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Graduation Thesis in Mathematics
by
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Asymptotic invariants of line bundles, semiampleness and finite generation

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Introduction

As the history of algebraic geometry shows, a mainstream problem has always been the classification of algebraic varieties. One of the most common ways to classify varieties is through studying algebraic objects associated to each variety.
Maybe the simplest example of this kind is given by the *coordinate ring* of a projective variety X. In this ring are contained basic geometrical properties of X such as its dimension and its degree. In particular, denoting by H a hyperplane section on X, we see that the coordinate ring of X is strictly related to

$$R(X, H) = \bigoplus_{m \geq 0} H^0(X, mH).$$

and in fact they have the same Hilbert polynomial.
Looking for other geometrical properties of X, this suggests us the idea of considering an arbitrary divisor D on X and defining the *graded ring associated* to D as

$$R(X, D) = \bigoplus_{m \geq 0} H^0(X, mD).$$

One of the purposes of this thesis is to study the relation between the geometrical characteristics of the couple (X, D) and the algebraic properties of the ring $R(X, D)$. For example, denoting by K_X the canonical divisor, we have that an n-dimensional variety X is of general type if and only if, asymptotically, $dim\ R(X, K_X)_m = O(m^n)$.
The most significant property that $R(X, D)$ might have is the finite generation as a $\mathbb{C}$-algebra and the divisor D itself is said to be finitely generated if its graded ring $R(X, D)$ is such.
In particular the finite generation of $R(X, K_X)$ has been considered for a long time, especially because it holds a particular role in the theory of minimal models. Given an algebraic variety X the idea is that a minimal model of X is the "simplest" variety birational to X. The existence of minimal models for surfaces was shown by the Italian school (Castelnuovo, Enriques, etc.) in the 30's. Passing to dimension n things get more complicated. However an important very recent theorem of Birkar, Cascini, Hacon and McKernan proves the existence of minimal models for smooth n-dimensional varieties

of general type and the finite generation of the graded ring $R(X, K_X)$ (see [BCHM07]). Moreover the importance of considering arbitrary divisors and not just the canonical one is given by the tight link between the finite generation of $R(X, K_X)$ and the properties of rings of kind $R(X, K_X + \Delta)$, for suitable divisors Δ.
Another remarkable historical reason that led to consider the rings $R(X, D)$ is the connection between their finite generation and Hilbert's fourteenth problem. We can formulate Hilbert's 14th problem in these terms: given a field k, a polynomial algebra $k[x_1, \ldots, x_n]$ and a subfield of the rational functions in n-variables over k, $K \subseteq k(x_1, \ldots, x_n)$, is the ring $K \cap k[x_1, \ldots, x_n]$ finitely generated as a k-algebra?
The answer is negative and, in a famous counter-example given in 1959, Nagata considers the surface X obtained by blowing up $\mathbb{P}^2$ in r general points $p_1, \ldots, p_r$ and finds a suitable divisor D on X such that $R(X, D)$ is not finitely generated and it is of kind $K \cap k[x_1, \ldots, x_n]$.

We now come to the description of the work of this thesis.
We study how the notion of finite generation can be related to different geometrical properties of the couple (X, D). Our approach will often be *asymptotic*, in the sense that we study a divisor looking at all its sufficiently large multiples and the graded ring $R(X, D)$ helps us in this sense, since, by definition, it depends on all the multiples of D.
A first step consists in analyzing the geometrical behaviour of (X, D) and observing how it reflects in the graded ring associated to D. With this goal in mind we first study divisors.
While in the first chapter we essentially follow [Har77] (except for the definition of intersection numbers, for whom we refer to [Kle66]), from the second one our point of view is that of [Laz04]. The reader will find in these books lacking proofs of some results presented and more explanations.
We begin with the classic theory of ample divisors before passing to the more recent one of nefness and we generalize the concept of *divisor* introducing $\mathbb{Q}$ and $\mathbb{R}$-divisors. Then we pass to the description of some important properties and notions, such as bigness, mainly developed in the last 30 years, closely linked to base loci and projective morphisms defined by divisors.
We begin by defining the stable base locus:

Definition 0.1. Let X be a variety and let D be a Cartier divisor on X. [Def. 3.5]
The *stable base locus* of D is the algebraic set

$$\mathbb{B}(D) = \bigcap_{m \geq 1} Bs(|mD|)$$

where we disregard the structures of schemes and consider the intersection just like an intersection of closed subsets.

Then we pass to the definition of semiampleness:

Definition 0.2. Let X be a variety. A Cartier divisor D on X is *semiample* if mD is globally generated for some $m \in \mathbb{N}$, or, equivalently, if $\mathbb{B}(D) = \emptyset$. [Def. 3.9]

Finally we introduce the notion of bigness:

Definition 0.3. Let X be a variety. A Cartier divisor D on X is *big* if there exists an ample $\mathbb{Q}$-divisor A, together with an effective $\mathbb{Q}$-divisor F, such that $D \sim_{\mathbb{Q}} A + F$. [Def. 3.19]

Note that if D is semiample then it is finitely generated and it is easy to check that every divisor on a curve is finitely generated. On the contrary, when passing to surfaces, a famous example of Zariski provides a big and nef divisor D whose graded ring is not finitely generated. Already in this example one can observe how the fundamental reason not allowing finite generation is that the multiplicity of a curve in the base locus of $|mD|$ is bounded. This drives us to the following definition:

Definition 0.4. Let X be a variety and let D be a Cartier divisor on X. Given a linear series $|V| \subseteq |D|$ and a point $x \in X$ we define the *multiplicity* of $|V|$ at x, denoted by $mult_x|V|$, as the multiplicity at x of a general divisor in $|V|$. [Def. 4.12]
Equivalently
$$mult_x|V| = \min_{D' \in |V|} \{mult_x D'\}.$$

Using this definition we prove the following original characterization of semiampleness in terms of finite generation and boundedness of the multiplicity at every point:

Theorem 0.5. Let X be a normal variety and let D be a Cartier divisor on X. [Theorem 4.16]
Then D is semiample if and only if D satisfies the following three conditions:

1. D is finitely generated.
2. $k(X, D) \geq 0$.
3. There exists a constant $C > 0$ such that for all $m > 0$, with $|mD| \neq \emptyset$, and for all $x \in X$, we have
$$mult_x|mD| \leq C.$$

Then we note that the multiplicity of a linear series is nothing but a particular discrete valuation on the function field $K(X)$. Hence the idea developed in the fifth chapter is to generalize the previous result in terms of valuations, along the lines of recent works of Ein, Lazarsfeld, Mustată, Nakamaye and Popa.
In these matters a relevant role is played by the restricted base locus:

Definition 0.6. Let X be a normal variety and let D be an $\mathbb{R}$-divisor on X. The *restricted base locus* of D is [Def. 5.1]

$$\mathbb{B}_-(D) = \bigcup_A \mathbb{B}(D + A),$$

where the union is taken over all ample $\mathbb{R}$-divisors A such that $D + A$ is a $\mathbb{Q}$-divisor.

Then, following [ELMNP06], we adopt the language of valuations to generalize the concept of multiplicity at a point of a linear series:

Definition 0.7. Let X be a variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$ with valuation ring R_v and center $\xi \in X$. [Def. 5.15 & 5.17]
If D is a Cartier divisor on X, with $|D| \neq \emptyset$, we put

$$v(|D|) = v((b(|D|)_\xi)_{R_v}),$$

where $b(|D|)$ is the base ideal of D and $((b(|D|)_\xi)_{R_v}$ is the ideal generated by the stalk $(b(|D|)_\xi$ in the ring R_v.

In analogy with Theorem 0.5 we define a v-bounded divisor, that is a divisor D such that the center of the valuation v is asymptotically contained in the base locus of mD with bounded multiplicity. In other words:

Definition 0.8. Let X be a normal variety and let v be a discrete valuation on $K(X)/\mathbb{C}$. If D is a Cartier divisor on X with $k(X, D) \geq 0$ we say that D is *v-bounded* if there exists a constant $C > 0$ such that [Def. 5.35]

$$v(|pD|) \leq C$$

for every $p > 0$ such that $|pD| \neq \emptyset$.

Moreover we generalize the concept of semiample divisor with the weaker one of v-semiample, that is a divisor not containing the center of v in its stable base locus:

Definition 0.9. Let X be a normal variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$, having center $Z_v \subseteq X$, and let D be a Cartier divisor on X with $k(X, D) \geq 0$. [Def. 5.39]
Then D is *v-semiample* if there exists a positive integer l_0 such that $|l_0 D| \neq \emptyset$ and $v(|l_0 D|) = 0$, or equivalently if $Z_v \not\subseteq \mathbb{B}(D)$.

These last two notions are not equivalent but they are linked by the following result:

Theorem 0.10. Let X be a normal variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$ and let D be a Cartier divisor on X with $k(X, D) \geq 0$. [Pr. 5.41 & 5.42]

- If D is v-semiample, then D is v-bounded.
- If D is finitely generated and v-bounded, then D is v-semiample.

Actually for the second statement it is not necessary to take D v-bounded, but it suffices the hypothesis of "v-sublinearity", that is $v(|mD|)$ can go to infinity but in a "sublinear" way with respect to m (see Proposition 5.42). As a corollary, considering all the discrete valuations on $K(X)/\mathbb{C}$ together and noting that D is semiample if and only if D is v-semiample for every valuation v, we can generalize Theorem 0.5 with a different characterization of semiampleness involving v-boundedness:

Theorem 0.11. Let X be a normal variety with function field $K = K(X)$ and let D be a Cartier divisor on X with $k(X, D) \geq 0$. [Theorem 5.46]
Then the following statements are equivalent:

1. D is semiample
2. D is finitely generated and there exists a constant $C > 0$ such that
$$mult_x|mD| \leq C$$
for every $m > 0$ such that $|mD| \neq \emptyset$, for every $x \in X$.
3. D is finitely generated and, for every discrete valuation v on $K/\mathbb{C}$, D is v-bounded.

On the other hand we extend the notion of asymptotic order of vanishing of a divisor along a valuation v, defined in [ELMNP06] only in the big case, to non-big divisors:

Definition 0.12. Let X be a normal variety with function field $K = K(X)$, let D be a Cartier divisor on X with $k(X, D) \geq 0$ and let v be a discrete valuation on $K/\mathbb{C}$. [Def. 5.20]
We define the *exponent* of D as

$$e(D) = \text{g.c.d.}\{m \in \mathbb{N} \ : \ |mD| \neq \emptyset\}.$$

If $e = e(D)$ is the exponent of D, the *asymptotic order of vanishing* of D along v is

$$v(||D||) = \lim_{m \to \infty} \frac{v(|meD|)}{me}.$$

We can simply extend this definition to $\mathbb{Q}$-divisors. In particular we fix as the right work environment the set of $\mathbb{Q}$-linearly effective $\mathbb{Q}$-divisors:

Definition 0.13. Let X be a variety and let D be a $\mathbb{Q}$-divisor on X. We define the $\mathbb{Q}$-*linear series* $|D|_{\mathbb{Q}}$ as the set of all the effective $\mathbb{Q}$-divisors that are $\mathbb{Q}$-linearly equivalent to D. [Def. 5.23]
D is $\mathbb{Q}$-*linearly-effective* if $|D|_{\mathbb{Q}} \neq \emptyset$.

Finally, we present the main result in [ELMNP06, §2] about the asymptotic order of vanishing of big $\mathbb{Q}$-divisors.

Theorem 0.14. Let X be a smooth variety with function field $K = K(X)$ and let v be a discrete valuation on $K/\mathbb{C}$, having center Z_v on X. [Theorem 5.36]
If D is a big $\mathbb{Q}$-divisor on X, then the following conditions are equivalent:

1. D is v-bounded;
2. $v(||D||) = 0$;
3. $Z_v \not\subseteq \mathbb{B}_-(D)$.

One of the aims of this thesis is, using Definition 0.12, to study which implications of the above theorem are still true in the non-big case. In particular, if D is a $\mathbb{Q}$-linearly effective $\mathbb{Q}$-divisor, we first observe that

(i) D is v-bounded $\Rightarrow v(||D||) = 0$;

(ii) $v(||D||) = 0 \Rightarrow Z_v \not\subseteq \mathbb{B}_-(D)$

are still true, using the same argument of [ELMNP06].
On the other hand the reverse implication of **(ii)** does not hold and we provide a counter-example in Section 5.6.
Finally we outline the questions that, at the moment, remain unsolved:

Open questions:

- We do not know whether D is v-bounded whenever the asymptotic order of vanishing of D along v is zero. However it is true if D is a divisor on a curve or a normal surface (see Remark 5.44).
- It remains an open problem how we can weaken the hypothesis of the second statement of Theorem 0.10 in order to have the viceversa.

Remark 0.15. Besides Theorem 0.5, the original results in this thesis are Theorem 0.10 and Theorem 0.11 (which are inspired by Theorem 0.5) and the example in Section 5.6.

Notation and conventions

- We always work throughout over the field of complex numbers $\mathbb{C}$.
- When speaking about a *scheme* we mean a projective scheme over $\mathbb{C}$, that is a scheme with a closed immersion $i : X \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$.
- By *variety* we mean an irreducible, reduced (projective) scheme.
- Unless clearly specified, a *point* of a scheme or a variety is a closed point.
- A *curve* is a variety of dimension one, a *surface* is a variety of dimension two.
- With the word *ring* we mean a commutative ring with the identity element.
- When speaking about a *divisor*, unless clearly specified, we always mean a Cartier divisor (see Definition 1.1).
- We denote by "$\mathbb{R}^+$" (respectively "$\mathbb{Q}^+$","$\mathbb{N}$") the set

$$\{x \in \mathbb{R} \text{ (respectively } \mathbb{Q}, \mathbb{Z}) : \ x > 0\}$$

- Given a function $f : \mathbb{N} \to \mathbb{R}$ and an integer $k \geq 0$, we say that $f(m) = O(m^k)$ if

$$\limsup_{m \to \infty} \frac{|f(m)|}{m^k} < \infty$$

Chapter 1

Divisors, line bundles and linear series

1.1 Divisors and line bundles

Cartier divisors

Definition 1.1. Let X be a scheme. For every open set $U \subseteq X$, let $S(U) = \{s \in \Gamma(U, \mathcal{O}_X) : \text{s is not a zero-divisor in } \mathcal{O}_{X,p}\ ,\ \forall x \in U\} \subseteq \Gamma(U, O_X)$. Let $\mathcal{K}_X$ be the sheaf of total quotient rings of $\mathcal{O}_X$, that is the sheaf associated to the presheaf $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$.
Let $\mathcal{K}_X^*$ be the subsheaf of invertible elements in $\mathcal{K}_X$ and let $\mathcal{O}_X^*$ be the subsheaf of invertible elements in $\mathcal{O}_X$.
A *Cartier divisor* on X is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$.

We observe that a Cartier divisor can be described by giving a collection $\{(U_i, f_i)\}_{i\in I}$, where $\{U_i\}$ is an open covering of X and $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$ for all $i \in I$, such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$ $\forall i, j \in I$.
In order to describe the operation of the group of Cartier divisors we find additive notation more comfortable. For this reason, from now on, we use the following convention:
If D_1 and D_2 are such that $D_1 \leftrightarrow \{(U_i, f_i)\}_{i\in I}$, $D_2 \leftrightarrow \{(V_j, g_j)\}_{j\in J}$
We put

$$D_1 + D_2 \longleftrightarrow \{(U_i \cap V_j, f_i g_j)\}$$

$$D_1 - D_2 \longleftrightarrow \{(U_i \cap V_j, \frac{f_i}{g_j})\}.$$

The group of Cartier divisors is denoted by $Div(X)$.

Definition 1.2. A Cartier divisor is *principal* if it is in the image of the natural map $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.
We say that two Cartier divisors D, D' are linearly equivalent (written $D \sim D'$) if their difference $D - D'$ is a principal divisor.

$\sim$ is an equivalence relation, whence we can consider the quotient group $Div(X)/\sim$.

Weil divisors

Definition 1.3. Let X be a scheme of dimension n. A *k-cycle* on X is a finite $\mathbb{Z}$-linear combination of subvarieties of dimension k.
$Z_k(X)$ is the group of all k-cycles.
A *Weil divisor* on X is an $(n-1)$-cycle.
We will denote the group of Weil divisors by $WDiv(X)$.

Definition 1.4. Let X be a variety of dimension n, regular in codimension one and let $f \in K$, the function field of X.
We define the Weil divisor of f, denoted by (f), by

$$(f) = \sum ord_Y(f)Y$$

where the sum is taken on all irreducible subvarieties of codimension one.

It is a well defined divisor because the sum is finite for every $f \in K$. Divisors of functions are called principal.
The subset of all principal Weil divisors is a subgroup of $WDiv(X)$ and, as for Cartier case, we can consider the quotient group $WDiv(X)/\sim$.
Now we observe that there is a cycle map

$$Div(X) \longrightarrow WDiv(X)$$

$$D \longmapsto \sum ord_v(D)V$$

where the sum is taken on all irreducible codimension one subvarieties and $ord_V(D)$is the order of D along V.
In particular:

Theorem 1.5. *For any normal variety X the cycle map above is injective. It is an isomorphism if X is non-singular.*

Proof. See [Har77, II, 6.11]. $\square$

Moreover, when X is regular in codimension one, since the cycle map sends principal Cartier divisors to principal Weil divisors, we can use it to define a cycle map of isomorphism classes that satisfies again the theorem.

Line bundles

For any divisor D on a scheme X we can consider a line bundle (that is a locally free $\mathcal{O}_X$-module of rank 1) associated to it in the following sense:

Definition 1.6. Let D be a (Cartier) divisor represented by $\{(U_i, f_i)\}_{i \in I}$. The *line bundle associated* to D, denoted by $\mathcal{O}_X(D)$, is the subsheaf of $\mathcal{K}_X$ generated by f_i^{-1} on any U_i.

Since f_i/f_j is an invertible element, f_i and f_j determine the same module on $U_i \cap U_j$, then $\mathcal{O}_X(D)$ is well-defined.
Moreover linearly equivalent divisors give rise to isomorphic line bundles.

Theorem 1.7. *For any (projective) scheme X, if we denote by $Pic(X)$ the group of isomorphism classes of line bundles, the correspondence*

$$Div(X) \longrightarrow Pic(X)$$

$$D \longmapsto \mathcal{O}_X(D)$$

is surjective and it determines an isomorphism $\frac{Div(X)}{\sim} \simeq Pic(X)$.

Proof. See [Har77, II, 6.13]. □

1.2 Linear series

Definition 1.8. Let X be a scheme, let $\mathcal{L}$ be a line bundle on X and let $V \subseteq H^0(X, \mathcal{L})$ be a vector subspace.
We denote by $|V| = \mathbb{P}(V)$ the projective space of all one-dimensional subspaces of V (if $\mathcal{L} = \mathcal{O}_X(D)$ we use to write $|\mathcal{L}|$ or simply $|D|$ in place of $|H^0(X, \mathcal{L})|$).
With this convention $|V|$ is called a *linear series*, $|D|$ is a *complete linear series*.

Now let X be a nonsingular variety.
In this case there is a natural correspondence between global sections of $\mathcal{L} = \mathcal{O}_X(D)$ and effective divisors linearly equivalent to D.
In fact, for each global section s in $H^0(X, \mathcal{O}_X(D))$ we can define the effective divisor $(s)_0$ as follows:
For every open set $U \subseteq X$ such that $\mathcal{O}_X(D)$ is trivial on U, let $\phi(s)$ be the image of s under the isomorphism $\phi : \mathcal{O}_X(D)_{|U} \to \mathcal{O}_U$.
$(s)_0$ is the effective divisor determined by the collection $\{(U, \phi(s))\}$, as U ranges over a covering of X.
Through the map $s \mapsto (s)_0$ we can identify $|D|$ with the set of the effective divisors linearly equivalent to D, equipping, in a such a way, this set with a structure of projective space.
Using the same map, for every $V \subseteq H^0(X, \mathcal{O}_X(D))$, $|V|$ will be identified with a subspace of this set.

Base points

Let X be a scheme, let $\mathcal{L}$ be a line bundle on X and let $V \subseteq H^0(X, \mathcal{L})$ be a subspace.
Then V determines a morphism

$$V \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \mathcal{L}$$

locally defined by

$$V \otimes \mathcal{O}_{X,p} \longrightarrow \mathcal{L}_p$$

$$s \otimes f_p \longmapsto s_p \cdot f_p.$$

Tensoring by the inverse line bundle $\mathcal{L}^*$ we find a morphism

$$e_V : V \otimes \mathcal{L}^* \longrightarrow \mathcal{O}_X$$

Definition 1.9. The *base ideal* of $|V|$, denoted by $b(|V|)$, is the ideal subsheaf of $\mathcal{O}_X$ image of the map e_V.
The *base locus* of $|V|$, written $Bs(|V|)$, is the closed subset of X determined by the ideal sheaf $b(|V|)$.
Each point $p \in Bs(|V|)$ is called a *base point* of $|V|$.

Equivalently p is a base point of $|V|$ if and only if all sections of V vanish in p.
Note that we put $Bs(|V|) = X$ if and only if $|V| = \emptyset$.

Proposition 1.10. *Let X be a scheme. If $D, E \in Div(X)$ are such that $|D| \neq \emptyset$, $|E| \neq \emptyset$, then there is an inclusion of ideal sheaves*

$$b(|D|) \cdot b(|E|) \subseteq b(|D + E|).$$

Moreover equality holds if the natural map

$$\mu_{D,E} : H^0(X, \mathcal{O}_X(D)) \otimes H^0(X, \mathcal{O}_X(E)) \longrightarrow H^0(X, \mathcal{O}_X(D + E))$$

determined by multiplication of sections is surjective.

Proof. See [Laz04, I, 1.1.9]

□

Definition 1.11. A linear series $|V|$ is *free*, or *base-point free*, if its base locus is empty. A divisor D (or its line bundle $\mathcal{O}_X(D)$) is free, or base-point free, if the corresponding linear series $|D|$ is such.

