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On the correspondence between Ulrich bundles and curves on surfaces, with applications

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A mio zio Mirko
ed a mio nonno Tiberio,
i miei cari angeli.

Contents

Introduction

From local algebra to geometry: the origin of Ulrich bundles In 1984, Bernd Ulrich published the landmark paper *Gorenstein Rings and Modules with High Numbers of Generators* ([Ulr]). Just a few years later, in 1987, together with Joseph P. Brennan and Jürgen Herzog, he further developed these ideas in the influential work *Maximally Generated Cohen-Macaulay Modules* ([BHU]).

These seminal works introduced a foundational problem in local algebra, focusing on a specific class of modules – finitely generated Cohen-Macaulay modules of the maximal possible dimension over a Cohen-Macaulay local or homogeneous ring (MCM). For such modules, the number of generators is always bounded by their multiplicity. This led to a central question in the theory: *under what conditions do these modules achieve the absolute maximum number of generators permitted by this theoretical bound?* Modules satisfying this extremal condition are also known as Ulrich modules.

Although this algebraic framework is rich and profound, we will not pursue its intricate depths here; instead, we shall focus on its corresponding geometric counterpart. We find that this very abstract setup is hiding a remarkably concrete geometric problem, which had already been studied almost a century prior to Ulrich's algebraic perspective. In its simplest form, it asks: *when can a homogeneous polynomial be written as the determinant of a matrix of linear forms?*

The classical question: determinantal representations of varieties The concept of representing geometric objects using determinants has roots stretching back to the mid-19th century. During this period, specific determinantal representations for objects like cubic surfaces were already known and other examples of curves and surfaces were also treated in this period (by [Gra] in 1855 and by [Sch] in 1881). The representation of the plane quartic as a symmetric determinant, for instance, can be traced to 1855, credited to O. Hesse [Hes]. Cubic and quartic surfaces have been also studied early; see [Cay].

The early 20th century saw a more systematic development of explicit determinantal representations for projective varieties. In 1902, Arthur Lee Dixon extended the idea to plane curves of arbitrary degree [Dix]. Subsequently, Leonard Eugene Dickson provided insights into expressing general homogeneous forms as linear determinants [Dic]. While classical efforts largely focused on curves and surfaces, modern research continues to explore these representations, examining which general homogeneous forms admit such structures.

Ulrich bundles and determinantal representations of hypersurfaces Building upon these classical insights, a more refined question emerges in modern algebraic geometry concerning the determinantal representation of hypersurfaces. Specifically, *given a smooth projective hypersurface X, can its defining equation be expressed as the determinant of a matrix of linear forms?*

While this question yields positive answers for certain low-dimensional cases, a critical challenge arises for higher-dimensional smooth hypersurfaces. A crucial observation is that a hypersurface defined by the determinant of a matrix of linear forms, say $\det(L_{ij}) = 0$, is necessarily singular along the locus where the rank of the matrix (L_{ij}) drops significantly (by at least two). This geometric constraint implies that a smooth hypersurface of dimension three or higher generally cannot be represented as a simple determinant of linear forms [Bea2, §2]. This highlights the necessity of moving beyond straightforward determinantal equations to more abstract structures for a comprehensive understanding.

To overcome this limitation, one can pose a weaker, yet profoundly insightful question: *can a smooth projective hypersurface be defined set-theoretically by a linear determinant?*

This question reveals a deep equivalence with the existence of specific vector bundles, as formalized by Beauville in [Bea2]:

Proposition 0.1. *Let $X \subset \mathbb{P}^N$ be a smooth hypersurface of degree d, given by an equation $F = 0$ and let $r \geq 1$ be an integer. The following conditions are equivalent:*

(i) $F^r = \det(L_{ij})$, where (L_{ij}) is an $rd \times rd$ matrix of linear forms on $\mathbb{P}^N$.

(ii) There exists a rank r vector bundle $\mathcal{E}$ on X and an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N}^{\oplus rd}(-1) \xrightarrow{L} \mathcal{O}_{\mathbb{P}^N}^{\oplus rd} \to \mathcal{E} \to 0.$$

For a detailed proof of this proposition, we refer the reader to [Bea2, Proposition 2.1].

Remark 0.2 (Illustrative case: a plane curve)**.** To see concretely how condition (ii) forces the determinantal equation in (i) in Proposition 0.1, consider a smooth plane curve $C \subset \mathbb{P}^2$ of degree d, defined by F. If there exists a line bundle $\mathcal{E}$ on C and an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}^{\oplus d}(-1) \xrightarrow{L} \mathcal{O}_{\mathbb{P}^2}^{\oplus d} \to \mathcal{E} \to 0$$

then at each point $x \in \mathbb{P}^2$ one checks easily that

$$F(x) = 0 \iff x \in C \iff \det(L(x)) = 0.$$

The vector bundles $\mathcal{E}$ appearing in condition (ii) are key objects in this geometric approach and are precisely what are known as *Ulrich bundles.* While condition (ii) offers a powerful characterization of Ulrich bundles on hypersurfaces (see Proposition 2.2), their more common formal definition, which we will use in this work, relies on the vanishing of certain cohomology groups:

$$H^i(\mathcal{E}(-p)) = 0 \text{ for all } i \geq 0 \text{ and } 1 \leq p \leq \dim X.$$

Two fundamental problems Having established the precise connection between determinantal representations and Ulrich bundles, a natural and fundamental line of inquiry arises: *do such Ulrich bundles always exist on a given projective variety?* The problem of their existence is a central and actively researched area in algebraic geometry. While Ulrich bundles are known to exist on various important classes of varieties, such as curves, K3 surfaces, hypersurfaces and smooth complete intersections, their existence on arbitrary projective varieties remains a significant open problem.

Furthermore, for a given projective variety where Ulrich bundles do exist, a related and equally important question concerns their minimum rank: *what is the smallest possible rank r for which an Ulrich bundle can be found?* Also is this case, determining this minimum rank for various classes of varieties is an active area of research, with known bounds and precise values established only for certain specific cases.

A Curve-Based construction of Ulrich bundles on surfaces As previously mentioned, the existence problem for Ulrich bundles on a given smooth projective variety remains open, even for varieties of low dimension, such as surfaces. Addressing this, significant progress has been made in characterizing such bundles; for instance, Casnati presented a key result that characterize Ulrich bundles on surfaces, providing criteria based on their Chern classes and cohomology (see Section 3.1).

Building upon such foundational contribution, this work primarily focuses on studying Ulrich bundles on smooth projective surfaces. The main result of this thesis, which is original, gives a precise correspondence between Ulrich bundles and curves on surfaces. Specifically, it is proved that an Ulrich bundle on a surface exists if and only if it arises as a *Lazarsfeld-Mukai bundle* (see Section 3.2) associated with a specific triple of geometric data. The main theorem (Theorem 3.7) establishing this correspondence is stated as follows:

Theorem 0.3. *Let $S \subset \mathbb{P}^N$ be a smooth projective surface of degree $d \geq 2$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$. Then, there exists an Ulrich bundle $\mathcal{E}$ of rank r on S if and only if there exists a smooth (possibly disconnected) curve $C \subset S$ of genus g together with a pair $(W, \mathcal{L})$, where $\mathcal{L}$ is a line bundle on C and $W \subseteq H^0(\mathcal{L})$ is a $r-$dimensional base-point free linear series, such that:*

(i) $H^1(C, \mathcal{L}(K_S + H)) = 0$;

(ii) the multiplication map

$$\varphi : W \otimes H^0(S, \mathcal{O}_S(K_S + 2H)) \to H^0(\mathcal{L}(K_S + 2H))$$

is injective;

(iii) $\deg(C) = \frac{r}{2}(K_S + 3H) \cdot H$ *and*
$\deg(\mathcal{L}) = r\chi(\mathcal{O}_S) + g - 1 - C \cdot K_S - rH^2$.

A natural question arising from this theorem is: *when is the curve C connected?* This question will be extensively addressed in Section 3.4. The answer is provided by the following result (Corollary 3.23):

Corollary 0.4. *Let $S \subset \mathbb{P}^N$ be a smooth projective surface of degree $d \geq 2$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$. Let $\mathcal{E}$ be the Ulrich bundle on S corresponding to the triple $(C, W, \mathcal{L})$, as in Theorem 3.7. Then C is irreducible if and only if one of the following cases arises:*

(i) $(S, \mathcal{O}_S(1), \mathcal{E}) \ncong (\mathbb{P}\mathcal{F}, \mathcal{O}_{\mathbb{P}\mathcal{F}}(1), \pi^*(\mathcal{G}(\det(\mathcal{F})))$ *where $\mathcal{F}$ is a rank 2 very ample vector bundle over a smooth curve B of genus g, $\mathcal{G}$ is a rank r vector bundle on B such that $H^q(\mathcal{G}) = 0$ for $q \geq 0$;*

(ii) $(S, \mathcal{O}_S(1), \mathcal{E}) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0))$ *or*
$(S, \mathcal{O}_S(1), \mathcal{E}) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1))$.

The insights derived from our main theorem are then applied to specific cases, particularly to surfaces in $\mathbb{P}^3$, offering new results about Ulrich bundles on them and their relation with Noether-Lefschetz loci (Chapter 4). To support these discussions, the preceding chapters provide the necessary background: Chapter 1 introduces general notions and tools from the theory of vector bundles and projective geometry, while Chapter 2 focuses on Ulrich bundles, presenting some known results and examples relevant to our study.

Chapter 1

Preliminaries

1.1 Riemann-Roch theorems

Theorem 1.1 (Riemann-Roch for line bundles on smooth curves). *Let D be a divisor on a smooth curve C of genus g. Then*

$$\chi(\mathcal{O}_C(D)) = \deg(D) + 1 - g.$$

Proof. See [Har, Theorem IV.1.3]. □

Theorem 1.2 (Riemann-Roch formula for line bundles on smooth surfaces). *Let S be a smooth projective surface and let D be a divisor on S. Then*

$$\chi(\mathcal{O}_S(D)) = \frac{1}{2}D \cdot (D - K_S) + \chi(\mathcal{O}_S).$$

Proof. See [Har, Theorem V.1.6]. □

Theorem 1.3 (Riemann-Roch formula for vector bundles on smooth surfaces). *Let S be a smooth projective surface and let $\mathcal{E}$ be a rank r vector bundle on S. Then the Euler characteristic of $\mathcal{E}$ is given by:*

$$\chi(\mathcal{E}) = r\chi(\mathcal{O}_S) + \frac{1}{2}c_1(S)c_1(\mathcal{E}) + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})).$$

Proof. It follows directly from the Hirzebruch-Riemann-Roch theorem [Har, App. A, Theorem 4.1]. □

1.2 Castelnuovo-Mumford regularity

Definition 1.4. Let $\mathcal{F}$ be a coherent sheaf on the projective space $\mathbb{P}^N$, and let m be an integer. $\mathcal{F}$ is *m-regular in the sense of Castelnuovo-Mumford* if

$$H^i(\mathbb{P}^N, \mathcal{F}(m-i)) = 0 \text{ for all } i > 0.$$

Theorem 1.5 (**Mumford's Theorem, I**). *Let $\mathcal{F}$ be an m-regular sheaf on $\mathbb{P}^N$. Then for every $k \geq 0$:*

(i) *$\mathcal{F}(m+k)$ is globally generated;*

(ii) *The natural maps*

$$H^0(\mathbb{P}^N, \mathcal{F}(m)) \otimes H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(\mathbb{P}^N, \mathcal{F}(m+k))$$

are surjective;

(iii) *$\mathcal{F}$ is $(m+k)$-regular.*

Proof. See [Laz, Theorem 1.8.3] □

This notion of regularity on projective space is in fact a special case of a more general definition, which applies to coherent sheaves over arbitrary projective varieties equipped with a globally generated ample line bundle.

Definition 1.6. Let X be a projective variety and let $\mathcal{L}$ be a globally generated ample line bundle on X. Let $\mathcal{F}$ be a coherent sheaf on X and let m be an integer. $\mathcal{F}$ is m-regular with respect to $\mathcal{L}$ if

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes(m-i)}) = 0 \text{ for all } i > 0.$$

The following result shows that the main properties of Castelnuovo-Mumford regularity extend naturally to this more general setting.

Theorem 1.7 (**Mumford's Theorem, II**). *Let $\mathcal{F}$ be an m-regular sheaf on a projective variety X, with respect to $\mathcal{L}$. Then for every $k \geq 0$:*

(i) *$\mathcal{F} \otimes \mathcal{L}^{\otimes(m+k)}$ is globally generated;*

(ii) *The natural maps*

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \otimes H^0(X, \mathcal{L}^{\otimes k}) \to H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes(m+k)})$$

are surjective;

(iii) *$\mathcal{F}$ is $(m+k)$-regular with respect to $\mathcal{L}$.*

Proof. See [Laz, Theorem 1.8.5] □

1.3 Chern classes

Here, we summarize the foundational elements of Chern classes, following the established framework provided by Eisenbud and Harris in [EH].

Definition 1.8 (The Chow ring). Let X be a variety.

(i) A *cycle of codimension* r on X is an element of the free abelian group generated by the closed irreducible subvarieties of X of codimension r.

(ii) For each r, we define $A^r(X)$ to be the group of cycles of codimension r on X, modulo rational equivalence (for the definition of *rational equivalence*, see [EH, §1.3.2])

(iii) We denote by $A(X)$ the graded group $\bigoplus_{r=0}^{n} A^r(X)$.

We present below Theorem 5.3 from [EH], which essentially provides the definition of Chern classes.

Theorem 1.9. *Let X be a smooth variety and let $\mathcal{E}$ be a vector bundle on X. Then there is a unique way to assign to $\mathcal{E}$ a class*

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \ldots \in A(X)$$

which satisfies the following conditions:

(i) if $\mathcal{L}$ is a line bundle on X then the Chern class of $\mathcal{L}$ is $1 + c_1(\mathcal{L})$ where $c_1(\mathcal{L})$ is the class of the divisor of zeros minus the divisor of poles of any rational section of $\mathcal{L}$;

(ii) if $\tau_0, \ldots, \tau_{r-i}$ are global sections of $\mathcal{E}$ and the degeneracy locus D where they are dependent has codimension i, then the i-th Chern class of $\mathcal{E}$ is

$$c_i(\mathcal{E}) = [D] \in A^i(X);$$

(iii) (Whitney's formula) Given a short exact sequence of vector bundles

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

the total Chern class satisfies

$$c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G});$$

(iv) (Functoriality) For any morphism $\varphi : X \to Y$ between smooth verieties, the Chern classes respect pullbacks:

$$\varphi^*(c(\mathcal{E})) = c(\varphi^*(\mathcal{E}))$$

Proof. See [EH, Chapter 5, §9] □

Definition 1.10 (Chern classes). Given a vector bundle $\mathcal{E}$ over a smooth variety X, its *total Chern Class* is the unique class

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \ldots \in A(X)$$

satisfying the four conditions of Theorem 1.9.

The *i-th Chern class* of $\mathcal{E}$ is then defined as the degree i component of $c(\mathcal{E})$, that is, $c_i(\mathcal{E}) \in A^i(X)$.

Remark 1.11. One can get a more concrete idea of Chern classes by thinking in terms of the behavior of global sections. The first Chern class can be interpreted as a way to measure how far a vector bundle is from being trivial. For line bundles, this idea is simple: if the first Chern class vanishes, the bundle has a nowhere-vanishing section and it is thus trivial. In higher rank, one generalizes this by looking at the loci where a set of global sections fails to be linearly independent.To simplify the discussion, let us assume $\mathcal{E}$ to be rank r vector bundle on a variety X generated by its global sections. Choose r general global sections $s_0, \ldots, s_{r-1}$. These define a bundle map

$$\varphi : \mathcal{O}_X^{\oplus r} \to \mathcal{E}$$

which fails to be surjective exactly along a codimension 1 locus defined by the vanishing of $\det \varphi$. In this setup, $\det \varphi$ is a general global section of the determinant line bundle $\bigwedge^r \mathcal{E}$. Its zero scheme is a divisor whose class in $A^1(X)$ is, by definition, the first Chern class of $\mathcal{E}$, $c_1(\mathcal{E})$.

More generally, taking only $r - i + 1$ general sections yields a section of $\bigwedge^{r-i+1}(\mathcal{E})$ whose vanishing defines a codimension i degeneracy locus; the class of this locus is, by definition, the i-th Chern class of $\mathcal{E}$, $c_i(\mathcal{E})$.

To facilitate future applications, we summarize here some of the standard identities and structural features satisfied by Chern classes.

Lemma 1.12. *Let X be a smooth variety and $\mathcal{E}$ a vector bundle on X.*

(i) (Splitting principle) Any identity among Chern classes of bundles that is true for bundles that are direct sum of line bundles is true in general;

(ii) $c_1(\det(\mathcal{E})) = c_1(\mathcal{E})$*;*

(iii) $c_i(\mathcal{E}^*) = (-1)^i c_i(\mathcal{E})$*;*

(iv) if $rk(\mathcal{E}) = r$ and $\mathcal{L}$ is a line bundle on X, then

$$c_k(\mathcal{E} \otimes \mathcal{L}) = \sum_{i=0}^{k} \binom{r-k+i}{i} c_1(\mathcal{L})^i c_{k-i}(\mathcal{E});$$

(v) If $rk(\mathcal{E}) = r$ and $\mathcal{F}$ is a rank s vector bundle on X, then

$$c_1(\mathcal{E} \otimes \mathcal{F}) = s \cdot c_1(\mathcal{E}) + r \cdot c_1(\mathcal{F}).$$

Proof. For point (i), see [EH, Theorem 5.11]. Points (ii) and (iii) follows at once from Whitney's formula together with the splitting principle, as showed in Example 5.14 and 5.15 in [EH]. For (iv) see [EH, Proposition 5.17] and for (v) refer to [EH, Proposition 5.18]. □

1.4 Degeneracy loci

This section introduces degeneracy loci of morphisms between vector bundles, following Ottaviani's exposition in [Ott2, §2]. Our primary objective is to clearly state Banica's Theorem, which provides precise information regarding the codimension of these loci.

Definition 1.13. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on a variety X and let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism. For each point $x \in X$ the morphism φ induces a linear map of fibers

$$\varphi(x) : \mathcal{E}(x) \to \mathcal{F}(x),$$

where $\mathcal{E}(x)$ and $\mathcal{F}(x)$ denote the fibers of $\mathcal{E}$ and $\mathcal{F}$ at x, respectively. We define the *k-th degerenacy locus* of φ as

$$D_k(\varphi) = \{x \in X : \ rk(\varphi(x)) \leq k\}.$$

If $k = \min\{rk(\mathcal{E}), rk(\mathcal{F})\} - 1$, we refer to $D_k(\varphi)$ as the *maximal degeneracy locus.*

Remark 1.14. For a morphism φ between vector bundles on a variety X, the degeneracy locus $D_k(\varphi)$ admits a natural structure of closed subscheme of X. It is defined scheme-theoretically by the vanishing of all $(k+1)\times(k+1)$ minors of a local presentation matrix of φ. In other words, $D_k(\varphi)$ arises as the ideal sheaf generated by these minors.

Remark 1.15. We start by analyzing a well-understood and accessible context. Let us consider the vector space of all $m \times n$ matrices $\mathrm{Mat}_{m\times n}(\mathbb{C}) \cong \mathbb{C}^{mn}$. We define M_k as the subset of matrices in $\mathrm{Mat}_{m\times n}(\mathbb{C})$ whose rank is less than or equal to k, that is,

$$M_k = \{x \in \mathrm{Mat}_{m\times n}(\mathbb{C}) : \ rk(x) \le k\}.$$

This particular case aligns with the general definition through a specific construction. Let V and W be finite-dimensional complex vector spaces of dimensions n and m respectively; from these, we construct two trivial vector bundles over the base space $\mathrm{Hom}(V, W)$, which we denote by $\mathcal{V}$ and $\mathcal{W}$, whose fibers are canonically identified with the original vector spaces.

Now, we define a canonical morphism $\psi : \mathcal{V} \to \mathcal{W}$. This morphism is defined pointwise: for any given point $x \in \mathrm{Hom}(V, W)$ (which represents a specific $m \times n$ matrix, which we denote by M_x), the linear map $\psi_x : V \to W$ is simply given by $\psi_x(v) = x(v)$, where $v \in V$. Under this construction, the set M_k is precisely the degeneration locus $D_k(\psi)$, infact, $rk(\psi_x) \le k$ if and only $rk(M_x) \le k$, by construction.

In addition, we have the following result.

Lemma 1.16. *M_k is an irreducible algebraic subvariety of $Mat_{m\times n}(\mathbb{C})$ with codimension $(m-k)(n-k)$.*

Furthermore, $Sing(M_k) = M_{k-1}$.

Proof. See [Ott2, Theorem 2.1] □

In the more general context, Lemma 1.16 extends as follows.

Lemma 1.17. *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles of rank m and n on a variety X and let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism. Then*

$$codim_X(D_k(\varphi)) \le (m-k)(n-k).$$

Proof. See [Ott2, Lemma 2.7] □

Definition 1.18. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles of rank m and n on a variety X and let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism. We say that $D_k(\varphi)$ has the *expected codimension* if

$$\mathrm{codim}_X(D_k(\varphi)) = (m-k)(n-k).$$

Theorem 1.19 (Banica). *Let X be a variety and let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles of rank m and n on X such that $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is globally generated. If φ is a general morphism of $Hom(\mathcal{E}, \mathcal{F}) = H^0(\mathcal{E}^* \otimes \mathcal{F})$, then for each k the degeneracy locus $D_k(\varphi)$ is either empty or has the expected codimension $(m-k)(n-k)$.*

In addition, $Sing(D_k(\varphi)) \subset D_{k-1}(\varphi)$.

