ON THE PROJECTIVE NORMALITY OF ENRIQUES SURFACES

(with an appendix by ANGELO FELICE LOPEZ and ALESSANDRO VERRA)

LUIS GIRALDO*
Departamento de Álgebra
Universidad Complutense de Madrid
Avenida Complutense, s/n
28040 Madrid, Spain
e-mail `giraldo@eucmax.sim.ucm.es`

ANGELO FELICE LOPEZ**
Dipartimento di Matematica
Università di Roma Tre
Largo San Leonardo Murialdo 1
00146 Roma, Italy
e-mail `lopez@matrm3.mat.uniroma3.it`

AND

ROBERTO MUÑOZ*
ESCET
Universidad Rey Juan Carlos
Campus de Móstoles - C. Tulipán, s/n
28933 Móstoles (Madrid), Spain
e-mail `r.munoz@escet.urjc.es`

1. INTRODUCTION

One of the basic but often difficult tasks in algebraic geometry is to describe the equations of a given smooth projective variety $X \subset \mathbb{P}^N$ in terms of its intrinsic and extrinsic geometry. In particular no general formula is known for the number of generators of the homogeneous ideal of X. Many authors from classical to nowadays, have therefore concentrated their attention on finding sufficient conditions for X to be projectively normal, that is such that the natural restriction maps $H^0(\mathcal{O}_{\mathbb{P}^N}(j)) \to H^0(\mathcal{O}_X(j))$ are surjective for every $j \geq 0$, for then Riemann-Roch and (often) vanishing theorems answer the question. In the case of curves many results are known, starting with Castelnuovo's [Ca] projective normality of linearly normal curves of genus g and degree at least $2g+1$ (with modern generalization by Mumford [Mu1]) and culminating with Green's result [G], that if a linearly normal curve

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of genus g has degree at least $2g+1+p$ then it satisfies property N_p [GL2], that is it is projectively normal, its homogeneous ideal is generated by quadrics, the relations among them are generated by linear ones and so on until the p-th syzygy module. In recent years Mukai interpreted this fact as suggesting that line bundles on X of type $K_X \otimes A^n$ should satisfy property N_p for $n \geq p+4$ when X is a surface (often called Mukai's conjecture) and that similar results should hold for higher dimensional varieties. Again many results have been proved in this direction. We mention here for example the results of Ein and Lazarsfeld [EL] for varieties of any dimension and the more precise results on syzygies or projective normality of surfaces: Pareschi [P1] proved Mukai's conjecture for abelian varieties, Butler [Bu] dealt with the ruled case, Homma [H1,2] settled Mukai's conjecture for $p=0$ on elliptic ruled surfaces and Gallego and Purnaprajna [GP1,2] gave several results on projective normality and syzygies of elliptic ruled surfaces, surfaces of general type and Enriques surfaces. The latter case has been the one of interest to us for at least three reasons. For K3 surfaces it follows by Noether's theorem and by a theorem of Saint-Donat [SD] that any linearly normal K3 surface is projectively normal and its ideal is generated by quadrics and cubics. In this case the general hyperplane section is a canonical curve which is not too far from Prym-canonical curves, like Enriques surface hyperplane sections. One is then naturally led to wonder if some kind of results of this type also hold for Enriques surfaces. On the other hand, despite of all the work done, the question of projective normality of Enriques surfaces had not been settled yet (to our knowledge the best results are the partial results of Gallego and Purnaprajna [GP1,2]). The third reason was that we had started the study of projective threefolds whose general hyperplane section is an Enriques surface, and for our methods it was important to know projective normality.

Let now $S \subset \mathbb{P}^{g-1}$ be a smooth linearly normal Enriques surface. As it is well known (or see section 3) we have $g \geq 6$ and already in the first case there are explicit examples of non projectively normal Enriques surfaces $S \subset \mathbb{P}^5$, as by the Riemann-Roch theorem this is equivalent to the fact that the surface lies on a quadric (the embedding is then called a Reye polarization; these cases are classified [CD1, Prop. 3.6.4]). On the other hand we have been able to prove that in fact the above are the only examples.

Theorem (1.1). *Let $S \subset \mathbb{P}^{g-1}$ be a linearly normal smooth irreducible Enriques surface.*
(1.2) If $g = 6$ and $\mathcal{O}_S(1)$ is a Reye polarization then S is j-normal for every $j \geq 3$ and its homogeneous ideal is generated by quadrics and cubics;
(1.3) If either $g \geq 7$ or $g = 6$ and $\mathcal{O}_S(1)$ is not a Reye polarization, then S is 3-regular in the sense of Castelnuovo-Mumford. In particular S is projectively normal and its homogeneous ideal is generated by quadrics and cubics.

In fact the theorem holds in many cases also when S is normal; see Remark (3.10).
The study of the projective normality of $S \subset \mathbb{P}^{g-1}$ can of course be reduced to the same for an hyperplane section C. In the case of an Enriques surface we have deg$C = 2g-2$ hence, by the theorem of Green and Lazarsfeld [GL2] (also in [KS]), C is projectively normal unless it has low Clifford index. Whence it becomes important to study curves with low Clifford index (or gonality) on an Enriques surface. We do this with the nowadays standard vector bundles techniques of Green, Lazarsfeld and Tyurin ([GL1], [L], [T]), proving results that are very close in spirit with the ones of [GL1], [P2], [Re1], [Ma], [Z]. We choose to state them here as they are of independent interest, since it is in general useful to know whether various specific curves can lie on an Enriques surface. Moreover they have applications in the study of projective threefolds whose general hyperplane section is an Enriques surface [GLM].
We first recall an important result about the Enriques lattice that will be also used extensively later. Let B be a nef line bundle on S with $B^2 > 0$ and set

$$\Phi(B) = \inf\{B \cdot E : |2E| \textit{ is a genus one pencil}\}.$$

Then by [CD1, Cor. 2.7.1, Prop. 2.7.1 and Thm. 3.2.1] (or [Co, 2.11]) we have $\Phi(B) \leq [\sqrt{B^2}]$, where $[x]$ denotes the integer part of a real number x. In particular if $C \subset S$ is a smooth irreducible curve of genus $g \geq 4$ and gonality k, choosing a genus one pencil calculating $\Phi(C)$, we get $g \geq \frac{k^2}{8}+1$. When g is slightly larger we can give some information on the geometry of C. Given an integer $k \geq 3$ set

$$f(k) = \begin{cases} 6 & \text{if } k = 3 \\ 2k+1 & \text{if } 4 \leq k \leq 6 \\ \frac{k^2+2k+5}{4} & \text{if } k \geq 7 \end{cases}, \quad f_a(k) = \begin{cases} 2k & \text{if } 3 \leq k \leq 6 \\ \frac{k^2+2k+5}{4} & \text{if } k \geq 7 \end{cases}.$$

Then we have

Theorem (1.4). *Let S be a smooth Enriques surface, $C \subset S$ a smooth irreducible curve of genus g and suppose that C has gonality $k \geq 3$. We have*
(1.5) if $g > \frac{k^2}{4} + k + 2$ then k is even and every g^1_k on C is cut out by a genus one pencil $|2E|$ on S;
(1.6) if k is even, $g = \frac{k^2}{4} + k + 2$ and there is no genus one pencil on S cutting out a g^1_k on C, then either there exist two genus one pencils $|2E_1|, |2E_2|$ with $E_1 \cdot E_2 = 1$ such that C is numerically equivalent to $(\frac{k}{2} + 1)(E_1 + E_2)$ or there exist a genus one pencil $|2E|$, a nodal curve R with $E \cdot R = 1$, such that C is numerically equivalent to $(\frac{k}{2} + 1)(2E + R + K_S)$;
(1.7) let $C_\eta \in |C|$ be a general element and suppose that C_η has also gonality $k \geq 3$ and that either $g > f(k)$ or C is very ample, $g > f_a(k)$ and, when $k = 6, g = 13$, that $\Phi(C) \geq 4$. Then k is even and every g^1_k on C_η is cut out by a genus one pencil $|2E|$ on S unless $k = 6, g = 13$ and C is numerically equivalent to $2E_1 + 2E_2 + 2E_3$, where $|2E_i|$ are genus one pencils and $E_i \cdot E_j = 1$ for $i \neq j$;
(1.8) if C is very ample and $k = 4$ then $g \leq 10$, and for $g = 9, 10$ the general element $C_\eta \in |C|$ has gonality at least 5;
(1.9) suppose that C is very ample. If $g \geq 18$ (respectively $g \geq 14$) and $k = 6$ (respectively $\mathrm{gon}(C_\eta) = 6$) then $S \subset \mathbb{P}H^0(\mathcal{O}_S(C))$ contains a plane cubic curve. The converse holds for C (resp. C_η) for $g \geq 14$ (resp. $g \geq 11$).

One of the nice consequences of the result of Green and Lazarsfeld in [GL1] is that a smooth plane curve of degree at least 7 cannot lie on a K3 surface ([Ma], [Re1]). As the above theorem shows the vector bundle techniques work quite well to study curves on an Enriques surface having low gonality with respect to the genus. Therefore it is not surprising that they also allow to study the existence of curves with given Clifford dimension. We recall that the Clifford index of a line bundle L on a curve C is $\mathrm{Cliff}(L) = \deg L - 2h^0(L) + 2$ and that the Clifford index of C is defined by $\mathrm{Cliff}(C) = \min\{\mathrm{Cliff}(L) : h^0(L) \geq 2, h^1(L) \geq 2\}$. For most curves the Clifford index is computed by a pencil, but there are exceptional ones, for example smooth plane curves. In [ELMS] Eisenbud, Lange, Martens and Schreyer studied curves whose Clifford index is not computed by a pencil and defined the Clifford dimension of a curve C by $\text{Cliffdim}(C) = \min\{h^0(L) - 1 : \text{Cliff}(L) = \text{Cliff}(C), h^0(L) \geq 2, h^1(L) \geq 2\}$. As it turns out curves with Clifford dimension two are just plane curves, while curves with higher Clifford dimension are quite sparse (see the conjecture and results in [ELMS]). We have

Corollary (1.10). *Let S be a smooth Enriques surface, $C \subset S$ a smooth irreducible curve of genus g and suppose that C has Clifford index $e \geq 1$ and Clifford dimension at least 2. We have*

(1.11)
$$g \leq \frac{e^2 + 10e + 29}{4};$$

(1.12) suppose that either $g > f(e+3)$ or C is very ample, $g > f_a(e+3)$ and, when $e = 3, g = 13$, that $\Phi(C) \geq 4$. Then for the general curve $C_\eta \in |C|$ we have either Cliffdim$(C_\eta) = 1$ or Cliff$(C_\eta) \neq e$, unless $e = 3, g = 13$ and C is numerically equivalent to $2E_1 + 2E_2 + 2E_3$ as in (1.7);

(1.13) S does not contain any curve isomorphic to a smooth plane curve of degree $d \geq 9$;

(1.14) the general curve $C_\eta \in |C|$ is not isomorphic to a smooth plane curve of degree 7 and 8.

We remark that Zube in [Z] has several claims about plane curves or curves of higher Clifford dimension on an Enriques surface, but almost all the proofs are incorrect.