We can observe that a divisor is free if and only if its line bundle is globally generated.

Definition 1.12. Let X be a variety and $D \in Div(X)$. Then, looking at D as a Weil divisor on X, we define $Supp(D)$ as the union of all subvarieties of codimension one Y_i such that $ord_{Y_i}(D) > 0$.

Remark 1.13. A point p in a variety X is a base point of a linear series $|V|$ if and only if $p \in Supp(D)$ for all $D \in |V|$.

Rational map defined by a linear series

Assume now $V \subseteq H^0(X, \mathcal{L})$ such that $dim(V) > 0$ and $B = Bs(|V|)$.
Then $|V|$ determines a morphism

$$\phi = \phi_{|V|} : X \setminus B \to \mathbb{P}(V) = \mathbb{P}^r.$$

If we choose a basis $\{s_0, s_1, \ldots, s_r\}$ of V, we can describe this map, using homogeneous coordinates, with the expression

$$\phi(x) = [s_0(x), \ldots, s_r(x)] \qquad \forall x \in X \setminus B.$$

If $|V|$ is free, then ϕ is defined on all X. In this case there is an isomorphism

$$\phi^*(\mathcal{O}(1)) \simeq \mathcal{L}.$$

Conversely if $\phi : X \to \mathbb{P}^r$ is a morphism, then $\phi^*(\mathcal{O}(1))$ is a free line bundle on X generated by the the linear series $|V| =< \phi^*(x_0), \ldots, \phi^*(x_r) >$.
Furthermore ϕ is identified with the morphism $\phi_{|V|}$.

Remark 1.14 (Projection)**.** Assume now $V \subseteq H^0(X, \mathcal{L})$ is a linear series, $W \subseteq V$ is a linear subspace.
Then $Bs(|V|) \subseteq Bs(|W|)$, so that $\phi_{|V|}$ and $\phi_{|W|}$ are both defined on $X \setminus Bs(|W|)$.
Considering them as morphisms on this set, one has the relation

$$\phi_{|W|} = \pi \circ \phi_{|V|}$$

where

$$\pi : \mathbb{P}(V) \setminus \mathbb{P}(V/W) \longrightarrow \mathbb{P}(W)$$

is the linear projection centered along the subspace $\mathbb{P}(V/W) \subseteq \mathbb{P}(V)$. Moreover if $|W|$ (whence also $|V|$) is free, then $\pi|_{\phi_{|V|}(X)}$ is finite.

1.3 Intersection numbers

Let X be a variety. Given $D_1, \ldots, D_k \in Div(X)$ and V a subvariety of X of dimension k, we want to define an integer number $(D_1 \cdot \cdots \cdot D_k \cdot V)$, called intersection number of $D_1, \ldots, D_k$ along V.
We start with a definition.

Definition 1.15. Let X be a scheme and let $\mathcal{F}$ be a coherent sheaf on X. The *Euler characteristic* of $\mathcal{F}$ is

$$\chi(\mathcal{F}) = \sum_{i=1}^{dim X} (-1)^i \cdot h^i(X, \mathcal{F}).$$

Before giving the next theorem, that will allow us to define intersection numbers in a very general setting, we recall that:

- Given a coherent sheaf $\mathcal{F}$ on a scheme X, by $Supp(\mathcal{F})$ we mean the subset of $p \in X$ such that the stalk $\mathcal{F}_p \neq 0$.
- A polynomial f with rational coefficients is *numerical* if $f(n_1, \dots, n_k)$ is an integer whenever $n_i \in \mathbb{Z}$ for every $i = 1, \dots k$.

Theorem 1.16 (Snapper)**.** *Let X be a scheme, let $\mathcal{F}$ be a coherent sheaf on X with $dim(Supp(\mathcal{F})) = s$ and let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be line bundles on X. Then the Euler characteristic $\chi(\mathcal{F} \otimes \mathcal{L}_1^{n_1} \otimes \dots \otimes \mathcal{L}_k^{n_k})$ is a numerical polynomial in $n_1, \dots, n_k$ of degree s.*

Proof. See [Kle66, I, §1]. □

Definition 1.17. Let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be line bundles on a scheme X and let $\mathcal{F}$ be a coherent sheaf on X such that $dim(Supp(\mathcal{F})) \leq k$. We define the *intersection number*

$$(\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_k \cdot \mathcal{F})$$

as the coefficient of the monomial $n_1 \cdots n_k$ in $\chi(\mathcal{F} \otimes \mathcal{L}_1^{n_1} \otimes \dots \otimes \mathcal{L}_k^{n_k})$.

Thanks to Snapper's theorem and to the properties of numerical polynomials we have that $(\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_k \cdot \mathcal{F})$ is always an integer.
Moreover we have:

Proposition 1.18. $(\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_k \cdot \mathcal{F}) = 0$ *if* $dim(Supp(\mathcal{F})) < k$.

Proof. See [Kle66, I, §2, Prop. 1]. □

Definition 1.19. Let $V \subseteq X$ be a subvariety of dimension $s \leq k$ and let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be line bundles on X. We define

$$(\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_k \cdot V) = (\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_k \cdot \mathcal{O}_V).$$

Similarly if $D_1, \dots, D_k \in Div(X)$ we define

$$(D_1 \cdot \dots \cdot D_k \cdot V) = (\mathcal{O}_X(D_1) \cdot \dots \cdot \mathcal{O}_X(D_k) \cdot V).$$

For every s-cycle $Y \in Z_s(X)$ the intersection number $(D_1 \cdot \dots \cdot D_k \cdot Y)$ is defined, in obvious way, by linearity.
Thanks to Proposition 1.18, if $Y \in Z_s(X)$ we have $(D_1 \cdot \dots \cdot D_k \cdot Y)$=0 whenever $s < k$, thus the only interesting case is when $s = k$.

Notation. Let X be an n-dimensional variety. Then we will write

- $(D_1 \cdot \dots \cdot D_n) = (D_1 \cdot \dots \cdot D_n \cdot X)$,
- $(D^n) = (\underbrace{D \cdot \dots \cdot D}_{\text{n times}})$.

Theorem 1.20 (Properties of intersection numbers)**.** *Let X be an n-dimensional variety, let $D_1, \dots, D_k \in Div(X)$ and let $Y \in Z_k(X)$. Then*

1. *$(D_1 \cdot \dots \cdot D_k \cdot Y)$ is a symmetric function of the D_i and it is multilinear with respect to divisors and k-cycles.*
2. *$(D_1 \cdot \dots \cdot D_k \cdot Y)$ depends only on the linear equivalence class of the D_i.*
3. *If $D_1, \dots, D_n$ are effective divisors meeting transversely, then*
$$(D_1 \cdot \dots \cdot D_n) = Card\{D_1 \cap \dots \cap D_n\}.$$
4. *If $f : Y \to X$ is a finite surjective morphism, then*
$$(f^*(D_1) \cdot \dots \cdot f^*(D_n))_Y = deg(f) \cdot (D_1 \cdot \dots \cdot D_n)_X.$$

Proof. See [Kle66, I, §2] and [Laz04, I, 1.1.13] □

1.4 Numerical equivalence

Definition 1.21. Let X be a scheme. Two divisors D_1, D_2 on X are *numerically equivalent* (we write $D_1 \equiv D_2$) if

$$(D_1 \cdot C) = (D_2 \cdot C)$$

for every irreducible curve $C \subseteq X$.
A divisor is said to be *numerically trivial* if it is numerically equivalent to zero. $Num(X) \subseteq Div(X)$ is the subgroup of all numerically trivial divisors.

Remark 1.22. Note that:

- We can deal with linear equivalence classes in the definition thanks to the properties of intersection numbers.

- Equivalently we could define $D_1 \equiv D_2$ if $(D_1 \cdot \gamma) = (D_2 \cdot \gamma), \quad \forall \gamma \in Z_1(X)$.
- We can define numerical equivalence classes of line bundles in the same way.

Definition 1.23. The *Neron-Severi group* of a scheme X is

$$N^1(X) = Div(X)/Num(X).$$

Theorem 1.24. *For any X scheme, $N^1(X)$ is a free abelian group of finite rank.*

Proof. See [Laz04, I, 1.1.16]. □

Lemma 1.25. *Let X be a scheme and let $D_1, \dots, D_k, D'_1, \dots, D'_k \in Div(X)$ be such that $D_i \equiv D'_i$ for all $i = 1, \dots, k$. Then*

$$(D_1 \cdot \dots \cdot D_k \cdot Y) = (D'_1 \cdot \dots \cdot D'_k \cdot Y)$$

for every k-cycle $Y \in Z_k(X)$.

Proof. See [Laz04, I, 1.1.18]. □

Thanks to this lemma we can define intersection numbers of numerical equivalence classes:

Definition 1.26. Let $Y \in Z_k(X)$ and let $\delta_1, \dots, \delta_k \in N^1(X)$ be numerical equivalence classes of divisors, the *intersection number*

$$(\delta_1 \cdot \dots \cdot \delta_k \cdot Y)$$

is the intersection number of any representatives of this classes.

Chapter 2

Ample and nef divisors

2.1 Ample divisors

Definition 2.1. Let X be a scheme. A line bundle $\mathcal{L}$ on X is *very ample* if there exists a closed immersion $i : X \hookrightarrow \mathbb{P}^r$ for some r such that

$$\mathcal{L} = i^*(\mathcal{O}(1))$$

In other words $\mathcal{L}$ is very ample if and only if the morphism $\phi_{|\mathcal{L}|} : X \to \mathbb{P}(H^0(X, \mathcal{L}))$ defined by $\mathcal{L}$ is a closed immersion.

Definition 2.2. Let X be a scheme. A line bundle $\mathcal{L}$ on X is *ample* if for any coherent sheaf $\mathcal{F}$ on X, there exists a positive integer $m(\mathcal{F}) > 0$ such that $\mathcal{F} \otimes \mathcal{L}^m$ is globally generated for any $m > m(\mathcal{F})$.
A divisor D is ample if $\mathcal{O}_X(D)$ is such.

The following theorem gives different characterizations of ampleness, using cohomology and very ample line bundles:

Theorem 2.3. *Let X be a scheme and let $\mathcal{L}$ be a line bundle on X. Then the following statements are equivalent:*

- *$\mathcal{L}$ is ample.*
- *For any coherent sheaf $\mathcal{F}$ on X, there exists a positive integer $m_1(\mathcal{F})$ such that for every $m > m_1(\mathcal{F})$ we get $H^i(X, \mathcal{F} \otimes \mathcal{L}^m) = 0$ for all $i > 0$.*
- *There exists a positive integer $m_2 > 0$ such that $\mathcal{L}^m$ is very ample for every $m > m_2$.*
- *There exists a positive integer $m_3 > 0$ such that $\mathcal{L}^{m_3}$ is very ample.*

Proof. See [Laz04, I, 1.2.6]. $\square$

Corollary 2.4. *Let X be a scheme, let D be an ample divisor on X and let $E \in Div(X)$ be an arbitrary divisor. Then there exists a positive integer $m_0 > 0$ such that $mD + E$ is very ample for all $m \geq m_0$.*

Proof. See [Laz04, I, 1.2.10]. □

The next proposition shows how ampleness is preserved when passing to reduced and irreducible components or restricting to subvarieties:

Proposition 2.5. *Let $f : Y \to X$ be a finite morphism of schemes and let $\mathcal{L}$ be a line bundle on X.*
Then

- *If $\mathcal{L}$ is ample on X, then $f^*(\mathcal{L})$ is ample on Y.*

  In particular for any Y subscheme of X, the restriction $L_{|_Y}$ is ample on Y.

- *$\mathcal{L}$ is ample on X if and only if $\mathcal{L}_{red}$ is ample on X_{red}.*

- *$\mathcal{L}$ is ample on X if and only if the restriction of $\mathcal{L}$ to each irreducible component of X is ample.*

Proof. See [Laz04, I, 1.2.13 and 1.2.16]. □

Theorem 2.6 (Nakay-Moishezon)**.** *Let X be an n-dimensional scheme and let $D \in Div(X)$.*
Then D is ample if and only if for all $k = 1, \dots, n$ we have

$$(D^k \cdot V) > 0$$

for every $V \subseteq X$ irreducible k-dimensional subvariety.

Proof. See [Laz04, I, 1.2.23]. □

Corollary 2.7. *Ampleness of divisors only depends on numerical equivalence classes.*

Proposition 2.8. *Let X be a scheme and let D and E be ample divisors on X.*
Then there is a positive integer m_0 such that the natural maps

$$H^0(X, \mathcal{O}_X(aD)) \otimes H^0(X, \mathcal{O}_X(bE)) \longrightarrow H^0(X, \mathcal{O}_X(aD + bE))$$

defined by multiplication of sections are surjective whenever $a, b \geq m_0$.
More generally, for any coherent sheaves $\mathcal{F}, \mathcal{G}$ on X, there is a positive integer $m_1(\mathcal{F})$ such that the maps

$$H^0(X, \mathcal{F} \otimes \mathcal{O}_X(aD)) \otimes H^0(X, \mathcal{G} \otimes \mathcal{O}_X(bE)) \longrightarrow$$
$$\longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{G} \otimes \mathcal{O}_X(aD + bE))$$

are surjective for all $a, b \geq m_1(\mathcal{F})$.

Proof. See [Laz04, I, 1.2.22]. □

Asymptotic Riemann-Roch

Definition 2.9. Let X be an n-dimensional variety and let $\mathcal{F}$ be a coherent sheaf on X.
The *rank* of $\mathcal{F}$ is

$$rank(\mathcal{F}) = dim_{\mathbb{C}(X)} \mathcal{F} \otimes \mathbb{C}(X).$$

Theorem 2.10 (Asymptotic Riemann-Roch)**.** *Let X be a variety of dimension n and let $D \in Div(X)$. Then*

$$\chi(\mathcal{O}_X(mD)) = \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

Moreover, for any coherent sheaf $\mathcal{F}$ on X we have

$$\chi(\mathcal{F} \otimes \mathcal{O}_X(mD)) = rank(\mathcal{F}) \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

In particular in the case of ample divisors, thanks to Theorem 2.3, we can re-formulate the theorem as follows:

Theorem 2.11 (Ample asymptotic Riemann-Roch)**.** *Let X be a variety of dimension n and let $D \in Div(X)$ be an ample divisor, then*

$$h^0(X, \mathcal{O}_X(mD)) = \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

Moreover, for any coherent sheaf $\mathcal{F}$ on X

$$h^0(X, \mathcal{F} \otimes \mathcal{O}_X(mD)) = rank(\mathcal{F}) \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

Corollary 2.12. *Let X be an n-dimensional variety and let $E \in Div(X)$. Then there exists a constant $C > 0$ such that*

$$h^0(X, \mathcal{O}_X(mE)) \leq C \cdot m^n$$

for all $m > 0$.

Castelnuovo-Mumford regularity

Definition 2.13. Let X be a variety and let $\mathcal{L}$ be an ample globally generated line bundle on X.
A coherent sheaf $\mathcal{F}$ on X is *m-regular* with respect to $\mathcal{L}$ if

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{m-i}) = 0$$

for all $i > 0$.

With this definition we have the following theorem:

Theorem 2.14 (Mumford)**.** *Let $\mathcal{L}$ be an ample globally generated line bundle on a variety X. If $\mathcal{F}$ is an m-regular sheaf with respect to $\mathcal{L}$, then $\mathcal{F} \otimes \mathcal{L}^{m+k}$ is globally generated for every $k \geq 0$.*

Proof. See [Laz04, I, 1.8.5]. □

2.2 $\mathbb{Q}$-Divisors

Definition 2.15. Let X be a scheme. We define a $\mathbb{Q}$-divisor as an element of the $\mathbb{Q}$-vector space $Div_{\mathbb{Q}}(X) = Div(X) \otimes \mathbb{Q}$.

We can represent any $\mathbb{Q}$-divisor D as a finite sum

$$D = \sum c_i D_i$$

for suitable $c_i \in \mathbb{Q}$, $D_i \in Div(X)$.
Clearing denominators we can always find $c \in \mathbb{Q}$ and $E \in Div(X)$ such that $D = cE$.

Definition 2.16. Let $D \in Div_{\mathbb{Q}}(X)$ be a $\mathbb{Q}$-divisor. Then:

- D is *integral* if $D \in Div(X) \subseteq Div_{\mathbb{Q}}(X)$.
- D is *effective* if there exist $c_i \in \mathbb{Q}^+$ and D_i effective integral divisors such that $D = \sum c_i D_i$.

We can naturally extend to $\mathbb{Q}$-divisors operations and properties defined in the integral case:

- $D_1, D_2 \in Div(X)$ are $\mathbb{Q}$-*linearly equivalent* if there is a positive integer r such that rD_1 and rD_2 are integral and linearly equivalent in $Div(X)$, that is $r(D_1 - D_2)$ is principal as a $\mathbb{Z}$-divisor.
- Given $D_1, \dots, D_k \in Div(X)$ and V irreducible k-dimensional subvariety of X, the *intersection number* $(D_1 \cdot \dots \cdot D_k \cdot V)$ is defined via $\mathbb{Q}$-multilinearity from the same operation in $Div(X)$.
- $D_1, D_2 \in Div_{\mathbb{Q}}(X)$ are numerically equivalent ($D_1 \equiv D_2$) if

  $$(D_1 \cdot C) = (D_2 \cdot C)$$

  for any irreducible curve $C \subseteq X$.
  The group of numerical equivalence classes of $\mathbb{Q}$-divisors is denoted by $N^1(X)_{\mathbb{Q}}$. In particular it is a finite-dimensional $\mathbb{Q}$-vector space.
- Let $f : Y \to X$ be a finite morphism and let $D \in Div_{\mathbb{Q}}(X)$.
  We define the pullback divisor $f^*(D) \in Div_{\mathbb{Q}}(Y)$ by $\mathbb{Q}$-linearity from the usual pullback of the integral divisors in a representation of D.

Now, we can also extend to the $\mathbb{Q}$-divisors environment the notion of ampleness:

Definition 2.17. A $\mathbb{Q}$-divisor $D \in Div_{\mathbb{Q}}(X)$ is *ample* if any of the three following equivalent statements is satisfied:

1. $D = \sum c_i A_i$, with $c_i \in \mathbb{Q}^+$, A_i ample integral divisors.

2. There exists a positive integer r such that $r \cdot D$ is ample and integral.

3. For every irreducible subvariety $V \subseteq X$ of dimension k

$$(D^k \cdot V) > 0,$$

that is D satisfies Nakay-Moishezon criterion.

As for $\mathbb{Z}$-divisors, thanks to the third characterization, we observe that ampleness is preserved by numerical equivalence, thus it makes sense to speak of ample classes in $N^1(X)_{\mathbb{Q}}$.
The next proposition shows how, in the context of $\mathbb{Q}$-divisors, ampleness is preserved by small perturbations:

Proposition 2.18. *Let X be a variety, let H be an ample $\mathbb{Q}$-divisor and E an arbitrary $\mathbb{Q}$-divisor.*
Then, for all sufficiently small rational numbers $0 < |\epsilon| << 1$, we have that $H + \epsilon E$ is ample.
More generally, if $E_1, \ldots, E_r \in Div_{\mathbb{Q}}(X)$ are arbitrary divisors,

$$H + \sum_{i=1}^{r} \epsilon_i E_i$$

is ample for all sufficiently small rational numbers $0 < |\epsilon_i| << 1$.

Proof. See [Laz04, I, 1.3.7]. □

2.3 $\mathbb{R}$-Divisors

The construction of the group $Div_{\mathbb{R}}(X)$ of $\mathbb{R}$-divisors faithfully follows that of $Div_{\mathbb{Q}}(X)$. In particular:

Definition 2.19. $Div_{\mathbb{R}}(X) = Div(X) \otimes \mathbb{R}$.

Concretely, we can represent any $\mathbb{R}$-divisor D by a finite sum $D = \sum c_i D_i$ for suitable $c_i \in \mathbb{R}$, $D_i \in Div(X)$ for all i.
D is *effective* if there exist $c_i \geq 0$ and D_i effective integral divisors such that $D = \sum c_i D_i$.
Intersection numbers and pullbacks are defined by linearity as before. Moreover, in obvious way, we can extend to $\mathbb{R}$-divisors the relation of numerical equivalence $\equiv$ and we can consider the group of numerical equivalence classes, denoted by $N^1(X)_{\mathbb{R}}$. As before, $N^1(X)_{\mathbb{R}}$ is a finite-dimensional $\mathbb{R}$-vector space. Furthermore we have the following lemma:

Lemma 2.20. *Let X be a scheme. Then, for all $D \in Div_{\mathbb{R}}(X)$, with $D \equiv 0$, there exist $r_i \in \mathbb{R}$ and D_i integral divisors, with $D_i \equiv 0$ for all $i = 1, \dots, m$, such that*

$$D = \sum_{i=1}^{m} r_i D_i.$$

In other words we have an isomorphism

$$N^1(X)_{\mathbb{R}} \simeq N^1(X) \otimes \mathbb{R}.$$

Proof. See [Laz04, I, 1.3.10 and proof of Prop. 1.3.13]. □

Definition 2.21. Let X be a scheme. An $\mathbb{R}$-divisor D is *ample* if we can represent it by

$$D = \sum c_i A_i$$

with $c_i \in \mathbb{R}^+$, A_i ample $\mathbb{Z}$-divisors for all i.

In this case it is not easy to understand if we can characterize ample $\mathbb{R}$-divisors by using a Nakay-Moishezon-type criterion. In particular if $D \in Div_{\mathbb{R}}(X)$ is ample, then $(D^{dim V} \cdot V) > 0$ for every $V \subseteq X$ irreducible subvariety, but the opposite implication is not obvious because D is not, in general, proportional to an integral divisor. However a theorem of Campana and Peternell shows that, even for $\mathbb{R}$-divisors, positivity of intersection with every irreducible subvariety of X gives ampleness (see [Laz04, I, 2.3.18]).

Proposition 2.22. *Let $D_1, D_2 \in Div_{\mathbb{R}}(X)$ such that $D_1 \equiv D_2$.*
Then D_1 is ample if and only if D_2 is such.

