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Extremality criteria for pseudoeffective cones of divisors in algebraic varieties

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Introduction

This thesis, as its title suggests, has been realized with the intent to collect and expand the most important convex geometry's criteria used to determine when subcones of a salient convex cone are extremal. However, a considerable part of this paper is concentrated on algebraic geometry. Indeed, these criteria can be applied to the pseudoeffective cone of divisors of a generic algebraic variety, hence an entity which belongs to the classical theory of positivity.
This theory and its keystones are largely developed and discussed in the first chapter of this work. It begins first of all with the basic definitions of line bundles and Cartier divisors, which are the only kind of divisors we will treat throughout. Once seen that a line bundle is characterized by isomorphisms of structure sheaves of a certain covering of the variety, we will underline how these functions are related to the ones which define a Cartier divisor. For completion we will also follow Hartshorne's discussion about linear systems to express how, on a nonsingular projective variety, global sections of a line bundle are connected to a set of effective divisors all linearly equivalent to each other.
Another weaker but fundamental equivalence between divisors is the numerical one. In order to describe it, we will recall the definition of the intersection number of a set of divisors and that of a divisor and a generic subvariety, and explain briefly how it can be calculated. Its interpretation is really intuitive when we intersect a divisor D with a curve $C \subseteq X$ not contained in D, since it reflects the simple idea of counting with multiplicity the points of intersection of D and C. The numerical equivalence between two divisors then holds if their intersection numbers with any irreducible curve or any one-cycle of the variety are equivalent. The group of numerical equivalence classes of divisors is called Néron-Severi group. Its dual space will be the group of numerical equivalence classes of one-cycles, as better discussed in Section 1.6.
As a natural progression, we will focus on the notion of amplitude, which, according to [L], represents the property of a line bundle to possess a tensor power which is very ample, i.e. it is isomorphic to the twisting sheaf of Serre of a projective space. Amplitude has two important characterizations, cohomological and numerical. The first one is proven in the theorem of Cartan-Serre-Grothendieck 1.4.2. The Nakai-Moishezon-Kleiman criterion represents instead the numerical characterization of amplitude, which is the most useful to our purposes. According to this criterion a line bundle on a projective scheme is ample if and only if the intersection number of any positive-dimensional and irreducible subvariety of the scheme with the line bundle raised to the dimension of the subvariety is positive. This means that the amplitude of a line bundle is determined only upon its numerical equivalence class.
The other remarkable property of line bundles and divisors we have to deal with is

numerical effectiveness. It is weaker than amplitude since a line bundle is numerical effective or shortly nef, if its intersection number with any irreducible curve of a complete variety is non negative. Kleiman's criterion extends this definition to any irreducible subvariety. Note that a nef divisor has non negative self-intersection.
Before proceeding in defining the cones of divisors of a complete variety, $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors will be introduced, because of their behavior with respect to amplitude. Roughly, these two types of divisors can be represented with a sum of integral divisors with rational or real coefficients respectively. Their properties are similar to those of integral divisors, for instance the amplitude of a $\mathbb{R}$-divisor is determined only upon its numerical equivalence class. Moreover, both $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors preserve the property of being ample on a suitable neighborhood. This is, in a few words, the reason why the cones we will describe in this work lie in the Néron-Severi group of the $\mathbb{R}$-divisors of a complete variety.
In this real vector space the classes of the ample divisors span the first cone studied here, the so called ample cone. A cone strongly related to this one is made up by the classes of the numerical effective divisors and it is called nef cone. The Kleiman's theorem shows how the ample cone is the interior of the nef cone, while the nef cone is the closure of the ample cone. The nef cone is also the cone dual to the closed cone of curves, which is the closure of the cone spanned by the classes of all the effective one-cycles of the variety, called simply cone of curves. This cone and its closure lie in the real vector space dual to the Néron-Severi group, formed by all the numerical equivalence classes of the one-cycles. Analogously to what we just said about one-cycles, effective cone and pseudoeffective cone of divisors can be defined in the Néron-Severi group. To complete our list of cones we will first need to introduce another class of divisors, the big divisors, i.e. those of maximal Iitaka dimension. These divisors are characterized by the property of being linearly and numerically equivalent to a positive linear combination of an ample divisor and an effective divisor. The classes of all the big divisors span the big cone in the Néron-Severi group, bond to the pseudoeffective cone of divisors in the same fashion as the ample cone is related to the nef one.
We will end our overview of the classical algebraic geometry theory with some examples of the construction of the cones defined so far. What is interesting to notice is that these cones can assume pretty different structures depending on the variety on which they are defined. For example, consider a smooth projective curve with a vector bundle of rank two on it. Then the projective space bundle of the vector bundle is a ruled surface where the effective cone is not closed. If instead we consider an elliptic curve E, then $E \times E$ is an abelian variety, that is a variety with a structure of abelian group. In this setting the nef cone coincides with the closed cone of curves, and it is circular. The last example proposed is about the blowing-up of the projective plane at ten or more very general points. In this remarkable case the cone of curves has rays clustering towards $K_X^{\perp}$.
Before explaining the details of our results about extremality it is appropriate to highlight its importance. The cone theorem and the contraction theorem discussed in [D] imply that we can find a morphism which contracts a curve that generates an extremal ray and whose intersection with the canonical divisor of the variety is negative. If we keep contracting we shall obtain a variety with nef canonical divisor.

This variety is called *minimal model* and it is crucial in the birational classification of varieties. It is conjectured that this operation can be performed only if the Kodaira's dimension of the variety is non negative. Such conjecture is called Mori's program or Minimal Model Program. By now, it has been proven only for general type varieties and this result is due to the Birkar-Cascini-Hacon-McKernan's theorem, which can be found in [BCHM]. Refer to [KM] for a deeper study of the Mori's program, and look at [CC] for another recent work about extremality.
Once our cone theory has been established, we come to the second chapter, which is the heart of this composition. We will approach the cones more generically from the convex geometry point of view. We will only consider closed convex salient cones, namely those that do not contain lines. These cones also have an interesting property which assure that we can always find a linear function that is positive on the nonzero elements of the cone. It is a well-known fact that the pseudoeffective cone of divisors is salient and then it is in the scope of our results.
In order to better illustrate these results, we briefly recall what a face of a cone is. Faces are characterized by the property that if we take two points of the cone with their sum being an element of the face, then they must lie in the face. A face of dimension 1 is called extremal ray. If a face has some additional properties, i.e. there exist linear functions that vanish on the face and keep the cone in the intersection of their non negative half-spaces, then we will call it a perfect face, or simply an edge if it has dimension 1. To better understand the geometrical difference between an extremal ray and an edge, we shall observe that the cone can be rounded near an extremal ray, but not near an edge.

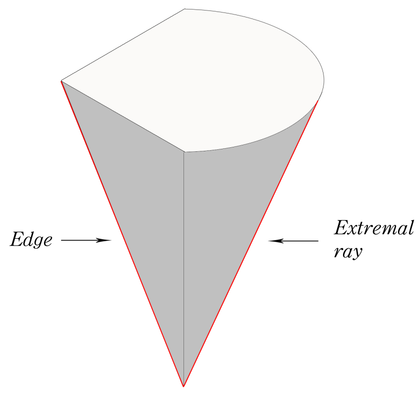

Figure 1: Geometrical difference between an edge and an extremal ray

The first criterion we discuss is useful to determine when a subcone of a given salient cone is a face. This criterion comes directly from a lemma which allows us to decompose a generic element of a cone with respect to a certain subset of vectors. Indeed in this criterion we assume that the cone is generated by a set of vectors and then we consider a subset $\{v_1, \ldots, v_s\}$. We also suppose that there exists s linear functions σ_i which satisfy two conditions. The first one is that $\sigma_i(v_j) = 0$ if $i \neq j$ and $\sigma_i(v_j) < 0$ if $i = j$. The second one is that all these linear functions are non negative

on the subset of vectors which generates the cone and that are not contained in the subcone generated by $v_1, \ldots, v_s$. One application of this criterion allows to prove very quickly that on a smooth projective surface the class of an irreducible curve with negative self-intersection spans an extremal ray of the closed cone of curves (cfr. [D, 6.2, (b)]) for the classical proof).
The second criterion is a new and original result that we introduce for the first time, realized as a generalization of the one provided in [O] (that corresponds to the case $s = 1$). It exposes a way to determine when a subcone generated by a subset of elements of the cone is a perfect face.

Theorem 0.1 (Perfect faces criterion). *Consider a subset $I \subseteq V$ such that $K := K(I)$ is salient. Let $v_1, \ldots, v_s \in I$, and suppose there exist s linear functions $\sigma_i : V \to \mathbb{R}$ such that*

(a) $\sigma_i(v_j) \begin{cases} = 0 & \text{if } i \neq j \\ < 0 & \text{if } i = j \end{cases}$

(b) $\sigma_i(w) \geq 0 \ \forall w \in I \setminus F(v_1, \ldots, v_s)$ *and* $\forall i$.

Then $F(v_1, \ldots, v_s)$ is a perfect face of $K(I)$.

The hypotheses for this criterion are the same as the previous, this means basically that we strengthened it, since in our construction a perfect face is also a face for the cone. The proof is based on two essential facts. The first one is that a cone K is salient if and only if there exists an affine hyperplane H which intersects the cone in a bounded, nonempty set. The second one is that we can find a specific basis for the real vector space in which the cone lies. This basis is made up of the elements which generate the subcone and elements $\{x_1, \ldots, x_c\}$ (where c is the codimension of the subcone) contained in the intersection T of the subspaces in which the σ_i vanish. Moreover, the intersection of the non negative half-spaces of the dual functions of the x_i contains the intersection between K, H and T.
Our conclusion is dedicated to convert this result to a criterion especially addressed to algebraic geometry. We put ourselves in a smooth projective variety. If we can find an arbitrary number of effective divisors with no intersection between each other and if moreover any of these divisors admit a covering of irreducible curves all numerically equivalent to each other and negative on the divisor, then the divisors generate a perfect face of the pseudoeffective cone.

Corollary 0.2 (Perfect faces criterion in algebraic geometry). *Let X be a smooth projective variety. Given irreducible effective divisors $D_1, \ldots, D_s \in \mathrm{Pseff}^1(X)$ with $D_i \cap D_j = \emptyset$ while $i \neq j$ and collections of curves $\{C_{i,t}\}_{t \in T_i}$ so that for each i we have $\cup_t \{C_{i,t}\}_{t \in T_i} = D_i$ and that $C_{i,t} \cdot D_i < 0$ for some $t \in T_i$, then the divisors $D_1, \ldots, D_s$ generate a perfect face of the pseudoeffective cone of X.*

A practical example in which this situation can be realized is the blowing-up of a smooth projective variety at the union of closed smooth subvarieties of arbitrary dimension with no intersection between each other.

Example 0.3 (Perfect faces criterion applied to a practical case). Let X be a smooth projective variety, and let $Z_1, \dots, Z_s$ be closed smooth subvarieties of X of arbitrary dimensions. Suppose also that $Z_i \cap Z_j = \emptyset$ if $i \neq j$. Now consider the blowing-up of X with center $Z_1 \cup \dots \cup Z_s$, and let $E_1, \dots, E_s$ be the exceptional divisors generated by the Z_i, giving the diagram

$$\begin{array}{ccc} \mathrm{Bl}_{\cup_{i=1}^s Z_i}(X) & \xrightarrow{\varphi} & X \\ \cup\mid & & \cup\mid \\ E_i & \longmapsto & Z_i \end{array}$$

In this situation, $E_i \cap E_j = \emptyset$ if $i \neq j$ since $\varphi^{-1}(Z_i) = E_i \ \forall i$. We should also recall that $E_i \cong \mathbb{P}(N^*_{Z_i}) = \mathbf{P}(\mathscr{E}_i) =: P_i$ and that $E_{i|E_i} \cong \mathscr{O}_{P_i}(-1)$ (cfr. [H, II,8.24]). This property lets us found families of irreducible curves $\{C_{i,j}\}_{j \in J}$, with $\cup_{j \in J} C_{i,j} = E_i$ and $C_{i,j} \cdot E_i < 0 \ \forall j$. The first assertion is clear if we regard the E_i as unions of projective spaces and we think about the curves represented by the inverse images of the points of each Z_i. The second one can be seen in the following manner:

$$C_{i,j} \cdot E_i = C_{i,j} \cdot E_{i|E_i} = C_{i,j} \cdot \mathscr{O}_{P_i}(-1) < 0.$$

We now find ourselves in the hypotheses of Corollary 0.2. This means that the E_i generates a perfect face of the pseudoeffective cone of $\mathrm{Bl}_{\cup_{i=1}^s Z_i}(X)$.

Notation and Conventions

We will generally adopt the notation established in [H] and [L]. This is a list of our most significant conventions:

- In this thesis we will work over the complex numbers $\mathbb{C}$, if not specified otherwise while dealing with more general results.
- A *scheme* is a separated algebraic scheme of finite type over an algebraically closed field k. A *variety* is an integral (reduced and irreducible) scheme.
- A *divisor* is a Cartier divisor, as better specified in Section 1.1.

Chapter 1

Ample and Nef Line Bundles

This chapter is essentially devoted to recollect some fundamental facts of classical theory about positivity in algebraic geometry. Starting from the basic definitions of line bundles and divisors, we will move on and expose that, with good hypotheses, they are in one-to-one correspondence. Moreover, we will follow Hartshorne's discussion about linear systems to show how global sections of a line bundle are connected to a set of effective divisors all linearly equivalent to each other. A brief overview of intersection theory is provided in the second section, because it is required to better understand the last sections of the chapter. In particular, the notion of numerical equivalence between divisors is pivotal. In fact, as we will see in the fourth section, numerical equivalence preserves amplitude of divisors. This property holds not only for integral divisors, but for $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors as well. Once the definition of numerical effectiveness for $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors is given, we can finally talk about cones of divisors, which belong to the Néron-Severi group. Ample cone, nef cone, big cone and pseudoeffective cone of divisors will be introduced, with detailed results about their bonds. We will conclude the chapter providing some examples of how these cones are constructed in special and remarkable cases.

1.1 Preliminaries: Divisors and Line Bundles

In the following we will give our standard definitions for line bundles and divisors along with some important properties. At the end of this section we will show that under certain hypotheses there exists a bijection between line bundles and divisors on a scheme. Once this correspondence is shown it is clear that the results exposed in this work regarding divisors hold for line bundles as well.

Definition 1.1.1 (Line Bundles). Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules on X. We say that $\mathscr{F}$ is *free* if it is isomorphic to a direct sum of copies of $\mathscr{O}_X$. If there exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that $\mathscr{F}|_{U_i} \cong \mathscr{O}_{U_i}^r$, $\forall i \in I$ then $\mathscr{F}$ is *locally free* and r will be its *rank* on the open covering $\mathcal{U}$. Note that if X is connected the rank will be the same on every open covering. A *line bundle* is a locally free sheaf of rank 1.

Lemma 1.1.2 (Gluing property). *Let X be a topological space, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering with a sheaf $\mathscr{F}_i$ assigned on U_i $\forall i \in I$ such that there* *exist isomorphisms*

$$\varphi_{ij} : \mathscr{F}_j|_{U_i \cap U_j} \longrightarrow \mathscr{F}_i|_{U_i \cap U_j}$$

with the following properties:

(i) $\varphi_{ii} = 1_{\mathscr{F}_i}$, $\forall i \in I$

(ii) $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$, $\forall i, j, k \in I$ *on* $U_i \cap U_j \cap U_k$.

