big) \otimes \mathcal{O}_E(q-1)\Big] =$$

$$= N_C \otimes \frac{\mathcal{I}_C^{q-1}}{\mathcal{I}_C^q}(qd-4)$$

$$N_C \otimes \frac{\mathcal{I}_C^{q-1}}{\mathcal{I}_C^q}(qd-4) \xrightarrow{\varphi_4} g_*(F(q-1)) \otimes \mathcal{O}_{\mathbf{P}^3}(qd-4).$$

CLAIM - $\varphi_1, \varphi_2, \varphi_3$ are all surjective in global sections.
In fact $\operatorname{Coker}\varphi_1 \subset H^1(T_{\mathbf{P}^3} \otimes \mathcal{I}_C^q(qd-4)) = 0$ by (f);
$\operatorname{Coker}\varphi_2 \subset H^1\left(T_C \otimes \dfrac{\mathcal{I}_C^{q-1}}{\mathcal{I}_C^q}(qd-4)\right) = 0$ by (h).
To see $\operatorname{Coker}\varphi_3$ let's go back to the exact sequence defining F.
Tensoring it by $\mathcal{O}_E(q-1)$ gives

$$0 \longrightarrow \mathcal{O}_E(q-2) \longrightarrow g^*N_C \otimes \mathcal{O}_E(q-1) \longrightarrow F \otimes \mathcal{O}_E(q-1) \longrightarrow 0$$

hence after taking g_* we see that

$$\operatorname{Coker}\varphi_3 \subset H^1\Big(g_*(\mathcal{O}_E(q-2)) \otimes \mathcal{O}_{\mathbf{P}^3}(qd-4)\Big) = 0$$

since

$$g_*(\mathcal{O}_E(q-2)) = \begin{cases} 0 & \text{if } q=1 \\ \mathcal{I}_C^{q-2}/\mathcal{I}_C^{q-1} & \text{if } q \geq 2 \end{cases}$$

so that for $q=1$ there is nothing to prove, for $q=2$ because $\mathcal{O}_C(2d-4)$ is non-special and for $q \geq 3$ by (a).

This proves the claim and therefore completes the proof of proposition 3.2.

We now deal with property (1) of the definition of sufficiently ample.

PROPOSITION 3.4. –

$$H^0\Big(\widetilde{\mathbb{P}}^3, T_{\widetilde{\mathbf{P}}^3}(-d\widetilde{H}+E)\Big) = H^1\Big(\widetilde{\mathbb{P}}^3, T_{\widetilde{\mathbf{P}}^3}(-d\widetilde{H}+E)\Big) = 0.$$

Proof. –

$$H^0\Big(T_{\widetilde{\mathbf{P}}^3}(-d\widetilde{H}+E)\Big)=H^0\Big(\Omega^2_{\widetilde{\mathbf{P}}^3}(4\widetilde{H}-E-d\widetilde{H}+E)\Big)=H^3\Big(\Omega^1_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H})\Big)^*=0$$

by the proof of proposition 3.2.

Similarly $H^1\Big(\widetilde{\mathbb{P}}^3, T_{\widetilde{\mathbf{P}}^3}(-d\widetilde{H}+E)\Big) = H^2\Big(\Omega^1_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H})\Big)^*$ and to see that it is 0 it will be enough (by Leray) to prove

$$H^1\Big(R^1 f_*(\Omega^1_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H}))\Big) = H^2\Big(f_*(\Omega^1_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H}))\Big) = 0\,.$$

By the sequence (•) we get

$$0\longrightarrow f^*\Omega^1_{\mathbf{P}^3}\otimes \mathcal{O}_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H})\longrightarrow \Omega^1_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H})\longrightarrow \Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3}((d-4)\widetilde{H})\longrightarrow 0\,.$$

Now observe that

$$R^1 f_*\Big(f^*\Omega^1_{\mathbf{P}^3}\otimes \mathcal{O}_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H})\Big) = R^1 f_*\Big(\mathcal{O}_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H})\Big)\otimes \Omega^1_{\mathbf{P}^3} = 0$$

since, with the same proof as in lemma 3.3, $R^1 f_*\Big(\mathcal{O}_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H})\Big) = 0$. Thus

$$H^1\Big(R^1 f_*(\Omega^1_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H}))\Big) = H^1\Big(R^1 f_*(\Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3}((d-4)\widetilde{H}))\Big) =$$
$$= H^1\Big(R^1 g_*(\Omega_{E/C})\otimes \mathcal{O}_{\mathbf{P}^3}(d-4)\Big) = H^1\Big(\mathcal{O}_C(d-4)\Big) = 0$$

as usual.

Also taking f_* of the above exact sequence we get

$$0\longrightarrow \Omega^1_{\mathbf{P}^3}(d-4)\longrightarrow f_*\Big(\Omega^1_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H})\Big)\longrightarrow f_*\Big(\Omega_{\widetilde{\mathbf{P}}^3/\mathbf{P}^3}((d-4)\widetilde{H})\Big)\longrightarrow 0$$

and this implies $H^2\Big(f_*(\Omega^1_{\widetilde{\mathbf{P}}^3}((d-4)\widetilde{H}))\Big) = 0$. ■

Finally let's prove that property (2) holds.

