UNIVERSITÀ DEGLI STUDI ROMA TRE
FACOLTÀ DI SCIENZE MM. FF. NN.

Graduation Thesis in Mathematics
by

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Some Issues about the Extendability of Projective Surfaces

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ACADEMIC YEAR 2003 - 2004
JULY 2004

AMS Classification: primary 14J30, secondary 14J26
Key Words: Algebraic Varieties, Extendability, Gaussian Maps, Ruled Surfaces

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Introduction

The main problem in algebraic geometry is the problem of the classification of algebraic varieties, that is giving a rational criterion for classifying all the algebraic varieties up to isomorphism. Actually this is a huge problem and nobody expects to solve it completely in a reasonable amount of time. Nevertheless this is certainly a goose that lays the golden egg or, more prosaically and respectfully, a guiding problem, that offers impulse to further research in geometry and that allows geometers to measure their achievements.

The central core in the classification problem is, of course, the study of the geometry of algebraic varieties and of their subvarieties. Clearly less the dimension, easier the study. Thus it is not weird at all, considering a variety, to analyze its hyperplane sections to get a good grasp over its basic components. But even the opposite approach can throw some light on the problem. It is just a change of point of view and it leads to the notion of extendability. Taking a nondegenerate variety X in $\mathbb{P}^r$ we can ask whether there exists a nondegenerate variety Y in $\mathbb{P}^{r+1}$ such that X is one of its hyperplane sections. Stated exactly in this way, the question is easy to answer, but useless: actually we can build a cone in $\mathbb{P}^{r+1}$ over every variety in $\mathbb{P}^r$. Thus we have to expunge the cone from the list of our possible extensions, and if there still exists Y as before but different from a cone then X is said to be *extendable*. In this way, for example, having to handle threefolds in $\mathbb{P}^{r+1}$, one can zero in on surfaces in $\mathbb{P}^r$ and then asking himself if they are extendable: if one is not extendable, say X, we are forced to conclude that a threefold different from a cone with X as one of its hyperplane sections simply *does not exist*.

The first thing to note is that the notion of extendability is not an intrinsic notion of a variety, but it deeply depends on the immersion of the variety in the projective space. The second thing to note is that *not all the varieties are extendable*. Intuitively if a variety in $\mathbb{P}^r$ is too much 'bent' one cannot hope to extend it: even considering the cone over it leads nowhere, since the vertex of the cone is too entangled and thus it cannot be round off. We see that one way of dealing with the problem of extendability is to study the complexity of the shape of the variety in its immersion, that is to study the normal directions to the variety at every point. Luckily cohomology comes to our aid: the most general theorem about extendability, Zak's theorem, simply states that if the dimension of $H^0(\mathcal{N}_{X/\mathbb{P}^r}(-1))$ (where $\mathcal{N}_{X/\mathbb{P}^r}$ is the normal sheaf of X in $\mathbb{P}^r$) is minimal then X is not extendable at all. It is an elegant theorem that does not set a limit to the dimensions of the variety and to the possible extensions. We will prove a simplified version of Zak's theorem in the first section of the first chapter.

Unluckily Zak's theorem is very difficult to apply in practice. However, in the case of curves, Wahl linked the dimension of the cohomology group aforesaid to the corank of Gaussian maps, the dimension being minimal when a certain Gaussian map is surjective. We will give the basic definitions and the proof of Wahl's theorem in 1.2. In 1.3 we will discuss the surjectivity of the Guassian maps, especially stating a theorem of Bertram, Ein, Lazarsfeld. But we are mainly interested in the extendability of surfaces, primarily because this study is related to the classification of threefolds, while the classification of surfaces has already been carried out to some extent. In order to say something about the extendability of surfaces we cannot use Wahl's theorem (that is valid only for curves) but we need some other tools: in the last section of the first chapter we will state a theorem by Knutsen, Lopez and Muñoz that, generalizing Wahl's theorem, gives some sufficient conditions for a surface to be non-extendable.

The rest of the thesis will be devoted to the application of Knutsen-Lopez-Muñoz's theorem (KLM) in some particular cases, that is in the cases of rational ruled surfaces. To apply KLM it is necessary to have a good knowledge of the geometry of the surface, e.g. its canonical divisor, its line bundles and their cohomology, the gonality of the curves lying on it. For this purpose it is necessary first of all to study (rational) ruled surfaces in depth. This will be the task of the second chapter: in section 1 we will first introduce ruled surfaces, that can be essentially seen as surfaces swept out by a moving line along a base curve. Then we will compute their Picard groups in terms of the Picard group of the underlying curve, then we will look at them as projective bundles. In the second section we will specialize to rational ruled surfaces, i.e. ruled surfaces where the base curve is a line, classifying base-point free, ample and very ample linear systems. We will follow [7].

In the last chapter we will deepen the study of rational ruled surfaces investigating the cohomology of the line bundles and the gonality of the curves. Again this could not have been possible without a prior study of the divisors on the surface. Eventually, at the end of the chapter, having translated, in the case of rational ruled surfaces, KLM's conditions in a number of equations, we will solve the problem of extendability of rational ruled surfaces in many cases. This is the original result of this thesis.

Remark: throughout this thesis *variety* will mean an integral, separated scheme of finite type over the field of complex numbers $\mathbb{C}$.

Chapter 1

Extendable varieties and Wahl-Gaussian maps

The aim of this chapter is to develop some tools to establish if a certain projective variety is extendable or not, or better, to find out some sufficient conditions that allow us to conclude that a projective variety, with its embedding in the projective space, is not extendable. Be careful! it is always possible, passing to a higher dimension, to build a cone over a projective variety $X \subset \mathbb{P}^r$, but in some cases this is the only geometric object which has X has one of its hyperplane sections. This means that in these cases it is useless to try to deform the cone to extend X, even if this sounds like a good idea.

Perhaps the most general result about extendability is a theorem of the Russian mathematician F.L.Zak. It gives a unique, easy-to-state, sufficient condition for X not to be extendable. The condition is about the dimension of the vector space $H^0(\mathcal{N}_{X/\mathbb{P}^r}(-1))$ ($\mathcal{N}_{X/\mathbb{P}^r}$ is the normal sheaf), that does not have to be higher than $r+1$. Zak's theorem does not put limitations on the geometry of the extension, but in section 1 we will give a simplified proof of the theorem (due to A.F. Lopez) only for nonsingular extensions.

Unluckily, as it often happens in mathematics, a general, elegant theorem is very difficult to apply in practice. And that is the case for Zak's theorem. How can we manage to compute the dimension of $H^0(\mathcal{N}_{X/\mathbb{P}^r}(-1))$? In the case of a curve C the answer was given by J. Wahl, who related this dimension to the corank of a particular map, called Gaussian map. In the second section we will give the basic definition of a Gaussian map and we will prove Wahl's theorem: if the Gaussian map is surjective then C is not extendable. Note that this theorem does not have a converse as it is, but Wahl (see section 2) showed that under some other conditions his theorem has actually a converse.

Thus, at least in the case of curves, the problem has been reduced to show if a Gaussian map is or not surjective. This probably explains the great amount of work that numerous mathematicians spent in studying Gaussian maps. Fruitful work, we can say, since there are lots of results about the surjectivity of Gaussian maps, some of which are stated in section 2. The two we are more interested in (for a reason that will be clear afterwards) are a theorem of Bertram, Ein, Lazarsfeld and a theorem of Tendian.

Finally we will go to the study of extendability of surfaces. We will state, without proof, a theorem of A.L. Knutsen, A.F. Lopez and R. Muñoz that gives sufficient conditions for a surface in $\mathbb{P}^r$ not to be extendable. The final part of this thesis will be devoted to the application of the theorem in particular cases.

1.1 Introduction to Extendability

Let X be a smooth nondegenerate (i.e. not contained in any hyperplane) projective variety in $\mathbb{P}^r$. We say that X is extendable if there exists a non-degenerate variety $Y \subset \mathbb{P}^{r+1}$ such that:

1. $X = Y \cap H$, where $H = \mathbb{P}^r$ is a hyperplane in $\mathbb{P}^{r+1}$.
2. Y is not a cone over X.

Condition (2) is essential: otherwise all X would be extendable.

Example 1.1.1. A nondegenerate curve in $\mathbb{P}^2$ is always extendable. In fact a nondegenerate curve is the zero-locus $Z(F)$ of a homogeneous polynomial $F(X_0, X_1, X_2)$ of degree at least 2. Now consider $G := F + X_3^{\deg F}$. G is homogeneous and $Z(G) \cap \{X_3 = 0\} = Z(F)$. Moreover G is not a cone because G is not singular at all outside from $Z(F)$. Actually every complete intersection variety (i.e. varieties of dimension r in $\mathbb{P}^n$ whose ideal is generated by $n - r$ elements) is extendable.

Before going on, we recall some basic definitions and propositions:

1. ([7],II,8.17). Let Y be a nonsingular subvariety of a nonsingular variety X, defined by a sheaf of ideals $\mathcal{I}$. We call $\mathcal{I}/\mathcal{I}^2$ the conormal sheaf of Y in X. It is a locally free sheaf of rank $\operatorname{codim}(Y,X)$. Its dual $\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is called the normal sheaf of Y in X. Moreover there is an exact sequence of sheaves:
$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_X \otimes \mathcal{O}_Y \longrightarrow \Omega_Y \longrightarrow 0 \tag{1.1}$$

2. Let X be a nonsingular variety. We define the tangent sheaf of X to be $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$, i.e. the dual of the sheaf of differentials.

3. Taking the dual of (1.1) we obtain:
$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X \otimes \mathcal{O}_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0 \tag{1.2}$$

4. *Euler's sequence* ([7],II,8.20.1). There is an exact sequence of sheaves:
$$0 \longrightarrow \Omega_{\mathbb{P}^r} \longrightarrow \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow 0 \tag{1.3}$$
Taking its dual we obtain:
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus(r+1)} \longrightarrow \mathcal{T}_{\mathbb{P}^r} \longrightarrow 0 \tag{1.4}$$
If X is a nonsingular subvariety of $\mathbb{P}^r$ then, tensoring by $\mathcal{O}_X$ we obtain:
$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{\oplus(r+1)} \longrightarrow \mathcal{T}_{\mathbb{P}^r|_X} \longrightarrow 0 \tag{1.5}$$

5. *Serre's duality theorem* ([7],III,7.8 and 7.12). If X is a projective nonsingular variety of dimension n then for every locally free sheaf $\mathcal{F}$ on X we have $H^i(X,\mathcal{F}) \cong (H^{n-i}(X,\mathcal{F}^* \otimes \omega_X))^*$, where $*$ means 'dual' and ω_X is the canonical sheaf of X.

6. *Serre's vanishing theorem* ([7],III,5.2). Let X be a projective variety in $\mathbb{P}^r$. Let $\mathcal{F}$ be a coherent sheaf on X. Then there is an integer n_0 depending on $\mathcal{F}$, such that for each $i > 0$ and each $n \geq n_0$, $H^i(X,\mathcal{F} \otimes \mathcal{O}_X(n)) = 0$.

From now on X will be a projective, smooth, nondegenerate variety in $\mathbb{P}^r$ and Y a nonsingular extension of X in $\mathbb{P}^{r+1}$. We suppose also that $\operatorname{codim} X \geq 2$.

Lemma 1.1.2. *$\mathcal{N}_{X/\mathbb{P}^r}(-1)$ is globally generated. Moreover it has at least $r+1$ global sections.*

Proof. We recall that a sheaf of $\mathcal{O}_X$-modules is said to be globally generated, or generated by global sections, if there is a set of global sections such that for every stalk their images generate the stalk as an $\mathcal{O}_{X,x}$-module. Note that if a sheaf can be written as a quotient of a free sheaf, it is globally generated. From (1.5) and (1.2), tensoring by $\mathcal{O}_X(-1)$ we indeed have a surjection: $\mathcal{O}_X^{\oplus(r+1)} \to \mathcal{T}_{\mathbb{P}^r|X}(-1) \to \mathcal{N}_{X/\mathbb{P}^r}(-1)$.

Since $H^0(\mathcal{O}_X(-1)) = 0$ and $H^0(\mathcal{T}_X(-1)) = 0$ again from (1.5) and (1.2) we obtain two injections: $H^0(\mathcal{O}_X^{\oplus(r+1)}) \to H^0(\mathcal{T}_{\mathbb{P}^r|X}(-1))$ and $H^0(\mathcal{T}_{\mathbb{P}^r|X}(-1)) \to H^0(\mathcal{N}_{X/\mathbb{P}^r}(-1))$. Therefore $h^0(\mathcal{N}_{X/\mathbb{P}^r}(-1)) \geq h^0(\mathcal{O}_X^{\oplus(r+1)})$ $= r+1$. □

Lemma 1.1.3. *If $h^0(\mathcal{N}_{X/\mathbb{P}^r}(-1)) = r+1$ then $H^0(\mathcal{N}_{X/\mathbb{P}^r}(-i)) = 0$ for all $i \geq 2$.*

Proof. We use induction on i. First of all suppose that $h^0(\mathcal{N}_{X/\mathbb{P}^r}(-2)) > 0$, i.e. there exists a global section s that is not the null section. Since $X \subset \mathbb{P}^r$ we know that there is a vector space $V \subset H^0(\mathcal{O}_X(1))$ such that $\dim V = r+1$. Let $t_0, \ldots, t_r$ be a basis for V. For all i, $s \otimes t_i \in H^0(\mathcal{N}_{X/\mathbb{P}^r}(-1))$ and, in addition, they are $r+1$ and linearly independent, thus they form a basis for $H^0(\mathcal{N}_{X/\mathbb{P}^r}(-1))$. Since $\mathcal{N}_{X/\mathbb{P}^r}(-1)$ is globally generated (1.1.2), the sections $s \otimes t_i$ generate the stalk of $\mathcal{N}_{X/\mathbb{P}^r}(-1)$ at every point x. But then $s_x \otimes (t_i)_x$ should generate $\mathcal{N}_{X/\mathbb{P}^r}(-1)_x$ as a $\mathcal{O}_{X,x}$-module. But $\mathcal{N}_{X/\mathbb{P}^r}(-1)_x$ is a free $\mathcal{O}_{X,x}$-module of rank $\operatorname{codim} X \neq 1$, while $\mathcal{O}_X(1)_x$ is a free $\mathcal{O}_{X,x}$-module of rank 1 and therefore also the $\mathcal{O}_{X,x}$-module generated by $s_x \otimes (t_i)_x$ if it is free has rank 1. Contradiction.

Now that the basis of the induction has been proved let us proceed with the induction step. Consider a hyperplane H in $\mathbb{P}^r$ and its defining sequence on X:

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X \cap H} \longrightarrow 0 \tag{1.6}$$

Tensor it by $\mathcal{N}_{X/\mathbb{P}^r}(-i)$, we obtain:

$$0 \longrightarrow \mathcal{N}_{X/\mathbb{P}^r}(-i-1) \longrightarrow \mathcal{N}_{X/\mathbb{P}^r}(-i) \longrightarrow \mathcal{N}_{X/\mathbb{P}^r}(-i)_{|X\cap H} \longrightarrow 0 \qquad (1.7)$$

Taking cohomology we have $H^0(\mathcal{N}_{X/\mathbb{P}^r}(-i-1)) \hookrightarrow H^0(\mathcal{N}_{X/\mathbb{P}^r}(-i))$. But $H^0(\mathcal{N}_{X/\mathbb{P}^r}(-i))$ is zero by the induction hypothesis and hence $H^0(\mathcal{N}_{X/\mathbb{P}^r}(-i-1))$ is zero too. □

Lemma 1.1.4. *Let $\mathcal{N}_{Y/\mathbb{P}^{r+1}}$ be the normal sheaf of Y in $\mathbb{P}^{r+1}$. Then $\mathcal{N}_{Y/\mathbb{P}^{r+1}} \otimes \mathcal{O}_X = \mathcal{N}_{Y/\mathbb{P}^{r+1}|X}$ is isomorphic to $\mathcal{N}_{X/\mathbb{P}^r}$.*

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_X & \longrightarrow & \mathcal{T}_{\mathbb{P}^r|X} & \longrightarrow & \mathcal{N}_{X/\mathbb{P}^r} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_{Y|X} & \longrightarrow & \mathcal{T}_{\mathbb{P}^{r+1}|X} & \longrightarrow & \mathcal{N}_{Y/\mathbb{P}^{r+1}|X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_{X/Y} & \longrightarrow & \mathcal{N}_{\mathbb{P}^r/\mathbb{P}^{r+1}|X} & \longrightarrow & \mathcal{CK} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{CK}$ is the coker of the two maps shown and where the first and the second row and the first and the second column come from (1.2) and the inclusions $X \subset \mathbb{P}^r$, $Y \subset \mathbb{P}^{r+1}$, $X \subset Y$, $\mathbb{P}^r \subset \mathbb{P}^{r+1}$, respectively. Since $\mathcal{N}_{X/Y} \cong \mathcal{O}_X(1)$, $\mathcal{N}_{\mathbb{P}^r/\mathbb{P}^{r+1}} \cong \mathcal{O}_{\mathbb{P}^r}(1)$ and $\mathcal{O}_{\mathbb{P}^r}(1) \otimes \mathcal{O}_X \cong \mathcal{O}_X(1)$, then $\mathcal{N}_{X/Y} \cong \mathcal{N}_{\mathbb{P}^r/\mathbb{P}^{r+1}|X}$, therefore $\mathcal{CK} = 0$. That is: $\mathcal{N}_{Y/\mathbb{P}^{r+1}|X}$ is isomorphic to $\mathcal{N}_{X/\mathbb{P}^r}$.