Then there exists a sheaf $\mathscr{F}$ *on* X *along with isomorphisms* $\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$, $\forall i \in I$ *such that*

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}, \quad \forall i, j \in I.$$

$\mathscr{F}$ *and the isomorphisms* φ_i *are unique up to unique isomorphism.*

Proof. Given any open subset $U \subset X$ we define

$$\Gamma(U, \mathscr{F}) = \Big\{(\sigma_i) \in \prod_i \Gamma(U_i \cap U, \mathscr{F}_i) : \varphi_{ij}(\rho^{U_j \cap U}_{U_i \cap U_j \cap U}(\sigma_j)) = \rho^{U_i \cap U}_{U \cap U_j \cap U}(\sigma_i) \quad \forall i, j \in I\Big\}$$

It is easy to verify that $\mathscr{F}$ is a sheaf (cf. [S, Lemma 15.1]). The isomorphisms φ_i are defined in the natural way

$$\varphi_i(U) : \Gamma(U, \mathscr{F}|_{U_i}) \to \Gamma(U, \mathscr{F}_i) \qquad (\sigma_h) \mapsto \sigma_i.$$

Now suppose $\mathscr{G}$ is another sheaf provided with isomorphisms

$$\psi_i : \mathscr{G}|_{U_i} \to \mathscr{F}_i$$

such that $\varphi_{ij} = \psi_i \circ \psi_j^{-1}$. We can then uniquely determine isomorphisms

$$\varphi_i^{-1} \circ \psi_i : \mathscr{G}|_{U_i} \to \mathscr{F}|_{U_i}$$

that glue to a isomorphism $\mathscr{G} \cong \mathscr{F}$ uniquely determined by the data $\{\mathscr{F}_i, \varphi_{ij}\}$. $\square$

This result gives us a way to assign a line bundle on a scheme X. Given an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ and isomorphisms

$$\varphi_{ij} : \mathscr{O}_{U_i \cap U_j} \longrightarrow \mathscr{O}_{U_i \cap U_j} \quad \forall i, j \in I$$

that satisfy conditions (i) and (ii) of Lemma 1.1.2, we have a uniquely determined line bundle $\mathscr{L}$ on X. These isomorphisms are called *transition functions* of $\mathscr{L}$. The viceversa is still true, so every line bundle $\mathscr{L}$ on X has its own transition functions on a suitable open covering. In general the φ_{ij} are represented with a multiplication by a non-zero regular function over $U_i \cap U_j$, in other words $\varphi_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}_X^*)$, where $\mathscr{O}_X^*$ is the sheaf of germs of regular functions that do not vanish at any point.

Now we will consider the set of isomorphism classes of line bundles, describing it with some fundamental results. We will see that this set has a group structure under the tensor product and how it is useful to express the relation between line bundles and divisors.

Proposition 1.1.3. *If $\mathscr{L}$ and $\mathscr{M}$ are two line bundles on a ringed space X, so is $\mathscr{L} \otimes \mathscr{M}$. If $\mathscr{L}$ is a line bundle on X, then there exists a line bundle $\mathscr{L}^{-1}$ on X such that $\mathscr{L} \otimes \mathscr{L}^{-1} \cong \mathscr{O}_X$.*

Proof. Since $\mathscr{L}$ and $\mathscr{M}$ are both locally free of rank 1 and $\mathscr{O}_X \otimes \mathscr{O}_X \cong \mathscr{O}_X$, the first statement is proved. For the second, given $\mathscr{L}$ any line bundle on X, we take $\mathscr{L}^{-1}$ to be the dual sheaf $\mathscr{L}^\vee = \mathscr{H}om(\mathscr{L}, \mathscr{O}_X)$. Then $\mathscr{L}^\vee \otimes \mathscr{L} \cong \mathscr{O}_X$ (see [H, II, Exercise 5.1]). □

Definition 1.1.4 (Picard Group). For any ringed space X we define the *Picard group* of X, denoted by $\operatorname{Pic}(X)$, as the set of isomorphism classes of line bundles on X. Proposition 1.1.3 shows that $\operatorname{Pic}(X)$ is a group under the tensor product.

In order to describe $\operatorname{Pic}(X)$ properly, we give a result which characterizes the isomorphism class of a line bundle using its transition functions.

Lemma 1.1.5. *Let $\mathscr{L}$ and $\mathscr{M}$ be two line bundles on a ringed space X, defined on the same open covering $\mathcal{U} = \{U_i\}_{i \in I}$ by transition functions $\{\varphi_{ij}\}$ and $\{\psi_{ij}\}$ respectively. Then $\mathscr{L} \cong \mathscr{M} \Leftrightarrow$ there exist $a_i \in \Gamma(U_i, \mathscr{O}_X^*)$ such that*

$$\psi_{ij} = a_i^{-1} \varphi_{ij} a_j, \qquad \forall i, j \in I. \tag{1.1}$$

Proof. $(\Rightarrow)$: Suppose that ρ is the isomorphism between $\mathscr{L}$ and $\mathscr{M}$. We define

$$a_i = \varphi_i \circ \rho_{|U_i} \circ \psi_i^{-1}, \qquad \forall i \in I.$$

The a_i defined above satisfy condition (1.1).
$(\Leftarrow)$: Conversely, suppose that the a_i exist. Then for every $i \in I$ we define an isomorphism

$$\rho_i : \mathscr{M}|_{U_i} \to \mathscr{L}|_{U_i}$$

putting $\rho_i = \varphi_i^{-1} \circ (a_i \cdot \psi_i)$. The ρ_i glue to an isomorphism $\rho : \mathscr{M} \to \mathscr{L}$. □

Theorem 1.1.6. *Let X be a ringed space. There exists an isomorphism*

$$\operatorname{Pic}(X) \cong H^1(X, \mathscr{O}_X^*). \tag{1.2}$$

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. The transition functions $\{\varphi_{ij}\}$ of any line bundle on X satisfy (i) and (ii) of Lemma 1.1.2, this means that $\{\varphi_{ij}\}$ is a 1-cocycle of Čech for the sheaf $\mathscr{O}_X^*$ towards the covering $\mathcal{U}$, so $\{\varphi_{ij}\} \in \check{Z}^1(\mathcal{U}, \mathscr{O}_X^*) := \operatorname{Ker}(\delta^1)$. The result of Lemma 1.1.5 shows that two different 1-cocycles $\{\varphi_{ij}\}, \{\psi_{ij}\}$ are isomorphic if and only if

$$\{\psi_{ij}\} = \{\varphi_{ij}\} \delta^0(\{a_i\}).$$

We then have a bijection between isomorphism classes of line bundles defined with transition functions on $\mathcal{U}$ and $\check{H}^1(\mathcal{U}, \mathscr{O}_X^*)$. Since this isomorphism is preserved if we refine our open covering and since we can apply this proof to any open covering we conclude that

$$\operatorname{Pic}(X) \cong \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathscr{O}_X^*).$$

The thesis follows immediately by [S, Lemma 11.7]. □

We will now introduce divisor theory. There are two main notions of divisors, Weil divisors and Cartier divisors. Weil divisors are the easiest to understand geometrically, but they are defined uniquely on certain schemes. In the rest of our work we will then treat only Cartier divisors, since they give a more generalized notion of divisors on an arbitrary scheme, and from now on we will refer to them merely with the word *divisors*.

We begin our construction by considering a reduced and irreducible scheme X and its field of rational functions K, with $K^* = K\setminus\{0\} \subset K$ being the subgroup of invertible elements. We define a constant sheaf $\mathcal{K}_X$ on X by putting, for any open subset $\emptyset \neq U \subset X$

$$\Gamma(U, \mathcal{K}_X) = K.$$

We denote with $\mathcal{K}_X^* \subset \mathcal{K}_X$ the subsheaf defined by

$$\Gamma(U, \mathcal{K}_X^*) = K^*.$$

Observe that $\mathscr{O}_X^* \subset \mathscr{O}_X$ is also a subsheaf of $\mathcal{K}_X^*$.

Definition 1.1.7 (Cartier divisors). A *Cartier divisor* on an integral scheme X is a global section of the sheaf $\mathcal{K}_X^*/\mathscr{O}_X^*$. In other words, every Cartier divisor can be defined giving a suitable open covering $\mathcal{U} = \{U_i\}_{i\in I}$ and, for every $i \in I$, an element $f_i \in \Gamma(U_i, \mathcal{K}_X^*) = K^*$, such that for every $i, j \in I$ one has $f_i f_j^{-1} \in \Gamma(U_i \cap U_j, \mathscr{O}_X^*)$. A Cartier divisor is *principal* if it belongs to the image of the natural map

$$\Gamma(X, \mathcal{K}_X^*) \longrightarrow \Gamma(X, \mathcal{K}_X^*/\mathscr{O}_X^*).$$

Cartier divisors form an abelian group $\mathrm{Div}(X)$, we will denote the operation defined on it additively. The principal divisors form a subgroup of $\mathrm{Div}(X)$, denoted with $\mathrm{Pr}(X)$. Two Cartier divisors D_1,D_2 are *linearly equivalent* if their difference D_1-D_2 is principal. In that case we will write $D_1 \equiv_{\mathrm{lin}} D_2$. A Cartier divisor D is *effective* if there exists an open covering $\mathcal{V} = \{V_i\}_{i\in I}$ in which D can be defined by $\{f_i \in \Gamma(V_i, \mathscr{O}_X)\}$, we indicate this property writing $D \succcurlyeq 0$. The notation $D \succcurlyeq D'$ indicates that $D - D'$ is effective.

We can now give a fundamental result about the correspondence between line bundles and divisors.

Proposition 1.1.8. *Let X be a reduced and irreducible scheme. Then there exists an isomorphism*

$$\frac{\mathrm{Div}(X)}{\mathrm{Pr}(X)} \cong \mathrm{Pic}(X). \tag{1.3}$$

Proof. Consider the exact sequence of sheaves on X:

$$0 \longrightarrow \mathscr{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^*/\mathscr{O}_X^* \longrightarrow 0$$

Since $\mathcal{K}_X^*$ is constant then it is also flasque and so the following exact sequence of abelian groups is obtained:

$$0 \longrightarrow \Gamma(X, \mathscr{O}_X^*) \longrightarrow K^* \xrightarrow{\psi} \mathrm{Div}(X) \xrightarrow{\delta} H^1(X, \mathscr{O}_X^*) \longrightarrow 0.$$

Since $\mathrm{Pr}(X) = \mathrm{Im}(\psi) = \mathrm{Ker}(\delta)$ and since δ is surjective we have $\frac{\mathrm{Div}(X)}{\mathrm{Pr}(X)} \cong H^1(X, \mathscr{O}_X^*)$. We conclude the proof using Theorem 1.1.6 . ☐

We can explicitly construct the isomorphism (1.3) in the following manner. Consider $D \in \mathrm{Div}(X)$ defined by a system $\{f_i \in \Gamma(U_i, \mathcal{K}_X^*)\}$. We can associate a line bundle $\mathscr{L}(D)$ to D by defining its transition function

$$\varphi_{ij} = f_i f_j^{-1} \in \Gamma(U_i \cap U_j, \mathscr{O}_X^*)$$

that clearly satisfy conditions (i) and (ii) of Lemma 1.1.2. Moreover, given a divisor $D \in \mathrm{Pr}(X)$ represented by $f \in K^*$ then $\mathscr{L}(D)$ is defined by transition functions $f_i f_j^{-1} = ff^{-1} = 1$, so $\mathscr{L}(D) = \mathscr{O}_X$ and then $\mathrm{Pr}(X) \subseteq \mathrm{Ker}(D \mapsto \mathscr{L}(D))$. For the converse, suppose $D \in \mathrm{Ker}(D \mapsto \mathscr{L}(D))$ defined by the system $\{f_i, U_i\}_{i \in I}$.This means that $\mathscr{L}(D) \cong \mathscr{O}_X$ and so, thanks to Lemma 1.1.5, there exist $a_i \in \Gamma(U_i, \mathscr{O}_X^*)$ such that $f_i f_j^{-1} = a_i a_j^{-1}$ over $U_i \cap U_j$ for every i, j. We now have $\frac{f_i}{a_i} = \frac{f_j}{a_j}$ over $U_i \cap U_j$, these functions glue to an $f \in K^*$ that represents D, so $D \in \mathrm{Pr}(X)$.

1.2 Intersection theory

The purpose of this section is to define the intersection number of n divisors or that of a divisor and a subvariety of X. In particular, as we will see, this notion is really simple when we intersect a divisor D with a curve $C \subseteq X$ not contained in D, since it reflects the simple idea of counting with multiplicity the points of intersection of D and C. The basic facts of intersection theory introduced here will lead to the definition of numerical equivalence between divisors. Before we explain how the intersection number of a set of divisors can be calculated (following [D] construction), we shall begin with a couple of basic definitions.

Definition 1.2.1 (Euler characteristic of a sheaf). Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum_i (-1)^i \dim_k H^i(X, \mathscr{F}).$$

Definition 1.2.2 (Hilbert polynomial). Let S be the polynomial ring $k[x_0, \dots, x_n]$, and let M be a finitely generated graded S-module. The *Hilbert polynomial* of M is the polynomial $P_M(z) \in \mathbb{Q}[z]$ such that $\varphi_M(l) = P_M(l)$ for all $l \gg 0$, where $\varphi_M(l)$ is the Hilbert function of M, given by $\varphi_M(l) = \dim_k M_l$. This polynomial exists and it is unique (see [H, I, Theorem 7.5]). Now let $X \subseteq \mathbb{P}^n$ be a projective scheme of dimension r. The *Hilbert polynomial* of X is the Hilbert polynomial P_X of its homogeneous coordinate ring $S(X)$. We define the *degree* of X to be $r!$ times the leading coefficient of P_X.

Remark 1.2.3. Thanks to the definition above and to the characterization of amplitude exposed in Theorem 1.4.2, if X is a subscheme of $\mathbb{P}_k^N$ of dimension r, then the function $l \mapsto \chi(X, \mathscr{O}_X(l))$ is polynomial of degree r for $l \gg 0$.

Theorem 1.2.4. *Let $D_1, \dots, D_r$ be divisors on a proper scheme X, and let $\mathscr{F}$ be a coherent sheaf on X. The function*

$$(m_1, \dots, m_r) \mapsto \chi(X, \mathscr{F}(m_1 D_1 + \dots + m_r D_r))$$

takes the same values on $\mathbb{Z}^r$ as a polynomial with rational coefficients of degree at most the dimension of the support of $\mathscr{F}$.

Proof. See [D, Theorem 1.5]. □

Definition 1.2.5 (Intersection number). Let $D_1, \dots, D_r$ be divisors on a proper scheme X, with $r \geq \dim(X)$. The intersection number

$$D_1 \cdots D_r$$

is the coefficient of $m_1 \cdots m_r$ in the polynomial $\chi(X, m_1 D_1 + \cdots + m_r D_r)$.

[D, Proposition 1.8] shows that this number is an integer, that is multilinear and symmetric as a function of its arguments, while Theorem 1.2.4 shows that it vanishes for $r > \dim(X)$. [D, Proposition 1.8] also states that if D_r is effective with associated subscheme Y, one has

$$D_1 \cdots D_r = D_1 \cdots D_{r-1} \cdot Y.$$

Note that $D_1 \cdots D_r$ depends only on the linear equivalence classes of the D_i. If $D_1, \dots, D_r$ are effective and meet properly in a finite number of points, the intersection number does have a geometric interpretation as the number of points in $\{D_1 \cap \dots \cap D_r\}$, counted with multiplicity. If Y is a subscheme of X of dimension at most s, we set

$$D_1 \cdots D_s \cdot Y = D_{1|Y} \cdots D_{s|_Y}.$$

Once we have Kleiman's characterization of ample divisors in terms of their intersection numbers with one-cycles (Theorem 1.4.5 and Corollary 1.4.6) we will in most cases intersect a divisor D with a curve $C \subseteq X$. In this case things are very simple: when D is an hypersurface that does not contain C, the intersection number counts with multiplicity the number of points of intersection of D and C.

Proposition 1.2.6 (Projection Formula). *Let $\pi : Y \to X$ be a surjective morphism between proper varieties. Let $D_1, \dots, D_r$ be divisors on X with $r \geq \dim(Y)$. We have*

$$\pi^* D_1 \cdots \pi^* D_r = \deg(\pi)(D_1 \cdots D_r).$$

Proof. See [D, Proposition 1.10] for the complete proof. □

Now we assume that X is a complete algebraic scheme over $\mathbb{C}$. We want to define the numerical equivalence between two divisors, which is the weakest natural equivalence relation on $\mathrm{Div}(X)$.

Definition 1.2.7 (Numerical equivalence). Two divisors $D_1, D_2 \in \mathrm{Div}(X)$ are *numerically equivalent*, written $D_1 \equiv_{\mathrm{num}} D_2$, if $D_1 \cdot C = D_2 \cdot C$ for every irreducible curve $C \subseteq X$, or equivalently if $D_1 \cdot \gamma = D_2 \cdot \gamma$ for all one-cycles γ on X. Numerical equivalence of line bundles is defined in the analogous manner. A divisor is *numerically trivial* if it is numerically equivalent to zero, and $\mathrm{Num}(X) \subseteq \mathrm{Div}(X)$ is the subgroup consisting of all numerically trivial divisors.

Definition 1.2.8 (Néron-Severi Group). The *Néron-Severi group* of X is the group

$$N^1(X) = \mathrm{Div}(X)/\mathrm{Num}(X)$$

of numerical equivalence classes of divisors on X.