Proof. See [Ott2, Theorem 2.8] □

1.5 Torsion sheaves

Definition 1.20 (Torsion modules). Let A be a commutative ring and let M be an A-module.

(i) An element $m \in M$ is called a *torsion element* if there exists a nonzero $a \in A$ such that $am = 0$.

(ii) The *torsion submodule* of M, denoted $\mathrm{Tors}(M)$ is

$$\mathrm{Tors}(M) = \{m \in M \mid \exists 0 \neq a \in A : \ a \cdot m = 0\}.$$

One checks directly that $\mathrm{Tors}(M)$ naturally forms an A-submodule of M;

(iii) M is a *torsion module* if $\mathrm{Tors}(M) = M$, i.e., every element of M is torsion;

(iv) M is *torsion-free* if $\mathrm{Tors}(M) = 0$.

Lemma 1.21. *Let A be a domain and let M be a torsion A-module. Then,*

$$Hom_A(M, A) = 0.$$

Proof. Let $\varphi : M \to A$ be a morphism in $\mathrm{Hom}_A(M, A)$. Fix $m \in M$; since M is a torsion module, there exists $0 \neq a \in A$ such that $a \cdot m = 0$. Hence,

$$a\varphi(m) = \varphi(am) = \varphi(0) = 0$$

and, since $a \neq 0$ and A is supposed to be a domain, one has $\varphi(m) = 0$. Therefore, for each $m \in M$, $\varphi(m) = 0$, that is, $\varphi \equiv 0$. □

Definition 1.22 (Torsion sheaves). Let X be a scheme and $\mathcal{F}$ a quasi-coherent sheaf on X. We define

$$\mathcal{F}(U)_{tors} = \{s \in \mathcal{F}(U) \mid \exists 0 \neq f \in \mathcal{O}_X(U) : \ f \cdot s = 0\}$$

One can show that there exists a unique subsheaf $\mathcal{F}_{tors}$ such that $\mathcal{F}_{tors}(U) = \mathcal{F}(U)_{tors}$ for every affine open subset U of X and that $\mathcal{F}_{tors}$ is a quasi-coherent sheaf on X (see [Liu, Exercise 1.14]). We say that

(i) $\mathcal{F}$ is a *torsion sheaf* if $\mathcal{F} = \mathcal{F}_{tors}$;

(ii) $\mathcal{F}$ is *torsion-free* if $\mathcal{F}_{tors} = 0$.

Lemma 1.23. *Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety X of dimension n. If*

$$\dim(Supp(\mathcal{F})) < n,$$

then $\mathcal{F}$ is a torsion sheaf.

Proof. Let U =Spec$(A) \subseteq X$ be an affine open subset such that $\mathcal{F}_U \cong \widetilde{M}$ for a finitely generated A-module M. Since the support of $\mathcal{F}$ has dimension strictly less than n, the set of points where the stalk $M_{\mathfrak{p}_x}$ is nonzero is a proper subset of Spec(A). In particular, there exists a prime ideal $\mathfrak{p}_{x_0}$ ∈Spec(A) such that

$$M_{\mathfrak{p}_{x_0}} = 0 \text{ and } A_{\mathfrak{p}_{x_0}} \neq 0.$$

For any element $m \in M$, the vanishing of the localized element $\frac{m}{1} \in M_{\mathfrak{p}_{x_0}}$ implies the existence of an element $s \in A \backslash \mathfrak{p}_{x_0}$ such that $s \cdot m = 0$, showing that each $m \in M$ is a torsion element. Hence, for any affine open subset U = Spec$(A) \subseteq X$, the associated A-module $\mathcal{F}(U) \cong M$ is torsion; this implies that $\mathcal{F}(U) = \mathcal{F}(U)_{\text{tors}}$ for all such U. Since $\mathcal{F}_{\text{tors}}$ is defined as the unique subsheaf whose sections over any affine open subset are precisely the torsion elements of $\mathcal{F}(U)$, we conclude that $\mathcal{F} = \mathcal{F}_{\text{tors}}$. Therefore, $\mathcal{F}$ is a torsion sheaf.

□

Lemma 1.24. *Let $\mathcal{F}$ be a torsion coherent sheaf on a smooth projective variety X. Then, its dual $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ vanishes.*

Proof. On any affine open U = Spec$(A) \subset X$, write $\mathcal{F}_{|U} \simeq \widetilde{M}$, where M is a finitely generated torsion A-module.

Then by definition one has

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)(U) = Hom_{\mathcal{O}_U}(\mathcal{F}_{|U}, \mathcal{O}_U) = Hom_{\mathcal{O}_U}(\widetilde{M}, \mathcal{O}_U).$$

By [Har, Exercise II.5.3], we have that $Hom_{\mathcal{O}_U}(\widetilde{M}, \mathcal{O}_U)$ is naturally isomorphic to $\mathrm{Hom}_A(M, A)$ and, since M is torsion, Lemma 1.21 gives $\mathrm{Hom}_A(M, A) = 0$. It follows that $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X) = 0$ on every affine open U and, therefore, $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ is the zero sheaf.

□

1.6 Auxiliary results and techniques

Lemma 1.25 (**Coherent sheaves of constant rank**). *Let X be a complete scheme. Let $\mathcal{F}$ be a coherent sheaf of costant rank r on X then $\mathcal{F}$ is locally free of rank r.*

Proof. Fix a point $x_0 \in X$. By hypothesis, the stalk $\mathcal{F}_{x_0}$ is a free $\mathcal{O}_{X,x_0}$-module of rank r. Choose an isomorphism

$$\varphi_{x_0} : \mathcal{O}_{X,x_0}^{\oplus r} \to \mathcal{F}_{x_0}.$$

Since $\mathcal{F}$ is coherent, there exists an affine neighborhood U =Spec(A) of x_0 and a finitely generated A-module M such that $\mathcal{F}_U \cong \widetilde{M}$. By shrinking U further if necessary, we may assume that the map φ_{x_0} is induced by a morphism of sheaves

$$\psi : \mathcal{O}_U^{\oplus r} \to \mathcal{F}_U.$$

On global sections this corresponds to an A-linear map

$$\psi_U : A^{\oplus r} \to M, \quad (a_1, \ldots, a_r) \to \sum_{i=1}^{r} a_i m_i,$$

for some chosen $m_1, \ldots, m_r \in M$.

Next, we show that ψ is surjective on a neighbourhood of x_0. Since M is finitely generated, we choose a full generating set $\{x_1, \ldots, x_s\}$. The surjectivity of ψ_{x_0} on the stalk on x_0 implies that, in the localized module $M_{\mathfrak{p}_{x_0}}$, each generator x_j can be written as an $A_{\mathfrak{p}_{x_0}}$-linear combination of the image of the m_i. Concretely, there exist elements $a_{ij} \in A$, $s_{ij} \notin \mathfrak{p}_{x_0}$ such that

$$\frac{x_j}{1} = \sum_{i=1}^{r} \frac{a_{ij}}{s_{ij}} \frac{m_i}{1} = \frac{a'_{1j} m_1 + \ldots + a'_{rj} m_r}{s_j}$$

for suitable $a'_{ij} \in A$, $s_j \notin \mathfrak{p}_{x_0}$. Clearing denominators then produces an element $t_j \notin \mathfrak{p}_{x_0}$ satisfying

$$t_j(s_j x_j - (a'_{1j} m_1 + \ldots + a'_{rj} m_r)) = 0.$$

Setting $f_j = t_j s_j$, we see that over the open set $U_{f_j} \subset U$ the element x_j indeed lies in the image of ψ. Taking the product $f = f_1 \ldots f_s$, we conclude that ψ is surjective on the smaller open set U_f.

We now turn to injectivity. Let $\mathcal{K} := \ker(\psi)$. This is a coherent subsheaf of $\mathcal{O}_U^{\oplus r}$ and by construction its stalk at x_0 vanishes, that is, $\mathcal{K}_{x_0} = 0$. Hence there is a finitely generated A-module N such that $\mathcal{K}_{|U} \cong \widetilde{N}$. Writing N as generated by $n_1, \ldots, n_t$, for each n_k there exists $s_k \notin \mathfrak{p}_{x_0}$ with $s_k n_k = 0$. Taking $g = s_1 \ldots s_t$, we get $gN = 0$, and, on the open set U_g, the element g is invertible, forcing $N = 0$. Hence, $\mathcal{K}_{|U_g} = 0$.

We conclude that ψ is simultaneously surjective and has trivial kernel on the intersection $U_f \cap U_g = U_{fg}$. Since the choice of x_0 was arbitrary, this proves that $\mathcal{F}$ is locally free of rank r.

□

Proposition 1.26 (Injectivity of the evaluation map). *Let $\mathcal{E}$ be locally free sheaf of rank r over a smooth projective variety X and let V be a subspace of $H^0(\mathcal{E})$ of dimension $s \leq r$. Then, the natural evaluation map*

$$\varphi_V : V \otimes \mathcal{O}_X \to \mathcal{E}$$

is injective as a morphism of sheaves.

Proof. Let $x \in X \setminus D_{s-1}(\varphi_V)$, so that the evaluation map has rank s at that point. Consider the stalk of the map φ_V at x

$$\varphi_{V,x} : \mathcal{O}_{X,x}^{\oplus s} \to \mathcal{E}_x \cong \mathcal{O}_{X,x}^{\oplus r}.$$

Denoting $A := \mathcal{O}_{X,x}$ and $\mathfrak{m} = \mathfrak{m}_x$, this map becomes $\varphi_{V,x} : A^{\oplus s} \to A^{\oplus r}$.

Then, $\varphi_{V,x}$ is represented by an $r \times s$ matrix $[a_{ij}]$ with entries in A. Tensoring the map with the residue field $K(x) = A/\mathfrak{m}$ we obtain the induced linear map

$$\varphi_V(x) : K(x)^{\oplus s} \to K(x)^{\oplus r},$$

represented by the matrix $[\bar{a}_{ij}]$, where $\bar{a}_{ij}$ denotes the image of a_{ij} modulo $\mathfrak{m}$. Since $x \notin D_{s-1}(\varphi_V)$ this map has rank s, meaning that the image of the sections in $V \subseteq H^0(\mathcal{E})$ remain linearly independent in the fiber $\mathcal{E}(x) \cong K(x)^{\oplus r}$. Therefore, the matrix $[\bar{a}_{ij}]$ has rank s. In particular, there exists an $s \times s$ submatrix M of $[\bar{a}_{ij}]$ such that $\det(M) \neq 0$ in $K(x)$. This means that the corresponding determinant $\det(M') \in A$, where M' is the lift of M in $[a_{ij}]$, is not contained in $\mathfrak{m}$. Hence, $\det(M')$ is an invertible element of A, implying that $[a_{ij}]$ has rank at least s over A. Thus, the map $\varphi_{V,x}$ is injective. Now, since the point x was chosen outside the degeneracy locus $D_{s-1}(\varphi_V)$, and injectivity holds there, we conclude that the kernel sheaf $\mathcal{K} = \ker(\varphi_V)$ vanishes on a nonempty open subset of X. Hence, $\mathcal{K}$ is torsion by Lemma 1.23.

But $\mathcal{K}$ is a subsheaf of $\mathcal{O}_X^{\oplus s}$, which is locally free and hence torsion-free. Therefore, $\mathcal{K}$ is a torsion sheaf inside a torsion-free sheaf, and must be zero everywhere. This implies that φ_V is injective as a morphism of sheaves.

□

Lemma 1.27 (Base-point free pencil trick). *Let C be a smooth curve, let $\mathcal{L}$ be a line bundle on C and let $\mathcal{F}$ be a torsion free $\mathcal{O}_C$-module. Let $s_1, s_2 \in H^0(\mathcal{L})$ be linearly independent sections of $\mathcal{L}$, and denote by W the subspace of $H^0(\mathcal{L})$. Then the kernel of the multiplication map*

$$\varphi : W \otimes H^0(\mathcal{F}) \to H^0(\mathcal{F} \otimes \mathcal{L})$$

is isomorphic to $H^0(\mathcal{F} \otimes \mathcal{L}^{-1}(B))$ where B is the base locus of the pencil spanned by s_1 and s_2.

Proof. See [ACGH, Chapter 3, §3] □

1.7 Ruled Surfaces

In this section we briefly recall the basic properties of ruled surfaces which will be essential for the arguments in Section 3.4. Our presentation follows the treatment in [Har, V.2].

Definition 1.28. A *ruled surface* on a smooth curve B is a surface S, together with a surjective morphism $\pi : S \to B$, such that $\pi^{-}1(p) \cong \mathbb{P}^1$ for every point $p \in B$ and such that π admits a section, that is, a morphism $\sigma : B \to S$ such that $\pi \circ \sigma = \mathrm{id}_B$.

Throughout this section, whenever we refer to a ruled surface S, we fix the surjective morphism $\pi : S \to B$ onto the nonsingular base curve B and the section $\sigma : B \to S$ provided by the definition. In addition, we denote by f a general fiber of π and we set $B_0 = \sigma(B)$.

Lemma 1.29. *Let S be a ruled surface and let D be a divisor on S. It follows that:*

(i) any two fibers of π are algebraically equivalent divisors on S. In particular, $D \cdot f$ is independent of the choice of the fiber;

(ii) if $D \cdot f = n \geq 0$ then $\pi_*(\mathcal{O}_S(D))$ is a locally free sheaf of rank $n + 1$ on B. In particular, $\pi_*(\mathcal{O}_S) = \mathcal{O}_B$.

Proof. See [Har, Lemma V.2.1] □

Proposition 1.30. *Let S be a ruled surface. Then:*

(i) there exists a locally free sheaf $\mathcal{F}$ of rank 2 on B, such that $S \cong \mathbb{P}(\mathcal{F})$. Conversely, every such $\mathbb{P}(\mathcal{F})$ is a ruled surface over B;

(ii) if $\mathcal{F}$ and $\mathcal{F}'$ are two locally free sheaves of rank 2 on B, then $\mathbb{P}(\mathcal{F})$ and $\mathbb{P}(\mathcal{F}')$ are isomorphic as ruled serfaces over B if and only if there is an invertible sheaf $\mathcal{L}$ on B such that $\mathcal{F}' \cong \mathcal{F} \otimes \mathcal{L}$.

Proof. See [Har, Proposition V.2.2] □

Lemma 1.31. *Let S be a ruled surface. Then $B_0 \cdot f = 1$ and $f^2 = 0$.*

Proof. Observe first that B_0 meets each fiber f in exactly one point, and does so transversely; hence their intersection number is $B_0 \cdot f = 1$. Meanwhile, any two distinct fibres are disjoint by definition of the ruling, so a fiber cannot intersect itself, giving $f^2 = 0$. □

Lemma 1.32. *Let S be a ruled surface on a smooth curve B and let D be a divisor on S. If $D \cdot f \geq 0$, then*

$$H^i(S, \mathcal{O}_S(D)) \cong H^i(B, \pi_*\mathcal{O}_S(D)) \text{ for all } i > 0.$$

Proof. See [Har, Lemma V.2.4] □

Lemma 1.33. *Let $S \cong \mathbb{P}(\mathcal{F})$ be a ruled surface over B and let H be a divisor in $|\mathcal{O}_S(1)|$. Then, the canonical divisor K_S on S is given by*

$$K_S \sim -2H + \pi^*(K_B + \det \mathcal{F}) \tag{1.1}$$

Proof. See [Har, Lemma 2.10] and use [Har, Exercise III.8.4(b)]

□

1.8 Nef line bundles

Definition 1.34 (Nef line bundles). Let X be a smooth projective variety. Let D be a Cartier divisor on X, and let $\mathcal{L} = \mathcal{O}_X(D)$ be the associated line bundle. We say that $\mathcal{L}$ (equivalently, D) is *nef* if for every irreducible curve $C \subset X$ one has

$$D \cdot C \ \geq \ 0.$$

In other words, the degree of $\mathcal{L}$ restricted to any curve is non-negative.

Definition 1.35 (Nef vector bundles). Let X be a smooth projective variety. A vector bundle $\mathcal{E}$ on X is said to be nef if the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on the projectivization $\mathbb{P}(\mathcal{E})$ is nef.

Lemma 1.36. *Let X be a smooth projective variety and let $\mathcal{L}$ be a globally generated line bundle on X. Then $\mathcal{L}$ is nef.*

Proof. Fix an irreducible curve $C \subseteq X$. Because $\mathcal{L}$ is globally generated, its base locus is empty. Now consider the short exact sequence of sheaves

$$0 \to \mathcal{I}_{C/X} \otimes \mathcal{L} \to \mathcal{L} \to \mathcal{L}_{|C} \to 0.$$

Passing to global sections gives

$$0 \to H^0(\mathcal{I}_{C/X} \otimes \mathcal{L}) \to H^0(\mathcal{L}) \to H^0(\mathcal{L}_{|C}).$$

If $\deg(\mathcal{L}_{|C}) < 0$, then $H^0(\mathcal{L}_{|C}) = 0$, and exactness forces

$$H^0(\mathcal{I}_{C/X} \otimes \mathcal{L}) \cong H^0(\mathcal{L}).$$

In other words, every global section of $\mathcal{L}$ vanishes identically along C. This means, C is contained in the base locus of $\mathcal{L}$, contradicting global generation. Therefore, $\deg(\mathcal{L}_{|C}) \geq 0$ for every curve $C \subseteq X$. Hence, $\mathcal{L}$ is nef. □

Corollary 1.37. *Let X be a smooth projective variety and let $\mathcal{E}$ be a globally generated vector bundle on X. Then* $\det(\mathcal{E})$ *is nef.*

Proof. If $\mathcal{E}$ is a globally generated vector bundle on X, then its determinant $\det(\mathcal{E})$ is a globally generated line bundle on X, and hence nef by the previous lemma. □

Lemma 1.38. *Let X be a smooth projective variety of dimension n and let $\mathcal{L} = \mathcal{O}_X(D)$ be a nef line bundle on X. Then $D^n \geq 0$.*

Proof. By [Laz, Theorem 1.4.9], for every irreducible subvariety $V \subseteq X$ of dimension k, one has $D^k \cdot V \geq 0$. Taking $k = n$ and $V = X$ gives $D^n = D^n \cdot [X] \geq 0$, as desired. □

1.9 Big line bundles

Definition 1.39 (**Big line bundles**)**.** Let X be a smooth projective variety of dimension n and let $\mathcal{L}$ be a line bundle on X. For any integer $m \geq 1$ such that $H^0(\mathcal{L}^{\otimes m}) \neq 0$, consider the rational map

$$\phi_m : X \dashrightarrow \mathbb{P}H^0(\mathcal{L}^{\otimes m})$$

induced by the complete linear system $|\mathcal{L}^{\otimes m}|$.

(i) The *Kodaira dimension* of $\mathcal{L}$, denoted by $k(X, \mathcal{L})$ is defined as

$$k(X, \mathcal{L}) := \begin{cases} -\infty & \text{if } H^0(X, \mathcal{L}^{\otimes m}) = 0 \text{ for all } m \geq 1 \\ \max\limits_{\substack{m \geq 1 \\ H^0(X, \mathcal{L}^{\otimes m}) \neq 0}} \dim \overline{\phi_m(X)} & \text{otherwise} \end{cases}$$

(ii) The line bundle $\mathcal{L}$ is said to be *big* if

$$k(X, \mathcal{L}) = \dim X = n.$$

Definition 1.40. Let X be a smooth projective variety of dimension n. A vector bundle $\mathcal{E}$ on X is said to be *big* if the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on the projectivization $\mathbb{P}(\mathcal{E})$ is big.

Lemma 1.41. *Let X be a smooth projective variety of dimension n and let $\mathcal{L} = \mathcal{O}_X(D)$ be a nef line bundle on X. Then*

$$\mathcal{L} \text{ is big} \iff D^n > 0.$$

Proof. See [Laz, Theorem 2.2.16]. □

Lemma 1.42 (Kawamata–Viehweg vanishing theorem for nef and big line bundles). *Let X be a smooth projective variety of dimension n and let $\mathcal{L}$ be a nef and big line bundle on X. Then*

$$H^i(\mathcal{L}(K_X)) = 0 \text{ for every } i > 0$$

Proof. See [Laz, Theorem 4.3.1] □

Remark 1.43. Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be a nef vector bundle on X. If $c_1(\mathcal{E})^n = 0$ then $\mathcal{E}$ is not big.