□

Theorem 1.1.5 (Zak)**.** *Let $X \subset \mathbb{P}^r$ be a smooth irreducible nondegenerate variety of codimension at least 2. If $h^0(\mathcal{N}_{X/\mathbb{P}^r}(-1)) = r+1$ then X is not extendable.*

Proof. We will only prove that, under the hypotheses of the theorem, X is not extendable to a nonsingular variety. Actually we will give a simplified proof by A. F. Lopez, but the theorem is still valid, even if we allow the extension to be singular. Thus, let us suppose that X is extendable and let us call $Y \subset \mathbb{P}^{r+1}$ the nonsingular extension of X. Recall that X is a hyperplane section of Y by hypothesis, hence the defining sequence of X in Y is:

$$0 \longrightarrow \mathcal{O}_Y(-1) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{1.8}$$

Let $i \in \mathbb{Z}$. If we tensor (1.8) with $\mathcal{N}_{Y/\mathbb{P}^{r+1}}(-i)$ we obtain:

$$0 \longrightarrow \mathcal{N}_{Y/\mathbb{P}^{r+1}}(-1-i) \longrightarrow \mathcal{N}_{Y/\mathbb{P}^{r+1}}(-i) \longrightarrow \mathcal{O}_X \otimes \mathcal{N}_{Y/\mathbb{P}^{r+1}}(-i) \longrightarrow 0 \tag{1.9}$$

By 1.1.4, $\mathcal{O}_X \otimes \mathcal{N}_{Y/\mathbb{P}^{r+1}}(-i) \cong \mathcal{N}_{X/\mathbb{P}^r}(-i)$. By 1.1.3, if $i \geq 2$ then $H^0(\mathcal{N}_{X/\mathbb{P}^r}(-i)) = 0$, therefore $\forall i \geq 2$ $H^0(\mathcal{N}_{Y/\mathbb{P}^{r+1}}(-1-i)) \cong H^0(\mathcal{N}_{Y/\mathbb{P}^{r+1}}(-i))$. By Serre's duality theorem, $H^0(\mathcal{N}_{Y/\mathbb{P}^{r+1}}(-i)) \cong H^{\dim Y}(\mathcal{N}^*_{Y/\mathbb{P}^{r+1}} \otimes \omega_Y \otimes \mathcal{O}_Y(i))$ and the second is zero if i is sufficiently large by Serre's vanishing theorem.

Let us consider (1.9) again, with $i = 1$. Since by the discussion above $H^0(\mathcal{N}_{Y/\mathbb{P}^{r+1}}(-2)) = 0$, taking cohomology we have an injection: $H^0(\mathcal{N}_{Y/\mathbb{P}^{r+1}}(-1)) \hookrightarrow H^0(\mathcal{N}_{X/\mathbb{P}^r}(-i))$. But this is impossible because by 1.1.2 $h^0(\mathcal{N}_{Y/\mathbb{P}^{r+1}}(-1)) \geq r+2$ while, by hypothesis, $h^0(\mathcal{N}_{X/\mathbb{P}^r}(-i)) = r+1$. $\square$

Thus the problem is now how to compute the dimension of $H^0(\mathcal{N}_{X/\mathbb{P}^r}(-1))$. In the case of X curve one possible answer was given by Jonathan Wahl. But before going on we need some basic definitions.

1.2 Wahl - Gaussian maps and Extendability

Let X be a smooth projective variety and $\Omega_X = \Omega_{X/\mathbb{C}}$ its sheaf of differentials. Let L be a line bundle on X. Let $s \in H^0(X, L)$ be a global section. We would like to define a differential for s, $\mathrm{d}s$. Since L is an invertible sheaf, locally (if $U \subset X$ is an open set) $L_{|U}$ is a free $\mathcal{O}_U$-module of rank 1. If σ is a generator for $H^0(U, L_{|U})$ as a $H^0(U, \mathcal{O}_U)$-module we can write $s = f\sigma$ with $f \in H^0(U, \mathcal{O}_U)$. Thus it seems natural to define locally $\mathrm{d}s := \mathrm{d}f \otimes \sigma$, to obtain a global section of $\Omega_X \otimes L$. Unluckily it clearly turns out that this definition does not work because it depends on the generator σ: in fact if ρ is another local generator, $\sigma = g\rho$, then $s = fg\rho$ and $\mathrm{d}s := (g\mathrm{d}f + f\mathrm{d}g) \otimes \rho = \mathrm{d}f \otimes \sigma + f\mathrm{d}g \otimes \rho \neq \mathrm{d}f \otimes \sigma$. But this computation suggests a way to avoid the problem.

Lemma 1.2.1. *Let s, t be two global sections of L. Suppose that locally $s = f\sigma$ and $t = h\sigma$. Then $W_L(s,t) = ds \otimes t - dt \otimes s$ is a well-defined section of $H^0(\Omega_X \otimes L \otimes L)$.*

Proof. From the discussion above we have only to show that the definition of $W_L(s,t)$ is independent from the local trivialization σ. Suppose that ρ is another local trivialization and that $\sigma = g\rho$. Then, depending on the trivialization, $\mathrm{d}s \otimes t$ is equal to $h\mathrm{d}f \otimes \sigma \otimes \sigma$ or $g^2 h\mathrm{d}f \otimes \rho \otimes \rho + ghf\mathrm{d}g \otimes \rho \otimes \rho = h\mathrm{d}f \otimes \sigma \otimes \sigma + ghf\mathrm{d}g \otimes \rho \otimes \rho$. Analogously $\mathrm{d}t \otimes s$ is equal to $f\mathrm{d}h \otimes \sigma \otimes \sigma$ or $f\mathrm{d}h \otimes \sigma \otimes \sigma + ghf\mathrm{d}g \otimes \rho \otimes \rho$. Now it is clear that $W_L(s,t)$ is well-defined. □

Definition 1.2.2.

$$\begin{aligned} \Phi_L : \quad \wedge^2 H^0(X, L) &\longrightarrow H^0(X, \Omega_X \otimes L^{\otimes 2}) \\ s \wedge t &\longmapsto s\mathrm{d}t - t\mathrm{d}s \end{aligned}$$

is called the Gaussian map relative to the line bundle L.

Note that this definition is correct because of the lemma above and because $W_L(s,t)$ is skew-symmetric.

Let us now extend this definition to two line bundles L, M on a smooth projective variety X. Denoted by $\mu_{L,M}$ the multiplication map $H^0(X, L) \otimes H^0(X, M) \longrightarrow H^0(X, L \otimes M)$ we define $R(L, M)$ as the space of relations between L and M, i.e. $R(L, M) := \mathrm{Ker}\ \mu_{L,M}$. If we choose two trivialization on an open set U (σ for L and ρ for M) then locally we can define

$$\begin{aligned} \Phi_{L,M} : R(L, M) &\longrightarrow H^0(X, \Omega_X \otimes L \otimes M) \\ s \otimes t &\longmapsto s \otimes \mathrm{d}t - t \otimes \mathrm{d}s \end{aligned}$$

if, as above, we interpret $\mathrm{d}s$ as $\mathrm{d}f \otimes \sigma$ ($s = f\sigma$) and similarly for $\mathrm{d}t$. More in detail if on U we have $r = \Sigma_{i=1}^n f_i g_i \sigma \otimes \rho$ then $\Phi_{L,M}(r) = (f_i \mathrm{d}g_i - g_i \mathrm{d}f_i) \cdot \sigma \otimes \rho$. This formula is well-defined. In fact it is straightforward to show that, choosing different trivializations, the expression differs from the previous only by terms in which there appear sums of $f_i g_i$ that are null by hypothesis.
It is also clear that if $L = M$ then Φ_L, previously defined, essentially coincides with $\Phi_{L,L}$. In fact $H^0(L) \otimes H^0(L) \cong \wedge^2 H^0(L) \oplus S^2 H^0(L)$, where S^2 is

the second symmetric power. By definition, $s \wedge t = s \otimes t - t \otimes s$, hence $\wedge^2 H^0(L) \subset R(L,L)$ and $\Phi_{L,L}(s \wedge t) = 2\Phi_L(s \wedge t)$ whilst $\Phi_{L,L}$ on symmetric elements is the null map.

Now let us come back to the problem left unsolved at the end of section 1.

Theorem 1.2.3 (Wahl)**.** *Let X be a smooth, irreducible, linearly normal (i.e.: embedded with a complete linear system) curve in $\mathbb{P}^r$ that is not a conic. Then $h^0(\mathcal{N}_{X/\mathbb{P}^r}(-1)) = r + 1 +$ corank $\Phi_{\omega_X,\mathcal{O}_X(1)}$.*

Before proceeding with the proof of the theorem we previously need a lemma:

Lemma 1.2.4. *Let X a smooth, irreducible, linearly normal curve in $\mathbb{P}^r$ of degree d and genus g that is not a conic. Then:*

(a) $h^0(\mathcal{T}_X(-1)) = 0$

(b) $h^0(\mathcal{T}_{\mathbb{P}^r|X}(-1)) = r + 1$.

Proof.

(a) Remember that we are considering a curve, hence by definition $\Omega_X = \omega_X$. But $\mathcal{T}_X$ is the dual of Ω_X, therefore $\mathcal{T}_X \cong \mathcal{O}_X(-K_X)$, where K_X is the canonical divisor. Moreover there exists a divisor D such that $\mathcal{O}_X(1) \cong \mathcal{O}_X(D)$, with $\deg D$ equal to d, the degree of the embedded curve. Consequently $\deg(\mathcal{T}_X(-1)) = \deg(-K_X - D) = -2g + 2 - d$, ([7],IV,1.3.3). Since X is not a conic $-2g + 2 - d < 0$ and hence $H^0(\mathcal{T}_X(-1)) = 0$.

(b) Let us consider (1.5). After tensoring it by $\mathcal{O}_X(-1)$ we obtain:

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X^{\oplus(r+1)} \longrightarrow \mathcal{T}_{\mathbb{P}^r|X}(-1) \longrightarrow 0 \qquad (1.10)$$

Since $H^0(\mathcal{O}_X(-1)) = 0$, $H^0(\mathcal{O}_X) \cong \mathbb{C}$, $H^0(\mathcal{O}_X^{\oplus r+1}) = \mathbb{C}^{r+1}$, taking cohomology we have:

$$0 \to \mathbb{C}^{r+1} \to H^0(\mathcal{T}_{\mathbb{P}^r|X}(-1)) \to H^1(\mathcal{O}_X(-1)) \xrightarrow{\psi} H^1(\mathcal{O}_X^{\oplus(r+1)}) \qquad (1.11)$$

Thus it remains to prove only that ψ is injective: in this case $h^0(\mathcal{T}_{\mathbb{P}^r|X}(-1))$ would be $r + 1$. Now this is equivalent to show that

$\psi^* : (H^1(\mathcal{O}_X(-1)))^* \to (H^1(\mathcal{O}_X^{\oplus(r+1)}))^*$ is surjective. By Serre's duality theorem $\psi^* : \bigoplus_{i=1}^{r+1} H^0(\omega_X) \to H^0(\omega_X(1))$. Since $\mathcal{O}_X(1)$ is very ample, we can rewrite this map as a multiplication map from $H^0(\mathcal{O}_X(1)) \otimes H^0(\omega_X)$ to $H^0(\omega_X(1))$, and we know that this is surjective by Arbarello-Petri-Sernesi's theorem ([1]).

□

Proof of 1.2.3. Consider (1.2). Tensoring it by $\mathcal{O}_X(-1)$ we obtain:

$$0 \to \mathcal{T}_X(-1) \to \mathcal{T}_{\mathbb{P}^r|_X}(-1) \to \mathcal{N}_{X/\mathbb{P}^r}(-1) \to 0 \tag{1.12}$$

Taking cohomology, by 1.2.4 we have:

$$0 \to \mathbb{C}^{r+1} \to H^0(\mathcal{N}_{X/\mathbb{P}^r}(-1)) \to H^1(\mathcal{T}_X(-1)) \xrightarrow{\phi} H^1(\mathcal{T}_{\mathbb{P}^r|_X}(-1)) \to \dots \tag{1.13}$$

Therefore by the rank-nullity theorem and the exactness of the sequence $h^0(\mathcal{N}_{X/\mathbb{P}^r}(-1)) = r + 1 + \dim Ker\phi$. Also $\dim Ker\phi = \dim Coker\phi^*$, but ϕ^* is exactly the Gaussian map $\Phi_{\omega_X,\mathcal{O}_X(1)}$. In fact by Serre's duality
$H^1(\mathcal{T}_{\mathbb{P}^r|_X}(-1))^* \cong H^0(\Omega_{\mathbb{P}^r|_X}(1) \otimes \omega_X)$ and $H^1(\mathcal{T}_X(-1))^* \cong H^0(\Omega_X(1) \otimes \omega_X)$.

□

Corollary 1.2.5. *Let X be a smooth, irreducible, linearly normal curve in $\mathbb{P}^r$ that is not a conic. If $\Phi_{\omega_X,\mathcal{O}_X(1)}$ is surjective then X is not extendable.*

Proof. Simply merge together 1.1.5 and 1.2.3. □

It is a natural question to ask whether this result has a converse, that is: if X is not extendable, the Gaussian map should be surjective? The answer is in general negative, but in a recent paper [11], J. Wahl has proved that under certain circumstances the extendability of a curve is equivalent to the non-surjectivity of the Gaussian map. He has studied only canonical curves, concluding that:

Theorem 1.2.6. *Let $X \subset \mathbb{P}^{g-1}$ be a canonical curve (i.e.: embedded with the canonical sheaf). Suppose that $H^1(\mathbb{P}^{g-1}, \mathcal{I}_X^2(k)) = 0$ for all $k \neq 2$, where $\mathcal{I}$ is the ideal sheaf of X in $\mathbb{P}^{g-1}$. Then X is extendable if and only if the Wahl map Φ_{ω_X} is not surjective.*

Remark 1.2.7. For $k = 2$, $H^1(\mathbb{P}^{g-1}, \mathcal{I}_X^2(2))$ is frequently the kernel of the Gaussian map of ω_X and hence is rarely zero.

Given an embedding $X \subset \mathbb{P}^r$ (i.e. given a very ample line bundle L), in which cases $H^1(\mathbb{P}^r, \mathcal{I}_X^2(k)) = 0$ for all k but 2? In the same paper Wahl has verified that this group is zero for

1. $X = \mathbb{P}^n$ for every L (though it is surprisingly non-trivial).

2. whichever X and L any sufficiently high power of a very ample line bundle.

3. the canonical embedding of a general non-hyperelliptic curve of genus $g \geq 3$.

1.3 Surjectivity of Gaussian Maps on Curves

From 1.1.5 and 1.2.3 we are led back the study of the extendability of a smooth projective curve $C \subset \mathbb{P}^r$ of genus g, to the study of the surjectivity of the Gaussian map. But when is $\Phi_{\omega_C, \mathcal{O}_C(1)}$ surjective? Or more generally if L, M are line bundles on C when is $\Phi_{L,M}$ surjective? A number of theorems in this direction have already appeared. First Ciliberto, Harris, Miranda showed in [4] that if $L = M$ and $\deg L \geq 5g + 2$ then Φ_L is surjective. Wahl proved in [10] that $\Phi_{L,M}$ is surjective provided that $\deg L \geq 5g + 1$ and $\deg M \geq 2g + 2$. He also proved that $\Phi_{\omega_X, L}$ is surjective if $\deg L \geq 5g + 2$. A. Bertram, L. Ein, R. Lazarsfeld strengthened these results proving two theorems (see [2]).

Theorem 1.3.1 (B-E-L 1). *Let L, M have degree d and e respectively. Assume that both $d, e \geq 2g + 2$. Then*

1. *If $d + e \geq 6g + 3$ then $\Phi_{L,M}$ is surjective.*

2. *If C is non-hyperelliptic, and $d + e \geq 6g + 2$ then $\Phi_{L,M}$ is surjective.*

3. *If C is hyperrelliptic, then given L of degree $2g + 2 \leq d \leq 4g$ there exists a line bundle M on C of degree $6g + 2 - d$ for which $\Phi_{L,M}$ fails to be surjective.*

But the main result in [2] we are primarily concerned with, is the second theorem (1.3.5).

Definition 1.3.2. Let C be a smooth projective curve. Then we define the *Clifford index* for C, Cliff C, to be $\min\{\deg(L) - 2r(L) : h^0(\mathcal{O}_C(L)) \geq 2, h^1(\mathcal{O}_C(L)) \geq 2\}$, where L varies in Pic C and $r(L) = h^0(L) - 1$.

Theorem 1.3.3 (Coppens-Martens [5]). *Let C be a smooth projective curve. Then Cliff $C \geq gon(C) - 3$.*

Remark 1.3.4. $gon(C)$ stands for gonality of C: see 3.2.1. Hyperelliptic and trigonal mean $gon(C) = 2, 3$ respectively.

Theorem 1.3.5 (B-E-L 2). *Assume that C is neither hyperelliptic, trigonal nor a plane quintic. If $deg(L) \geq 4g + 1 - 2$Cliff C, then $\Phi_{\omega_C,L}$ is surjective.*

If $C \subset \mathbb{P}^r$ is trigonal $\Phi_{\omega_C,\mathcal{O}_C(1)}$ could be surjective in any case. Tendian, in fact, in an unpublished paper, has showed that:

Theorem 1.3.6 (Tendian). *Assume C trigonal. Let $\mathcal{A}$ be an invertible sheaf that defines the map of degree 3 onto $\mathbb{P}^1$. Suppose $g \geq 5$. If*

1. $h^0(\omega_C^{\otimes 2} \otimes \mathcal{O}_C(-1)) \leq 1$
2. $h^0(\omega_C^{\otimes 3} \otimes \mathcal{A}^{\otimes(4-g)} \otimes \mathcal{O}_C(-1)) = 0$
3. $h^1(\mathcal{O}_C(1)) = 0$

then $\Phi_{\omega_C,\mathcal{O}_C(1)}$ is surjective.