Proposition 1.2.9 (Theorem of the base). *The Néron-Severi group $N^1(X)$ is a free abelian group of finite rank.*

Proof. See [L, Proposition 1.1.16] for the integral proof. □

Definition 1.2.10 (Picard number). The rank of $N^1(X)$ is called the *Picard number* of X, written $\rho(X)$.

1.3 Linear Systems and Projective Morphisms

We now consider a nonsingular projective variety X over an algebraically closed field k. In this setting the isomorphism (1.3) holds and for every line bundle $\mathscr{L}$ on X, the global sections $\Gamma(X, \mathscr{L})$ form a finite-dimensional k-vector space (see [H, II, Theorem 5.19]). What we want to show in the following is that giving a certain set of global sections of $\mathscr{L}$ is the same as giving a set of effective divisors all linearly equivalent to each other, which is the historical notion of linear system.

Definition 1.3.1 (Divisor of zeros of a global section of a line bundle). Let $\mathscr{L}$ be a line bundle on X, and let $s \in \Gamma(X, \mathscr{L})$ be a nonzero section of $\mathscr{L}$. We define an effective divisor $D = (s)_0$, the *divisor of zeros* of s, as follows. Given an open covering of X where $\mathscr{L}$ is trivial and φ_i isomorphisms between $\mathscr{L}|_{U_i}$ and $\mathscr{O}_{U_i}$ we clearly have $\varphi_i(s) \in \Gamma(U_i, \mathscr{O}_{U_i})$. The collection $\{\varphi_i(s), U_i\}$ determines an effective divisor D on X. Indeed, φ_i is determined up to multiplication by an element of $\Gamma(U_i, \mathscr{O}^*_{U_i})$ so we get a well-defined divisor.

Proposition 1.3.2. *Let X be a variety as defined above. Let D_0 be a divisor on X, and let $\mathscr{L} \cong \mathscr{L}(D_0)$ be its corresponding line bundle. Then:*

(a) for each nonzero $s \in \Gamma(X, \mathscr{L})$, the divisor of zeros $(s)_0$ is an effective divisor linearly equivalent to D_0;

(b) every effective divisor linearly equivalent to D_0 is $(s)_0$ for some $s \in \Gamma(X, \mathscr{L})$;

(c) two sections $s, s' \in \Gamma(X, \mathscr{L})$ have the same divisor of zeros if and only if there is a $\lambda \in k^*$ such that $s' = \lambda s$.

Proof. (a) Regarding $\mathscr{L}(D_0)$ as a subsheaf of $\mathcal{K}_X$ we see that s corresponds to a rational function $f \in K$. If D_0 is locally defined by $\{f_i, U_i\}$, with $f_i \in K^*$, then $\mathscr{L}(D_0)$ is locally generated by f_i^{-1}, so the isomorphism $\varphi_i : \mathscr{L}(D_0)|_{U_i} \to \mathscr{O}_{U_i}$ is obtained by multiplying by f_i. So $D = (s)_0$ is locally defined by $f_i f$. Thus $D = D_0 + (f)$, which means that $D \equiv_{\mathrm{lin}} D_0$.
(b) If $D \succcurlyeq 0$ and $D = D_0 + (f)$ then $(f) \succcurlyeq -D_0$. Thus f gives a global section of $\mathscr{L}(D_0)$ whose divisor of zeros is D.

(c) Using the same construction seen in (a), if $(s)_0 = (s')_0$ then s and s' correspond to rational functions $f, f' \in K^*$ such that $(f/f') = 0$. Therefore $f/f' \in \Gamma(X, \mathscr{O}_X^*)$. Since X is a scheme over k algebraically closed then $\Gamma(X, \mathscr{O}_X) = k$ (see [H, I, Theorem 3.4]) and then $f/f' \in k^*$. □

Definition 1.3.3 (Complete linear system). A *complete linear system* on a nonsingular projective variety is defined as the set (maybe empty) of all effective divisors linearly equivalent to some given divisor D_0. It is denoted by $|D_0|$. Proposition 1.3.2 shows that there is a one-to-one correspondence between $(\Gamma(X, \mathscr{L}) - \{0\})/k^*$ and $|D_0|$. This gives to the complete linear system a structure of the set of closed points of a projective space over k.

Definition 1.3.4 (Linear system). A *linear system* δ on X is a subset of a complete linear system $|D_0|$, which is a linear subspace, since $|D_0|$ has a projective space structure. Thus δ corresponds to a sub-vector space $V \subseteq \Gamma(X, \mathscr{L})$, where $V = \{s \in \Gamma(X, \mathscr{L}) | (s)_0 \in \delta\} \cup \{0\}$. The *dimension* of the linear system δ is its dimension as a linear projective variety. Hence $\dim \delta = \dim V - 1$.

Remark 1.3.5. The dimensions of linear systems are finite because, as we outlined in the introduction, $\Gamma(X, \mathscr{L})$ is a finite-dimensional vector space.

Definition 1.3.6 (Base point of a linear system). A point $p \in X$ is a *base point* of a linear system δ if $p \in \mathrm{Supp}(D)$ for all $D \in \delta$. $\mathrm{Supp}(D)$ means the union of the prime divisors of D (see [H, II, Section 6] to better understand the notion of prime divisor of a Weil divisor, that in this situation corresponds to a Cartier divisor).

We now recall the definition of globally and locally generated sheaf in order to characterize base points and to start talking about projective morphisms.

Definition 1.3.7 (Globally and locally generated sheaf). Let X be a scheme, and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is *globally generated* (or *generated by its global sections*) if there is a family of global sections $\{s_i\}_{i \in I} \in \Gamma(X, \mathscr{F})$ such that for each $p \in X$ the images of s_i in the stalk $\mathscr{F}_p$ generate it as an $\mathscr{O}_p$-module.
We say that $\mathscr{F}$ is *locally generated* at a point $p \in X$ if the stalk $\mathscr{F}_p$ is generated as an $\mathscr{O}_p$-module by the images of a family of global sections of $\mathscr{F}$.

Lemma 1.3.8. *Let δ be a linear system on X corresponding to the sub-space $V \subseteq \Gamma(X, \mathscr{L})$. Then a point $p \in X$ is a base point of δ if and only if $s_p \in \mathfrak{m}_p \mathscr{L}_p$ for all $s \in V$. In particular δ is base-point-free if and only if $\mathscr{L}$ is globally generated by the sections in V.*

Proof. This follows immediately from the fact that for any $s \in \Gamma(X, \mathscr{L})$, the support of the divisor of zeros $(s)_0$ is the complement of X_s (in [H]'s notation this is the open set of points $p \in X$ where $s_p \notin \mathfrak{m}_p \mathscr{L}_p$). □

The section is concluded with a result on how a morphism of a scheme X to a projective space can be determined giving a line bundle $\mathscr{L}$ on X and a set of its global sections.

Theorem 1.3.9. *Let A be a ring, and let X be a scheme over A.*

(a) *If* $\varphi : X \to \mathbb{P}^n_A = \text{Proj}A[x_0, \dots, x_n]$ *is an* A*-morphism then* $\varphi^*(\mathscr{O}(1))$ *is a line bundle on* X*, which is globally generated by* $s_i = \varphi^*(x_i)$*,* $i = 0, 1, \dots, n$*.*

(b) *Conversely, if* $\mathscr{L}$ *is a line bundle on* X*, and if* $s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$ *are global sections which generate* $\mathscr{L}$*, then there exists a unique* A*-morphism* $\varphi : X \to \mathbb{P}^n_A$ *such that* $\mathscr{L} \cong \varphi^*(\mathscr{O}(1))$ *and* $s_i = \varphi^*(x_i)$ *under this isomorphism.*

Proof. (a): On $\mathbb{P}^n_A$ we have the line bundle $\mathscr{O}(1)$, and the homogeneous coordinates $x_0, \dots, x_n$ give rise to global sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n_A, \mathscr{O}(1))$. The sheaf $\mathscr{O}(1)$ is globally generated by these global sections, this means that the images of these sections generate the stalk $\mathscr{O}(1)_p$ as a module over the local ring $\mathscr{O}_p$, for each point $p \in \mathbb{P}^n_A$. Then $\mathscr{L} = \varphi^*(\mathscr{O}(1))$ is a line bundle on X, and the global sections $s_0, \dots, s_n$ where $s_i = \varphi^*(x_i), s_i \in \Gamma(X, \mathscr{L})$, generate the sheaf $\mathscr{L}$.
(b): Suppose given $\mathscr{L}$ and the global sections $s_0, \dots, s_n$ which generate it. For each i, let $X_i = \{p \in X \mid (s_i)_p \notin \mathfrak{m}_p \mathscr{L}_p\}$. These are open subsets of X and since the s_i generate $\mathscr{L}$, the X_i must cover X. We define a morphism from X_i to the standard open set $U_i = \{x_i \neq 0\}$ of $\mathbb{P}^n_A$ as follows. Recall that $U_i \cong \text{Spec}A[y_0, \dots, y_n]$ where $y_j = x_j/x_i$, with $y_i = 1$ omitted. We can then define a ring homomorphism $A[y_0, \dots, y_n] \to \Gamma(X_i, \mathscr{O}_{X_i})$ by sending $y_j \mapsto s_j/s_i$ and making it A-linear. s_j/s_i is a well-defined element of $\Gamma(X_i, \mathscr{O}_{X_i})$ since for each $p \in X_i, (s_i)_p \notin \mathfrak{m}_p \mathscr{L}_p$ and $\mathscr{L}$ is locally free of rank 1. Now by ([H, II, Exercise 2.4]) this ring homomorphism gives rise to a morphism of schemes (over A) $X_i \to U_i$. These morphisms glue, so we obtain a morphism $\varphi : X \to \mathbb{P}^n_A$. It is clear from the construction that φ is an A-morphism, that $\mathscr{L} \cong \varphi^*(\mathscr{O}(1))$, that the sections s_i correspond to $\varphi^*(x_i)$ and that φ is unique. □

1.4 Amplitude of Line Bundles

In this section we will introduce one of the most important concepts of this work, that is the notion of amplitude of line bundles. Once given the standard definitions (following [L] construction, which slightly differs from [H]'s one), we will characterize the amplitude from different point of views. Cartan-Serre-Grothendieck's theorem provides a cohomological characterization, while the Nakai-Moishezon's criterion establishes a relation between amplitude and the intersection number of the line bundle. This criterion directly implies that the amplitude of a line bundle depends only on its numerical class, which is the concluding and most remarkable result of this section.

Definition 1.4.1 (Ample and very ample line bundles and divisors on a complete scheme). Let X be a complete scheme, and $\mathscr{L}$ a line bundle on X.

(i) $\mathscr{L}$ is *very ample* if there exists a closed embedding $X \subseteq \mathbb{P}$ of X into some projective space $\mathbb{P} = \mathbb{P}^N$ such that

$$\mathscr{L} = \mathscr{O}_X(1) := \mathscr{O}_{\mathbb{P}^N}(1)|_X.$$

(ii) $\mathscr{L}$ is *ample* if $\mathscr{L}^{\otimes m}$ is very ample for some $m > 0$.

A divisor D on X is *ample* or *very ample* if the corresponding line bundle $\mathscr{L}(D)$ is so.

Theorem 1.4.2 (Cartan-Serre-Grothendieck theorem). *Let $\mathscr{L}$ be a line bundle on a complete scheme X. The following are equivalent:*

(i) *$\mathscr{L}$ is ample.*

(ii) *Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_1 = m_1(\mathscr{F})$ having the property that*

$$H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes m}) = 0 \quad \textit{for all } \; i > 0, \quad m \geq m_1(\mathscr{F}).$$

(iii) *Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_2 = m_2(\mathscr{F})$ such that $\mathscr{F} \otimes \mathscr{L}^{\otimes m}$ is generated by its global sections for all $m \geq m_2(\mathscr{F})$.*

(iv) *There is a positive integer $m_3 > 0$ such that $\mathscr{L}^{\otimes m}$ is very ample for every $m \geq m_3$.*

Proof. (i) $\Rightarrow$ (ii). We assume to begin with that $\mathscr{L}$ is very ample, defining an embedding of X into some projective space $\mathbb{P}$. In this case, extending $\mathscr{F}$ by zero to a coherent sheaf on $\mathbb{P}$, we are reduced to the vanishing of $H^i(\mathbb{P}, \mathscr{F}(m))$, for $m \gg 0$, which is the content of [H, III, Theorem 5.2]. In general, when $\mathscr{L}$ is merely ample, fix m_0 such that $\mathscr{L}^{\otimes m_0}$ is very ample. Then apply the case already treated to each of the sheaves $\mathscr{F}, \mathscr{F} \otimes \mathscr{L}, \ldots, \mathscr{F} \otimes \mathscr{L}^{\otimes m_0 - 1}$.
(ii) $\Rightarrow$ (iii). Fix a point $x \in X$, and denote by $\mathfrak{m}_x \subset \mathscr{O}_X$ the maximal ideal sheaf of x. By (ii) there is an integer $m_2(\mathscr{F}, x)$ such that

$$H^1(X, \mathfrak{m}_x \cdot \mathscr{F} \otimes \mathscr{L}^{\otimes m}) = 0 \quad \text{for } m \geq m_2(\mathscr{F}, x).$$

It then follows from the exact sequence

$$0 \longrightarrow \mathfrak{m}_x \cdot \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathfrak{m}_x \cdot \mathscr{F} \longrightarrow 0$$

upon twisting by $\mathscr{L}^{\otimes m}$ and taking cohomology that $\mathscr{F} \otimes \mathscr{L}^{\otimes m}$ is globally generated in a neighborhood of x for every $m \geq m_2(\mathscr{F}, x)$. By quasi-compactness we can then choose a single natural number $m_2(\mathscr{F})$ that works for all $x \in X$.
(iii) $\Rightarrow$ (iv). It follows first of all from (iii) that there exists a positive integer p_1 such that $\mathscr{L}^{\otimes m}$ is globally generated for all $m \geq p_1$. Denote by

$$\Phi_m : X \longrightarrow \mathbb{P}H^0(X, L^{\otimes m})$$

the corresponding map to projective space. We need to show that we can arrange for Φ_m to be an embedding by taking $m \gg 0$, for which it is sufficient to prove that Φ_m is one-to-one and unramified ([H, II, Proposition 7.3]). To this end, consider the set

$$U_m = \{y \in X \mid L^{\otimes m} \otimes \mathfrak{m}_y \text{ is globally generated}\}.$$

This is an open set [L, Example 1.2.9], and $U_m \subset U_{m+p}$ for $p \geq p_1$ thanks to the fact that $\mathscr{L}^{\otimes p}$ is generated by its global sections. Given any point $x \in X$ we can find by

(iii) an integer $m_2(x)$ such that $x \in U_m$ for all $m \geq m_2(x)$, and therefore $X = \cup U_m$. By quasi-compactness there is a single integer $m_3 \geq p_1$ such that $\mathscr{L}^{\otimes m} \otimes \mathfrak{m}_x$ is generated by its global sections for every $x \in X$ whenever $m \geq m_3$. But the global generation of $\mathscr{L}^{\otimes m} \otimes \mathfrak{m}_x$ implies that $\Phi_m(x) \neq \Phi_m(x')$ for all $x \neq x'$, and that Φ_m is unramified at x. Thus Φ_m is an embedding for all $m \geq m_3$ as required.
(iv) ⇒ (i). It follows immediately from the definition. □

Proposition 1.4.3 (Finite pullbacks). *Let $f : Y \to X$ be a finite mapping of complete schemes, and $\mathscr{L}$ an ample line bundle on X. Then $f^*\mathscr{L}$ is an ample line bundle on Y. In particular, if $Y \subseteq X$ is a subscheme of X, then the restriction $\mathscr{L}|_Y$ of $\mathscr{L}$ to Y is ample.*

Proof. Let $\mathscr{F}$ be a coherent sheaf on Y. Then $f_*(\mathscr{F} \otimes f^*\mathscr{L}^{\otimes m}) = f_*\mathscr{F} \otimes \mathscr{L}^{\otimes m}$ by the projection formula, and $R^j f_*(\mathscr{F} \otimes f^*\mathscr{L}^{\otimes m}) = 0$ for $j > 0$ thanks to the finiteness of f. Therefore

$$H^i(Y, \mathscr{F} \otimes f^*\mathscr{L}^{\otimes m}) = H^i(Y, f_*\mathscr{F} \otimes \mathscr{L}^{\otimes m}) \quad \forall i,$$

and the statement then follows from the characterization (ii) of Theorem 1.4.2. □

Corollary 1.4.4 (Globally generated line bundles). *Suppose that $\mathscr{L}$ is globally generated, and let*

$$\Phi = \Phi_{|\mathscr{L}|} : X \to \mathbb{P} = \mathbb{P}H^0(X, \mathscr{L})$$

be the resulting map to projective space defined by the complete linear system $|\mathscr{L}|$. Then $\mathscr{L}$ is ample if and only if Φ is a finite mapping, or equivalently if and only if

$$\mathscr{L} \cdot C > 0$$

for every irreducible curve $C \subseteq X$.