Proof. See [Laz04, I, 1.3.13]. □

Also for $\mathbb{R}$-divisors ampleness is an open condition, that is it is not affected by small perturbations:

Proposition 2.23. *Let X be a variety, let $H \in Div_{\mathbb{R}}(X)$ be ample and let $E_1, \dots, E_r$ be arbitrary $\mathbb{R}$-divisors.*
Then for all sufficiently small real numbers $0 < |\epsilon_i| << 1$

$$H + \sum_{i=1}^{r} \epsilon_i E_i$$

is ample.

Proof. See [Laz04, I, 1.3.14]. □

Corollary 2.24. *The finite-dimensional vector space $N^1(X)_{\mathbb{R}}$ is spanned by the classes of ample $\mathbb{R}$-divisors.*

Proof. See [Laz04, I, 1.3.15] □

2.4 Nef divisors

Definition 2.25. Let X be scheme. A divisor $D \in Div(X)$, $Div_{\mathbb{Q}}(X)$ or $Div_{\mathbb{R}}(X)$ is *nef* if

$$(D \cdot C) \geq 0$$

for every irreducible curve $C \subseteq X$.
A line bundle $\mathcal{L}$ is nef if there is a nef $\mathbb{Z}$-divisor D such that $\mathcal{L} = \mathcal{O}_X(D)$.

As this definition only depends on numerical equivalence classes we can speak of nefness of classes in $N^1(X)$, $N^1(X)_{\mathbb{Q}}$ and $N^1(X)_{\mathbb{R}}$.

Proposition 2.26 (Properties of nefness)**.** *Let X be a scheme, let $\mathcal{L}$ be a line bundle on X and let $f : Y \to X$ be a morphism.*
Then

- *If $\mathcal{L}$ is nef on X, then $f^*(\mathcal{L})$ is nef on Y.*

  In particular for any Y subscheme of X, the restriction $L_{|_Y}$ is nef on Y.

- *If f is surjective and $f^*(\mathcal{L})$ is nef, then $\mathcal{L}$ is nef.*

- *$\mathcal{L}$ is nef if and only if $\mathcal{L}_{red}$ is nef on X_{red}.*

- *$\mathcal{L}$ is nef if and only if the restriction of $\mathcal{L}$ to each irreducible component of X is nef.*

Proof. See [Laz04, I, 1.4.4]. □

Theorem 2.27 (Kleiman)**.** *Let X be a scheme and let D be a nef divisor. Then*

$$(D^k \cdot V) \geq 0$$

for every irreducible subvariety V of X of dimension k.

Proof. See [Laz04, I, 1.4.9]. □

Theorem 2.28 (Fujita's vanishing theorem)**.** *Let X be a scheme and let $H \in Div(X)$ be an ample divisor.*
For any coherent sheaf $\mathcal{F}$ on X, there exists an integer $m(\mathcal{F}, H)$ such that

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for all $i > 0$, $m \geq m(\mathcal{F}, H)$ and for every nef integral divisor D on X.

The essential point is that the integer $m(\mathcal{F}, H)$ does not depend on the nef divisor D.

Proof. See [Laz04, I, 1.4.35]. □

Proposition 2.29. *Let X be a variety of dimension n and let D be a nef divisor on X. Then*

$$h^i(X, \mathcal{O}_X(mD)) = O(m^{n-i}).$$

Proof. See [Laz04, 1.4.40]. □

Corollary 2.30 (Nef asymptotic Riemann-Roch)**.** *Let X be a variety of dimension n and let D be a nef divisor on X. Then*

$$h^0(X, \mathcal{O}_X(mD)) = \frac{(D^n)}{n!}m^n + O(m^{n-1}).$$

2.5 Ample and nef cones

Definition 2.31. Let V be a finite-dimensional real vector space. A subset $K \subseteq V$ is a *cone* if positive linear combinations of vectors of K are still in K.

Denoting by $Amp(X)$ the set of all ample $\mathbb{R}$-divisor classes in $N^1(X)_{\mathbb{R}}$, and by $Nef(X)$ the set of all nef $\mathbb{R}$-divisor classes, we observe that $Nef(X)$ and $Amp(X)$ are cones.
The following theorem shows how these two cones are closely related.

Theorem 2.32. *Let X be a scheme, then*

1. $Nef(X) = \overline{Amp(X)}$.
2. $Amp(X) = Int(Nef(X))$.

Proof. See [Laz04, I, 1.4.23]. □

Definition 2.33. Let X be a scheme, we define the $\mathbb{R}$-vector space of real 1-cycles on X by

$$Z_1(X)_{\mathbb{R}} = Z_1(X) \otimes \mathbb{R}.$$

For any $\gamma \in Z_1(X)_{\mathbb{R}}$, we can write γ as a finite linear combination $\gamma = \sum a_i \cdot C_i$, with $a_i \in \mathbb{R}$ and C_i irreducible curves on X.
We say that two one-cycles γ_1 and γ_2 are *numerically equivalent* if

$$(D \cdot \gamma_1) = (D \cdot \gamma_2)$$

for any $D \in Div_{\mathbb{R}}(X)$.

Definition 2.34. $N_1(X)_{\mathbb{R}}$ is the real vector space of numerical equivalence classes of real one cycles.

By construction there is a perfect pairing

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$
$$(\delta, \gamma) \longmapsto (\delta \cdot \gamma).$$

In particular $N_1(X)_{\mathbb{R}}$ is a finite dimensional real vector space on which we put the Euclidean topology.

Definition 2.35. Let X be a scheme. We define the *cone of curves* $NE(X) \subseteq N_1(X)_{\mathbb{R}}$ as the cone spanned by the classes of effective one cycles on X. Its closure $\overline{NE}(X)$ is the closed cone of curves in X.

In other words each element $\gamma \in NE(X)$ is of the form

$$\gamma = \sum a_i[C_i]$$

with $a_i \in \mathbb{R}^+ \cup \{0\}$ and C_i irreducible curves.

Proposition 2.36. *For any variety X we have*

$$\overline{NE}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} \quad | \quad (\delta \cdot \gamma) \geq 0 \quad \forall\, \delta \in Nef(X)\}.$$

We can express this property saying that $\overline{NE}(X)$ is the closed cone dual to $Nef(X)$.

Proof. See [Laz04, I, 1.4.28]. □

Now for any $\mathbb{R}$-divisor $D \in Div_{\mathbb{R}}(X)$, not numerically trivial, we can define a linear map

$$\phi_D : N_1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$
$$\gamma \longmapsto (D \cdot \gamma).$$

We put

$$D^{\perp} = Ker(\phi_D) = \{\gamma \in N_1(X)_{\mathbb{R}} \quad | \quad (D \cdot \gamma) = 0\}.$$
$$D_{>0} = \{\gamma \in N_1(X)_{\mathbb{R}} \quad | \quad (D \cdot \gamma) > 0\}.$$

$D_{\geq 0}$ is the disjoint union of $D^{\perp}$ and $D_{>0}$.
$D_{<0}$ and $D_{\leq 0}$ are defined similarly.
Now, after observing that, as $N_1(X)_{\mathbb{R}} \cong \mathbb{R}^n$ as a vector space, any two norms on it are equivalent, we fix a norm $|| \cdot ||$ and we put

$$S = \{\gamma \in N_1(X)_{\mathbb{R}} \quad | \quad ||\gamma|| = 1\}.$$

With this notation we have the following theorem:

Theorem 2.37 (Kleiman)**.** *Let X be a variety and let $D \in Div_{\mathbb{R}}(X)$. Then*

$$D \text{ is ample} \Longleftrightarrow \overline{NE}(X) \setminus \{0\} \subseteq D_{>0}.$$

Equivalently

$$D \text{ is ample} \Longleftrightarrow \overline{NE}(X) \cap S \subseteq D_{>0} \cap S.$$

Proof. See [Laz04, I, 1.4.29]. □

Chapter 3

Semiample and big line bundles

3.1 Iitaka dimension and stable base locus

Definition 3.1. Let X be a variety and let $\mathcal{L}$ be a line bundle on X. The *semigroup* of $\mathcal{L}$ is

$$\mathbb{N}(\mathcal{L}) = \mathbb{N}(X, \mathcal{L}) = \{m \in \mathbb{N} \mid H^0(X, \mathcal{L}^m) \neq 0\} \cup \{0\}.$$

The semigroup $\mathbb{N}(D)$ of a divisor D is the semigroup of $\mathcal{O}_X(D)$.

$\mathbb{N}(\mathcal{L})$ is actually a semigroup because if $0 \neq s \in H^0(X, \mathcal{L}^m)$, and $0 \neq t \in H^0(X, \mathcal{L}^n)$, then $H^0(X, \mathcal{L}^{m+n})$ contains the section $s \cdot t \neq 0$ obtained by multiplication.

Definition 3.2. Assuming $\mathbb{N}(\mathcal{L}) \neq 0$, the *exponent* of $\mathcal{L}$, denoted by e or $e(\mathcal{L})$ is the g.c.d. of all the elements of $\mathbb{N}(\mathcal{L})$.
The exponent $e(D)$ of $D \in Div(X)$ is defined again by passing to the line bundle $\mathcal{O}_X(D)$.

Obviously all elements in $\mathbb{N}(\mathcal{L})$ are multiples of $e(\mathcal{L})$, while, on the other hand, all sufficiently large multiples of $e(\mathcal{L})$ appear in the semigroup of $\mathcal{L}$.

Notation. Let X be a variety, let $\mathcal{L}$ be a line bundle on X and let $m \in \mathbb{N}(\mathcal{L})$. Then we can view the morphism associated to the complete linear series $|\mathcal{L}^m|$, that, from now on, we denote by

$$\phi_m : X \setminus Bs(|\mathcal{L}^m|) \longrightarrow \mathbb{P}(H^0(X, \mathcal{L}^m))$$

as a rational mapping

$$\phi_m : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{L}^m)).$$

Moreover, for the rest of this chapter, we denote by Y_m or $\phi_m(X)$ the closure of the image of ϕ_m.

Definition 3.3. Let X be a variety and let $\mathcal{L}$ be a line bundle on X. We define the *Iitaka dimension* of $\mathcal{L}$, denoted by $k(X, \mathcal{L})$, as follows:

If X is normal

- If $\mathbb{N}(\mathcal{L}) \neq 0$

$$k(X, \mathcal{L}) = \max_{m \in \mathbb{N}(\mathcal{L})} \{dim\ \phi_m(X)\}.$$

- If $\mathbb{N}(\mathcal{L}) = 0$ (that is $H^0(X, \mathcal{L}^m) = 0$ for all $m > 0$)

$$k(X, \mathcal{L}) = -\infty.$$

If X is not normal, we consider $\nu : X' \to X$ to be the normalization of X and set

$$k(X, \mathcal{L}) = k(X', \nu^*(\mathcal{L})).$$

Finally, we can define the Iitaka dimension of a divisor $D \in Div(X)$ as

$$k(X, D) = k(X, \mathcal{O}_X(D)).$$

Proposition 3.4. *Let X be a normal variety. If $k(X, \mathcal{L}) = k \in \mathbb{N}$, then there exists a positive integer m_0 such that $dim\ \phi_m(X) = k$, $\forall m \geq m_0$ in $\mathbb{N}(\mathcal{L})$.*

Proof. As all sufficiently large $m \in \mathbb{N}(\mathcal{L})$ are multiples of the exponent $e(\mathcal{L})$, we can replace $\mathcal{L}$ by $\mathcal{L}^{e(\mathcal{L})}$, so that we have $e(\mathcal{L}) = 1$.
Then there exists a positive integer p_0 such that $H^0(X, \mathcal{L}^p) \neq 0$ for all $p \geq p_0$. Now we fix $h \in \mathbb{N}(\mathcal{L})$ such that $dim\ \phi_h(X) = k$ and we consider for every $p \geq p_0$ the embedding

$$H^0(X, \mathcal{L}^h) \subseteq H^0(X, \mathcal{L}^{h+p})$$

determined by a non-zero section in $H^0(X, \mathcal{L}^p)$.
This in turn gives rise to a factorization $\phi_h = \nu_p \circ \phi_{h+p}$, where ν_p is the rational mapping arising from the linear projection

$$\mathbb{P}H^0(X, \mathcal{L}^{h+p}) \setminus \mathbb{P}(\frac{H^0(X, \mathcal{L}^{h+p})}{H^0(X, \mathcal{L}^h)}) \longrightarrow \mathbb{P}H^0(X, \mathcal{L}^h)$$

associated with the above embedding itself.
Therefore we can find an open subset $U \subseteq Y_{h+p}$ such that

$$\nu_p^* = \nu_{p_{|U}} : U \longrightarrow Y_h$$

is a dominant morphism.
Then $dimY_{h+p} \geq dimY_h = k$ and the reverse inequality holds by definition.

$\square$

Definition 3.5. Let X be a variety and let D be an integral divisor on X. The *stable base locus* of D is the algebraic set

$$\mathbb{B}(D) = \bigcap_{m \geq 1} Bs(|mD|)$$

where we disregard the structures of schemes and consider the intersection just like an intersection of closed subsets.

Proposition 3.6. *Let X be a variety and let D be an integral divisor on X. Then:*

1. *The stable base locus $\mathbb{B}(D)$ is the unique minimal element of the family of closed sets $\{Bs(|mD|)\}_{m>0}$.*
2. *There exists a positive integer n_0 such that, for all $k \in \mathbb{N}$, we have*

$$\mathbb{B}(D) = Bs(|kn_0D|).$$

Proof.
1. We have that X (if only considered as a topological space) is a noetherian space, thus every non-empty family of closed subsets in X has a minimal element. Then in particular the same holds for the family $\{Bs(|mD|)\}_{m>0}$. On the other hand, given any natural numbers m, l, there is a set-theoretic inclusion

$$Bs(|lmD|) \subseteq Bs(|mD|)$$

arising from the reverse inclusion $b(|mD|)^l \subseteq b(|lmD|)$ on base ideals (Proposition 1.10). Then the minimal element is unique, because if $Bs(|pD|)$ and $Bs(|qD|)$ are each minimal they both coincide with $Bs(|pqD|)$; therefore, by definition, it must coincide with $\mathbb{B}(D)$.
2. Let $m_0 \in \mathbb{N}$ be such that $Bs(|m_0D|) = \mathbb{B}(D)$. Then for all $k \in \mathbb{N}$ we get

$$Bs(|km_0D|) \subseteq Bs(|m_0D|) = \mathbb{B}(D)$$

and the reverse inclusion follows by definition.

$\square$

Corollary 3.7. *Let X be a variety and let $D \in Div(X)$. Then*

$$\mathbb{B}(pD) = \mathbb{B}(D)$$

for all $p \in \mathbb{N}$.

Proof. By the second statement of Proposition 3.6 there exists a positive integer m_0 such that $\mathbb{B}(D) = Bs(|km_0D|)$ for all $k \in \mathbb{N}$. Analogously we can find $m_1 \in \mathbb{N}$ such that $\mathbb{B}(pD) = Bs(|k'm_1pD|)$ for all $k' \in \mathbb{N}$. In particular
$$\mathbb{B}(pD) = Bs(|m_0m_1pD|) = \mathbb{B}(D).$$
□

The previous corollary suggests a natural definition of stable base locus of a $\mathbb{Q}$-divisor:

Definition 3.8. Let D be a $\mathbb{Q}$-divisor on a variety X. We define $\mathbb{B}(D) = \mathbb{B}(kD)$, where $k \in \mathbb{N}$ is such that kD is integral.

3.2 Semiample line bundles

Definition 3.9. Let X be a scheme. A line bundle $\mathcal{L}$ on X is *semiample* if $\mathcal{L}^m$ is globally generated for some $m \in \mathbb{N}$.
A divisor D is semiample if $\mathcal{O}_X(D)$ is such.

Definition 3.10. Let $\mathcal{L}$ be a semiample line bundle on a scheme X, we denote by $M(X, \mathcal{L}) = M(\mathcal{L}) \subseteq \mathbb{N}(\mathcal{L})$ the sub-semigroup
$$M(\mathcal{L}) = \{m \in \mathbb{N}(\mathcal{L}) \mid \mathcal{L}^m \text{ is globally generated}\}.$$
The *exponent* of $M(\mathcal{L})$, denoted by f or $f(\mathcal{L})$, is the g.c.d. of all the elements of $M(\mathcal{L})$.

Note that, for any line bundle $\mathcal{L}$, we conventionally put $\mathcal{L}^0 = \mathcal{O}_X$, whence we always have $0 \in M(\mathcal{L})$.
Besides, as for the exponent $e(\mathcal{L})$, we see that every sufficiently large multiple of f appear in $M(\mathcal{L})$, that is $\mathcal{L}^{kf}$ is free for every $k >> 0$.

Definition 3.11. Let X and Y be varieties. An *algebraic fibre space* is a surjective morphism $f : X \to Y$ such that $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Remark 3.12. Note that, given any surjective morphism of varieties $g : V \to W$, the Stein factorization expresses g as a composition
$$V \xrightarrow{a} Z \xrightarrow{b} W$$
where a is an algebraic fibre space and b is finite (see [Har77, III, 11.5]). Thus g itself is an algebraic fibre space if and only if the finite part of its Stein factorization is an isomorphism.

In particular, all the fibres of an algebraic fibre space f are connected. Conversely, if Y is normal, then any surjective morphism $f : X \to Y$ with connected fibres is an algebraic fibre space.
Moreover if X is normal, $f : X \to Y$ is an algebraic fibre space and $\mu : X' \to X$ is a birational mapping, then the composition $f \circ \mu : X' \to Y$ is again an algebraic fibre space (see [Har77, III, 11.3, 11.4]).

The following lemma gives a very useful property of algebraic fibre spaces:

Lemma 3.13. *Let X and Y be varieties, let $f : X \to Y$ be an algebraic fibre space and $\mathcal{L}$ a line bundle on Y.*
Then for every $m > 0$ we have

$$H^0(X, f^*\mathcal{L}^m) = H^0(Y, \mathcal{L}^m).$$

In particular $k(Y, \mathcal{L}) = k(X, f^*\mathcal{L})$.

Proof. In general we have

$$H^0(X, f^*\mathcal{L}^m) = H^0(Y, f_*(f^*\mathcal{L}^m)).$$

But, thanks to the projection formula

$$f_*(f^*\mathcal{L}^m) = f_*\mathcal{O}_X \otimes \mathcal{L}^m$$

and $f_*\mathcal{O}_X = \mathcal{O}_Y$ since f is a fibre space.

□

Corollary 3.14. *Let X and Y be varieties and $f : X \to Y$ an algebraic fibre space. Then the induced homomorphism*

$$f^* : Pic(Y) \longrightarrow Pic(X)$$

is injective.

Proof. Let $\mathcal{L}$ be a line bundle on Y such that $f^*\mathcal{L} \simeq \mathcal{O}_X$. Then, thanks to Lemma 3.13

$$H^0(Y, \mathcal{L}) = H^0(X, f^*\mathcal{L}) = H^0(X, \mathcal{O}_X)$$

and similarly

$$H^0(Y, \mathcal{L}^{-1}) = H^0(X, f^*(\mathcal{L}^{-1})) = H^0(X, (f^*\mathcal{L})^{-1}) = H^0(X, \mathcal{O}_X).$$

Therefore $H^0(Y, \mathcal{L}) \neq 0$ and $H^0(Y, \mathcal{L}^{-1}) \neq 0$, thus $\mathcal{L} \simeq \mathcal{O}_X$. □

Proposition 3.15. *Let $f : X \to Y$ be an algebraic fibre space.*
If X is normal, then Y is such.

Proof. Let $\nu : Y' \to Y$ be the normalization of Y. Then, as ν is a finite morphism and the algebraic fibre space f factors through ν, by Remark 3.12 we must have that ν is an isomorphism. $\square$

From now on, given a line bundle $\mathcal{L}$ on a variety X, adopting a slight abuse of notation, we consider ϕ_m as a dominant rational map

$$\phi_m : X \dashrightarrow Y_m = \overline{\phi_m(X)}.$$

In particular, if $\mathcal{L}$ is semiample, for $m \in M(\mathcal{L})$, ve can view ϕ_m as a surjective morphism

$$\phi_m : X \longrightarrow Y_m.$$

The following theorem shows how, in this case, ϕ_m stabilizes to a constant algebraic fibre space ϕ for sufficiently large $m \in M(\mathcal{L})$.

Theorem 3.16 (Semiample fibrations)**.** *Let X be a normal variety and let $\mathcal{L}$ be a semiample line bundle on X.*
Then there exists a variety Y and an algebraic fibre space $\phi : X \to Y$ such that for any sufficiently large $m \in M(\mathcal{L})$

$$Y_m = Y \qquad \textit{and} \qquad \phi_m = \phi.$$

Furthermore there is an ample line bundle $\mathcal{A}$ on Y such that

$$\phi^*(\mathcal{A}) = \mathcal{L}^{f(\mathcal{L})},$$

where $f(\mathcal{L})$ is the exponent of $\mathcal{L}$.

Proof. We prove the theorem in four steps.