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2. LINEAR SYSTEMS ON CURVES ON ENRIQUES SURFACES

The goal in this section will be to study when a line bundle on a given curve lying on an Enriques surface S and calculating the gonality (or the Clifford index) of the curve is restriction of a line bundle on S. The methods employed are the usual vector bundle techniques of Green, Lazarsfeld and Tyurin ([GL1], [L], [T]). We denote by $\sim$ (respectively $\equiv$)

the linear (respectively numerical) equivalence of divisors on S. Unless otherwise specified for the rest of the article we will denote by E (or E_1 etc.) divisors such that $|2E|$ is a genus one pencil on S, while nodal curves will be denoted by R, R_1 etc.. We recall that for a divisor D on S we have $D \equiv 0$ if and only if $D \sim 0$ or $D \sim K_S$. We collect what we need in the ensuing

Lemma (2.1). *Let S be a smooth irreducible Enriques surface and $C \subset S$ a smooth irreducible curve of genus g. Let $|A|$ be a base-point free g^1_k on C, let $\mathcal{F}_{C,A}$ be the kernel of the evaluation map $H^0(A) \otimes \mathcal{O}_S \to A \to 0$ and set $\mathcal{E} = \mathcal{E}_{C,A} = \mathcal{F}^*_{C,A}$. Then $\mathcal{E}$ is a rank two vector bundle sitting in an exact sequence*

$$0 \to H^0(A)^* \otimes \mathcal{O}_S \xrightarrow{\phi} \mathcal{E} \to \mathcal{O}_C(C) \otimes A^{-1} \to 0 \tag{2.2}$$

and satisfying

(2.3) $c_1(\mathcal{E}) = C,\ c_2(\mathcal{E}) = k,\ \Delta(\mathcal{E}) = c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) = 2g - 2 - 4k.$

Suppose that $g \geq 2k + 1$. Then there is an exact sequence

$$0 \to M \to \mathcal{E} \to \mathcal{I}_Z \otimes L \to 0 \tag{2.4}$$

where L, M are line bundles and Z is a zero-dimensional subscheme of S such that:

(2.5) $C \sim M + L,\ k = M \cdot L + deg(Z),\ (M - L)^2 = 2g - 2 - 4k + 4deg(Z)$;

(2.6) $|L|$ is base-component free, nontrivial and $L^2 \geq 0$;

(2.7) if $g > 2k + 1$ (respectively $g = 2k + 1$) then $M - L$ lies in the positive cone of S (respectively in its closure) and, in both cases, $M \cdot L \geq L^2$;

(2.8) if $L^2 = 0$ and k is the gonality of C then $L \sim 2E$ is a genus one pencil on S cutting out $|A|$ on C;

(2.9) if $Z = \emptyset$ and $H^1(M - L) = 0$ then the base locus of $|L|$ is contained in C.

Proof. It is well known that the vector bundles $\mathcal{E}$ as above satisfy (2.2) and (2.3) ([GL1], [L], [T], [P2]). A standard Chern class calculation shows that (2.4) implies (2.5). If $g > 2k+1$ then $\Delta(\mathcal{E}) > 0$ and $\mathcal{E}$ is Bogomolov unstable ([Bo], [L], [R], [Re2]), hence we get (2.4) in this case and the first part of (2.7). Suppose that $g = 2k + 1$ and that $\mathcal{E}$ is H-stable with respect to some ample divisor H. By a well-known argument (see e.g. [L, proof of Prop. 3.4.1]) it follows that $h^0(\mathcal{E} \otimes \mathcal{E}^*) = 1$ and $h^2(\mathcal{E} \otimes \mathcal{E}^*) = h^0(\mathcal{E} \otimes \mathcal{E}^*(K_S)) \leq 1$ (the latter because both $\mathcal{E}$ and $\mathcal{E}(K_S)$ are H-stable with the same determinant). But the Riemann-Roch theorem gives $\chi(\mathcal{E} \otimes \mathcal{E}^*) = 4$, whence a contradiction. This establishes (2.4). The instability condition means $(M-L) \cdot H \geq 0$, hence $M-L$ lies in the closure of the positive cone of S. To see (2.6) notice that $h^0(\mathcal{O}_C(C) \otimes A^{-1}) = h^1(\mathcal{O}_C(K_S) \otimes A) \neq 0$, else by the Riemann-Roch theorem we get the contradiction $0 \leq h^0(\mathcal{O}_C(K_S) \otimes A) = k - g + 1$. Since $h^1(\mathcal{O}_S) = 0$ we get by (2.2) that $\mathcal{E}$ is globally generated away from a finite set and so is L by (2.4). Note that L is not trivial: In fact by (2.2) we have $h^0(\mathcal{E}(-C)) = 0$, while if L were trivial then $C \sim M$ by (2.5) and (2.4) would imply $h^0(\mathcal{E}(-C)) \geq h^0(\mathcal{O}_S) = 1$. Then $L^2 \geq 0$ by [CD1, Prop. 3.1.4]. Now both $M-L$ and L lie in the closure of the positive cone of the Neron-Severi group of S, hence the signature theorem implies that $(M-L) \cdot L \geq 0$ ([BPV, VIII.1]), that is (2.7). To see (2.8) notice that if $L^2 = 0$ by (2.6) and [CD1, Prop. 3.1.4] we have $L \sim 2hE$ for some $h \geq 1$. Also $h^0(\mathcal{O}_S(2E-C)) = 0$, else by (2.5) and (2.6) we get $0 \leq (2E-C) \cdot C = \frac{L \cdot M}{h} - C^2 \leq \frac{k}{h} - 2g + 2 < 0$. Therefore $|2E|$ cuts out a pencil on C and hence

$$k = \text{gon}(C) \leq 2E \cdot C = \frac{L \cdot M}{h} \leq \frac{k}{h} \leq k$$

that is $h = 1, L \cdot M = k$. In particular we have $h^0(\mathcal{O}_S(-M)) = 0$, as L is nef. By (2.4) we have $h^0(\mathcal{E}(-M)) \geq 1$ and (2.2) gives $h^0(L_{|C} \otimes A^{-1}) \geq h^0(\mathcal{E}(-M)) \geq 1$. But we also have $\deg L_{|C} \otimes A^{-1} = 0$ hence (2.8) is proved. Under the hypotheses of (2.9) we have $\mathcal{E} \cong L \oplus M$ hence in particular the map ϕ of (2.2) clearly drops rank on the base points of L, that is these points belong to C. ■

We will apply the above technique to study curves with low gonality on an Enriques surface. In view of the applications in the forthcoming article [GLM], we give a result in greater generality than the one needed for the aim of the present paper.

Proof of Theorem (1.4). Suppose first $g \geq \frac{k^2}{4} + k + 2$. Since $k \geq 3$ we have $g > 2k + 1$. Let $|A|$ be a (necessarily) base-point free g^1_k on C and apply Lemma (2.1). Set $x = M \cdot L$ and $L^2 = 2y$. By the Hodge index theorem, (2.5) and (2.7), we have

$$(2g - 2 - 4k)2y \leq (M-L)^2 L^2 \leq ((M-L) \cdot L)^2 = (x - 2y)^2 \leq (k - 2y)^2$$

therefore, if $y \geq 1$, we get $g \leq \frac{k^2}{4y} + k + y + 1$ and $x \geq 2y + 1$. In particular $y \leq \frac{k-1}{2}$ hence $g = \frac{k^2}{4} + k + 2$. Thus if $g > \frac{k^2}{4} + k + 2$ then $L^2 = 0$ and we get (1.5) by (2.8).

Suppose now that k is even and $g = \frac{k^2}{4} + k + 2$. By the above argument and the hypothesis in (1.6) we get $y = 1, x = k$. Moreover we have equality in the Hodge index theorem, hence $(M-L)^2 L \equiv ((M-L)\cdot L)(M-L)$, that is $M \equiv \frac{k}{2} L$ and $C \equiv (\frac{k}{2}+1)L$. Since $L^2 = 2$ by [CD1, Prop. 3.1.4 and Cor. 4.5.1 of page 243] we have either $L \sim E_1 + E_2$ with $E_1 \cdot E_2 = 1$ or $L \sim 2E + R + K_S$ with $E \cdot R = 1$ (note that the case $L \sim 2E + R$ is excluded since it has a base component). This proves (1.6).

To see (1.7) let $|A|$ be a g^1_k on C_η. Applying Lemma (2.1) to $|A|$ we get the decomposition (2.5). By (2.8) we will be done if we prove that $L^2 = 0$. Suppose first $g > f(k)$ and $L^2 \geq 2$. The Hodge index theorem applied to $M - L$ and L implies that the only case possible is $L^2 = 2, Z = \emptyset$. Then the base locus of $|L|$ consists of two points by [CD1, Thm. 4.4.1 and Prop. 4.5.1]. Note that C_η is not hyperelliptic, hence $|C|$ is base-point free and $\Phi(C) \geq 2$ by [CD1, Cor. 4.5.1 of page 248 and Prop. 4.5.1]. Now we are going to prove that C_η must contain the base points of $|L|$. As this kind of line bundles are countably many, we get a contradiction.

To see that $\mathrm{Bs}|L| \subset C_\eta$ we use (2.9). Suppose that $h^1(M-L) \geq 1$. By (2.5) $C \cdot (M - L) = 2g - 6 - 2k > 0$, hence $h^2(M-L) = 0$. Also $(M-L)^2 = 2g - 2 - 4k$, hence $h^0(M-L) = g - 2k + h^1(M-L) \geq g - 2k + 1$. Note that $g > 2k+1$ unless $k = 3, g = 7$. Therefore $|M - L|$ is not base-component free unless $k = 3, g = 7$, for [CD1, Cor. 3.1.3] implies $h^1(M-L) = 0$. When $k = 3, g = 7$ if $|M-L|$ is base-component free by [CD1, Prop. 3.1.4] we have $M - L \sim 2hE$ and we get the contradiction $2 = C \cdot (M-L) = C \cdot 2hE \geq 4$. Therefore $M - L \sim F + \mathcal{M}$ where F is the nonempty base component and $|\mathcal{M}|$ is base-component free. In particular $h^0(\mathcal{M}) = h^0(M-L) \geq g - 2k + 1 \geq 2$ and hence $h^2(\mathcal{M}) = 0$. If $\mathcal{M}^2 \geq 2$ by [CD1, Cor. 3.1.3] we have $h^1(\mathcal{M}) = 0$ and the Riemann-Roch theorem gives $h^0(\mathcal{M}) = 1 + \frac{1}{2}\mathcal{M}^2 \geq g - 2k + 1$, that is $\mathcal{M}^2 \geq 2g - 4k$. Also $C \cdot \mathcal{M} \leq C \cdot (M-L) = 2g - 6 - 2k$. But the Hodge index theorem applied to C and $\mathcal{M}$ contradicts the inequalities on g. Now by [CD1, Prop. 3.1.4] we must have that $\mathcal{M} \sim 2hE$. Moreover notice that, unless $k = 3, g = 7$, we have $(M-L)^2 > 0$ and in this case the proof of [CD1, Cor. 3.1.2] implies $h = 1, h^1(M-L) = 0$. Therefore we are left with the case $k = 3, g = 7$ and $\mathcal{M} \sim 2hE$. Again this is impossible since $2 = C \cdot (M-L) = C \cdot F + 2hC \cdot E \geq 4$.