Proof. The preceding proposition shows that if Φ is finite, then $\mathscr{L}$ is ample. In this case evidently $\mathscr{L} \cdot C > 0$ for every irreducible curve $C \subseteq X$. Conversely, if Φ is not finite then there is a subvariety $Z \subseteq X$ of positive dimension that is contracted by Φ to a point. Since $\mathscr{L} = \Phi^*(\mathscr{O}_{\mathbb{P}}(1))$, we see that $\mathscr{L}$ restricts to a trivial line bundle on Z. In particular $\mathscr{L}|_Z$ is not ample, and so thanks again to the previous proposition, neither is $\mathscr{L}$. Moreover, if $C \subseteq Z$ is any irreducible curve, then $\mathscr{L} \cdot C = 0$. □

Theorem 1.4.5 (Nakai-Moishezon-Kleiman criterion). *Let $\mathscr{L}$ be a line bundle on a projective scheme X. Then $\mathscr{L}$ is ample if and only if*

$$\mathscr{L}^{\dim(V)} \cdot V > 0 \tag{1.4}$$

for every positive-dimensional irreducible subvariety $V \subseteq X$ (including the irreducible components of X).

Proof. See [L, Theorem 1.2.23]. □

Corollary 1.4.6 (Numerical nature of amplitude). *If $D_1, D_2 \in \mathrm{Div}(X)$ are numerically equivalent divisors on a projective variety or scheme X then D_1 is ample if and only if D_2 is so.*

Proof. The thesis comes directly from the theorem above, along with the result exposed in [L, Lemma 1.1.18]. □

1.5 $\mathbb{Q}$-Divisors and $\mathbb{R}$-Divisors

$\mathbb{Q}$-divisors and especially $\mathbb{R}$-divisors are essential to develop the theory exposed throughout our work. In this section we will explain how these divisors are defined and constructed, proceeding then to analyze some interesting properties. In particular, it is useful to point out how both these types of divisors preserve the property of being ample on a suitable neighborhood. Moreover, the amplitude of a $\mathbb{R}$-divisor is determined only upon its numerical equivalence class.

Definition 1.5.1 ($\mathbb{Q}$-divisors). Let X be an algebraic variety or scheme.
A $\mathbb{Q}$-*divisor* on X is an element of the $\mathbb{Q}$-vector space

$$\mathrm{Div}_{\mathbb{Q}}(X) \quad := \quad \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We represent a $\mathbb{Q}$-divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ as a finite sum, called *representation*,

$$D = \sum c_i \cdot A_i,$$

where $c_i \in \mathbb{Q}$ and $A_i \in \mathrm{Div}(X)$. By clearing denominators we can also write $D = cA$ for a single rational number c and integral divisor A, and if $c \neq 0$ then $cA = 0$ if and only if A is a torsion element of $\mathrm{Div}(X)$. A $\mathbb{Q}$-divisor D is *integral* if it lies in the image of the natural map $\mathrm{Div}(X) \to \mathrm{Div}_{\mathbb{Q}}(X)$.The $\mathbb{Q}$-divisor D is *effective* if it is of the form $D = \sum c_i A_i$ with $c_i \geq 0$ and A_i effective.

Definition 1.5.2 (Supports of $\mathbb{Q}$-divisors). Let $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ be a $\mathbb{Q}$-divisor. A codimension one subset $E \subseteq X$ *supports* D, or *is a support* of D, if D admits a representation in which the union of the supports of the A_i is contained in E.

Definition 1.5.3 (Equivalences and operations on $\mathbb{Q}$-divisors). We assume that X is complete.

(i) Given a subvariety or subscheme $V \subseteq X$ of pure dimension k, a $\mathbb{Q}$-valued intersection product

$$\begin{aligned} \mathrm{Div}_{\mathbb{Q}}(X) \times \cdots \times \mathrm{Div}_{\mathbb{Q}}(X) &\to \mathbb{Q} \\ (D_1, \dots, D_k) &\mapsto D_1 \cdot \cdots \cdot D_k \cdot [V] \end{aligned}$$

is defined via extension of scalars from the analogous product on $\mathrm{Div}(X)$.

(ii) Two $\mathbb{Q}$-divisors $D_1, D_2 \in \mathrm{Div}_{\mathbb{Q}}(X)$ are *numerically equivalent*, written $D_1 \equiv_{\mathrm{num}} D_2$ (or $D_1 \equiv_{\mathrm{num},\mathbb{Q}} D_2$ when confusion is possible) if

$$D_1 \cdot C = D_2 \cdot C$$

for every curve $C \subseteq X$. We denote by $N^1(X)_{\mathbb{Q}}$ the resulting finite-dimensional $\mathbb{Q}$-vector space of numerical equivalence classes of $\mathbb{Q}$-divisors.

(iii) Two $\mathbb{Q}$-divisors $D_1, D_2 \in \mathrm{Div}_{\mathbb{Q}}(X)$ are *linearly equivalent*, written $D_1 \equiv_{\mathrm{lin}} D_2$ (or $D_1 \equiv_{\mathrm{lin},\mathbb{Q}} D_2$) if there is an integer r such that rD_1 and rD_2 are integral and linearly equivalent in the usual sense, i.e. if $r(D_1 - D_2)$ is the image of a principal divisor in $\mathrm{Div}(X)$.

(iv) If $f : Y \to X$ is a morphism such that the image of every associated subvariety of Y meets a support of $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ properly, then $f^*D \in \mathrm{Div}_{\mathbb{Q}}(Y)$ is defined by extension of scalars from the corresponding pullback of integral divisors (this property is independent from the representation of D).

(v) If $f : Y \to X$ is an arbitrary morphism of complete varieties or projective schemes, extension of scalars give rise to a functorial induced homomorphism $f^* : N^1(X)_{\mathbb{Q}} \to N^1(Y)_{\mathbb{Q}}$ compatible with the divisor-level pullback defined in (iv).

Definition 1.5.4 (Amplitude for $\mathbb{Q}$-divisors). A $\mathbb{Q}$-divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ is *ample* if and only if one of the following three equivalent conditions is satisfied:

(i) D is of the form $D = \sum c_i A_i$ where $c_i > 0$ is a positive rational number and A_i is an ample divisor, for each $i \in I$.

(ii) There is a positive integer $r > 0$ such that $r \cdot D$ is integral and ample.

(iii) D satisfies the Nakai's criterion, i.e.

$$D^{\dim V} \cdot V > 0$$

for every irreducible subvariety $V \subseteq X$ of positive dimension.

As in the case of classical Cartier divisors, amplitude is preserved by numerical equivalence, so we speak of ample classes in $N^1(X)_{\mathbb{Q}}$.

Proposition 1.5.5 (Openness of amplitude for $\mathbb{Q}$-divisors). *Let X be a projective variety, H an ample $\mathbb{Q}$-divisor on X, and E an arbitrary $\mathbb{Q}$-divisor. Then $H + \varepsilon E$ is ample for all sufficiently small rational numbers $0 \leq |\varepsilon| \ll 1$. More generally, given finitely many $\mathbb{Q}$-divisors $E_1, \dots, E_r$ on X,*

$$H + \varepsilon_1 E_1 + \cdots + \varepsilon_r E_r \tag{1.5}$$

is ample for all sufficiently small rational numbers $0 \leq |\varepsilon_i| \ll 1$

Proof. Clearing denominators, we may assume that H and each E_i are integral. By taking $m \gg 0$ we can arrange for each of the $2r$ divisors $mH \pm E_1, \dots, mH \pm E_r$ to be ample [L, Example 1.2.10]. Now provided that $|\varepsilon_i| \ll 1$ we can write any divisor of the form (1.5) as a positive $\mathbb{Q}$-linear combination of H and some of the $\mathbb{Q}$-divisors $H \pm \frac{1}{m} E_i$. In fact, if we set $a_i = m|\varepsilon_i|$, $q = 1 - \sum_{i=1}^r a_i$, if we choose the ε_i in order to satisfy $|\varepsilon_i| < \frac{1}{rm}$ $\forall i$ and if we assume that

$$\mathrm{sgn}(x) = \begin{cases} 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}, \quad x \in \mathbb{R},$$

then we have:

$$\begin{aligned}
H+\varepsilon_1E_1+\cdots+\varepsilon_rE_r &= H+\frac{m}{m}(\varepsilon_1E_1+\cdots+\varepsilon_rE_r)=\\
=&H+m\Big(\operatorname{sgn}(\varepsilon_1)\frac{|\varepsilon_1|}{m}E_1+\cdots+\operatorname{sgn}(\varepsilon_r)\frac{|\varepsilon_r|}{m}E_r\Big)=\\
=&H+\sum_{i=1}^r a_i\Big(\frac{\operatorname{sgn}(\varepsilon_i)}{m}E_i\Big)=\Big(1+\sum_{i=1}^r a_i-\sum_{i=1}^r a_i\Big)H+\sum_{i=1}^r a_i\Big(\frac{\operatorname{sgn}(\varepsilon_i)}{m}E_i\Big)=\\
=&\Big(1-\sum_{i=1}^r a_i\Big)H+\sum_{i=1}^r a_i\Big(H+\frac{\operatorname{sgn}(\varepsilon_i)}{m}E_i\Big)=qH+\sum_{i=1}^r a_i\Big(H+\frac{\operatorname{sgn}(\varepsilon_i)}{m}E_i\Big).
\end{aligned}$$

Since $q>0$, we have obtained a positive linear combination of ample $\mathbb{Q}$-divisors that is therefore ample. □

The definition of $\mathbb{R}$-divisors proceeds in an exactly analogous fashion. Thus one defines the real vector space

$$\operatorname{Div}_{\mathbb{R}}(X)=\operatorname{Div}(X)\otimes\mathbb{R}$$

of $\mathbb{R}$-divisors on X. Supposing X is complete, there is an associated $\mathbb{R}$-valued intersection theory, giving rise in particular to the notion of numerical equivalence. Very concretely, an $\mathbb{R}$-divisor D is represented by a finite sum $D=\sum c_iA_i$ where $c_i\in\mathbb{R}$ and and $A_i\in\operatorname{Div}(X)$. It is numerically trivial if and only if $\sum c_i(A_i\cdot C)=0$ for every curve $C\subseteq X$. The resulting vector space of equivalence classes is denoted by $N^1(X)_{\mathbb{R}}$. We say that D is *effective* if $D=\sum c_iA_i$ with $c_i\geq 0$ and A_i effective. Pullbacks and supports of $\mathbb{R}$-divisors are likewise as before.

Definition 1.5.6 (Amplitude for $\mathbb{R}$-divisors). Assume that X is complete. An $\mathbb{R}$-divisor D on X is *ample* if it can be expressed as a finite sum $D=\sum c_iA_i$ where $c_i>0$ is a positive real number and A_i is an ample divisor.

Proposition 1.5.7 (Ample classes of $\mathbb{R}$-divisors). *The amplitude of an $\mathbb{R}$-divisor depends only upon its numerical equivalence class.*

Proof. It is sufficient to show that if D and B are $\mathbb{R}$-divisors, with D ample and $B\equiv_{\text{num}}0$, then $D+B$ is again ample. To this end, observe first that B is an $\mathbb{R}$-linear combination of numerically trivial integral divisors. Indeed, the condition that an $\mathbb{R}$-divisor

$$B=\sum r_iB_i,\qquad r_i\in\mathbb{R},\ B_i\in\operatorname{Div}(X)$$

be numerically trivial is given by finitely many integer linear equations on the r_i, determined by intersecting with a set of generators of the subgroup of $H_2(X,\mathbb{Z})$ spanned by algebraic 1-cycles on X. The assertion then follows from the fact that any real solution to these equations is an $\mathbb{R}$-linear combination of integral ones. We are now reduced to showing that if A and B are integral divisors, with A ample and $B\equiv_{\text{num}}0$, then $A+rB$ is ample for any $r\in\mathbb{R}$. If r is rational we already know this. In general, we can fix rational numbers $r_1<r<r_2$, together with a real number $t\in[0,1]$, such that $r=tr_1+(1-t)r_2$. Then

$$A+rB=t(A+r_1B)+(1-t)(A+r_2B),$$

exhibiting $A+rB$ as a positive $\mathbb{R}$-linear combination of ample $\mathbb{Q}$-divisors. □

The openness of amplitude for $\mathbb{Q}$-divisors seen in Proposition 1.5.5 holds for $\mathbb{R}$-divisors as well.

Proposition 1.5.8 (Openness of amplitude for $\mathbb{R}$-divisors). *Let X be a projective variety and H an ample $\mathbb{R}$-divisor on X. Given finitely many $\mathbb{R}$-divisors $E_1, \dots, E_r$, the $\mathbb{R}$-divisor*

$$H + \varepsilon_1 E_1 + \cdots + \varepsilon_r E_r$$

is ample for all sufficiently small real numbers $0 \leq |\varepsilon_i| \ll 1$.

Proof. When H and each E_i are rational this follows from the proof of Proposition 1.5.5, and one reduces the general case to this one. To begin with, since each E_j is a finite $\mathbb{R}$-linear combination of integral divisors, there is no loss of generality in assuming at the outset that all of the E_j are integral. Now write $H = \sum c_i A_i$ with $c_i > 0$ and A_i ample and integral, and fix a rational number $0 < c < c_1$. Then

$$H + \sum \varepsilon_j E_j = (cA_1 + \sum \varepsilon_j E_j) + (c_1 - c)A_1 + \sum_{i \geq 2} c_i A_i.$$

The first term on the right is governed by the case already treated, and the remaining summands are ample. □

1.6 Nef Divisors and cones

This section introduces cones for the first time in this work and it is the heart of the first chapter. After giving the definition of numerically effectiveness, both for divisors and one-cycles, and once seen a couple of properties regarding numerical effective divisors, we will introduce the ample and the nef cone in the real vector space $N^1(X)_{\mathbb{R}}$. Thanks to Kleiman's result we will show that the ample cone is the interior of the nef cone, and that the closure of the ample cone is the nef cone. We will also define effective and pseudoeffective cones of one-cycles which are also known as the cone of curves and the closed cone of curves. These two structures are in the real vector space $N_1(X)_{\mathbb{R}}$ and in particular, the pseudoeffective cone is the dual cone to the nef one.

Definition 1.6.1 (Nef line bundles and divisors). Let X be a complete variety or scheme. A line bundle $\mathscr{L}$ on X is *numerically effective*, or *nef*, if

$$\mathscr{L} \cdot C \geq 0$$

for every irreducible curve $C \subseteq X$. Similarly, a divisor D on X (with $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ coefficients) is *nef* if

$$D \cdot C \geq 0$$

for all irreducible curves $C \subset X$.

Example 1.6.2. If X is a surface and $C \subseteq X$ is an irreducible curve, then C is nef if and only if $C^2 \geq 0$. If C is nef then $C^2 \geq 0$ by definition. For the converse, given any irreducible curve $C' \subseteq X$, if $C \neq C'$ then C has non-negative intersection with C' and so $C \cdot C' \geq 0$. If $C = C'$ then by hypothesis $C \cdot C' = C^2 \geq 0$.

Theorem 1.6.3 (Kleiman's Theorem). *Let X be a complete variety (or scheme). If D is a nef $\mathbb{R}$-divisor on X, then*

$$D^k \cdot V \geq 0$$

for every irreducible subvariety $V \subseteq X$ of dimension k. Similarly,

$$\mathscr{L}^{\dim(V)} \cdot V \geq 0$$

for every nef line bundle on X.

Proof. See [L, Theorem 1.4.9]. □

Corollary 1.6.4. *Let X be a projective variety or scheme, and D a nef $\mathbb{R}$-divisor on X. If H is any ample $\mathbb{R}$-divisor on X, then*

$$D + \varepsilon \cdot H$$

is ample for every $\varepsilon > 0$. Conversely, if D and H are any two divisors such that $D + \varepsilon H$ is ample for all sufficiently small $\varepsilon > 0$, then D is nef.