Proof. See [LM, Remark 2.2] □

Chapter 2

Ulrich bundles

2.1 Definition and characterizations

Definition 2.1. Let $X \subseteq \mathbb{P}^N$ be a smooth projective variety and let $\mathcal{E}$ be a vector bundle on X. $\mathcal{E}$ is said to be *Ulrich* if it satisfies

$$H^i(X, \mathcal{E}(-p)) = 0 \text{ for all } i \geq 0 \text{ and } 1 \leq p \leq \dim X.$$

Proposition 2.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth hypersurface of degree d and let $\mathcal{E}$ be a rank r vector bundle on X. The following conditions are equivalent:*

(i) $\mathcal{E}$ is Ulrich;

(ii) There exists an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N}^{\oplus rd}(-1) \xrightarrow{L} \mathcal{O}_{\mathbb{P}^N}^{\oplus rd} \to \mathcal{E} \to 0.$$

Proof. See [Bea2, Proposition 2.2] □

Theorem 2.3. *Let $X \subseteq \mathbb{P}^N$ be a smooth projective variety and let $\mathcal{E}$ be a vector bundle on X. The following conditions are equivalent:*

(i) There exists a linear resolution

$$0 \to \mathcal{L}_c \to \mathcal{L}_{c-1} \to \ldots \to \mathcal{L}_0 \to \mathcal{E} \to 0$$

with $c = codim(X, \mathbb{P}^N)$ and $\mathcal{L}_i = \mathcal{O}_{\mathbb{P}^N}(-i)^{\oplus b_i}$;

(ii) $\mathcal{E}$ is an Ulrich bundle on X;

(iii) if $\pi : X \to \mathbb{P}^{\dim X}$ is a finite linear projection, the vector bundle $\pi_*\mathcal{E}$ is trivial.

Proof. See [Bea2, Theorem 2.3] □

Definition 2.4 (**Ulrich complexity**). Let $X \subseteq \mathbb{P}^N$ be a smooth variety. The *Ulrich complexity* of X is the integer

$$uc(X) := \min\{r \geq 1 \ : \ X \text{ carries an Ulrich vector bundle of rank } r\}.$$

If no Ulrich vector bundle exists on X, we set $uc(X) := \infty$.

2.2 Main properties

Lemma 2.5. *Let $\mathcal{E}$ be a rank r Ulrich bundle on a smooth projective variety $X \subseteq \mathbb{P}^N$ of dimension n and degree d. Then*

- *(i) $\mathcal{E}$ is 0-regular and globally generated;*
- *(ii) $H^i(X, \mathcal{E}) = 0$ for all $i > 0$;*
- *(iii) $H^i(X, \mathcal{E}(j)) = 0$ for every integer j and $0 < i < n$;*
- *(iv) $\chi(\mathcal{E}(m)) = rd\binom{m+n}{n}$;*
- *(v) $h^0(X, \mathcal{E}) = rd$.*

Proof. (i) Fix a positive integer p. For $p > n$, then $H^p(X, \mathcal{E}(-p)) = 0$; for $p \leq n$, since $\mathcal{E}$ is Ulrich, $H^i(X, \mathcal{E}(-p)) = 0$ for all $i \geq 0$, in particular for $i = p$. Hence, $\mathcal{E}$ is 0-regular and by Theorem 1.7(i), it is also globally generated.

(ii) By point (i) and Theorem 1.7(iii), $\mathcal{E}$ is k-regular for all $k \geq 0$, that is,

$$H^p(X, \mathcal{E}(k-p)) = 0 \text{ for all } k \geq 0 \text{ and } p > 0.$$

Taking $k = p$, one has

$$H^p(X, \mathcal{E}) = 0 \text{ for all } p > 0,$$

that is, (ii).

(iii) By Theorem 2.3(iii), we have $H^i(X, \mathcal{E}(j)) = H^i(\mathbb{P}^N, \pi_*\mathcal{E}(j))$, which vanishes for $1 \leq i \leq n-1$ and all integers j.

(iv) By [Laz, Theorem 1.1.24] and [Har, Appendix A, Theorem 4.1], the Euler characteristic $\chi(X, \mathcal{E}(m))$ is given by a polynomial $P(m) \in \mathbb{Z}[m]$ of degree n, whose leading coefficient is

$$\frac{r \cdot H^n}{n!},$$

where $H \in |\mathcal{O}_X(1)|$ denotes a hyperplane section of X, and $r = \mathrm{rk}(\mathcal{E})$.

Since the degree of X is defined as $d := H^n$, the leading coefficient of $P(m)$ becomes $\frac{rd}{n!}$. Since $\mathcal{E}$ is Ulrich, $\chi(X, \mathcal{E}(t)) = 0$ for all $-n \leq t \leq -1$. Hence, $P(m)$ vanishes for $m = -1, \dots, -n$, so that $P(m)$ must be of the form

$$P(m) = \frac{rd}{n!}(m+1)(m+2)\cdots(m+n),$$

that is, $\chi(\mathcal{E}(m)) = rd\binom{m+n}{n}$.

(v) Evaluating the expression in (iv) at $m = 0$, we find

$$rd = \chi(X, \mathcal{E}) = h^0(X, \mathcal{E}),$$

where the last equality follows from (ii).

□

Lemma 2.6 (**[Bea2, (3.4)]**). *Let $\mathcal{E}$ be an Ulrich bundle on a smooth projective variety X of dimension n and let Y be a hyperplane section of X. Then, $\mathcal{E}_{|Y}$ is an Ulrich bundle on Y.*

Proof. Consider the short exact sequence

$$0 \to \mathcal{E}(-1) \to \mathcal{E} \to \mathcal{E}_{|Y} \to 0.$$

Tensoring with $\mathcal{O}_X(-j)$ yields

$$0 \to \mathcal{E}(-1-j) \to \mathcal{E}(-j) \to \mathcal{E}_{|Y}(-j) \to 0.$$

From this, we obtain the long exact sequence in cohomology

$$\dots \to H^i(\mathcal{E}(-j)) \to H^i(\mathcal{E}_{|Y}(-j)) \to H^{i+1}(\mathcal{E}(-1-j)) \to \dots$$

Since $\mathcal{E}$ is Ulrich, we have that

$$H^i(\mathcal{E}(-j)) = 0 \text{ for } i \geq 0 \text{ and } 1 \leq j \leq n,$$
$$H^{i+1}(\mathcal{E}(-1-j)) = 0 \text{ for } i \geq 0 \text{ and } 0 \leq j \leq n-1.$$

Hence, by exactness,

$$H^i(\mathcal{E}_{|Y}(-j)) = 0 \text{ for all } i \geq 0 \text{ and } 1 \leq j \leq n-1 = \dim Y,$$

that is, $\mathcal{E}_{|Y}$ is Ulrich. □

Lemma 2.7. *Let X be a smooth projective variety of dimension n such that $(X, \mathcal{O}_X(1)) \ncong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and let $\mathcal{E}$ be an Ulrich vector bundle on X. Then*

$$H^0(\mathcal{E}^*) = 0.$$

Proof. If $n = 1$ we have three cases, according to the genus g of X.

If $g = 1$, then $K_X = 0$, so that $h^0(\mathcal{E}^*) = h^1(\mathcal{E}) = 0$, by Serre's duality and Lemma 2.5(ii).

If $g \geq 2$, choosing $D \in |K_X|$, we have the exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}(K_X) \to \mathcal{E}(K_X)_{|D} \to 0$$

which gives

$$\ldots \to H^1(\mathcal{E}) \to H^1(\mathcal{E}(K_X)) \to H^1(\mathcal{E}(K_X)_{|D}) \to \ldots$$

Since $H^1(\mathcal{E}) = 0$ by Lemma 2.5(ii) and $H^1(\mathcal{E}(K_X)_{|D}) = 0$, the exactness of the sequence implies that $h^1(\mathcal{E}(K_X)) = 0$, that is, $h^0(\mathcal{E}^*) = 0$, by Serre's duality.

If $g = 0$, then $X \cong \mathbb{P}^1$ and then $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$. Now, $\mathcal{E}$ is Ulrich if and only if $\mathcal{O}_{\mathbb{P}^1}(a_i)$ is Ulrich for each i with $1 \leq i \leq r$. Since $\mathcal{E}$ is globally generated by Lemma 2.5(i), it follows that $a_i \geq 0$ for all i. Suppose, for contradiction, that $h^0(\mathcal{E}^*) \neq 0$, then there is an i such that $a_i = 0$. Since by assumption $(X, \mathcal{O}_X(1)) \ncong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, we have that $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^1}(a)$ for some integer $a \geq 2$. Since $a_i = 0$ and $\mathcal{O}_{\mathbb{P}^1}$ is Ulrich, we have $H^1(\mathcal{O}_{\mathbb{P}^1}(-a)) = 0$ for some $a \geq 2$, which is clearly a contradiction.

This completes the proof for $n = 1$.

Moreover, observe that if $(X, \mathcal{O}_X(1)) \ncong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ then $H^0(\mathcal{E}^*(-\ell)) = 0$ for all $\ell \geq 0$.

We now claim that, if $(X, \mathcal{O}_X(1)) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, then $H^0(\mathcal{E}^*(-\ell)) = 0$ for every $\ell \geq 0$. This is in fact equivalent to the statement. For $n = 1$, the statement holds by the above argument. For $n \geq 2$, we proceed by induction. Hence, Let H be a divisor in the linear system $|\mathcal{O}_X(1)|$ and assume that $(X, H) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Consider a smooth irreducible hyperplane section $Y \in |\mathcal{O}_X(1)|$. Then $(Y, H_{|Y}) \neq (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$, for otherwise $(X, H) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. It follows by Lemma 2.6 that $\mathcal{E}_{|Y}$ is an Ulrich bundle with respect to $(Y, H_{|Y})$, hence $H^0(\mathcal{E}^*_{|Y}(-\ell)) = 0$ for every $\ell \geq 0$ by induction. From the exact sequence

$$0 \to \mathcal{E}^*(-\ell - 1) \to \mathcal{E}^*(-\ell) \to \mathcal{E}^*_{|Y}(-\ell) \to 0$$

we deduce that $h^0(\mathcal{E}^*(-\ell))) = h^0(\mathcal{E}^*(-\ell - 1))$ for every $\ell \geq 0$. Since $h^0(\mathcal{E}^*(-\ell))) = 0$ for $\ell \gg 0$ by Serre vanishing, we get that $H^0(\mathcal{E}^*(-\ell)) = 0$ for every $\ell \geq 0$. □

Lemma 2.8 (**[Bea2, (3.6)]**). *Let X and Y be smooth projective varieties and let $\pi : X \to Y$ be a finite surjective morphism. Let $\mathcal{L}$ be a very ample line bundle on Y and let $\mathcal{E}$ be a vector bundle on X. Then $\mathcal{E}$ is an Ulrich bundle for $X, \pi^*\mathcal{L}$ if and only if $\pi_*\mathcal{E}$ is an Ulrich bundle for $(Y, \mathcal{L})$.*

Proof. The claim follows directly from the isomorphism

$$H^i(Y, \pi_*\mathcal{E} \otimes \mathcal{L}^{\otimes -k}) \cong H^i(X, \mathcal{E} \otimes \pi^*\mathcal{L}^{\otimes -k})$$

which holds for all $i \geq 0$. □

2.3 Stability

Definition 2.9. Let $\mathcal{E}$ be a rank r vector bundle on a smooth projective variety X.

(i) $\mathcal{E}$ is said to be *semistable* if for every nonzero coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ we have the inequality

$$P_{\mathcal{F}}/rk(\mathcal{F}) \leq P_{\mathcal{E}}/rk(\mathcal{E})$$

where $P_{\mathcal{F}}$ and $P_{\mathcal{E}}$ are the Hilbert polynomials of the sheaves.

(ii) $\mathcal{E}$ is said to be *stable* if the inequality above is strict for every nonzero proper coherent subsheaf of $\mathcal{E}$.

Lemma 2.10. *Let $\mathcal{E}$ be a rank r Ulrich bundle on a smooth projective variety X. Then $\mathcal{E}$ is semi-stable.*

Proof. See [CHGS, Theorem 2.9] □

Lemma 2.11 (Bogomolov's inequality). *Let $\mathcal{E}$ be a semistable rank $r \geq 2$ vector bundle on a smooth projective surface S. Then the following inequality holds*

$$2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2 \geq 0$$

Proof. See [Gie, Theorem 0.3] □

Lemma 2.12. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety and let $\mathcal{E}$ be an Ulrich bundle of minimal rank r on X, that is, $uc(X) = r$. Then, $\mathcal{E}$ is stable.*

Proof. It follows from [CHGS, Theorem 2.9], by applying an argument analogous to that in [CG, Proof of Lemma 2.3]. □

Definition 2.13. Let $X \subseteq \mathbb{P}^N$ be a smooth variety and let $\mathcal{E}$ be a vector bundle on X. We say that $\mathcal{E}$ is *simple* if

$$h^0(X, \mathcal{E} \otimes \mathcal{E}^*) = 1,$$

or, equivalently, $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \cong \mathbb{C}$.

Lemma 2.14. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety and let $\mathcal{E}$ be an Ulrich bundle of minimal rank on X. Then, $\mathcal{E}$ is a simple vector bundle.*

Proof. It follows from Lemma 2.12 by using [HL, Corollary 1.2.8].

□

2.4 Notable cases and explicit constructions

Lemma 2.15 (**[Bea2, (3.3)]**). *Let $C \subseteq \mathbb{P}^N$ be a projective curve. Then Ulrich bundles on C are the bundles $\mathcal{E}(1)$ where $\mathcal{E}$ is a vector bundle on C with vanishing cohomology.*

Proof. Set $\mathcal{F} = \mathcal{E}(1)$, where $\mathcal{E}$ is a vector bundle on C with vanishing cohomology and set $p = 1$. Then, for all $i \geq 0$,

$$H^i(\mathcal{F}(-p)) = H^i(\mathcal{E}) = 0,$$

that is $\mathcal{F}$ is Ulrich. □

Lemma 2.16. *Let $Q \subset \mathbb{P}^{n+1}$ be a smooth quadric.*

(i) If n is odd, there is exactly one indecomposable Ulrich bundle on Q, of rank $2^{\frac{n-1}{2}}$, the spinor bundle.

(ii) If n is even, there are exactly two indecomposable Ulrich bundles on Q, of rank $2^{\frac{n-2}{2}}$, the spinor bundles.

(iii) Ulrich vector bundles on quadrics are direct sums of spinor bundles.

Proof. For (i) and (ii), see [Bea2, Proposition 2.5]. Point (iii) is a direct consequence of previous points. □

Remark 2.17. For a more detailed discussion on spinor bundles on quadrics, see [Ott1]. For applications relevant to this work, the case $n = 2$ is particularly noteworthy: here, the quadric Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and in this case, the spinor bundles correspond precisely to the pullbacks of the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ along the two natural projection morphisms ([Bea2, Remark 2.6]).

Proposition 2.18. *$(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$ admits an Ulrich bundle of rank $n!$.*

Proof. See [Bea2, Proposition 3.1] □

Corollary 2.19. *Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be a rank r Ulrich bundle on X. Then, for every $d \geq 1$, there exists an Ulrich bundle $\mathcal{E}$ of rank $rn!$ for $(X, \mathcal{O}_X(d))$*

Proof. The claim follows directly from Proposition 2.18 and Lemma 2.8. □

Proposition 2.20 (**Ulrich line bundles**). *(i) Let X be a projective variety of degree $d > 1$ and assume $Pic(X) = \mathbb{Z}H$, where $H \in |\mathcal{O}_X(1)|$. Then, there exists no Ulrich line bundle on X;*

(ii) Let $S \subset \mathbb{P}^N$ be a del Pezzo surface, that is, $K_S = -H$ where $H \in |\mathcal{O}_X(1)|$. Let $\mathcal{L} = \mathcal{O}_S(D)$ be a line bundle on S satisfying $D^2 = -2$ and $D \cdot K_S = 0$. Then, $\mathcal{L}(1)$ is an Ulrich line bundle on S.

In particular, these conditions are satisfied by taking $\mathcal{L} = \mathcal{O}_S(\ell - \ell')$ where ℓ and ℓ' are two disjoint lines.

Proof. (i) Suppose, for the sake of contradiction, that $\mathcal{O}_X(k)$ is an Ulrich bundle on X. Then,

$$h^0(\mathcal{O}_X(k-1)) = 0 \text{ and } h^0(\mathcal{O}_X(k)) = d$$

by Lemma 2.5(v). These cohomological conditions imply that k must be zero. Indeed, the vanishing $h^0(\mathcal{O}_X(k-1)) = 0$ forces $k \leq 0$, while $h^0(\mathcal{O}_X(k)) = d > 0$ forces $k \geq 0$.

Consequently, $d = 1$, which is a contradiction.

(ii) See [Bea2, Proposition 4.1(i)] □

Chapter 3

Ulrich bundles on surfaces

This chapter delves into the study of Ulrich bundles specifically on smooth projective surfaces. Building upon the foundational concepts of Ulrich bundles introduced in Chapter 2, we explore their characteristics and properties in the two-dimensional setting. We begin by presenting Casnati's Theorem (Section 3.1), a central result which provides a key characterization of Ulrich bundles on surfaces in terms of their Chern classes and only two cohomology vanishing conditions. Following this, in Section 3.2, we will introduce the concept of Lazarsfeld-Mukai bundles and establish important relationships between such bundles and Ulrich bundles. These relationships, explored further in Section 3.3, will be crucial for understanding the correspondence between curves on a surface and Ulrich bundles on surfaces, which constitutes the primary objective of this treatment. Section 3.4 then investigates specific conditions under which certain curves related to Ulrich bundles exhibit connectedness properties.

3.1 Casnati's Theorem

Theorem 3.1 (**Casnati, [Cas, Proposition 2.1]**). *Let $S \subseteq \mathbb{P}^N$ be a smooth projective surface of degree d, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$, and let $\mathcal{E}$ be a vector bundle of rank r on S. Then, the following statements are equivalent:*

(i) $\mathcal{E}$ is Ulrich;

(ii) The following conditions hold:

(1) $H^0(\mathcal{E}(-1)) = H^2(\mathcal{E}(-2)) = 0$;

(2) $c_1(\mathcal{E}) \cdot H = \frac{r}{2}(K_S + 3H) \cdot H$ and

$$c_2(\mathcal{E}) = r\chi(\mathcal{O}_S) + \tfrac{1}{2}c_1(S)c_1(\mathcal{E}) + \tfrac{1}{2}c_1(\mathcal{E})^2 - rH^2.$$

Before proceeding with the proof of Theorem 3.1, let us first prove the following lemma.

Lemma 3.2. *Let $S \subseteq \mathbb{P}^N$ be a smooth projective surface of degree d, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$, and let $\mathcal{E}$ be a vector bundle of rank r on S. Then*

$$\begin{cases} \chi(\mathcal{E}(-1)) = 0 \\ \chi(\mathcal{E}(-2)) = 0 \end{cases} \iff \begin{cases} c_1(\mathcal{E}) \cdot H = \frac{r}{2}(K_S + 3H) \cdot H \\ c_2(\mathcal{E}) = r\chi(\mathcal{O}_S) + \frac{1}{2}c_1(S)c_1(\mathcal{E}) + \frac{1}{2}c_1(\mathcal{E})^2 - rH^2 \end{cases}$$

Proof. Observe that, if $\mathcal{L}$ is any line bundle on S, so that $\mathcal{L} = \mathcal{O}_S(D)$ for some divisor D, recalling [Prop.5.17 EH], one has the standard Chern class identities for $\mathcal{E} \otimes \mathcal{L}$

$$c_1(\mathcal{E} \otimes \mathcal{L}) = c_1(\mathcal{E}) + rc_1(\mathcal{L}),$$
$$c_2(\mathcal{E} \otimes \mathcal{L}) = c_2(\mathcal{E}) + (r-1)c_1(\mathcal{L})c_1(\mathcal{E}) + \tbinom{r}{2}c_1(\mathcal{L})^2$$

(Here, $c_1(\mathcal{L}) = [D]$ denotes the divisor class of D.)

We now apply these identities with $\mathcal{L} = \mathcal{O}_S(-1)$ and $\mathcal{L} = \mathcal{O}_S(-2)$. To streamline notation, write $c_1 := c_1(S)$, $d_1 := c_1(\mathcal{E})$, $d_2 := c_2(\mathcal{E})$. A direct computation then shows

$$c_1(\mathcal{E}(-k)) = d_1 - krH,$$
$$c_2(\mathcal{E}(-k)) = d_2 - k(r-1)d_1H + k^2\tbinom{r}{2}H^2,$$

for $k = 1, 2$. Substituting each of these into the Riemann–Roch formula of Theorem 1.3, produces two explicit formulas for $\chi(\mathcal{E}(-1))$ and $\chi(\mathcal{E}(-2))$. Requiring both Euler characteristics to vanish is therefore equivalent to the system

$$\begin{cases} r\chi(\mathcal{O}_S) + \frac{1}{2}c_1d_1 - \frac{r}{2}c_1H + \frac{1}{2}d_1^2 - d_2 - d_1H + \frac{r}{2}H^2 = 0 \\ r\chi(\mathcal{O}_S) + \frac{1}{2}c_1d_1 - rc_1H + \frac{1}{2}d_1^2 - d_2 - 2d_1H + 2rH^2 = 0 \end{cases}$$

that is, subtracting the first equation from the second one,

$$\begin{cases} d_2 = r\chi(\mathcal{O}_S) + \frac{1}{2}c_1d_1 - \frac{r}{2}c_1H + \frac{1}{2}d_1^2 - d_1H + \frac{r}{2}H^2 \\ d_1H = -\frac{r}{2}c_1H + \frac{3}{2}rH^2 = \frac{r}{2}(-c_1 + 3H)H \end{cases}$$

Noting that $c_1 = c_1(S) = -K_S$, the second equality immediately rewrites as $c_1(\mathcal{E}) \cdot H = \frac{r}{2}(K_S + 3H) \cdot H$ and substituting this back into the first equation yields

$$d_2 = r\chi(\mathcal{O}_S) + \frac{1}{2}c_1d_1 - \frac{r}{2}c_1H + \frac{1}{2}d_1^2 - \frac{r}{2}(-c_1 + 3H)H + \frac{r}{2}H^2$$

that is exactly $c_2(\mathcal{E}) = r\chi(\mathcal{O}_S) + \frac{1}{2}c_1(S)c_1(\mathcal{E}) + \frac{1}{2}c_1(\mathcal{E})^2 - rH^2$.