Suppose now that $L^2 \geq 2$, C is very ample, $g > f_a(k)$ and, when $k = 6, g = 13$, that

$\Phi(C) \geq 4$. Of course we just need to do the case $4 \leq k \leq 6, g = 2k+1$. By (2.5) the Hodge index theorem applied to $M - L$ and L implies that the only cases possible are: $L^2 = 2, k = 6, \deg Z = 1$; $Z = \emptyset$ and either $L^2 = 2, 4$ or $L^2 = k = 6$. Moreover when $L^2 = k$ we have $M \equiv L$ hence $C \equiv 2L$ and by [CD1, Lemma 3.6.1] $\Phi(L) \leq 2$; but by hypothesis $3 \leq \Phi(C) = 2\Phi(L)$, hence $\Phi(L) = 2$. If in addition $k = 6$ then by [CD1, Prop. 3.1.4 and Prop. 3.6.3] we conclude that $L \equiv E_1 + E_2 + E_3$, hence $C \equiv 2E_1 + 2E_2 + 2E_3$ as in (1.7) (here we use the fact that C is very ample).

In the case $L^2 = 2, k = 6, \deg Z = 1$ we have $M \cdot L = 5, C \cdot L = 7$. By [CD1, Prop. 3.1.4 and Cor. 4.5.1 of page 243] we have either $L \sim E_1 + E_2$ with $E_1 \cdot E_2 = 1$ or $L \sim 2E + R + K_S$ with $E \cdot R = 1$, and the hypothesis $\Phi(C) \geq 4$ gives $C \cdot L \geq 8$, a contradiction.

When $Z = \emptyset$ and $L^2 = 2, 4$ we will prove that $h^1(M - L) = 0$ unless $k = 6$ and $C \sim 2E_1 + 2E_2 + 2E_3$ as in (1.7), $L \sim E_2 + E_3$. Excluding this exception, if $L^2 = 2$ or $L^2 = 4$ and $\Phi(L) = 1$, the base locus of $|L|$ consists of two points and, as above, we will get a contradiction. Set then $L^2 = 2y, y = 1, 2$. Since $Z = \emptyset$ we have $M \cdot L = k, (M - L)^2 = 0$ by (2.5). If $k = 4, y = 2$ we already know that $h^1(M - L) = 0$. Suppose now that, in the remaining cases for k, y, we have $h^1(M - L) \geq 1$. As $C \cdot (M - L) = 2k - 4y > 0$ we get $h^2(M - L) = 0$. By the Riemann-Roch theorem we have $h^0(M - L) = 1 + h^1(M - L) \geq 2$. If $|M - L|$ is base-component free by [CD1, Prop. 3.1.4] we have $M - L \sim 2hE_1$. Therefore $2k - 4y = C \cdot (M - L) = 2hC \cdot E_1 \geq 6h$ and we have necessarily $y = h = 1, k = 5, 6$. If $k = 5$ we have $3 = (M - L) \cdot L = 2E_1 \cdot L$, a contradiction. If $k = 6$ note that it cannot be $L \sim 2E + R + K_S$ (because $\Phi(C) \geq 4$ gives $8 = C \cdot L \geq 9$), therefore by [CD1, Prop. 3.1.4 and Cor. 4.5.1 of page 243] we have $L \sim E_2 + E_3$ with $E_2 \cdot E_3 = 1$. Now $4 = (M - L) \cdot L = 2E_1 \cdot (E_2 + E_3)$ implies $E_1 \cdot E_2 = E_1 \cdot E_3 = 1$ (else $E_1 \cdot E_2 = 0, E_1 \cdot E_3 = 2$, but then $E_1 \equiv E_2$ contradicting $E_2 \cdot E_3 = 1$). Therefore $C \sim M + L \sim 2E_1 + 2E_2 + 2E_3$ as in (1.7). Suppose now that $M - L \sim F + \mathcal{M}$ where F is the nonempty base component and $|\mathcal{M}|$ is base-component free. If $C \cdot F = 1$ then F is a line, $F^2 = -2$ and $1 = C \cdot F = 2L \cdot F - 2 + \mathcal{M} \cdot F$ implies that $\mathcal{M} \cdot F$ is odd and at least 1. In particular $0 = (M - L)^2 = -2 + \mathcal{M}^2 + 2\mathcal{M} \cdot F \geq \mathcal{M}^2$.

Going back to the general case, we have $h^0(\mathcal{M}) = h^0(M - L) \geq 2$. If $\mathcal{M}^2 \geq 2$ we have $C \cdot F \geq 2$ and hence $C \cdot \mathcal{M} \leq 2k - 2 - 4y$. But the Hodge index theorem applied to

C and $\mathcal{M}$ gives a contradiction. Therefore $\mathcal{M}^2 = 0$ and by [CD1, Prop. 3.1.4] we have $\mathcal{M} \sim 2hE_1$. As $C \cdot \mathcal{M}$ is now even we also get $C \cdot F \geq 2$. From $2k - 4y = C \cdot (M - L) = C \cdot F + 2hC \cdot E_1 \geq 2 + 2h\Phi(C)$ we get $1 \leq h \leq \frac{k-1-2y}{\Phi(C)}$, again a contradiction.

We are then left with the case $L^2 = 4$ and $\Phi(L) = 2$. Moreover, as we have seen above, we have $M \cdot L = k, Z = \emptyset, (M - L)^2 = 0$ and $h^1(M - L) = h^1(M - L + K_S) = 0$ (the latter because the proof of $h^1(M - L) = 0$ depends only on the numerical class of $M - L$ and the first because the exception $C \equiv 2E_1 + 2E_2 + 2E_3$ does not occur when $L^2 = 4$). Recall that we have also proved that, when $k = 4$, then $M \equiv L, C \equiv 2L$. Observe now that it cannot be $k = 5$, else $C \cdot (M - L) = 2$. But then $h^2(M - L) = 0$ and $h^0(M - L) = 1$, by the Riemann-Roch theorem. This is not possible since then $|M - L|$ contains a conic, but for a conic $F \subset S$ the only possible F^2 are $-2, -4, -8$.

Suppose then $k = 4, 6$. First we prove that $H^1(-M) = 0$. By [CD1, Prop. 3.1.4 and Thm. 4.4.1] $|L|$ is base-point free and $H^1(L) = H^1(L + K_S) = 0$ by [CD1, Cor. 3.1.3]. Let $D \in |L|$ be a general member. Then D is smooth irreducible of genus 3 and the exact sequence

$$0 \to \mathcal{O}_S(M - L + K_S) \to \mathcal{O}_S(M + K_S) \to \mathcal{O}_D(M + K_S) \to 0$$

shows that $H^1(-M) = H^1(M + K_S) = 0$ if $k = 6$ since $M \cdot D = 6 > 2g(D) - 2$. If $k = 4$ we have $H^1(-M) = 0$ since $M \equiv L$. Similarly $H^1(M) = 0$.

Then $h^0(L) = h^0(L_{|C_\eta}) = 3$. Note now that by (2.2) and (2.4) we have $h^0(L_{|C_\eta} \otimes A^{-1}) = h^0(\mathcal{E}(-M)) \geq 1$. The linear system $|L|$ defines a surjective morphism $\phi_L : S \to \mathbb{P}^2$ of degree 4 by [CD1, Thm. 4.6.3]. Let $\Delta \in |L_{|C_\eta} \otimes A^{-1}|$ be an effective divisor on C_η of degree 4. For every $B \in |A|$ we have $\Delta + B \in |L_{|C_\eta}|$, hence we can find a line $L_B \subset \mathbb{P}^2$ such that $\phi_L(\Delta + B) \subset L_B$. But we can also find $B' \in |A|$ such that $L_B \neq L_{B'}$, hence $\phi_L(\Delta)$ must be a point in $\mathbb{P}^2$, that is either $\Delta = \phi_L^{-1}(\phi_L(x))$ for some $x \in S$ such that $\dim\phi_L^{-1}(\phi_L(x)) = 0$, or Δ is contained on a one-dimensional fiber of ϕ_L. We will therefore be done if we show that C_η does not contain any scheme-theoretic zero-dimensional fiber of ϕ_L nor shares four points with any one-dimensional fiber of ϕ_L, for every L as above.

Note that the second case does not occur if $k = 4$ because we have $C \equiv 2L$, hence L is ample and base-point free, therefore all the fibers of ϕ_L are zero-dimensional.

Consider now the incidence correspondence

$$J_L = \{(x, H) : \dim \phi_L^{-1}(\phi_L(x)) = 0, \phi_L^{-1}(\phi_L(x)) \subset H\} \subset S \times |C|,$$

together with its two projections π_i. We claim that $\dim \pi_1^{-1}(x) \leq g-4$ for every $x \in S$ such that $\dim \phi_L^{-1}(\phi_L(x)) = 0$. Of course this gives $\dim J_L \leq g-2$ and π_2 is not dominant. As the possible L are at most countably many we get the first result needed.

Now let $W = \phi_L^{-1}(\phi_L(x))$ be zero-dimensional and let $D, D' \in |L|$ be two general divisors passing through x so that $W = D \cap D'$ and $\pi_1^{-1}(x) = I\!\!P H^0(\mathcal{J}_{W/S}(C))$. In the exact sequence

$$0 \to \mathcal{J}_{D/S}(C) \to \mathcal{J}_{W/S}(C) \to \mathcal{J}_{W/D}(C) \to 0$$

we have $\mathcal{J}_{D/S}(C) = M$, hence $h^0(\mathcal{J}_{D/S}(C)) = k-1, h^1(\mathcal{J}_{D/S}(C)) = 0$. Also $h^0(\mathcal{J}_{W/D}(C)) = h^0(\mathcal{O}_D(C-W)) = h^0(M_{|D})$. But for $k=6$ we have $h^1(M_{|D}) = 0$, while for $k=4$ we get $h^1(M_{|D}) \leq 1$, hence $h^0(M_{|D}) \leq k-1$ and $h^0(\mathcal{J}_{W/S}(C)) \leq g-3$.

We now deal with the case of one-dimensional fibers. We have then $k=6$. Let G be any effective divisor on S such that $L \cdot G = 0, G^2 \leq -2$. Set $x = C \cdot G = M \cdot G \geq 1, G^2 = -2y, y \geq 1$. The Hodge index theorem applied to M and $-3xL + 2G$ gives the inequality $2y \geq x^2$. In particular if $G^2 = -2$ then $C \cdot G = 1$. This fact implies that there is no nodal curve R such that $L \cdot R = 0, h^0(L-2R) \geq 2$ because then $C \cdot (L-2R) = 8, (L-2R)^2 = -4$, hence certainly $L - 2R \sim F_1 + \mathcal{M}$ has a base component F_1 and $|\mathcal{M}|$ is base-component free, $h^0(\mathcal{M}) = h^0(L-2R) \geq 2$. As usual either $\mathcal{M} \sim 2hE$, but this gives the contradiction $8 = C \cdot (L-2R) = C \cdot F_1 + 2hC \cdot E \geq 9$, or $\mathcal{M}^2 \geq 2, C \cdot \mathcal{M} \leq 7$. By the Hodge index theorem applied to C and $\mathcal{M}$ we get $\mathcal{M}^2 = 2, C \cdot \mathcal{M} = 7$. By [CD1, Prop. 3.1.4 and Cor. 4.5.1 of page 243] we have either $\mathcal{M} \sim E_1 + E_2$ with $E_1 \cdot E_2 = 1$ or $\mathcal{M} \sim 2E + R + K_S$ with $E \cdot R = 1$, and the hypothesis $\Phi(C) \geq 4$ gives $C \cdot \mathcal{M} \geq 8$, a contradiction.