Proof. If $D + \varepsilon H$ is ample for $\varepsilon > 0$, then

$$D \cdot C + \varepsilon(H \cdot C) = (D + \varepsilon H) \cdot C > 0$$

for every irreducible curve C. Letting $\varepsilon \to 0$ it follows that $D \cdot C \geq 0$, and hence that D is nef. Assume conversely that D is nef and H is ample. Replacing εH by H, it suffices to show that $D + H$ is ample. To this end, the main point is to verify that $D + H$ satisfies the Nakai's criterion stated in Definition 1.5.4 (iii). Provided that $D + H$ is (numerically equivalent to) a rational divisor, this will establish that is ample; the general case will follow by an approximation argument.
So fix an irreducible subvariety $V \subseteq X$ of dimension $k > 0$. Then

$$(D + H)^k \cdot V = \sum_{s=0}^{k} \binom{k}{s} H^s \cdot D^{k-s} \cdot V. \tag{1.6}$$

Since H is a positive $\mathbb{R}$-linear combination of integral ample divisors, the intersection $H^s \cdot V$ is represented by an effective real $(k - s)$-cycle. Applying Kleiman's theorem to each of the components of this cycle, it follows that $H^s \cdot D^{k-s} \cdot V \geq 0$. Thus each of the terms in (1.6) is non-negative, and the last intersection number $H^k \cdot V$ is strictly positive. Therefore $(D + H)^k \cdot V > 0$ for every V, and in particular if $D + H$ is rational then it is ample.
It remains to prove that $D + H$ is ample even when it is irrational. To this end, choose ample divisors $H_1, \ldots, H_r$ whose classes span $N^1(X)_{\mathbb{R}}$. By the open nature of amplitude seen in Proposition 1.5.8, the $\mathbb{R}$-divisor

$$H(\varepsilon_1, \ldots, \varepsilon_r) = H - \varepsilon_1 H_1 - \cdots - \varepsilon_r H_r$$

remains ample for all $0 \leq \varepsilon_i \ll 1$. But the classes of these divisors fill up an open subset of $N^1(X)_{\mathbb{R}}$, and consequently there exist $0 < \varepsilon_i \ll 1$ such that $D' = D + H(\varepsilon_1, \ldots, \varepsilon_r)$ represents a rational class in $N^1(X)_{\mathbb{R}}$. The case of the corollary already treated shows that D' is ample. Consequently so too is

$$D + H = D' + \varepsilon_1 H_1 + \cdots + \varepsilon_r H_r$$

as required. □

Definition 1.6.5 (Numerical equivalence classes of curves). Let X be a complete variety. We denote by $Z_1(X)_{\mathbb{R}}$ the $\mathbb{R}$-vector space of *real one-cycles* on X, consisting of all finite $\mathbb{R}$-linear combinations of irreducible curves on X. An element $\gamma \in Z_1(X)_{\mathbb{R}}$ is thus a formal sum

$$\gamma = \sum_i a_i \cdot C_i$$

where $a_i \in \mathbb{R}$ and $C_i \subset X$ is an irreducible curve. Two one-cycles $\gamma_1, \gamma_2 \in Z_1(X)_{\mathbb{R}}$ are *numerically equivalent* if

$$D \cdot \gamma_1 = D \cdot \gamma_2$$

for every $D \in \mathrm{Div}_{\mathbb{R}}(X)$. The corresponding vector space of *numerical equivalence classes* of one-cycles is written $N_1(X)_{\mathbb{R}}$. Thus by construction one has a perfect pairing

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R} \qquad , \qquad (\delta, \gamma) \mapsto \delta \cdot \gamma \in \mathbb{R}.$$

In particular $N_1(X)_{\mathbb{R}}$ is a finite dimensional real vector space on which we put the standard Euclidean topology.

Definition 1.6.6 (Intersection form associated to a divisor). Let X be a scheme, and let $D \in \mathrm{Div}_{\mathbb{R}}(X)$. The *intersection form* associated to D is the function

$$\begin{aligned} \varphi_D : N_1(X) &\to \mathbb{R} \\ \gamma &\mapsto D \cdot \gamma. \end{aligned}$$

Proposition 1.6.7 (Continuity of the intersection form). *The intersection form associated to an $\mathbb{R}$-divisor D on a scheme X is continuous.*

Proof. We will show that the intersection form associated to D is a linear function, and therefore is continuous. Assuming $N_1(X)$ has dimension s, we can find a basis $\{\gamma_1, \dots, \gamma_s\}$ and then write a generic element $\gamma \in N_1(X)$ as $\gamma = \sum_i x_i \gamma_i$, $x_i \in \mathbb{R}$. Now

$$\varphi_D(\gamma) = \varphi_D\Big(\sum_{i=1}^s x_i \gamma_i\Big) = \Big(\sum_{i=1}^s x_i \gamma_i\Big) \cdot D = \sum_{i=1}^s x_i \gamma_i \cdot D.$$

□

Before introducing ample and nef cones, we clarify that if V is a finite-dimensional real vector space, by a *cone* we understand a set $K \subseteq V$ stable under multiplication by positive scalars and we do not require that cones contain the origin.

Definition 1.6.8 (Ample and nef cones). The *ample cone*

$$\mathrm{Amp}(X) \subset N^1(X)_{\mathbb{R}}$$

of X is the convex cone of all ample $\mathbb{R}$-divisor classes on X. The *nef cone*

$$\mathrm{Nef}(X) \subset N^1(X)_{\mathbb{R}}$$

is the convex cone of all nef $\mathbb{R}$-divisor classes.

Theorem 1.6.9 (Kleiman). *Let X be any projective variety or scheme.*

(i). *The nef cone is the closure of the ample cone:*

$$\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}.$$

(ii). *The ample cone is the interior of the nef cone:*

$$\mathrm{Amp}(X) = \mathrm{int}\Big(\mathrm{Nef}(X)\Big).$$

Proof. Thanks to the continuity of the intersection form shown in Proposition 1.6.7, the nef cone is closed, and it follows from Proposition 1.5.8 that $\mathrm{Amp}(X)$ is open. This gives the inclusions

$$\overline{\mathrm{Amp}(X)} \subseteq \mathrm{Nef}(X) \qquad \text{and} \qquad \mathrm{Amp}(X) \subseteq \mathrm{int}(\mathrm{Nef}(X)).$$

The remaining two inclusions

$$\overline{\mathrm{Amp}(X)} \supseteq \mathrm{Nef}(X) \qquad \text{and} \qquad \mathrm{Amp}(X) \supseteq \mathrm{int}(\mathrm{Nef}(X))$$

are consequences of Corollary 1.6.4. In fact let H be an ample divisor on X. If D is any nef $\mathbb{R}$-divisor then Corollary 1.6.4 shows that $D + \varepsilon H$ is ample for all $\varepsilon > 0$. Therefore D is a limit of ample divisors, establishing the inclusion $\overline{\mathrm{Amp}(X)} \supseteq \mathrm{Nef}(X)$. For $\mathrm{Amp}(X) \supseteq \mathrm{int}(\mathrm{Nef}(X))$, observe that if the class of D lies in the interior of $\mathrm{Nef}(X)$, then $D - \varepsilon H$ remains nef for $0 < \varepsilon \ll 1$. Consequently

$$D = (D - \varepsilon H) + \varepsilon H$$

is ample thanks again to Corollary 1.6.4. □

Proposition 1.6.10. *If $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is nef then $D^2 \geq 0$.*

Proof. Thanks to Theorem 1.6.9, since D is nef there exists $\{A_m\}_m \subseteq \mathrm{Amp}(X)$ such that $\lim_{m\to\infty} A_m = D$. If this implies that $D^2 = \lim_{m\to\infty} A_m^2$ then the proof is concluded, in fact $A_m^2 > 0\ \forall m$. Let $\{D_1, \ldots, D_\rho\}$ be a basis of $N^1(X)_{\mathbb{R}}$. Then we can write $D = \sum x_i D_i$, $x_i \in \mathbb{R}$ and $A_m = \sum x_{i,m} D_i$, $x_{i,m} \in \mathbb{R}$. This means that $A_m \to D \Leftrightarrow \lim_{m\to\infty} x_{i,m} = x_i\ \forall i$. We can now write

$$A_m^2 = (\sum_i x_{i,m} D_i)(\sum_j x_{j,m} D_j) = \sum_{i,j} x_{i,m} x_{j,m} D_i D_j \to \sum_{i,j} x_i x_j D_i D_j = D^2.$$

□

Definition 1.6.11 (Cone of curves). Let X be a complete variety. The *cone of curves*

$$\mathrm{Eff}_1(X) \subseteq N_1(X)_{\mathbb{R}}$$

is the cone spanned by the classes of all effective one-cycles on X. Concretely,

$$\mathrm{Eff}_1(X) = \Big\{ \sum_i a_i[C_i] \mid C_i \subset X \text{ an irreducible curve}, a_i \geq 0 \Big\}.$$

Its closure

$$\mathrm{Pseff}_1(X) \subseteq N_1(X)_{\mathbb{R}}$$

is the *closed cone of curves* on X.

Proposition 1.6.12. $\mathrm{Pseff}_1(X)$ *is the closed cone dual to* $\mathrm{Nef}(X)$, *i.e.*

$$\mathrm{Pseff}_1(X) = \Big\{\gamma \in N_1(X)_{\mathbb{R}} \mid \delta \cdot \gamma \geq 0 \ \text{ for all } \delta \in \mathrm{Nef}(X)\Big\}.$$

Proof. This is a consequence of the theory of duality for cones. Specifically, suppose that $K \subseteq V$ is a closed convex cone in a finite-dimensional real vector space. Recall that the dual of K is defined to be the cone in V^* given by

$$K^* = \Big\{\varphi \in V^* \mid \varphi(x) \geq 0 \quad \forall x \in K\Big\}.$$

The duality theorem for cones states that under the natural identification of V^{**} with V, one has $K^{**} = K$. In the situation at hand take

$$V = N_1(X)_{\mathbb{R}} \qquad , \qquad K = \mathrm{Pseff}_1(X).$$

Then $\mathrm{Nef}(X) = \mathrm{Pseff}_1(X)^*$ by definition. Consequently

$$\mathrm{Pseff}_1(X) = \mathrm{Nef}(X)^*,$$

which is the assertion of the proposition. □

Continue to assume that X is complete, and fix a divisor $D \in \mathrm{Div}_{\mathbb{R}}(X)$, not numerically trivial. We denote by

$$\varphi_D : N_1(X)_{\mathbb{R}} \to \mathbb{R}$$

the linear functional determined by intersection with D, and we set

$$D^{\perp} = \Big\{\gamma \in N_1(X)_{\mathbb{R}} \mid D \cdot \gamma = 0\Big\}, \qquad D_{>0} = \Big\{\gamma \in N_1(X)_{\mathbb{R}} \mid D \cdot \gamma > 0\Big\}.$$

Thus $D^{\perp} = \mathrm{Ker}\varphi_D$ is a hyperplane and $D_{>0}$ an open half-space in $N_1(X)_{\mathbb{R}}$. One can define $D_{\geq 0}, D_{\leq 0}, D_{<0} \subset N_1(X)_{\mathbb{R}}$ similarly.

Theorem 1.6.13 (Amplitude via cones). *Let X be a projective variety (or scheme), and let D be an $\mathbb{R}$-divisor on X. Then D is ample if and only if*

$$\mathrm{Pseff}_1(X) - \{0\} \subseteq D_{>0}.$$

Equivalently, choose any norm $\| \cdot \|$ on $N_1(X)_{\mathbb{R}}$, and denote by

$$S = \{\gamma \in N_1(X)_{\mathbb{R}} \mid \|\gamma\| = 1\}$$

the "unit sphere" of classes in $N_1(X)_{\mathbb{R}}$ of length 1. Then D is ample if and only if

$$\Big(\mathrm{Pseff}_1(X) \cap S\Big) \subseteq \Big(D_{>0} \cap S\Big).$$

Proof. See [L, Theorem 1.4.29]. □

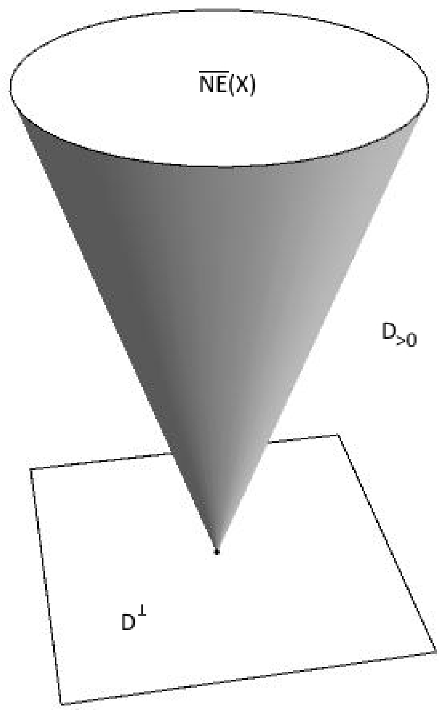

Figure 1.1: Test for amplitude via the cone of curves

1.7 Semiample line bundles

In this section we will first of all recall some basic definitions of asymptotic theory, needed to understand the notion of semiample line bundle. These particular line bundles have at least one tensor power that is globally generated and on a normal projective variety the property of being semiample implies that the line bundle is also globally generated.

Definition 1.7.1 (Semigroup and exponent of a line bundle). Let $\mathscr{L}$ be a line bundle on the irreducible projective variety X. The *semigroup* of $\mathscr{L}$ consists of those non-negative powers of $\mathscr{L}$ that have a non-zero section:

$$\mathbb{N}(\mathscr{L}) = \mathbb{N}(X, \mathscr{L}) = \{m \geq 0 \mid H^0(X, \mathscr{L}^{\otimes m}) \neq 0\}.$$

Assuming $\mathbb{N}(\mathscr{L}) \neq (0)$, all sufficiently large elements of $\mathbb{N}(X, \mathscr{L})$ are multiples of a largest single natural number $e = e(\mathscr{L}) \geq 1$, which we may call the *exponent* of $\mathscr{L}$. The exponent e is the g.c.d. of all the elements of $\mathbb{N}(\mathscr{L})$. The semigroup $\mathbb{N}(X, D)$ and exponent $e = e(D)$ of a divisor D are defined analogously.

Given $m \in \mathbb{N}(X, \mathscr{L})$, consider the rational mapping

$$\Phi_m = \Phi_{|\mathscr{L}^{\otimes m}|} : X \dashrightarrow \mathbb{P}H^0(X, \mathscr{L}^{\otimes m})$$

associated to the complete linear system $|\mathscr{L}^{\otimes m}|$. We denote by

$$Y_m = \Phi_m(X) \subseteq \mathbb{P}H^0(X, \mathscr{L}^{\otimes m})$$

the closure of its image, i.e. the image of the closure of the graph of Φ_m.

Definition 1.7.2 (Iitaka dimension). Assume that X is normal. Then the *Iitaka dimension* of $\mathscr{L}$ is defined to be

$$\kappa(\mathscr{L}) = \kappa(X, \mathscr{L}) = \max_{m \in \mathbb{N}(\mathscr{L})} \{\dim \Phi_m(X)\},$$

provided that $\mathbb{N}(\mathscr{L}) \neq (0)$. If $H^0(X, \mathscr{L}^{\otimes m}) = 0$ for all $m > 0$, one puts $\kappa(X, \mathscr{L}) = -\infty$. For a divisor D one takes $\kappa(X, D) = \kappa(X, \mathscr{O}_X(D))$.

Definition 1.7.3 (Algebraic fibre space). An *algebraic fibre space* is a surjective projective mapping $f : X \to Y$ of reduced and irreducible varieties such that $f_* \mathscr{O}_X = \mathscr{O}_Y$.

Definition 1.7.4 (Section ring associated to a line bundle). Given a line bundle $\mathscr{L}$ on a projective variety X, the *graded ring* or *section ring* associated to $\mathscr{L}$ is the graded $\mathbb{C}$-algebra

$$R(\mathscr{L}) = R(X, \mathscr{L}) = \bigoplus_{m \geq 0} H^0(X, \mathscr{L}^{\otimes m}).$$

The graded ring $R(D) = R(X, D)$ associated to a divisor D is defined similarly.

Definition 1.7.5 (Finitely generated line bundles and divisors). A line bundle $\mathscr{L}$ on a projective variety X is *finitely generated* if its section ring $R(X, \mathscr{L})$ is a finitely generated $\mathbb{C}$-algebra. A divisor D is finitely generated if $\mathscr{O}_X(D)$ is so.