Hence the two vanishing conditions on $\chi(\mathcal{E}(-1))$ and $\chi(\mathcal{E}(-2))$ are equivalent to the stated Chern-class equalities.

□

We now proceed to prove Theorem 3.1.

Proof of Theorem 3.1. Suppose $\mathcal{E}$ is an Ulrich bundle on S. By definition, this means $H^i(\mathcal{E}(-p)) = 0$ for all i and for $p = 1, 2$. In particular, this immediately gives $H^0(\mathcal{E}(-1)) = H^2(\mathcal{E}(-2)) = 0$, so condition (1) is satisfied. Moreover, the total cohomology vanishing forces $\chi(\mathcal{E}(-1)) = \chi(\mathcal{E}(-2)) = 0$ and, by the previous lemma, this simultaneous vanishing is exactly equivalent to the Chern–class relations in (2). Hence both (1) and (2) hold, completing the proof of (ii).

Conversely, suppose (1) and (2) both hold. Then, by the previous lemma, we again conclude that $\chi(\mathcal{E}(-1)) = \chi(\mathcal{E}(-2)) = 0$. Since moreover $H^0(\mathcal{E}(-2)) \subseteq H^0(\mathcal{E}(-1)) = 0$ and $H^2(\mathcal{E}(-2)) = 0$, it follows at once that $H^i(\mathcal{E}(-2)) = 0$ for all i.

For $\mathcal{E}(-1)$, consider the short exact sequence of sheaves coming from restriction to the hyperplane section H

$$0 \to \mathcal{E}(-2) \to \mathcal{E}(-1) \to \mathcal{E}(-1)|_H \to 0.$$

Passing to cohomology gives the exact segment

$$\dots \to H^2(\mathcal{E}(-2)) \to H^2(\mathcal{E}(-1)) \to H^2(\mathcal{E}(-1)|_H) \to \dots$$

Since $H^2(\mathcal{E}(-2)) = 0$ and $H^2(\mathcal{E}(-1)|_H) = 0$, exactness forces $H^2(\mathcal{E}(-1)) = 0$. Finally, $\chi(\mathcal{E}(-1)) = 0$ gives $H^1(\mathcal{E}(-1)) = 0$.

Thus $H^i(\mathcal{E}(-p)) = 0$ for all i and $p = 1, 2$, hence $\mathcal{E}$ is Ulrich.

□

3.2 Lazarsfeld-Mukai bundles

In the correspondence we aim to establish between Ulrich bundles and curves on a surface, Lazarsfeld–Mukai bundles play a central role. Specifically, in our construction, Ulrich bundles on the surface S will arise precisely as Lazarsfeld–Mukai bundles associated with certain curves $C \subset S$ satisfying suitable conditions. This section reviews the construction of Lazarsfeld–Mukai bundles and highlights the properties that make them central to our approach.

Definition 3.3 (**Vector bundle** $K_{C,W,\mathcal{L}}$)**.** Let $S \subseteq \mathbb{P}^N$ be a smooth projective surface of degree d, with $H \in |\mathcal{O}_S(1)|$, and let $C \subset S$ be a smooth curve. Let $\mathcal{L}$ be a line bundle on C and $W \subseteq H^0(\mathcal{L})$ a r-dimensional base-point-free linear series. Consider the canonical surjective evaluation map

$$e_{C,W,\mathcal{L}} : W \otimes \mathcal{O}_S \twoheadrightarrow \mathcal{L}.$$

We define

$$K_{C,W,\mathcal{L}} \colon = \ker(e_{C,W,\mathcal{L}})$$

to be its kernel. Its dual $K^*_{C,W,\mathcal{L}}$ is the *Lazarsfeld–Mukai bundle* associated to the triple $(C, W, \mathcal{L})$.

Remark 3.4. Set $\mathcal{K} := K_{C,W,\mathcal{L}}$. Then

(i) One has by construction the exact sequence

$$0 \to \mathcal{K} \to W \otimes \mathcal{O}_S \to \mathcal{L} \to 0; \tag{3.1}$$

(ii) $\mathcal{K}$ is a rank r vector bundle on S;

(iii) $c_1(\mathcal{K}) = -[C]$;

(iv) $c_2(\mathcal{K}) = \deg(\mathcal{L})$.

Proof. *(i)* Trivial

(ii) This can be checked locally: since $\mathcal{L}$ is a line bundle on a smooth curve, we may assume $\mathcal{L} \cong \mathcal{O}_C$ in a neighborhood. Then the evaluation map decomposes as the direct sum of the canonical surjection $\mathcal{O}_S \to \mathcal{O}_C$ and $r-1$ copies of the zero map $\mathcal{O}_S \to 0$.

Indeed, if $x \notin C$, then the stalk $\mathcal{L}_x = 0$, so that the evaluation map $e_{C,W,\mathcal{L},x} : W \otimes \mathcal{O}_{S,x} \to \mathcal{L}_x$ is the zero map. Consequently, its kernel is $W \otimes \mathcal{O}_{S,x} \cong \mathcal{O}_{S,x}^{\oplus r}$.

If $x \in C$, then the stalk $\mathcal{L}_x$ is isomorphic to $\mathcal{O}_{C,x}$, a local ring and a free module of rank 1 over itself. The evaluation map at the stalk level becomes a surjective map

$$W \otimes \mathcal{O}_{S,x} \to \mathcal{L}_x \cong \mathcal{O}_{C,x}$$

which, by definition, is given by the composition of $W \otimes \mathcal{O}_{S,x} \to W \otimes \mathcal{O}_{C,x}$ and $W \otimes \mathcal{O}_{C,x} \to \mathcal{O}_{C,x}$. Since $\mathcal{O}_{C,x}$ is a free module over itself, it is in particular a projective module. A standard result in commutative algebra states that any surjection onto a projective module splits (see [Lan, Chapter III, §4]); that is, there exists a section $\sigma : \mathcal{O}_{C,x} \to W \otimes \mathcal{O}_{C,x}$ such that the composition is the identity on $\mathcal{O}_{C,x}$. It follows that $W \otimes \mathcal{O}_{C,x} \cong \mathcal{O}_{C,x} \oplus \ker(W \otimes \mathcal{O}_{C,x} \to \mathcal{O}_{C,x})$, by [Lan, Proposition III.3.2]. Hence, $\ker(W \otimes \mathcal{O}_{C,x} \to \mathcal{O}_{C,x}) = \mathcal{O}_{C,x}^{\oplus r-1}$ and the map $W \otimes \mathcal{O}_{C,x} \to \mathcal{O}_{C,x}$ splits.

Lifting the decomposition from C to S is straightforward. Select the distinguished copy of $\mathcal{O}_{C,x} \subset W \otimes \mathcal{O}_{C,x}$ on which the evaluation is the identity and take any element of $W \otimes \mathcal{O}_{S,x}$ mapping onto it. The $\mathcal{O}_{S,x}$–summand generated by this element still surjects onto $\mathcal{O}_{C,x}$, while the entire preimage of the complementary summand, already annihilated after restriction to C, is sent to zero. Thus, the evaluation splits locally as

$$\mathcal{O}_{S,x} \twoheadrightarrow \mathcal{O}_{C,x}, \quad \mathcal{O}_{S,x}^{\oplus(r-1)} \to 0,$$

so that globally the map $W \otimes \mathcal{O}_S \to \mathcal{L}$ is the direct sum of the canonical surjection $\mathcal{O}_S \to \mathcal{O}_C$ with $r-1$ copies of the zero map, as claimed.

Its kernel is thus locally isomorphic to $\mathcal{O}_S(-C) \oplus \mathcal{O}_S^{\oplus r}$, which is locally free of rank r. Therefore, $\mathcal{K}_{C,W,\mathcal{L}}$ is a vector bundle of rank r on S.

(iii) Outside the curve C, the kernel $\mathcal{K}$ has rank r, implying that the map $\mathcal{K} \to W \otimes \mathcal{O}_S$ maintains full rank in this region. However, along C, this map drops rank, as $\mathcal{L}$ is supported on C. Consequently, the determinant line bundle $\det(\mathcal{K})$ vanishes along C, yielding the isomorphism $\det(\mathcal{K}) = \mathcal{O}_S(-C)$, hence, $c_1(\mathcal{K}) = -[C]$.

(iv) Let $s \in W$ be a generic section with divisor D, and let $W' := W/ < s >$. Then we have a map $\mathcal{K} \to W' \otimes \mathcal{O}_S$ which drops rank exaclty along $D \subseteq S$. Hence, $c_2(\mathcal{K}) = [D] = \deg(\mathcal{L})$. □

Lemma 3.5. *Let $S \subseteq \mathbb{P}^N$ be a smooth projective surface of degree d, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$, and let $C \subset S$ be a smooth curve. Let $\mathcal{L}$ be a line bundle on C, $W \subseteq H^0(\mathcal{L})$ a base-point-free linear series of dimension r and let $\mathcal{E}$ be the Lazarsfeld–Mukai bundle on S associated to $(C, W, \mathcal{L})$, i.e., $\mathcal{E}^* = K_{C,W,\mathcal{L}}$. Then*

(i) $h^0(\mathcal{E}(-1)) = h^1(\mathcal{L}(K_S + H))$

(ii) $H^2(\mathcal{E}(-2)) = \ker\varphi$, *where φ is the multiplication map,*

$$\varphi : W \otimes H^0(S, \mathcal{O}_S(K_S + 2H)) \to H^0(\mathcal{L}(K_S + 2H)).$$

$$\textit{(iii)} \quad \begin{cases} \chi(\mathcal{E}(-1)) = 0 \\ \chi(\mathcal{E}(-2)) = 0 \end{cases} \iff \begin{cases} \deg(C) = \frac{r}{2}(K_S + 3H) \cdot H \\ \deg(\mathcal{L}) = r\chi(\mathcal{O}_S) + g - 1 - C \cdot K_S - rH^2 \end{cases}$$

where g is the genus of the curve C.

Proof. *(i)* We start by tensoring the exact sequence

$$0 \to \mathcal{E}^* \to W \otimes \mathcal{O}_S \to \mathcal{L} \to 0, \tag{3.2}$$

given by Remark 3.4(i), with $\mathcal{O}_S(K_S + H)$, which gives

$$0 \to \mathcal{E}^*(K_S + H) \to W \otimes \mathcal{O}_S(K_S + H) \to \mathcal{L}(K_S + H) \to 0.$$

Taking cohomology of the above sequence yields the segment

$$\begin{gathered} ... \to W \otimes H^1(\mathcal{O}_S(K_S + H)) \to H^1(\mathcal{L}(K_S + H)) \to H^2(\mathcal{E}^*(K_S + H)) \to \\ W \otimes H^2(\mathcal{O}_S(K_S + H)) \to ... \end{gathered}$$

By Kodaira's vanishing theorem, we know that $H^1(\mathcal{O}_S(K_S + H)) = H^2(\mathcal{O}_S(K_S + H)) = 0$. Therefore, we conclude that

$$W \otimes H^1(\mathcal{O}_S(K_S + H)) = 0 \text{ and } W \otimes H^2(\mathcal{O}_S(K_S + H)) = 0,$$

that is, by the exactness of (3.2), $H^1(\mathcal{L}(K_S + H)) \cong H^2(\mathcal{E}^*(K_S + H))$.

Hence, $h^0(\mathcal{E}(-1)) \cong h^1(\mathcal{L}(K_S + H))$, by Serre's duality.

(ii) Begin by observing that Serre's duality on the surface S identifies $H^2(\mathcal{E}(-2H)) \cong H^0(\mathcal{E}^*(K_S + 2H))^*$. In particular, vanishing of $H^2(\mathcal{E}(-2H))$ is equivalent to the statement $H^0(\mathcal{E}^*(K_S + 2H)) = 0$.

On the other hand,tensoring the exact sequence (3.2) by $\mathcal{O}_S(K_S + 2H)$ yields

$$0 \to \mathcal{E}^*(K_S + 2H) \to W \otimes \mathcal{O}_S(K_S + 2H) \to \mathcal{L}(K_S + 2H) \to 0$$

On cohomology this becomes

$$\begin{gathered} 0 \to H^0(\mathcal{E}^*(K_S + 2H)) \to W \otimes H^0(\mathcal{O}_S(K_S + 2H)) \xrightarrow{\varphi} H^0(\mathcal{L}(K_S + 2H)) \to \\ H^1(\mathcal{E}^*(K_S + 2H)) \to ... \end{gathered}$$

By exactness, the space $H^0(\mathcal{E}^*(K_S + 2H))$ is precisely the kernel of the multiplication map φ.

(iii) From Remark 3.4(iii),(iv) and the basic properties of the Chern classes of the dual bundle (see also 1.12), it follows that

$$c_1(\mathcal{E}) = [C] \text{ and } c_2(\mathcal{E}) = \deg(\mathcal{L})$$

According to Lemma 3.2, we have

$$\begin{cases} \chi(\mathcal{E}(-1)) = 0 \\ \chi(\mathcal{E}(-2)) = 0 \end{cases} \iff \begin{cases} c_1(\mathcal{E}) \cdot H = \frac{r}{2}(K_S + 3H) \cdot H \\ c_2(\mathcal{E}) = r\chi(\mathcal{O}_S) + \frac{1}{2}c_1(S)c_1(\mathcal{E}) + \frac{1}{2}c_1(\mathcal{E})^2 - rH^2 \end{cases}$$

$$\iff \begin{cases} \deg(C) = \frac{r}{2}(K_S + 3H) \cdot H \\ \deg(\mathcal{L}) = r\chi(\mathcal{O}_S) + \frac{1}{2}c_1(S)c_1(\mathcal{E}) + \frac{1}{2}c_1(\mathcal{E})^2 - rH^2 = \\ \qquad = r\chi(\mathcal{O}_S) + \frac{1}{2}(C^2 - C.K_S) - rH^2 \end{cases}$$

since $c_1(\mathcal{E}) = C$ and $c_1(S) = -K_S$.

The claim follows from the adjunction formula, which implies that

$$\frac{1}{2}(C^2 - C \cdot K_S) = \frac{1}{2}(C^2 + C \cdot K_S) - C \cdot K_S = g - 1 - C \cdot K_S.$$

Substituting this expression into the formula for $\deg(\mathcal{L})$, we find that

$$\deg(\mathcal{L} = r\chi(\mathcal{O}_S) + \frac{1}{2}(C^2 - C.K_S) - rH^2$$

is equivalent to $\deg(\mathcal{L}) = r\chi(\mathcal{O}_S) + g - 1 - C \cdot K_S - rH^2$.

□

3.3 Ulrich bundles and curves on surfaces

Having introduced the necessary tools, we now turn to the core topic of this chapter: the interplay between Ulrich bundles and curves on surfaces. Building on the previous sections, we describe how certain curves give rise to Ulrich bundles, and conversely, how geometric properties of Ulrich bundles reflect the structure of the underlying curves. We begin by considering a simple case in the next Remark 3.6. This will be followed by Theorem 3.7, which represents the true core of our discussion.

Remark 3.6. Let $\mathcal{E}$ be a rank r vector bundle on $\mathbb{P}^2$. Then

$$\mathcal{E} \text{ is Ulrich respect to } \mathcal{O}_{\mathbb{P}^2}(1) \iff \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}^{\oplus r}.$$

Proof. It follows directly from Theorem 2.3. Indeed, when $X = \mathbb{P}^2 \subseteq \mathbb{P}^2$, we have $\operatorname{codim}(X, \mathbb{P}^2) = 0$, so that the linear resolution in point (i) of Theorem 2.3 reduces to

$$0 \to \mathcal{L}_0 \to \mathcal{E} \to 0$$

with $\mathcal{L}_0 = \mathcal{O}_{\mathbb{P}^2}^{\oplus r}$. Hence, $\mathcal{E}$ is Ulrich if and only if $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}^{\oplus r}$, by the equivalence of (i) and (ii) in Theorem 2.3. □

We have thus fully described the case when $(S, \mathcal{O}_S(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. For this reason, we shall systematically exclude it from our subsequent analysis. The intrinsic justification lies in the fact that the "curve" associated with this case is actually empty (as shown in the proof of Theorem 3.7), which naturally leads us to treat it as an isolated instance.

Theorem 3.7. *Let $S \subset \mathbb{P}^N$ be a smooth projective surface of degree $d \geq 2$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$. Then, there exists an Ulrich bundle $\mathcal{E}$ of rank r on S if and only if there exists a smooth (possibly disconnected) curve $C \subset S$ of genus g together with a pair $(W, \mathcal{L})$, where $\mathcal{L}$ is a line bundle on C and $W \subseteq H^0(\mathcal{L})$ is a $r-$dimensional base-point free linear series, such that:*

(i) $H^1(C, \mathcal{L}(K_S + H)) = 0$;

(ii) the multiplication map

$$\varphi : W \otimes H^0(S, \mathcal{O}_S(K_S + 2H)) \to H^0(\mathcal{L}(K_S + 2H))$$

is injective;

(iii) $\deg(C) = \frac{r}{2}(K_S + 3H) \cdot H$ *and*
$\deg(\mathcal{L}) = r\chi(\mathcal{O}_S) + g - 1 - C \cdot K_S - rH^2$.

Proof. Let us assume that $\mathcal{E}$ is a rank r Ulrich vector bundle on S. Observe that $\mathcal{E}$ is globally generated by Lemma 2.5(i); therefore, there exists a general subspace V of $H^0(\mathcal{E})$ of dimension r. Hence, the evaluation map, which is injective by Proposition 1.26,

$$\varphi_V : V \otimes \mathcal{O}_S \to \mathcal{E}$$

is general in $\mathcal{H}om(V \otimes \mathcal{O}_S, \mathcal{E})$.

In addition, observe that $\mathcal{H}om(V \otimes \mathcal{O}_S, \mathcal{E}) \cong (V^* \otimes \mathcal{O}_S) \otimes \mathcal{E} \cong \mathcal{O}_S^{\oplus r} \otimes \mathcal{E}$, hence $\mathcal{H}om(V \otimes \mathcal{O}_S, \mathcal{E})$ is globally generated, since it is the tensor product of two globally generated vector bundles. We now consider the $(r-1)$-st degeneracy locus of φ_V, $D_{r-1}(\varphi_V)$. According to Banica's theorem (Theorem 1.19), this locus is either empty or his codimension is given by

$$\text{codim}(D_{r-1}(\varphi_V)) = (rk(\mathcal{O}_S^{\oplus r}) - (r-1))(rk(\mathcal{E}) - (r-1)) = 1.$$

We now claim that if $D_{r-1}(\varphi_V) = \emptyset$ then $(S, \mathcal{O}_S(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, which is excluded by the assumptions.

In fact, if $D_{r-1}(\varphi_V) = \emptyset$, then $rk(\varphi(x)) > r-1 \ \forall x \in S$, that is, $rk(\varphi(x)) = r \ \forall x \in S$. Hence, φ_V is surjective at every point of S. It follows that $\mathcal{E} \cong \mathcal{O}_S^{\oplus r}$. In particular, since $\mathcal{O}_S^{\oplus r}$ is Ulrich, so is O_S. Recall that, by Lemma 2.5(v), one has that $h^0(\mathcal{E}) = rd$. In particular, for $\mathcal{O}_S$, we get $h^0(\mathcal{O}_S) = d$, since $rk(\mathcal{O}_S) = 1$. Thus, it follows that $d = 1$. Therefore S is a variety of degree 1, implying that S is isomorphic to $\mathbb{P}^2$. Moreover, S is embedded in $\mathbb{P}^N$ through an immersion of degree 1. Consequently, we conclude that $\mathcal{O}_S(1) \cong \mathcal{O}_{\mathbb{P}^2}(1)$, and therefore we obtain the isomorphism of pairs $(S, \mathcal{O}_S(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

Hence, the case $D_{r-1}(\varphi_V) = \emptyset$ cannot occur, so that $D_{r-1}(\varphi_V) \neq \emptyset$ and, therefore, it has the expected codimension one. In particular, it defines a curve on S. Let

$$C := D_{r-1}(\varphi_V).$$

By Banica's Theorem 1.19, Sing(C)=$D_{r-2}(\varphi_V)$. Moreover, according to the same theorem, if $D_{r-2}(\varphi_V)$ were non empty, it would have codimension 4 in a 2-dimensional variety, which is clearly a contraddiction. Therefore, $D_{r-2}(\varphi_V) = \emptyset$, and consequently, Sing(C)= $\emptyset$, implying that C is smooth. We now proceed to define an appropriate line budle $\mathcal{L}$ on C, as well as a suitable r-dimensional base-point free subspace $W \subseteq H^0(\mathcal{L})$, which we we later show satisfies the required properties (i), (ii) and (iii). Let us consider the following short exact sequence of vector bundle

$$0 \to \mathcal{O}_S^{\oplus r} \xrightarrow{\varphi_V} \mathcal{E} \to \mathcal{L}' := \text{Coker}(\varphi_V) \to 0.$$

Observe that $\mathcal{L}'$ is supported at the points of S where φ_V is not surjective, that is, at the point $x \in S$ such that rk($\varphi_V(x)$)$< r$. Therefore, $\mathcal{L}'$ is supported on $D_{r-1}(\varphi_V) = C$. Since $D_{r-2}(\varphi_V) = \emptyset$, for every point $x \in C$, Im($\varphi_V(x)$) has rank exactly $r-1$ and, hence, the stalk $\mathcal{L}'_x$ has rank 1. It follows that the pullback $i^*\mathcal{L}'$, where $i: C \hookrightarrow S$ is the inclusion, and which we will simply denote by $\mathcal{L}'$ by ease of notation, is a coherent sheaf on C of constant rank one. Therefore, by Lemma 1.25, $\mathcal{L}'$ is a line bundle on C. Dualizing the above exact sequence, we obtain the following dual exact sequence of vector bundles:

$$0 \to (\mathcal{L}')^* \to \mathcal{E}^* \to V^* \otimes \mathcal{O}_S \to \mathcal{L}: \ = \mathcal{E}xt^1(\mathcal{L}', \mathcal{O}_S) \to 0$$

observing that $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_S) = 0$, since $\mathcal{E}$ and $\mathcal{O}_S$ are both vector bundles on S.