PROPOSITION 3.5. –

$$H^0\Big(\widetilde{\mathbb{P}}^3,\mathcal{O}_{\widetilde{\mathbf{P}}^3}(d\widetilde{H}-E)\Big)\otimes H^0\Big(\widetilde{\mathbb{P}}^3,\mathcal{O}_{\widetilde{\mathbf{P}}^3}((qd-4)\widetilde{H}-(q-1)E)\Big)\longrightarrow\!\!\!\!\rightarrow$$
$$\longrightarrow\!\!\!\!\rightarrow H^0\Big(\widetilde{\mathbb{P}}^3,\mathcal{O}_{\widetilde{\mathbf{P}}^3}([(q+1)d-4]\widetilde{H}-qE)\Big)\qquad \forall q\geq 1\,.$$

Proof. - Suppose first $q = 1$. Then the statement is equivalent to

$$H^0\Big(\mathbb{P}^3, \mathcal{I}_C(d)\Big) \otimes H^0\Big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-4)\Big) \twoheadrightarrow H^0\Big(\mathbb{P}^3, \mathcal{I}_C(2d-4)\Big)$$

which is true by the hypothesis on d.

Now for $q \geq 2$ let S be a generic surface in the linear system $|\mathcal{O}_{\widetilde{\mathbb{P}}^3}(d\widetilde{H} - E)|$ and let's look at the following diagram

$$\begin{array}{ccc} H^0\Big(\mathcal{O}_{\widetilde{\mathbb{P}}^3}(d\widetilde{H}-E)\Big) \otimes H^0\Big(\mathcal{O}_{\widetilde{\mathbb{P}}^3}((qd-4)\widetilde{H}-(q-1)E)\Big) & \longrightarrow & H^0\Big(\mathcal{O}_{\widetilde{\mathbb{P}}^3}([(q+1)d-4]\widetilde{H}-qE)\Big) \\ \Big\downarrow r_{q,S} & & \Big\downarrow r_{q+1,S} \\ H^0\Big(\mathcal{O}_S(d\widetilde{H}-E)\Big) \otimes H^0\Big(\mathcal{O}_S((qd-4)\widetilde{H}-(q-1)E)\Big) & \xrightarrow{\varphi_{q,S}} & H^0\Big(\mathcal{O}_S([(q+1)d-4]\widetilde{H}-qE)\Big) \end{array}$$

CLAIM - It's enough to see that $\varphi_{q,S}$ is surjective.
In fact $\operatorname{Ker} r_{q+1,S} = H^0\Big(\mathcal{O}_{\widetilde{\mathbb{P}}^3}((qd-4)\widetilde{H} - (q-1)E)\Big)$ so the claim will be proved if $r_{q,S}$ is surjective.

But $r_{q,S}$ is tensor of two restrictions which are both surjective: in fact $H^1(\widetilde{\mathbb{P}}^3, \mathcal{O}_{\widetilde{\mathbb{P}}^3}) = 0$ since as usual $R^1 f_*(\mathcal{O}_{\widetilde{\mathbb{P}}^3}) = 0 \implies H^1(\widetilde{\mathbb{P}}^3, \mathcal{O}_{\widetilde{\mathbb{P}}^3}) = H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$ and

$$H^1\Big(\widetilde{\mathbb{P}}^3, \mathcal{O}_{\widetilde{\mathbb{P}}^3}([(q-1)d-4]\widetilde{H}-(q-2)E)\Big) = H^1\Big(\widetilde{\mathbb{P}}^3, \Omega^3_{\widetilde{\mathbb{P}}^3}([(q-1)d]\widetilde{H}-(q-1)E)\Big) = 0$$

by proposition 3.2 (here $q \geq 2$).

Now let C' be a general curve of the linear system $|\mathcal{O}_S \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^3}(d\widetilde{H} - E)|$, then since $K_S \cong \mathcal{O}_S \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^3}((d-4)\widetilde{H})$ and $K_{C'} \cong \mathcal{O}_{C'} \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^3}((2d-4)\widetilde{H} - E)$ we have a diagram

$$\begin{array}{ccc} H^0\Big(\mathcal{O}_S(d\widetilde{H}-E)\Big) \otimes H^0\Big(K_S \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^3}((q-1)d\widetilde{H}-(q-1)E)\Big) & \xrightarrow{\varphi_{q,S}} & H^0\Big(K_S \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^3}(qd\widetilde{H}-qE)\Big) \\ \Big\downarrow r_{q-1,C'} & & \Big\downarrow r_{q,C'} \\ H^0\Big(\mathcal{O}_{C'}(d\widetilde{H}-E)\Big) \otimes H^0\Big(K_{C'}((q-2)d\widetilde{H}-(q-2)E)\Big) & \xrightarrow{\varphi_{q-2,C'}} & H^0\Big(K_{C'}((q-1)d\widetilde{H}-(q-1)E)\Big) \end{array}$$

CLAIM -
(i) $\varphi_{q-2,C'}$ surjective $\Longrightarrow$ $\varphi_{q,S}$ surjective
(ii) $\varphi_{q-2,C'}$ is surjective.