Let now F be a scheme-theoretic one-dimensional fiber of ϕ_L, with irreducible components F_i's. Then $L \cdot F_i = 0$ for every i and the Hodge index theorem shows that $F^2 \leq -2, F_i^2 = -2$. Let $z = \phi_L(F) \in I\!\!P^2$ and take a pencil of lines L_t through z. Then $\phi_L^*(L_t) = F + D_t \in |L|$ for some divisors D_t. In particular $h^0(L-F) \geq 2$. This shows that all the F_i's occur with multiplicity one in F, else $h^0(L-2F_i) \geq 2$, which we have have proved impossible.

If F is connected then $p_a(F) \geq 0$, hence $F^2 = -2$ and, as we have seen above, $C \cdot F = 1$, the desired result. Now by [CD1, proof of Lemma 4.6.3 and Cor. 4.3.1] we see that a fiber of ϕ_L must be connected unless $L \sim 2E + R_1 + R_2 + K_S$ with $E \cdot R_1 = E \cdot R_2 = 1, R_1 \cdot R_2 = 0$. In the latter case setting $G = R_1 + R_2$ we get $x \leq 2$. But C is very ample, hence $x = 2$ and we have equality in the Hodge index theorem, that is $2M \equiv 3L - R_1 - R_2$ and then $C \equiv 5E + 2R_1 + 2R_2$. But in this case any nodal curve R different from R_1 and R_2 is not contracted by ϕ_L, else $L \cdot R = 0$, hence $E \cdot R = R_1 \cdot R = R_2 \cdot R = 0$, but then $C \cdot R = 0$, a contradiction. Therefore the only curves contracted by ϕ_L in this case are R_1 and R_2 and $C \cdot R_1 = C \cdot R_2 = 1$.

Alternatively we can avoid the use of [CD1, proof of Lemma 4.6.3 and Cor. 4.3.1] in the following way. If F has a unique irreducible component R, by the above we have $F = R$ and $C \cdot F = 1$. If not let R_1, R_2 be two distinct irreducible components of F. As $(R_1 + R_2)^2 \leq -2$ we have $0 \leq R_1 \cdot R_2 \leq 1$. Set $G = R_1 + R_2$. If $R_1 \cdot R_2 = 1$ then $G^2 = -2$ hence $C \cdot G = 1$, a contradiction. Therefore $R_1 \cdot R_2 = 0$ and, as above, we get $2M \equiv 3L - R_1 - R_2$ and then $2C \equiv 5L - R_1 - R_2$. Now if R is another irreducible component of F we have $R \cdot L = R \cdot R_1 = R \cdot R_2 = 0$, hence $C \cdot R = 0$, a contradiction. Therefore $F = R_1 + R_2$ and $C \cdot F = 2$. The proof of (1.7) is then complete.

Now (1.8) follows from (1.5) and (1.7) since, if C is very ample it cannot be $2E \cdot C = 4$, otherwise E is a conic, in contradiction with $E^2 = 0$. Similarly for (1.9), since (1.5) and (1.7) give $E \cdot C = 3$, that is E is a plane cubic. On the other hand if there is a plane cubic E then $C \cdot E = 3$ and by [CD1, Thm. 3.2.1, Prop. 3.1.2 and Prop. 3.1.4] the system $|2E|$ is a genus one pencil which cuts out a g^1_6 on C. Then (1.5) and (1.7) imply that the gonality is 6. ■

Remark (2.10). In the case C very ample and $k = 5, g \geq 11$ a more precise result holds. In fact the above proof shows that there exists a countable family $\{Z_n, n \in \mathbb{N}\}$ of zero dimensional subschemes $Z_n \subset S$ of degree two, such that if $C' \in |C|$ does not contain Z_n for every n, then $\mathrm{gon}(C') \geq 6$. This remark will be useful in [GLM].

We now deal with the existence of curves on an Enriques surface with low Clifford dimension.

Proof of Corollary (1.10). By a result of Coppens and Martens [CM, Thm. 2.3] we have

$k = \text{gon}(C) = e + 3$ and there is a one dimensional family of g^1_k's. Let $|A|$ be a general g^1_k. Of course $|A|$ cannot be cut out by a line bundle on S. Whence $g \le \frac{e^2+10e+29}{4}$ by (1.5). Similarly (1.12) follows by (1.7). Finally (1.13) and (1.14) are easy consequences of (1.11), (1.12) by taking into account the fact that a smooth plane curve of degree $d \ge 5$ has Clifford dimension 2 and Clifford index $d - 4$. ∎

3. CLIFFORD INDEX AND PROJECTIVE NORMALITY OF CURVES ON ENRIQUES SURFACES

We henceforth let $S \subset I\!P^{g-1}$ be a smooth linearly normal Enriques surface and C be a general hyperplane section of S of genus g. Note that necessarily $g \ge 6$ since, as C is very ample, we have $3 \le \Phi(C) \le [\sqrt{2g-2}]$.

We start the study of projective normality with a special case that appears to escape the vector bundle methods of section 2 and needs to be done in another way. In fact we do not know if this case really occurs (see also Remark (3.9)).

Lemma (3.1). *Let $S \subset I\!P^9$ be a smooth linearly normal Enriques surface such that its general hyperplane section C is isomorphic to a smooth plane sextic. Then S is 2-normal, that is $H^1(\mathcal{J}_S(2)) = 0$.*

Proof. Of course we have $g = 10$ and $C^2 = 18$ hence $3 \le \Phi(C) \le 4$. We first exclude the case $\Phi(C) = 3$. To this end let $|2E|$ be a genus one pencil such that $C \cdot E = 3$. Set $L = 2E, M = C - 2E$. Observe that $C \cdot L = 6, C \cdot M = 12$, hence $H^2(M) = H^0(-M) = 0$ and there is an exact sequence

$$0 \to \mathcal{O}_S(-M) \to \mathcal{O}_S(L) \to \mathcal{O}_S(L)_{|C} \to 0$$

whence we will be done if we prove that

(3.2) $\quad H^1(-M) = 0$

for then $|L_{|C}|$ is a base-point free complete g^1_6 on C, but this is not possible on a smooth plane sextic, as any such g^1_6 is contained in the linear series cut out by the lines (this is a well-known fact, see for example [LP]). To see (3.2) first notice that since $M^2 = 6$ by the Riemann-Roch theorem $h^0(M + K_S) \ge 4$. Suppose first that $M + K_S$ is base-component free. Then it is nef, hence so is M and therefore (3.2) follows by [CD1, Cor.

3.1.3]. Otherwise set $M + K_S \sim F + \mathcal{M}$ where F is the nonempty base component and $|\mathcal{M}|$ is base-component free. Note that $h^0(\mathcal{M}) = h^0(M + K_S) \geq 4$ hence $h^2(\mathcal{M}) = 0$. By [CD1, Prop. 3.1.4] we have either $\mathcal{M} \sim 2hE_1$ or $\mathcal{M}^2 > 0$. In the first case notice that the proof of [CD1, Cor. 3.1.2] gives $h = 1, (M + K_S)^2 = 2$, a contradiction. If $\mathcal{M}^2 > 0$, since $\mathcal{M}$ is nef we get $h^1(\mathcal{M}) = 0$ by [CD1, Cor. 3.1.3], hence $4 \leq h^0(\mathcal{M}) = 1 + \frac{1}{2}\mathcal{M}^2$, that is $\mathcal{M}^2 \geq 6$. The Hodge index theorem gives then $C \cdot \mathcal{M} \geq 11$, whence necessarily $C \cdot \mathcal{M} = 11, C \cdot F = 1, \mathcal{M}^2 = 6$. But then F is a line, $F^2 = -2$ and $M^2 = 6$ gives $F \cdot \mathcal{M} = 1$. Therefore $(M + K_S) \cdot F = -1$ and $H^1((M + K_S)_{|F}) = 0$. On the other hand $H^1(M + K_S - F) = H^1(\mathcal{M}) = 0$ which, together with the previous vanishing, implies (3.2) by Serre duality. We now suppose $\Phi(C) = 4$ and let $|2E|$ be a genus one pencil such that $C \cdot E = 4$. We are going to prove first that there are three possible cases for C:

(3.3) $C \sim 2E + E_1 + E_2$ with $E \cdot E_1 = E \cdot E_2 = 2, E_1 \cdot E_2 = 1;$

(3.4) $C \sim 2E + E_1 + E_2 + F$ with $E \cdot E_1 = E \cdot F = E_1 \cdot E_2 = E_1 \cdot F = 1,$
$E \cdot E_2 = 2, F \cdot E_2 = 0;$

(3.5) $C \sim 2E + E_1 + E_2 + R_1 + R_2$ with $E \cdot E_1 = E \cdot E_2 = E_1 \cdot E_2 = E \cdot R_1 = E \cdot R_2 =$
$= E_1 \cdot R_2 = E_2 \cdot R_1 = 1, E_1 \cdot R_1 = E_2 \cdot R_2 = R_1 \cdot R_2 = 0$

where $|2E_1|, |2E_2|$ are genus one pencils, F, R_1, R_2 are nodal curves.

Setting $L = 2E, M = C - 2E$ we have $C \cdot M = 10, M^2 = 2$ and $h^2(M) = 0, h^0(M) \geq 2$ by the Riemann-Roch theorem. First suppose that M is base-component free. Then by [CD1, Prop. 3.1.4 and Cor. 4.5.1 of page 243] we have that either $M \sim E_1 + E_2$ or $M \sim 2E_1 + R + K_S$ where $E_1 \cdot E_2 = E_1 \cdot R = 1$. We start by excluding the second case. In fact then $10 = C \cdot M = 2C \cdot E_1 + C \cdot R$ and $C \cdot R \geq 1, C \cdot E_1 \geq 4$ (recall the hypothesis $\Phi(C) = 4$) imply $4 = C \cdot E_1 = 2E \cdot E_1 + 1$, a contradiction. If $M \sim E_1 + E_2$, by the same argument we must have, without loss of generality, either $C \cdot E_1 = 4, C \cdot E_2 = 6$ or $C \cdot E_1 = C \cdot E_2 = 5$. The first case is not possible since then $4 = C \cdot E_1 = 2E \cdot E_1 + 1$. Therefore $5 = C \cdot E_1 = 2E \cdot E_1 + 1$, that is $E \cdot E_1 = 2$, similarly $E \cdot E_2 = 2$ and we are in case (3.3). Now suppose that M has a nonempty base component F and set $M \sim F + \mathcal{M}$, with $|\mathcal{M}|$ base-component free and $h^0(\mathcal{M}) = h^0(M) \geq 2$. We claim that in this case $\mathcal{M}^2 = 2$. If not, as above we get that either $\mathcal{M} \sim 2E_1$ or $\mathcal{M}^2 \geq 4$. In the latter case, since $10 = C \cdot F + C \cdot \mathcal{M}$, we have $C \cdot \mathcal{M} \leq 9$ and the Hodge index theorem implies