Definition 1.7.6 (Semiample line bundles and divisors). A line bundle $\mathscr{L}$ on a complete scheme is *semiample* if $\mathscr{L}^{\otimes m}$ is globally generated for some $m > 0$. A divisor D is *semiample* if the corresponding line bundle is so.

Fixing a semiample line bundle $\mathscr{L}$, we denote by $M(X, \mathscr{L}) \subseteq \mathbb{N}(X, \mathscr{L})$ the subsemigroup

$$M(X, \mathscr{L}) = \{m \in \mathbb{N} \mid \mathscr{L}^{\otimes m} \text{ is free}\}.$$

We write $f = f(\mathscr{L})$ for the *exponent* of $M(X, \mathscr{L})$, i.e. the largest natural number such that every element of $M(X, \mathscr{L})$ is a multiple of f (so that in particular $\mathscr{L}^{\otimes kf}$ is free for $k \gg 0$).

Theorem 1.7.7 (Semiample fibrations). *Let X be a normal projective variety, and let $\mathscr{L}$ be a semiample line bundle on X. Then there is an algebraic fibre space*

$$\Phi : X \to Y$$

having the property that for any sufficiently large integer $m \in M(X, \mathscr{L})$,

$$Y_m = Y \qquad \textit{and} \qquad \Phi_m = \Phi.$$

Furthermore there is an ample line bundle A on Y such that $\Phi^* A = \mathscr{L}^{\otimes f}$, where $f = f(\mathscr{L})$ is the exponent of $M(X, \mathscr{L})$.

In other words, for $m \gg 0$ the mappings Φ_m stabilize to define a fibre space structure on X.

Proof. See [L, Theorem 2.1.27] for the full proof. □

Theorem 1.7.8. *Let $\mathscr{L}$ be a semiample line bundle on a normal projective variety X. Then $\mathscr{L}$ is finitely generated.*

Proof. See [L, Example 2.1.30] for the full proof. □

1.8 Big divisors

In this section we study a particularly important class of line bundles, namely those of maximal Iitaka dimension. We will first state the basic facts about those divisors, then we discuss the big and pseudoeffective cones of divisors. These two cones have a tie which is comparable to the one the ample and the nef cone have. In fact, the big cone is the interior of the pseudoeffective cone, furthermore the pseudoeffective cone is the closure of the big cone.

Definition 1.8.1 (Big). A line bundle $\mathscr{L}$ on an irreducible projective variety X is *big* if $\kappa(X, \mathscr{L}) = \dim X$. A divisor D on X is *big* if $\mathscr{O}_X(D)$ is so.

Lemma 1.8.2. *Assume that X is a projective variety of dimension n. A divisor D on X is big if and only if there is a constant $C > 0$ such that*

$$h^0(X, \mathscr{O}_X(mD)) \geq C \cdot m^n$$

for all sufficiently large $m \in \mathbb{N}(X, D)$.

Proof. See [L, Lemma 2.2.3] for the complete proof. □

Proposition 1.8.3 (Kodaira's Lemma). *Let D be a big divisor and F an arbitrary effective divisor on X. Then*

$$H^0(X, \mathscr{O}_X(mD - F)) \neq 0$$

for all sufficiently large $m \in \mathbb{N}(X, D)$.

Proof. Suppose that $\dim X = n$, and consider the exact sequence

$$0 \to \mathscr{O}_X(mD - F) \to \mathscr{O}_X(mD) \to \mathscr{O}_F(mD) \to 0.$$

Since D is big, there is a constant $c > 0$ such that $h^0(X, \mathscr{O}_X(mD)) \geq c \cdot m^n$ for sufficiently large $m \in \mathbb{N}(X, D)$. On the other hand, F being a scheme of dimension $n-1$, $h^0(F, \mathscr{O}_F(mD))$ grows at most like m^{n-1} (see [L, Example 1.2.20]). Therefore

$$h^0(X, \mathscr{O}_X(mD)) > h^0(F, \mathscr{O}_F(mD))$$

for large $m \in \mathbb{N}(X, D)$, and the assertion follows from the displayed sequence. □

Corollary 1.8.4 (Characterization of big divisors). *Let D be a divisor on an irreducible projective variety X. Then the following are equivalent:*

(i) *D is big.*

(ii) *For any ample integer divisor A on X, there exists a positive integer $m > 0$ and an effective divisor N on X such that $mD \equiv_{\text{lin}} A + N$.*

(iii) *Same as in* (ii) *for some ample divisor A.*

(iv) *There exists an ample divisor A, a positive integer $m > 0$, and an effective divisor N such that $mD \equiv_{\text{num}} A + N$.*

Proof. Assuming that D is big, take $r \gg 0$ so that $rA \equiv_{\text{lin}} H_r$ and $(r+1)A \equiv_{\text{lin}} H_{r+1}$ are both effective. Apply Lemma 1.8.3 with $F = H_{r+1}$ to find a positive integer m and an effective divisor N' with

$$mD \equiv_{\text{lin}} H_{r+1} + N' \equiv_{\text{lin}} A + (H_r + N').$$

Taking $N = H_r + N'$ gives (ii). The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) being trivial, we assume (iv) and deduce (i). If $mD \equiv_{\text{num}} A + N$, then $mD - N$ is numerically equivalent to an ample divisor, and hence ample by Corollary 1.4.6. So after possibly passing to an even larger multiple of D we can assume that $mD \equiv_{\text{lin}} H + N'$, where H is very ample and N' is effective. But then

$$\kappa(X, D) \geq \kappa(X, H) = \dim X,$$

so D is big. □

Corollary 1.8.5 (Numerical nature of bigness). *The bigness of a divisor D depends only on its numerical equivalence class.*

Proof. This follows from statement (iv) in the previous corollary. □

Definition 1.8.6 (Big $\mathbb{Q}$-divisors). A $\mathbb{Q}$-divisor is *big* if there is a positive integer $m > 0$ such that mD is integral and big.

Definition 1.8.7 (Big $\mathbb{R}$-divisors). An $\mathbb{R}$-divisor on an irreducible projective variety X is *big* if it can be written in the form

$$D = \sum_i a_i \cdot D_i$$

where each D_i is a big integral divisor and a_i is a positive real number.

Proposition 1.8.8 (Formal properties of big $\mathbb{R}$-divisors). *Let D and D' be $\mathbb{R}$-divisors on X.*

(i) *If $D \equiv_{\text{num}} D'$, then D is big if and only if D' is big.*

(ii) *D is big if and only if $D \equiv_{\text{num}} A + N$ where A is an ample $\mathbb{R}$-divisor and N is an effective $\mathbb{R}$-divisor.*

Proof. See [L, Proposition 2.2.22] for a sketch of the proof. □

Example 1.8.9 (Big and nef $\mathbb{R}$-divisors). Let D be a nef and big $\mathbb{R}$-divisor. Then there is an effective $\mathbb{R}$-divisor N such that $D - \frac{1}{k}N$ is an ample $\mathbb{R}$-divisor for every $k \in \mathbb{N}, k \geq 1$.

Proof. Proposition 1.8.8 (ii) lets us find an ample $\mathbb{R}$-divisor A and an effective $\mathbb{R}$-divisor N such that $D \equiv_{\text{num}} A + N$. Given $k \in \mathbb{N}$, this numerical equivalence can be written as $kD \equiv_{\text{num}} (k-1)D + A + N$. If we take $k \geq 1$ then $(k-1)D + A$ is a sum of a nef and an ample $\mathbb{R}$-divisor, hence it is ample by Corollary 1.6.4. Now we have that

$$D - \frac{1}{k}N \equiv_{\text{num}} \frac{1}{k} \cdot ((k-1)D + A) \in \text{Amp}(X), \qquad \forall k \geq 1.$$

The conclusion follows from Corollary 1.4.6. □

Corollary 1.8.10. *Let $D \in \mathrm{Div}_{\mathbb{R}}(X)$ be a big $\mathbb{R}$-divisor, and let $E_1, \ldots, E_t \in \mathrm{Div}_{\mathbb{R}}(X)$ be arbitrary $\mathbb{R}$-divisors. Then*

$$D + \varepsilon_1 E_1 + \cdots + \varepsilon_t E_t$$

remains big for all sufficiently small real numbers $0 \leq |\varepsilon_i| \ll 1$.

Proof. This follows from statement (ii) of the previous proposition thanks to the open nature of amplitude (Proposition 1.5.8). □

Definition 1.8.11 (Big and pseudoeffective cones). The *big cone*

$$\mathrm{Big}(X) \subseteq N^1(X)_{\mathbb{R}}$$

is the convex cone of all big $\mathbb{R}$-divisor classes on X. The *pseudoeffective cone*

$$\mathrm{Pseff}^1(X) \subseteq N^1(X)_{\mathbb{R}}$$

is the closure of the convex cone spanned by the classes of all the effective $\mathbb{R}$-divisors. A divisor $D \in \mathrm{Div}(X)_{\mathbb{R}}$ is *pseudoeffective* if its class lies in the pseudoeffective cone.

Lemma 1.8.12. *Let $X = \mathbb{R}^n$ with the standard topology, A an open convex subset of X, and Y a subset of X such that $\overline{Y} \subseteq \overline{A}$. Then $\mathrm{int}(\overline{Y}) \subseteq A$.*

Proof. By hypothesis we have $\mathrm{int}(\overline{Y}) \subseteq \overline{Y} \subseteq \overline{A}$. Since $\mathrm{int}(\overline{Y})$ is open we must have $\mathrm{int}(\overline{Y}) \subseteq \mathrm{int}(\overline{A})$. Thanks to [R, Theorem 6.3] we also have $\mathrm{int}(\overline{A}) = \mathrm{int}(A)$ and then we conclude $\mathrm{int}(\overline{Y}) \subseteq \mathrm{int}(A) = A$. □

Theorem 1.8.13. *The big cone is the interior of the pseudoeffective cone and the pseudoeffective cone is the closure of the big cone:*

$$\mathrm{Big}(X) = \mathrm{int}\Big(\mathrm{Pseff}^1(X)\Big) \quad , \quad \mathrm{Pseff}^1(X) = \overline{\mathrm{Big}(X)}.$$

Proof. The pseudoeffective cone is closed by definition, the big cone is open by Corollary 1.8.10, and $\mathrm{Big}(X) \subseteq \mathrm{Pseff}^1(X)$ thanks to Proposition 1.8.8 (ii). It remains to establish the inclusions

$$\mathrm{Pseff}^1(X) \subseteq \overline{\mathrm{Big}(X)} \quad , \quad \mathrm{int}\Big(\mathrm{Pseff}^1(X)\Big) \subseteq \mathrm{Big}(X).$$

To prove the first of these, consider $\nu \in \mathrm{Pseff}^1(X)$. Then one can write ν as the limit $\nu = \lim_k \nu_k$ of the classes of effective divisors. Fixing an ample class $\alpha \in N^1(X)_{\mathbb{R}}$ one has

$$\nu = \lim_{k \to \infty} (\nu_k + \frac{1}{k}\alpha).$$

Each of the classes $\nu_k + \frac{1}{k}\alpha$ is big thanks to Proposition 1.8.8 (ii), so ν is a limit of big classes. For the second one, once the first is shown, we can apply Lemma 1.8.12 by taking $A = \mathrm{Big}(X)$ and $Y = \mathrm{Eff}^1(X)$. □

1.9 Examples of cones

We end our introductory chapter with some examples of the construction of the cones defined so far. What is interesting to notice, is that these cones can assume pretty different structures depending on the variety on which they are defined. We will see cones that are not closed, circular cones and cones with clustering rays.

1.9.A Ruled surface where $\mathrm{Eff}_1(X)$ is not closed

Let E be a smooth projective curve of genus g, let U be a vector bundle on E of rank two, and set $X = \mathbb{P}(U)$ with

$$\pi : X = \mathbb{P}(U) \to E$$

the bundle projection. We can assume, after twisting by a suitable divisor and without loss of generality that $\deg U = 0$. In this setting $N^1(X)_{\mathbb{R}}$ is generated by the two classes

$$\xi = [D] \quad , \quad f = [F],$$

where F is a fibre of π and D is the divisor associated to the line bundle $\mathscr{O}_{\mathbb{P}(U)}(1)$. The intersection form on X is determined by the relations

$$\xi^2 = \deg U = 0 \quad , \quad \xi \cdot f = 1 \quad , \quad f^2 = 0.$$

In particular $((af + b\xi)^2) = 2ab$. If we represent the class $(af + b\xi)$ by the point (a, b) in the $f - \xi$ plane, it follows that the nef cone $\mathrm{Nef}(X)$ must lie within the first quadrant $a, b \geq 0$. In fact, since $(a\varepsilon + bf)$ is nef and f is effective, we have $0 \leq (a\varepsilon + bf) \cdot f = a$. Thanks again to the nefness of $(a\varepsilon + bf)$, along with the continuity of the intersection form, we can also conclude that $0 \leq (a\varepsilon + bf)^2 = 2ab$. So if $a > 0$ it must be $b \geq 0$. On the other hand, if $a = 0$, then bf is our nef divisor and then $b \geq 0$, because given any ample divisor A one has $f \cdot A > 0$ (Theorem 1.4.5) and $bf \cdot A \geq 0$. We have seen that the nef cone lies within the first quadrant, now also note that the fibre F is clearly nef (see [L, Example 1.4.6]). Therefore the non-negative f-axis forms one of the two boundaries of $\mathrm{Nef}(X)$. The second one depends on the geometry of U. Consider the case in which U is semistable, which means, by definition, that it does not admit any quotients of negative degree. When U is semistable all its symmetric powers $S^m U$ are so (cfr. [L, 6.4.14]). In the present situation this implies that if A is a line bundle of degree a such that $H^0(E, S^m U \otimes A) \neq 0$, then $a \geq 0$. Now suppose that $C \subset X$ is an effective curve. Then C arises as a section of $\mathscr{O}_{\mathbb{P}(U)}(m) \otimes \pi^* A$ for some integer $m \geq 0$ and some line bundle A on E. On the other hand,

$$H^0(\mathbb{P}(U), \mathscr{O}_{\mathbb{P}(U)}(m) \otimes \pi^* A) = H^0(E, S^m U \otimes A),$$

so by what we have just said $a = \deg A \geq 0$. In other words, the class $(af + m\xi)$ of C lies in the first quadrant. So in this case $\mathrm{Nef}(X) = \mathrm{Pseff}_1(X)$ and the cones in question fill up the first quadrant of the $f - \xi$ plane. Now we ask whether the positive ξ-axis $\mathbb{R}^{\geq 0}\xi$ actually lies in the cone $\mathrm{Eff}_1(X)$ of effective curves, or merely in its closure. In other words we ask whether there exists an irreducible curve $C \subset X$ with $[C] = m\xi$ for some $m \geq 1$. The presence of such a curve is equivalent to the existence of a line bundle A of degree 0 on E such that $H^0(E, S^mU \otimes A) \neq 0$, which implies that S^mU is semistable but not strictly stable. Thanks to ([H1], 1.10.5) if E has genus $g \geq 2$ then there exist bundles U of degree 0 on E having the property that

$$H^0(E, S^mU \otimes A) = 0 \quad \text{for all} \quad m \geq 1$$

whenever $\deg A \leq 0$: in fact this holds for a sufficiently general semistable bundle U. Thus there is no effective curve C on the resulting surface $X = \mathbb{P}(U)$ with class $[C] = m\xi$, and therefore the positive ξ-axis does not itself lie in the cone of effective curves.

1.9.B Products of curves

Before we start discussing our next example, we will expose some remarkable properties of abelian varieties. Recall that a variety A is *abelian* if it has a structure of an abelian group. We will denote its binary operation with $+$. Thanks to this group structure we can define translations of a subvariety $Z \subseteq A$,

$$t_a : Z \to Z + a \qquad z \mapsto z + a.$$

Proposition 1.9.1. *If Z is an irreducible divisor of an abelian variety A then for every $x \in A$ there exists $a \in A$ such that $x \notin Z + a$.*

Proof. Suppose that $x \in Z + a$ for every $a \in A$, then $x = z + a$ for some $z \in Z$ depending on a. Then for all $a \in A$ and for some $z \in Z$ one should have $a = x - z \in (x - Z) \cong Z$. This is absurd, because since Z has codimension 1 then it must exist $a' \in A \backslash (x - Z)$. □

Proposition 1.9.2. *Let A be an abelian variety. If $Z \in \mathrm{Div}(A)$ is irreducible then it is nef.*

Proof. Suppose that Z is not nef, then we can find an irreducible curve $C \subseteq A$ such that $Z \cdot C < 0$. This implies that $C \subseteq Z$, otherwise $Z \cdot C = \#\{Z \cap C\} \geq 0$. Let $x \in C$. Then, for Proposition 1.9.1, we can find an element $a \in A$ such that $x \notin Z + a$. This means that $C \nsubseteq (Z + a)$, so $(Z + a) \cdot C \geq 0$. On the other hand, $Z \equiv_{\mathrm{num}} (Z + a)$. The conclusion $0 > Z \cdot C = (Z + a) \cdot C \geq 0$ is clearly a contradiction. □

Proposition 1.9.3. *If A is an abelian variety, then* $\mathrm{Pseff}^1(A) = \mathrm{Nef}(A)$.