The proof of (i) is similar as the one of the claim.

In fact $\operatorname{Ker} r_{q,C'} = H^0\left(K_S \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^3}((q-1)d\widetilde{H}-(q-1)E)\right)$ and $r_{q-1,C'}$ is surjective since tensor of two surjective maps:

from $$0 \longrightarrow \mathcal{O}_{\widetilde{\mathbb{P}}^3}(-d\widetilde{H}+E) \longrightarrow \mathcal{O}_{\widetilde{\mathbb{P}}^3} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

and the fact that $H^1(\widetilde{\mathbb{P}}^3, \mathcal{O}_{\widetilde{\mathbb{P}}^3}) = 0$ and

$$H^2(\widetilde{\mathbb{P}}^3, \mathcal{O}_{\widetilde{\mathbb{P}}^3}(-d\widetilde{H}+E)) = H^1\left(\mathcal{O}_{\widetilde{\mathbb{P}}^3}((d-4)\widetilde{H})\right)^* = 0$$

(by the usual proof) we see that $H^1(S, \mathcal{O}_S) = 0$ so the first map is surjective; as for the second map its cokernel is contained in $H^1\left(\mathcal{O}_S((qd-4)\widetilde{H}-(q-1)E)\right)$ which is 0 by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{\mathbb{P}}^3}\left([(q-1)d-4]\widetilde{H}-(q-2)E\right) \longrightarrow \mathcal{O}_{\widetilde{\mathbb{P}}^3}((qd-4)\widetilde{H}-(q-1)E) \longrightarrow$$
$$\longrightarrow \mathcal{O}_S((qd-4)\widetilde{H}-(q-1)E) \longrightarrow 0$$

where

$$H^1\left(\mathcal{O}_{\widetilde{\mathbb{P}}^3}((qd-4)\widetilde{H}-(q-1)E)\right) = H^1\left(\Omega^3_{\widetilde{\mathbb{P}}^3}(qd\widetilde{H}-qE)\right) = 0$$

by prop. 3.2 and

$$H^2\left(\mathcal{O}_{\widetilde{\mathbb{P}}^3}([(q-1)d-4]\widetilde{H}-(q-2)E)\right) = H^1\left(\Omega^3_{\widetilde{\mathbb{P}}^3}((q-1)d\widetilde{H}-(q-1)E)\right) = 0$$

for $q \geq 2$ by prop 3.2 and for $q = 1$ by $H^1\left(\widetilde{\mathbb{P}}^3, \Omega^3_{\widetilde{\mathbb{P}}^3}\right) = H^2\left(\widetilde{\mathbb{P}}^3, \mathcal{O}_{\widetilde{\mathbb{P}}^3}\right) = H^2\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}\right) = 0$.

To see (ii) we use the following classical lemma ([1]).

LEMMA. – *Let C a smooth irreducibile non-rational curve and $|D|$ a base point free birational linear system on C. Then $\forall q \geq 0$*

$$H^0(C, \mathcal{O}_C(D)) \otimes H^0(C, K_C(qD)) \twoheadrightarrow H^0(C, K_C((q+1)D)) .$$

Now the lemma applies to C' since it is not rational (in fact C' is the strict transform in the blow-up of $\mathbb{P}^3$ along C of the generic residual Γ of C in the complete intersection of two surfaces of degree d containing C and if Γ were rational it would follow that every surface of degree $2d-4$ containing C contains Γ, against the hypothesis on d and the linear system $|\mathcal{O}_{C'} \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^3}(d\widetilde{H} - E)|$ is base point free and birational since by the hypothesis $d \geq \alpha(C) + 1$ it separates points.

Thus (ii) is proved and so is proposition 3.5.

Then collecting the results of propositions 3.2, 3.4 and 3.5 we see that $L = \mathcal{O}_{\widetilde{\mathbb{P}}^3}(d\widetilde{H} - E)$ is sufficiently ample and so, by the argument in the beginning of section 3, theorem 1.1 is proved.

Finally I would like to make some remarks on the hypothesis $d \geq \deg C + 1$.

First of all if C is any curve the best one can do to improve it, by using the same proof, is to decrease it by 3, i.e. $d \geq \deg C - 2$ since there are curves with $H^1(\mathcal{I}_C(\deg C - 3)) \neq 0$. On the other hand by making special hypotheses on the curve one can get better results: for example with the same proof it is easy to prove

PROPOSITION 3.6. – *Let C be a complete intersection curve in $\mathbb{P}^3$ of two surfaces of degree a and b with (a,b) different from* (1,1), (1,2). *Then theorem* 1.1 *is true for C for any $d \geq 2a + 2b - 3$.*

But again all of these hypotheses on d seem to be fairly stronger than they should be (it seems to me that it should be wenough to assume $d \geq \alpha(C)$ or something similar) so it is still interesting to find a proof that uses more the geometry of the problem (so not only to drop the useless hypotheses) and this is what I hope to do by using deformation theory.

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