$\mathcal{M}^2 = 4, C \cdot \mathcal{M} = 9, C \cdot F = 1$ and as above $F^2 = -2, F \cdot \mathcal{M} = 0$ (from $M^2 = 2$). But then $1 = C \cdot F = 2E \cdot F - 2$, a contradiction. If $\mathcal{M} \sim 2E_1$ by $10 = C \cdot F + 2C \cdot E_1$ we must have $C \cdot F = 2, C \cdot E_1 = 4$. Now F is a conic (possibly non reduced), F^2 can be only $-2, -4$ or -8 and $2 = M^2 = F^2 + 4F \cdot E_1$ implies $F^2 = -2, F \cdot E_1 = 1$. But this contradicts $4 = C \cdot E_1 = 2E \cdot E_1 + 1$. Now let us consider the case $\mathcal{M}^2 = 2$. Again either $\mathcal{M} \sim E_1 + E_2$ or $\mathcal{M} \sim 2E_1 + R + K_S$ with $E_1 \cdot E_2 = E_1 \cdot R = 1$. In the second case we have $10 = C \cdot M = C \cdot F + C \cdot \mathcal{M}$ and $C \cdot \mathcal{M} = 2C \cdot E_1 + C \cdot R \geq 9$ hence $C \cdot E_1 = 4, C \cdot R = 1, C \cdot \mathcal{M} = 9, C \cdot F = 1, F^2 = -2$ and $F \cdot \mathcal{M} = 1$ (from $M^2 = 2$). Also $1 = F \cdot \mathcal{M} = 2E_1 \cdot F + R \cdot F$ implies $R \neq F$, hence necessarily $E_1 \cdot F = 0$ (recall that E_1 is nef since $2E_1$ is). Now $C \sim 2E + 2E_1 + R + F + K_S$ and we get the contradiction $4 = C \cdot E_1 = 2E \cdot E_1 + 1$. If $\mathcal{M} \sim E_1 + E_2$, since $C \cdot F + C \cdot \mathcal{M} = 10$, without loss of generality we can assume that either $C \cdot E_1 = C \cdot E_2 = 4$ or $C \cdot E_1 = 4, C \cdot E_2 = 5$. First we prove that if $C \cdot E_1 = 4, C \cdot E_2 = 5$ we are in case (3.4). In fact then $C \cdot F = 1, F^2 = -2$ and $F \cdot \mathcal{M} = 1$. The latter gives $1 = F \cdot E_1 + F \cdot E_2$ hence $0 \leq F \cdot E_1 \leq 1$ and the first implies $E \cdot F = 1$. From $C \cdot E_1 = 4$ we get $3 = 2E \cdot E_1 + F \cdot E_1$ hence $E \cdot E_1 = F \cdot E_1 = 1, F \cdot E_2 = 0$. Finally $C \cdot E_2 = 5$ gives $E \cdot E_2 = 2$ and we are in case (3.4). It remains to see that, if $C \cdot E_1 = C \cdot E_2 = 4$, then we are in case (3.5). To this end notice that $C \cdot F = 2$ and F is a conic. Recall that $2 = M^2$ gives $F^2 + 2F \cdot \mathcal{M} = 0$. If $F = 2R$, with R a line, then $F^2 = -8, R \cdot \mathcal{M} = 2$ and $1 = C \cdot R = 2E \cdot R - 2$, a contradiction. If F is irreducible or union of two distinct meeting lines then $F^2 = -2, F \cdot \mathcal{M} = 1$, but this contradicts $2 = C \cdot F = 2E \cdot F - 1$. Therefore F must be union of two disjoint lines R_1, R_2 and $F^2 = -4, F \cdot \mathcal{M} = 2$. Hence $(E_1 + E_2) \cdot R_1 + (E_1 + E_2) \cdot R_2 = 2$ and in particular $0 \leq (E_1 + E_2) \cdot R_1 \leq 2$. On the other hand by $1 = C \cdot R_1 = 2E \cdot R_1 + (E_1 + E_2) \cdot R_1 - 2$ we must have $(E_1 + E_2) \cdot R_1 = 1$ and $E \cdot R_1 = 1$ and similarly $(E_1 + E_2) \cdot R_2 = E \cdot R_2 = 1$. From $C \cdot E = C \cdot E_1 = C \cdot E_2 = 4$ we have then $E \cdot E_1 + E \cdot E_2 = 2, 3 = 2E \cdot E_1 + R_1 \cdot E_1 + R_2 \cdot E_1, 3 = 2E \cdot E_2 + R_1 \cdot E_2 + R_2 \cdot E_2$. It follows that $0 \leq E \cdot E_i \leq 1, i = 1, 2$. If $E \cdot E_1 = 0$ then $E \equiv E_1$ but this contradicts the first of the three equalities above. Similarly we cannot have $E \cdot E_2 = 0$. Therefore $E \cdot E_1 = E \cdot E_2 = 1, R_1 \cdot E_1 + R_2 \cdot E_1 = R_1 \cdot E_2 + R_2 \cdot E_2 = 1$, and again $0 \leq E_1 \cdot R_1 \leq 1$. Swapping R_1 with R_2 we can assume $E_1 \cdot R_1 = 0$ and we get $E_1 \cdot R_2 = 1, E_2 \cdot R_1 = 1$ (from $(E_1 + E_2) \cdot R_1 = 1$), $E_2 \cdot R_2 = 0$, hence we are in case (3.5).

Finally we prove that the linear systems (3.3), (3.4) and (3.5) are 2-normal. In all cases we will apply the following easy

Claim (3.6). *Write* $C \sim B_1 + B_2$ *with* $|B_1|, |B_2|$ *base-point free linear systems such that* $H^1(B_1) = H^2(B_1 - B_2) = H^1(2B_2) = H^2(2B_2 - B_1) = 0$. *Then* S *is* 2*-normal, that is the multiplication map* $H^0(\mathcal{O}_S(C)) \otimes H^0(\mathcal{O}_S(C)) \to H^0(\mathcal{O}_S(2C))$ *is surjective.*

Proof of Claim (3.6). This is similar to [GP1, Lemma 2.6]. We have a diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_S(B_1)) \otimes H^0(\mathcal{O}_S(B_2)) \otimes H^0(\mathcal{O}_S(C)) & \to & H^0(\mathcal{O}_S(C)) \otimes H^0(\mathcal{O}_S(C)) \\ \downarrow \mathrm{id} \otimes \mu & & \downarrow \\ H^0(\mathcal{O}_S(B_1)) \otimes H^0(\mathcal{O}_S(B_2 + C)) & \xrightarrow{\nu} & H^0(\mathcal{O}_S(2C)) \end{array}$$

where the maps μ, ν are surjective by Castelnuovo-Mumford and Claim (3.6) is proved.

We now set $B_i = E + E_i, i = 1, 2$ in case (3.3). To see that B_1 is base-point free notice that certainly B_1 is nef and $B_1^2 = 4$, hence by [CD1, Prop. 3.1.6] B_1 has no base component unless $B_1 \sim 2E' + R$ with $|2E'|$ a genus one pencil, R a nodal curve and $E' \cdot R = 1$. In that case $E' \cdot E + E' \cdot E_1 = 1$ hence either $E' \cdot E = 0, E' \cdot E_1 = 1$ but then $E' \equiv E, E' \cdot E_1 = 2$ or $E' \cdot E_1 = 0, E' \cdot E = 1$ but then $E' \equiv E_1, E' \cdot E = 2$. Now by [CD1, Prop. 3.1.4 and Thm. 4.4.1] B_1 is base-point free unless $\Phi(B_1) = 1$, which we have just excluded. Similarly B_2 is base-point free. Moreover $H^1(B_1) = 0$ by [CD1, Cor. 3.1.3]. Also $B_1 - B_2 = E_1 - E_2$ and $C \cdot (E_2 - E_1 + K_S) = 0$ whence if $H^2(B_1 - B_2) = H^0(E_2 - E_1 + K_S)^* \neq 0$, then $E_2 \sim E_1 + K_S$, but this contradicts $E_1 \cdot E_2 = 1$. Now $2B_2$ is nef, $(2B_2)^2 = 16$ hence as usual $H^1(2B_2) = 0$. Also $C \cdot (E_1 - E - 2E_2 + K_S) = -9$ hence $H^2(2B_2 - B_1) = H^0(E_1 - E - 2E_2 + K_S)^* = 0$ and we are done with case (3.3). We now proceed similarly in the other two cases. In case (3.4) set $B_1 = E + E_2, B_2 = E + E_1 + F$. Note that both B_1 and B_2 are nef (since F is irreducible). Now exactly by the same argument of case (3.3) B_1 is base-point free and $H^1(B_1) = 0$. As for B_2, if there exists a genus one pencil $|2E'|$ such that $E' \cdot B_2 = 1$ then $E' \cdot E + E' \cdot E_1 + E' \cdot F = 1$ hence either $E' \cdot E = 1, E' \cdot E_1 = E' \cdot F = 0$ and $E' \equiv E_1$ but then $E' \cdot F = 1$, or $E' \cdot E_1 = 1, E' \cdot E = E' \cdot F = 0$ and $E' \equiv E$ but then $E' \cdot F = 1$, or $E' \cdot F = 1, E' \cdot E = E' \cdot E_1 = 0$ and $E' \equiv E \equiv E_1$ but then $E \cdot E_1 = 0$. Hence B_2 is base-point free. Now $B_1 - B_2 = E_2 - E_1 - F$ and $C \cdot (E_1 + F - E_2 + K_S) = 0$ whence if $H^2(B_1 - B_2) = H^0(E_1 + F - E_2 + K_S)^* \neq 0$, then $E_1 + F \sim E_2 + K_S$, but this gives $E_2^2 = 1$. Also $2B_2$ is nef, $(2B_2)^2 = 16$ hence as usual $H^1(2B_2) = 0$. Since $C \cdot (E_2 - E - 2E_1 - 2F + K_S) = -9$ we get $H^2(2B_2 - B_1) = H^0(E_2 - E - 2E_1 - 2F + K_S)^* = 0$ and we are done with case (3.4). In case (3.5) set $B_1 = E + E_1 + R_2, B_2 = E + E_2 + R_1$. Again both B_1 and B_2 are nef and let us show that they are base-point free and $H^1(B_1) = 0$. In fact if there exists a genus one pencil $|2E'|$ such that $E' \cdot B_1 = 1$ then $E' \cdot E + E' \cdot E_1 + E' \cdot R_2 = 1$ hence either $E' \cdot E = 1, E' \cdot E_1 = E' \cdot R_2 = 0$ and $E' \equiv E_1$ but then $E' \cdot R_2 = 1$, or $E' \cdot E_1 = 1, E' \cdot E = E' \cdot R_2 = 0$ and $E' \equiv E$ but then $E' \cdot R_2 = 1$, or $E' \cdot R_2 = 1, E' \cdot E = E' \cdot E_1 = 0$ and $E' \equiv E \equiv E_1$ but then $E \cdot E_1 = 0$. Hence B_1 is base-point free and so is B_2 by symmetry. Now $B_1 - B_2 = E_1 + R_2 - E_2 - R_1$ and $C \cdot (E_2 + R_1 - E_1 - R_2 + K_S) = 0$ whence if $H^2(B_1 - B_2) = H^0(E_2 + R_1 - E_1 - R_2 + K_S)^* \neq 0$, then $E_2 + R_1 \sim E_1 + R_2 + K_S$, but this gives $(E_2 + R_1) \cdot R_1 = 0$, a contradiction. Also $2B_2$ is nef, $(2B_2)^2 = 16$ hence as usual $H^1(2B_2) = 0$. Since $C \cdot (E_1 + R_2 - E - 2E_2 - 2R_1 + K_S) = -9$ we get $H^2(2B_2 - B_1) = H^0(E_1 + R_2 - E - 2E_2 - 2R_1 + K_S)^* = 0$ and we are done with case (3.5). ∎

In the case of a Reye polarization of genus 6 we do not have projective normality, however we can still decide j-normality for $j \geq 3$ and the generation of the ideal.