1. *Preliminary observations.*
Let $m > 0$ such that $\mathcal{L}^m$ is globally generated and let k be a positive integer. Then we can consider the two morphisms

$$\phi_m : X \longrightarrow Y_m \subseteq \mathbb{P}H^0(X, \mathcal{L}^m) = \mathbb{P}^r$$

$$\phi_{km} : X \longrightarrow Y_{km} \subseteq \mathbb{P}H^0(X, \mathcal{L}^{km}).$$

Moreover we have a map

$$\nu_k : Y_m \longrightarrow \nu_k(Y_m) \subseteq \mathbb{P}H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k))$$

obtained by restricting the morphism on $\mathbb{P}^r$ determined by the free linear series $|\mathcal{O}_{\mathbb{P}^r}(k)|$, that is $\nu_k(Y_m)$ is the k^{th} Veronese re-embedding of Y_m.
Now, denoting by $S^k H^0(X, \mathcal{L}^m)$ the k^{th} symmetric power of $H^0(X, \mathcal{L}^m)$, we have a map

$$S^k H^0(X, \mathcal{L}^m) \longrightarrow V_k \subseteq H^0(X, \mathcal{L}^{km})$$

$$s_1 \otimes \cdots \otimes s_k \longmapsto s_1 \cdot \cdots \cdot s_k$$

where V_k is simply defined as the image of the map.
In particular, using the definition of Veronese embedding, one can easily check that $\nu_k(Y_m) \simeq \phi_{|V_k|}(X)$.
Hence we have the following situation

$$\begin{tikzcd}
X \arrow[r, "\phi_m"] \arrow[rrrd, "\phi_{km}"'] & Y_m \arrow[r, "\sim"] & \nu_k(Y_m) \arrow[r, "\sim"] & \phi_{|V_k|}(X) \arrow[r, hook] & \mathbb{P}(V_k) \\
& & & Y_{km} \arrow[u, "\pi_{V_k|_{Y_{km}}}"] \arrow[r, hook] & \mathbb{P}H^0(X, \mathcal{L}^{mk}) \arrow[u, dashed, "\pi_{V_k}"']
\end{tikzcd}$$

where, using Remark 1.14, the projection π_{V_k} follows by the inclusion $V_k \subseteq H^0(X, \mathcal{L}^{mk})$ and $\pi_{V_k|_{Y_{km}}}$ is actually a morphism and makes the diagram commute. Moreover, again by Remark 1.14, it is finite and, looking at it as a morphism on Y_m, we find a factorization of ϕ_m given by

$$\phi_m = \pi_k \circ \phi_{km}$$

where we define $\pi_k = \pi_{V_k|_{Y_{km}}} : Y_{km} \to Y_m$.
Note also that Y_m carries a very ample line bundle $\mathcal{A}_m$, given by the restriction of the hyperplane bundle on $\mathbb{P}^r = \mathbb{P}H^0(X, \mathcal{L}^m)$, such that $\phi_m^*(\mathcal{A}_m) = \mathcal{L}^m$ and $H^0(X, \mathcal{L}^m) = H^0(Y_m, \mathcal{A}_m)$.

2. Lemma. *Let $m \in M(X, \mathcal{L})$, then for all sufficiently large integers $k > 0$, the composition*

$$X \xrightarrow{\phi_{km}} Y_{km} \xrightarrow{\pi_k} Y_m$$

gives the Stein factorization of ϕ_m, so that ϕ_{km} is an algebraic fibre space. In particular, Y_{km} and ϕ_{km} are independent of k for sufficiently large $k \in \mathbb{N}$.
To prove the lemma let

$$X \xrightarrow{\psi} V \xrightarrow{\mu} Y_m$$

be the Stein factorization of ϕ_m, so that, in particular, ψ is a fibre space, V is normal (see Proposition 3.15) and μ is finite. Moreover we have just proved the existence of a very ample line bundle $\mathcal{A}_m$ on Y_m that pulls back to $\mathcal{L}^m$ on X. Since μ is finite, thanks to Proposition 2.5, we have that $\mathcal{B} = \mu^*(\mathcal{A}_m)$ is an ample line bundle on V. Thus $\mathcal{B}^k$ is very ample for all sufficiently large $k \in \mathbb{N}$.
On the other hand

$$\psi^*(\mathcal{B}^k) = \mathcal{L}^{km}$$

and, being ψ an algebraic fibre space, thanks to Lemma 3.13, we find

$$\mathbb{P}H^0(X, \mathcal{L}^{km}) = \mathbb{P}H^0(V, \mathcal{B}^k) \overset{def}{=} \mathbb{P}.$$

Hence, we have that

$$\phi_{km}^*(\mathcal{O}_{\mathbb{P}}(1)) = \mathcal{L}^{km} = \psi^*(\mathcal{B}^k) = (i \circ \psi)^*(\mathcal{O}_{\mathbb{P}}(1))$$

where $i : V \hookrightarrow \mathbb{P}H^0(V, \mathcal{B}^k)$ is the embedding defined by $\mathcal{B}^k$.
Then we obtain $\phi_{km} = i \circ \psi$; therefore V is the image of X under the morphism ϕ_{km}, that is $Y_{km} = V$ and $\phi_{km} = \psi$. This proves the lemma.

Replacing, if necessary, $\mathcal{L}$ by $\mathcal{L}^{f(\mathcal{L})}$, we assume from now on in the proof of the theorem that $f(\mathcal{L}) = 1$, so that every sufficiently large multiple of $\mathcal{L}$ is free.

3. *We define ϕ and produce an ample line bundle $\mathcal{A}$ that pulls back to $\mathcal{L}$.*
Thanks to the Lemma we can fix two relatively prime positive integers p and q such that for all $k \geq 1$ we have $Y_{kp} = Y_p$, $\phi_{kp} = \phi_p$, $Y_{kq} = Y_q$, $\phi_{kq} = \phi_q$. Thus in particular

$$Y_p = Y_{pq} = Y_q \stackrel{def}{=} Y \quad \text{and} \quad \phi_p = \phi_{pq} = \phi_q \stackrel{def}{=} \phi : X \longrightarrow Y.$$

Now, thanks to the first step of the proof, Y carries very ample line bundles $\mathcal{A}_p$ and $\mathcal{A}_q$ such that $\phi^*\mathcal{A}_p = \mathcal{L}^p$ and $\phi^*\mathcal{A}_q = \mathcal{L}^q$. But, since p and q are relatively prime, we can find $r, s \in \mathbb{Z}$ such that $1 = rp + sq$.
Then we define

$$\mathcal{A} = \mathcal{A}_p^r \otimes \mathcal{A}_q^s.$$

We easily see that $\phi^*\mathcal{A} = \mathcal{L}$. Furthermore, as ϕ is a fibre space, then $\phi^* : Pic(Y) \to Pic(X)$ is injective by Corollary 3.14, whence $\mathcal{A}_p = \mathcal{A}^p$ and $\mathcal{A}_q = \mathcal{A}^q$. Therefore $\mathcal{A}$ is ample because it has a very ample positive multiple.

4. *We show that $Y_m = Y$ and $\phi_m = \phi$ for all $m >> 0$.*
Fix positive integers $c, d \geq 1$. Then the product $S^cH^0(Y, \mathcal{A}^p) \otimes S^dH^0(Y, \mathcal{A}^q)$ determines a free linear subseries of

$$H^0(Y, \mathcal{A}^{cp+dq}) = H^0(X, \mathcal{L}^{cp+dq}).$$

Arguing as in the first step of the proof we get that ϕ factors as the composition of ϕ_{cp+dq} with a finite map. Thus, being ϕ a fibre space, it follows that $\phi = \phi_{cp+dq}$. But, as any sufficiently large integer m is of the form $cp + dq$ for suitable $c, d \in \mathbb{N}$, we obtain the assert.

□

We have a birational analogue of the last theorem considering all $m \in \mathbb{N}(\mathcal{L})$:

Theorem 3.17 (Iitaka fibrations)**.** *Let $\mathcal{L}$ be a line bundle on a normal variety X with $k(X, \mathcal{L}) > 0$.*
Then there exists an algebraic fibre space of normal varieties

$$\phi_\infty : X_\infty \longrightarrow Y_\infty$$

such that $\phi_k : X \dashrightarrow Y_k$ is birationally equivalent to ϕ_∞ for all sufficiently large $k \in \mathbb{N}(\mathcal{L})$ and the restriction of $\mathcal{L}$ to a very general fibre of ϕ_∞ has Iitaka dimension zero.

Corollary 3.18. *Let $\mathcal{L}$ be a line bundle on a normal variety X and set $k = k(X, \mathcal{L})$.*
Then there are two constants $c, C > 0$ such that for all sufficiently large $m \in \mathbb{N}(\mathcal{L})$

$$c \cdot m^k \ \leq \ h^0(X, \mathcal{L}^m) \ \leq \ C \cdot m^k.$$

3.3 Big line bundles and divisors

Definition 3.19. Let X be a variety. A line bundle $\mathcal{L}$ on X is *big* if $k(X, \mathcal{L}) = dim X$. A divisor $D \in Div(X)$ is big if $\mathcal{O}_X(D)$ is such.

The following very useful theorem gives different characterizations of big divisors.

Theorem 3.20. *Let X be a variety and let $D \in Div(X)$.*
Then the following conditions are equivalent:

1. *D is big.*
2. *For **any** ample divisor $A \in Div(X)$ there exists a positive integer m and an effective divisor N such that*
$$mD \sim A + N.$$
3. *There **exists** an ample divisor $A \in Div(X)$, a positive integer m and an effective divisor N such that*
$$mD \equiv A + N.$$

In particular, since the third characterization of the theorem is numerical, we see that bigness of a divisor only depends on its numerical equivalence class.

Proof. **(1 $\Rightarrow$ 2)** Let H be a very ample divisor on X and let $k \in \mathbb{N}$ be such that $dim\ Y_k = n$. Then, in particular, $\phi_k(H)$ is strictly contained in $Y_k \subseteq \mathbb{P}^N$ for a suitable $N \in \mathbb{N}$. Hence there exists a hypersurface of $\mathbb{P}^N$ containing $\phi_k(H)$ and not containing Y_k. Namely there exists a positive integer α such that $|\alpha k D - H| \neq \emptyset$. Thus for every very ample divisor H on X there exists an effective divisor E and a positive integer $m = \alpha k$, depending on H, such that

$$mD \sim H + E.$$

Now, if A is ample, we can choose an integer $r_0 > 0$ such that rA is very ample for any $r \geq r_0$. Then, in particular, for a suitable $m \in \mathbb{N}$, there exists an effective divisor E such that

$$mD \sim (r+1)A + E \sim A + (rA + E).$$

But, since rA is very ample it is effective. Therefore the assert follows.

(2 $\Rightarrow$ 3) Trivial.

(3 $\Rightarrow$ 1) $mD \equiv A + N \Rightarrow mD - N \equiv A$, with A ample. Then $mD - N$ is ample itself, because ampleness only depends by numerical equivalence classes (see Corollary 2.7). After possibly passing to an even larger multiple of D we can assume that

$$mD \sim H + N'$$

with H very ample and N' effective. Then

$$k(X, D) \geq k(X, mD) = k(X, H + N') \geq k(X, H) = dim\ X.$$

Therefore D is big.

□

Corollary 3.21. *If D is big, then the exponent $e(D) = 1$.*

In other words, if D is big, then every sufficiently large multiple mD is linearly equivalent to an effective divisor.

Proof. Thanks to Corollary 2.4 we can choose a very ample divisor H on X such that $H - D$ is very ample, so that $H - D \sim H_1$ is effective. By Theorem 3.20 there is an integer $m > 0$ and an effective divisor N such that

$$mD \sim H + N \sim N'$$

with N' effective. On the other hand

$$(m-1)D \sim (H - D) + N \sim H_1 + N$$

with $H_1 + N$ effective.
In other words the two consecutive integers $m - 1$ and m both lie in the semigroup $\mathbb{N}(D)$. Then

$$e(D) = \text{g.c.d}\ \{n > 0 \mid n \in \mathbb{N}(D)\} = 1.$$

□

Now, using Theorem 3.20 and Corollary 3.21 we can characterize bigness of D through birationality of the map defined by a multiple of D, and studying the asymptotic behaviour of the rings of global sections of mD:

Proposition 3.22. *Let X be an n-dimensional variety, let D be a divisor on X and let $\mathcal{L} = \mathcal{O}_X(D)$. Then the following statements are equivalent:*

1. *D is big.*
2. *The rational map ϕ_m defined by $\mathcal{L}^m$ is birational onto its image for some $m > 0$.*
3. *There exists a constant $C > 0$ such that for all sufficiently large $m \in \mathbb{N}$ we have*
$$h^0(X, \mathcal{O}_X(mD)) \geq C \cdot m^n.$$

Proof.

(1 ⇒ 2)
Let H be a very ample divisor. By Theorem 3.20, since D is big, there exists an integer $m > 0$ and $E \in Div(X)$ effective such that

$$mD \sim H + E.$$

Hence ϕ_m is a birational mapping because it gives an inclusion on $X \setminus Supp(E)$.

(2 ⇒ 1) Trivial.

(1 ⇒ 3) As above we choose an integer $m_0 > 0$ such that $m_0 D \sim A + E$, with A ample and E effective.
Then we can fix a positive integer $k_0(X, D)$ such that for all $k \geq k_0(X, D)$ we have
$$h^0(X, \mathcal{O}_X(km_0D)) \geq h^0(X, \mathcal{O}_X(kA)) \geq C'k^n$$
for a suitable positive constant C' only depending on X and D.
Moreover, by Corollary 3.21, there exists a positive integer $m_1(X, D)$ such that for all $m \geq m_1(X, D)$, we have

$$h^0(X, \mathcal{O}_X(mD)) \neq 0.$$

Hence there exists an integer $s_0 > 0$ such that for all $s > s_0$ we can write

$$s = qm_0 + r = (q - a_0)m_0 + a_0m_0 + r$$

with

- $0 \leq r < m_0$,
- $a_0m_0 + r \geq m_1(X, D)$,
- $q - a_0 \geq k_0(X, D)$.

Therefore

$$h^0(X, \mathcal{O}_X(sD)) \geq h^0(X, \mathcal{O}_X((q - a_0)m_0D)) \geq C'(q - a_0)^n \geq Cs^n$$

for a suitable positive constant $C > 0$.

(3 $\Rightarrow$ 1) Let D satisfy the hypothesis and let F be an effective divisor on X. Then for all sufficiently large integers m we have

$$h^0(X, \mathcal{O}_X(mD - F)) \neq 0 :$$

In fact consider the exact sequence

$$0 \to \mathcal{O}_X(mD - F) \to \mathcal{O}_X(mD) \to \mathcal{O}_F(mD) \to 0.$$

By hypothesis there exists a positive constant C such that $h^0(X, \mathcal{O}_X(mD)) \geq C \cdot m^n$ for any sufficiently large $m \in \mathbb{N}$.
On the other hand, since F, considered with its structure of closed subscheme, has dimension $n - 1$, then, thanks to Corollary 2.12, we have that $h^0(F, \mathcal{O}_F(mD)) = O(m^{n-1})$. Therefore

$$h^0(X, \mathcal{O}_X(mD)) > h^0(F, \mathcal{O}_F(mD)),$$

and, using to the exact sequence above, we find that $h^0(X, \mathcal{O}_X(mD-F) \neq 0$.
In particular, choosing F very ample, there exists an effective divisor E such that

$$mD \sim F + E.$$

Therefore D is big by Theorem 3.20. $\square$

Now, in order to extend the notion of bigness to $\mathbb{Q}$-divisors, we note that a $\mathbb{Z}$-divisor $D \in Div(X)$ is big if and only if kD is big for some integer $k > 0$. Thus we introduce the following definition:

Definition 3.23. A $\mathbb{Q}$-divisor D on a variety X is *big* if there exists an integer $n > 0$ such that nD is integral and big.

We have again that bigness of a $\mathbb{Q}$-divisor D only depends by its numerical equivalence class.
The next theorem gives a useful characterization of bigness for nef divisors.

Theorem 3.24. *Let D be a nef divisor on an n-dimensional variety X. Then*

$$D \text{ is big} \iff (D^n) > 0.$$

Proof. Thanks to Proposition 3.22 we have that D is big if and only if there exists a constant $C > 0$ such that

$$h^0(X, \mathcal{O}_X(mD)) = C \cdot m^n + O(m^{n-1}).$$

But, as D is nef, using Corollary 2.30, we have that

$$h^0(X, \mathcal{O}_X(mD)) = \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

Therefore D is big if and only if $(D^n) > 0$.

□

3.4 Pseudoeffective and big cones

We begin this section extending the notion of bigness to $\mathbb{R}$-divisors.

Definition 3.25. Let X be a variety, let $D \in Div_{\mathbb{R}}(X)$.
D is *big* if it can be written in the form

$$D = \sum a_i \cdot D_i$$

where each D_i is an integral big divisor and $a_i \in \mathbb{R}^+$.

We can give a numerical characterization of bigness also for $\mathbb{R}$-divisors:

Proposition 3.26. *Let D and D' be $\mathbb{R}$-divisors on a variety X. Then*

1. *If $D \equiv D'$, D is big if and only if D' is such.*
2. *D is big if and only if there exist an ample $\mathbb{R}$-divisor A and an effective $\mathbb{R}$-divisor N such that $D \equiv A + N$.*

Proof.

1. It suffices to show that if D is a big $\mathbb{R}$-divisor and B is a numerically trivial $\mathbb{R}$-divisor, then $D + B$ is big. But, thanks to Lemma 2.20, we can write $B = \sum r_i B_i$ with $r_i \in \mathbb{R}$ and B_i integral divisors such that $B_i \equiv 0$ for all i. Thus, if we prove that for any integral divisors A, B, with A big and $B \equiv 0$, the $\mathbb{R}$-divisor $A + rB$ is big for all $r \in \mathbb{R}$, then the requested statement follows by induction.
Let A and B be as above and $r \in \mathbb{Q}$, then $A + rB$ is big because bigness of $\mathbb{Q}$-divisors only depends by numerical equivalence classes. In general, we can fix two rational numbers r_1, r_2, with $r_1 < r < r_2$, together with a real number $t \in [0, 1]$ such that $r = tr_1 + (1 - t)r_2$. Then

$$A + rB = t(A + r_1 B) + (1 - t)(A + r_2 B),$$

that is $A + rB$ is a positive $\mathbb{R}$-linear combination of big $\mathbb{Q}$-divisors, therefore $A + rB$ is a big $\mathbb{R}$-divisor.

2. If D is big, then $D = \sum_{i=1}^{n} a_i D_i$, with $a_i \in \mathbb{R}^+$ and D_i big integral divisors.
By Theorem 3.20 for all $i = 1, \ldots, n$ there exists a positive integer m_i such that $m_i D_i \equiv A_i + N_i$ (with A_i ample and N_i effective). Then, denoting by m the product $m = m_1 \cdot \cdots \cdot m_n$ we have that

$$mD \equiv \sum_{i=1}^{n} a'_i (A_i + N_i)$$

where a'_i is the positive real number defined by $a'_i = a_i m_1 \cdot \cdots \cdot m_{i-1} m_{i+1} \cdot \cdots \cdot m_n$.
Therefore

$$D = \sum_{i=1}^{n} \frac{a'_i}{m} \cdot A_i + \sum_{i=1}^{n} \frac{a'_i}{m} \cdot N_i$$

exhibits D as a sum of an ample $\mathbb{R}$-divisor and an effective one.
For the converse we reduce to show that if B and N are integral divisors, with B big and N effective, and $s \in \mathbb{R}^+$, then $B + sN$ is big (combined with the hypothesis, this easily implies the assert). If $s \in \mathbb{Q}$ there exists a multiple of $B + sN$ that is a sum of an integral ample divisor and an integral effective divisor, then we conclude thanks to Theorem 3.20.
In general we can choose two positive rational numbers s_1, s_2 and $t \in [0, 1]$, such that $s_1 < s < s_2$ and $s = ts_1 + (1 - t)s_2$. Then we proceed as in the proof of the first point and write $B + sN$ as a positive linear combination of big $\mathbb{Q}$-divisors, whence $B + sN$ is big.

□

As a corollary we find that bigness is an open condition. In other words:

Corollary 3.27. *Let D be a big $\mathbb{R}$-divisor on a variety X and let $E_1, \ldots, E_r$ be arbitrary $\mathbb{R}$-divisors on X. Then*

$$D + \sum_{i=1}^{r} \epsilon_i E_i$$

is big for all sufficiently small numbers $0 < |\epsilon_i| << 1$.

Proof. This follows from the second statement of the previous proposition thanks to the open nature of ampleness (Proposition 2.23).

□

Now, thanks to Proposition 3.26 it makes sense to talk about a big $\mathbb{R}$-divisor class in $N^1(X)_{\mathbb{R}}$. Moreover we see that the subset of all numerical classes of big $\mathbb{R}$-divisors is a cone and we adopt the following definition:

Definition 3.28. The *big cone*

$$Big(X) \subseteq N^1(X)_{\mathbb{R}}$$

is the convex cone of all big $\mathbb{R}$-divisor classes in $N^1(X)_{\mathbb{R}}$.
The *pseudoeffective cone*

$$\overline{Eff}(X) \subseteq N^1(X)_{\mathbb{R}}$$

is the closure of the cone of all effective $\mathbb{R}$-divisors.
A divisor $D \in Div_{\mathbb{R}}(X)$ is said to be *pseudoeffective* if its numerical equivalence class lies in $\overline{Eff}(X)$.

The next theorem shows the relation between the two cones just defined.

Theorem 3.29. *Let X be a variety. Then*

1. $Big(X) = int(\overline{Eff}(X))$.
2. $\overline{Eff}(X) = \overline{Big(X)}$.

Proof. We begin by observing that the pseudoeffective cone is closed by definition, the big cone is open by Corollary 3.27 and $Big(X) \subseteq \overline{Eff}(X)$ because every big $\mathbb{R}$-divisor is a sum of an effective divisor and an ample one (whence it is effective) (Proposition 3.26). It remains to establish the inclusions

$$\overline{Eff}(X) \subseteq \overline{Big(X)},$$
$$int(\overline{Eff}(X)) \subseteq Big(X).$$

For the first, given $\eta \in \overline{Eff}(X)$, we can write η as a limit $\eta = \lim_k \eta_k$ of classes of effective divisors. Then, fixing an ample class $\alpha \in N^1(X)_{\mathbb{R}}$, we have

$$\eta = \lim_{k \to \infty} (\eta_k + \frac{1}{k}\, \alpha).$$

Since each of the classes $\eta_k + \frac{1}{k}\, \alpha$ is big thanks to Proposition 3.26, then η is a limit of big classes, whence $\eta \in \overline{Big(X)}$.
The first inclusion implies that $int(\overline{Eff}(X)) \subseteq int(\overline{Big(X)})$, so that, in order to prove the second one, it suffices to show that

$$int(\overline{Big(X)}) \subseteq Big(X).$$

By absurd let $x \in int(\overline{Big(X)}) \setminus Big(X)$ and fix a norm $|| \cdot ||$ on $N^1(X)_{\mathbb{R}}$. Then, as $int(\overline{Big(X)})$ is open, there exists an $\varepsilon > 0$ such that the ball of center x and radius ε

$$D_\varepsilon(X) \subseteq int(\overline{Big(X)}) \Longrightarrow D_\varepsilon(X) \subseteq \overline{Big(X)}.$$

Moreover, as $Big(X)$ is dense in its closure and $D_\varepsilon(X)$ is a non-empty open subset of $\overline{Big(X)}$, we have

$$Big(X) \cap D_\varepsilon(X) = A$$

where A is a non-empty open set.
Now let

$$A' = \{2x - y \mid y \in A\}.$$

We observe that A' is open and $A' \subseteq D_\varepsilon(X) \subseteq \overline{Big(X)}$; moreover we claim that

$$A' \cap Big(X) = \emptyset$$

so that we obtain a contradiction because $Big(X)$ intersects all the open sets contained in its closure.
To prove the claim, if by absurd there is an element $y' = 2x - y \in A' \cap Big(X)$, then, as $Big(X)$ is a cone, we get $\frac{1}{2}y + \frac{1}{2}y' = x \in Big(X)$, leading again to a contradiction.