It should be noted that $\mathcal{L}'$, being supported exclusively on the proper subvariety $C \subset S$, is a torsion sheaf on S, hence its dual $\mathcal{L}'$ vanishes, as estabilished in Lemma 1.24. Finally, we hence obtain the following exact sequence:

$$0 \to \mathcal{E}^* \to V^* \otimes \mathcal{O}_S \to \mathcal{L} \to 0.$$

Now observe that $\mathcal{L} \cong \mathcal{N}_{C/S} \otimes \mathcal{L}'^*$ (see proof of [GL, Proposition 1.1]), hence, $\mathcal{L}$ is a line bundle on C.

The above short exact sequence induces the long exact sequence in cohomology

$$0 \to H^0(\mathcal{E}^*) \to V^* \otimes H^0(\mathcal{O}_S) \xrightarrow{\psi} H^0(\mathcal{L}) \to H^1(\mathcal{E}^*) \to ...$$

Since $H^0(\mathcal{E}^*) = 0$ by Lemma 2.7, ψ is necessarily injective. Moreover, because $H^0(\mathcal{O}_S) \cong \mathbb{C}$, we identify $V^* \otimes H^0(\mathcal{O}_S) \cong V^*$.

Setting

$$W := \mathrm{Im}(\psi : V^* \to H^0(\mathcal{L})),$$

then gives $W \cong V^*$ and hence $\dim(W) = r$. At each $p \in C$, the fibre map $\psi_p : V^* \to \mathcal{L}_p$ is nonzero, so some element of $W = \mathrm{Im}(\psi)$ remains nonvanishing at p. Hence, W is base-point free on C. Notice that $(C, W, \mathcal{L})$ meets exactly the conditions laid out in Lemma 3.5-namely $C \subset S$ is smooth, $\mathcal{L}$ is a line bundle on C, $W \subseteq H^0(\mathcal{L})$ is an $r-$dimensional base-point-free linear series, and $\mathcal{E}^* = K_{C,W,\mathcal{L}}$. Since $\mathcal{E}$ is Ulrich, we have

$$H^0(\mathcal{E}(-1)) = 0, \quad H^2(\mathcal{E}(-2)) = 0 \quad \text{and} \quad \chi(\mathcal{E}(-1)) = \chi(\mathcal{E}(-2)) = 0.$$

An application of Lemma 3.5 at once establishes that $H^1(\mathcal{L}(K_S+H)) = 0$, that the map $W \otimes \mathcal{O}_S \to H^0(\mathcal{L}(K_S + H)) = 0$ is injective and that the vanishing of those two Euler characteristics is equivalent to the two numerical identities

$$\deg(C) = \tfrac{r}{2}(K_S + 3H) \cdot H \quad \text{and} \quad \deg(\mathcal{L}) = r\chi(\mathcal{O}_S) + g - 1 - C \cdot K_S - rH^2.$$

Thus the triple $(C, W, \mathcal{L})$ clearly fulfills conditions (i),(ii) and (iii), completing the first part of the proof.

Conversely, suppose one is given a smooth curve $C \subset S$, a line bundle $\mathcal{L}$ on C, and an $r-$dimensional, base-point-free subspace $W \subseteq H^0(\mathcal{L})$ satisfying conditions (i), (ii) and (iii). Set

$$\mathcal{E} := K^*_{C,W,\mathcal{L}}$$

so that by Remark 3.4(ii) the kernel bundle $K_{C,W,\mathcal{L}}$ is locally free of rank r and hence $\mathcal{E}$ is indeed a vector bundle of rank r on S. Moreover, $\mathcal{E}^* = K_{C,W,\mathcal{L}}$.

Then, by Lemma 3.5(i), the vanishing $H^1(\mathcal{L}(K_S + H)) = 0$ is equivalent to $H^0(\mathcal{E}(-1)) = 0$ and Lemma 3.5(ii) shows that the injectivity of

$$W \otimes H^0(\mathcal{O}_S(K_S + 2H)) \to H^0(\mathcal{L}(K_S + 2H))$$

is exactly the same as $H^2(\mathcal{E}(-2)) = 0$.

Finally, Lemma 3.5(iii) and Lemma 3.2 identifies the two numerical equalities on $\deg(C)$ and $\deg(\mathcal{L})$ with the Chern-class conditions

$$c_1(\mathcal{E}) \cdot H = \frac{r}{2}(K_S + 3H) \cdot H, \quad c_2(\mathcal{E}) = r\chi(\mathcal{O}_S) + \frac{1}{2}c_1(S)c_1(\mathcal{E}) + \frac{1}{2}c_1(\mathcal{E})^2 - rH^2.$$

But these four facts are exactly the hypotheses of Casnati's theorem characterizing Ulrich bundles on smooth projective surfaces. Hence, $\mathcal{E}$ is an Ulrich bundle of rank r on S, completing the proof.

□

Remark 3.8. Let $S \subset \mathbb{P}^N$ be a smooth projective surface of degree $d \geq 2$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$. In the setting of Theorem 3.7, let $\mathcal{E}$ be be an Ulrich bundle on S with associated triple $(C, W, \mathcal{L})$, or conversely let $(C, W, \mathcal{L})$ be any triple satisfying (i), (ii) and (iii) with associated Ulrich bundle $\mathcal{E}$. Let g be the genus of C. Then

(i) $g \leq \frac{r^2}{8d} K_S \cdot H(K_S \cdot H + 6d) + \frac{1}{2} C \cdot K_S + \frac{9r^2 d}{8} + 1$;

(ii) $g \geq \frac{r+1}{2} C \cdot K_S + r^2(d - \chi(\mathcal{O}_S)) + 1$.

Proof. (i) From the Hodge index theorem [Har, Theorem V.1.9], one has

$$C^2 H^2 \leq (C \cdot H)^2.$$

Observe that $C^2 = 2g - 2 - C \cdot K_S$ by adjunction and $C \cdot H = \frac{r}{2}(K_S + 3H) \cdot H$ by condition (iii) of Theorem 3.7. Hence

$$(2g - 2 - C \cdot K_S)d \leq \frac{r^2}{4}(K_S \cdot H + 3d)^2,$$

and rearranging the inequality yields the bound in (i).

(ii) Observe that $\mathcal{E}$ is an Ulrich vector bundle, hence, it is semi-stable. Therefore, the Bogomolov's inequality (Lemma 2.11) applies:

$$2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2 \geq 0. \tag{3.3}$$

Recalling that $c_1(\mathcal{E}) = C$ and that $c_2(\mathcal{E}) = \deg \mathcal{L} = r\chi(\mathcal{O}_S) + \frac{1}{2}C^2 - \frac{1}{2}C \cdot K_S - rd$ (by Remark 3.4(iii) and (iv) and condition (iii) of Theorem 3.7 together with the adjunction), (3.3) yields

$$2r^2\chi(\mathcal{O}_S) + C^2 - rC \cdot K_S - 2r^2 d \geq 0.$$

Again by adjunction one has $C^2 = 2g - 2 - C \cdot K_S$, so that

$$2r^2\chi(\mathcal{O}_S) + 2g - 2 - (r+1)C \cdot K_S - 2r^2 d \geq 0,$$

that is, rearranging, (ii).

□

Remark 3.9. Once again, within the framework of Theorem 3.7, let $\mathcal{E}$ be be an Ulrich bundle on S with associated triple $(C, W, \mathcal{L})$, or conversely let $(C, W, \mathcal{L})$ be any triple satisfying (i), (ii) and (iii) with associated Ulrich bundle $\mathcal{E}$. Then the multiplication map

$$\varphi : W \otimes H^0(S, \mathcal{O}_S(K_S + 2H)) \to H^0(\mathcal{L}(K_S + 2H))$$

is not merely injective but in fact an isomorphism.

Proof. Observe that if $\mathcal{E}$ is an Ulrich bundle on S with associated triple $(C, W, \mathcal{L})$, then $\mathcal{E}^* \cong K_{C,W,\mathcal{L}}$; conversely, given any triple $(C, W, \mathcal{L})$ meeting conditions (i), (ii) and (iii), then the Ulrich bundle $\mathcal{E}$ produced by that data is such that $\mathcal{E}^* \cong K_{C,W,\mathcal{L}}$. Note that Serre's duality identifies $\chi(\mathcal{E}^*(K_S+2H)) = \chi(\mathcal{E}(-2H))$, hence, in either case, $\chi(\mathcal{E}^*(K_S + 2H)) = 0$ since $\mathcal{E}$ is an Ulrich bundle. Moreover, by Remark 3.4(i), we have the following exact sequence

$$0 \to \mathcal{E}^* \to W \otimes \mathcal{O}_S \to \mathcal{L} \to 0,$$

which, after tensoring with $\mathcal{O}_S(K_S + 2H)$, yields

$$0 \to \mathcal{E}^*(K_S + 2H) \to W \otimes \mathcal{O}_S(K_S + 2H) \to \mathcal{L}(K_S + 2H) \to 0.$$

Therefore, by additivity of the Euler characteristic in the exact sequence, one has

$$r\chi(\mathcal{O}_S(K_S + 2H)) = \chi(\mathcal{L}(K_S + 2H)),$$

where $r = \dim W$. Now, observe that $\chi(\mathcal{O}_S(K_S+2H)) = h^0(\mathcal{O}_S(K_S+2H))$, by Kodaira's vanishing theorem. In addition, from the following short exact sequence

$$0 \to \mathcal{L}(K_S + H) \to \mathcal{L}(K_S + 2H) \to \mathcal{L}(K_S + 2H)_{|H_C} \to 0$$

and since $H^1(\mathcal{L}(K_S+H)) = 0$ by condition (i) of Theorem 3.7 and $H^1(\mathcal{L}(K_S+2H)_{|H_C})$ vanishes, as $\mathcal{L}(K_S + 2H)_{|H_C}$ is supported on finitely many points, one has $H^1(\mathcal{L}(K_S + 2H)) = 0$. Hence, $\chi(\mathcal{L}(K_S + 2H)) = h^0(\mathcal{L}(K_S + 2H))$.

Therefore, $rh^0(\mathcal{O}_S(K_S + 2H)) = h^0(\mathcal{L}(K_S + 2H))$, which exactly means that the multiplication map φ is not only injective (by assumption) but also an isomorphism.

□

Remark 3.10. Recall that the multiplication map φ appearing in point (ii) of Theorem 3.7 is defined by restriction to C; in other words, $\varphi = \psi \circ \rho$ where

$$\psi : W \otimes H^0(\mathcal{O}_S(K_S + 2H)) \to W \otimes H^0(\mathcal{O}_S(K_S + 2H)_{|C})$$

and ρ is the natural map $W \otimes H^0(\mathcal{O}_S(K_S + 2H)_{|C}) \to H^0(\mathcal{L}(K_S + H))$.

Under the hypotheses of Theorem 3.7 (in particular $d \geq 2$) and assuming $r \geq 2$, the map ψ is always injective.

Proof. Since $\ker \psi \cong W \otimes H^0\big(\mathcal{O}_S(K_S + 2H - C)\big)$, it suffices to show

$$\big(K_S + 2H - C\big) \cdot H < 0,$$

so that $\mathcal{O}_S(K_S + 2H - C)$ has no nonzero global sections.

Using $\deg(C) = C \cdot H = \frac{r}{2}\,(K_S + 3H) \cdot H$, we compute

$$\begin{aligned}\big(K_S + 2H - C\big) \cdot H &= K_S \cdot H + 2\,H^2 - \deg(C)\\ &= K_S \cdot H + 2d - \frac{r}{2}\big(K_S \cdot H + 3d\big)\\ &= -\tfrac{1}{2}\Big((r-2)\,K_S \cdot H + (3r-4)\,d\Big)\\ &= -\tfrac{1}{2}\Big((r-2)(K_S \cdot H + 2d) + r\,d\Big)\\ &= -\tfrac{1}{2}\Big((r-2)\big(2g(H) - 2 + d\big) + r\,d\Big),\end{aligned}$$

where we used the adjunction formula $K_S \cdot H + H^2 = 2g(H) - 2$. Since $r \geq 2$ and $d \geq 2$, one checks

$$(r-2)\big(2g(H) - 2 + d\big) + r\,d \;\geq\; r\,d > 0,$$

hence

$$\big(K_S + 2H - C\big) \cdot H < 0.$$

Thus, ψ is injective.

□

Remark 3.11. Let $S \subset \mathbb{P}^N$ be a smooth projective surface of degree $d \geq 2$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$, and assume that the canonical divisor satisfies $K_S = mH$ for some integer m. In the setting of Theorem 3.7, there exists a rank r Ulrich bundle on S if and only if there exists a triple $(C, W, \mathcal{L})$ satisfying conditions (i), (ii) and (iii) of Theorem 3.7 with $W = H^0(\mathcal{L})$.

A surface satisfying such a condition is called *subcanonical*. Examples include smooth surfaces of degree d in $\mathbb{P}^3$, for which $K_S = (d-4)H$, and $K3$ surfaces, where $K_S = 0$.

Proof. Suppose $\mathcal{E}$ is a rank r Ulrich vector bundle on S, and let us consider the associated triple $(C, W, \mathcal{L})$ given by Theorem 3.7. From the construction in the proof of Theorem 3.7, we obtain a short exact sequence

$$0 \to \mathcal{E}^* \to W \otimes \mathcal{O}_S \to \mathcal{L} \to 0$$

and, passing to cohomology, we get the long exact sequence

$$0 \to H^0(\mathcal{E}^*) \to W \otimes H^0(\mathcal{O}_S) \xrightarrow{\psi} H^0(\mathcal{L}) \to H^1(\mathcal{E}^*) \to \dots$$

Since $\mathcal{E}$ is Ulrich and $d \geq 2$, we know, by Lemma 2.7, $H^0(\mathcal{E}^*) = 0$ and, by Lemma 2.5(iii), we also have $h^1(\mathcal{E}^*) = h^1(\mathcal{E}(K_S)) = h^1(\mathcal{E}(m)) = 0$. Hence, the map ψ is an isomorphism, so that $W = H^0(\mathcal{L})$.

□

3.3.1 Ulrich bundles of rank 1 and rank 2 on surfaces

We now examine how Theorem 3.7 applies in the low-rank cases $r = 1$ and $r = 2$. We will first analyze these low-rank situations in general, and later return to them in the context of surfaces in $\mathbb{P}^3$.

Corollary 3.12 (Ulrich line bundles on surfaces). *Let $S \subset \mathbb{P}^N$ be a smooth projective surface of degree $d \geq 2$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$. Then, there exists an Ulrich line bundle $\mathcal{E}$ on S if and only if there exists a smooth (possibly disconnected) curve $C \subset S$ of genus g such that:*

(i) $H^1(\mathcal{O}_C(K_S + H)) = 0$;

(ii) $H^0(\mathcal{O}_S(K_S + 2H - C)) = 0$

(iii) $\deg(C) = \frac{1}{2}(K_S + 3H) \cdot H$ *and*
$g = C \cdot K_S + 1 + d - \chi(\mathcal{O}_S)$.

(see also [CFK, Proposition 2.4])

Proof. We first observe that if $\mathcal{L} \cong \mathcal{O}_C$ and $W \subseteq H^0(\mathcal{O}_C)$ is a base-point free linear series of dimension 1 then the multiplication map supplied by Theorem 3.7 factors as

$$\varphi : W \otimes H^0(\mathcal{O}_S(K_S+2H)) \xrightarrow{\psi} W \otimes H^0(\mathcal{O}_C(K_S+2H)) \xrightarrow{\rho} H^0(\mathcal{O}_C(K_S+2H)).$$

and we have that

$$\varphi \text{ is injective} \iff \psi \text{ is injective} \tag{3.4}$$

Indeed, ρ acts by multiplication with the generator σ of $W \subseteq H^0(\mathcal{O}_C)$.

On each connected component C_i the generator σ restricts to a constant c_i. The base-point free condition means that σ vanishes nowhere on C: if even one c_i were zero, σ would vanish at every point of that component, producing a base point and contradicting the hypothesis. Hence $c_i \neq 0$ for every i.

The map ρ therefore acts on each C_i simply by multiplying sections by the nonzero scalar c_i; such multiplication is injective – and in fact bijective – so that ρ is injective on the whole curve C. Because ρ is already injective, the injectivity of the composite φ is equivalent to the injectivity of ψ.

Now, let $\mathcal{E}$ be an Ulrich line bundle on S, and let $(C, W, \mathcal{L})$ be the triple associated to it by Theorem 3.7. Then, $W \subseteq H^0(\mathcal{L})$ is a base-point free linear series of dimension 1, hence, $\mathcal{L} \cong \mathcal{O}_C$. By condition (i) of Theorem 3.7, this directy implies that $H^1(\mathcal{O}_C(K_S + H)) = 0$. Moreover, since $\deg(\mathcal{L}) = 0$, condition (iii) follows from condition (iii) of Theorem 3.7. Moreover, the vanishing in (ii) follows directly from the injectivity of φ guaranteed by point (ii) of Theorem 3.7.

On the other hand, let $C \subset S$ be a smooth curve satisfying conditions (i)-(iii). Then the triple $(C, W, \mathcal{O}_C)$, where $W = \langle \sigma \rangle$ with $0 \neq \sigma \in H^0(\mathcal{O}_C)$, meets immediately conditions (i) and (iii) of Theorem 3.7, whereas condition (ii), together with the equivalence (3.4), provides the injectivity of φ, required in point (ii) of the same theorem, thereby guaranteeing the existence of an Ulrich line bundle on S.

□

Corollary 3.13 (Ulrich bundles of rank 2 on surfaces). *Let $S \subset \mathbb{P}^N$ be a smooth projective surface of degree $d \geq 2$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$. Then, there exists an Ulrich vector bundle $\mathcal{E}$ of rank 2 on S if and only if there exists a smooth (possibly disconnected) curve $C \subset S$ of genus g together with a pair $(W, \mathcal{L})$, where $\mathcal{L}$ is a line bundle on C and $W \subseteq H^0(\mathcal{L})$ is a 2−dimensional base-point free linear series, such that:*

(i) $H^1(C, \mathcal{L}(K_S + H)) = 0$;

(ii) the multiplication map

$$\varphi : W \otimes H^0(S, \mathcal{O}_S(K_S + 2H)) \to H^0(\mathcal{L}(K_S + 2H))$$

is injective;

(iii) $\deg(C) = (K_S + 3H) \cdot H$ *and*
$\deg(\mathcal{L}) = 2\chi(\mathcal{O}_S) + g - 1 - C \cdot K_S - 2d.$

Proof. The corollary is an immediate consequence of Theorem 3.7 in the case $r = 2$. □

Remark 3.14 ([Bea2, Remark 5.1]). Let $S \subset \mathbb{P}^N$ be a smooth projective surface. A natural strategy for constructing Ulrich bundles on S is to look for bundles $\mathcal{E}$ such that $\mathcal{E}(-1)$ and $\mathcal{E}(-2)$ are Serre dual, that is,

$$\mathcal{E}(-2)^* \otimes \omega_S \cong \mathcal{E}(-1). \tag{3.5}$$

In this way, the vanishing of the cohomology of $\mathcal{E}(-1)$ automatically ensures the vanishing for $\mathcal{E}(-2)$ as well. For bundles of rank 2, this Serre duality condition is satisfied when the determinant of $\mathcal{E}$ equals $K_S + 3H$. In fact, under this condition we have the isomorphism $\mathcal{E}^* = \mathcal{E}(-K_S - 3H)$ which yields the identification (3.5).

This motivates the following definition of a special Ulrich vector bundle.

Definition 3.15. A rank 2 Ulrich bundle $\mathcal{E}$ on a smooth projective variety $X \subseteq \mathbb{P}^N$ of dimension n is *special* if $\det(\mathcal{E}) = K_X + (n+1)H$.

The next result, which appears as Proposition 6.2 in [ESW], provides a precise description of special Ulrich bundles on a smooth surface. We revisit both its statement and proof from the point of view of Theorem 3.7.

Lemma 3.16 (Special Ulrich bundles on surfaces, [ESW, Proposition 6.2]). *Let $S \subset \mathbb{P}^N$ be a smooth projective surface of degree $d \geq 2$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$. Then, there exists a special Ulrich bundle $\mathcal{E}$ on S if and only if there exists a smooth curve $C \subset S$ of class $K_S + 3H$ together with a pair $(W, \mathcal{L})$, where $\mathcal{L}$ is a line bundle on C and $W \subseteq H^0(\mathcal{L})$ is a $2-$dimensional base-point free linear series, such that:*

(i) $H^1(C, \mathcal{L}(K_S + H)) = 0$;

(ii) $\deg(\mathcal{L}) = \frac{1}{2} H \cdot (5H + 3K_S) + 2\chi(\mathcal{O}_S)$.