Lemma (3.7). *Let $S \subset \mathbb{P}^5$ be a linearly normal smooth irreducible Enriques surface embedded with a Reye polarization. Then S is j-normal for every $j \geq 3$ and its homogeneous ideal is generated by quadrics and cubics.*

Proof. By definition S lies on a quadric in $\mathbb{P}^5$. In fact by [CD2] (as mentioned in section 1 of [DR]) the quadric must be nonsingular and, under its identification with the Grassmann variety $\mathbb{G} = \mathbb{G}(1,3)$, S is equal to the Reye congruence of some web of quadrics. We apply then the results of Arrondo-Sols [ArSo]. Setting Q for the universal quotient bundle on $\mathbb{G}$, by [ArSo, 4.3] we have an exact sequence

$$(3.8) \qquad 0 \to S^2 Q^* \to \mathcal{O}_{\mathbb{G}}^{\oplus 4} \to \mathcal{I}_{S/\mathbb{G}}(3) \to 0$$

whence $H^1(\mathcal{I}_{S/\mathbb{G}}(3)) = 0$ (since $H^1(\mathcal{O}_{\mathbb{G}}) = H^2(S^2 Q^*) = 0$ by [ArSo, 1.4] or Bott vanishing) and then of course $H^1(\mathcal{I}_{S/\mathbb{P}^5}(3)) = 0$. It follows that $\mathcal{I}_{S/\mathbb{P}^5}$ is 4-regular in the sense of Castelnuovo-Mumford and hence in particular $H^1(\mathcal{I}_{S/\mathbb{P}^5}(j)) = 0$ for every $j \geq 3$. To see the generation of the homogeneous ideal $\bigoplus_{j \geq 0} H^0(\mathcal{I}_{S/\mathbb{P}^5}(j))$ it is again enough to show that the multiplication maps $H^0(\mathcal{O}_{\mathbb{G}}(1)) \otimes H^0(\mathcal{I}_{S/\mathbb{G}}(j)) \to H^0(\mathcal{I}_{S/\mathbb{G}}(j+1))$ are surjective for every $j \geq 3$. The latter in turn follows by the Euler sequence of $\mathbb{G} \subset I\!P^5$ from the vanishing $H^1(\Omega^1_{I\!P^5}{}_{|\mathbb{G}} \otimes \mathcal{J}_{S/\mathbb{G}}(j)) = 0$ for every $j \geq 4$. Tensoring (3.8) with $\Omega^1_{I\!P^5}{}_{|\mathbb{G}}(j-3)$ we see that we just need $H^1(\Omega^1_{I\!P^5}{}_{|\mathbb{G}}(j-3)) = H^2(S^2Q^* \otimes \Omega^1_{I\!P^5}{}_{|\mathbb{G}}(j-3)) = 0$. The first follows by the Euler sequence and the second by tensoring the Euler sequence with S^2Q^* and [ArSo, 1.4] (or Bott vanishing). ∎

We are now ready to prove the main result of this article.

Proof of Theorem (1.1). By Lemma (3.7) we have to prove (1.3). Notice that we just need to show that $H^1(\mathcal{J}_S(2)) = 0$ because the other two vanishings $H^2(\mathcal{J}_S(1)) = H^1(\mathcal{O}_S(1)) = 0$ and $H^3(\mathcal{J}_S) = H^2(\mathcal{O}_S) = 0$ are already given. The other conclusions of the theorem all follow by Castelnuovo-Mumford regularity ([Mu2, page 99], [EG, Thm. 1.2]). The case $g = 6$ being already mentioned in the introduction and the cases $g = 7, 8$ being handled in the appendix, we suppose henceforth $g \geq 9$. Let now C be a general hyperplane section of S. Of course, as S is linearly normal, it is equivalent to prove that C is 2-normal, as it can be readily seen from the exact sequence

$$0 \to \mathcal{J}_{S/I\!P^{g-1}}(1) \to \mathcal{J}_{S/I\!P^{g-1}}(2) \to \mathcal{J}_{C/I\!P^{g-2}}(2) \to 0.$$

Since $h^1(\mathcal{O}_S) = 0$ we know that C is linearly normal and we can apply [GL2, Thm. 1] (or [KS]), that is we need to show that $\deg(C) \geq 2g + 1 - 2h^1(\mathcal{O}_C(1)) - \mathrm{Cliff}(C)$. Now $\mathcal{O}_C(1) \cong \omega_C(K_S)$ hence $\deg(C) = 2g - 2, h^1(\mathcal{O}_C(1)) = h^0(\mathcal{O}_C(K_S)) = 0$. Therefore we will be done if we show that $\mathrm{Cliff}(C) \geq 3$. Notice that by [CD1, Thm. 4.5.4] C is not hyperelliptic, that is $\mathrm{Cliff}(C) \geq 1$. As it is well known $\mathrm{Cliff}(C) = 1$ if and only if either $\mathrm{gon}(C) = 3$ or C is isomorphic to a smooth plane quintic. The latter have genus 6 and the first are excluded by (1.5). Again we know that $\mathrm{Cliff}(C) = 2$ if and only if either $\mathrm{gon}(C) = 4$ or C is isomorphic to a smooth plane sextic. The latter being done in Lemma (3.1) we are left with the case $\mathrm{gon}(C) = 4$ which is excluded by (1.8). ∎

Remark (3.9). In the case of genus 9 when $C \sim 2L + K_S$ the line bundle L is not very ample, hence the results of [BEL], [AnSo], do not apply. Moreover note that this case is exactly below the application of Thm. 2.14 of [GP2] (where it is required $L^2 \geq 6$; note that this hypothesis is missing both in Thm. 0.3 and in Cor. 2.15 of [GP2] because of a misprint). In the case of genus 10 we suspect, but have been unable to prove, that there is no Enriques surface embedded in $\mathbb{P}^9$ so that the general hyperplane section is isomorphic to a smooth plane sextic. By introducing the vector bundle $\mathcal{E}$ associated to a g^1_5 we can only prove that we have a contradiction if $h^1(\mathcal{E} \otimes \mathcal{E}^*) \neq 0$. It is likely that the case $h^1(\mathcal{E} \otimes \mathcal{E}^*) = 0$ can be done using the characterization of exceptional bundles of Kim [K].

Remark (3.10). It is not difficult to see that the proof of Theorem (1.1) holds, with simple modifications, in many cases, also for normal Enriques surfaces. Precisely we have that a globally generated line bundle $\mathcal{L}$ on an Enriques surface S with $\mathcal{L}^2 = 2g-2$ and $\Phi(\mathcal{L}) \geq 3$ (that is when the image $\phi_{\mathcal{L}}(S)$ is normal [CD1, Thm. 4.6.1]) is normally generated in the following cases: $g = 6$ and $\mathcal{L}$ is not a Reye polarization; $g = 9$ or $g \geq 11$; $g = 10$ and the general curve $C \in |\mathcal{L}|$ is not isomorphic to a smooth plane sextic.

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APPENDIX

ANGELO FELICE LOPEZ AND ALESSANDRO VERRA

Dipartimento di Matematica
Università di Roma Tre
Largo San Leonardo Murialdo 1
00146 Roma, Italy
e-mail lopez@matrm3.mat.uniroma3.it verra@matrm3.mat.uniroma3.it

In this note we complement the result of Giraldo-Lopez-Muñoz on the question of projective normality of Enriques surfaces by proving the following

Theorem (A.1). *For $g = 7, 8$ let $S \subset I\!P^{g-1}$ be a linearly normal smooth irreducible Enriques surface. Then S is 3-regular in the sense of Castelnuovo-Mumford. In particular S is projectively normal and its ideal is generated by quadrics and cubics.*

We denote by $\sim$ (respectively $\equiv$) the linear (respectively numerical) equivalence of divisors on S. Unless otherwise specified we will denote by E (or E_1 etc.) divisors such that $|2E|$ is a genus one pencil on S, while nodal curves will be denoted by R, R_1 etc..

Our first task will be to use a deep result about lattices [CD] to characterize the possible linear systems for $g = 7, 8$.

Lemma (A.2). *Let C be a hyperplane section of S. For $g = 7$ we have*

$(A.3)$ $C \sim 2E + F + K_S$

where $|2E|$ is a genus one pencil, F is an isolated curve with $E \cdot F = 3, F^2 = 0$.

For $g = 8$ the possible linear systems are:

$(A.4)$ $C \sim 2E + E_1 + E_2 + K_S$ *with* $E \cdot E_1 = E_1 \cdot E_2 = 1, E \cdot E_2 = 2;$

$(A.5)$ $C \sim 2E + 2E_1 + R$ *with* $E \cdot E_1 = E \cdot R = E_1 \cdot R = 1;$

$(A.6)$ $C \sim 2E + 2E_1 + R + K_S$ *with* $E \cdot E_1 = E \cdot R = E_1 \cdot R = 1;$

$(A.7)$ $C \sim 2E + 2E_1 + R_1 + R_2$ *with* $E \cdot E_1 = E \cdot R_2 = E_1 \cdot R_1 = R_1 \cdot R_2 = 1$

$E \cdot R_1 = E_1 \cdot R_2 = 0;$

$(A.8)$ $C \sim 2E + 2E_1 + R_1 + R_2 + K_S$ *with* $E \cdot E_1 = E \cdot R_2 = E_1 \cdot R_1 = R_1 \cdot R_2 = 1$

$E \cdot R_1 = E_1 \cdot R_2 = 0;$

$(A.9)$ $C \sim 2E + E_1 + E_2 + R_1 + R_2 + K_S$ *with* $E_2 \equiv E, E \cdot E_1 = E \cdot R_1 = E \cdot R_2 = 1$

$$E_1 \cdot R_1 = E_1 \cdot R_2 = R_1 \cdot R_2 = 0;$$

$$(A.10) \quad C \sim 2E+E_1+E_2+R+K_S \quad \textit{with} \ E \cdot E_1 = E \cdot E_2 = E \cdot R = E_1 \cdot E_2 = E_2 \cdot R = 1$$
$$E_1 \cdot R = 0,$$

where $|2E|, |2E_1|$ *and* $|2E_2|$ *are genus one pencils,* R, R_1, R_2 *are nodal curves.*