$\square$

Chapter 4

The graded ring R(X,D)

4.1 Algebraic preliminaries

Definition 4.1. A *graded ring* is a ring R, together with a decomposition $R = \bigoplus_{i \geq 0} R_i$, with the R_i subgroups of R for all $i \geq 0$, such that

$$R_m R_n \subseteq R_{m+n}$$

for all n, $m \geq 0$.

Thus R_0 is a subring of R (in particular it contains the identity element) and each R_n is an R_0-module.

Definition 4.2. Given a graded ring R, a *graded R-module* is an R-module M, together with a decomposition $M = \bigoplus_{i \geq 0} M_i$, with the M_i subgroups of M for all $i \geq 0$, such that

$$R_m M_n \subseteq M_{m+n}$$

for all m, $n \geq 0$.
An element $x \in M$ is homogeneous if $x \in M_n$ for some n (n=degree of x).

We observe that each element $y \in M$ can be written uniquely as a sum $y = \sum y_n$, where $y_n \in M_n$ for all $n \geq 0$ and all but a finite number of the y_n are 0.

Definition 4.3. Let $R = \bigoplus_{i \geq 0} R_i$ be a graded ring.
For all $p \geq 1$ we define the *Veronese subring*

$$R^{(p)} = \bigoplus_{i \geq 0} R_{ip}.$$

Lemma 4.4. *Let A be a graded ring such that A is generated by A_1 as an A_0-algebra (in other words $A = A_0[A_1]$), let M be a graded A-module and let $\{y_i\}_{i \in I}$ be a system of homogeneous generators of M such that $deg(y_i) \leq n_0$ $\forall i \in I$.*
Then, for all $n \geq n_0$, for all $k \geq 0$

$$M_{n+k} = A_k M_n.$$

Proof. Let $n \geq n_0$, $k > 0$ and $x \in M_{n+k}$. Since the y_i generate M, there exists a finite set $I \subseteq \mathbb{N}$ and a family $\{a_i\}_{i \in I}$ of elements of A such that $x = \sum_{i \in I} a_i y_i$; we can further suppose that each a_i is homogeneous and of degree $n + k - deg(y_i)$. As $A = A_0[A_1]$ and $deg(a_i) > 0$, we can write each a_i as a sum of elements of the form bb', with $b \in A_1$ and $b' \in A$, thus $x \in A_1 M_{n+k-1}$.
Therefore

$$M_{n+k} = A_1 M_{n+k-1}$$

and the lemma follows by induction on k.

□

Lemma 4.5. *Let A be a graded ring such that $A = A_0[A_1]$ and let $S = \bigoplus_{i \geq 0} S_i$ be a graded A-algebra, which is finitely generated as an A-module. Then there exists an $n_0 \in \mathbb{N}$ such that*

1. *$S_{n+k} = S_k \cdot S_n$ for all $n \geq n_0$ and $k \geq 0$.*
2. *$S^{(d)} = S_0[S_d]$ for all $d \geq n_0$.*

Proof.

1. We can always suppose that S is generated by a finite number of homogeneous elements. Hence, applying Lemma 4.4, there exists an integer $n_0 > 0$ such that, for $n > n_0$ and $k \geq 0$, $S_{n+k} = A_k S_n$. Then

$$S_k S_n \subseteq S_{n+k} = A_k S_n \subseteq S_k S_n \Longrightarrow S_{n+k} = S_k S_n.$$

2. For $d \geq n_0$ and $m > 0$, we have that $S_{md} = (S_d)^m$, as follows by induction on m applying the first statement. This implies that $S^{(d)} = S_0[S_d]$.

□

Theorem 4.6. *Let $R = \bigoplus_{i \geq 0} R_i$ be a graded ring, which is a finitely generated R_0-algebra.*
Then there exists an integer $l \geq 1$ such that the Veronese subalgebra $R^{(ml)}$ is generated by R_{ml} as an R_0-algebra for all $m \geq 1$.
In other words $R^{(ml)} = R_0[R_{ml}]$ $\forall m \geq 1$.

Proof. Let $\{x_j\}_{1\leq j\leq s}$ be a set of generators of R as an R_0-algebra. Without loss of generality we can assume $\forall j = 1, \dots, s$ that x_j is an homogeneous element and $h_j = deg(x_j) > 0$. Let $q = m.c.m.\{h_j\}$ and $q_j = q/h_j$ for all $j = 1, \dots, s$.
Let B be the graded R_0-subalgebra of R generated by all the $x_j^{q_j}$. All these elements have degree q, thus $B_i = 0$ if i is not a multiple of q, so that $R^{(q)}$ is a graded B-module.
In order to have a ring generated by its part of degree one, we define the graded ring A by putting $A_i = B_{iq}$. With the same spirit we consider S as the graded ring such that $S_i = R_{iq}$. In other words A (respectively S) and B (respectively $R^{(q)}$) contain the same elements, but we have changed the graduation so that $A = A_0[A_1]$ and S is a graded A-module.
Now we consider all the elements of R of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_s^{\alpha_s}$ such that

- $0 \leq \alpha_j \leq q_j \qquad$ for $1 \leq j \leq s$,
- $\alpha_1 h_1 + \cdots + \alpha_s h_s \equiv 0 \quad (mod\ q)$.

The number of elements of this type is finite and we claim that they generate $R^{(q)}$ as a B-module. In fact it is enough to show that every element of $R^{(q)}$ of the form $x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s}$ is a B-linear combination of the above elements. But, for every $j = 1, \dots, s$, we can write $n_j = k_j q_j + r_j$, for suitable non negative integers k_j, r_j, with $r_j < q_j$. Then we have

$$x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s} = (x_1^{q_1})^{k_1} \cdots (x_s^{q_s})^{k_s} \cdot (x_1^{r_1} \dots x_s^{r_s}).$$

As these elements are in $R^{(q)}$, their degree is a multiple of q, whence

$$\sum_{i=1}^{s} h_i q_i k_i + \sum_{i=1}^{s} r_i h_i = q(\sum_{i=1}^{s} k_i) + \sum_{i=1}^{s} r_i h_i \equiv 0 \quad (mod\ q),$$

so that we have

$$\sum_{i=1}^{s} r_i h_i \equiv 0\ (mod\ q).$$

This proves our claim because the $x_j^{q_j}$ belong to B by definition.
Therefore, since $R^{(q)}$ is a finitely generated B-algebra, analogously S is a finitely generated A-algebra. Applying Lemma 4.5, we find an $n_0 \in \mathbb{N}$ such that $S^{(d)} = S_0[S_d]$ for all $d \geq n_0$, that is $R^{(qd)} = R_0[R_{qd}]$. The theorem follows by taking $l = qn_0$. □

4.2 Finitely generated line bundles

Definition 4.7. Let X be a variety and let $\mathcal{L}$ be a line bundle on X.

The *graded ring associated* to $\mathcal{L}$ is the graded $\mathbb{C}$-algebra

$$R(X, \mathcal{L}) = \bigoplus_{m \geq 0} H^0(X, \mathcal{L}^m).$$

Analogously, the graded ring associated to a divisor D is

$$R(X, D) = \bigoplus_{m \geq 0} H^0(X, mD) = R(X, \mathcal{O}_X(D)).$$

Definition 4.8. A line bundle $\mathcal{L}$ on a variety X is *finitely generated* if its graded ring $R(X, \mathcal{L})$ is finitely generated as a $\mathbb{C}$-algebra. A divisor D is finitely generated if the line bundle $\mathcal{O}_X(D)$ is such.

Lemma 4.9. *Let X be a normal variety and let $\mathcal{L}$ be a globally generated line bundle on X.*
Then there exists an integer $m_0(\mathcal{L}) > 0$ such that $\forall\, a, b \geq m_0(\mathcal{L})$ the mappings

$$H^0(X, \mathcal{L}^a) \otimes H^0(X, \mathcal{L}^b) \longrightarrow H^0(X, \mathcal{L}^{a+b})$$

determined by multiplication of sections are surjective. More generally, for any coherent sheaf $\mathcal{F}$ on X, there exists an integer $m_0(\mathcal{F}, \mathcal{L}) > 0$ such that for any $a, b \geq m_0(\mathcal{F}, \mathcal{L})$

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^a) \otimes H^0(X, \mathcal{L}^b) \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}^{a+b})$$

is surjective.

Proof. Thanks to Theorem 3.16 there is an algebraic fibre space $\phi : X \to Y \subseteq \mathbb{P}^r$ and an ample line bundle $\mathcal{A}$ on Y such that $\mathcal{L} = \phi^*\mathcal{A}$, whence we have $H^0(X, \mathcal{L}^m) = H^0(X, \phi^*(\mathcal{A}^m)) = H^0(Y, \mathcal{A}^m)$.
Thus, for the first statement, we can suppose $\mathcal{L}$ ample, so that the assertion follows by Proposition 2.8.
For the second, using projection formula, we observe that $H^0(X, \mathcal{F} \otimes \mathcal{L}^a) = H^0(X, \mathcal{F} \otimes \phi^*(\mathcal{A}^a)) = H^0(Y, \phi_*\mathcal{F} \otimes \mathcal{A}^a)$, where $\phi_*\mathcal{F}$ is a coherent sheaf on Y. Therefore we reduce again to the ample case and use Proposition 2.8. $\square$

Theorem 4.10. *Let X be a normal variety, let $\mathcal{L}$ be a line bundle on X. If $\mathcal{L}$ is semiample, then it is finitely generated.*

Proof. As $\mathcal{L}$ is semiample there exists an integer $k > 0$ such that $\mathcal{L}^k$ is free. Hence, applying Lemma 4.9, we obtain that the Veronese subalgebra

$$R(X, \mathcal{L})^{(k)} = \bigoplus_{m \geq 0} H^0(X, \mathcal{L}^{mk})$$

is finitely generated. In fact there exists an integer m_0 such that it is generated by the generators of $H^0(X, \mathcal{L}^{ik})$, for $i = 0, \dots, m_0$.
Now we use the second statement of the same lemma (taking $\mathcal{F}$ in turn to be each of the sheaves $\mathcal{L}$, $\mathcal{L}^2$, ..., $\mathcal{L}^{k-1}$) to find the finite generation of the graded ring $R(X, \mathcal{L})$ itself. $\square$

Theorem 4.11 (Wilson)**.** *Let X be a normal variety and let $D \in Div(X)$ be a big and nef divisor.*
Then there exists an effective divisor N and a natural number m_0 such that for any $m > m_0$

$$|mD - N|$$

is a free linear series.

Proof. Let $dim(X) = n$. Since D is big, thanks to Theorem 3.20, for every very ample line bundle B there is an integer $m_0 > 0$ and an effective divisor N such that

$$m_0 D \sim (n+1)B + N.$$

Then

$$mD - N \sim (m - m_0)D + (n+1)B \sim B + nB + kD,$$

where $k = m - m_0 > 0$, whence kD is nef.
Thus, for all $i = 1, \dots, n$, we get

$$mD - N - iB \sim kB + P$$

with $k > 0$ and P nef. Hence, using Fujita's vanishing theorem (2.28), we have that
$H^i(X, \mathcal{O}_X(mD - N - iB)) = 0 \quad \forall i > 0.$
In other words $mD - N$ is 0-regular with respect to B, whence it is globally generated by Theorem 2.14, that is $|mD - N|$ is a free linear series.

$\square$

Definition 4.12. Let X be a variety and let $D \in Div(X)$. Given a linear series $|V| \subseteq |D|$ and a point $x \in X$ we define the *multiplicity* of $|V|$ at x, denoted by $mult_x|V|$, as the multiplicity at x of a general divisor in $|V|$.
Equivalently

$$mult_x|V| = \min_{D' \in |V|} \{mult_x D'\}.$$

Note that, in the above definition, we use the convention that, denoting by 0 the trivial divisor on X, $mult_x|0| = mult_x 0 = 0$ for all $x \in X$.
Note also that, being $\mathcal{L}$ a line bundle on X and $|V| \subseteq |\mathcal{L}|$ a linear series, a point $x \in X$ is a base point of $|V|$ if and only if $mult_x|V| > 0$. In particular $\mathcal{L}$ is globally generated if and only if $mult_x|\mathcal{L}| = 0$ for all $x \in X$.
Using Wilson's theorem we can give a bound on multiplicity of big and nef divisors at every point:

Corollary 4.13. *Let D be a nef and big divisor on a variety X. Then there exists a constant $C > 0$ (not depending on m and x) such that*

$$mult_x|mD| \leq C$$

$\forall\, x \in X$, $\forall\, m \in \mathbb{N}(D)$.

Proof. Wilson's theorem allows us to take an effective divisor N such that $mD - N$ is free $\forall m > m_0$.
Thus, for all $x \in X$, $mult_x|mD - N| = 0$, whence $mult_x|mD| \leq mult_x N$ for all $m > m_0$. The assertion follows because the multiplicity of a single divisor at any point is finite.

□

Lemma 4.14. *Let X be a normal variety and let $D \in Div(X)$.*
If $k(X,D) \geq 0$ and D is finitely generated, then there exists a positive integer $n \in \mathbb{N}(D)$ such that

$$mult_x|knD| = k \cdot mult_x|nD|$$

$\forall k \geq 1$, $\forall x \in X$.

Proof.

($\leq$) For this inequality we do not need the hypothesis of finite generation and it holds for all $n \in \mathbb{N}(D)$. In fact let $x \in X$ and let $E \in |nD|$ be an effective divisor such that $mult_x|nD| = mult_x E$. Then, for every integer $k > 0$, $kE \in |knD|$, whence

$$mult_x|knD| \leq mult_x(kE) = k \cdot mult_x E = k \cdot mult_x|nD|.$$

($\geq$) D finitely generated means that the graded ring $R(X,D)$ is finitely generated as an $R(X,D)_0$-algebra. By Theorem 4.6 there is an integer $l > 0$ such that $H^0(X, \mathcal{O}_X(mlD))$ generates the Veronese subring $R(X,D)^{(ml)}$ $\forall m \geq 1$. Moreover since $k(X,D) \geq 0$ we have that $\mathbb{N}(D) \neq (0)$. Thus, being $e = e(D)$ the exponent of D, $ae \in \mathbb{N}(D)$ $\forall a >> 0$.
Now, if h is a sufficiently large natural number, and putting $n = hel$, we have that $H^0(X, \mathcal{O}_X(nD))$ generates the Veronese subring $R(X,D)^{(n)}$ and $n \in \mathbb{N}(D)$ (so that, in particular, $|knD| \neq \emptyset$ for all $k \geq 1$).
Then, for all $k > 0$, we can write every section $f \in H^0(X, \mathcal{O}_X(knD))$ as $f = f_1 \cdot \cdots \cdot f_k$, for suitable $f_i \in H^0(X, \mathcal{O}_X(nD))$.
Thus for each $E \in |knD|$ there exist $F_1, \ldots, F_k \in |nD|$ such that $E = \sum F_i$, whence

$$mult_x E \geq \sum_{i=1}^{k} mult_x F_i \geq k \cdot mult_x|nD|$$

for all $x \in X$. Therefore

$$mult_x|knD| \geq k \cdot mult_x|nD|$$

for all $k \geq 1$, for all $x \in X$.

$\square$

Corollary 4.13 and Lemma 4.14 allow us to give a partial converse of Theorem 4.10 in the case big and nef.

Theorem 4.15. *Let X be a normal variety and let $D \in Div(X)$ be a big and nef divisor. Then D is finitely generated if and only if it is semiample.*

Proof. If D is semiample it is finitely generated by Theorem 4.10.
For the converse as D is big we have that $k(X, D) \geq 0$, then we can apply Lemma 4.14 to find that there exists $n \in \mathbb{N}(D)$ such that

$$mult_x|knD| = k \cdot mult_x|nD|$$

for all $k \geq 1$, for all $x \in X$.
But, since D is big and nef, $\forall\ x \in X$ the left-hand side is bounded when $k \to \infty$ (Corollary 4.13), therefore $mult_x|nD| = 0$, that is x is not in the base locus of $|nD|$, whence $|nD|$ is free.

$\square$

The following theorem gives a characterization of semiampleness in a general setting. A possible generalization, involving discrete valuations, is presented in Chapter 5, Theorem 5.46.

Theorem 4.16. *Let X be a normal variety and let D be a divisor on X. Then D is semiample if and only if D satisfies the following three conditions:*

1. *D is finitely generated.*
2. *$k(X, D) \geq 0$.*
3. *There exists a constant $C > 0$ such that $\forall\ m \in \mathbb{N}(D)$, $\forall\ x \in X$, we have*
$$mult_x|mD| \leq C.$$

Proof.
($\Leftarrow$) As D satisfies the hypothesis of Lemma 4.14, we can find a positive integer $n \in \mathbb{N}(D)$ such that

$$mult_x|knD| = k \cdot mult_x|nD|$$

for all $k \geq 1$, for all $x \in X$.
But the third hypothesis assures that the left-hand side is bounded as $k \to \infty$, whence we must have $mult_x|nD| = 0$. Therefore nD is globally generated, that is D is semiample.

($\Rightarrow$) D is finitely generated thanks to Theorem 4.10.
$k(X,D) \geq 0$ because we can consider $l \in \mathbb{N}$ such that lD is globally generated, so that $H^0(X, \mathcal{O}_X(lD)) \neq 0$.
We concentrate on the third statement.
As D is semiample there exists an integer l_0 such that $l_0 D$ is free.
Now let $r \in \{0, \dots, l_0 - 1\}$, so that, thanks to Lemma 4.9, there exists an integer $m_0(r,D)$ such that whenever $a,\, b \geq m_0(r,D)$ the multiplication map

$$H^0(X, (al_0 + r)D) \otimes H^0(X, bl_0 D) \longrightarrow H^0(X, ((a+b)l_0 + r)D)$$

is surjective.
We define

$$X_D = \{r \in \{0, \dots, l_0 - 1\} \text{ such that } |(m_0(r,D)l_0 + r)D| \neq \emptyset\}$$

while we denote by m_1 the integer $m_1 = \max_{0 \leq r < l_0}\{m_0(r,D)\}$.
Hence, for all $r \in X_D$ we can choose an effective divisor $D_r \in |(m_0(r,D)l_0 + r)D|$.
Let

$$C' = \max\{mult_x\ D_r \mid r \in X_D;\ \ x \in X\},$$

$$C'' = \max\{mult_x|mD| \ \mid m \in \mathbb{N}(D),\, m < (2m_1 + 1)l_0;\ \ x \in X\},$$

$$C = \max\{C',\, C''\}.$$

Now we fix $x \in X$, $m \in \mathbb{N}(D)$, and we want to show that $mult_x|mD| \leq C$.
If $m < (2m_1 + 1)l_0$, then $mult_x|mD| \leq C'' \leq C$.
If $m \geq (2m_1 + 1)l_0$ we can write $m = ql_0 + r$, with $q \in \mathbb{N}$ and $0 \leq r \leq l_0 - 1$, or equivalently $m = (q - m_0(r,D))l_0 + m_0(r,D)l_0 + r$, where $q - m_0(r,D) \geq m_0(r,D)$. In fact

$$q - m_0(r,D) \geq \frac{m - m_0(r,D)l_0 - l_0}{l_0} \geq \frac{m_0(r,D)l_0}{l_0} = m_0(r,D).$$

Then we can apply Lemma 4.9 to the map

$$H^0(X, (m_0(r,D)l_0 + r)D) \otimes H^0(X, (q - m_0(r,D))l_0 D) \longrightarrow H^0(X, mD)$$

and we find it is surjective.
Hence, being $H^0(X, mD) \neq 0$, we get $H^0(X, (m_0(r,D)l_0 + r)D) \neq 0$, that is $r \in X_D$.
Let $T \in |(q - m_0(r,D))l_0 D|$ be a general divisor, then $mult_x T = 0$ because $|(q - m_0(r,D))l_0 D|$ is a free linear series. Moreover $T + D_r \sim mD$ and, in particular, as T and D_r are effective divisors, $T + D_r \in |mD|$.
Therefore

$$mult_x|mD| \leq mult_x(T + D_r) = mult_x(D_r) \leq C' \leq C.$$

$\square$

4.3 Curves

Let us consider the case of a smooth curve X, let D be a divisor on X. Properties like semiampleness and finite generation of D are easy to verify and they are very closely related to its degree and Iitaka dimension.
We can consider three cases:

$\boldsymbol{degD > 0}$:
One easily checks

$$degD > 0 \iff D \text{ ample} \iff D \text{ big} \iff k(X, D) = 1.$$

In particular thanks to ampleness D is nef and semiample, so that it is finitely generated.

$\boldsymbol{degD = 0}$:
Certainly we have that D is nef and it is not ample nor big.
In order to study semiampleness we must distinguish two cases:

- $k(X, D) = 0$.

This is equivalent to say that D is a torsion divisor, that is there is an $m > 0$ such that $mD \sim 0$.
D is semiample because mD is free, then it is finitely generated.