Proof. Observe that if $C \subset S$ is a smooth curve of class $K_S + 3H$, then the condition

$$\deg(\mathcal{L}) = 2\chi(\mathcal{O}_S) + \frac{1}{2} C \cdot (C - K_S) - 2H^2 \tag{3.6}$$

is equivalent to (ii). Indeed, one computes

$$\frac{1}{2} C \cdot (C - K_S) - 2H^2 = \frac{1}{2}(K_S + 3H) \cdot 3H - 2H^2 = \frac{1}{2} H \cdot (5H + 3K_S).$$

and putting these together yields (3.6).

Now let $\mathcal{E}$ be a special Ulrich vector bundle on S and let $(C, W, \mathcal{L})$ be the triple associated to $\mathcal{E}$ by Corollary 3.13. By that construction, we already know $H^1(\mathcal{L}(K_S + H)) = 0$, so condition (i) is immediate. Moreover, by the proof of Theorem 3.7, we know that $\det \mathcal{E} = c_1(\mathcal{E}) = C$ and, since $\mathcal{E}$ is assumed special, we have that C lies in the linear system $|K_S + 3H|$. Finally, point (iii) of Corollary 3.13 together with adjunction gives (3.6), hence, (ii). Therefore, every special Ulrich bundle produces a curve $C \in |K_S + 3H|$ and a pair $(W, \mathcal{L})$ satisfying (i) and (ii).

On the other hand, suppose that there exists a smooth curve $C \subset S$ of class $K_S + 3H$ together with a pair $(W, \mathcal{L})$, where $\mathcal{L}$ is a line bundle on C and $W \subseteq H^0(\mathcal{L})$ is a $2-$dimensional base-point free linear series, satisfying (i) and (ii). Then, hypothesis (i) directly implies condition (i) of Corollary 3.13. Moreover, since $C \in |K_S + 3H|$, identity (3.6) together with the adjunction formula yields exactly the numerical condition on $\deg(\mathcal{L})$ of Corollary 3.13. Hence, condition (iii) of Corollary 3.13 is satisfied.

We now prove that the multiplication map

$$\varphi : W \otimes H^0(\mathcal{O}_S(K_S + 2H)) \to H^0(\mathcal{L}(K_S + 2H))$$

is injective. Recall that φ is defined by restriction to C, that is, $\varphi = \psi \circ \rho$ where

$$\psi : W \otimes H^0(\mathcal{O}_S(K_S + 2H)) \to W \otimes H^0(\mathcal{O}_S(K_S + 2H)_{|C}) \text{ and}$$
$$\rho : W \otimes H^0(\mathcal{O}_S(K_S + 2H)_{|C}) \to H^0(\mathcal{L}(K_S + H)).$$

Observe that, since $r := \dim W = 2$, the map ψ is injective by Remark 3.10, so it suffices to show the injectivity of ρ in order to obtain the injectivity of φ. Let D be a divisor on C such that $\mathcal{L} = \mathcal{O}_C(D)$. Using the base-point free pencil trick (Lemma 1.27), one has that $\ker \rho = H^0(O_C(K_S + 2H - D))$. Recalling that $K_C = (K_S + C)_{|C} = (2K_S + 3H)_{|C}$ and using Serre's duality, one has

$$h^0(\mathcal{O}_C(K_S + 2H - D)) = h^1(\mathcal{O}_C(2K_S + 3H - K_S - 2H + D)) =$$
$$= h^1(\mathcal{O}_C(K_S + H + D)) = h^1(\mathcal{L}(K_S + H)).$$

Hence, $\ker \rho = 0$ by (i), i.e., ρ is injective. It follows that the triple $(C, W, \mathcal{L})$ satisfies conditions (i), (ii) and (iii) of Corollary 3.13. Hence Corollary 3.13 produces a rank 2 Ulrich vector bundle $\mathcal{E}$ on S whose determinant is precisely C. Since C was chosen in $|K_S + 3H|$, one obtains $\det(\mathcal{E}) = K_S + 3H$. In particular, $\mathcal{E}$ is special.

□

3.4 When is the curve in Theorem 3.7 connected?

We are interested in understanding whether the curve C from Theorem 3.7 is connected or, equivalently, irreducible. When $\mathcal{E}$ is an Ulrich bundle with $c_1(\mathcal{E})^2 > 0$, the answer is straightforward: the curve is connected. The interesting and more subtle situation arises when $c_1(\mathcal{E})^2 = 0$, a case which has been studied in detail by López and Muñoz in [LM], and whose results we now recall.

Theorem 3.17 (Lopez-Muñoz, [LM, Theorem 1]). *Let $S \subseteq \mathbb{P}^N$ be a smooth irreducible complex surface, and let $\mathcal{E}$ be a rank r Ulrich vector bundle on S. Then $\mathcal{E}$ is not big if and only if the triple $(S, \mathcal{O}_S(1), \mathcal{E})$ is one of the following:*

(i) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}^{\oplus r})$;

(ii) $(\mathbb{P}\mathcal{F}, \mathcal{O}_{\mathbb{P}\mathcal{F}}(1), \pi^*(\mathcal{G} \otimes \det \mathcal{F}))$*, where* $\mathcal{F}$ *is a rank 2 very ample vector bundle over a smooth curve* B *and* $\mathcal{G}$ *is a rank* r *vector bundle on* B *such that* $H^q(\mathcal{G}) = 0$ *for* $q \geq 0$.

Proof. See [LM, *Proof of Theorem 1*]. □

The following corollary does not appear explicitly in [LM], but it is a straightforward consequence of their results. We present it here in this form, as it will be useful in what follows.

Corollary 3.18 (Lopez-Muñoz). *Let* $S \subseteq \mathbb{P}^N$ *be a smooth projective surface, and let* H *be a very ample line bundle on* S*. Suppose that* $\mathcal{E}$ *is an Ulrich bundle with respect to* (S, H)*. Then the following conditions are equivalent:*

(i) $(S, \mathcal{O}_S(1), \mathcal{E})$ *is isomorphic either to* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}^{\oplus r})$ *or to* $(\mathbb{P}\mathcal{F}, \mathcal{O}_{\mathbb{P}\mathcal{F}}(1), \pi^*(\mathcal{G}(\det(\mathcal{F}))))$ *where* $\mathcal{F}$ *is a rank 2 very ample vector bundle over a smooth curve* B *and* $\mathcal{G}$ *is a rank* r *vector bundle on* B *such that* $H^q(\mathcal{G}) = 0$ *for* $q \geq 0$*;*

(ii) $c_1(\mathcal{E})^2 = 0$*;*

(iii) the bundle $\mathcal{E}$ *is not big.*

Proof. Assume (i) holds. If $(S, \mathcal{O}_S(1), \mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}^{\oplus r})$, then $c_1(\mathcal{E})^2 = 0$. Otherwise, set $\mathcal{M} := \mathcal{G} \otimes \det(\mathcal{F})$, so that $\mathcal{E} = \pi^*(\mathcal{M})$, and $\mathcal{L} := \det(\mathcal{M})$. Since $\pi : S \to B$ is a ruled surface on a curve, we have

$$c_1(\mathcal{E})^2 = (\det(\mathcal{E}))^2 = (\det(\pi^*\mathcal{M}))^2 = (\pi^*(\det \mathcal{M}))^2 = (\pi^*\mathcal{L})^2 = \pi^*(\mathcal{L}^2) = 0$$

because the self-intersection number of a line bundle on a curve is always zero. Hence, condition (ii) follows.

Now assume (ii) holds. According to Remark 1.43, if an Ulrich bundle $\mathcal{E}$ satisfies $c_1(\mathcal{E})^2 = 0$ then $\mathcal{E}$ cannot be big, thus condition (iii) follows.

Finally, if (iii) holds, then Theorem 3.17 asserts that $\mathcal{E}$ must be of the specific form described in (i).

Therefore, the three statements are equivalent. □

We proceed by presenting the following corollary of Theorem 3.7.

Corollary 3.19. *Let* $S = \mathbb{P}\mathcal{F} \subset \mathbb{P}^N$ *be a ruled surface of degree* $d \geq 2$ *over a smooth curve* B*, with* $H \in |\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)|$*. Then there exists a rank-r Ulrich vector bundle* $\mathcal{E}$ *on* S *with* $c_1(\mathcal{E})^2 = 0$ *if and only if* S *contains a smooth curve* C*, which is the disjoint union of*

$$t = r\,(g(B) - 1 + d)$$

fibers (lines) of S, together with a base point free r-dimensional subspace $W \subseteq H^0(\mathcal{O}_C)$ such that the multiplication map

$$W \otimes H^0(\mathcal{O}_S(K_S + 2H)) \to H^0(\mathcal{O}_C)$$

is injective.

For clarity, we isolate the following technical observations.

Lemma 3.20. *Let $S \cong \mathbb{P}\mathcal{F} \subset \mathbb{P}^N$ be a ruled surface of degree $d \geq 2$ over a smooth curve B, with $H \in |\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)|$. Let C be disjoint union of t fibers of S. Then*

(i) $r\chi(\mathcal{O}_S) + g(C) - 1 - C \cdot K_S - rH^2 = t - r(g(B) - 1 + d)$

(ii) Let $\mathcal{L}$ be a line bundle on C. Then $\mathcal{L} \cong \bigoplus_{i=1}^t \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some integers a_i.
In particular, $H^1(\mathcal{L}(K_S + H)) = 0$ if and only if $a_i \geq 0$ for every $i = 1, \ldots, t$

Proof. (i) By adjunction, one has

$$r\chi(\mathcal{O}_S) + g(C) - 1 - C \cdot K_S - rH^2 = r\chi(\mathcal{O}_S) + \frac{1}{2}(C^2 - C \cdot K_S) - rd.$$

Since each fiber F satisfies $F^2 = 0$ and $g(F) = 0$, adjunction gives $F \cdot K_S = -2$. It follows that $C \cdot K_S = -2t$. In addition, by [Har, V.2.4] and Riemann–Roch on curves (Theorem 1.1), one has $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_B) = 1 - g(B)$. Observing that $C^2 = 0$, yields

$$r\chi(\mathcal{O}_S) + g(C) - 1 - C \cdot K_S - rH^2 = r(1 - g(B)) + t - rd = t - r(g(B) - 1 + d)$$

(ii) The first part of the proof is immediate. For the second, observe that on each fiber F one has $(K_S + H) \cdot F = K_S \cdot F + H \cdot F = -1$, so that

$$\mathcal{L}(K_S + H)_{|F} \cong \mathcal{O}_{\mathbb{P}^1}(a_i - 1),$$

and hence

$$H^1(\mathcal{L}(K_S + H)) = \bigoplus_{i=1}^{t} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i - 1)).$$

Since, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(l)) = 0$ iff $l \geq -1$, we obtain

$$H^1(\mathcal{L}(K_S + H)) = 0 \iff a_i \geq 0 \quad \forall i.$$

□

We now return to the proof of Corollary 3.19.

Proof of Corollary 3.19. Suppose that $\mathcal{E}$ is an Ulrich bundle with $c_1(\mathcal{E})^2 = 0$ and let $(C, W, \mathcal{L})$ denote the triple arising from Theorem 3.7. Then, by Corollary 3.18, $\mathcal{E} \cong \pi^*\mathcal{G}(\det(\mathcal{F}))$ where $\mathcal{G}$ is a rank r vector bundle on B with vanishing cohomology. In the proof of Theorem 3.7, we saw that the curve C is a divisor in the $\det(\mathcal{E})$. Observe that one has the natural identification

$$\det(\mathcal{E}) = \det(\pi^*\mathcal{G}') = \pi^* \det(\mathcal{G}')$$

where we set $\mathcal{G}' = \mathcal{G}(\det(\mathcal{F}))$ and use the fact that pullback commutes with determinant. But $\det(\mathcal{G}')$ is a divisor on the base curve B and therefore corresponds to an effective divisor $D = P_1 + \ldots + P_t$ on B. Pulling back, it follows that

$$C \in |\pi^* D| = |\pi^* P_1 + \ldots + \pi^* P_t|.$$

Since each $\pi^* P_i$ is exactly the fiber $F_i \cong \mathbb{P}^1$ we conclude that C is linearly equivalent to the sum of these fibers:

$$C \sim F_1 + \ldots + F_t.$$

Finally, because C is smooth by Theorem 3.7, no two fibers F_i can coincide or meet nontrivially. Hence,

$$C = \bigsqcup_{i=1}^{t} F_i$$

is the disjoint union of t distinct copies of $\mathbb{P}^1$, which, specifically, are fibers of the ruling π.

In particular, observe that

$$t = \deg(c_1(\mathcal{G}')) = \deg(c_1(\mathcal{G} \otimes \det(\mathcal{F}))).$$

By Lemma 1.12 (iv),

$$\deg(c_1(\mathcal{G} \otimes \det(\mathcal{F}))) = \deg(c_1(\mathcal{G})) + r \deg(c_1(\mathcal{F})).$$

Since $H^i(\mathcal{G}) = 0$ for all i, Riemann–Roch on the curve B gives

$$0 = \chi(\mathcal{G}) = \deg(\mathcal{G}) + r(1 - g(B))$$

hence $\deg(\mathcal{G}) = r(g(B) - 1)$, that is, $\deg(c_1(\mathcal{G})) = r(g(B) - 1)$.

Grothendieck's relation in the Chow ring of $\mathbb{P}\mathcal{F}$ [Har, Appendix A.3] states

$$\sum_{j=0}^{2} (-1)^j \pi^* c_j(\mathcal{F}) H^{2-j} = 0$$

and we have $c_0(\mathcal{F}) = 1$ and $c_2(\mathcal{F}) = 0$, so that the above reduces to $\pi^* c_0(\mathcal{F})H^2 - \pi^* c_1(\mathcal{F})H = 0$, that is, $d = \pi^* c_1(\mathcal{F})H$. But $\pi^* c_1(\mathcal{F})$ is the sum of $\deg(c_1(\mathcal{F}))$ distinct fibers of π, and each fiber meets H at one point. Hence, $d = \deg(c_1(\mathcal{F}))$.

It follows that

$$t = \deg(c_1(\mathcal{G})) + r\deg(c_1(\mathcal{F})) = r(g(B) - 1) + rd = r(g(B) - 1 + d)$$

as required.

Next, we show that $\mathcal{L}$ is trivial. Since condition (iii) of Theorem 3.7 occurs, the preceding computation of t together with Lemma 3.20(i) immediately yield that

$$\deg(\mathcal{L}) = t - r(g(B) - 1 + d) = 0.$$

Now, $C = \bigsqcup_{i=1}^{t} F_i$ and $H^1(\mathcal{L}(K_S+H)) = 0$ from condition (i) of Theorem 3.7. Hence, according to Lemma 3.20(ii), $\mathcal{L} \cong \bigoplus_{i=1}^{t} \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $\sum_i a_i = \deg(\mathcal{L}) = 0$ and $a_i \geq 0$ for each i. Therefore, $a_i = 0$ for every i, hence $\mathcal{L} \cong \mathcal{O}_C$.

Finally, the validity of condition (ii) of Theorem 3.7 ensures that the multiplication map

$$W \otimes H^0(\mathcal{O}_S(K_S + 2H)) \to H^0(\mathcal{O}_C(K_S + 2H))$$

is injective. This concludes the first part of the proof, simply by observing that $(K_S + 2H).C = 0$ and, hence, $H^0(\mathcal{O}_C(K_S + 2H)) = H^0(\mathcal{O}_C)$.

Conversely, suppose C is the disjoint union of $t = r(g(B) - 1 + d)$ fibers of S, $\mathcal{L} = \mathcal{O}_C$ and $W \subseteq H^0(\mathcal{L})$ is a base–point–free subspace of dimension r such that the multiplication map $W \otimes H^0(\mathcal{O}_S(K_S + 2H)) \to H^0(\mathcal{O}_C)$ is injective. By Lemma 3.20(i), one has

$$r\chi(\mathcal{O}_S) + g(C) - 1 - C \cdot K_S - rH^2 = 0$$

and since $\deg(\mathcal{L}) = 0$, it follows that

$$\deg(\mathcal{L}) = r\chi(\mathcal{O}_S) + g(C) - 1 - C \cdot K_S - rH^2.$$

Lemma 3.20(ii) then gives $H^1(\mathcal{L}(K_S+H)) = 0$. Recalling that $K_S = -2H + \pi^*(K_B + \det \mathcal{F})$ and that $\deg(\det \mathcal{F}) = d$ by the first part of the proof, we compute

$$\begin{aligned} \tfrac{r}{2}(K_S + 3H) \cdot H = \tfrac{r}{2}(H^2 + \deg(K_B + \det \mathcal{F})) = \tfrac{r}{2}(d + 2g(B) - 2 + d) = \\ = r(g(B) - 1 + d) = t = \deg(C). \end{aligned}$$

Therefore, the triple $(C, W, \mathcal{L})$ satisfies conditions (i), (ii) and (iii) of Theorem 3.7, giving rise to an Ulrich bundle $\mathcal{E}$ on S such that $c_1(\mathcal{E})^2 = C^2 = 0$ and thus concluding the proof. □

Remark 3.21. We offer an alternative proof of the first assertion of the proposition by exploiting the fact that the curve C constructed in the theorem is precisely the $(r-1)$-st degeneracy locus of the map φ_V, namely $D_{r-1}(\varphi_V)$, where $V \subseteq H^0(\mathcal{E})$ is a r-dimensional general subspace. Our goal is to show that C is a disjoint union of fibers of the projection $\pi : \mathbb{P}\mathcal{F} \to B$. To this end, it suffices to prove that the rank of φ_V remains constant along each fiber of π. This will imply that if $x \in C$, $\mathrm{rk}(\varphi_V(x)) = r - 1$, so that the entire fiber $\pi^{-1}(\pi(x))$ is contained in C; conversely, if $x \notin C$, then $\mathrm{rk}(\varphi_V(x)) = r$ and hence the fiber $\pi^{-1}(\pi(x))$ meets C in no points.

To make this precise, fix a point $b \in B$ and choose an open neighbourhood $U \ni b$ over which the bundle $\mathcal{G}' = \mathcal{G}(\det(\mathcal{F}))$ is trivial.

Observe that $\mathcal{E}_{|\pi^{-1}(U)} \cong \mathcal{O}^{\oplus r}_{|\pi^{-1}(U)}$. Indeed, we have the following commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xhookrightarrow{\ j\ } & \mathbb{P}\mathcal{F} \\ {\scriptstyle \pi_1 := \pi|_{\pi^{-1}(U)}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ U & \xhookrightarrow{\ i\ } & B \end{array}$$

so that,

$$\mathcal{E}_{|\pi^{-1}(U)} \cong j^*\mathcal{E} \cong j^*(\pi^*\mathcal{G}') \cong \pi_1^*(i^*\mathcal{G}') \cong \pi_1^*(\mathcal{G}'_{|U}) \cong \pi_1^*(\mathcal{O}_U^{\oplus r}) \cong \mathcal{O}^{\oplus r}_{\pi^{-1}(U)}.$$

Now let F be any fiber of π. The restriction φ_V becomes a morphism

$$\mathcal{O}_F^{\oplus r} \to \mathcal{O}_F^{\oplus r}$$

which is equivalent to choosing r global sections of $\mathcal{O}_F^{\oplus r}$. But $F \cong \mathbb{P}^1$ and $H^0(\mathbb{P}^1) \cong \mathbb{C}$ so each section may be identified with an r-tuple of constants in $\mathbb{C}$. Hence, the matrix representing φ_V has constant entries, independent of the point $x \in F$. In particular, its rank is the same at every point of the fiber.

Remark 3.22. In the context of Corollary 3.19, assume $\deg(S) = d \geq 2$. One easily checks that

$$t = 1 \iff r = 1,\ d = 2,\ g(B) = 0,$$

that is, $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. On $\mathbb{P}^1 \times \mathbb{P}^1$ every Ulrich vector bundle splits as

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)^{\oplus \alpha} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)^{\oplus \beta},$$

for some integers $\alpha, \beta \geq 0$ [Lemma 2.16]. Since here $r = 1$, the only possibilities for $\mathcal{E}$ are

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0) \text{ or } \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1).$$

Moreover, this special case is covered by the Lopez–Muñoz theorem (Theorem 3.17), by taking

$$\mathcal{F} = O_{\mathbb{P}^1}(1)^{\oplus 2},\ H = \mathcal{O}_{\mathbb{P}\mathcal{F}}(1),\ \mathcal{G} = O_{\mathbb{P}^1}(-1).$$

The following corollary finally provides a characterization of the connectedness of the curve in Theorem 3.7.

Corollary 3.23. *Let $S \subset \mathbb{P}^N$ be a smooth projective surface of degree $d \geq 2$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$. Let $\mathcal{E}$ be the Ulrich bundle on S corresponding to the triple $(C, W, \mathcal{L})$, as in Theorem 3.7. Then C is irreducible if and only if one of the following cases arises:*

(i) $(S, \mathcal{O}_S(1), \mathcal{E}) \not\cong (\mathbb{P}\mathcal{F}, \mathcal{O}_{\mathbb{P}\mathcal{F}}(1), \pi^*(\mathcal{G}(\det(\mathcal{F})))$ *where $\mathcal{F}$ is a rank 2 very ample vector bundle over a smooth curve B of genus g, $\mathcal{G}$ is a rank r vector bundle on B such that $H^q(\mathcal{G}) = 0$ for $q \geq 0$;*

(ii) $(S, \mathcal{O}_S(1), \mathcal{E}) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0))$ *or*
$(S, \mathcal{O}_S(1), \mathcal{E}) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1))$.