Proof of Lemma (A.2). By [CD, Cor. 2.7.1, Prop. 2.7.1 and Thm. 3.2.1] (or [Co, 2.11]) we know that if we set $\Phi(C) = \inf\{C \cdot E : |2E| \text{ is a genus one pencil}\}$ then $3 \leq \Phi(C) \leq [\sqrt{2g-2}]$, where $[x]$ denotes the integer part of a real number x. Hence in our case $\Phi(C) = 3$ and there is a genus one pencil $|2E|$ such that $C \cdot E = 3$. We set $M = C - 2E + K_S$. Suppose first that $g = 7$. We have $M^2 = 0, C \cdot M = 6$ hence $h^2(M) = 0, h^0(M) \geq 1$. Note that $|M|$ cannot be base-component free, else by [CD, Prop. 3.1.4] we have $M \sim 2hE_1$. But then $C \sim 2E + 2hE_1 + K_S$ and this contradicts $C \cdot E = 3$. Set then $M \sim F + \mathcal{M}$ where F is the nonempty base component and $|\mathcal{M}|$ is base-component free. Note that $h^0(\mathcal{M}) = h^0(M) \geq 1$. We are going to prove that $\mathcal{M}$ is trivial. In fact if not then by [CD, Prop. 3.1.4] either $\mathcal{M} \sim 2hE_1$ or $\mathcal{M}^2 > 0$. In the first case we get the contradiction $6 = C \cdot M = C \cdot F + 2hC \cdot E_1 \geq 7$ since $C \cdot F \geq 1, C \cdot E_1 \geq 3$. In the second case by $6 = C \cdot M = C \cdot F + C \cdot \mathcal{M}$ we get $C \cdot \mathcal{M} \leq 5$ and the Hodge index theorem gives $12\mathcal{M}^2 \leq (C \cdot \mathcal{M})^2 \leq 25$ hence $\mathcal{M}^2 = 2, C \cdot \mathcal{M} = 5, C \cdot F = 1$, that is F is a line and $F^2 = -2$. Also $M^2 = 0$ gives $F \cdot \mathcal{M} = 0$. By [CD, Prop. 3.1.4 and Cor. 4.5.1 of page 243] we have that either $\mathcal{M} \sim E_1 + E_2$ or $\mathcal{M} \sim 2E_1 + R + K_S$ with $E_1 \cdot E_2 = E_1 \cdot R = 1$ (note that the case $\mathcal{M} \sim 2E_1 + R$ is excluded since it has a base component). Now the first case is excluded by $5 = C \cdot \mathcal{M} = C \cdot E_1 + C \cdot E_2 \geq 6$, while the second is excluded by $5 = C \cdot \mathcal{M} = 2C \cdot E_1 + C \cdot R \geq 7$. Therefore for $g = 7$ we see that (A.3) holds.

We now consider the case $g = 8$. We have $M^2 = 2, C \cdot M = 8$ hence $h^2(M) = 0, h^0(M) \geq 2$. If $|M|$ is base-component free by [CD, Prop. 3.1.4 and Cor. 4.5.1 of page 243] we have that either $M \sim E_1 + E_2$ or $M \sim 2E_1 + R + K_S$ with $E_1 \cdot E_2 = E_1 \cdot R = 1$. In the first case we have $8 = C \cdot M = C \cdot E_1 + C \cdot E_2$ hence either $C \cdot E_1 = 3, C \cdot E_2 = 5$ and we get case (A.4) or $C \cdot E_1 = C \cdot E_2 = 4$, but this is not possible since it gives that $4 = C \cdot E_1 = 2E \cdot E_1 + 1$. In the second case from $8 = 2C \cdot E_1 + C \cdot R$ we get $C \cdot E_1 = 3, C \cdot R = 2$. The latter implies $E \cdot R = 1$, the first $E \cdot E_1 = 1$ and we get case (A.5). Now suppose instead that $M \sim F + \mathcal{M}$ where F is the nonempty base component and $|\mathcal{M}|$ is base-component free.

Note that $h^0(\mathcal{M}) = h^0(M) \geq 2$. By [CD, Prop. 3.1.4 and Cor. 3.1.2] we have that either $\mathcal{M} \sim 2E_1$ or $\mathcal{M}^2 > 0$. In the first case we claim that we get the linear systems (A.6) and (A.8).

To see this note that $8 = C \cdot M = C \cdot F + 2C \cdot E_1$ gives as usual $C \cdot F = 2, C \cdot E_1 = 3$. In particular F is a conic and hence the possible values of F^2 are $-2, -4, -8$. On the other hand from $2 = M^2 = F^2 + 4F \cdot E_1$ we get $F^2 = -2, F \cdot E_1 = 1$. Now $C \cdot F = 2$ implies $E \cdot F = 1$ and $C \cdot E = 3$ gives $E \cdot E_1 = 1$. If F is irreducible we get case (A.6). If $F = R_1 + R_2$ is union of two meeting lines then $1 = C \cdot R_i, i = 1, 2$ gives $1 = E \cdot R_i + E_1 \cdot R_i$. Also $1 = E \cdot F = E \cdot R_1 + E \cdot R_2$ hence without loss of generality we can assume $E \cdot R_1 = 0$ and therefore $E \cdot R_2 = 1, E_1 \cdot R_1 = 1, E_1 \cdot R_2 = 0$ and we are in case (A.8).

Now suppose $\mathcal{M}^2 > 0$. We have $8 = C \cdot F + C \cdot \mathcal{M}$ hence $C \cdot \mathcal{M} \leq 7$ and the Hodge index theorem gives $14\mathcal{M}^2 \leq (C \cdot \mathcal{M})^2 \leq 49$, therefore necessarily $\mathcal{M}^2 = 2, C \cdot \mathcal{M} = 6, 7$. If $C \cdot \mathcal{M} = 7$ it follows that $C \cdot F = 1$, that is F is a line and $F^2 = -2$. Also $M^2 = 2$ gives $F \cdot \mathcal{M} = 1$. By [CD, Prop. 3.1.4 and Cor. 4.5.1 of page 243] we have that either $\mathcal{M} \sim E_1 + E_2$ or $\mathcal{M} \sim 2E_1 + R + K_S$ with $E_1 \cdot E_2 = E_1 \cdot R = 1$. If $\mathcal{M} \sim E_1 + E_2$ without loss of generality we can assume $C \cdot E_1 = 3, C \cdot E_2 = 4$. Now $1 = E_1 \cdot F + E_2 \cdot F$ hence $0 \leq F \cdot E_1 \leq 1$ and from $C \cdot F = 1$ we get $E \cdot F = 1$. Also $C \cdot E_1 = 3$ gives $2E \cdot E_1 + F \cdot E_1 = 2$ and it cannot be $E \cdot E_1 = 0, F \cdot E_1 = 2$, therefore we have $E \cdot E_1 = 1, F \cdot E_1 = 0, E_2 \cdot F = 1$ and $C \cdot E_2 = 4$ gives $E \cdot E_2 = 1$. This is now case (A.10).

When $\mathcal{M} \sim 2E_1 + R + K_S$ we must have $C \cdot E_1 = 3, C \cdot R = 1$. Now $1 = F \cdot \mathcal{M} = 2E_1 \cdot F + R \cdot F$ gives $E_1 \cdot F = 0, R \cdot F = 1$. Also $C \cdot F = 1$ implies $E \cdot F = 1$; $C \cdot R = 1$ implies $E \cdot R = 0$ and $C \cdot E_1 = 3$ gives $E \cdot E_1 = 1$. Thus we get case (A.7).

Finally we deal with the case $C \cdot \mathcal{M} = 6, C \cdot F = 2$ and F is a conic. It cannot be $\mathcal{M} \sim 2E_1 + R + K_S$, else $6 = 2C \cdot E_1 + C \cdot R \geq 7$. Hence $\mathcal{M} \sim E_1 + E_2, C \cdot E_1 = C \cdot E_2 = 3$. From $M^2 = 2$ we get $F^2 + 2F \cdot \mathcal{M} = 0$. If $F^2 = -2$ then $F \cdot \mathcal{M} = 1$, but this contradicts $2 = C \cdot F = 2E \cdot F - 1$. If $F = 2R$ with R a line, then $F^2 = -8$ and $R \cdot \mathcal{M} = 2$, but this contradicts $1 = C \cdot R = 2E \cdot R - 2$. It remains the case $F = R_1 + R_2$ with R_1, R_2 two lines and $R_1 \cdot R_2 = 0$. Now $F^2 = -4$ hence $F \cdot \mathcal{M} = 2$, that is $(E_1 + E_2) \cdot R_1 + (E_1 + E_2) \cdot R_2 = 2$. In particular $0 \leq (E_1 + E_2) \cdot R_1 \leq 2$. On the other hand $1 = C \cdot R_1 = 2E \cdot R_1 - 2 + (E_1 + E_2) \cdot R_1$ implies $(E_1 + E_2) \cdot R_1 = E \cdot R_1 = 1$ and similarly $(E_1 + E_2) \cdot R_2 = E \cdot R_2 = 1$. From

$3 = C \cdot E = E \cdot E_1 + E \cdot E_2 + 2$ we deduce, without loss of generality $E \cdot E_2 = 0, E \equiv E_2, E \cdot E_1 = E_2 \cdot R_1 = E_2 \cdot R_2 = 1$ and then $E_1 \cdot R_1 = E_1 \cdot R_2 = 0$ and we are in case (A.9). ■

Before proving the theorem we record the following easy ad hoc modification of Green's H^0-Lemma to the case of Gorenstein curves (this is inspired by the work of Franciosi [F]).

Lemma (A.11). *Let D be a Gorenstein curve, $\mathcal{L}, \mathcal{M}$ be two base-point free line bundles on D. Suppose that either*

(A.12) $h^0(\omega_D \otimes \mathcal{M}^{-1} \otimes \mathcal{L}) = 0$, *or*

(A.13) $h^0(\omega_D \otimes \mathcal{M}^{-1} \otimes \mathcal{L}) = 1, h^0(\mathcal{L}) = 4$ *and there is an irreducible component Z of D such that Im* $\{H^0(D, \omega_D \otimes \mathcal{M}^{-1} \otimes \mathcal{L}) \to H^0(Z, (\omega_D \otimes \mathcal{M}^{-1} \otimes \mathcal{L})_{|Z})\} \neq 0$ *and* $H^0(D, \mathcal{L}) \to H^0(Z, \mathcal{L}_{|Z})$ *is injective,*

then the multiplication map $H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L} \otimes \mathcal{M})$ *is surjective.*