1.4 Extendable Surfaces

Since now we have only discussed some general questions about extendability and the extendability of curves. Though the problem of extendability of surfaces is more complex, even in this case we can say something. In fact A.L. Knutsen, A.F. Lopez and R. Muñoz proved a general theorem on the extendability of surfaces ([8]), that is actually a generalization of 1.2.3:

Theorem 1.4.1 (Knutsen, Lopez, Muñoz)**.** *Let X be a smooth surface in $\mathbb{P}^r$, embedded with a complete linear system related to a very ample divisor L (i.e.: $\mathbb{P}^r \cong \mathbb{P}H^0(\mathcal{O}_X(L))$). If there exists an invertible sheaf $\mathcal{O}_X(D)$ such that for a generic $C \in |D|$ the following hypotheses are verified,*

1. *$|D|$ is base-point free*
2. *$D^2 > 0$ and $g(C) > 0$*
3. *$H^1(\mathcal{O}_X(L - D)) = 0$*
4. *$H^1(\mathcal{O}_X(L - 2D)) = 0$ and $(L - D).D \geq 2g(C) + 1$*
5. *$L.D > 2D^2$*
6. *$\Phi_{\omega_C, L|_C}$ is surjective*

then X is not extendable.

Remark 1.4.2. Actually condition 4 is a simplification of another weaker but less calculable condition, that is the surjectivity of the multiplication map $\mu : V_C \otimes H^0(\omega_C) \to H^0(\omega_C \otimes \mathcal{O}_X(L-C) \otimes \mathcal{O}_C)$, where V_C is defined as follow. Consider the defining sequence for C,

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0 \tag{1.14}$$

Tensor it by $\mathcal{O}_X(L - C)$, obtaining

$$0 \longrightarrow \mathcal{O}_X(L - 2C) \longrightarrow \mathcal{O}_X(L - C) \longrightarrow \mathcal{O}_C(L - C) \longrightarrow 0 \tag{1.15}$$

Taking cohomology we have:

$$\ldots \to H^0(\mathcal{O}_X(L-C)) \xrightarrow{\pi} H^0(\mathcal{O}_C(L-C)) \to H^1(\mathcal{O}_X(L-C)) \to \ldots \tag{1.16}$$

Well, $V_C := \mathrm{Im}\pi$.

By Arbarello-Petri-Sernesi's theorem we know that μ is surjective if V_C is a complete linear system and it is very ample, that is: there exists a divisor E on C such that $V_C = H^0(\mathcal{O}_C(E))$ and $\mathcal{O}_C(E)$ is very ample. By (1.16) we see that if $H^1(\mathcal{O}_X(L - 2C)) = 0$ then π is surjective and therefore V_C is complete. Moreover if $(L - C).C \geq 2\mathrm{g}(C) + 1$ then $(L - C)|_C$ is very ample (see [7], IV, 3.2).

Chapter 2

Ruled Surfaces

Remark. Throughout this and the next chapter *surface* will mean a non-singular projective variety of dimension 2 over $\mathbb{C}$ (or, actually, any other algebraically closed field of characteristic 0). A *curve* on a surface will be an effective divisor. Therefore, if not clearly stated, a curve may be also reducible or singular or even have multiple components. A point will be a closed point.

In this chapter first of all we will study ruled surfaces and their geometry (Picard group, canonical divisor...) and in some way we will classify them, viewing them as projective space bundles. In particular our attention will be concerned with rational ruled surfaces, for which we will be able to say lots of things: we will compute the Picard group and we will classify all the ample and very ample divisors on them.

2.1 Ruled Surfaces

Definition 2.1.1. A *ruled surface* is a surface X together with a morphism $\pi : X \to C$, where C is a nonsingular curve, such that $\pi^{-1}(p) \cong \mathbb{P}^1$ for every $p \in X$ and such that there exists a section, that is: there exists a morphism $\sigma : C \to X$ that composed with π is the identity on C.

Notation 2.1.2. Every time we speak of ruled surfaces we will consider the surface endowed with the projection π on the curve C of genus g and the section σ like in the definition. We set $C_0 := \sigma(C)$. Moreover we will call f_p the fibre $\pi^{-1}(p)$. If it is not necessary to specify the point p, we will call the fibre f.

Remark 2.1.3. $f \cong \mathbb{P}^1$. C_0 is a curve on X that is isomorphic to C through $\pi|_{C_0}$. In fact $\pi|_{C_0} \circ \sigma = id_C$ by definition and $\sigma \circ \pi|_{C_0} = id_{C_0}$ because π is injective on C_0 and $\pi|_{C_0}(\sigma(\pi|_{C_0}(p))) = \pi|_{C_0}(p) \Rightarrow \sigma(\pi|_{C_0}(p)) = p$.

Definition 2.1.4. A *rational ruled surface* is a ruled surface with $C \cong \mathbb{P}^1$. The simplest example is $\mathbb{P}^1 \times \mathbb{P}^1$, which has two rulings associated with the projection on the first and on the second factor.

Lemma 2.1.5. *Let X be a ruled surface. All the fibres of X are numerically equivalent (i.e.: $D.f_p = D.f_q$ for all $p, q \in C$).*

Proof. For a general proof see [7], V, 2.1. However in the case of rational ruled surfaces the proof is very simple. In fact all the points on $\mathbb{P}^1$ are linearly equivalent and hence $f_p = \pi^*(p) \sim \pi^*(q) = f_q$. Therefore $D.f$ is independent on the fibre because the intersection pairing depends only on the linear equivalence class of divisors ([7],V,§1). □

Corollary 2.1.6. *Let D be a divisor on X. $D.f$ is independent on the choice of the fibre.*

Lemma 2.1.7. *Let X be a ruled surface and D a divisor. Suppose that $D.f = n \geq 0$. Then $\pi_*(\mathcal{O}_X(D))$ is a locally free sheaf on C of rank $n+1$. In particular $\pi_*\mathcal{O}_X = \mathcal{O}_C$.*

Proof. See [7], V, 2.1. □

Proposition 2.1.8. *If X is a ruled surface then there exists a locally free sheaf $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}(\mathcal{E})$ (for the definition of the projective space bundle $\mathbb{P}(\mathcal{E})$ see [7], II, §7). Conversely every such $\mathbb{P}(\mathcal{E})$ is a ruled surface X over C. Moreover if two ruled surfaces over the same curve are isomorphic then the sheaves are isomorphic up to the tensorization by an invertible sheaf, and vice versa.*

Proof. See [7], V, 2.2. □

Proposition 2.1.9. *Let X be a ruled surface. Pic $X \cong \mathbb{Z} \oplus \pi^*$Pic C, where $\mathbb{Z}$ is generated by C_0. Also Num $X \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by C_0 and f and satisfying $C_0.f = 1,\ f^2 = 0$.*

Proof. First of all we see that $C_0.f = 1$ because C_0 and f meet at only one point and they are transversal there. It is also clear that $f^2 = 0$ since two distinct fibres do not intersect.

Now π^* :Pic $C \to$ Pic X is injective, since $(\pi|_{C_0})^*$ is injective (because it is invertible) and if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ then $\mathcal{O}_X(D) \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_X(D') \otimes \mathcal{O}_{C_0}$.

Let D be a divisor on X and $n = D.f$. Then we can write $D = nC_0 + (D - nC_0)$. Let $D' = D - nC_0$. D' is such that $D'.f = 0$, hence, by 2.1.7, $\pi_*(\mathcal{O}_X(D'))$ is an invertible sheaf on C. But then $\pi^*\pi_*(\mathcal{O}_X(D')) \cong \mathcal{O}_X(D')$. Thus , since π^* is injective, π^* Pic $C = \{D \in X\ s.t.\ D.f = 0\}$, therefore
Pic $X \cong \mathbb{Z} \oplus\ \pi^*$Pic C, where $\mathbb{Z}$ is seen as the additive group generated by C_0. The conclusion about Num X is obvious, remembering that all the fibres are numerically equivalent. □

Remark 2.1.10. On the rational ruled surfaces, since all the fibres are linearly equivalent, we have Pic $X \cong \mathbb{Z} \oplus \mathbb{Z}$, where the first $\mathbb{Z}$ is generated by C_0 and the second $\mathbb{Z}$ is generated by f. Or, in another way: for every D in Pic X there exist $a, b \in \mathbb{Z}$ such that $D \sim aC_0 + bf$.

Lemma 2.1.11. *Let D be a divisor on X. Assume that $D.f \geq 0$. Then, for all i, $H^i(X, \mathcal{O}_X(D)) \cong H^i(C, \pi_*\mathcal{O}_X(D))$.*

Proof. See ([7], V, 2.4). □

Remark 2.1.12. Since C is a curve, by the vanishing theorem of Grothendieck ([7],III,2.7), for all sheaves $\mathcal{F}$ on C we have $H^2(C, \mathcal{F}) = 0$. Consequently for all D on X such that $D.f \geq 0$ we have $H^2(\mathcal{O}_X(D)) = 0$.

Corollary 2.1.13. *$h^0(\mathcal{O}_X) = 1$, $h^1(\mathcal{O}_X) = g$, $h^2(\mathcal{O}_X) = 0$.*

Proof. Use the lemma above ($0.f = 0$) remembering that $\pi_*\mathcal{O}_X \cong \mathcal{O}_C$ by 2.1.7. □

Proposition 2.1.14. *Let $\mathcal{E}$ be a locally free sheaf of rank 2 on C, and let X be the ruled surface $\mathbb{P}(\mathcal{E})$. Then there is a one-to-one correspondence between* *sections $\sigma : C \to X$ and surjections $\mathcal{E} \to \mathcal{L}$, where $\mathcal{L} = \sigma^*(\mathcal{O}_X(1))$. Under this correspondence if $\mathcal{N} = \ker(\mathcal{E} \to \mathcal{L})$ then $\mathcal{N}$ is an invertible sheaf on C and $\mathcal{N} \cong \pi_*(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-D))$, where $D = \sigma(C)$ and $\pi^*\mathcal{N} \cong \mathcal{O}_X(1) \otimes \mathcal{O}_X(-D)$.*

Proof. See [7], V, 2.6. □

Proposition 2.1.15. *If X is a ruled surface it is possible to write X as $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf of rank 2 on C such that $H^0(C, \mathcal{E}) \neq 0$, but for every invertible sheaf $\mathcal{L}$ on C of negative degree, $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$. In this case the integer $e = -\deg \mathcal{E} := -\deg \wedge^2 \mathcal{E}$ is an invariant of X. Additionally in this case we can choose a section $\sigma_0 : C \to X$ such that $\mathcal{O}_X(\sigma_0(C)) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.*

Proof. See [7], V, 2.8. □

Notation 2.1.16. From now on, when we will speak of a ruled surface we will intend the data of a locally free sheaf $\mathcal{E}$ as in the proposition above (i.e. $\mathcal{E}$ is *normalized*), a curve C of genus g, a projection map $\pi : X = \mathbb{P}(\mathcal{E}) \to C$, and a particular section σ_0, with its image C_0, as in the proposition above. By abuse of notation, if $\mathfrak{d}$ is a divisor on C we will write $\mathfrak{d}f$ for $\pi^*(\mathfrak{d})$.

Proposition 2.1.17. *Let $\sigma : C \to X$ be any section on X, corresponding (2.1.14) to a surjection $\mathcal{E} \to \mathcal{L}$, where $\mathcal{L} \cong \sigma^*(\mathcal{O}_X(1))$. Let $D = \sigma(C)$. Let $\mathfrak{d}$ be a divisor on C such that $\mathcal{O}_C(\mathfrak{d}) \cong \mathcal{L}$. Let $\mathfrak{e} = \wedge^2\mathcal{E}$. Then we have $D \sim C_0 + (\mathfrak{d} - \mathfrak{e})f$. Moreover $\deg \mathfrak{d} = C_0.D$ and, in particular, $C_0^2 = \deg \mathfrak{e} = -e$.*

Proof. D is a section, hence $D.f = 1$. Therefore by 2.1.9 $D \sim C_0 + \pi_*(D - C_0)f$. What is $\pi_*(D - C_0)$? Let $\mathcal{N}$ be the kernel of the surjection $\mathcal{E} \to \mathcal{L}$. From 2.1.15 we know that $\mathcal{N} \cong \pi_*(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-D))$. But, by the choice of C_0, $\mathcal{O}_X(1) \cong \mathcal{O}_X(C_0)$ and hence $\mathcal{N} \cong \pi_*(\mathcal{O}_X(C_0 - D))$. But $\mathcal{N} = \mathcal{O}_C(\mathfrak{e} - \mathfrak{d})$ by [7], 2, ex. 5.16d, thus the first part of the thesis follows.

$\mathcal{L} = \sigma^*(\mathcal{O}_X(1)) \cong \sigma^*\mathcal{O}_X(C_0)$. We consider $\sigma : C \to D$. Clearly $\sigma^*(\mathcal{O}_X(C_0)) = \sigma^*(\mathcal{O}_X(C_0) \otimes \mathcal{O}_D)$. Since C and D are nonsingular (C by hypothesis and D because it is isomorphic to C) we can apply [7],II,6.9: σ has degree 1, then $\deg \sigma^*(\mathcal{O}_X(C_0)) = \deg \mathcal{O}_X(C_0) \otimes \mathcal{O}_D = C_0.D$ by [7],V,§1.

In particular if $D = C_0$, $\deg \mathfrak{d} = C_0.C_0$ and $\mathcal{N} = \pi_*(\mathcal{O}_X) = \mathcal{O}_C$. Therefore $\mathcal{O}_C(\mathfrak{e} - \mathfrak{d}) = \mathcal{O}_C \Rightarrow \mathfrak{d} \sim \mathfrak{e}$. But then $C_0^2 = \deg \mathfrak{d} = \deg \mathfrak{e} = -e$. □

Theorem 2.1.18 (Adjunction Formula). *If C is a nonsingular curve of genus g on a surface Y we have $\omega_C \cong \omega_Y \otimes \mathcal{O}_Y(C) \otimes \mathcal{O}_C$. In particular if K is the canonical divisor on Y we have:*

$$2g - 2 = C.(C + K) \tag{2.1}$$

Theorem 2.1.19 (Riemann-Roch for curves). *If D is any divisor on a projective nonsingular curve Γ of genus g, then*

$$h^0(\mathcal{O}_\Gamma(D)) - h^1(\mathcal{O}_\Gamma(D)) = \deg D + 1 - g$$

Theorem 2.1.20 (Riemann-Roch for surfaces). *If D is any divisor on a surface Y and if K is the canonical divisor of Y, then:*

$$h^0(\mathcal{O}_Y(D)) - h^1(\mathcal{O}_Y(D)) + h^0(\mathcal{O}_Y(K - D)) = \frac{1}{2}D.(D - K) + \chi(\mathcal{O}_Y)$$

where $\chi(\mathcal{O}_Y)$ is the Euler characteristic of $\mathcal{O}_Y$, i.e. $\chi(\mathcal{O}_Y) = h^0(\mathcal{O}_Y) - h^1(\mathcal{O}_Y) + h^2(\mathcal{O}_Y)$.

Theorem 2.1.21 (Projection Formula). *If $\theta : (Z, \mathcal{O}_Z) \to (W, \mathcal{O}_W)$ is a morphism of ringed spaces, if $\mathcal{F}$ is an $\mathcal{O}_Z$-module and if $\mathcal{G}$ is a locally free $\mathcal{O}_W$-module of finite rank, then there is a natural isomorphism $\theta_*(\mathcal{F} \otimes_{\mathcal{O}_Z} \theta^*\mathcal{G}) \cong \theta_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G}$.*

Remark 2.1.22. Note that, by [7],II,7.11, $\pi_*(\mathcal{O}_X(1)) = \pi_*(\mathcal{O}_X(C_0)) = \mathcal{E}$.