- $k(X, D) = -\infty$.

This is the case of a non-torsion divisor, that is $mD \not\sim 0$ for any integer $m > 0$, then $H^0(X, \mathcal{O}_X(mD)) = 0 \quad \forall m > 0$.
In particular D cannot be semiample but it is finitely generated because $R(X, D) = \mathbb{C}$.

$\boldsymbol{degD < 0}$:
In this case we cannot find an effective divisor linearly equivalent to any multiple of D, then the Iitaka dimension $k(X, D) = -\infty$ and D is not semiample nor nef.
Anyway D is again finitely generated.
We observe that these four situations all really occur. In fact the following proposition assures the existence of a degree-zero non-torsion line bundle on a curve.

Proposition 4.17. *Let C be a smooth curve of genus $g \geq 1$ and let $Pic^0(C)$ be the group of isomorphism classes of line bundles on C of degree zero. Then $Pic^0(C)$ has a structure of abelian variety and all but numerably many classes of $Pic^0(C)$ are represented by non-torsion line bundles.*

Summarizing we have:

1. Any divisor D on a smooth curve X is finitely generated.

2. D is semiample if and only if $k(X, D) \geq 0$.

3. We can have a finitely generated nef divisor that is not semiample.

4.4 Zariski's construction

We have seen in the last section how properties like finite generation and semiampleness of a divisor D on a smooth irreducible curve are very easy to verify. In fact the first is trivial while the second holds whenever D has non-negative Iitaka dimension.
When passing to surfaces things become more complicated. In particular in this section we deal with an example, provided by Zariski, of a nef and big divisor on a surface, that is not semiample nor finitely generated.

Let us consider a nonsingular cubic plane curve C_0 and let l be a hyperplane section on $\mathbb{P}^2$. Thanks to Proposition 4.17 we can choose twelve points $P_1, \ldots, P_{12} \in C_0$ such that $\eta = \mathcal{O}_{C_0}(P_1 + \cdots + P_{12} - 4l)$ is a non-torsion line bundle of degree zero (in fact it suffices to fix $P_2, \ldots, P_{12} \in C_0$, let P_1 vary on all X and use the fact that two points on C_0 are linearly equivalent if and only if they are the same point to find a not numerable quantity of isomorphism classes of line bundles in $Pic^0(C)$). Let

$$\mu : X = Bl_{\{P_1 + \cdots + P_{12}\}}\mathbb{P}^2 \longrightarrow \mathbb{P}^2$$

be the blowing up of $\mathbb{P}^2$ in these twelve points.
Denoting by E the exceptional divisor, we can decompose it as $E = \sum_{i=1}^{12} E_i$, where E_i is the exceptional component over P_i.
We put $H = \mu^* l$, the pullback of the hyperplane section on $\mathbb{P}^2$, and we denote by C the proper transform of C_0 under μ, thus $C \simeq C_0$.
We have that

$$C \in |3H - E|$$

because $C \sim \mu^*(C_0) - E$ and $\mu^*(C_0) \sim 3H$ (in fact $3l \sim C_0$ on $\mathbb{P}^2$).
Now we consider on X the divisor

$$D = 4H - E.$$

Note that

- $(C^2) = ((3H - E)^2) = -3$,

- $(D \cdot C) = (H \cdot C) + (C^2) = (H \cdot (3H - E)) - 3 = 0.$

We will show that D is big and nef but it is not finitely generated (then, in particular, it is not semiample by Theorem 4.10).

***D* is big**: $D \sim H + C$, then it is a sum of a big and effective divisors, whence it is big.

***D* is nef**: We observe that $H = D - C$ is free because it is the pullback under μ of a hyperplane section on $\mathbb{P}^2$, so that it is nef.
If by absurd D is not nef, we can find an irreducible curve C' such that $(D \cdot C') < 0$. But $D = H + C$, thus we have $(H \cdot C') + (C \cdot C') < 0$. H being nef, we must have $(C \cdot C') < 0$, so that we get $C' = C$. Therefore we find the contradiction $(D \cdot C) < 0$.

***D* is not finitely generated**: We will prove that for all $m > 0$ the linear series $|mD|$ contains C in its base locus, but $|mD - C|$ is free.
This is enough to show that D is not finitely generated because, by Lemma 4.14, finite generation implies that the multiplicity of a point in a base curve of $|mD|$ must go to infinity with m.
$C \subseteq Bs(|mD|)$ because, identifying $C \simeq C_0$ through μ, we have $\mathcal{O}_C(D) = \eta^{-1}$, whence $\mathcal{O}_C(mD) = \eta^{-m}$ has no sections (because $deg\ \eta = 0$ and it is not a torsion line bundle).

To show that $|mD - C|$ is free we work by induction:
For $m = 1$ we have that $mD - C = D - C \sim H$ is globally generated.
We assume now $|(m-1)D - C|$ free and we prove $|mD - C|$ is such.
We begin observing that $mD - C \sim (m-1)D + H$, thus we can consider the exact sequence

$$0 \to \mathcal{O}_X((m-1)D - C + H) \to \mathcal{O}_X(mD - C) \to \mathcal{O}_C(mD - C) \to 0.$$

The line bundle on the right has degree three on the elliptic curve C, whence it is free. That on the left is free because both $(m-1)D - C$ and H are free. Thus, passing to cohomology, we find that $Bs(|mD - C|) \subseteq C$.
Now, if, by absurd, there exists $x \in Bs(|mD - C|)$, then, as $x \in C$, there exists a non-zero section $\tau \in H^0(C, \mathcal{O}_C(mD - C))$ such that $\tau(x) \neq 0$.
If we prove that the restriction map

$$\delta : H^0(X, \mathcal{O}_X(mD - C)) \longrightarrow H^0(C, \mathcal{O}_C(mD - C))$$

is surjective, we find that there exists a non-zero section $s \in H^0(X, \mathcal{O}_X(mD - C))$ not vanishing at x, and this leads to a contradiction.
The following lemma completes the proof assuring the surjectivity of δ.

Lemma 4.18. *Let X,D,C,H be as above.*
Then $H^1(X, \mathcal{O}_X((m-1)D - C + H)) = 0$ for all $m \geq 1$.

Proof. We begin observing that $C \sim -K_X$, where K_X is the canonical sheaf on X, then we must prove that $H^1(X, \mathcal{O}_X(K_X + (m-1)D + H)) = 0$.
We proceed by induction:
For $m = 1$, $H^1(X, \mathcal{O}_X(K_X + H)) = H^1(X, \mathcal{O}_X(-H))$ thanks to Serre's duality theorem. We use the exact sequence

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0.$$

Passing to cohomology and observing that $H^0(X, \mathcal{O}_X)$ goes isomorphically into $H^0(H, \mathcal{O}_H)$ and that $H^1(X, \mathcal{O}_X) \simeq H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$ we have the induction base.
Now, assume that the assertion holds for $m-2$. Thanks to the exact sequence

$$0 \to \mathcal{O}_X(K_X + (m-2)D + H) \to \mathcal{O}_X(K_X + (m-1)D + H) \to$$
$$\to \mathcal{O}_D(K_X + (m-1)D + H) \to 0$$

it is enough to show that $H^1(D, \mathcal{O}_D(K_X + (m-1)D + H)) = 0$.
Now, we observe that $Bs(|D|) = C$. In fact $C \subseteq Bs(|D|)$ and the other inclusion follows because $D = H + C$ and H is free.
Then, considering D as a closed subscheme, there exists $\Gamma \in |H|$ such that $D = \Gamma \cup C$ and $\Gamma \simeq \mathbb{P}^1$.
Hence there is an exact sequence

$$0 \to \mathcal{O}_\Gamma(-C) \to \mathcal{O}_D \to \mathcal{O}_C \to 0.$$

And tensoring by $K_X + (m-1)D + H$ we obtain

$$0 \to \mathcal{O}_\Gamma(2K_X + (m-1)D + H) \to \mathcal{O}_D(K_X + (m-1)D + H) \to$$
$$\to \mathcal{O}_C(K_X + (m-1)D + H) \to 0.$$

But

$$deg\ \mathcal{O}_\Gamma(2K_X + (m-1)D + H) = 4m - 9 > 2g(\Gamma) - 2 \quad \forall m > 1$$

and

$$deg\ \mathcal{O}_C(K_X + (m-1)D + H) = 6 > 2g(C) - 2 \quad \forall m > 1.$$

Therefore the assert follows because both the term on the left and that on the right have $h^1 = 0$.

□

Chapter 5

Valuations

5.1 Restricted base locus

Definition 5.1. Let X be a normal variety and let D be an $\mathbb{R}$-divisor on X. The *restricted base locus* of D is

$$\mathbb{B}_-(D) = \bigcup_A \mathbb{B}(D + A),$$

where the union is taken over all ample $\mathbb{R}$-divisors A such that $D + A$ is a $\mathbb{Q}$-divisor.

We recall that, given a $\mathbb{Q}$-divisor E, we denote by $\mathbb{B}(E)$ the stable base locus of E (see Def. 3.5 and Def. 3.8).

Remark 5.2. Note that if D is an $\mathbb{R}$-divisor and $c \in \mathbb{R}^+$, then $\mathbb{B}_-(D) = \mathbb{B}_-(cD)$, see [ELMNP06, 1.15].

Lemma 5.3. *Let X be a normal variety, let $||\cdot||$ be a norm on $N^1(X)_\mathbb{R}$ and let D be an $\mathbb{R}$-divisor on X. Then*

1. *There exists a sequence $\{A_m\}_{m\in\mathbb{N}}$ of ample $\mathbb{R}$-divisors such that*
   - $\lim_{m\to\infty} ||A_m|| = 0$,
   - *$D + A_m$ is a $\mathbb{Q}$-divisor for every $m \in \mathbb{N}$.*
2. *D is nef if and only if $\mathbb{B}_-(D) = \emptyset$.*

Proof.

1. Let A be an ample $\mathbb{R}-$divisor. Since the ample cone $Amp(X) \subseteq N^1(X)_\mathbb{R}$ is open by Proposition 2.23, we have that there exists an $\epsilon_0 > 0$ such that $D_{\epsilon_0}([A])$, the ball of radius ϵ_0 and centered in the numerical class of A, is contained in $Amp(X)$.
On the other hand for every $\epsilon > 0$ we have that

$$D_\epsilon([D+A]) \cap N^1(X)_{\mathbb{Q}} \neq \emptyset,$$

so that there exists a $\mathbb{Q}$-divisor D' such that $||D' - D - A|| < \epsilon_0$.
Hence $[D' - D] \in D_{\epsilon_0}([A]) \subseteq Amp(X)$, that is $A' = D' - D$ is ample and $D + A' = D'$ is a $\mathbb{Q}$-divisor.
Thus we can write

$$A' = \sum_{i=1}^{s} c_i A'_i$$

for suitable $c_i \in \mathbb{R}^+$ and A'_i ample integral divisors.
Moreover, for all $i = 1, \dots, s$, we can consider a sequence $\{q_{im}\}_{m \in \mathbb{N}}$ such that

- $q_{im} \in \mathbb{Q}^+ \quad \forall m \in \mathbb{N}$;
- $q_{im} < c_i \quad \forall m \in \mathbb{N}$;
- $\lim_{m \to \infty} q_{im} = c_i$.

We define

$$A_m = A' - \sum_{i=1}^{s} q_{im} A'_i = \sum_{i=1}^{s} (c_i - q_{im}) A'_i,$$

so that, for all $m \in \mathbb{N}$, we have that A_m is ample and $D + A_m$ is a $\mathbb{Q}$-divisor.
Moreover $||A_m|| \to 0$, so that the assertion follows.

2. **($\Rightarrow$)** Let A be an ample $\mathbb{R}$-divisor on X such that $D + A$ is a $\mathbb{Q}$-divisor.
By the nefness of D it follows that $D + A$ is ample, whence there exists an integer $m > 0$ such that $m(D + A)$ is integral and very ample.
Thus $\mathbb{B}(D + A) \subseteq Bs(|m(D + A)|) = \emptyset$.

($\Leftarrow$) Let D be an $\mathbb{R}$-divisor with $\mathbb{B}_-(D) = \emptyset$. Then $\mathbb{B}(D + A) = \emptyset$ for all ample $\mathbb{R}$-divisors A such that $D + A$ is a $\mathbb{Q}$-divisor.
Let $\{A_m\}_{m \in \mathbb{N}}$ be the sequence of ample $\mathbb{R}$-divisors found in the first part of the lemma. Then, for all $m \in \mathbb{N}$, we can choose an integer $b_m > 0$ such that $b_m(D + A_m)$ is integral and

$$\mathbb{B}(b_m(D + A_m)) = \mathbb{B}(D + A_m) = \emptyset.$$

Hence $b_m(D + A_m)$ is semiample, so that $D + A_m$ is nef for all $m > 0$.
Thus, for every irreducible curve C on X we have

$$((D + A_m) \cdot C) \geq 0.$$

In other words

$$(D \cdot C) \geq -(A_m \cdot C),$$

and, passing to limit, as $||A_m|| \to 0$, we obtain$(D \cdot C) \geq 0$.

$\square$

Proposition 5.4. *Let X be a normal variety, let $||\cdot||$ be a norm on $N^1(X)_{\mathbb{R}}$ and let D be an $\mathbb{R}$-divisor on X.*
If $\{A_m\}_{m\in\mathbb{N}}$ is a sequence of ample $\mathbb{R}$-divisors such that

- $\lim_{m\to\infty} ||A_m|| = 0$,
- *$D + A_m$ is a $\mathbb{Q}$-divisor for every $m \in \mathbb{N}$,*

then

$$\mathbb{B}_-(D) = \bigcup_m \mathbb{B}(D + A_m).$$

In particular $\mathbb{B}_-(D)$ is a countable union of Zariski closed subsets of X.

Proof. By definition we obviously have that $\mathbb{B}_-(D) \supseteq \bigcup_m \mathbb{B}(D + A_m)$.
For the converse let A be an ample divisor such that $D + A$ is a $\mathbb{Q}$-divisor. Then, as $||A_m|| \to 0$, there exists a sufficiently large $m \in \mathbb{N}$ such that $A-A_m$ is ample (Proposition 2.23). Hence, as $D + A = (D + A_m) + (A - A_m)$ and $\mathbb{B}(A - A_m) = \emptyset$, we get

$$\mathbb{B}(D + A) \subseteq \mathbb{B}(D + A_m) \cup \mathbb{B}(A - A_m) = \mathbb{B}(D + A_m).$$

We find the assert. □

5.2 Discrete valuations and linear series

We begin this section by recalling some basic algebraic notions about discrete valuations that we will need later.

Definition 5.5. Let B be an integral domain and let K be its field of fractions. B is a *valuation ring* of K if, for every $x \in K^*$, either $x \in B$ or $x^{-1} \in B$.

Definition 5.6. Let K be a field. A *discrete valuation* on K is an application $v : K^* \to \mathbb{Z}$ such that

- $v(xy) = v(x) + v(y) \quad \forall x, y \in K^*$,
- $v(x + y) \geq \min\{v(x), v(y)\} \quad \forall x, y \in K^*$ such that $x + y \neq 0$.

If $k \subseteq K$ is a subfield, a discrete valuation on K/k is a discrete valuation v on K such that $v(x) = 0$ for all $x \in k$.

Given a discrete valuation on K we observe that the set consisting of 0 and all $x \in K^*$ with $v(x) \geq 0$ is a valuation ring of K. This justifies the following:

Definition 5.7. Let v be a discrete valuation on a field K. The ring

$$R_v = \{f \in K^* : v(f) \geq 0\} \cup \{0\}$$

is called the *valuation ring* of v.

In the next proposition we list some basic properties of the ring R_v.

Proposition 5.8. *Let v be a discrete valuation on a field K and let R_v be the valuation ring of v. Then*

1. *R_v is a local domain with maximal ideal*

$$m_{R_v} = \{f \in K^* : v(f) > 0\} \cup \{0\}.$$

2. *R_v is a PID and each ideal is generated by any element of the ideal of minimum valuation.*

3. *Spec $R_v = \{(0), m_{R_v}\}$.*

Proof. See [AM00, Chapter 9, page 94]. □

Definition 5.9. Let v be a discrete valuation on a field K with valuation ring R_v. Let $a \subseteq R_v$ be an ideal, $a \neq (0)$. Then we define

$$v(a) = \min\{v(f) : f \in a,\ f \neq 0\}.$$

Equivalently, by Proposition 5.8 (2), we have $v(a) = v(f)$ for any f such that $a = (f)$.

Proposition 5.10. *Let K be a field, let v be a discrete valuation on K and let $a, b \subseteq R_v$ be non-zero ideals. Then*

1. *$a \subseteq b \Longrightarrow v(a) \geq v(b)$,*

2. *$v(ab) = v(a) + v(b)$,*

3. *$v(a) = 0 \Longleftrightarrow a = R_v$.*

Proof.

1. Let $a = (f)$ and $b = (g)$. Then $f \in (f) \subseteq (g)$, so that there exists a non-zero element $h \in R_v$ such that $f = hg$.
Hence $v(a) = v(f) = v(hg) = v(h) + v(g) \geq v(g) = v(b)$.

2. Let $f, g \in R_v$ be such that $a = (f)$ and $b = (g)$, then it is easy to see that $ab = (fg)$.
Hence $v(ab) = v(fg) = v(f) + v(g) = v(a) + v(b)$.

3. Let $a = (f)$. Then

$$v(a) = 0 \iff v(f) = 0 \iff (f) = R_v \iff a = R_v.$$

$\square$

From now on we deal with discrete valuations of the function field of a variety that are zero on the subfield of complex numbers. Passing through the notion of center we will be able to use them to define "valuations" of ideal sheaves and linear series.
We begin by recalling the relation of domination between two local rings:

Definition 5.11. Let A and B be local rings, respectively with maximal ideal m_A and m_B. We say that B *dominates* A if

- $A \subseteq B$,
- $m_A \subseteq m_B$.

Now, for any variety X, we can define the center of a valuation on the function field $K(X)$:

Definition 5.12. Let X be a variety, let $K = K(X)$ be the function field of X and let v be a discrete valuation on $K/\mathbb{C}$.
We say that v has *center* $\xi \in X$ if R_v dominates $\mathcal{O}_{X,\xi}$.

Proposition 5.13. *Let X be a variety with function field $K = K(X)$ and let v be a discrete valuation on $K/\mathbb{C}$. Then v has a unique center $\xi \in X$.*

Proof. See [Vaq00, Prop. 6.2 and Prop. 6.3] $\square$

From now on, with a slight abuse of notation, when speaking about the center of v we consider the subvariety $Z_v \subseteq X$ with generic point ξ.
Note that every subvariety Z of a variety X is the center of a discrete valuation on $K(X)/\mathbb{C}$.

Definition 5.14. Let X be a variety and let $\mathcal{I}$ be a non-zero quasi-coherent sheaf of ideals. We denote by $\mathcal{Z}(\mathcal{I})$ the closed subset of X defined by $\mathcal{I}$.

Definition 5.15. Let X be a variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$ with center $Z_v \subseteq X$ and let $\xi \in X$ be the generic point of Z_v.
If $\mathcal{I} \subseteq \mathcal{O}_X$ is a non-zero quasi-coherent sheaf of ideals, we put

$$v(\mathcal{I}) = v((\mathcal{I}_\xi)_{R_v}),$$

where $(\mathcal{I}_\xi)_{R_v}$ is the ideal generated by the set $\mathcal{I}_\xi$ in the ring R_v.

Note that the definition makes sense because $\mathcal{I}_\xi \subseteq \mathcal{O}_{X,\xi} \subseteq R_v$. Moreover $\mathcal{I}_\xi$ is an ideal of $\mathcal{O}_{X,\xi}$, but it is not, in general, an ideal of R_v, whence it is necessary to pass to $(\mathcal{I}_\xi)_{R_v}$.

Proposition 5.16. *Let X be a variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$ with center $Z_v \subseteq X$ and let $\mathcal{I}, \mathcal{I}' \subseteq \mathcal{O}_X$ be non-zero quasi-coherent ideal sheaves. Then*

1. $v(\mathcal{I}) = 0 \Longleftrightarrow Z_v \not\subseteq \mathcal{Z}(\mathcal{I})$,
2. $\mathcal{I} \subseteq \mathcal{I}' \Longrightarrow v(\mathcal{I}) \geq v(\mathcal{I}')$,
3. $v(\mathcal{I} \cdot \mathcal{I}') = v(\mathcal{I}) + v(\mathcal{I}')$.

Proof.

1. $v(\mathcal{I}) = 0$ if and only if, by definition, $v((\mathcal{I}_\xi)_{R_v}) = 0$. Thanks to Proposition 5.10 (3), this is equivalent to say that $(\mathcal{I}_\xi)_{R_v} = R_v$. Hence it remains to show that $(\mathcal{I}_\xi)_{R_v} = R_v$ if and only if $Z_v \not\subseteq \mathcal{Z}(\mathcal{I})$ or, equivalently, that $(\mathcal{I}_\xi)_{R_v} = R_v$ if and only if $\mathcal{I}_\xi = \mathcal{O}_{X,\xi}$.
Assuming $(\mathcal{I}_\xi)_{R_v} = R_v$, if by absurd $\mathcal{I}_\xi \neq \mathcal{O}_{X,\xi}$, then $\mathcal{I}_\xi \subseteq m_\xi \subseteq m_{R_v}$. Hence $(\mathcal{I}_\xi)_{R_v} \subseteq m_{R_v}$ and we find a contradiction.
On the contrary, if $\mathcal{I}_\xi = \mathcal{O}_{X,\xi}$, then $1 \in \mathcal{I}_\xi$, so that $(\mathcal{I}_\xi)_{R_v} = R_v$.

2. $\mathcal{I} \subseteq \mathcal{I}' \Longrightarrow \mathcal{I}_\xi \subseteq \mathcal{I}'_\xi \Longrightarrow (\mathcal{I}_\xi)_{R_v} \subseteq (\mathcal{I}'_\xi)_{R_v}$.
By Proposition 5.10 (1) it follows that $v(\mathcal{I}) \geq v(\mathcal{I}')$.