Proof. The inclusion $\supseteq$ holds for a generic variety X, since $\mathrm{Amp}(X) \subseteq \mathrm{Eff}^1(X)$ and then, switching to the closures, $\mathrm{Nef}(X) = \overline{\mathrm{Amp}}(X) \subseteq \mathrm{Pseff}^1(X)$. For the opposite inclusion, Proposition 1.9.2 states that $\mathrm{Eff}^1(A) \subseteq \mathrm{Nef}(A)$, so we can conclude simply by switching to the closures, since the nef cone is closed. □

Let E be a smooth irreducible complex projective curve of genus $g = g(E)$. We set $X = E \times E$, with projections $\mathrm{pr}_1, \mathrm{pr}_2 : X \to E$. Fixing a point $P \in E$, consider in $N^1(X)_{\mathbb{R}}$ the three classes

$$f_1 = [\{P\} \times E] \quad , \quad f_2 = [E \times \{P\}] \quad , \quad \delta = [\Delta],$$

where $\Delta \subset E \times E$ is the diagonal. Provided that $g(E) \geq 1$ these classes are independent, and if E has general moduli then it is known that they span $N_1(X)_{\mathbb{R}}$. Intersections among them are governed by the formulae

$$\delta \cdot f_1 = \delta \cdot f_2 = f_1 \cdot f_2 = 1 \quad , \quad f_1^2 = f_2^2 = 0 \quad , \quad \delta^2 = 2 - 2g$$

Assume that $g(E) = 1$. Then $X = E \times E$ is an abelian surface, and one has

Lemma 1.9.4. *Any effective curve on X is nef, and consequently*

$$\mathrm{Pseff}_1(X) = \mathrm{Nef}(X).$$

A class $\alpha \in N^1(X)_{\mathbb{R}}$ is nef if and only if

$$\alpha^2 \geq 0 \quad , \quad \alpha \cdot h \geq 0, \tag{1.7}$$

for some ample class h. In particular, if

$$\alpha = x \cdot f_1 + y \cdot f_2 + z \cdot \delta,$$

then α is nef if and only if

$$xy + xz + yz \geq 0, \quad x + y + z \geq 0. \tag{1.8}$$

Proof. To prove the first equality we observe that since X is a surface then $\mathrm{Pseff}^1(X) = \mathrm{Pseff}_1(X)$. X is also abelian, so we conclude using Proposition 1.9.3. Now, consider a class $\alpha \in \mathrm{Nef}(X)$. Then, for the previous equality, α is also effective, so by definition of nefness $\alpha^2 \geq 0$. Given an ample class h, it is also nef thanks to Theorem 1.6.9, and then we have $\alpha \cdot h \geq 0$. To end the proof we need to show that if $\alpha \in N^1(X)_{\mathbb{R}}$ satistifes conditions (1.7) then α is nef. Given a divisor A with $A^2 > 0$ and $A \cdot h > 0$ then $[A] = \alpha \in \mathrm{Eff}_1(X)$. The thesis will then follow by switching to closures. Given an integer $m \gg 0$ and applying Riemann-Roch to mA we obtain

$$h^0(mA) = h^1(mA) - h^2(mA) + \frac{1}{2}mA(mA - K_X) + \chi(X).$$

Clearly $h^1(mA) \geq 0$ and for $m \gg 0$ we have $\frac{1}{2}mA(mA - K_X) + \chi(X) \sim \frac{1}{2}m^2A^2$. Then $h^0(mA) \geq \frac{1}{2}m^2A^2 - h^2(mA)$. Thanks to Serre's duality $h^2(mA) \cong h^0(K_X - mA)$. This number must be zero, otherwise $|K_X - mA| \neq \emptyset$ and then there would exist an effective divisor $E \equiv_{\mathrm{lin}} (K_X - mA)$. Thus we would have

$$0 \leq E \cdot h = (K_X - mA) \cdot h.$$

On the other $A \cdot h > 0$ by hypothesis so if $m \gg 0$ we would necessarily have $(K_X - mA) \cdot h < 0$ that is evidently a contradiction. At this point we have shown that $h^0(mA) \geq \frac{1}{2}m^2A^2 > 0$. As we have seen before, this implies that $|mA| \neq \emptyset$ and so there exists an effective divisor $F \equiv_{\mathrm{lin}} mA$. We can now conclude that $\alpha = \frac{1}{m}[F] \in \mathrm{Eff}^1(X) \subseteq \mathrm{Eff}_1(X)$. In particular, if we regard α as an element of the space generated by f_1, f_2 and δ, we see with some easy computations that it is nef if and only if it lies in the circular cone described by conditions (1.8). $\square$

1.9.C Blow-ups of $\mathbb{P}^2$

Let X be the blowing up of the projective plane at ten or more very general points. Denote by $e_i \in N^1(X)$ the classes of the exceptional divisors, and let l be the pullback to X of the hyperplane class on $\mathbb{P}^2$. We may fix $0 < \varepsilon \ll 1$ such that $h := l - \varepsilon \cdot \sum e_i$ is an ample class. [H, V, Exercise 4.15] shows that we can find (-1)-curves of arbitrarily high degree on X. In other words there exists a sequence $C_i \subseteq X$ of smooth rational curves with

$$C_i \cdot C_i = -1 \quad \text{and} \quad C_i \cdot h \to \infty \quad \text{with } i.$$

By Corollary 2.3.1 each $[C_i]$ generates an extremal ray in $\mathrm{Pseff}_1(X)$. On the other hand, let K_X denote as usual the canonical divisor on X. Then $C_i \cdot K_X = -1$ does not grow with i.

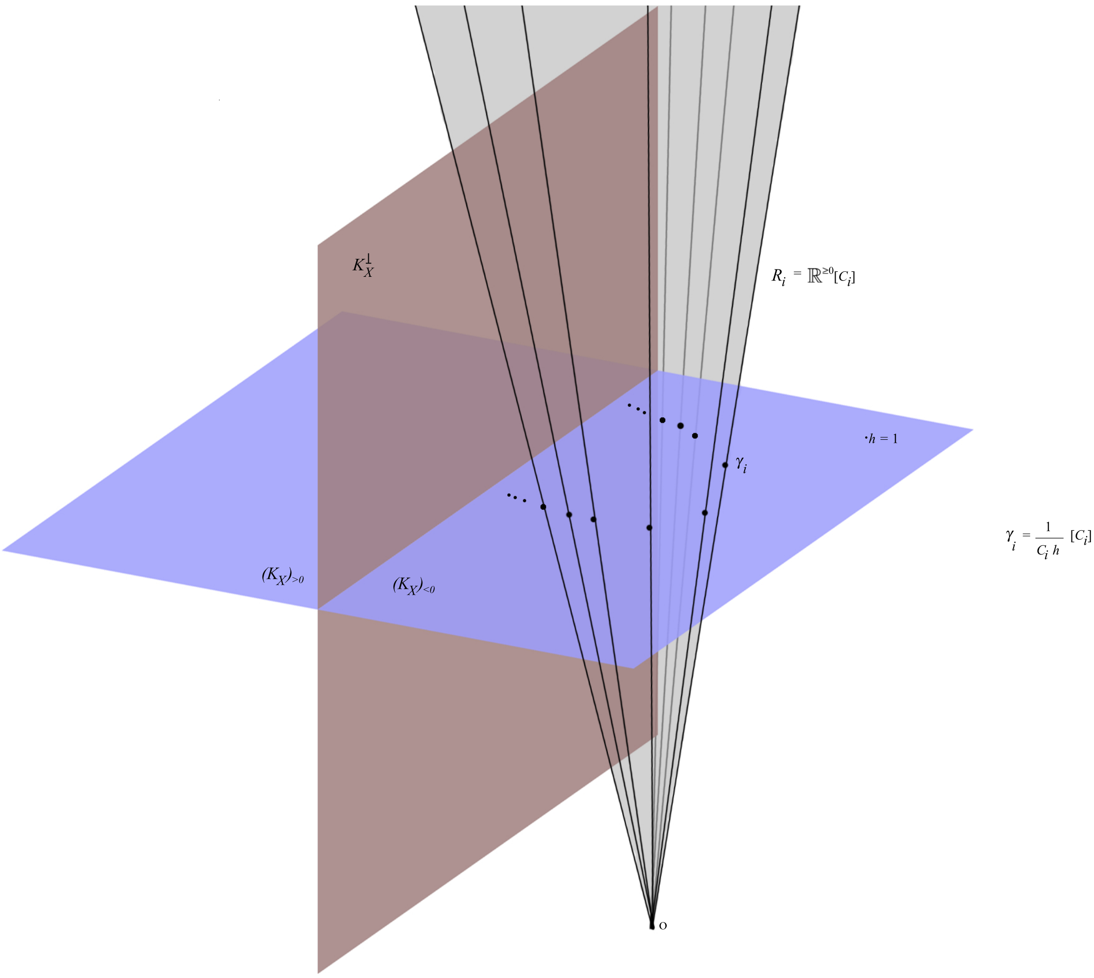

Figure 1.2: Cone of curves on blow-up of $\mathbb{P}^2$

The purple plane shown in figure is given by $\{\alpha \in N_1(X) : \alpha \cdot h = 1\}$. We can see that the points represent the extremal rays $\mathbb{R}^{\geq 0}[C_i]$ generated by the $[C_i]$. In particular if we put $a[C_i] \cdot h = 1$ with $a \geq 0$, we note that the extremal rays intersect our plane when $a = \frac{1}{[C_i]\cdot h}$. We also observe that the rays $\mathbb{R}^{\geq 0}[C_i]$ generated by the C_i cluster in $N_1(X)_{\mathbb{R}}$ towards the plane $K_X^{\perp}$ defined by the vanishing of K_X. In fact,

$\gamma_i \cdot K_X = -\frac{1}{C_i \cdot h} \to 0$, since $C_i \cdot h \to +\infty$ and $C_i \cdot K_X = -1$. It is conjectured that $\mathrm{Pseff}_1(X)$ is circular on the region $(K_X)_{>0}$ but this is not yet known.

Chapter 2

Criteria of extremality

This chapter differs from the previous, since it is not focused on algebraic geometry. Indeed, the results exposed in the following descend more generally from notions of convex geometry. The final purpose of this work is to apply these results to the entities introduced before. In fact, obviously the cones of divisors and curves defined in the first chapter can be studied from the convex geometry point of view. In particular, we are interested in finding criteria to establish when certain elements of these cones give rise to faces or extremal rays, which are faces of dimension 1. Faces are characterized by the property that if we take two points of the cone with their sum being an element of the face, then they must lie in the face. If a face has some additional properties, i.e. there exist linear functions that vanish on the face and keep the cone in the intersection of their non negative half-spaces, then we will call it a perfect face or simply an edge if it has dimension 1. To better understand the geometrical difference between an extremal ray and an edge, we shall observe that the cone can be rounded near an extremal ray, but not near an edge. In the second section we will expose a criterion to determine when a subcone generated by a subset of elements of the cone is a perfect face. We will conclude our thesis providing an example of a smooth projective variety in which the conditions required from the criterion are satisfied, in a few words a practical example of a variety in which we can determine the perfect faces of the pseudoeffective cone of divisors.

2.1 Preliminaries of convex geometry

In this section we will first of all recall basic definitions about convex geometry, which will be useful throughout the entire chapter. Then we will discuss a criterion to determine when a subcone of a given salient cone is a face. This criterion comes directly from a lemma which allows us to decompose a generic element of a cone with respect to a certain subset of vectors. In the following discussion V will always be a finite-dimensional $\mathbb{R}$-vector space.

Definition 2.1.1 (Convex cone). A subset $K \subseteq V$ is a *convex cone* if $\forall\ x, y \in K$ and $\forall\ \alpha, \beta \geq 0$ then $\alpha x + \beta y \in K$.

Definition 2.1.2 (Salient cone). A convex cone $K \subseteq V$ is *salient* if it does not contain subspaces of dimension ≥ 1. In other words, if $x, -x \in K$ then $x = 0$.

Definition 2.1.3 (Cone generated by a subset). Let I be a subset of V. The closed convex *cone generated by* I is

$$K(I) := \overline{\Big\{w \in V \mid w = \sum_{j=1}^{p} a_j v_j \ ,\ p \geq 1, a_j \geq 0, v_j \in I\Big\}}.$$

Given a subset $\{v_1, \dots, v_s\} \subseteq I$ we define the *subcone generated by* $v_1, \dots, v_s$ as follows

$$F(v_1, \dots, v_s) := \Big\{w \in V \mid w = \sum_{j=1}^{s} a_i v_i \ ,\ a_i \geq 0\Big\}.$$

Definition 2.1.4 (Extremal faces and rays). Let $K \subseteq V$ be a closed convex cone. An *extremal face* $F \subseteq K$ is a closed convex subcone having the property that if $v + w \in F$ for some vectors $v, w \in K$, then necessarily $v, w \in F$. An *extremal ray* of K is an extremal face of K of dimension 1.

Lemma 2.1.5. *Let I be a subset of V such that $K(I)$ is salient. Let $v_1, \dots, v_s \in I$, then every $w \in K(I)$ can be written as*

$$w = \sum_{i=1}^{s} a_i v_i + u \tag{2.1}$$

with $a_i \geq 0$ $\forall i$ and $u \in K(I \setminus F(v_1, \dots, v_s))$.

Proof. We can assume $v_i \neq 0$ $\forall i$. Let $w \in K(I)$. Then $w = \lim_{m\to\infty} w_m$, and by Definition 2.1.3 each of the w_m can be decomposed as

$$w_m = \sum_{i=1}^{s} a_{i,m} v_i + \sum_{j=1}^{s_m} b_{j,m} v_{j,m}, \quad v_{j,m} \in I \setminus F(v_1, \dots v_s),\ a_{j,m}, b_{j,m} \geq 0\ \forall j, m.$$

All we need to show is that the limits of the $a_{i,m}$ exist, the Lemma will follow by setting $a_i = \lim_m a_{i,m}$ and $u = \lim_m u_m$, with $u_m := \sum_{j=1}^{s_m} b_{j,m} v_{j,m}$. Observe that, choosing any $v_k \in \{v_1, \dots v_s\}$, w_m can be rewritten as

$$w_m = a_{k,m} v_k + u'_m \quad u'_m \in K(I),\ a_k \geq 0.$$

Since $K(I)$ is salient, we have that $K(I)^*$ spans V^* (cf. [D, Lemma 6.7 (a)]) and then there exists a linear function $\Phi : V \to \mathbb{R}$ such that $\Phi(z) > 0$ $\forall z \in K(I), z \neq 0$. This lets us prove that there exists $M > 0$ such that $a_{k,m} \leq M$, $\forall m$. In fact, since $a_{k,m}\Phi(v_i) \geq 0$ and $\Phi(u'_m) \geq 0$ we have that

$$a_{k,m}\Phi(v_k) \leq a_{k,m}\Phi(v_k) + \Phi(u'_m) = \Phi(w_m) \to \Phi(w)$$

and then necessarily $a_{k,m}$ is bounded. Once shown this, we may suppose, switching to subsequences if necessary, that $a_{k,m} \to_m a_k \geq 0$. We can repeat this argument for any i to conclude the proof.

□

Corollary 2.1.6. *Consider a subset $I \subseteq V$ such that $K := K(I)$ is salient. Let $v_1, \dots, v_s \in I$, and suppose there exist s linear functions $\sigma_i : V \to \mathbb{R}$ such that*

(a) $\sigma_i(v_j) \begin{cases} = 0 & \textit{if} \ \ i \neq j \\ < 0 & \textit{if} \ \ i = j \end{cases}$

(b) $\sigma_i(w) \geq 0 \ \forall w \in I \setminus F(v_1, \dots, v_s)$ *and* $\forall i$.