Lemma 2.1.23. *The canonical divisor K on X is given by $K \sim -2C_0 + (\mathfrak{l} + \mathfrak{e})f$, where $\mathfrak{l}$ is the canonical divisor on C.*

Proof. By 2.1.9 we know that there exists $\mathfrak{b}$ divisor on C such that $K \sim aC_0 + \mathfrak{b}f$. Using the adjunction formula for a fibre $f \cong \mathbb{P}^1$ we have: $-2 = (f + K).f = a$ (clearly $\pi^*(\mathfrak{b}).f = 0$). Now we use the adjunction formula for C_0 on X. $\omega_{C_0} \cong \omega_X \otimes \mathcal{O}_X(C_0) \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_X(K + C_0) \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_X(-C_0 + \mathfrak{b}f) \otimes \mathcal{O}_{C_0}$. $C_0 \cong C$ through π, hence we have $\omega_C \cong (\wedge^2\mathcal{E})^{-1} \otimes \mathcal{O}_C(\mathfrak{b})$, that is: $\mathfrak{l} \sim -\mathfrak{e} + \mathfrak{b}$. This implies that $\mathfrak{b} \sim \mathfrak{e} + \mathfrak{l}$. □

Corollary 2.1.24. *If X is rational ruled surface then the canonical divisor K on X is linearly equivalent to $-2C_0 + (-2 - e)f$.*

Proof. Since by definition a rational ruled surface is ruled over $\mathbb{P}^1$ we have $\deg \mathfrak{l} = \deg \omega_C = -2$. By 2.1.10 and 2.1.23 $(\mathfrak{l} + \mathfrak{e})f$ depends only on the degree of $\mathfrak{l} + \mathfrak{e}$. Since $\deg(\mathfrak{l} + \mathfrak{e}) = \deg \mathfrak{l} + \deg \mathfrak{e} = -2 - e$ we have $K \sim -2C_0 + (-2 - e)f$. □

Theorem 2.1.25. *Let X be a ruled surface. If $\mathcal{E}$ is decomposable (i.e., it is a direct sum of two invertible sheaves) then $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$, with* $\deg \mathcal{L} \leq 0$. *Therefore $e \geq 0$. If $\mathcal{E}$ is indecomposable, then $-2g \leq e \leq 2g - 2$.*

Proof. By definition if $\mathcal{E}$ is decomposable we can write $\mathcal{E} = \mathcal{O}_C(D_1) \oplus \mathcal{O}_C(D_2)$, where D_1 and D_2 are divisors on C. Since $\mathcal{E}$ is normalized, $\deg D_1 \leq 0$ and $\deg D_2 \leq 0$, otherwise, if for example $\deg D_1 > 0$, we would have $H^0(\mathcal{E} \otimes \mathcal{O}_C(-D_1)) = H^0(\mathcal{O}_C \oplus \mathcal{O}_C(D_2 - D_1)) = H^0(\mathcal{O}_C) \oplus H^0(\mathcal{O}_C(D_2 - D_1)) \neq 0$, while $\deg(-D_1) < 0$. However, again from the normalization of $\mathcal{E}$, $H^0(\mathcal{E}) = H^0(\mathcal{O}_C(D_1)) \oplus H^0(\mathcal{O}_C(D_2)) \neq 0$ and hence we can suppose that $H^0(\mathcal{O}_C(D_1)) \neq 0$. $H^0(\mathcal{O}_C(D_1)) \neq 0$ and $\deg D_1 \leq 0$ imply that $D_1 \sim 0$, i.e. $\mathcal{O}_C(D_1) \cong \mathcal{O}_C$. Thus we have the short exact sequence:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C \oplus \mathcal{O}_C(D_2) \longrightarrow \mathcal{O}_C(D_2) \longrightarrow 0 \tag{2.2}$$

Since the degree is additive for exact sequences we have $\deg \mathcal{E} = \deg D_2 \leq 0 \Rightarrow e = -\deg \mathcal{E} \geq 0$. Now suppose $\mathcal{E}$ is indecomposable. By 2.1.14 we have an exact sequence, corresponding to the section C_0:

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0 \tag{2.3}$$

By 2.1.14 $\mathcal{N} \cong \pi_*(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-D))$, where $D = \sigma(C) = C_0$. But by 2.1.16 $\mathcal{O}_X(1) \cong \mathcal{O}_X(C_0)$, hence $\mathcal{N} \cong \pi_*\mathcal{O}_X = \mathcal{O}_C$ by 2.1.7. Hence 2.3 becomes:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0 \tag{2.4}$$

This says that $\mathcal{E}$ is an extension of $\mathcal{L}$ by $\mathcal{O}_C$. It cannot be a trivial extension by hypothesis (trivial means $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$) hence this extension corresponds to a nonzero element $\xi \in \mathrm{Ext}^1(\mathcal{L}, \mathcal{O}_C) \cong H^1(\mathcal{L}^*)$ (see [7], III, §6 and ex. 6.1). In particular $H^1(\mathcal{L}^*) \neq 0$. Remember that $\mathcal{L}$ is an invertible sheaf on C curve, hence, by Serre's duality, $H^1(\mathcal{L}^*) = H^1(\mathcal{L}^{-1}) \cong H^0(\omega_C \otimes \mathcal{L})^* \neq 0$. Therefore $\deg(\omega_C \otimes \mathcal{L}) \geq 0 \Rightarrow 2g - 2 + \deg \mathcal{L} \geq 0 \Rightarrow \deg \mathcal{L} \geq 2 - 2g$. From

2.4, since the degree is an additive function for exact sequences, we have: $\deg \mathcal{E} = \deg \mathcal{O}_C + \deg \mathcal{L} \Rightarrow e := -\deg \mathcal{E} = -\deg \mathcal{L} \leq 2g - 2$.

On the other hand consider a point p on C. After tensoring 2.4 by $\mathcal{O}_C(-p)$ we have:

$$0 \longrightarrow \mathcal{O}_C(-p) \longrightarrow \mathcal{E} \otimes \mathcal{O}_C(-p) \longrightarrow \mathcal{L} \otimes \mathcal{O}_C(-p) \longrightarrow 0 \tag{2.5}$$

Taking cohomology, since $\mathcal{E}$ is normalized and thus $H^0(\mathcal{E} \otimes \mathcal{O}_C(-p)) = 0$, we obtain an injection $0 \to H^0(\mathcal{L} \otimes \mathcal{O}_C(-p)) \to H^1(\mathcal{O}_C(-p))$. Hence $h^0(\mathcal{L} \otimes \mathcal{O}_C(-p)) \leq h^1(\mathcal{O}_C(-p))$. But, by Riemann-Roch, since $h^0(\mathcal{O}_C(-p)) = 0$, $h^1(\mathcal{O}_C(-p)) = g$ and hence $h^0(\mathcal{L} \otimes \mathcal{O}_C(-p)) \leq g$. But, again by Riemann-Roch, $h^0(\mathcal{L} \otimes \mathcal{O}_C(-p)) \geq \deg \mathcal{L} - g$ (in fact $\deg(\mathcal{L} \otimes \mathcal{O}_C(-p)) = \deg \mathcal{L} + \deg \mathcal{O}_C(-p)$). Combining the two inequalities we have $\deg \mathcal{L} - g \leq g \Rightarrow \deg \mathcal{L} \leq 2g \Rightarrow e = -\deg \mathcal{L} \geq -2g$. □

Corollary 2.1.26. *If $g = 0$ (i.e. X is a rational ruled surface), then $e \geq 0$ and $\mathcal{E}$ is decomposable as $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$. Conversely for each $e \geq 0$ there is one rational ruled surface with invariant e, given by $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$.*

Proof. If $g = 0$ then $\mathcal{E}$ must be decomposable. Hence $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{F}$, where $\mathcal{F}$ is an invertible sheaf on $\mathbb{P}^1$ of degree $-e$. But on $\mathbb{P}^1$ all the invertible sheaves of the same degree d are isomorphic to $\mathcal{O}_{\mathbb{P}^1}(d)$, therefore $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$. On the other hand $\forall e \geq 0$ $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$ is normalized and hence $e = -\deg \mathcal{E}$. □

Corollary 2.1.27. *On any ruled surface X for which the normalized sheaf $\mathcal{E}$ is decomposable, $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$, we have:*

1. *if $e > 0$, the normalized sheaf is unique.*
2. *if $e = 0, \mathcal{L} \cong \mathcal{O}_C$, the normalized sheaf is unique.*
3. *if $e = 0, \mathcal{L} \ncong \mathcal{O}_C$, we have two choices for the normalized sheaf: namely $\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{L}^{-1}$.*

In particular for the rational ruled surfaces the normalized sheaf is always unique.

Proof. If $\mathcal{E}'$ is another normalized sheaf for X then by 2.1.8 $\mathcal{E}' = \mathcal{E} \otimes \mathcal{F}$, therefore $\mathcal{E}'$ is decomposable and $\mathcal{E}' = \mathcal{F} \oplus (\mathcal{L} \otimes \mathcal{F})$. By the theorem either $\mathcal{F} \cong \mathcal{O}_C$ (and thus $\mathcal{E}' \cong \mathcal{E}$) or $\mathcal{L} \otimes \mathcal{F} \cong \mathcal{O}_C \Rightarrow \mathcal{F} \cong \mathcal{L}^{-1}$. Again by the theorem, $\deg \mathcal{F} \leq 0$. But also $\deg \mathcal{L} \leq 0$, hence $\mathcal{F}$ can be isomorphic to $\mathcal{L}^{-1}$ only if $e = -\deg \mathcal{L} = 0$. In particular for the rational ruled surfaces the normalized sheaf is unique: in fact on $\mathbb{P}^1$ every invertible sheaf of degree 0 is isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. □

2.2 Rational Ruled Surfaces

Now we concentrate our attention on rational ruled surfaces. From the preceding paragraph we know that for all invariant $e \geq 0$ there exists one and only one rational ruled surface $X = \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is the unique normalized sheaf $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$.

Theorem 2.2.1. *Let X be the rational ruled surface with invariant e. Then:*

(a) there is a section $D \sim C_0 + nf$ if, and only if, $n = 0$ or $n \geq e$. In particular, there is a section $C_1 \sim C_0 + ef$ with $C_0 \cap C_1 = \emptyset$ and $C_1^2 = e$;

(b) the linear system $|C_0 + nf|$ is base-point free if and only if $n \geq e$;

(c) the linear system $|C_0 + nf|$ is very ample if and only if $n > e$.

Proof.

(a) If D is a section, by 2.1.17, $D \sim C_0 + (C_0.D + e)f$. Since both C_0 and D are isomorphic to $\mathbb{P}^1$, and hence irreducible, we have only two cases: $C_0 = D$ and C_0 and D without common irreducible components. In the first case $C_0.D = -e$, in the second case $C_0.D \geq 0$ by [7], V, 1.4.

Now let us prove the converse. We can suppose $n \geq e$, because if $n = 0$ we already know that C_0 is a section. By 2.1.14 and 2.1.17, we know that finding a section D linearly equivalent to $C_0 + nf$ is equivalent to finding a surjection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \to \mathcal{O}_{\mathbb{P}^1}(n - e)$. For this purpose let us consider $2n - e$ distinct points on $\mathbb{P}^1$, $p_1, \ldots, p_{n-e}$ and $q_1, \ldots, q_n$. We have:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-(n-e)) \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\{p_1,\ldots,p_{n-e}\}} \to 0 \tag{2.6}$$

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-n) \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\{q_1,\ldots,q_n\}} \to 0 \tag{2.7}$$

Tensor (2.6) and (2.7) by $\mathcal{O}_{\mathbb{P}^1}(n-e)$. We have:

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(n-e) \to \mathcal{O}_{\{p_1,\ldots,p_{n-e}\}} \to 0 \tag{2.8}$$

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-e) \to \mathcal{O}_{\mathbb{P}^1}(n-e) \to \mathcal{O}_{\{q_1,\ldots,q_n\}} \to 0 \tag{2.9}$$

Looking at the stalk of each exact sequence above, we notice that for every stalk, at least one of the maps $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(n-e)$ or $\mathcal{O}_{\mathbb{P}^1}(-e) \to \mathcal{O}_{\mathbb{P}^1}(n-e)$ is surjective. Therefore we can construct a surjective map $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \to \mathcal{O}_{\mathbb{P}^1}(n-e)$.

In particular if $n = e$ there is a section $C_1 \sim C_0 + ef$. $C_1^2 = e$. $C_0.C_1 = 0$, therefore by [7], V, 1.4, $C_0 \cap C_1 = \emptyset$.

(b) If $|C_0 + nf|$ is base-point free then, in particular, it contains an effective divisor $D \sim C_0 + nf$ such that D does not contain C_0 as one of its irreducible components. But then $D.C_0 \geq 0$ by [7],V,1.4. $D.C_0 = (n - e) \geq 0 \Rightarrow n \geq e$.

On the contrary suppose now $n \geq e$. Let p be a point of X, and f a fibre different from $\pi^{-1}(\pi(p))$. If p does not belong to C_0, then $C_0 + nf$ does not pass through p. If $p \in C_0$ then $C_1 + (n-e)f$ is an effective divisor that does not pass through p: in fact, by a), $C_1 \cap C_0 = \emptyset$. Thus whichever p we fix, there exists an effective divisor linearly equivalent to $C_0 + nf$ that does not contain p, i.e. $|C_0 + nf|$ is base-point free.

(c) First suppose that $|C_0 + nf|$ is very ample: remember that a linear system is very ample if and only if it separates points and tangent vectors ([7],II,7.8.2). In particular a very ample linear system is without base point then, by b), $n \geq e$. Furthermore n cannot be e: in fact if $E \in |C_0 + ef|$ then, since $C_0.(C_0 + ef) = 0$, either E contains C_0 as one of its irreducible components, or E is a curve that does not intersect C_0. We see that, in both cases, $C_0 + ef$ cannot separate points on C_0. Or, more easily: if $C_0 + nf$ is very ample then $(C_0 + nf).C_0 = n - e$ must be strictly greater than 0, because this number is the degree of C_0 in the projective embedding determined by $C_0 + nf$.

Suppose now $n > e$. We will show that, in this case, $D = C_0 + nf$ is very ample. Actually we will show that $|D|$ separates points and tangent vectors.

1. If $p \neq q$ are two points not both in C_0 and not in the same fibre, then $C_0 + nf_p$ and $C_0 + nf_q$ (both in $|D|$) separate them. (This is true also if $n = e$. We will use this result later).
2. Le p be a point and t a tangent vector at p. Suppose that p and t are neither both in C_0 neither both in any fibre. Then a divisor of the form $C_0 + \sum_{i=1}^{n} f_{p_i}$, for suitable p_i, will separate them.
3. Suppose now that p, q or p, t are both in C_0. Then $C_1 + \sum_{i=1}^{n-e} f_{p_i}$, for suitable p_i, will separate them.
4. Suppose now that p, q or p, t are both on the same fibre. Let us consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-f) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_f \longrightarrow 0 \tag{2.10}$$

   After tensoring it by $\mathcal{O}_X(C_0+nf)$, since $f \cong \mathbb{P}^1$ and $f.(C_0+nf) = 1$ we obtain:

$$0 \longrightarrow \mathcal{O}_X(C_0+(n-1)f) \longrightarrow \mathcal{O}_X(C_0+nf) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0 \tag{2.11}$$

   But $\mathcal{O}_{\mathbb{P}^1}(1)$ is very ample on $\mathbb{P}^1$, therefore it separates points and tangent vectors. It is clear that if $H^0(\mathcal{O}_X(C_0+nf)) \to H^0(\mathcal{O}_f(C_0+nf)) = H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ is surjective then $|C_0+nf|$ separates points and tangent vectors on the fibre f. Thus, taking cohomology, we have only to show that $H^1(\mathcal{O}_X(C_0 + (n-1)f)) = 0$. By 2.1.11, since $(C_0+(n-1)f).f = 1$, $H^1(\mathcal{O}_X(C_0+(n-1)f)) = H^1(\mathbb{P}^1, \pi_*(\mathcal{O}_X(C_0+(n-1)f)))$. But $\mathcal{O}_X(C_0+(n-1)f) = \mathcal{O}_X(C_0)\otimes\pi^*(\mathcal{O}_{\mathbb{P}^1}(n-1))$, thus, by the projection formula and by 2.1.22, $\pi_*(\mathcal{O}_X(C_0+(n-1)f)) = (\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-e))\otimes\mathcal{O}_{\mathbb{P}^1}(n-1) = \mathcal{O}_{\mathbb{P}^1}(n-1)\oplus\mathcal{O}_{\mathbb{P}^1}(n-e-1)$. Therefore $H^1(\mathcal{O}_X(C_0+(n-1)f)) = H^1(\mathcal{O}_{\mathbb{P}^1}(n-1)) + H^1(\mathcal{O}_{\mathbb{P}^1}(n-e-1)) = 0$ because $n > e$.

□

Corollary 2.2.2. *Let D be the divisor aC_0+bf on the rational ruled surface X with invariant $e \geq 0$. Then:*

a) D is very ample $\Leftrightarrow$ D is ample $\Leftrightarrow$ $a > 0$ and $b > ae$;

b) the linear system $|D|$ contains an irreducible nonsingular curve $\Leftrightarrow$ it contains an irreducible curve $\Leftrightarrow$ $a = 0, b = 1$ (i.e. f); or $a = 1, b = 0$ (i.e. C_0); or $a > 0, b \geq ae, e \neq 0$; or $a > 0, b > 0, e = 0$.

Proof.

a) If D is very ample then D is ample. If D is ample then there exists $n \in \mathbb{N}$ such that nD is very ample, therefore $nD.f > 0$ and $nD.C_0 > 0$, that is: $a > 0, b > ae$. Moreover if we write D as $(a-1)(C_0+ef)+(C_0+(b-ae+e)f)$ we see that if $a > 0, b > ae$ D is very ample: in fact, by the theorem, C_0+ef is base-point free and $C_0+(b-ae+e)f$ is very ample.

b) if $|D|$ contains an irreducible nonsingular curve then, in particular, it contains an irreducible curve. An irreducible curve in $|D|$ can be f ($a = 0, b = 1$), or can be C_0 ($a = 1, b = 0$). If it is not f then we have $D.f > 0 \Rightarrow a > 0$. Moreover if it is not C_0 we have also $D.C_0 \geq 0 \Rightarrow b \geq ae$. In addition if $e = 0$, $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, i.e. $|C_0|$ and $|f|$ play the same role: simply they are the two different rulings of X. Hence, in this case, if the irreducible curve is not C_0 we have $D.C_0 > 0 \Rightarrow b > 0$. Thus the restriction on a, b are necessary.