Proof. First suppose that C is irreducible. If $C^2 > 0$, then by Corollary 3.18 we cannot have $(S, \mathcal{O}_S(1), \mathcal{E}) \cong (\mathbb{P}\mathcal{F}, \mathcal{O}_{\mathbb{P}\mathcal{F}}(1), \pi^*(\mathcal{G}(\det(\mathcal{F})))$, so we are in case (i). On the other hand, if $C^2 = 0$, Corollary 3.18 forces exactly the situation excluded from (i), and by Corollary 3.19 and previous numerical remark (Remark 3.22) the only way for an irreducible C to arise is when $r = 1$, $g(B) = 0$, $d = 2$, in which case $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E}$ is one of the two line bundles $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)$ or $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)$ described in Remark 3.22, that is, (ii).

Conversely, assume that C is reducible. Consider the following short exact sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

and the induced long exact sequence in cohomology

$$0 \to H^0(\mathcal{O}_S(-C)) \to H^0(\mathcal{O}_S) \xrightarrow{\varphi} H^0(\mathcal{O}_C) \to H^1(\mathcal{O}_S(-C)) \to \dots\ .$$

Since C is smooth and reducible, we have $h^0(\mathcal{O}_C) \geq 2$, whereas $H^0(\mathcal{O}_S) \cong \mathbb{C}$. It follows that the map φ fails to be surjective, and hence $H^1(\mathcal{O}_S(-C)) \neq 0$. On the other hand, from the proof of Theorem 3.7 we know that $\det(\mathcal{E}) = c_1(\mathcal{E}) = C$, so that, by Serre's duality, we have

$$h^1(\mathcal{O}_S(K_S + \det(\mathcal{E}))) = h^1(\mathcal{O}_S(-C)) \neq 0.$$

Moreover, by Lemma 2.5(i) the bundle $\mathcal{E}$ is globally generated, whence $\det(\mathcal{E})$ is nef (Corollary 1.37). An application of the Kawamata–Viehweg vanishing theorem (Lemma 1.42) then yields $c_1(\mathcal{E})^2 = 0$. Hence, Corollary 3.18 forces $(S, \mathcal{O}_S(1), \mathcal{E}) = (\mathbb{P}\mathcal{F}, \mathcal{O}_{\mathbb{P}\mathcal{F}}(1), \pi^*(\mathcal{G}(\det(\mathcal{F}))))$, so that (i) fails. Condition (ii) can also be excluded. Indeed, assuming its validity leads to a contradiction: by Corollary 3.19 and Remark 3.22, C would have to be a line and, hence, irreducible, contradicting our assumptions.

Therefore, if C is reducible neither (i) nor (ii) can hold, which completes the proof. □

Chapter 4

Ulrich bundles on surfaces in $\mathbb{P}^3$

Building on the correspondence established in Chapter 3, this chapter turns to the most classical theatre for Ulrich theory: smooth degree-d surfaces in $\mathbb{P}^3$. Our point of departure is Theorem 3.7, which expresses an Ulrich bundle in terms of a Lazarsfeld-Mukai construction on a curve $C \subset S$ together with a linear series on C. When we work inside $\mathbb{P}^3$, these data simplify significantly. In particular, the connectedness criterion examined at the end of Chapter 3 (Corollary 3.23) collapses to a single exceptional configuration: except for the ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$ carrying spinorial Ulrich bundles, the associated curve is automatically connected.

Section 4.1 collects these explicit consequences. We show how Theorem 3.7 yields streamlined existence criteria, we record the lone disconnected case, and we illustrate the general bounds for the genus g of curve C.

Section 4.2 first reviews the definition and basic properties of Noether–Lefschetz loci for degree-d surfaces before proving that the subset of surfaces supporting an Ulrich line bundle coincides with an entire irreducible component of the Noether-Lefschetz locus (Theorem 4.20).

The final Section 4.5 turns to the special case of quartic surfaces. Starting from the general bounds established earlier, we prove a sharper lower estimate for the genus of the curve C when it arises from a minimal-rank Ulrich bundle.

4.1 Immediate applications of the results from Chapter 3

Remark 4.1. Let $S \subset \mathbb{P}^3$ a smooth projective surface of degree d, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$, and let $C \subset S$ be a smooth curve. To make the discussion clearer, we highlight the following computations:

(i) $K_S = (d-4)H$;

(ii) $K_S \cdot H = d(d-4)$;

(iii) $\chi(\mathcal{O}_S) = \binom{d-1}{3} + 1$.

Proof. (i) $K_S = (K_{\mathbb{P}^3} + S)_{|S} = (-4H_{\mathbb{P}^3} + dH_{\mathbb{P}^3})_{|S} = (d-4)H$;
(ii) it follows direcly from (i);
(iii) Recall that $\chi(\mathcal{O}_{\mathbb{P}^3}) = 1$ and that $\chi(\mathcal{O}_{\mathbb{P}^3}(-d)) = -\binom{d-1}{3}$. Then (iii) follows directly from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0$$

and additivity of the Euler characteristic for short exact sequences. □

Theorem 4.2. *Let $S \subset \mathbb{P}^3$ be a smooth projective surface of degree $d \geq 2$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$. Then, there exists an Ulrich bundle $\mathcal{E}$ of rank r on S if and only if there exists a smooth (possibly disconnected) curve $C \subset S$ of genus g and a line bundle $\mathcal{L}$ on C, with $h^0(\mathcal{L}) = r$, such that:*

(i) $H^1(C, \mathcal{L}((d-3)H)) = 0$*;*

(ii) the multiplication map

$$\varphi : H^0(\mathcal{L}) \otimes H^0(S, \mathcal{O}_S((d-2)H)) \to H^0(\mathcal{L}((d-2)H))$$

is injective;

(iii) $\deg(C) = \frac{r}{2}d(d-1)$ *and*
$\deg(\mathcal{L}) = \frac{r}{2}(2\binom{d-1}{3} - d(d-2)(d-3) + 2) + g - 1$.

Proof. The Theorem is a direct consequence of Theorem 3.7, Remark 3.11 and computations of Remark 4.1. □

Remark 4.3. The inequalities from Remark 3.8 concerning the genus g of the curve C of Theorem 4.2 take the following form:

(i) $g \leq \frac{r}{8}d(d-4)((r+2)d + 2(r-1)) + \frac{9r^2d}{8} + 1$

(ii) $g \geq \frac{r(r+1)}{4}d(d-1)(d-4) - r^2(\binom{d-1}{3} - d + 1) + 1$.

Corollary 4.4. *Let $S \subset \mathbb{P}^3$ be a smooth projective surface of degree $d \geq 2$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$. Let $\mathcal{E}$ be the Ulrich bundle on S corresponding to the pair $(C, \mathcal{L})$, as in Theorem 4.2. Then C is disconnected if and only if*

$$
\begin{gathered}
(S, \mathcal{O}_S(1), \mathcal{E}) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)^{\oplus r}) \text{ or} \\
(S, \mathcal{O}_S(1), \mathcal{E}) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)^{\oplus r})
\end{gathered}
\tag{4.1}
$$

with $r \geq 2$.

In particular, in this case C is the disjoint union of exactly r lines.

Proof. If (4.1) holds, then both (i) and (ii) of Corollary 3.23 fail. Indeed, (i) fails choosing

$$\mathcal{F} = O_{\mathbb{P}^1}(1)^{\oplus 2},\ B = \mathbb{P}^1,\ H = \mathcal{O}_{\mathbb{P}\mathcal{F}}(1),\ \mathcal{G} = O_{\mathbb{P}^1}(-1).$$

and (ii) fails since $r \geq 2$. Hence, C is disconnected and, in particular, by Lemma 3.19, it is a disjoint union of $t = r(g(B) - 1 + d) = r$ lines.

On the other hand, by Corollary 3.23, if the curve C is disconnected, S is a ruled surface

$$(S, \mathcal{O}_S(1), \mathcal{E}) \cong (\mathbb{P}\mathcal{F}, \mathcal{O}_{\mathbb{P}\mathcal{F}}(1), \pi^*(\mathcal{G}(\det(\mathcal{F}))),$$

where the data $(\mathcal{F}, B, \mathcal{G})$ are exactly those specified in Corollary 3.23(i), and, in particular, the triple $(S, \mathcal{O}_S(1), \mathcal{E})$ is not isomorphic to either of the spinorial configurations

$$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)), \quad (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)).$$

Comparing the two canonical expressions

$$K_S = (d-4)H \text{ and } K_S = -2H + \pi^*(K_B + \det(\mathcal{F})),$$

one has

$$(d-2)H = \deg(K_B + \det(\mathcal{F}))f,$$

where f is a generic fiber of S. Intersecting first with a fiber f then gives

$$(d-2)H \cdot f = \deg(K_B + \det(\mathcal{F}))f^2$$

and, recalling that $H \cdot f = 1$ and $f^2 = 0$, one has $d = 2$. Hence,
$\deg(K_B + \det(\mathcal{F})) = 0$, that is, $2g(B) - 2 + d = 0$, which implies $g(B) = 0$.

Thus, $B = \mathbb{P}^1$ and $S \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Since $H^q(B, \mathcal{G}) = 0$ for all $q \geq 0$ and every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles, we have that $\mathcal{G} \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(-1)$. Moreover $\deg(\det \mathcal{F}) = -\deg(K_B) = 2$, hence $\det \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1}(2)$. Tensoring gives

$$\mathcal{G} \otimes \det \mathcal{F} \ \cong\ \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}.$$

Hence, $\mathcal{E}$ coincides with the pull-back $\pi^*(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r})$, that is,

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)^{\oplus r} \text{ or } \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)^{\oplus r}.$$

and, since the single-spinor cases have been excluded, we must have $r \geq 2$.

This exactly gives the configuration in (4.1), completing the proof.

□

4.2 Noether-Lefschetz loci

Definition 4.5. The parameter space of curves of degree n and genus g in $\mathbb{P}^3$ is denoted $H_{n,g}$ and it is called the *Hilbert scheme* of curves of degree n and genus g.

One can verify that $H_{n,g}$ carries the natural structure of a projective variety (see [Har, Remark 9.8.1 and Ch. IV, §6]).

Definition 4.6. Let $d \geq 1$ be an integer.

(i) We denote by $\mathbb{P}^N = \mathbb{P}^{\binom{d+3}{3}-1}$ the projective space whose points correspond to surfaces of degree d in $\mathbb{P}^3$.

(ii) $\mathcal{U}_d \subseteq \mathbb{P}^N$ is defined as the open subset consisting of points corresponding to smooth surfaces.

Theorem 4.7 (**Noether-Lefschetz**)**.** *Let $S \subset \mathbb{P}^3$ be a very general surface of degree $d \geq 4$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$. Then*

$$Pic(S) = \mathbb{Z}H.$$

Proof. The first rigorous proof was given by Lefschetz in [Lef]; a modern, scheme-theoretic proof was later given by Grothendieck in [GR]. For yet another presentation, see [GH]. □

Definition 4.8 (**Noether-Lefschetz locus**)**.** We define the *Noether-Lefschetz locus* as

$$NL(d) = \{S \in \mathcal{U}_d \ : \ \operatorname{Pic}(S) \ncong \mathbb{Z}H\}.$$

Remark 4.9. Recall that a property holds for a very general point of a projective variety X when it is satisfied outside a countable union of proper closed subvarieties of X.

In particular, the Noether–Lefschetz Theorem (Theorem 4.7) implies that the exceptional set of degree-d surfaces whose Picard group is larger than $\mathbb{Z}H$, namely the locus $NL(d)$, is precisely such a countable union inside $\mathcal{U}_d$, that is,

$$NL(d) = \bigcup_{i\in\mathbb{Z}} W_i \subsetneq \mathcal{U}_d, \quad \text{with } W_i \subset \mathcal{U}_d \text{ proper closed subvariety.}$$

Lemma 4.10. *Let W be an irreducible component of $NL(d)$. Then one has the inequalities*

$$d - 3 \leq \operatorname{codim}_{\mathcal{U}_d} W \leq p_g(d) = \binom{d-1}{3}.$$

Proof. See [CGGH] or [Gre]. □

Definition 4.11. An irreducible component W of $NL(d)$ is called *general* if it has the maximal possible codimension in $\mathcal{U}_d$, namely

$$\operatorname{codim}_{\mathcal{U}_d} W = \binom{d-1}{3}.$$

Lemma 4.12. *Let W be a component of $H_{n,g}$ and consider the incidence correspondence*

$$\begin{array}{ccc} & \{(S, C) \ : \ C \subseteq S\} \subseteq \mathbb{P}^N \times W & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{P}^N & & W \end{array}$$

Set $W(d) := \operatorname{Im} \pi_1$ and let C be a general point of W. Assume that

(i) *C is smooth and irreducible;*

(ii) *the ideal sheaf $\mathcal{J}_C$ is $(d-1)$-regular;*

(iii) $H^1(\mathcal{J}_C(d-4)) = 0$.

Then $W(d)$ is a component of the Noether–Lefschetz locus $NL(d)$ and

$$\operatorname{codim}_{\mathcal{U}_d} W(d) = h^0(\mathcal{O}_C(d-4)) - \dim W + 4 \deg C.$$

Proof. See [CL, Lemma 1.2]. □

4.3 Arithmetically Cohen-Macaulay curves

Definition 4.13. A smooth projective curve $C \subseteq \mathbb{P}^N$ is said *arithmetically Cohen-Macaulay (ACM)* if it satisfies

$$H^1(\mathcal{I}_{C/\mathbb{P}^N}(j)) = 0 \text{ for all } j \in \mathbb{Z}.$$

Remark 4.14. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$, embedded by the linear system $|H|$, where $H \in |\mathcal{O}_S(1)|$.

Let $C \subset S$ be a smooth curve of genus g as in Corollary 3.12, that is,

(i) $H^1(\mathcal{O}_C((d-3)H)) = 0$;

(ii) $H^0(\mathcal{O}_S((d-2)H - C)) = 0$

(iii) $\deg(C) = \frac{1}{2}d(d-1)$ and
$g = \frac{1}{6}(d-2)(d-3)(2d+1)$.

Then:

(i) C is irreducible;

(ii) $\mathcal{O}_S(C)$ is an Ulrich line bundle on S;

(iii) C is ACM.

Proof. Item (i) follows directly from Corollary 4.4, simply by noting that $d \geq 4$, while (ii) is an immediate consequence of the constructions of Theorem 3.7 and Corollary 3.12.

To prove (iii), fix an integer $j \in \mathbb{Z}$ and consider the short exact sequence

$$0 \to \mathcal{I}_{S/\mathbb{P}^3}(j) \to \mathcal{I}_{C/\mathbb{P}^3}(j) \to \mathcal{I}_{C/S}(j) \to 0. \tag{4.2}$$

Observe that

$$H^1(\mathcal{I}_{S/\mathbb{P}^3}(j)) = H^1(\mathcal{O}_{\mathbb{P}^3}(-d+j)) = 0 \tag{4.3}$$

since $H^1(\mathcal{O}_{\mathbb{P}^3}(\ell)) = 0$ for all integers ℓ and $\mathcal{I}_{S/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-d)$.

Moreover, from Lemma 2.5(iii) and point (ii), we have that

$$h^1(\mathcal{O}_S(C)(\ell)) = 0 \text{ for all } \ell \in \mathbb{Z}.$$

It follows that,

$$0 = h^1(\mathcal{O}_S(C)(\ell)) = h^1(\mathcal{O}_S((d-4-\ell)H - C)) \text{ for all } \ell \in \mathbb{Z},$$

that is, $H^1(\mathcal{O}_S(aH - C)) = 0$ for all $a \in \mathbb{Z}$, thereby implying that

$$H^1(\mathcal{I}_{C/S}(j)) = h^1(\mathcal{O}_S(jH - C)) = 0. \tag{4.4}$$

From the exact sequence (4.2) and the vanishings in (4.3) and (4.4), we conclude that

$$H^1(\mathcal{I}_{C/\mathbb{P}^3}(j)) = 0 \text{ for all } j \in \mathbb{Z},$$

that is, C is ACM.

□

Lemma 4.15. *Let $C \subset \mathbb{P}^3$ be a smooth curve. Then C admits a minimal free resolution*

$$0 \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^3}(-m_i) \to \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^3}(-d_j) \to \mathcal{I}_{C/\mathbb{P}^3} \to 0 \tag{4.5}$$

if and only if C is ACM.

Moreover, one has

(i) $m_i > \min\{d_j$ *for* $j = 1, \dots, n+2\}$*;*

(ii) $\sum_{i=1}^{n+2} m_i = \sum_{j=1}^{n+2} d_j$*;*

(iii) $\deg(C) = \frac{1}{2}(\sum_{i=1}^{n+1} m_i^2 - \sum_{j=1}^{n+2} d_j^2)$*;*

(iv) $g(C) = 1 + \frac{1}{6}(\sum_{i=1}^{n+1} m_i^3 - \sum_{j=1}^{n+2} d_j^3) - 2\deg(C)$.

Proof. If C is ACM, the existence of the minimal free resolution (4.5) is proved in [Bea1, Theorem A]. On the other hand, observe that

$$H^1(\bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^3}(-d_j + k)) = H^2(\bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^3}(-m_i + k)) = 0$$

for all integres k.

Therefore, if C admits a minimal free resolution as in (4.5), we have that $H^1(\mathcal{I}_C(k)) = 0$ for every $k \in \mathbb{Z}$, that is, C is ACM.

In particular, write the first map in (4.5) as the matrix $[A_{ij}]$. Since the resolution is minimal, each entry satisfies

$$\deg A_{ij} = m_i - d_j \quad \text{if } m_i - d_j > 0, \qquad A_{ij} = 0 \quad \text{if } m_i - d_j \leq 0.$$

Suppose, on the contrary, that some i satisfies $m_i \leq \min\{d_j\}$. Then $m_i - d_j \leq 0$ for every j and consequently $A_{ij} = 0$ for all j. Hence, the i-th row of $[A_{ij}]$ is identically zero. Since the ideal $I_{C/\mathbb{P}^3}$ is generated by the $(n+1) \times (n+1)$ minors of this matrix, a zero row forces all those minors to vanish, so $I_{C/\mathbb{P}^3} = 0$, a contradiction. Hence the inequality must hold for every i, that is, (i).

Finally, for the equality in (ii), see [PS, §3], and for point (iii) and (iv) see [PS, Proposition 3.1].

□

Lemma 4.16. *Let $C \subset \mathbb{P}^3$ be a smooth ACM curve and let $d \geq 4$ be an integer. Then*

$$\mathcal{I}_C \text{ is } (d-1)\text{-regular} \iff H^1(\mathcal{O}_C(d-3)) = 0.$$

Proof. $\mathcal{I}_C$ is $(d-1)$-regular $\iff H^i(\mathcal{I}_C(d-1-i)) = 0$ for all $i > 0 \iff$

$$H^1(\mathcal{I}_C(d-2)) = 0, \quad H^2(\mathcal{I}_C(d-3)) = 0, \quad H^3(\mathcal{I}_C(d-4)) = 0,$$

that is, equivalently, since C is assumed to be ACM,

$$H^2(\mathcal{I}_C(d-3)) = 0, \quad H^3(\mathcal{I}_C(d-4)) = 0.$$

Now, consider the short exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}^3}(d-3)$ and passing to cohomology, one has

$$H^2(\mathcal{I}_C(d-3)) = 0 \iff H^1(\mathcal{O}_C(d-3)) = 0,$$

while, observing that $H^2(\mathcal{O}_C(d-4)) = 0$ and $H^3(\mathcal{O}_{\mathbb{P}^3}(d-4)) = 0$, the condition $H^3(\mathcal{I}_C(d-4)) = 0$ is automatically satisfied.

It follows that

$$H^2(\mathcal{I}_C(d-3)) = 0, \quad H^3(\mathcal{I}_C(d-4)) = 0 \iff H^1(\mathcal{O}_C(d-3)) = 0,$$

that is, the statement. □

Lemma 4.17. *Let $C \subset \mathbb{P}^3$ be a smooth irreducible ACM curve such that $\mathcal{I}_C$ is $(d-1)$-regular, with $d \geq 4$. Then*

(i) there exists a smooth surface S of degree d containing C;

(ii) for any smooth surface S lying in the linear system $|\mathcal{I}_C(d)|$, one has the isomorphism

$$H^0(\mathcal{I}_C((d-2)) \cong H^0(\mathcal{O}_S((d-2)H - C))$$

where $H \in |\mathcal{O}_S(1)|$.

Proof. (i) Observe that $\mathcal{I}_C(d-1)$ is 0-regular, hence, it is globally generated. Then, by Theorem 1.7(ii) the natural map

$$H^0(\mathcal{I}_C(d-1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{I}_C(d))$$

is surjective and, since $H^0(\mathcal{I}_C(d-1)) \neq 0$, one has $H^0(\mathcal{I}_C(d)) \neq 0$. Hence, there exists a surface $S \in |\mathcal{I}_C(d)|$, that is, a degree d surface containing C. Actually, one can prove the existence of a smooth degree d surface containing C – see for istance [BGL, Proof of Lemma 4.1].