Proof of Lemma (A.11). Given any pair of line bundles $\mathcal{A}, \mathcal{B}$ on D, we define in the usual way ([G], [L], [F]) the Koszul cohomology groups $\mathcal{K}_{p,q}(D, \mathcal{A}, \mathcal{B}) = \mathrm{Ker} d_{p,q}/\mathrm{Im} d_{p+1,q-1}$ where $d_{p,q} : \bigwedge^p H^0(\mathcal{B}) \otimes H^0(\mathcal{A} \otimes \mathcal{B}^q) \to \bigwedge^{p-1} H^0(\mathcal{B}) \otimes H^0(\mathcal{A} \otimes \mathcal{B}^{(q+1)})$. Then the Lemma is equivalent to the vanishing $\mathcal{K}_{0,1}(D, \mathcal{M}, \mathcal{L}) = 0$. Note that the duality theorem [G, Thm. 2.c.6] holds also in this setting (see [F]) and gives $\mathcal{K}_{0,1}(D, \mathcal{M}, \mathcal{L}) \cong \mathcal{K}_{r-1,1}(D, \omega_D \otimes \mathcal{M}^{-1}, \mathcal{L})^*$, where $h^0(\mathcal{L}) = r + 1$. Under hypothesis (A.12) we have clearly $\mathcal{K}_{r-1,1}(D, \omega_D \otimes \mathcal{M}^{-1}, \mathcal{L}) = 0$. If (A.13) holds we have $r = 3$ and if we denote by σ a generator of $H^0(\omega_D \otimes \mathcal{M}^{-1} \otimes \mathcal{L})$, by hypothesis we can choose general points $P_j \in Z, 1 \leq j \leq 4$ and a basis $\{s_1, \ldots, s_4\}$ of $H^0(\mathcal{L})$ such that $s_i(P_j) = \delta_{ij}, \sigma(P_j) \neq 0$ for all i, j. Now if $\alpha = \sum_{1 \leq i < j \leq 4} s_i \wedge s_j \otimes (\lambda_{ij} \sigma) \in \mathrm{Ker} d_{2,1}$ where $\lambda_{ij} \in \mathbb{C}$, then

$$0 = d_{2,1}(\alpha) = \sum_{1 \leq i < j \leq 4} [s_j \otimes (s_i \lambda_{ij} \sigma) - s_i \otimes (s_j \lambda_{ij} \sigma)]$$

whence the four equations

$$\sigma(-\lambda_{12} s_2 - \lambda_{13} s_3 - \lambda_{14} s_4) = 0; \quad \sigma(\lambda_{12} s_1 - \lambda_{23} s_3 - \lambda_{24} s_4) = 0$$

$$\sigma(\lambda_{13} s_1 + \lambda_{23} s_2 - \lambda_{34} s_4) = 0; \; \sigma(\lambda_{14} s_1 + \lambda_{24} s_2 + \lambda_{34} s_3) = 0.$$

Evaluating at the points P_j's we get $\lambda_{ij} = 0$ for all i, j, hence $\alpha = 0$. ■

Proof of Theorem (A.1). Let C be a general hyperplane section of S. Notice that we just need to show that $H^1(\mathcal{I}_S(2)) = 0$ because the other two vanishings $H^2(\mathcal{I}_S(1)) = H^1(\mathcal{O}_S(1)) = 0$ and $H^3(\mathcal{I}_S) = H^2(\mathcal{O}_S) = 0$ are already given. To prove the desired vanishing we set $E' = E + K_S$, where E is the plane cubic of Lemma (A.2) and choose a general divisor $F \in |C - E - E'|$. In particular $C' = E \cup E' \cup F$ is a hyperplane section of S. We are going to show that

$$(A.14) \qquad h^0(\mathcal{I}_{C'/\mathbb{P}^{g-2}}(2)) = \begin{cases} 3 & \text{if } g = 7 \\ 7 & \text{if } g = 8 \end{cases}.$$

Of course (A.14) suffices since by semicontinuity we get $h^0(\mathcal{I}_{C/\mathbb{P}^{g-2}}(2)) \leq \begin{cases} 3 & \text{if } g = 7 \\ 7 & \text{if } g = 8 \end{cases}$ hence $h^1(\mathcal{I}_{C/\mathbb{P}^{g-2}}(2)) = 0$ by the Riemann-Roch theorem and the same holds for S. First we prove

$$(A.15) \qquad h^1(\mathcal{O}_S(C - E - E')) = 0.$$

In case (A.3) it follows by the Riemann-Roch theorem since $h^0(\mathcal{O}_S(F)) = 1, F^2 = 0$. Notice now that in all cases (A.4) through (A.10) we have that $(C-E-E')^2 = 2$. In cases (A.4), (A.5) and (A.6) in fact $C-E-E'$ is nef, hence (A.15) follows by [CD, Cor. 3.1.3]. In cases (A.7) and (A.8) we have $(C-E-E') \cdot R_2 = -1$ hence $h^1(\mathcal{O}_{R_2}(C-E-E')) = 0$; moreover $C - E - E' - R_2$ is nef and $(C - E - E' - R_2)^2 = 2$, hence $h^1(\mathcal{O}_S(C - E - E' - R_2)) = 0$ by [CD, Cor. 3.1.3], therefore we get (A.15) by the exact sequence

$$0 \to \mathcal{O}_S(C - E - E' - R_2) \to \mathcal{O}_S(C - E - E') \to \mathcal{O}_{R_2}(C - E - E') \to 0.$$

In case (A.9) we have $(C - E - E') \cdot R_2 = -1, (C - E - E' - R_2) \cdot R_1 = -1$ and $C - E - E' - R_1 - R_2$ is nef, $(C - E - E' - R_1 - R_2)^2 = 2$ hence, as above, we get (A.15). Similarly in case (A.10) we have $(C-E-E') \cdot R = -1, C-E-E'-R$ is nef, $(C-E-E'-R)^2 = 2$, hence again (A.15) is proved.

Notice now that $C \cdot (C-E-E') > 0$ hence $h^2(\mathcal{O}_S(C-E-E')) = 0$ and the Riemann-Roch theorem together with (A.15) implies

$$(A.16) \qquad h^0(\mathcal{O}_S(C - E - E')) = h^0(\mathcal{O}_S(F)) = \begin{cases} 1 & \text{if } g = 7 \\ 2 & \text{if } g = 8 \end{cases}.$$

Another consequence of (A.15) that will be used later is that $h^1(\mathcal{O}_S(2C-E-E'))=0$, as it can be easily checked by restricting to C. Since $C\cdot(2C-E-E')=4g-10>0, (2C-E-E')^2=8(g-4)$ we also get $h^2(\mathcal{O}_S(2C-E-E'))=0$ and $h^0(\mathcal{O}_S(2C-E-E'))=4g-15$ by the Riemann-Roch theorem. Denote now by $<E>, <E'>$ the $\mathbb{P}^2$'s that are linear spans of the two plane cubics E, E'. By (A.16) we deduce $<E>\cap<E'>=\emptyset$ hence $h^0(\mathcal{J}_{E\cup E'/\mathbb{P}^{g-2}}(2)) = h^0(\mathcal{J}_{<E>\cup<E'>/\mathbb{P}^{g-2}}(2)) = \begin{cases} 9 & \text{if } g=7 \\ 16 & \text{if } g=8 \end{cases}$. Also from the exact sequence

$$0\to\mathcal{O}_S(C)\to\mathcal{O}_S(2C-E-E')\to\mathcal{O}_F(2C-E-E')\to 0$$

and what we proved above, we get that $h^0(\mathcal{O}_F(2C-E-E')) = \begin{cases} 6 & \text{if } g=7 \\ 9 & \text{if } g=8 \end{cases}$. Now by the exact sequence

$$0\to\mathcal{J}_{C'/\mathbb{P}^{g-2}}(2)\to\mathcal{J}_{E\cup E'/\mathbb{P}^{g-2}}(2)\to\mathcal{O}_F(2C-E-E')\to 0$$

we see that (A.14) will follow once we show that the map

$$r_F : H^0(\mathcal{J}_{E\cup E'/\mathbb{P}^{g-2}}(2))\to H^0(\mathcal{O}_F(2C-E-E'))$$

is surjective. To this end consider the natural restriction maps $r : H^0(\mathcal{J}_{E/\mathbb{P}^{g-2}}(1))\to H^0(\mathcal{O}_F(C-E)), r' : H^0(\mathcal{J}_{E'/\mathbb{P}^{g-2}}(1))\to H^0(\mathcal{O}_F(C-E'))$ and the diagram

$$\begin{array}{ccc} H^0(\mathcal{J}_{E/\mathbb{P}^{g-2}}(1))\otimes H^0(\mathcal{J}_{E'/\mathbb{P}^{g-2}}(1)) & \longrightarrow & H^0(\mathcal{J}_{E\cup E'/\mathbb{P}^{g-2}}(2)) \\ \downarrow r\otimes r' & & \downarrow r_F \\ H^0(\mathcal{O}_F(C-E))\otimes H^0(\mathcal{O}_F(C-E')) & \xrightarrow{\mu} & H^0(\mathcal{O}_F(2C-E-E')). \end{array}$$

Since $C-E-F\sim E+K_S$ we have $h^1(\mathcal{O}_S(C-E-F))=0$ and it follows that $h^1(\mathcal{J}_{E\cup F/\mathbb{P}^{g-2}}(1))=0$, hence r and similarly r' are surjective (in fact isomorphisms). Therefore we just need to prove that the multiplication map μ above is surjective. We apply now Lemma (A.11). To see that $\mathcal{L}$ and $\mathcal{M}$ are base-point free we use the exact sequence

$$(A.17)\qquad 0\to\mathcal{O}_S(C-E-F)\to\mathcal{O}_S(C-E)\to\mathcal{O}_F(C-E)\to 0.$$

Since $h^1(\mathcal{O}_S(C-E-F))=0$ we just need to show that $\mathcal{O}_S(C-E)$ is base-point free. The latter follows by applying [CD, Prop. 3.1.6, Prop. 3.1.4 and Thm. 4.4.1]. In fact a quick inspection of cases (A.3) through (A.10) shows that $C-E$ is nef and that $\Phi(C-E) \neq 1$ (in case (A.3) use also the fact that C is very ample). Similarly for $\mathcal{O}_F(C-E')$.

Now if $g=7$ we are in case (A.3) and we show that (A.12) holds. We have $\omega_F \otimes \mathcal{M}^{-1} \otimes \mathcal{L} \cong \mathcal{O}_F(F)$ hence (A.12) holds since $h^0(\mathcal{O}_S(F)) = 1$.

When $g = 8$ we will see that the hypotheses (A.13) hold. First $h^0(\mathcal{O}_S(F)) = 2$ by (A.16), hence $h^0(\omega_F \otimes \mathcal{M}^{-1} \otimes \mathcal{L}) = h^0(\mathcal{O}_F(F)) = 1$ and we can choose its generator σ to be $\tau_{|F}$ where $\tau \in H^0(\mathcal{O}_S(F))$. To compute $h^0((C-E)_{|F})$ first notice that $h^0(\mathcal{O}_S(C-E-F)) = 1$. Since $C \cdot (C-E) = 11$ we get $h^2(\mathcal{O}_S(C-E)) = 0$; moreover $C-E$ is nef, $(C-E)^2 = 8$ and therefore $h^1(\mathcal{O}_S(C-E)) = 0$ (by [CD, Cor. 3.1.3]), $h^0(\mathcal{O}_S(C-E)) = 5$ by the Riemann-Roch theorem. The exact sequence (A.17) then gives $h^0(\mathcal{O}_F(C-E)) = 4$. Now applying [CD, Prop. 3.1.4] and [CD, Prop. 3.1.6] in case (A.4), [CD, Cor. 3.1.4] in case (A.5), we see that F is irreducible in these cases, hence (A.13) holds. In case (A.6) we have $F = R \cup Z$ with Z general in $|2E_1|$. As $\tau_{|R} = 0$ (R is a base component) we get $\sigma_{|Z} = \tau_{|Z} \neq 0$. Moreover $(C-E-Z) \cdot R = -1$ hence $h^0((C-E)_{|R}(-Z)) = 0$ and (A.13) holds.

In the remaining cases (A.7) through (A.10) we will just limit ourselves to indicate the component Z to be chosen and leave the easy verification of (A.13) to the reader. We choose Z to be a general divisor in $|2E_1 + R_1 + K_S|$ in case (A.7), $|2E_1|$ in case (A.8) and $|E_1 + E_2|$ in cases (A.9) and (A.10). ■

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