On the contrary, if $a > 0, b > ae$ there exists an irreducible nonsingular curve in $|D|$, by Bertini's theorem, because D is very ample. If $a > 0, b = ae, e > 0$ then $|D|$ is without base-points ($D = a(C_0+ef)$ and C_0+ef is base-point free by 2.2.1). To apply Bertini's theorem again and thus to conclude that $|D|$ contains an irreducible nonsingular curve we have only to show that $|D|$ is not composite with a pencil. Let us call ϕ_D the morphism $X \to \mathbb{P}(H^0(\mathcal{O}_X(D))) = \mathbb{P}^r$ induced by $|D|$. To prove that $|D|$ is not composite with a pencil (i.e.: $\dim \phi_D(X) \geq 2$) it is enough to show that every fibre f has an image of dimension 1, because we already know that $|D|$ separates points that are not both on C_0 or on a given fibre f (see proof of 2.2.1). Therefore let us consider $\phi_{D|_f}$. We know that this morphism is given by the image of $H^0(\mathcal{O}_X(D)) \xrightarrow{\psi} H^0(\mathcal{O}_X(D)\otimes\mathcal{O}_f)$. But ψ is surjective: in fact $H^1(\mathcal{O}_X(D-f)) = H^1(\mathcal{O}_X(aC_0+(ae-1)f) = 0$ by 3.1.7. Moreover, since $D.f = a$, $H^0(\mathcal{O}_X(D)\otimes\mathcal{O}_f) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ and, since $a > 0$, $\mathcal{O}_{\mathbb{P}^1}(a)$ is very ample on $\mathbb{P}^1$. Therefore $\phi_{D|_f}$ is a closed immersion of f in $\mathbb{P}^r$.

We could have proved more easily that $|D|$ is not composite with a pencil simply noting that $D^2 = a^2e > 0$ (see 3.3.2).

□

Definition 2.2.3. Let X be a ruled surface embedded in $\mathbb{P}^r$. If for every p in C, $\pi^{-1}(p)$ has degree 1 (i.e.: every fibre has degree 1) then we call X a scroll.

Corollary 2.2.4. *For every $n > e \geq 0$, there is an embedding of the rational ruled surface X of invariant e as a rational scroll in $\mathbb{P}^{2n-e+1}$.*

Proof. Consider the divisor $D = C_0 + nf$. D is very ample by the previous corollary. Thus D defines an embedding in $\mathbb{P}^d$, where $d = h^0(\mathcal{O}_X(D)) - 1$. Since $D.f = 1$ the embedding is a scroll. Moreover by 2.1.11, the projection formula and 2.1.22, as already seen, we have:

$$H^0(\mathcal{O}_X(D)) \cong H^0(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(n)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(n-e)) \tag{2.12}$$

Since $n - e > 0$ then $h^0(\mathcal{O}_X(D)) = n + 1 + n - e + 1 = 2n - e + 2$. □

Chapter 3

Extendability of Rational Ruled Surfaces

The aim of this chapter is to study the extendability of rational ruled surfaces. We will use Knutsen-Lopez-Muñoz's theorem (1.4.1), together with B-E-L 2 (1.3.5) or Tendian (1.3.6) to render explicit the condition about the surjectivity of the Gaussian map. In order to apply the theorem we need to know something about cohomology of line bundles on rational ruled surfaces, i.e. we have to characterize those line bundles for which the first cohomology group is zero. The first section will be devoted to this problem. Afterwards, in section 2, we will compute the gonality of a generic irreducible curve in a base-point free linear system. We will see that, although gonality could depend *a priori* on the specific curve, actually gonality of curves on rational ruled surfaces depends only on the linear system. Eventually, in section 3, using the results of sections 1 and 2, we will translate 3.3.1 for rational ruled surfaces in a system of numerical conditions and we will solve it.

Remark: From now on X will be a rational ruled surface. We will use the notation of chapter 2.

3.1 Cohomology of Line Bundles on Rational Ruled Surfaces

Definition 3.1.1. A divisor D on a smooth surface is called *numerically effective (nef)* if $E.D \geq 0$ for every E effective divisor.

For example any fibre f of X is nef. Indeed if E is effective then $E \sim E' + nf$ with $n \geq 0$ and E' such that none of its irreducible components is linearly equivalent to f. But then $E.f = E'.f \geq 0$ by [7],V,1.4. On the contrary C_0 is not nef as soon as $e > 0$: indeed $C_0.C_0 = -e$.

Lemma 3.1.2. *If $a < 0$ then $H^0(\mathcal{O}_X(aC_0 + bf)) = 0$.*

Proof. Suppose by contradiction that $H^0(\mathcal{O}_X(aC_0 + bf)) \neq 0$. Then there exist $D \sim aC_0 + bf$ such that D is effective. However this is not possible since f is nef but $D.f = a < 0$. □

Let us start trying to characterize a, b for which $H^1(\mathcal{O}_X(aC_0 + bf)) = 0$. The case $a = -1$ will be treated separately. Then we will use induction to cover all the cases with $a \geq 0$. Then, by Serre's duality, we will extend the results to the cases with $a \leq -2$.

Proposition 3.1.3. *If $D = -C_0 + bf$ then $H^1(\mathcal{O}_X(D)) = 0$ for all $b \in \mathbb{Z}$*

Proof. By 3.1.2 $H^0(\mathcal{O}_X(D)) = 0$ and, using Serre's duality, also $H^2(\mathcal{O}_X(D)) = 0$. Hence we can use Riemann-Roch to compute $H^1(\mathcal{O}_X(D))$. Let K be the canonical divisor for X: $\chi(\mathcal{O}_X(D)) = \frac{1}{2}D.(D-K) + \chi(\mathcal{O}_X) = 0 \Rightarrow H^1(\mathcal{O}_X(D)) = 0$. □

Lemma 3.1.4. *If $b \geq -1$ then $H^1(\mathcal{O}_X(bf)) = 0$.*

Proof. After tensoring the defining exact sequence of C_0 by $\mathcal{O}_X(bf)$ we have:

$$0 \longrightarrow \mathcal{O}_X(-C_0 + bf) \longrightarrow \mathcal{O}_X(bf) \longrightarrow \mathcal{O}_X(bf) \otimes \mathcal{O}_{C_0} \longrightarrow 0 \tag{3.1}$$

Since $C_0 \cong \mathbb{P}^1$ and $\deg_{C_0}(\mathcal{O}_X(bf) \otimes \mathcal{O}_{C_0}) = C_0.bf = b$, taking cohomology we have:

$$\ldots \to H^1(\mathcal{O}_X(-C_0 + bf)) \to H^1(\mathcal{O}_X(bf)) \to H^1(\mathcal{O}_{\mathbb{P}^1}(b)) \to \ldots \tag{3.2}$$

$H^1(\mathcal{O}_X(-C_0 + bf)) = 0$ (3.1.3), $H^1(\mathcal{O}_{\mathbb{P}^1}(b)) = 0$ ($b \geq -1$) $\Rightarrow H^1(\mathcal{O}_X(bf)) = 0$. □

Proposition 3.1.5. *Let D be the divisor aC_0+bf, with $a \geq 0$. $H^1(\mathcal{O}_X(D)) = 0$ if, and only if, $b \geq ae-1$.*

Proof. Let $b \geq ae-1$. If $a=0$ then $H^1(\mathcal{O}_X(D)) = 0$ because of 3.1.4. We use induction on a: suppose $H^1(\mathcal{O}_X(D)) = 0$ for a fixed and for every $b \geq ae-1$. We will prove that $H^1(\mathcal{O}_X(D)) = 0$ for $a+1$ and $b \geq (a+1)e-1$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C_0) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X \otimes \mathcal{O}_{C_0} \longrightarrow 0 \tag{3.3}$$

Tensoring it with $\mathcal{O}_X((a+1)C_0+bf)$ we obtain:

$$0 \to \mathcal{O}_X(aC_0+bf) \to \mathcal{O}_X((a+1)C_0+bf) \to \mathcal{O}_X((a+1)C_0+bf) \otimes \mathcal{O}_{C_0} \to 0 \tag{3.4}$$

Taking cohomology:

$$\begin{aligned} \ldots \to H^1(\mathcal{O}_X(aC_0+bf)) \to H^1(\mathcal{O}_X((a+1)C_0+bf)) \to \\ \to H^1(\mathcal{O}_X((a+1)C_0+bf) \otimes \mathcal{O}_{C_0}) \to \ldots \end{aligned} \tag{3.5}$$

Now:

1. Since $C_0 \cong \mathbb{P}^1$,$H^1(\mathcal{O}_X((a+1)C_0+bf) \otimes \mathcal{O}_{C_0}) \cong$ $H^1(\mathcal{O}_{\mathbb{P}^1}(-(a+1)e+b) = 0$ since $b \geq (a+1)e-1$.

2. $H^1(\mathcal{O}_X(aC_0+bf)) = 0$ by the induction hypothesis.

Therefore $H^1(\mathcal{O}_X((a+1)C_0+bf)) = 0$ too, by (3.5), for $b \geq (a+1)e-1$.

Let us now conclude proving that if $a \geq 0$ and $b < ae-1$ then $H^1(\mathcal{O}_X(aC_0+bf)) \neq 0$. Consider the exact sequence (3.3) and, after tensoring it by $\mathcal{O}_X(aC_0+bf)$, consider its cohomology:

$$\begin{aligned} \ldots \to H^1(\mathcal{O}_X(aC_0+bf)) \to H^1(\mathcal{O}_{C_0}(aC_0+bf)) \to \\ \to H^2(\mathcal{O}_X((a-1)C_0+bf)) \to \ldots \end{aligned} \tag{3.6}$$

Now:

1. $H^1(\mathcal{O}_{C_0}(aC_0+bf)) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(b-ae)) \neq 0$ since $b-ae < -1$.

2. Since $-(a+1) < 0$, by 3.1.2 $H^0(\mathcal{O}_X(-(a+1)C_0 - (2+e+b)f) = 0$ and, by Serre's duality, it is isomorphic to $H^2(\mathcal{O}_X((a-1)C_0 + bf))^*$ that therefore is the null vector space.

Consequently $H^1(\mathcal{O}_X(aC_0 + bf))$ cannot be zero. $\square$

Corollary 3.1.6. *Let D be the divisor aC_0+bf, with $a \leq -2$. $H^1(\mathcal{O}_X(D)) = 0$ if, and only if, $b \leq e(a+1) - 1$*

Proof. Simply apply Serre's duality to 3.1.5. $\square$

To sum up what we have proved:

Proposition 3.1.7. *Let D be the divisor aC_0+bf on X. $H^1(\mathcal{O}_X(D)) = 0$ if, and only if, $a \geq 0$ and $b \geq ae-1$, or $a = -1$, or $a \leq -2$ and $b \leq e(a+1)-1$.*

We have computed what we needed in a relatively simple manner, using only basic facts about cohomology and geometry of ruled surfaces. Alternatively we could have used a more powerful tool (an application of Leray's spectral sequence) to obtain the same results, and much more, quicker and tidier. In fact if X is our ruled surface $\mathbb{P}(\mathcal{E})$, where $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$ on $\mathbb{P}^1$, and if π is the natural projection from X to $\mathbb{P}^1$ then, if $D = aC_0 + bf$ with $a \geq 0$, $H^i(\mathcal{O}_X(D)) \cong H^i(\mathbb{P}^1, \pi_*(\mathcal{O}_X(D)))\forall i$. Moreover $\pi_*(\mathcal{O}_X(D)) = \pi_*(\mathcal{O}_X(aC_0) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(b)) \cong S^a\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(b)$, where S^a is the a-th symmetric power. In general if $\mathcal{L}, \mathcal{F}$ are two line bundles $S^a(\mathcal{L} \oplus \mathcal{F}) \cong \bigoplus_{i=0}^a(\mathcal{L}^i \otimes \mathcal{F}^{a-i})$, therefore $S^a\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(b) \cong \bigoplus_{i=0}^a(\mathcal{O}_{\mathbb{P}^1}(-ie)) \otimes \mathcal{O}_{\mathbb{P}^1}(b)$. In this way the proposition 3.1.5 immediately follows. Moreover we can compute $H^0(\mathcal{O}_X(D))$ when requested and we see that $H^2(\mathcal{O}_X(D)) = 0$.

3.2 Gonality of Curves on Rational Ruled Surfaces

The aim of this section is to compute the gonality (see below) of a smooth, irreducible curve in the linear system $|D|$, where D is a divisor on a rational ruled surface X. Not every linear system contains an irreducible, nonsingular curve (see 2.2.2): if it does, we will first easily find an upper bound for the gonality, then we will show that in most cases this is the correct answer.

Before discussing the gonality of divisors on ruled surfaces we will compute the gonality of plane curves, as this is a paradigmatic example that it is not only interesting by itself, but it allows also to understand better the techniques of proof that come next.

Definition 3.2.1. Let C be an irreducible and nonsingular curve. We call *gonality of C (gon(C))* the least $n \geq 1$ for which there exists a finite morphism $\phi : \; C \longrightarrow \mathbb{P}^1$ of degree n

Example 3.2.2. The gonality of a hyperelliptic curve, i.e. a curve of genus ≥ 2 and with a finite morphism onto $\mathbb{P}^1$ of degree 2, is trivially 2.

Remark 3.2.3. Finding a finite morphism ϕ from C to $\mathbb{P}^1$ of degree k is the same as finding a vector space $V \subseteq H^0(\mathcal{O}_C(L))$ with L divisor on C, deg$L$=k, $|V|$ without base points and dim$V$=2.

Remark 3.2.4. With the notation as above, if gon(C)=k the linear system $\mathbb{P}V$ is complete. Indeed on the contrary $h^0(\mathcal{O}_C(L)) \geq 3$, hence if P is not a base-point, and the generic point is not since base-points are finite, $h^0(\mathcal{O}_C(L - P)) \geq 2$. Hence there would exist $W \subseteq H^0(\mathcal{O}_C(L - P))$ with dim$W$=2. It would define a rational map $C - -- > \mathbb{P}^1$ that can be extended to a finite morphism because of [7],I,6.8. But now deg$(L - P)$=$k - 1$. Contradiction.

Remark 3.2.5. Because of 3.2.3 and 3.2.4, using the same notation, gon(C) $= \min\{\deg L : L \in DivX, \dim |L| = 1\}$.

Theorem 3.2.6 (Bertini ([7],III, 10.9 and III, ex. 11.3)**.** *Let X be a smooth projective variety and let $|L|$ be a base-point free linear system. The generic divisor in $|L|$ is smooth. Moreover if $|L|$ is not composite with a pencil (i.e. if $f : X \to \mathbb{P}^n$ is the morphism determined by $|L|$ then* $\dim f(X) \geq 2$*) then every divisor in $|L|$ is connected, and thus the generic divisor in $|L|$ is smooth and irreducible.*

Proposition 3.2.7 (Gonality of plane curves)**.** *Let C be a smooth curve in $\mathbb{P}^2$ of degree $d \geq 3$. Then gon(C)=$d - 1$.*

Proof. Let P be a closed point on C. Projecting $C - \{P\}$ from this point to a line $l \cong \mathbb{P}^1$ we obtain a rational map $C - -- > \mathbb{P}^1$ that can be extended to a morphism ϕ.

Since C is projective and does not lie on a line, ϕ is a finite morphism (see [7], II, 6.8). By Bezout's theorem ([7], V, 1.4.2) the degree of ϕ is $d-1$. Thus gon(C)$\leq d-1$.

Let us now prove that gon(C) is exactly $d-1$. Suppose that $k = \text{gon}(C) \leq d-2$. Because of 3.2.5 there exists $L \in \text{Div}X$ such that $\dim|L| = 1$,$\deg L = k$, $|L|$ is base-point free. Using Bertini's theorem (3.2.6) we can suppose $L \sim P_1 + \ldots + P_k$ where $P_1, \ldots, P_k$ are distinct points.

Let K be the canonical divisor on C. Consider the following exact sequence, obtained by tensoring the defining sequence of $Z = \{P_1, \ldots, P_k\}$ by $\mathcal{O}_C(K)$:

$$0 \longrightarrow \mathcal{O}_C(K-L) \longrightarrow \mathcal{O}_C(K) \longrightarrow \mathcal{O}_C(K) \otimes \mathcal{O}_Z \longrightarrow 0 \qquad (3.7)$$

Since $\mathcal{O}_Z$ is a skyscraper sheaf sitting at the points $P_1, \ldots, P_k$ and $\mathcal{O}_C(K)$ is an invertible sheaf, tensoring by it does not affect $\mathcal{O}_Z$.

Taking cohomology:

$$0 \to H^0(\mathcal{O}_C(K-L)) \to H^0(\mathcal{O}_C(K)) \xrightarrow{\psi} H^0(\mathcal{O}_Z) \to \ldots \qquad (3.8)$$

Now $\psi : H^0(\mathcal{O}_C(K)) \to H^0(\mathcal{O}_Z) \cong \mathbb{C}^k$ sends sections s to $(s(P_1), \ldots, s(P_k))$, but while $h^0(\mathcal{O}_C(K)) = g$ (g is the genus of C), $h^0(\mathcal{O}_C(K - P_1 - \ldots - P_k) = g-k+1$ by Riemann-Roch ($h^0(\mathcal{O}_C(K-L) = h^1(\mathcal{O}_C(L)) = h^0(\mathcal{O}_C(L)) - \deg L + g - 1 = 2-k+g-1$). Therefore ψ is not surjective. Without loss of generality we can suppose that $(1, 0, \ldots, 0)$ does not belong to the image of ψ. That is: if a global section in the canonical sheaf of the curve is zero in the $k-1$ points $P_2, \ldots, P_k$ then it is zero in P_1 too. Or, analogously ([7],II, 7.7), if a divisor in $|K|$ has $P_2, \ldots, P_k$ in its support then it passes also through P_1. But this is not true. Indeed by the adjunction formula ([7],V, 1.5 and II, 8.20.1) $\mathcal{O}_C(K) \cong \mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(C) \otimes \mathcal{O}_C$. Since C has degree d, $\mathcal{O}_C(K) \cong \mathcal{O}_{\mathbb{P}^2}(d-3) \otimes \mathcal{O}_C$. When $k < d-2$ consider other different points $P_{k+1}, \ldots, P_{d-2}$. We can find $l_2, \ldots, l_{d-2}$ lines such that $P_i \in l_i$ but $P_1 \notin l_i$ for all i. Now $l_2 + \ldots + l_{d-2}$ restricted to C is in $|K|$ and passes through P_i for every i **except** i=1. □

We now go back to rational ruled surfaces. The way of computing the gonality for an irreducible nonsingular curve in $|D|$, where $D = aC_0 + bf$

on X, is similar to the one just discussed. In fact π, the projection to $\mathbb{P}^1$ with which every ruled surface is endowed by definition, will play the role of the projection onto a line in 3.2.7. And since we cannot speak of lines that separate points as in 3.2.7, we will speak instead of special divisors that, even if they are not very ample, separate the points that we are interested in. Therefore first of all we characterize base-point free linear systems.