3. By definition $v(\mathcal{I} \cdot \mathcal{I}') = v(((\mathcal{I} \cdot \mathcal{I}')_\xi)_{R_v})$. But $((\mathcal{I} \cdot \mathcal{I}')_\xi)_{R_v} = (\mathcal{I}_\xi \cdot \mathcal{I}'_\xi)_{R_v} = (\mathcal{I}_\xi)_{R_v} \cdot (\mathcal{I}'_\xi)_{R_v}$. Therefore, using Proposition 5.10 (2), we get

$$v(\mathcal{I} \cdot \mathcal{I}') = v((\mathcal{I}_\xi)_{R_v} \cdot (\mathcal{I}'_\xi)_{R_v}) = v((\mathcal{I}_\xi)_{R_v}) + v((\mathcal{I}'_\xi)_{R_v}) = v(\mathcal{I}) + v(\mathcal{I}').$$

□

Definition 5.17. Let X be a variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$ and let D be a divisor on X such that $|D| \neq \emptyset$. Then we put

$$v(|D|) = v(b(|D|)),$$

where $b(|D|)$ is the base ideal of the linear series $|D|$ (Definition 1.9).

Proposition 5.18. *Let X be a variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$ having center Z_v on X and let D and E be divisors on X such that $|D| \neq \emptyset$, $|E| \neq \emptyset$. Then*

1. $v(|D|) = 0 \Longleftrightarrow Z_v \not\subseteq Bs(|D|)$,

2. *$v(|D+E|) \leq v(|D|) + v(|E|)$ and equality holds if the map of multiplication of sections*

$$\mu_{D,E} : H^0(X, \mathcal{O}_X(D)) \otimes H^0(X, \mathcal{O}_X(E)) \longrightarrow H^0(X, \mathcal{O}_X(D+E))$$

is surjective.

3. *$Z_v \not\subseteq Bs(|E|) \Longrightarrow v(|D+E|) \leq v(|D|)$.*

Proof.

1. $v(|D|) = 0$ if and only if, by definition, $v(b(|D|)) = 0$. But, by Proposition 5.16 (1), this happens if and only if $Z_v \not\subseteq \mathcal{Z}(b(|D|)) = Bs(|D|)$.

2. By Proposition 1.10 we have that $b(|D|) \cdot b(|E|) \subseteq b(|D+E|)$. Hence, thanks to Proposition 5.16 (2),(3), we find

$$v(|D+E|) \leq v(b(|D|) \cdot b(|E|)) = v(b(|D|)) + v(b(|E|)) = v(|D|) + v(|E|).$$

Moreover, again by Proposition 1.10, if the map of multiplication of sections $\mu_{D,E}$ is surjective, then $b(|D|) \cdot b(|E|) = b(|D+E|)$, so that we get the assert.

3. Follows immediately combining the first and the second part. □

5.3 Asymptotic order of vanishing

Now, given an integral divisor D with non negative Iitaka dimension on a normal variety X, we will define an "asymptotic valuation" of D, that is a measure of the order of vanishing of $|mD|$ at the center Z_v of a valuation v as $m \to \infty$. Before carrying on we recall the following lemma:

Lemma 5.19. *Let $\{\alpha_m\}_{m \geq m_0}$ be a sequence of real numbers such that*

- $\alpha_m \geq 0 \quad \forall m \geq m_0$,
- $\alpha_{p+q} \leq \alpha_p + \alpha_q \quad \forall p, q \geq m_0$.

Then there exists the limit

$$\lim_{m \to \infty} \frac{\alpha_m}{m} = \inf_{m \geq m_0} \{\frac{\alpha_m}{m}\}.$$

Proof. See [Mus02, Lemma 1.4]. □

Definition 5.20. Let X be a normal variety with function field $K = K(X)$, let D be an integral divisor on X with $k(X, D) \geq 0$ and let v be a discrete valuation on $K/\mathbb{C}$.
If $e = e(D)$ is the exponent of D, we define the *asymptotic order of vanishing* of D along v as

$$v(||D||) = \lim_{m \to \infty} \frac{v(|meD|)}{me}.$$

In particular if D is big, since, by Corollary 3.21, D has exponent $e(D) = 1$, we have

$$v(||D||) = \lim_{m\to\infty} \frac{v(|mD|)}{m}.$$

Remark 5.21. The existence of the limit in Definition 5.20 follows by Proposition 5.18 and Lemma 5.19:
In fact, if D has exponent $e = e(D)$, then there exists an integer $m_0 > 0$ such that $|meD| \neq \emptyset$ for all $m \geq m_0$. Hence we can use Proposition 5.18 (2) to find that $v(|(p+q)eD|) \leq v(|peD|) + v(|qeD|)$ for all $p, q \geq m_0$.
Moreover, for all $m \geq m_0$, we have $v(|meD|) \geq 0$ by definition.
Therefore Lemma 5.19 applies to the sequence $\{v(|meD|)\}_{m\geq m_0}$, so that there exists

$$\lim_{m\to\infty} \frac{v(|meD|)}{me} = \frac{1}{e} \cdot \lim_{m\to\infty} \frac{v(|meD|)}{m} = \frac{1}{e} \cdot \inf_{m\geq m_0} \{\frac{v(|meD|)}{m}\}.$$

Thus the asymptotic order of vanishing $v(||D||)$ is well defined.

Proposition 5.22. *Let D be an integral divisor on a normal variety with $k(X, D) \geq 0$. Then for any $k \in \mathbb{N}$ we have*

$$v(||kD||) = k \cdot v(||D||).$$

Proof. Denoting by e_1 the exponent of D and by e_k the exponent of kD, we begin noting that ke_k/e_1 is an integer number:
In fact it is easy to see that $e_k = e_1/\text{g.c.d.}(e_1, k)$, whence

$$\frac{ke_k}{e_1} = \frac{k}{\text{g.c.d.}(e_1, k)} = n \in \mathbb{N}.$$

Thus, by the definition of $v(||D||)$ as a limit it follows that

$$v(||D||) = \lim_{m\to\infty} \frac{v(|e_1 mD|)}{e_1 m} = \lim_{m\to\infty} \frac{v(|e_1 \frac{ke_k}{e_1} mD|)}{e_1 m \frac{ke_k}{e_1}} =$$

$$= \frac{1}{k} \cdot \lim_{m\to\infty} \frac{v(|e_k mkD|)}{e_k m} = \frac{1}{k} \cdot v(||kD||).$$

□

The property of $\mathbb{Q}$-linearly-effectiveness described in the following definition has to be considered in analogy to the property of non negativity of the Iitaka dimension for integral divisors.

Definition 5.23. Let X be a variety and let $D \in Div_{\mathbb{Q}}(X)$, we define the $\mathbb{Q}$-*linear series* $|D|_{\mathbb{Q}}$ as the set of all the effective $\mathbb{Q}$-divisors that are $\mathbb{Q}$-linearly equivalent to D.
D is $\mathbb{Q}$-*linearly-effective* if $|D|_{\mathbb{Q}} \neq \emptyset$.

Thanks to Proposition 5.22 we can give a natural definition of asymptotic order of vanishing of $\mathbb{Q}$-linearly-effective $\mathbb{Q}$-divisors:

Definition 5.24. Let X be a normal variety with function field $K = K(X)$, let D be a $\mathbb{Q}$-linearly-effective $\mathbb{Q}$-divisor on X and let v be a discrete valuation on $K/\mathbb{C}$.
Then the *asymptotic order of vanishing* of D along v is

$$v(||D||) = \frac{1}{m} \cdot v(||mD||),$$

where $m \in \mathbb{N}$ is such that mD is integral.

As D is a $\mathbb{Q}$-linearly-effective $\mathbb{Q}$-divisor we have that $k(X, mD) \geq 0$ whenever mD is a $\mathbb{Z}$-divisor, so that $v(||mD||)$ is well defined. Moreover by Proposition 5.22 the definition does not depend on the integer m such that mD is integral.

Proposition 5.25. *If D and E are two $\mathbb{Q}$-linearly-effective $\mathbb{Q}$-divisors on a normal variety X, then*

$$v(||D + E||) \leq v(||D||) + v(||E||).$$

Proof. It is enough to check the statement for integral divisors.
Let $e(D)$ and $e(E)$ be the exponents of D and E respectively and let $e = e(D) \cdot e(E)$, so that eD, eE and $e(D + E)$ are all divisors with exponent one.
Thus using Proposition 5.18 (2), we have that

$$v(|pe(D + E)|) \leq v(|peD|) + v(|peE|)$$

for all $p >> 0$.
Dividing by p and passing to limit for $p \to \infty$ we obtain

$$v(||e(D + E)||) \leq v(||eD||) + v(||eE||),$$

so that the assert follows by Proposition 5.22. $\square$

5.4 Computation via multiplier ideals

In this section we will show how the asymptotic order of vanishing of divisors can be computed by using multiplier ideals.
Some important results about asymptotic multiplier ideals, that we will use in this chapter, are briefly presented in Appendix A, but for a complete treatment we refer to [Laz04, Part Three].

Remark 5.26. Note that if $f : X' \to X$ is a birational morphism of normal varieties with function fields $K(X) = K(X')$ and v is a discrete valuation on $K(X)/\mathbb{C}$, then for any $\mathbb{Q}$-linearly-effective $\mathbb{Q}$-divisor D on X we get

$$v(||D||) = v(||f^*(D)||):$$

In fact we restrict to the integral setting and we use the fact that for all $p \in \mathbb{N}(D)$ we have $H^0(X', \mathcal{O}_{X'}(p \cdot f^*(D))) = H^0(X, \mathcal{O}_X(pD))$ (see Remark 3.12 and Lemma 3.13), so that $v(|p \cdot f^*(D)|) = v(|pD|)$.
In particular by taking f such that X' is smooth we reduce the computation of the asymptotic order of vanishing along v to the case of a smooth variety. In this case we can make use of multiplier ideals.

Notation. Let X be a smooth variety with function field $K = K(X)$ and let v be a discrete valuation on $K/\mathbb{C}$ whose center is the subvariety Z_v with generic point $\xi \in X$.
Then if D is an integral divisor on X with $k(X, D) \geq 0$ we denote by $\mathcal{I}(X, ||D||)$ the asymptotic multiplier ideal associated to $|D|$ (see Definition A.6). We recall that $\mathcal{I}(X, ||pD||)$ is a non-zero quasi coherent sheaf of ideals for all $p > 0$ (see Remark A.7), so that it makes sense to consider $v(\mathcal{I}(X, ||pD||))$.
Moreover, for all $p \in \mathbb{N}$, we write

$$j_p = j_{pD} = (\mathcal{I}(X, ||pD||)_\xi)_{R_v},$$

so that by definition we get $v(\mathcal{I}(X, ||pD||)) = v(j_p)$.
Analogously, whenever $|pD| \neq \emptyset$, it will be useful to define

$$a_p = a_{pD} = (b(|pD|)_\xi)_{R_v},$$

so that, denoting by e the exponent $e(D)$, we have $v(|pD|) = v(a_p)$ and $v(||D||) = \lim_{p\to\infty} \frac{v(a_{ep})}{ep}$.

The following proposition translates in the language of valuations some important properties of asymptotic multiplier ideals (see Appendix A).

Proposition 5.27. *Let X be a nonsingular variety and let D be an integral divisor on X with $k(X, D) \geq 0$. Then*

1. $v(j_{p+q}) \geq v(j_p) + v(j_q) \quad \forall p, q \in \mathbb{N}$,
2. $v(j_p) \leq v(j_q) \quad$ *if* $p < q$,
3. $v(j_p) \leq v(a_p) \quad \forall p \in \mathbb{N}(D)$.

Proof.

1. By using Theorem A.9, we have

$$\mathcal{I}(||(p+q)D||) \subseteq \mathcal{I}(||pD||) \cdot \mathcal{I}(||qD||)$$

for every $p, q \in \mathbb{N}$. Hence by Proposition 5.16 (2),(3) we have

$$v(\mathcal{I}(||(p+q)D||) \geq v(\mathcal{I}(||pD||) \cdot \mathcal{I}(||qD||)) = v(\mathcal{I}(||pD||)) + v(\mathcal{I}(||qD||)),$$

or equivalently

$$v(j_{p+q}) \geq v(j_p) + v(j_q).$$

2. We use Theorem A.8 (1) to find that

$$\mathcal{I}(||pD||) \supseteq \mathcal{I}(||qD||)$$

whenever $p < q$. Thus the assert follows again by Proposition 5.16 (2).

3. Follows by Theorem A.8 (2) and Proposition 5.16 (2). □

Lemma 5.28. *Let $\{\beta_m\}_{m\in\mathbb{N}}$ be a sequence of real numbers such that*

- $\beta_m \geq 0 \quad \forall m \in \mathbb{N}$,
- $\beta_m \leq \beta_{m+1} \quad \forall m \in \mathbb{N}$,
- $\beta_{mp} \geq m\beta_p \quad \forall m, p \in \mathbb{N}$.

Then there exists the limit

$$\lim_{m\to\infty} \frac{\beta_m}{m} = \sup_{m\in\mathbb{N}} \{\frac{\beta_m}{m}\}.$$

Proof. See [Mus02, Lemma 2.2]. □

Remark 5.29. Combining Proposition 5.27 and Lemma 5.28 we deduce that for any integral divisor D with $k(X, D) \geq 0$ there exists

$$\lim_{p\to\infty} \frac{v(j_p)}{p} = \sup_{p\in\mathbb{N}} \{\frac{v(j_p)}{p}\}.$$

Proposition 5.30. *Let X be a smooth variety, let D be an integral divisor on X and let v be a discrete valuation on $K(X)/\mathbb{C}$. Then*

1. *if $k(X, D) \geq 0$ we have*

$$v(||D||) \geq \lim_{p\to\infty} \frac{v(j_p)}{p};$$

2. *if D is big we have*

$$v(||D||) = \lim_{p\to\infty} \frac{v(j_p)}{p}.$$

Proof.

1. Let us denote by e the exponent $e(D)$. Thus if p is sufficiently large then $ep \in \mathbb{N}(D)$, so that, thanks to Proposition 5.27 (3), we find

$$v(j_{ep}) \leq v(a_{ep})$$

for all $p >> 0$.
Now, by Remark 5.29 we have that there exists $\lim_{p\to\infty} \frac{v(j_p)}{p}$.
Therefore

$$\lim_{p\to\infty} \frac{v(j_p)}{p} = \lim_{p\to\infty} \frac{v(j_{ep})}{ep} \leq \lim_{p\to\infty} \frac{v(a_{ep})}{ep} = v(||D||).$$

2. If D is big, then by Theorem A.11, there exists an effective divisor E on X such that

$$\mathcal{I}(||pD||) \otimes \mathcal{O}_X(-E) \subseteq b(|pD|)$$

for all integers $p >> 0$. Hence, by using Proposition 5.16 (2),(3), we get

$$v(|pD|) \leq v(\mathcal{I}(X, ||pD||)) + v(\mathcal{O}_X(-E)) = v(j_p) + v(\mathcal{O}_X(-E)).$$

Then, as D has exponent $e(D) = 1$,

$$v(||D||) = \lim_{p\to\infty} \frac{v(|pD|)}{p} \leq \lim_{p\to\infty} \frac{v(j_p) + v(\mathcal{O}_X(-E))}{p} = \lim_{p\to\infty} \frac{v(j_p)}{p}.$$

The opposite inequality is given by the first part of the proposition. □

Remark 5.31. Let D be an integral big divisor. Then by Proposition 5.30 (2) and Remark 5.29 we find that $v(||D||) = 0$ if and only if $v(j_p) = 0$ for every $p \in \mathbb{N}$.
Hence, denoting by Z_v the center of the discrete valuation v, by Proposition 5.16 (1) we get

$$v(||D||) = 0 \iff Z_v \not\subseteq \mathcal{Z}(\mathcal{I}(X, ||pD||)) \quad \forall p \in \mathbb{N}.$$

Now let D be an integral divisor with $k(X, D) \geq 0$. By Proposition 5.30 (1) and Remark 5.29 we have that if $v(||D||) = 0$, then $v(j_p) = 0$ for every $p \in \mathbb{N}$. Thus, in this case, we can just say that

$$v(||D||) = 0 \Longrightarrow Z_v \not\subseteq \mathcal{Z}(\mathcal{I}(X, ||pD||)) \quad \forall p \in \mathbb{N}.$$

We are now able to show how the asymptotic order of vanishing of big divisors only depends on numerical equivalence classes.

Corollary 5.32. *Let X be a normal variety with function field $K = K(X)$, let D and E be big $\mathbb{Q}$-divisors on X such that $D \equiv E$ and let v be a discrete valuation on $K/\mathbb{C}$. Then*

$$v(||D||) = v(||E||).$$

Proof. First of all, by definition of asymptotic order of vanishing for a $\mathbb{Q}$-divisor, we can suppose D and E be numerically equivalent integral divisors. Now let $\mu : X' \to X$ be a resolution of singularities, so that, since μ is birational, we have that $\mu^*(D) \equiv \mu^*(E)$ and $v(||D||) = v(||\mu^*(D)||)$ (see Remark 5.26). Hence, without loss of generality, we can suppose that X is smooth. Thus, thanks to Theorem A.12, we have

$$\mathcal{I}(X, ||pD||) = \mathcal{I}(X, ||pE||)$$

for every integer $p > 0$. Therefore $v(j_{pD}) = v(j_{pE})$ for every $p \in \mathbb{N}$, so that, $v(||D||) = v(||E||)$ thanks to Proposition 5.30 (2). $\square$

Proposition 5.33. *Let X be a smooth variety with function field $K = K(X)$ and let v be a discrete valuation on $K/\mathbb{C}$, having center Z_v on X.*
If D is a $\mathbb{Q}$-linearly-effective $\mathbb{Q}$-divisor on X, then

$$v(||D||) = 0 \Longrightarrow Z_v \not\subseteq \mathbb{B}_-(D)$$

Proof. First of all note that, using Remark 5.2 and Proposition 5.22, we can assume, without loss of generality, that D is an integral divisor.
Now let $\xi \in X$ be the generic point of Z_v. Using Remark 5.31 we have that $Z_v \not\subseteq \mathcal{Z}(\mathcal{I}(X, ||pD||))$ for every $p \in \mathbb{N}$, whence $\mathcal{I}(X, ||pD||)_\xi = \mathcal{O}_{X,\xi}$.
On the other hand if we denote by n the dimension of X and by K_X be the canonical divisor on X, by Corollary 2.4 we can choose a suitable ample divisor A such that $G = K_X + (n+1)A$ is ample. Hence, thanks to Theorem A.10, we have that $\mathcal{I}(X, ||pD||) \otimes \mathcal{O}_X(G + pD)$ is globally generated for all $p \in \mathbb{N}$.
This shows that ξ is not contained in the base locus of $|G + pD|$, or, equivalently, that

$$Z_v \not\subseteq Bs(|G + pD|) \quad \forall p \in \mathbb{N}.$$

Now, setting $A_p = \frac{1}{p} \cdot G$, we have that the sequence of divisors $\{A_p\}_{p \in \mathbb{N}}$ satisfies the hypothesis of Proposition 5.4, so that

$$\mathbb{B}_-(D) = \bigcup_{p \in \mathbb{N}} \mathbb{B}(D + A_p) = \bigcup_{p \in \mathbb{N}} \mathbb{B}(G + pD),$$

using Corollary 3.7 for the last equality.
Therefore, if by absurd $Z_v \subseteq \mathbb{B}_-(D)$, then

$$Z_v = \bigcup_{p \in \mathbb{N}} (\mathbb{B}(pD + G) \cap Z_v),$$

whence there exists an integer $p_0 > 0$ such that $Z_v = \mathbb{B}(p_0 D + G) \cap Z_v$, that is $Z_v \subseteq \mathbb{B}(p_0 D + G) \subseteq Bs(|p_0 D + G|)$, so that we find a contradiction. $\square$

Remark 5.34. The opposite implication in the proposition above is not true in general (see Section 5.6 for a counter-example), but it holds if D is big (see Theorem 5.36).

5.5 v-bounded and v-semiample

Definition 5.35. Let X be a normal variety and let v be a discrete valuation on $K(X)/\mathbb{C}$. If D is an integral divisor on X with $k(X, D) \geq 0$ we say that D is *v-bounded* if there exists a constant $C > 0$ such that

$$v(|pD|) \leq C$$

for every $p \in \mathbb{N}(D)$.
If D is a $\mathbb{Q}$-linearly-effective $\mathbb{Q}$-divisor on X we say that D is *v-bounded* if there exists a constant $C > 0$ and an integer $n \in \mathbb{N}$ such that nD is integral and v-bounded.

Theorem 5.36. *Let X be a smooth variety with function field $K = K(X)$ and let v be a discrete valuation on $K/\mathbb{C}$, having center Z_v on X.*
If D is a big $\mathbb{Q}$-divisor on X, then the following conditions are equivalent:

1. *D is v-bounded;*
2. *$v(||D||) = 0$;*
3. *$Z_v \not\subseteq \mathbb{B}_-(D)$.*

Proof. First of all without loss of generality we assume that D is an integral divisor.

(1 $\Rightarrow$ 2) Trivial.

(2 $\Rightarrow$ 3) See Proposition 5.33.