(ii) Let $S \in |\mathcal{I}_C(d)|$ be a smooth surface. Then, one has the following short exact sequence

$$0 \to \mathcal{I}_{S/\mathbb{P}^3}(d-2) \to \mathcal{I}_{C/\mathbb{P}^3}(d-2) \to \mathcal{I}_{C/S}(d-2) \to 0 \tag{4.6}$$

Observe that $\mathcal{I}_{S/\mathbb{P}^3}(d-2) \cong \mathcal{O}_{\mathbb{P}^3}(-2)$ and, hence,

$$H^0(\mathcal{I}_{S/\mathbb{P}^3}(d-2)) = H^1(\mathcal{I}_{S/\mathbb{P}^3}(d-2)) = 0.$$

Then, from the cohomology of the exact sequence (4.6), one has

$$H^0(\mathcal{I}_C(d-2)) \cong H^0(\mathcal{I}_{C/S}(d-2)) \cong H^0(\mathcal{O}_S((d-2)H - C)),$$

that is, (ii).

□

4.4 Ulrich bundles and Noether Lefschetz Loci

Definition 4.18. Let d be a positive integer. We define

$$Ul_{r,d} := \{S \in \mathcal{U}_d \, : \, uc(S) = r\}$$

Remark 4.19. We claim that the locus $\mathrm{Ul}_{r,d}$ is constructible. For any projective space $\mathbb{P}^n$ with $n \geq 2$ and for any integers $d \geq 1$ and $r \geq 1$, let $V_{r,d}$ denote the locus of degree-d hypersurfaces that carry at least one Ulrich bundle of rank r. Write S_k for the homogeneous polynomials of degree k in $n+1$ variables, let

$$\alpha : \mathrm{Mat}_{r\times r}(S_1) \to S_{rd}, \quad [L_{ij}] \to \det([L_{ij}])$$

be the determinant map, and let

$$\beta : S_d \to S_{rd}, \quad F \longmapsto F^r.$$

Because both α and β are regular morphisms between affine varieties, Chevalley's theorem shows that $\beta^{-1}(\mathrm{Im}\,\alpha)$ is constructible; this set is exactly $V_{r,d}$, so $V_{r,d}$ is constructible. The subset in which the minimal Ulrich rank equals r,

$$\mathrm{Ul}_{r,d} = V_{r,d} \setminus \big(V_{1,d} \cup \cdots \cup V_{r-1,d}\big),$$

is therefore constructible as well, because finite unions and set-theoretic differences of constructible sets remain constructible.

In the setting of this paper we work with hypersurfaces in $\mathbb{P}^3$; thus $\mathrm{Ul}_{r,d}$ is viewed inside the parameter space of degree-d surfaces in $\mathbb{P}^3$, but the same reasoning carries over unchanged to projective spaces of any dimension.

Theorem 4.20. *Let $d \geq 4$ be an integer. Then, $Ul_{1,d}$ is the general component of the Noether-Lefschetz locus $NL(d)$ made of surfaces containing a smooth ACM curve whose ideal is given by the $(d-1) \times (d-1)$ minors of a $(d-1) \times d$ matrix of linear forms.*

Proof. Let $C \subset \mathbb{P}^3$ be a curve as in the statement. It is well known (see [PS, Theorem 6.2]) that C admits a minimal free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d)^{\oplus(d-1)} \to \mathcal{O}_{\mathbb{P}^3}(1-d)^{\oplus d} \to \mathcal{I}_C \to 0. \tag{4.7}$$

Observe that such a curve exists from [PS, Theorem 6.2].

Then, C is ACM by Lemma 4.15.

In addition, $\mathcal{I}_C$ is $(d-1)$-regular. Indeed, from the long exact sequence in cohomology induced by (4.7) twisted by $\mathcal{O}_{\mathbb{P}^3}(d-1-i)$, we have the segment

$$\ldots \to H^i(\mathcal{O}_{\mathbb{P}^3}(-i))^{\oplus d} \to H^i(\mathcal{I}_C(d-1-i)) \to H^{i+1}(\mathcal{O}_{\mathbb{P}^3}(-1-i))^{\oplus(d-1)} \to \ldots$$

Hence, it follows that $H^i(\mathcal{I}_C(d-1-i)) = 0$ for all $i > 0$, simply by observing that

$$H^{i+1}(\mathcal{O}_{\mathbb{P}^3}(-1-i)) = H^i(\mathcal{O}_{\mathbb{P}^3}(-i)) = 0 \text{ for } i = 1, 2, 3\ .$$

In addition, by point (iii) and (iv) of Lemma 4.15 we have that

$$d' := \deg(C) = \frac{1}{2}((d-1)d^2 - d(d-1)^2) = \frac{1}{2}d(d-1);$$

$$g := g(C) = 1 + \frac{1}{6}d(d-1)(2d-7) = \frac{1}{6}(d-2)(d-3)(2d+1).$$

We define

$$W := \overline{\{[C] \in H_{d',g}\ :\ \text{there exists minimal free resolution as in (4.7)}\}}. \tag{4.8}$$

Then, W is a component of $H_{n,g}$ and a general point $[C]$ of W is a curve that admits the minimal free resolution (4.7) (see [Ell]), hence, it is smooth and irreducible, ACM and such that $\mathcal{I}_C$ is $(d-1)$-regular. Therefore, by Lemma 4.12, $W(d) = Im(\pi_1)$ is a component of $NL(d)$.

Moreover, by Lemma 4.16, $H^1(\mathcal{O}_C(d-3)) = 0$.

Note that $H^0(\mathcal{O}_{\mathbb{P}^3}(-1)) = H^1(\mathcal{O}_{\mathbb{P}^3}(-2)) = 0$, hence, $H^0(\mathcal{I}_C(d-2)) = 0$. By Lemma 4.17, it follows that for any smooth surface $S \in |\mathcal{I}_C(d)|$ we have $H^0(\mathcal{O}_S((d-2)H - C)) = 0$, where $H \in |\mathcal{O}_S(1)|$. Hence, any $C \in W$ satisfies point (i), (ii) and (iii) of Remark 4.14, implying that, by Corollary 3.12, for any smooth surface $S \in |\mathcal{I}_C(d)|$, we have that $\mathcal{O}_S(C)$ is an Ulrich line bundle on S.

On the other hand, let $S \subset \mathbb{P}^3$ a smooth surface of degree d. Suppose that there exists a smooth irreducible curve $C \subset S$ satisfying condition (i), (ii) and (iii) of Remark 4.14. Then, by Remark 4.14, C is ACM and, by Lemma 4.16, $\mathcal{I}_C$ is $(d-1)$-regular.

Moreover, C admits a minimal free resolution

$$0 \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^3}(-m_i) \to \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^3}(-d_j) \to \mathcal{I}_{C/\mathbb{P}^3} \to 0.$$

by Lemma 4.15.

Observe that $H^0(\mathcal{I}_C(d-2)) = 0$ by Lemma 4.17 together with condition (ii) of Remark 4.14. Hence, $d_j = d-1$ for all $j = 1, \dots, n+2$.

Moreover, $n+2 = h^0(\mathcal{I}_C(d-1))$ which we now prove to be equal to d. From the short exact sequence

$$0 \to \mathcal{I}_{S/\mathbb{P}^3}(d-1) \to \mathcal{I}_{C/\mathbb{P}^3}(d-1) \to \mathcal{I}_{C/S}(d-1) \to 0 \tag{4.9}$$

we have that $h^0(\mathcal{I}_{C/\mathbb{P}^3}(d-1)) = h^0(\mathcal{I}_{C/S}(d-1)) = h^0(\mathcal{O}_S((d-1)H - C)) = h^2(O_S(C)(-3))$. Note that, by Remark 4.14, $\mathcal{O}_S(C)$ is an Ulrich line bundle on S. Hence, $h^0(\mathcal{O}_S(C)(-3)) \subseteq h^0(\mathcal{O}_S(C)(-2)) = 0$ and Lemma 2.5(iii) implies that $h^1(\mathcal{O}_S(C)(-3)) = 0$. In particular,

$$h^0(\mathcal{I}_{C/\mathbb{P}^3}(d-1)) = h^2(O_S(C)(-3)) = \chi(\mathcal{O}_S(C)(-3)) = d,$$

where the last equality follows Lemma 2.5(iv).

Therefore, by point (i) of Lemma 4.15 we have $m_i \geq d$ for all i, hence, point (ii) of the same lemma gives

$$d(d-1) \leq \sum_{i=1}^{n+1} m_i = d(d-1),$$

which implies that $m_i = d$ for all i.

Hence, C admits a minimal free resolution as is (4.7).

This exactly means that

$$S \in W(d) \iff S \text{ admits an Ulrich line bundle},$$

that is, $Ul_{1,d} = W(d)$, thereby implying that $Ul_{1,d}$ is a component of $NL(d)$.

We now prove that this component is *general*.

Observe that the curves constructed in W, satisfy the vanishing condition $H^1(N_C) = 0$, where N_C denotes the normal bundle of $C \subset \mathbb{P}^3$. The vanishing

$H^1(N_C) = 0$ can be checked explicitly using a version of Kleppe's Lemma (see [BGL, Lemma 4.2]), from which one has in particular

$$H^1(N_C) \cong Ext^2_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{I}_C, \mathcal{I}_C).$$

Applying the functor $\mathrm{Hom}(-, \mathcal{I}_C)$ to the exact sequence (4.7), we obtain the exact segment

$$\ldots \to Ext^1(\mathcal{O}_{\mathbb{P}^3}(-d), \mathcal{I}_C)^{\oplus(d-1)} \to Ext^2(\mathcal{I}_C, \mathcal{I}_C) \to Ext^2(\mathcal{O}_{\mathbb{P}^3}(1-d), \mathcal{I}_C)^{\oplus d} \to \ldots$$

Observe that

$$Ext^1(\mathcal{O}_{\mathbb{P}^3}(-d), \mathcal{I}_C) = Ext^1(\mathcal{O}_{\mathbb{P}^3}, \mathcal{I}_C(d)) = H^1(\mathcal{I}_C(d)) = 0 \quad \text{and}$$

$$Ext^2(\mathcal{O}_{\mathbb{P}^3}(1-d), \mathcal{I}_C) = Ext^2(\mathcal{O}_{\mathbb{P}^3}, \mathcal{I}_C(d-1)) = H^2(\mathcal{I}_C(d-1)) = 0$$

where the vanishing $H^1(\mathcal{I}_C(d)) = 0$ is given by the ACM property and $H^2(\mathcal{I}_C(d-1)) = 0$ is a consequence of Theorem 1.7.

Hence, $Ext^2(\mathcal{I}_C, \mathcal{I}_C) = 0$, that is, $H^1(N_C) = 0$.

This vanishing immediately yields the identity $\chi(N_C) = h^0(N_C)$. Now, according to [HE, Chapter 2.a] or [HM, §1.E] one has the inequality

$$\chi(N_C) \leq \dim W \leq h^0(N_C).$$

Since both bounds coincide in this case, it follows that

$$\dim W = \chi(N_C) = 4\deg(C).$$

For an explicit computation of $\chi(N_C)$, see Remark 4.21.

By applying Lemma 4.12, one concludes that the associated component $W(d) \subset \mathrm{NL}(d)$ in the Noether–Lefschetz locus is general, in the sense that it has maximal possible codimension

$$\operatorname{codim} W(d) = p_g(d) = \binom{d-1}{3}.$$

This follows from the identity

$$h^0(\mathcal{O}_C(d-4)) = h^0(\mathcal{O}_{\mathbb{P}^3}(d-4)) = p_g(d).$$

Moreover, this codimension is indeed the maximum allowed for a component of $NL(d)$, by Lemma 4.10.

□

Remark 4.21 (**Computation of** $\chi(N_C)$)**.** We now justify the identity $\chi(N_C) = 4\deg(C)$. Start with the exact tangent sequence

$$0 \to \mathcal{T}_C \to \mathcal{T}_{\mathbb{P}^3|_C} \to N_C \to 0,$$

so that

$$\chi(\mathcal{N}_C) = \chi(\mathcal{T}_{\mathbb{P}^3|_C}) - \chi(\mathcal{T}_C).$$

Restricting to C the standard short exact sequence in [Har, Example II.8.20.1] we obtain

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus 4} \to \mathcal{T}_{\mathbb{P}^3|_C} \to 0,$$

thereby implying

$$\chi(\mathcal{T}_{\mathbb{P}^3|_C}) = 4\chi(\mathcal{O}_C(1)) - \chi(\mathcal{O}_C) = 4(\deg C + 1 - g) - (1-g) = 4\deg C + 3 - 3g.$$

Recall that $\mathcal{T}_C \simeq \omega_C^{-1}$, then Riemann–Roch gives

$$\chi(\mathcal{T}_C) = \deg\bigl(\omega_C^{-1}\bigr) + 1 - g = -3g + 3.$$

Combining these two formulas gives $\chi(\mathcal{N}_C) = 4\deg C$, as claimed.

The results obtained throughout this chapter naturally converge to the following statement – essentially the content of [Bea1, Proposition 6.2] – which we include here for completeness.

Proposition 4.22 (**[Bea1, Proposition 6.2]**)**.** *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d and defined by an equation $F = 0$, and let $C \subset S$ be an ACM curve such that*

$$\deg C = \frac{1}{2}\, d(d-1), \qquad g(C) = \frac{1}{6}\,(d-2)(d-3)(2d+1).$$

Then the line bundle $\mathcal{O}_S(C)$ admits a linear determinantal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^3}^{\oplus d} \to \mathcal{O}_S(C) \to 0$$

with $\det M = F$.

Conversely, let $M \in \mathrm{M}_d(S^3)$ be a linear *matrix with* $\det M = F$. *Then the cokernel of*

$$M : \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus d}$$

is isomorphic to $\mathcal{O}_S(C)$, where $C \subset S$ is a smooth ACM curve with degree and genus specified above.

Proof. We first observe that the assumptions on the curve C in the statement are equivalent to conditions (i), (ii) and (iii) of Remark 4.14. Indeed, the degree and genus conditions assumed in the proposition match exactly those required in Remark 4.14(iii).

To verify condition (i) and (ii) of Remark 4.14 consider the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0. \tag{4.10}$$

Twisting (4.10) by $\mathcal{O}_{\mathbb{P}^3}(d-2)$, taking global sections and using the vanishing of
$H^1(\mathcal{I}_C(d-2))$, given by the ACM property, we obtain

$$h^0(\mathcal{I}_C(d-2)) = h^0(\mathcal{O}_{\mathbb{P}^3}(d-2)) - h^0(\mathcal{O}_C(d-2)) =$$

$$= \binom{d+1}{3} - ((d-2)\deg C - g(C) + 1 + h^1(\mathcal{O}_C(d-2))) = -h^1(\mathcal{O}_C(d-2)).$$

Hence, since $h^0(\mathcal{I}_C(d-2))$ and $h^1(\mathcal{O}_C(d-2))$ are both non-negative integers, we have $h^0(\mathcal{I}_C(d-2)) = 0$, which is precisely, by Lemma 4.17(ii), Remark 4.14(ii).

Repeating the same procedure twisting (4.10) by $\mathcal{O}_{\mathbb{P}^3}(d-3)$ we obtain

$$h^0(\mathcal{I}_C(d-3)) = h^0(\mathcal{O}_{\mathbb{P}^3}(d-3)) - h^0(\mathcal{O}_C(d-3)) =$$

$$= \binom{d}{3} - ((d-3)\deg C - g(C) + 1 + h^1(\mathcal{O}_C(d-3))) = -h^1(\mathcal{O}_C(d-3)),$$

which implies $h^1(\mathcal{O}_C(d-3)) = 0$, that is, Remark 4.14(i).

Conversely, if a curve fulfils conditions (i), (ii) and (iii) of Remark 4.14, then Remark 4.14 itself ensures that C is ACM.

Having established this equivalence, the statement becomes an immediate consequence of Proposition 2.2 together with Remark 4.14.

□

4.5 Refining the lower genus bound on quartic surfaces

Remark 4.23. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface. Then, from Remark 4.3, we have the following bounds for the genus g of the curve C of Theorem 4.2:

$$2r^2 + 1 \leq g \leq \frac{9}{2}r^2 + 1.$$

In what follows, we refine the lower bound for g when the curve C arises from a minimal rank Ulrich bundle on S.

Remark 4.24. Let $S \subset \mathbb{P}^N$ be a smooth surface of degree d and fix an Ulrich bundle $\mathcal{E}$ of minimal rank r on S. From Lemma 2.14, the existence of a non-scalar endomorphism of $\mathcal{E}$ – equivalently $h^0(S, \mathcal{E} \otimes \mathcal{E}^*) \geq 2$ – would force $S \notin Ul_{r,d}$. Computing $h^0(S, \mathcal{E} \otimes \mathcal{E}^*)$ directly is, however, seldom practical. It is easier to work with the Euler characteristic

$$\chi(\mathcal{E} \otimes \mathcal{E}^*) = h^0(\mathcal{E} \otimes \mathcal{E}^*) - h^1(\mathcal{E} \otimes \mathcal{E}^*) + h^2(\mathcal{E} \otimes \mathcal{E}^*).$$

Assume moreover that S is a $K3$ surface.

By Serre duality $h^2(S, \mathcal{E} \otimes \mathcal{E}^*) = h^0(S, \mathcal{E} \otimes \mathcal{E}^*)$; hence

$$2h^0(S, \mathcal{E} \otimes \mathcal{E}^*) = \chi(\mathcal{E} \otimes \mathcal{E}^*) + h^1(S, \mathcal{E} \otimes \mathcal{E}^*) \geq \chi(\mathcal{E} \otimes \mathcal{E}^*),$$

so that,

$$h^0(S, \mathcal{E} \otimes \mathcal{E}^*) \geq \frac{\chi(\mathcal{E} \otimes \mathcal{E}^*)}{2}.$$

In particular, if one can prove that $\chi(\mathcal{E} \otimes \mathcal{E}^*) \geq 4$, then $h^0 \geq 2$, contradicting the simplicity established in Lemma 2.14 for any minimal–rank Ulrich bundle. This observation justifies the computations in the following remark.

Remark 4.25. Let $S \subset \mathbb{P}^N$ be a smooth projective surface and let $\mathcal{E}$ be a rank r vector bundle on S. Then

(i) $c_1(\mathcal{E} \otimes \mathcal{E}^*) = 0$;

(ii) $c_2(\mathcal{E} \otimes \mathcal{E}^*) = 2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2$;

(iii) $\chi(\mathcal{E} \otimes \mathcal{E}^*) = r^2\chi(\mathcal{O}_S) - 2rc_2(\mathcal{E}) + (r-1)c_1(\mathcal{E})^2$.

Proof. (i) It follows directly from Lemma 1.12(v).

(ii) Observe that, from [Har, Appendix A, §4], we have

$$[\mathrm{ch}(\mathcal{E} \otimes \mathcal{E}^*)]_2 = \frac{1}{2}(c_1(\mathcal{E} \otimes \mathcal{E}^*)^2 - 2c_2(\mathcal{E} \otimes \mathcal{E}^*)),$$

that is, by point (i), $[\mathrm{ch}(\mathcal{E} \otimes \mathcal{E}^*)]_2 = -c_2(\mathcal{E} \otimes \mathcal{E}^*)$.

On the other hand, [EH, §5.5.2] gives

$$[\mathrm{ch}(\mathcal{E} \otimes \mathcal{E}^*)]_2 = [\mathrm{ch}(\mathcal{E})\mathrm{ch}(\mathcal{E}^*)]_2 =$$

$$= \left[\left(r + c_1(\mathcal{E}) + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E}))\right) \cdot \left(r + c_1(\mathcal{E}^*) + \frac{1}{2}(c_1(\mathcal{E}^*)^2 - 2c_2(\mathcal{E}^*))\right)\right]_2 =$$

$$= \left[\left(r + c_1(\mathcal{E}) + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E}))\right) \cdot \left(r - c_1(\mathcal{E}) + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E}))\right)\right]_2 =$$

$$= -2rc_2(\mathcal{E}) + (r-1)c_1(\mathcal{E})^2,$$

which directly implies (ii).

(iii) It is a direct consequence of the Riemann-Roch theorem (Theorem 1.3), together with (i) and (ii).

□

We are now ready to present the sharper lower bound announced in Remark 4.23.

Lemma 4.26. *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface of degree d and let $\mathcal{E}$ be a minimal Ulrich bundle of rank r on S. Denote by $(C, \mathcal{L})$ the pair associated with $\mathcal{E}$ via Theorem 4.2. Then, the genus g of C satisfies*

$$g \geq 3r^2.$$

Proof. Observe that, by Remark 4.25(iii), we have

$$\chi(\mathcal{E} \otimes \mathcal{E}^*) = 2r^2 - 2rc_2(\mathcal{E}) + (r-1)c_1(\mathcal{E})^2,$$

For the Ulrich bundle $\mathcal{E}$ one has $c_1(\mathcal{E}) = C$ with $C^2 = 2g - 2$, and $c_2(\mathcal{E}) = g - 1 - 2r$. Substituting these expressions gives

$$\chi(\mathcal{E} \otimes \mathcal{E}^*) = 6r^2 - 2g + 2.$$

Assume, for contradiction, that $g \leq 3r^2 - 1$. Then

$$\chi(\mathcal{E} \otimes \mathcal{E}^*) \geq 4$$

which, by Remark 4.24, contradict the simplicity of the minimal-rank Ulrich bundle $\mathcal{E}$. Consequently, we must have $g \geq 3r^2$.

□

Remark 4.27. Note that for $r = 1$ the inequality coincides with the previous bound, whereas for $r \geq 2$ it is already strictly stronger.

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