Lemma 3.2.8. *Let D be a divisor on a smooth projective surface S. If $|D|$ is base-point free then D is nef.*

Proof. Suppose that D is not nef. Thus there exists an effective irreducible divisor E for which $D.E < 0$. Therefore for every $D' \in |D|$, since $D'.E = D.E < 0$, by [7],V,1.4, E and D' have a common irreducible component, that is: E is an irreducible component of D'. But this means that E is a base-component for $|D|$. Contradiction. □

Lemma 3.2.9. *Let D be the divisor $aC_0 + bf$ on the surface X. $|D|$ is base-point free if, and only if, $a \geq 0, b \geq ae$.*

Proof. If $|D|$ is base-point free then by 3.2.8 $D.f \geq 0$ and $D.C_0 \geq 0$, that is: $a \geq 0$ and $b \geq ae$.

If $a \geq 0$ and $b \geq ae$ then $D = a(C_0 + ef) + (b - ae)f$ with $b - ae \geq 0$. $|C_0 + ef|$ is base-point free by 2.2.1 and $|kf|$ ($k \geq 0$) is base-point free since all the fibres are linearly equivalent. Therefore $|D|$ is base-point free. □

Lemma 3.2.10. *Let D be the divisor $aC_0 + bf$ on the rational ruled surface X. Every irreducible, nonsingular curve C in $|D|$ has gonality at most* $\max\{1, a\}$.

Proof. From 2.2.2 we already know that $|D|$ contains an irreducible and nonsingular curve C if and only if $a = 0, b = 1$ or $a = 1, b = 0$ or $e = 0, a > 0, b > 0$, or $e > 0, a > 0, b \geq ae$. C is a projective curve. Let us now consider $\pi \circ i : C \hookrightarrow X \to \mathbb{P}^1$, where i is the inclusion. If it is constant then $C \subseteq f$, where f is a fibre. But then $C = f \cong \mathbb{P}^1$ and therefore $a = 0, b = 1$ and gon(C)=1.

If $\pi \circ i$ is not constant then it is a finite morphism ([7], II, 6.8), and $a \geq 1$. By [7],II,6.9 if p is a point in $\mathbb{P}^1$ the degree of this morphism is $\deg(\pi \circ i)^*(p) = C.f = a$. □

Corollary 3.2.11. *If $a = 1$ then gon(C)= 1.*

Lemma 3.2.12. *Let D be the divisor $aC_0 + bf$ on the rational ruled surface X. Let C be an irreducible, nonsingular curve in $|D|$. Then C is rational if, and only if, either $a = 1$, or $a = 0, b = 1$, or $e = 0, b = 1$, or $e = 1, a = 2, b = 2$.*

Proof. C is rational if, and only if, $g(C) = 0$. By 3.3.3 we have $g(C) = 0 \Leftrightarrow a = 1$, or $b = \frac{ae}{2} + 1$. But by 2.2.2 we know that if $a \neq 0, 1$ and $e > 0$ then $a > 0$ and $b \geq ae$. Thus $b = \frac{ae}{2} + 1$ if and only if $e = 2, a = 1, b = 2$, or $a = 0, b = 1$, or $e = 0, b = 1$, or $e = 1, a = 2, b = 2$. □

Lemma 3.2.13. *Let $e \neq 0$. Let D be the divisor $aC_0 + bf$ on the rational ruled surface X. Suppose $e \neq 1$ if $a = b$. Then every irreducible, nonsingular curve C in $|D|$ has gonality at least a.*

Proof. By the previous lemma we can take $a \geq 3$. Suppose that gon(C)= $k < a$, i.e. there exists a finite morphism ϕ from C to $\mathbb{P}^1$ of degree k. From 3.2.3 and 3.2.4 there exists a divisor L on C such that $\mathcal{O}_C(L) \cong \phi^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and $h^0(\mathcal{O}_C(L)) = 2$ and $\deg(L) = k$ and $|L|$ is base-point free. Since $C_0 \neq C$ ($a > 1$), $C_0 \cap C$ is a finite number of points. Therefore by Bertini's theorem we can suppose that $L \sim P_1 + \ldots + P_k$ where $P_1, \ldots, P_k$ are distinct points not lying on C_0.

Let K be the canonical divisor on C. Following exactly the same proof of 3.2.7 we obtain the same conclusion: if a divisor in $|K|$ has $P_2, \ldots, P_k$ in its support then it passes also through P_1. As usual, if $k < a - 1$ consider other different points $P_{k+1}, \ldots, P_{a-1}$. Now we will find an effective divisor N such that $P_1 \notin \text{supp}(N)$ and $P_i \in \text{supp}(N)$, for all $2 \leq i \leq a - 1$ and such that $N \sim K$, in order to arrive to a contradiction. In fact we will find divisors on X and then we will restrict them to C.

At first we have to compute, by adjunction formula, the canonical divisor on C: by 2.1.24 the canonical divisor on X is linearly equivalent to $-2C_0 + (-2 - e)f$, hence $\mathcal{O}_C(K) \cong \mathcal{O}_X((a - 2)C_0 + (b - 2 - e)f) \otimes \mathcal{O}_C$.

Let $E = C_0 + ef$ and $2 \leq i \leq a - 1$. If P_1 and P_i are not on the same fibre, since they does not belong to C_0, $M_i = C_0 + e\pi^{-1}(\pi(P_i))) \in |E|$ separates them. Instead if P_1 and P_i are on the same fibre f consider this exact sequence:

$$0 \to H^0(\mathcal{O}_X(C_0 + (e-1)f) \to H^0(\mathcal{O}_X(C_0 + ef)) \xrightarrow{\psi}$$
$$\xrightarrow{\psi} H^0(\mathcal{O}_f(C_0 + ef)) \to H^1(\mathcal{O}_X(C_0 + (e-1)f)) \to \dots \qquad (3.9)$$

First notice that ψ is surjective because $H^1(\mathcal{O}_X(C_0 + (e-1)f)) = 0$ by 3.1.7.

Since ψ is surjective and $\mathcal{O}_f(C_0+ef) \cong \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\mathbb{P}^1}(1)$ is very ample, there exists $M_i \in |E|$ such that $P_i \in \mathrm{supp}(M_i)$ but $P_1 \notin \mathrm{supp}(M_i)$.

Let us now consider the divisor $(b - ae + e - 2)f$. By 3.2.9 this divisor is base-point free if and only if $b \geq ae - e + 2$, but since we are considering only the cases with $a > 0, b \geq ae$ and $e \neq 0$ and $e \neq 1$ if $a = b$, this divisor is always base-point free. Let $T \in |(b - ae + e - 2)f|$ be an effective divisor that does not pass through P_1.

Let $F = M_2 + \dots M_{a-1} + T$. F is an effective divisor linearly equivalent to $(a-2)C_0 + (b-2-e)f$ that passes through every point P_i except P_1. Let $N = F \mid_C$. □

In the case $e = 0$, $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $|C_0|$ and $|f|$ are the two rulings. Therefore, in fact, C_0 and f play the same role and besides the projection π for which f is the fibre there is a projection π' for which C_0 is the fibre. Then it is natural to argue that $\mathrm{gon}(aC_0 + bf)$ is $\min\{a, b\}$. In fact:

Lemma 3.2.14. *Let $e = 0$. Let D be the divisor $aC_0 + bf$ on the rational ruled surface X, with $a > 0, b \geq a$. Every irreducible, nonsingular curve C in $|D|$ has gonality a.*

Proof. We already know that the gonality is at most a (3.2.10). Following the proof of 3.2.13 we only have to find another E, since $|C_0|$ does not separate points in different fibres. Let $E = C_0 + f$. This is a very ample divisor by 2.2.2, thus in particular it separates points. Therefore it is possible to find $M_2, \dots, M_{a-1}$ that behave as in the proof of 3.2.13. Since $b \geq a$, $|(b-a)f|$ is base-point free and we can find also T. By construction $F = M_2 + \dots M_{a-1} + T$ is an effective divisor linearly equivalent to K. We can conclude. □

Corollary 3.2.15. *Let $e = 0$. Let D be the divisor $aC_0 + bf$ on the rational ruled surface X, with $a > 0, b > 0$. Every irreducible, nonsingular curve C in $|D|$ has gonality* $\min\{a, b\}$.

Proof. if $b \geq a$ the result follows from the preceding lemma. If $b < a$ consider the isomorphism $\theta : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ such that $\theta(x, y) = (y, x)$. Naturally gon(C)=gon($\theta(C)$)= b, since θ simply exchanges the two rulings. □

Now it remains only the case $e = 1, a = b$. We could not use the proof in 3.2.13 since $H^0(\mathcal{O}_X(-f)) = 0$, hence there are not effective divisors in $|-f|$. But we can adapt that proof to show that gon(C) is not lower than $a - 1$: we simply forget the point P_{a-1} and the associated divisor M_{a-1}. As T we take an effective divisor in $|C_0|$ that does not pass through P_1: it is possible since C_0 itself does not pass through P_1.

Lemma 3.2.16. *Let $e = 1$. Let D be the divisor $aC_0 + af$ on the rational ruled surface X, with $a > 1$. Every irreducible, nonsingular curve C in $|D|$ has gonality $a - 1$.*

Proof. The idea of the proof is to find a linear system on C of dimension 2 determined by a divisor of degree a, and then remove a non-base-point. It is similar to what we have done for plane curves of degree d: the linear system determined by all the intersections of lines in $\mathbb{P}^2$ with the curve was a g^2_d.

As usual let us first consider the defining sequence of C:

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0 \tag{3.10}$$

Tensoring it with $\mathcal{O}_X(C_0 + f)$ and taking cohomology:

$$\begin{aligned} 0 \to H^0(\mathcal{O}_X(C_0 + f - C)) \to H^0(\mathcal{O}_X(C_0 + f)) \to \\ \to H^0(\mathcal{O}_X(C_0 + f) \otimes \mathcal{O}_C)) \to H^1(\mathcal{O}_X(C_0 + f - C)) \to \dots \end{aligned} \tag{3.11}$$

Since $C_0 + f - C \sim (1 - a)C_0 + (1 - a)f$ its H^0 and its H^1 are zero, hence $|(C_0 + f)\ |_C\ |$ has the same dimension as $|C_0 + f|$ that is 2. Moreover $\deg((C_0 + f)\ |_C) = (C_0 + f).C = a$. Note that $|C_0 + f|$ separates points that are not on C_0 (the proof is analogous to 3.2.13). But $C \cap C_0 = \emptyset$ since $C.C_0 = 0$ and they are different irreducible curves. It follows that no points in C are base points and that $|(C_0 + f)_{|C} - P|$ (where P is any point on C) is a complete linear system without base-points: hence it is a g^1_{a-1}. □

Summing up what we have proved:

Proposition 3.2.17. *Let C be an irreducible, non singular curve on the rational ruled surface X, linearly equivalent to $aC_0 + bf$, with $a \geq 2$. Then*

1. $e = 0 \Rightarrow gon(C) = \min\{a, b\}$
2. $e = 1, a = b \Rightarrow gon(C) = a - 1$
3. *in the other cases* $gon(C) = a$

3.3 Extendability

Having all the information we need it is now time to try to answer the question we are interested in. That is: if we have a rational ruled surface $X = \mathbb{P}(\mathcal{E})$ ($\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e), e \geq 0$) embedded in $\mathbb{P}^n$ with a very ample divisor L, when is it extendable? What we can do is to find some examples in which X turns out to be non-extendable and other examples in which X turns out to be extendable. Certainly it is much more difficult to conclude that a surface is not extendable, since in this case we need a correct logical proof and not only a specific example. Luckily, as we have already seen in the previous chapter, there are many results that help us. Combining Knutsen-Lopez-Muñoz's theorem with B-E-L 2 we can say that:

Theorem 3.3.1. *Let S be a smooth projective surface in $\mathbb{P}^r$, embedded with a complete linear system related to a very ample divisor L (i.e.: $\mathbb{P}^r \cong \mathbb{P}H^0(\mathcal{O}_S(L))$). If there exists an invertible sheaf $\mathcal{O}_S(D)$ such that for a generic $C \in |D|$ the following hypotheses are verified,*

1. $|D|$ *is base-point free*
2. $D^2 > 0$ *and* $g(C) > 0$
3. $H^1(\mathcal{O}_S(L - D)) = 0$
4. $H^1(\mathcal{O}_S(L - 2D)) = 0$ *and* $(L - D).D \geq 2g(C) + 1$
5. $L.D > 2D^2$
6. $gon(C) \geq 4$, $C \ncong$ *plane quintic and* $L.D \geq 4g(C) + 7 - 2gon(C)$

then S is not extendable.

Remark 3.3.2. Since $|D|$ is base-point free, by Bertini's theorem it follows that the generic $C \in |D|$ is nonsingular. It is also irreducible because $D^2 > 0$ implies that $|D|$ is not composite with a pencil. In fact if it were we can find two disjoint divisors in $|D|$ and thus D^2 would be 0 by [7], V, 1.4.

From now on, as in the preceding paragraphs, we will consider only rational ruled surfaces. Hence X will be the rational ruled surface $\mathbb{P}(\mathcal{E})$ ($\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e),\ e \geq 0$), L will be a very ample divisor linearly equivalent to $\alpha C_0 + \beta f$. Since L is very ample on X then $\alpha > 0, \beta > \alpha e$ (2.2.2). K will be the canonical divisor on X. For each e, α, β, i.e. for every rational ruled surface and for every embedding, we can ask if such a D exists, that is: if there exist $a, b \in \mathbb{Z}$ such that $D \sim aC_0 + bf$ verifies all the hypothesis. Let C be a generic smooth and irreducible curve in $|D|$.

Lemma 3.3.3. *The genus of C is $\frac{(a-1)(-ae+2(b-1))}{2}$.*

Proof. By the adjunction formula ([7],V,1.5) $2\mathrm{g}(C) - 2 = C.(C + K)$. Since we already know that $K \cong -2C_0 + (-2 - e)f$, we have: $2\mathrm{g}(C) = (aC_0 + bf).((a-2)C_0 + (b-e-2)f) + 2 = (a-1)(-ae+2(b-1))$. □

Lemma 3.3.4. *If $a \geq 3$ conditions 1 and 2 of 3.3.1 are satisfied if and only if $e > 1, b \geq ae$ or $e = 0, b \geq 2$.*

Proof. $|D|$ is base-point free $\Leftrightarrow a \geq 0, b \geq ae$ (3.2.9). In this case $D^2 = -a^2e + 2ab = a(2b - ae) > 0 \Leftrightarrow b > \frac{ae}{2}$. By 3.3.3, since $a \geq 3, g(C) > 0 \Leftrightarrow b > \frac{ae}{2} + 1$. If $e > 0$ the condition $b > \frac{ae}{2} + 1$ is already implied by $b \geq ae$ and $a \geq 3$, if $e = 0$ it gives an additional information: $b > 1$. □

Lemma 3.3.5. *A curve C on X is isomorphic to a smooth plane quintic if, and only if, $e = 1$ and C is linearly equivalent to $4C_0 + 5f$ or $5C_0 + 5f$.*

Proof. A plane quintic has genus 6 and gonality 4 (3.2.7). If $C \sim aC_0 + bf$ is isomorphic to a plane quintic then, looking at its gonality, by 2.2.2 and by 3.2.17 we have four possibilities: $e = 0, \min\{a, b\} = 4$ or $e = 1, a = 4, b > 4$ or $e = 1, a = 5, b = 5$ or $e \geq 2, a = 4, b \geq ae$. Now, looking at the genus, by 3.3.3 we see that in the first case $\mathrm{g}(C) \geq 9 > 6$; in the second case $\mathrm{g}(C) = 6 \Leftrightarrow b = 5$; in the third case $\mathrm{g}(C) = 6$; in the fourth case $\mathrm{g}(C) \geq 9 > 6$. Thus the conditions on e, a, b are necessary. Now we will see that actually, if $e = 1$, every smooth irreducible curve C in $|D|$ where $D = 4C_0 + 5f$ or $D = 5C_0 + 5f$ is isomorphic to a plane quintic. In fact consider the divisor $C_0 + f$ over X, that is base-point free by 3.2.9. Clearly $E = (C_0 + f)|_C$ is a base-point free linear system on C. The degree of the linear system is $(C_0+f).C = 5$. Moreover $h^0((C_0+f)\otimes\mathcal{O}_C) = 3$ (the proof is analogous to other proofs already examined: consider the defining sequence for C, tensor it by $\mathcal{O}_X(C_0+f)$ and pass to cohomology). Therefore $|E|$ is a g^2_5 over C. It is also very ample: in fact if it is not, by [7], IV, 3.1, there exist P, Q points on C (possibly coincident) such that $\dim |E - P - Q| = \dim |E| - 1$. But then $\dim |E-P-Q| = 1$, while $\deg(E-P-Q) = 3$. This is impossible because C is not trigonal. □

Lemma 3.3.6. *On the rational ruled surface X conditions 1 ... 6 of 3.3.1 are equivalent to:*

(i) $e = 0, a \geq 4, b \geq 4$ *or* $e > 0, a \geq 4, b \geq ae$

(ii) if $e = 1, a = b$ *then* $a \geq 5$

(iii) if $e = 1$ *then* $(a, b) \neq (4, 5)$ *and* $(a, b) \neq (5, 5)$

(iv) $H^1(\mathcal{O}_X(L - 2D)) = 0$

(v) $L.C > 2D^2$

Proof. By 3.2.17 and 3.3.4 and 3.3.5 if there exists D such that the generic smooth curve $C \in |D|$ satisfies 1 ... 6 of 3.3.1, then D satisfies also (i), ..., (v).