(2 $\Rightarrow$ 1) Let $dim\, X = n$ and let A be a suitable very ample divisor such that $G = K_X + (n+1)A$ is ample. Repeating the first part of the proof of Proposition 5.33 we see that

$$Z_v \not\subseteq Bs(|G + pD|) \quad \forall p \in \mathbb{N}.$$

On the other hand, since D is big, by Theorem 3.20, there exist an integer $p_0 > 0$ and an integral effective divisor E such that $p_0 D \sim G + E$, so that, for every $p > p_0$, $pD \sim (p - p_0)D + G + E$.
Note that, for every $p \in \mathbb{N}$, $|G + pD| \neq \emptyset$ because $Z_v \not\subseteq Bs(|G + pD|)$, whence, by Proposition 5.18 (2), for every $p > p_0$, we have

$$v(|pD|) \leq v(|(p - p_0)D + G|) + v(|E|).$$

Now, by Proposition 5.18 (1), we have that $v(|(p - p_0)D + G|) = 0$, so that $v(|pD|) \leq v(|E|)$, that is D is v-bounded

(3 $\Rightarrow$ 2) Since D is big, by Theorem 3.20, we can find an integer $p_0 > 0$ and integral divisors A and E, with A ample and E effective, such that $p_0 D$ is linearly equivalent to $A + E$, so that, for every $p > p_0$, we get $pD \sim (p - p_0)D + A + E$.
Hence, by Proposition 5.25, we have

$$v(||pD||) \leq v(||(p - p_0)D + A||) + v(||E||).$$

Now, for $p > p_0$, we put $A_p = \frac{1}{p-p_0} A$. Since $\lim_{p\to\infty} ||A_p|| = 0$, we can use Proposition 5.4 to find that

$$\mathbb{B}_-(D) = \bigcup_{p>p_0} \mathbb{B}(D + A_p) = \bigcup_{p>p_0} \mathbb{B}((p - p_0)D + A).$$

Thus, using the hypothesis, we get that $Z_v \not\subseteq \mathbb{B}((p - p_0)D + A)$, for every $p > p_0$.
By Proposition 3.6, we find that there exists an integer $m_0 \in \mathbb{N}((p-p_0)D+A)$ such that for all $k \geq 1$, $Z_v \not\subseteq Bs(|km_0[(p-p_0)D+A]|)$. Hence, by Proposition 5.18 (1), $v(|km_0[(p - p_0)D + A]|) = 0$ for all $k \in \mathbb{N}$, so that

$$v(||m_0[(p - p_0)D + A]||) = 0.$$

Then, thanks to Proposition 5.22, we get

$$v(||(p - p_0)D + A||) = \frac{1}{m_0} \cdot v(||m_0[(p - p_0)D + A]||) = 0.$$

Therefore

$$v(||pD||) \leq v(||E||) \quad \forall p > p_0,$$

or, equivalently, using again Proposition 5.22, $v(||D||) \leq \frac{v(||E||)}{p}$ for every $p > p_0$, that is $v(||D||) = 0$. $\square$

As a corollary we find the following characterization of the restricted base locus of a big divisor:

Corollary 5.37. *Let X be a smooth variety and let D be a big integral divisor. Then we have the equality of sets*

$$\mathbb{B}_-(D) = \bigcup_{m\in\mathbb{N}} \mathcal{Z}(\mathcal{I}(X, ||mD||)).$$

Proof. Let $x \in X$, then there exists a valuation v_x on $K(X)/\mathbb{C}$ such that $x = Z_{v_x}$.
Hence, by Theorem 5.36, $x \notin \mathbb{B}_-(D)$ if and only if $v_x(||D||) = 0$. Using Remark 5.31, this is equivalent to say that $Z_{v_x} \not\subseteq \mathcal{Z}(\mathcal{I}(X, ||mD||))$ for all $m \in \mathbb{N}$, that is $x \notin \bigcup_m \mathcal{Z}(\mathcal{I}(X, ||mD||))$.

$\square$

Note that the corollary does not hold if D is not big (see Remark 5.47).

Remark 5.38 (Non-big divisors)**.** If we remove the hypothesis of bigness in Theorem 5.36 only some implications survive.
In particular, if D is a $\mathbb{Q}$-linearly effective $\mathbb{Q}$-divisor, we have that:

• D is v-bounded $\Rightarrow v(||D||) = 0$, trivially.

• $v(||D||) = 0 \Rightarrow Z_v \not\subseteq \mathbb{B}_-(D)$, as we have shown in Proposition 5.33.

• $v(||D||) = 0 \overset{?}{\Rightarrow} D$ is v-bounded.
We do not know whether D is v-bounded whenever $v(||D||) = 0$. However it is true if D is a divisor on a curve or a normal surface (see Remark 5.44).

• $Z_v \not\subseteq \mathbb{B}_-(D) \not\Rightarrow v(||D||) = 0$.
See Section 5.6 for a counter-example.

Definition 5.39. Let X be a normal variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$, having center $Z_v \subseteq X$, and let D be an integral divisor on X with $k(X, D) \geq 0$.
Then D is *v-semiample* if there exists a positive integer $l_0 \in \mathbb{N}(D)$ such that $v(|l_0D|) = 0$, or equivalently if $Z_v \not\subseteq \mathbb{B}(D)$.

Remark 5.40. Note that if D is v-semiample, that is $v(|l_0D|) = 0$, then, thanks to Proposition 5.18 (2), we have that $v(|kl_0D|) = 0$ for all $k \in \mathbb{N}$.
Note also that D is semiample if and only if D is v-semiample for every discrete valuation v on $K(X)/\mathbb{C}$.

Proposition 5.41. *Let X be a normal variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$ and let D be an integral divisor on X with $k(X, D) \geq 0$.*
If D is v-semiample, then D is v-bounded.

Proof. Let $l_0 \in \mathbb{N}(D)$ be such that $v(|l_0D|) = 0$ and let $e = e(D)$ be the exponent of D. Then, by definition, there exists an integer $m_0 > 0$ such that, for all $n \geq m_0e$,

$$n \in \mathbb{N}(D) \iff e \mid n,$$

and, in particular, for all $r \in \{0, \dots, l_0-1\}$, we have that $(m_0l_0+r)e \in \mathbb{N}(D)$. Hence we can define

$$C' = \max\{v(|mD|) \mid m \in \mathbb{N}(D),\ m < (m_0l_0 + l_0)e\}$$

$$C'' = \max_{0 \leq r < l_0} \{v(|(m_0l_0 + r)eD|)\}$$

$$C = \max\{C', C''\}.$$

We will show that, for every $m \in \mathbb{N}(D)$, $v(|mD|) \leq C$.
If $m < (m_0l_0 + l_0)e$, then $v(|mD|) \leq C' \leq C$.

If $m \geq (m_0 l_0 + l_0)e$, then, as $m \in \mathbb{N}(D)$, there exists an integer $k \geq m_0 l_0 + l_0$ such that $m = ke$.
Thus there exist $q \in \mathbb{N}$ and $r \in \{0, \ldots, l_0 - 1\}$ such that $k = q l_0 + r = l_0(q - m_0) + m_0 l_0 + r$, so that

$$mD = keD = l_0(q - m_0)eD + (m_0 l_0 + r)eD.$$

Note that $(m_0 l_0 + r)e \in \mathbb{N}(D)$, while $v(|l_0(q - m_0)eD|) = 0$ because $v(|l_0 D|) = 0$ and $q - m_0 > 0$:
In fact

$$q - m_0 = \frac{k - r}{l_0} - m_0 > \frac{k - l_0}{l_0} - m_0 \geq \frac{m_0 l_0}{l_0} - m_0 = 0.$$

Therefore, by Proposition 5.18 (2), we have $v(|mD|) \leq v(|l_0(q - m_0)eD|) + v(|(m_0 l_0 + r)eD|) \leq v(|(m_0 l_0 + r)eD|) \leq C'' \leq C$.

□

The following proposition describes what happens when we add the hypothesis of finite generation:

Proposition 5.42. *Let X be a normal variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$ and let D be an integral divisor on X with $k(X, D) \geq 0$. If D is finitely generated, then*

1. *there exists an integer $n \in \mathbb{N}(D)$ such that*

$$v(||D||) = \frac{v(|nD|)}{n}.$$

2. *$v(||D||) = 0$ if and only if D is v-semiample.*

Proof.

1. D is finitely generated if and only if the graded ring $R(X, D)$ is such. Hence, by Theorem 4.6, we have that there exists an integer $l > 0$ such that $H^0(X, O_X(mlD))$ generates the Veronese subring $R(X, D)^{(ml)}$ for all $m \geq 1$. Now if $e = e(D)$ is the exponent of D and h is a sufficiently large natural number, we put $n = hel$, so that $n \in \mathbb{N}(D)$ and $H^0(X, O_X(nD))$ generates the Veronese subring $R(X, D)^{(n)}$. Thus we find that for all $k > 1$ the map of multiplication of sections

$$\mu : H^0(X, O_X((k-1)nD)) \otimes H^0(X, O_X(nD)) \longrightarrow H^0(X, O_X(knD))$$

is surjective.
By Proposition 5.18 (2) we get

$$v(|(k-1)nD|) + v(|nD|) = v(|knD|)$$

for all $k > 1$, so that, working by induction, we find

$$v(|knD|) = k \cdot v(|nD|).$$

Thus

$$v(||D||) = \frac{1}{n} \cdot v(||nD||) = \frac{1}{n} \cdot \lim_{k\to\infty} \frac{v(|knD|)}{k} = \frac{v(|nD|)}{n}.$$

2. If $v(||D||) = 0$, then it follows immediately from the first part of the lemma that D is v-semiample.
For the opposite implication we do not need the hypothesis of finite generation: In fact if D is v-semiample, then there exists an integer $l \in \mathbb{N}(D)$ such that $v(|lD|) = 0$. Hence, noting that lD has exponent $e(lD) = 1$, thanks to Proposition 5.22 we have that

$$v(||D||) = \frac{1}{l} \cdot v(||lD||) = \frac{1}{l} \cdot \lim_{m\to\infty} \frac{v(|mlD|)}{m} = 0.$$

□

Remark 5.43. Note that, in particular, it follows by Proposition 5.42 and Proposition 5.41 that, under the hypothesis of finite generation, a divisor D with non-negative Iitaka dimension is v-bounded if and only if D is v-semiample if and only if $v(||D||) = 0$.

Remark 5.44. Using Remark 5.43 we find that, if X is a normal surface and D is an integral divisor on X with $k(X, D) \geq 0$, then $v(||D||) = 0$ if and only if D is v-bounded.
In fact, considering a suitable birational modification of X, we can suppose that X is smooth. Now a theorem of Zariski (see [Bad01, Theorem 14.19]) states that if D is a divisor on a smooth surface such that $k(X, D) \leq 1$, then D is finitely generated. Hence, if $v(||D||) = 0$ and $k(X, D) \leq 1$, then D is v-bounded by Remark 5.43. On the other hand if $k(X, D) = 2$, then D is big and the assertion follows by Theorem 5.36.

Proposition 5.45. *Let X be a normal variety, let Z be a subvariety of X and let D be a big integral divisor on X.*
If D is finitely generated and $Z \not\subseteq \mathbb{B}_-(D)$, then $Z \not\subseteq \mathbb{B}(D)$.

Proof. Let v be a discrete valuation on $K(X)/\mathbb{C}$ such that v has center Z. Then, by Theorem 5.36 we have that D is v-bounded. Hence, thanks to Remark 5.43, we find that D is v-semiample, that is $Z \not\subseteq \mathbb{B}(D)$. □

The following theorem generalizes Theorem 4.16 giving a different characterization of semiampleness.

Theorem 5.46. *Let X be a normal variety with function field $K = K(X)$ and let D be an integral divisor on X with $k(X, D) \geq 0$.*
Then the following statements are equivalent:

1. *D is semiample*
2. *D is finitely generated and there exists a constant $C > 0$ such that*

$$mult_x|mD| \leq C$$

for every $m \in \mathbb{N}(D)$, for every $x \in X$.

3. *D is finitely generated and, for every discrete valuation v on $K/\mathbb{C}$, D is v-bounded.*

Proof.

(1 ⇔ 2) See Theorem 4.16.

(1 ⇒ 3) D is finitely generated by Theorem 4.16. Moreover, as D is semiample, D is v-semiample for every discrete valuation v on $K/\mathbb{C}$. Thus, by Proposition 5.41, D is v-bounded.

(3 ⇒ 2) Follows immediately from the fact that for every $x \in X$ we have $mult_x|mD| = v_x(|mD|)$, where v_x is the discrete valuation whose center is the point x. □

5.6 An example

In the following example we provide an effective, non-big divisor D on a smooth ruled surface X and a discrete valuation v on $K(X)/\mathbb{C}$, such that $Z_v \not\subseteq \mathbb{B}_-(D)$ but $v(||D||) > 0$. This shows how the hypothesis of bigness in Theorem 5.36 is essential (see also Remark 5.38).

Let C be a nonsingular curve of genus $g \geq 1$ and let P be a non-torsion divisor of degree zero on C (whose existence is assured by Proposition 4.17). Let $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{O}_C(P)$. Then $\mathcal{E}$ is a locally free sheaf of rank two on C and we consider the ruled surface $X = \mathbb{P}(\mathcal{E})$ (see [Har77, V, §2]).
Note that $\mathcal{E}$ is normalized, that is $H^0(C, \mathcal{E}) \neq 0$ and $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for every line bundle $\mathcal{L}$ of negative degree. Hence, by [Har77, V, 2.8], there exists a section D such that $\mathcal{O}_X(D) \simeq \mathcal{O}_X(1)$, so that D is nonsingular and $(D^2) = deg\, \mathcal{E} = 0$.
Now, let $x \in D$ and let v be the discrete valuation on $K(X)/\mathbb{C}$ whose center is $Z_v = \{x\}$.
We will show that $\mathbb{B}_-(D) = \emptyset$ (so that, in particular, $Z_v = \{x\} \not\subseteq \mathbb{B}_-(D)$), while $v(||D||) = 1$.

$B_-(D) = \emptyset$: We will prove that D is nef, so that the assert follows by Lemma 5.3.

We denote by f a fibre of the morphism $\pi : X \to C$. It follows by [Har77, V, 2.20] that, if Y is an irreducible curve on X, then $Y \equiv aD + bf$, with $b \geq 0$. Thus

$$(D \cdot Y) = (D \cdot (aD + bf)) = b \geq 0.$$

$\boldsymbol{h^0(X, \mathcal{O}_X(kD)) = 1}$ for all $k \geq 1$: We denote by $S^k(\mathcal{E})$ the k^{th} symmetric power of $\mathcal{E}$. By [Har77, II, 7.11], for all $k \geq 1$ we have that $\pi_*(\mathcal{O}_X(k)) \simeq S^k(\mathcal{E})$. Hence

$$H^0(X, \mathcal{O}_X(k)) = H^0(C, S^k(\mathcal{E})) = \bigoplus_{t=0}^{k} H^0(C, \mathcal{O}_C(tP)) = H^0(C, \mathcal{O}_C).$$

Thus

$$h^0(X, \mathcal{O}_X(kD)) = h^0(X, \mathcal{O}_X(k)) = h^0(C, \mathcal{O}_C) = 1$$

for all $k \geq 1$. Note that, in particular, this implies that D is not big.

$\boldsymbol{v(||D||) = 1}$: Since $h^0(X, \mathcal{O}_X(kD)) = 1$ for all $k \geq 1$, we have that the exponent $e(D) = 1$, while $dim_{\mathbb{C}}|kD| = h^0(X, \mathcal{O}_X(kD)) - 1 = 0$, that is $|kD| = \{kD\}$.
Hence, for all $k \geq 1$, we get

$$v(|kD|) = mult_x|kD| = mult_x\,(kD) = k.$$

Therefore

$$v(||D||) = \lim_{m \to \infty} \frac{v(|mD|)}{m} = 1.$$

Remark 5.47. This example also proves that Corollary 5.37 does not work if the divisor is not big. In fact, for all $m \geq 1$, we easily check that $\mathcal{I}(X, ||mD||) = \mathcal{O}_X(-mD)$. Then, set-theoretically, we have

$$\bigcup_{m \in \mathbb{N}} \mathcal{Z}(\mathcal{I}(X, ||mD||)) = D,$$

while $\mathbb{B}_-(D) = \emptyset$.

Appendix A

Asymptotic multiplier ideals

Throughout this appendix, unless clearly specified, X denotes a nonsingular variety.
Moreover, given a birational morphism $\mu : X' \to X$, we denote by $exc(\mu)$ the sum of the exceptional divisors of μ.
We begin by giving some preliminary definitions that we will need to define the asymptotic multiplier ideal associated to a linear series.

Definition A.1. *Let $D = \sum a_i D_i$ be a $\mathbb{Q}$-divisor on X, with D_i prime divisors. The integral part $[D]$ of D is the integral divisor*

$$[D] = \sum [a_i] D_i,$$

where, for any $x \in \mathbb{Q}$, we denote by $[x]$ the greatest integer $\leq x$.

Definition A.2. Let D_i be distinct prime divisors on X and let $D = \sum D_i$ be an effective reduced divisor on X. D has *simple normal crossings* (and D is an *SNC divisor*) if each D_i is smooth and if D is defined in a neighborhood of any point by an equation in local analytic coordinates of the type

$$z_1 \cdot \dots \cdot z_k = 0$$

for some $k \leq dim\ X$.
A $\mathbb{Q}$-divisor $E = \sum a_i D_i$ has *simple normal crossing support* if the underlying reduced divisor $\sum D_i$ is an SNC divisor.

Definition A.3. Let D be an integral divisor on X and let $|V| \subseteq |D|$ be a non-empty linear series. A *log resolution* of $|V|$ is a birational map $\mu : X' \to X$, with X' nonsingular, such that

$$\mu^*|V| = |W| + F,$$

where $F + exc(\mu)$ is a divisor with SNC support, and $|W|$ is a free linear series.

Given a non-empty linear series $|V|$, the existence of a log resolution of V is assured by Hironaka's theorem (see [Laz04, I, 4.1.3]).
Moreover, given a log resolution $\mu : X' \to X$, we denote by

$$K_{X'/X} = K_{X'} - \mu^* K_X$$

the *relative canonical divisor of X' over X.*

Definition A.4. Let D be a divisor on X, let $|V| \subseteq |D|$ be a non-empty linear series and let $\mu : X' \to X$ be a log resolution of $|V|$ with

$$\mu^*|V| = |W| + F.$$

Given a rational number $c > 0$, the *multiplier ideal* associated to c and $|V|$ is

$$\mathcal{I}(c \cdot |V|) = \mathcal{I}(X, c \cdot |V|) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [c \cdot F]).$$

If $|V| = \emptyset$ we put $\mathcal{I}(c \cdot |V|) = 0$ for every $c > 0$.

The definition makes sense because the multiplier ideal does not depend on the log resolution considered (see [Laz04, II, 9.2.18]).

Lemma A.5. *Let D be a divisor on X with $k(X, D) \geq 0$, let $c \in \mathbb{Q}^+$ and let $p \in \mathbb{N}$. Then*

1. *For every integer $k > 0$ we have*
$$\mathcal{I}\big(\frac{c}{p} \cdot |pD|\big) \subseteq \mathcal{I}\big(\frac{c}{pk} \cdot |pkD|\big).$$
2. *The family of ideals*
$$\Big\{\mathcal{I}\big(\frac{c}{p} \cdot |pD|\big)\Big\}_{p \geq 1}$$
has a unique maximal element.

Proof. See [Laz04, II, 11.1.1] □

We are now able to define the asymptotic multiplier ideal associated to a linear series:

Definition A.6. Let D be an integral divisor on X with $k(X, D) \geq 0$ and let $c \in \mathbb{Q}^+$.
The *asymptotic multiplier ideal* associated to c and $|D|$,

$$\mathcal{I}(c \cdot ||D||) = \mathcal{I}(X, c \cdot ||D||) \subseteq \mathcal{O}_X,$$

is defined as the unique maximal element of the family of ideals $\Big\{\mathcal{I}\big(\frac{c}{p} \cdot |pD|\big)\Big\}$.

Remark A.7. Note that for every divisor D on X with $k(X, D) \geq 0$ the asymptotic multiplier ideal $\mathcal{I}(X, ||D||)$ is a non-zero quasi-coherent sheaf of ideals.

We recall here some important results concerning the sheaves of ideals just defined.

Theorem A.8. *Let D be an integral divisor on X with $k(X, D) \geq 0$, and let $c > 0$ be a fixed rational number. Then*

1. *For every $m \in \mathbb{N}$ we have*

$$\mathcal{I}(c \cdot ||mD||) \supseteq \mathcal{I}(c \cdot ||(m+1)D||),$$

   that is the ideals $\mathcal{I}(c \cdot ||mD||)$ form a decreasing sequence in m.

2. *For every $m \in \mathbb{N}(D)$ we have*

$$b(|mD|) \subseteq \mathcal{I}(X, ||mD||).$$

Proof. See [Laz04, II, 11.1.8]. □

Theorem A.9 (Subadditivity)**.** *Let D be a divisor on X with $k(X, D) \geq 0$ and let $c \in \mathbb{Q}^+$. Then*

$$\mathcal{I}(c \cdot ||(m+k)D||) \subseteq \mathcal{I}(c \cdot ||mD||) \cdot \mathcal{I}(c \cdot ||kD||).$$

for any integers $m, k > 0$.

Proof. See [Laz04, II, 11.2.4]. □

Theorem A.10. *Let X be an n-dimensional nonsingular variety and let D, B, A be integral divisors on X such that D has non-negative Iitaka dimension, B is globally generated and ample and A is big and nef.*
Then, if we denote by K_X the canonical divisor on X, we have

1. $H^i(X, \mathcal{O}_X(K_X + mD + A) \otimes \mathcal{I}(||mD||)) = 0$ *for any integers* $i, m > 0$.

2. $\mathcal{O}_X(K_X + nB + A + mD) \otimes \mathcal{I}(||mD||)$ *is globally generated for all* $m \in \mathbb{N}$.

Proof. See [Laz04, II, Theorem 11.2.12 and Corollary 11.2.13]. □

Theorem A.11. *Let D be a big integral divisor on X. Then there exists an effective divisor E on X and a positive integer $t_0(D)$ such that*

$$\mathcal{I}(||mD||) \otimes \mathcal{O}_X(-E) \subseteq b(|mD|)$$

for every $m \geq t_0(D)$.

Proof. See [Laz04, II, 11.2.21]. □

Theorem A.12. *Let D_1 and D_2 be numerically equivalent big divisors on X. Then*

$$\mathcal{I}(X, c \cdot ||D_1||) = \mathcal{I}(X, c \cdot ||D_2||)$$

for every $c \in \mathbb{Q}^+$.

Proof. See [Laz04, II, 11.3.12]. □

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