On the contrary: suppose that there exists D such that the generic smooth curve $C \in |D|$ satisfies (i), ..., (v). Then D, C satisfy 1,2 of 3.3.1 and the condition about gonality by 3.2.17 and 3.3.4. Furthermore, by 3.3.5, C is not isomorphic to a plane quintic.

Moreover$(L - D).D \geq 2g(C) + 1$. In fact $(L - D).D = L.C - C^2$, while $2g(C)+1$ by the adjunction formula is $C.(C+K)+3$. From (v) $L.C - C^2 > C^2$. But $C.K + 3 = ae - 2b - 2a + 3 \leq 0$ by (i), therefore the thesis follows.

Moreover $H^1(\mathcal{O}_X(L-D)) = 0$. In fact let us consider this exact sequence:

$$0 \longrightarrow \mathcal{O}_X(L-2C) \longrightarrow \mathcal{O}_X(L-C) \longrightarrow \mathcal{O}_X(L-C) \otimes \mathcal{O}_C \longrightarrow 0 \quad (3.12)$$

Taking cohomology:

$$\ldots \to H^1(\mathcal{O}_X(L-2C)) \to H^1(\mathcal{O}_X(L-C)) \to H^1(\mathcal{O}_C(L-C)) \to \ldots \quad (3.13)$$

We prove that $H^1(\mathcal{O}_C(L-C)) = 0$. In fact $(L-C)|_C$ has degree $(L-C).C$ on C and $L.C - C^2 \geq 2g(C) + 1 > 2g(C) - 2$. Since $L - C$ on C has degree greater than $2g(C) - 2$, by [7], IV, 1.3.4 its H^1 is zero. But $H^1(\mathcal{O}_X(L-2C))$ is zero too by (iv), hence $H^1(\mathcal{O}_X(L-C)) = 0$.

Moreover $L.D \geq 4\text{g}(C) + 7 - 2\text{gon}(C)$. In fact $4g(C) + 7 - 2\text{gon}(C)$ is at most $2C^2 + 2K.C - 2a + 13$. Since $L.C > 2C^2$, we have only to show that $2K.C - 2a + 13 = 2ae - 4b - 6a + 13$ is ≤ 0. But this is true by (i).

Eventually note that if $e = 1$ then by (i), since $b \geq a$, conditions (ii) and (iii) can be stated as $b \geq 6$.

□

Lemma 3.3.7. *Let X be a rational ruled surface embedded in $\mathbb{P}^r$ with a very ample line bundle $L \sim \alpha C_0 + \beta f$. Let D be the divisor $aC_0 + bf$. Conditions 1 ... 6 of 3.3.1 are equivalent to four systems of equations:*

1. *(a)* $e = 0, a \geq 4, b \geq 4$ *or* $e > 0, a \geq 4, b \geq ae$
   (b) if $e = 1$ *then* $b \geq 6$
   (c) $a \leq \frac{\alpha}{4}$
   (d) $b \leq -\frac{\alpha e}{2} + ae + \frac{1}{2} + \frac{\beta}{2}$
   (e) $b(\alpha - 4a) > a(\alpha e - \beta - 2ae)$

2. *(a)* $e = 0, a \geq 4, b \geq 4$ *or* $e > 0, a \geq 4, b \geq ae$
   (b) if $e = 1$ *then* $b \geq 6$
   (c) $\frac{\alpha}{4} < a \leq \frac{\alpha}{2}$
   (d) $b \leq -\frac{\alpha e}{2} + ae + \frac{1}{2} + \frac{\beta}{2}$
   (e) $b < \frac{a}{4a-\alpha}(\beta - \alpha e + 2ae)$

3. (a) $e = 0, a \geq 4, b \geq 4$ *or* $e > 0, a \geq 4, b \geq ae$

   (b) *if* $e = 1$ *then* $b \geq 6$

   (c) $a = \frac{\alpha}{2} + \frac{1}{2}$

   (d) $b = b$

   (e) $b < \frac{a}{4a-\alpha}(\beta - \alpha e + 2ae)$

4. (a) $e = 0, a \geq 4, b \geq 4$ *or* $e > 0, a \geq 4, b \geq ae$

   (b) *if* $e = 1$ *then* $b \geq 6$

   (c) $a \geq \frac{\alpha}{2} + 1$

   (d) $b \geq -\frac{\alpha e}{2} + ae + \frac{1}{2} + \frac{\beta}{2} - \frac{e}{2}$

   (e) $b < \frac{a}{4a-\alpha}(\beta - \alpha e + 2ae)$

Proof. Conditions (a),(b) of all the systems come directly from 3.3.6 (i), (ii), (iii). Conditions (c) and (d) are the translations into equations of the condition $H^1(\mathcal{O}_X(L - 2D)) = 0$, using 3.1.7. Condition (e) is $L.C > 2D^2 \Leftrightarrow b(\alpha - 4a) > a(\alpha e - \beta - 2ae)$. Note that $\alpha - 4a < 0 \Leftrightarrow a > \frac{\alpha}{4}$. □

Given $e, \alpha, \beta > \alpha e$, we can surely conclude that X embedded in $\mathbb{P}^n$ with L is not extendable at all as soon as we can find a solution of one of the four systems. Hence the next step is to say when the four systems have solutions.

Lemma 3.3.8. *System (1) of 3.3.7 has a solution if, and only if, either* $e \geq 2, \alpha \geq 16$ *or* $e = 0, \alpha \geq 16, \beta \geq 7$ *or* $e = 1, \alpha \geq 20$ *or* $e = 1, 16 \leq \alpha < 20, \beta > \alpha + 2$.

Proof. The condition $\alpha \geq 16$ cannot be eliminated in all the cases: otherwise a should be lower than 4. Moreover condition (e) is always verified since the left hand term is ≥ 0 while the right hand term is strictly negative.

If $e \geq 2$ then $a = 4, b = 4e + 1$ is a solution (note that in (d) $\frac{\beta - \alpha e}{2} \geq \frac{1}{2}$, since $\beta > \alpha e$ by hypothesis).

If $e = 1$ and $16 \leq \alpha < 20$ then $a = 4$. Therefore by (b) $b \geq 6$. By (d) this implies $\beta \geq \alpha + 3$. Actually in this case $a = 4, b = 6$ is a solution.

If $e = 1$ and $\alpha \geq 20$ then $a = 5$, $b = 6$ is a solution.

If $e = 0$ $\beta \geq 7$ is a necessary condition, since from (a) and (d) we have that $4 \leq \frac{\beta+1}{2} \Leftrightarrow \beta \geq 7$. But this is also a sufficient condition: in fact if $\beta \geq 7$, $a = 4, b = 4$ is a solution. □

Lemma 3.3.9. *System (2) of 3.3.7 has a solution if, and only if, either* $e \geq 2, \alpha \geq 8$ *or* $e = 0, \alpha = 8, \beta \geq 9$ *or* $e = 0, \alpha = 9, \beta \geq 8$ *or* $e = 0, \alpha \geq 10, \beta \geq 7$ *or* $e = 1, \alpha = 8, \beta \geq 13$ *or* $e = 1, \alpha = 9, \beta \geq 12$ *or* $e = 1, \alpha = 10, \beta \geq 13$ *or* $e = 1, \alpha \geq 11$.

Proof. If $e \geq 1$, $\alpha \geq 8$ is a necessary and sufficient condition for the system to be solved. In fact (a)+(c) $\Rightarrow \alpha \geq 8$. Moreover $a = \left[\frac{\alpha}{2}\right], b = ae$ is a solution: in fact, when $e \geq 2$, (e) $\Leftrightarrow e < \frac{\beta - \alpha e + 2ae}{4a - \alpha} \Leftrightarrow \beta > 2ae$, but by hypothesis $\beta > \alpha e \geq 2\left[\frac{\alpha}{2}\right] e = 2ae$.

If $e = 0$ it is necessary that $\beta \geq 7$ (3.3.8). Another necessary condition is (a)+(e) $\Rightarrow \beta > 16 - 4\frac{\alpha}{a}$. If $\alpha = 8$ then $a = 4$, therefore $\beta > 8$. In such cases $a = 4, b = 4$ is a solution. If $\alpha = 9$ then $a = 4$, therefore $\beta > 7$. In such cases $a = 4, b = 4$ is a solution. If $\alpha \geq 10$ for every $\beta \geq 7$, $a = 4, b = 4$ is a solution.

If $e = 1$ and $\alpha \geq 12$ then $a = \left[\frac{\alpha}{2}\right], b = a$ is a solution (note that $a \geq 6$). First note that if $b \geq 6$ then, by (d) and by (e) we get

$$\beta \geq 11 - 2a + \alpha \tag{3.14}$$

$$\beta > 24 - 2a + \alpha - 6\alpha/a \tag{3.15}$$

If $8 \leq \alpha < 12$, a is at most 5, thus, by (b), b shall be at least 6. By (c), if $\alpha = 8, 9$ then $a = 4$, if $\alpha = 10, 11$ then $a = 4$ or $a = 5$. Looking at all the possibilities in (3.14), (3.15) we have the following necessary conditions for the system to have a solution: $\alpha = 8 \Rightarrow \beta \geq 13; \alpha = 9 \Rightarrow \beta \geq 12; \alpha = 10 \Rightarrow \beta \geq 13; \alpha = 11 \Rightarrow \beta \geq 12$ (this is already implied by $\beta > \alpha e$). Actually if $\alpha = 8, 9$ then $a = 4, b = 6$ is a solution; if $\alpha = 10, 11$ then $a = 5, b = 6$ is a solution. □

Lemma 3.3.10. *System (3) of 3.3.7 has a solution if, and only if, either* $e \geq 2, \alpha \geq 7, \alpha$ *odd*, $\beta \geq \alpha e + e + 1$ *or* $e = 0, \alpha \geq 9, \alpha$ *odd*, $\beta \geq 9$ *or* $e = 0, \alpha = 7, \beta \geq 10$ *or* $e = 1, \alpha \geq 11, \alpha$ *odd*, $\beta \geq \alpha + 2$ *or* $e = 1, \alpha = 7, 9, \beta \geq 13$.

Proof. If $e \geq 2$ a necessary condition is $\alpha \geq 7$. Also (a)+(e) $\Rightarrow ae < \frac{a}{4a - \alpha}(\beta - \alpha e + 2ae)$, that is $\beta > 2ae$, hence $\beta > \alpha e + e$. In such cases $a = \frac{\alpha + 1}{2}, b = ae$ is a solution.

If $e = 0$ a necessary condition is (a)+(e) $\Rightarrow 4 < \frac{1}{2}\frac{\alpha+1}{\alpha+2}\beta$, that is: $\beta > 8\frac{\alpha+2}{\alpha+1}$. If $\alpha = 7$ this condition is equivalent to $\beta > 9$, if $\alpha \geq 8$ it is equivalent to $\beta > 8$. In both cases $a = \frac{\alpha+1}{2}, b = 4$ is a solution.

If $e = 1$ and $\alpha \geq 11$ then $a = \frac{\alpha+1}{2}, b = a$ is a solution as soon as $\beta \geq \alpha+2$ (a is at least 6). As we have already seen this is also a necessary condition. If $\alpha = 7$ then $a = 4$ but then necessarily by (b) $b \geq 6$, therefore by (e) $\Rightarrow 6 < \frac{4}{16-7}(\beta - 7 + 8)$, that is $\beta \geq 13$. In such cases $a = 4, b = 6$ is a solution. If $\alpha = 9$ then $a = 5$, therefore $b \geq 6 \Rightarrow 6 < \frac{5}{11}(\beta + 1) \Rightarrow \beta \geq 13$. In such cases $a = 5, b = 6$ is a solution. □

Lemma 3.3.11. *If for given α, β, e with $\alpha \geq 7$ system (4) has a solution, then for the same α, β, e one of the previous systems has a solution too.*

Proof. (a) + (e) implies that if $e \neq 0$ then $\beta > 2ae$, if $e = 0$ then $\beta > 16 - 4\frac{\alpha}{a}$. If $e \neq 0$ by (c) we get $\beta > \alpha e + 2e$ and $\beta > 10e$ if $\alpha = 7$, since $a \geq 5$.

If $e = 0$ then $\beta > 16 - 8\frac{\alpha}{\alpha+2}$, that is: $\beta > 8$ in general and $\beta \geq 10$ if $\alpha = 7$.

If $e = 1$, as already said,

$$\beta > 2a \tag{3.16}$$

This implies $\beta > \alpha + 2$. But in this case we have also $b \geq 6$. This with (e) implies

$$\beta > 24 + \alpha - 2a - 6\alpha/a \tag{3.17}$$

Combining (3.16) and (3.17), we have: if $\alpha = 7, 8$ then $\beta \geq 13$ (by (3.16) if $a \geq 6$, and by (3.17) if $a = 5$). Moreover if $\alpha \geq 9$ then, by (c), $a \geq 6 \Rightarrow \beta \geq 13$. □

Lemma 3.3.12. *If $\alpha \leq 6$ then system (4) has no solutions.*

Proof. A necessary condition for system (4) to have a solution is (d)+(e) $\Rightarrow -\frac{\alpha e}{2} + ae + \frac{1}{2} + \frac{\beta}{2} - \frac{e}{2} < \frac{a}{4a-\alpha}(\beta - \alpha e + 2ae)$. This is equivalent to

$$(a - \frac{\alpha}{2})\beta + \frac{\alpha}{2}(e-1) + 2e\left(a^2 - a\alpha - a + \frac{\alpha^2}{4}\right) + 2a < 0 \tag{3.18}$$

By (c), $a \geq \frac{\alpha}{2} + 1$ and we know that $\beta > \alpha e$ hence $2a \geq \alpha + 2$ and $\left(a - \frac{\alpha}{2}\right)\beta \geq \alpha e$. Thus 3.18 implies that $\frac{\alpha}{2}(e+1) + 2e\left(a^2 - a\alpha - a + \frac{\alpha^2}{4} + \frac{\alpha}{2}\right) +$

$2 < 0$. But $\frac{\alpha}{2}(e+1) \geq 0$ and, since $a \geq 4$, $a^2 - a(\alpha+1) + \frac{\alpha^2}{4} + \frac{\alpha}{2} \geq 0$ for all $\alpha \leq 6$. Therefore, if $\alpha \leq 6$, (3.18) is never verified. □

To sum up what we have proved:

Proposition 3.3.13. *Let $X = \mathbb{P}(\mathcal{E})$ ($\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e), e \geq 0$) be a rational ruled surface embedded in $\mathbb{P}^r$ with a complete linear system related to a very ample divisor $L \sim \alpha C_0 + \beta f$ with $\alpha > 0$ and $\beta > \alpha e$. In all the following cases the embedding of X is not extendable:*

1. $e \geq 2$, $\alpha \geq 8$
2. $e \geq 2$, $\alpha = 7$, $\beta \geq 8e + 1$
3. $e = 0$, $\alpha \geq 10$, $\beta \geq 7$
4. $e = 0$, $\alpha = 9$, $\beta \geq 8$
5. $e = 0$, $\alpha = 8$, $\beta \geq 9$
6. $e = 0$, $\alpha = 7$, $\beta \geq 10$
7. $e = 1$, $\alpha \geq 11$
8. $e = 1$, $\alpha = 10$, $\beta \geq 13$
9. $e = 1$, $\alpha = 9$, $\beta \geq 12$
10. $e = 1$, $\alpha = 8$, $\beta \geq 13$
11. $e = 1$, $\alpha = 7$, $\beta \geq 13$

Remark 3.3.14. Notice that almost all the cases listed in 3.3.13 (except when $\alpha = 7$) actually are solutions of system 2.

We can now try to add items to the list above by allowing a to be lower than 4. Actually in the case of trigonal curves we can use 1.3.6 instead of 1.3.5 to cope with the surjectivity of the Gaussian map. Thus let us now begin the study of the conditions of 1.3.6, keeping in mind that every irreducible non singular curve C in $|3C_0 + bf|$ ($b \geq 3e$ if $e \neq 0$, or $b > 0$ if $e = 0$) has gonality 3 as soon as $e \geq 2$, or $e = 1, b \geq 4$, or $e = 0, b \geq 3$ and that in all these cases $|f_{|C}|$ is the linear system that defines the finite map of degree 3 of C onto $\mathbb{P}^1$ (see 3.2.10).

Lemma 3.3.15. *Let* $X = \mathbb{P}(\mathcal{E})$ $(\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e), e \geq 0)$ *be a rational ruled surface embedded in* $\mathbb{P}^r$ *with a complete linear system related to a very ample divisor* $L \sim \alpha C_0 + \beta f$ *with* $\alpha \geq 4$ *and* $\beta > \alpha e$. *Let* D *be the divisor* $3C_0 + bf$, *with* $b \geq 3e$ *if* $e \geq 2$, *or* $b \geq 4$ *if* $e = 1$, *or* $b \geq 3$ *if* $e = 0$. *Let* C *in* $|D|$ *be an irreducible nonsingular curve. Let* g *be the genus of* C *and* K_C *its canonical divisor. Then* $H^0(C, \mathcal{O}_C(3K_C - (g-4)f_{|C} - L_{|C})) = 0$ *and* $H^1(\mathcal{O}_C(L_{|C})) = 0$.

Proof. By the adjunction formula we know that $K_C = (K + C)|_C$, where K is the canonical divisor of X, i.e. $K \sim -2C_0 + (-2-e)f$. Therefore let us consider the defining sequence of C. After tensoring it by $\mathcal{O}_X(3K + 3C + -(g-4)f - L)$ and considering its cohomology we have:

$$\begin{array}{rcl} H^0(\mathcal{O}_X((3-\alpha)C_0 + (d+b)f)) & \rightarrow & H^0(\mathcal{O}_C(3K_C - (g-4)f_{|C} - L_{|C})) \rightarrow \\ \rightarrow H^1(\mathcal{O}_X(-\alpha C_0 + df)) & \rightarrow & \ldots \end{array} \qquad (3.19)$$

where $d = -2 - 3e + 2b - g - \beta = -\beta$ by 3.3.3.

Now, since $\alpha \geq 4$, $H^0(\mathcal{O}_X((3-\alpha)C_0 + (d+b)f)) = 0$ by 3.1.2. Moreover $d \leq (-\alpha+1)e - 1$, because $\beta > \alpha e$. Therefore $H^1(\mathcal{O}_X(-\alpha C_0 + df)) = 0$ too by 3.1.7. By (3.19), $H^0(\mathcal{O}_C(3K_C - (g-4)f_{|C} - L_{|C})) = 0$.

Analogously, let us consider the defining sequence of C and tensor it by $\mathcal{O}_X(L)$. Taking cohomology we have:

$$\ldots \rightarrow H^1(\mathcal{O}_X(L)) \rightarrow H^1(\mathcal{O}_C(L_{|C})) \rightarrow H^2(\mathcal{O}_X(L - C)) \rightarrow \ldots \qquad (3.20)$$

$H^1(\mathcal{O}_X(L)) = 0$ by 3.1.7; $H^2(\mathcal{O}_X(L - C)) = 0$ by 2.1.12. Thus $H^1(\mathcal{O}_C(L_{|C})) = 0$ by (3.20). □

Lemma 3.3.16. *Using the hypotheses and notation above,* $g(C) \geq 5$ *if and only if* $b \geq \frac{3e+7}{2}$.

Proof. Simply apply 3.3.3. □

Lemma 3.3.17. *Using the hypotheses and notation above, if* $b(8-\alpha) \leq 3(\beta - \alpha e) + 12e + 14$, *then* $h^0(\mathcal{O}_C(2K_C - L_{|C})) \leq 1$.

Proof. Remember that C is trigonal, therefore if $\deg_C(2K_C - L_{|C}) \leq 2$ then $h^0(\mathcal{O}_C(2K_C - L_{|C})) \leq 1$. In fact, on the contrary, we could find a linear system of dimension 1 and degree ≤ 2.

Now $\deg_C(2K_C - L_{|C}) = (2(K+C)-L).C = 3(\alpha e - \beta) - 12 - 12e + b(8-\alpha)$. Hence $\deg_C(2K_C - L_{|C}) \leq 2 \Leftrightarrow b(8-\alpha) \leq 3(\beta - \alpha e) + 12e + 14$. □

Now to improve the list of 3.3.13, we use again K-L-M but combined with Tendian's theorem instead of B-E-L 2. By 3.3.15, 3.3.16, 3.3.17 we have to solve again four system of equations as in 3.3.7 but in which, using the same notation, $\alpha \geq 4$ and condition (a) and (b) are now:

$$\text{(a) } e = 0, a = 3, b \geq 3 \text{ or } e > 0, a = 3, b \geq 3e$$

$$\text{(b) if } e = 1 \text{ then } b \geq 4$$

and in which we add conditions (f) and (g):

$$\text{(f) } b \geq \frac{3e+7}{2}$$

$$\text{(g) } \beta \geq \frac{b}{3}(8-\alpha) - 4e - \frac{14}{3} + \alpha e$$

Note that if $e \geq 3$ then (f) follows from (a). Note also that condition (e), except for system 1, can be rewritten as:

$$\text{(e) } \beta > (\alpha - 6)e + b(4 - \frac{\alpha}{3})$$

Note eventually that condition (g) is always verified if $\alpha \geq 8$ (in fact $\beta > \alpha e$).

Now, since a is fixed, the strategy of resolution of these new four systems is much more simple: in every system condition (c) determines the values of α for which the system may have solutions, conditions (a),(b),(f) determine the minimal possible value of b (that is always the best value to take) and consequently condition (d),(e),(g) determine necessary and sufficient conditions on β for the system to be solved.

Lemma 3.3.18. *The modified system 1 in 3.3.7 does not improve the list 3.3.13.*

Proof. By (c) $\alpha \geq 12$. If $e = 0$ then (f)+(d) implies $\beta \geq 7$. □

Lemma 3.3.19. *The modified system 2 in 3.3.7 has a solution if and only if either* $e \geq 3, 6 \leq \alpha < 12$, *or* $e = 2, \alpha = 6, \beta > 14$, *or* $e = 2, 7 \leq \alpha < 12$, *or* $e = 1, \alpha = 6, \beta > 10$ *or* $e = 1, 7 \leq \alpha < 12, \beta \geq \alpha + 3$, *or* $e = 0, \alpha = 6, \beta > 8$ *or* $e = 0, 7 \leq \alpha < 12, \beta \geq 7$.

Proof. By (c) $6 \leq \alpha < 12$.

Let $e \geq 3$. Then (a) implies $b \geq 3e$. We take $b = 3e$. Note that (d) and (e) are always verified because $\beta > \alpha e \geq 6e$. But also (g) is always verified: trivially if $\alpha \geq 8$; if $6 \leq \alpha \leq 7$ the condition is $\beta \geq 4e - \frac{14}{3}$. But $\beta > \alpha e > 4e - \frac{14}{3}$.

Let $e = 2$. Then (f) implies $b \geq 7$. We take $b = 7$. (d) is verified ($\beta - \alpha e \geq 1$). (g) is verified. Thus in these cases the system has a solution if and only if (e): $\beta > 16 - \frac{\alpha}{3}$. But for $7 \leq \alpha < 12$ (e) is always verified ($\beta > \alpha e$), while for $\alpha = 6$ we have (e) $\Leftrightarrow \beta > 14$.

Let $e = 1$. Then (f) implies $b \geq 5$. We take $b = 5$. (g) is verified. Thus in these cases the system has a solution if and only if (d)+(e): $\beta \geq \alpha + 3, \beta > 14 - \frac{2}{3}\alpha$. Note that if $\alpha = 6$, (e) $\Rightarrow$ (d); if $\alpha \geq 7$ the converse is true.

Let $e = 0$. Then (f) implies $b \geq 4$. We take $b = 4$. (g) is verified. Thus in these cases the system has a solution if and only if (d)+(e): $\beta \geq 7, \beta > 16 - \frac{4}{3}\alpha$. Note that if $\alpha = 6$, (e) $\Rightarrow$ (d); if $\alpha \geq 7$ the converse is true. $\square$

Lemma 3.3.20. *The modified system 3 in 3.3.7 has a solution if and only if either* $e \geq 3, \alpha = 5, \beta > 6e$, *or* $e = 2, \alpha = 5, \beta \geq 15$, *or* $e = 1, \alpha = 5, \beta \geq 11$, *or* $e = 0, \alpha = 5, \beta \geq 10$.

Proof. By (c) $\alpha = 5$.

Let $e \geq 3$. As before $b = 3e$ and (g) is verified. (d) is verified. Thus in these cases the system has a solution if and only if (e): $\beta > 6e$.

Let $e = 2$. As before $b = 7$. (d) is verified . (g) is verified. Thus in these cases the system has a solution if and only if (e): $\beta \geq 15$.

Let $e = 1$. As before $b = 5$. (d) is verified. (g) is verified. Thus in these cases the system has a solution if and only if (e): $\beta \geq 11$.

Let $e = 0$. As before $b = 4$. (d) is verified. (g) is verified. Thus in these cases the system has a solution if and only if (e): $\beta \geq 10$. $\square$

Lemma 3.3.21. *The modified system 4 in 3.3.7 has no solutions.*

Proof. Recall that $\alpha \geq 4$. By (c), $\alpha \leq 4 \Rightarrow \alpha = 4$. (d)+(e) $\Rightarrow$ (3.18) $\Rightarrow$ $\beta < 2e - 4$. Contradiction. $\square$

Putting together 3.3.13 and these last results we can conclude that:

Theorem 3.3.22. *Let $X = \mathbb{P}(\mathcal{E})$ ($\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e), e \geq 0$) be a rational ruled surface embedded in $\mathbb{P}^r$ with a complete linear system related to a very ample divisor $L \sim \alpha C_0 + \beta f$ with $\alpha > 0$ and $\beta > \alpha e$. In all the following cases the embedding of X is not extendable:*

1. $e \geq 2$, $\alpha \geq 7$
2. $e \geq 3$, $\alpha = 6$
3. $e = 2$, $\alpha = 6$, $\beta \geq 15$
4. $e \geq 3$, $\alpha = 5$, $\beta \geq 6e + 1$
5. $e = 2$, $\alpha = 5$, $\beta \geq 15$
6. $e = 0$, $\alpha \geq 7$, $\beta \geq 7$
7. $e = 0$, $\alpha = 6$, $\beta \geq 9$
8. $e = 0$, $\alpha = 5$, $\beta \geq 10$
9. $e = 1$, $\alpha \geq 11$
10. $e = 1$, $7 \leq \alpha \leq 10$, $\beta \geq \alpha + 3$
11. $e = 1$, $\alpha = 6$, $\beta \geq 11$
12. $e = 1$, $\alpha = 5$, $\beta \geq 11$

To conclude let us find some examples of embedded rational ruled surfaces that actually are extendable. For this purpose we will consider only rational scrolls, that is: our rational ruled surface X will be embedded in $\mathbb{P}^r$ with a very ample divisor $D = C_0 + nf$ ($n > e$). In this embedding every fibre f has degree $D.f = 1$, i.e. it is a line in the projective space.

A rational normal scroll S of dimension d in $\mathbb{P}^r$ can be defined with the following standard construction (see [3] or [6], 8.26). Choose $a_1, \ldots, a_d$

non-negative integers with $a_i \geq a_{i+1}$ and $\sum_{i=1}^{d} a_i = r - d + 1$. Choose d complementary linear subspaces $\Lambda_i \subset \mathbb{P}^r$ each of dimension a_i, rational normal curves $C_i \subset \Lambda_i$ and isomorphisms $\phi_i : \mathbb{P}^1 \to C_i$. Then define $S := \bigcup_{p \in \mathbb{P}^1} \overline{\phi_1(p), \ldots, \phi_d(p)}$, where $\overline{\phi_1(p), \ldots, \phi_d(p)}$ denotes the span of these points in $\mathbb{P}^r$. The scroll S, which we also denote $S(a_1, \ldots, a_d)$, is determined up to projective equivalence by the numbers a_i which are called the invariants of the scroll.

First of all we state a definition and a result from [3] that we will use to extend our scrolls.

Definition 3.3.23. Let $S = S(a_1, \ldots, a_d)$ be a d-dimensional rational normal scroll defined by the invariants $a_1, \ldots, a_d$. We define the *index of relative balance of* S, $r(S)$, to be $\min \left\{ n \in \mathbb{N} : (d-n)a_n \leq \sum_{i=n}^{d} a_i \right\}$.

Theorem 3.3.24. *Let $S = S(a_1, \ldots, a_d)$ be a d-dimensional rational normal scroll in $\mathbb{P}^r$ with index of relative balance $r(S) = k$. Then a general hyperplane section of S is a $(d-1)$-dimensional rational normal scroll with invariants b_i ($b_i \geq b_{i+1}$) satisfying:*

1. $\sum_{i=1}^{d-1} b_i = \sum_{i=1}^{d} a_i$
2. $b_k \leq b_{d-1} + 1$
3. $b_i = a_i \quad \forall i, 1 \leq i \leq k-1$

Now we have to show that our rational ruled surface $X = \mathbb{P}(\mathcal{E})$, with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$, embedded in $\mathbb{P}^{2n-e+1}$ with the complete linear system $C_0 + nf$ ($n > e$) (see 2.2.4) is actually a rational normal scroll $S(b_1, b_2)$. The next step is then to find a scroll $S(a_1, a_2, a_3)$ different from a cone that has $S(b_1, b_2)$ as one of its hyperplane sections. We will use 3.3.24.

Lemma 3.3.25. *Let $X = \mathbb{P}(\mathcal{E})$, with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$, embedded in $\mathbb{P}^{2n-e+1}$ with the very ample line bundle $L = \mathcal{O}_X(C_0 + nf)$ ($n > e$). Then X is a rational normal scroll of invariants $(n, n-e)$, i.e. $X \cong S(n, n-e)$, where in this case $\cong$ means projectively equivalent.*

Proof. Let us consider the two sections C_0 and C_1 as in 2.2.1. Considering

$$0 \to \mathcal{O}_X(L - C_1) \to \mathcal{O}_X(L) \to \mathcal{O}_{C_1}(L) \to 0 \tag{3.21}$$

$$0 \to \mathcal{O}_X(L - C_0) \to \mathcal{O}_X(L) \to \mathcal{O}_{C_0}(L) \to 0 \tag{3.22}$$

we see that $|L_{|C_1}|$ has dimension n and degree n. Therefore C_1 is a rational normal curve lying in the linear subspace $\Lambda_1 \cong \mathbb{P}^n \subset \mathbb{P}^{2n-e+1}$. Analogously $|L_{|C_0}|$ has dimension $n-e$ and degree $n-e$, therefore C_0 is a rational normal curve lying in the linear subspace $\Lambda_2 \cong \mathbb{P}^{n-e} \subset \mathbb{P}^{2n-e+1}$. Note that $\Lambda_1 \cap \Lambda_2 = \emptyset$. In fact $X \subset \overline{\Lambda_1, \Lambda_2}$, but $\overline{X}$ is $\mathbb{P}^{2n-e+1}$ because X is nondegenerate. Hence $\overline{\Lambda_1, \Lambda_2} = \mathbb{P}^{2n-e+1}$, thus $2n - e + 1 = \dim(\Lambda_1) + \dim(\Lambda_2) - \dim(\Lambda_1 \cap \Lambda_2)$ $\Rightarrow \dim(\Lambda_1 \cap \Lambda_2) = -1 \Rightarrow \Lambda_1 \cap \Lambda_2 = \emptyset$. Now consider the morphisms $\phi_1 = \pi_{|C_1}$ and $\phi_2 = \pi_{|C_0}$ and the rational normal scroll of dimension 2, $S = S(n, n-e)$, build as explained above, starting from the two rational normal curves C_1, C_0. $X \subset S$. Since both are irreducible we have $X = S$. □

Proposition 3.3.26. *Let $X = \mathbb{P}(\mathcal{E})$, with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$, embedded in $\mathbb{P}^{2n-e+1}$ with the very ample line bundle $L = \mathcal{O}_X(C_0 + nf)$ ($n \geq e+2$). Then X is extendable.*

Proof. By the previous lemma we know that X is projectively equivalent to $S(n, n-e) \subset \mathbb{P}^{2n-e+1}$. Thus we have only to find $S = S(a_1, a_2, a_3) \subset \mathbb{P}^{2n-e+2}$ with all the $a_i \neq 0$ (in this case S is not a cone) such that $S(n, n-e)$ is a hyperplane section of S.

Let $e = 0$. Let $a_1 = n, a_2 = \left[\frac{n+1}{2}\right], a_3 = \left[\frac{n}{2}\right]$. Since $n \geq 2$ we have $a_1 \geq a_2 \geq a_3 \geq 1$. We have also $a_1 + a_2 + a_3 = 2n$. Moreover $r(S) = 1$. Therefore by 3.3.24 the general hyperplane section of S is $S(b_1, b_2)$ such that $b_1 + b_2 = 2n$, $b_1 \geq b_2$, $b_1 \leq b_2 + 1$. The only possibility is $b_1 = n, b_2 = n$.

Let $e \geq 1$. Let $a_1 = n, a_2 = n - e - 1, a_3 = 1$. Since $n \geq e + 2$ we have $a_1 \geq a_2 \geq a_3 \geq 1$. We have also $a_1 + a_2 + a_3 = 2n - e$. Moreover $r(S) = 2$. Therefore by 3.3.24 the general hyperplane section of S is $S(b_1, b_2)$ such that $b_1 + b_2 = 2n - e$, $b_1 \geq b_2$, $b_1 = a_1 = n$. The only possibility is $b_1 = n, b_2 = n - e$. □

Remark 3.3.27. Also if $e = 1$ and $n = 2$ then X is extendable. In fact let $a_1 = a_2 = a_3 = 1$. We have $a_1 + a_2 + a_3 = 3$. In this case $r(S) = 1$, and therefore the only possibility is $b_1 = 2, b_2 = 1$.

Eventually if $e = 0$ and $n = 1$ then X is the quadric in $\mathbb{P}^3$, and therefore it is extendable.

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