Then $F(v_1, \dots, v_s)$ *is an extremal face of* K.

Proof. To prove that $F(v_1, \dots, v_s)$ is an extremal face for K we need to show that chosen $z_1, z_2 \in K$ such that $z_1 + z_2 = \sum_{j=1}^{s} \alpha_j v_j$ with $\alpha_j \geq 0 \ \forall j$ then $z_1, z_2 \in F(v_1, \dots, v_s)$. Since $z_1, z_2 \in K$, thanks to Lemma 2.1.5 we can write

$$z_i = z_i' + \sum_{k=1}^{s} a_{k,i} v_k,$$

with $a_{k,i} \geq 0, \ z_i' \in K(I \setminus F(v_1, \dots, v_s))$. We can then rewrite $z_1 + z_2 = \sum_{k=1}^{s} \alpha_k v_k$ as

$$z_1 + z_2 = \sum_{k=1}^{s} a_{k,1} v_k + z_1' + \sum_{k=1}^{s} a_{k,2} v_k + z_2' = \sum_{k=1}^{s} \alpha_k v_k,$$

having then

$$z_1' + z_2' = \sum_{k=1}^{s} (\alpha_k - a_{k,1} - a_{k,2}) v_k. \tag{2.2}$$

Observe that, thanks to (b), $\sigma_k(z_i') \geq 0 \ \forall i, k$, so $\forall k'$ one has, applying (a) and (2.2),

$$0 \leq \sigma_{k'}(z_1' + z_2') = \sigma_{k'}\Big(\sum_{k=1}^{s} (\alpha_k - a_{k,1} - a_{k,2}) v_k \Big) = (\alpha_{k'} - a_{k',1} - a_{k',2}) \sigma_{k'}(v_{k'}).$$

Again by (a) holds $\sigma_{k'}(v_{k'}) < 0$, so it must be $\alpha_k - a_{k,1} - a_{k,2} \leq 0 \ \forall k$.
On the other hand, using (2.2) again and taking $\Phi \in \mathrm{Int}(K(I)^*)$, the conditions

$$0 \leq \Phi(z_1' + z_2') = \sum_{k=1}^{s} (\alpha_k - a_{k,1} - a_{k,2}) \Phi(v_k) \qquad \text{and} \qquad \Phi(v_k) > 0 \quad \forall k$$

let us conclude that $\alpha_k - a_{k,1} - a_{k,2}$ must be 0 $\forall k$. So $z_1' + z_2' = 0$ and this implies, since K is a salient cone, that $z_1' = z_2' = 0$, which gives us the Corollary. □

2.2 Perfect faces of convex cones

This section is completely dedicated to perfect faces of convex cones. Once given the definitions, we will first show that a perfect face of dimension 1, which is called an edge, is also an extremal ray for the cone. This may not be true for perfect faces of higher dimension, that is why we will in general assume that perfect faces are also faces. The rest of the section contains a result which gives conditions for a subcone generated by a set of vectors to be a perfect face of a salient cone. In the next section we will then see how this result of convex geometry can be applied to algebraic geometry. In the following, as in the previous section, we set V as a $\mathbb{R}$-vector space of dimension N, with $N < +\infty$.

Definition 2.2.1 (Edges and perfect faces of a convex cone). Given a convex cone $K \subseteq V$ and a face $F \subseteq K$, we say that F is a *perfect face of* K if $F = K$ or if there exist linear functions $h_1, \dots, h_c$, where $c \geq 1$ is the codimension of F in V, so that

$$\langle F \rangle = \bigcap_{i=1}^{c} \{h_i = 0\} \qquad \text{and} \qquad K \subseteq \bigcap_{i=1}^{c} \{h_i \geq 0\}. \tag{2.3}$$

If F is one-dimensional and perfect, then F is called an *edge* for K.

We shall observe that in general the definition above cannot be weakened. In fact, if we take F as a subcone of dimension 1 which satisfies (2.3) this automatically implies that it is an extremal ray (the proof is contained in the next lemma), but the same is not true in higher dimensions. Since (2.3) is not sufficient to guarantee that the subcone will be a face, we give it as a requirement in the general definition of perfect face. See 2.2.3 for a counter-example.

Lemma 2.2.2. *Let K be a convex cone in V, and let $v \in K$ be an element so that*

$$\langle v \rangle = \bigcap_{i=1}^{N-1} \{h_i = 0\} \qquad \text{and} \qquad K \subseteq \bigcap_{i=1}^{N-1} \{h_i \geq 0\},$$

then either $-v \in K$ and $\mathbb{R}v$ is extremal, or $-v \notin K$ and $\mathbb{R}^{\geq 0}v$ is extremal.

Proof. Suppose at first that $-v \in K$. Then what we need to prove is that given $z_1, z_2 \in K$ with $z_1 + z_2 = \alpha v,\ \alpha \in \mathbb{R}$ then $z_1, z_2 \in \mathbb{R}v$. Applying the functions h_i we have $0 = \alpha h_i(v) = h_i(z_1) + h_i(z_2)$ for each $i \leq N-1$. Since $z_i \in K,\ h_i(z_i) \geq 0$ by Definition 2.2.1. This means that necessarily $h_i(z_1) = h_i(z_2) = 0,\ \forall i$. Hence, thanks again to Definition 2.2.1 $z_1, z_2 \in \mathbb{R}v$ as desired. Now suppose that $-v \notin K$. Then what we need to prove is that given $z_1, z_2 \in K$ with $z_1 + z_2 = \alpha v,\ \alpha \geq 0$ then $z_1, z_2 \in \mathbb{R}^{\geq 0}v$. Repeating the steps of the first case we obtain once again $z_1, z_2 \in \mathbb{R}v$. We conclude observing that $z_1, z_2 \in K \cap \mathbb{R}v = \mathbb{R}^{\geq 0}v$. ☐

Example 2.2.3. Suppose that V has dimension ≥ 3. Let $K \subseteq V$ be a convex cone with triangular section and let the origin be its vertex. Choose two linearly independent vectors $v_1, v_2 \in F$, with F being one of the faces of K. Suppose also that v_1, v_2 are not extremal. In this setting there exist $h_1, \dots, h_{N-2}$ linear functions which satisfy (2.3) with respect to $F(v_1, v_2)$. At the same time, according to our construction, $F(v_1, v_2)$ is not a face of K.

Proposition 2.2.4. *Let $K \subseteq V$ be a closed convex cone of dimension ≥ 2. Then K is salient if and only if there exists an affine hyperplane $H \subseteq V$ such that $0 \notin H$ and $H \cap K$ is nonempty and bounded.*

Proof. Suppose K is salient in the first place. As seen in the proofs of the previous section, since K is salient there exists a linear function $\Phi : V \to \mathbb{R}$ such that $\Phi(z) > 0$ $\forall z \in K, z \neq 0$, according to [D, Lemma 6.7 (a)]. We define $H := \{w \in V \mid \Phi(w) = 1\}$. With this construction $0 \notin H$ and $H \cap K \neq \emptyset$. In fact, given $v \in K,\ v \neq 0$, we have $\Phi(v) > 0$ so if we set $\lambda = \frac{1}{\Phi(v)}$, clearly $\lambda v \in H \cap K$. At this point only the boundedness of $H \cap K$ is left to be seen. Suppose by absurd that there exists a sequence $w_m \in H \cap K$ such that $\|w_m\| \to +\infty$. This implies that there exists $m_0 \in \mathbb{N}$ such that if $m \geq m_0$ then $\|w_m\| > 0$, which means

$$\frac{w_m}{\|w_m\|} \in S^{N-1} \quad \forall m \geq m_0.$$

This sequence has a subsequence that admits a limit $w \in S^{N-1}$. We reach a contradiction by observing that

$$\Phi(w) \leftarrow \Phi\Big(\frac{w_m}{\|w_m\|}\Big) = \frac{1}{\|w_m\|} \to 0,$$

and then $\Phi(w) = 0$. This implies that $w = 0$, and this is absurd since we had $w \in S^{N-1}$.
Now assume that there exists an affine hyperplane $H \subseteq V$ such that $0 \notin H$ and $H \cap K$ is nonempty and bounded. Suppose by absurd that K is not salient, then there exists $x \neq 0$ such that $x, -x \in K$. We will reach the absurd by finding an unbounded sequence of points which lies in $H \cap K$ in every possible case. Without loss of generality we can represent $H = \{w \in V \mid g(w) = 1\}$, with $g : V \to \mathbb{R}$ a linear function. Suppose at first that $g(x) = 0$. Since $H \cap K \neq \emptyset$, we can choose a point $y \in K$ with $g(y) = 1$. Consider the sequence $y + \lambda x,\ \lambda \geq 0$. Observe that $g(y + \lambda x) = g(y) + \lambda g(x) = 1$, and since $\lambda \geq 0$ then $y + \lambda x \in H \cap K,\ \forall \lambda$. Moreover

$$\|y + \lambda x\|^2 = \lambda^2\|x^2\| + 2\lambda\langle x, y\rangle + \|y^2\| \to +\infty$$

which gives us the absurd. Now, we have obtained $g(x) \neq 0$. This implies that there exists a multiple λx with $g(\lambda x) = 1$, since $x, -x \in K$, and we replace x with it in the following. We now claim that we can find an element $y \in H \cap K$ linearly independent from x. In fact, since K has dimension ≥ 2, we can choose $y \in K$ linearly independent from x. If $g(y) = 0$, then we can consider the sequence $\lambda y + x,\ \lambda \geq 0$. This sequence lies in $H \cap K$ and

$$\|\lambda y + x\|^2 = \lambda^2\|y\|^2 + 2\lambda\langle x, y\rangle + \|x\|^2 \to +\infty,$$

it then contradicts the boundedness of $H \cap K$. If instead we assume that $g(y) < 0$, we will consider the sequence $\lambda_1 x + \lambda_2 y$, with $\lambda_1 > 1$ and $\lambda_2 = \frac{\lambda_1 - 1}{|g(y)|}$. This sequence lies in $H \cap K$ and

$$\Big\|\lambda_1 x + \frac{\lambda_1 - 1}{|g(y)|}y\Big\|^2 = \lambda_1^2\Big\|x + \frac{y}{|g(y)|}\Big\|^2 - 2\frac{\lambda_1}{|g(y)|}\langle x + \frac{y}{|g(y)|}, y\rangle + \frac{1}{|g(y)|^2}\|y\|^2 \to +\infty.$$

This is again a contradiction. We are now sure that there exists an element $y \in K$ linearly independent from x with a multiple y' which lies in $H \cap K$. We have then $g(y') = 1$ and $g(x) = 1$. Consider the points $\lambda_1 y' + \lambda_2(-x)$, with $\lambda_1 > 1$ and $\lambda_2 = \lambda_1 - 1$. Since $-x \in K$, these points still lie in K, but this means that once again we have found an unbounded sequence also lying in H, because

$$g(\lambda_1 y' + \lambda_2(-x)) = \lambda_1 - \lambda_1 + 1 = 1,$$

and

$$\|\lambda_1 y' - (\lambda_1 - 1)x\|^2 = \lambda_1^2\|y' - x\|^2 + 2\lambda_1\langle y' - x, x\rangle + \|x\|^2 \to +\infty.$$

□

Lemma 2.2.5. *Consider a subset $I \subseteq V$ such that $K := K(I)$ is salient. Let $v_1, \dots, v_s \in I$, and suppose there exist s linear functions $\sigma_i : V \to \mathbb{R}$ such that*

(a) $\sigma_i(v_j) \begin{cases} = 0 & \text{if } i \neq j \\ < 0 & \text{if } i = j \end{cases}$

(b) $\sigma_i(w) \geq 0 \ \forall w \in I \setminus F(v_1, \dots, v_s)$ *and* $\forall i$.

Then $F(v_1, \dots, v_s)$ is a perfect face of $K(I)$.

Proof. Before we begin, note that we are under the same hypotheses of Corollary 2.1.6 and then $F(v_1, \dots, v_s)$ is a face for $K(I)$. From now on we will prove that $F(v_1, \dots, v_s)$ is perfect. In the first place, we should observe that the vectors $v_1, \dots v_s$ are linearly independent, and so are the linear functions $\sigma_1, \dots, \sigma_s$. To see this, consider a linear combination $\sum_i a_i v_i = 0$. Apply each of the σ_i to this combination to obtain, thanks to (a),

$$a_i \sigma_i(v_i) = 0 \ \forall i.$$

Again by (a) we have $\sigma_i(v_i) < 0$ so it must be $a_i = 0, \ \forall i$. In almost the same fashion we consider a linear function $\sum_i b_i \sigma_i = 0$. Apply this function to each of the v_i to get once again by (a)

$$b_i \sigma_i(v_i) = 0 \ \forall i$$

and conclude as above. Now the first case to discuss is when $s = N$. In this situation $\langle v_1, \dots, v_s \rangle = V$ and we would like to prove that $F(v_1, \dots, v_s) = K(I)$. The Lemma will then follow from Definition 2.2.1. Obviously $F(v_1, \dots, v_s) \subseteq K(I)$ since it is a subcone of $K(I)$. What we need to show is that $K(I) \subseteq F(v_1, \dots, v_s)$. Consider $w \in K(I)$. Using Lemma 2.1.5 along with the fact that the v_i are a basis for V we have

$$\sum_{j=1}^{s} \beta_j v_j = w = \sum_{j=1}^{s} \alpha_j v_j + u,$$
$$\beta_j \in \mathbb{R}, \ \alpha_j \geq 0, \ u \in K(I \setminus F(v_1, \dots, v_s)).$$

We can then rewrite u as

$$u = \sum_{j=1}^{s} (\beta_j - \alpha_j) v_j.$$

Since $u \in K(I \setminus F(v_1, \dots, v_s))$ we have from (b) that $\sigma_j(u) \geq 0, \ \forall j$ and then, thanks to (a),

$$(\beta_j - \alpha_j)\sigma_j(v_j) \geq 0 \qquad \forall j.$$

This means, again by (a), that $\beta_j - \alpha_j \leq 0, \ \forall j$. On the other hand, since $K(I)$ is salient there exists a linear function $\Phi : V \to \mathbb{R}$ such that $\Phi(z) > 0 \ \forall z \in K(I), z \neq 0$, according to [D, Lemma 6.7 (a)]. Then it must be $\Phi(u) \geq 0$ and $\Phi(v_j) > 0, \ \forall j$. In other words

$$\sum_{j=1}^{s} (\beta_j - \alpha_j)\Phi(v_j) \geq 0, \quad \Phi(v_j) > 0, \ \forall j.$$

This lets us conclude that $\beta_j - \alpha_j = 0,\ \forall j$ and then $u = 0$. This proves that given any $w \in K(I)$,

$$w = \sum_{j=1}^{s} \alpha_j v_j \in F(v_1, \dots, v_s),$$

which means that $K(I) \subseteq F(v_1, \dots, v_s)$. We move on to the case $s < N$. We assume throughout that I contains at least two points that are not multiples of each other, since the lemma is clear when I is a ray or line. With this assertion along with the fact that $K(I)$ is salient, it can also be supposed that there exists an affine hyperplane $H \subseteq V$ such that $0 \notin H$ and $H \cap K(I)$ is nonempty and bounded, which is the statement of Lemma 2.2.4. Without loss of generality we can set $H := \{z \in V \mid \Phi(z) = 1\}$, with $\Phi \in K(I)^*$ which is nonempty for [D, Lemma 6.7 (a)] as seen before. For any $x \in I,\ x \neq 0$, some positive multiple of x lies in H, since setting $\lambda = \frac{1}{\Phi(x)}$, we have $\lambda x \in H$ and $\lambda > 0$ thanks to the properties of Φ. Let $B := K(I) \cap H$. We have shown that B is closed, bounded, and convex, and that

$$K(I) = \{\lambda x \mid x \in B, \lambda \geq 0\}.$$

Now let T denote the subspace $\{y \in V \mid \sigma_i(y) = 0\ \forall i\} \subseteq V$, where σ_i are supplied by (a). Since $s < N$, T is non-zero and it is $(N - s)$-dimensional because, as argued before, all the σ_i are linearly independent.

Claim 2.2.6. *Let c denote the codimension of $F(v_1, \dots, v_s)$ in V. There exists a basis $\{v_1, \dots, v_s, x_1, \dots, x_c\}$ for V so that, if we let h_i denote the coordinate function naturally associated to the element x_i of the basis $\{v_j, x_i\}$:*

(i) *$x_i \in T$ for $1 \leq i \leq c$.*

(ii) *$B \cap T \subseteq \bigcap_{i=1}^{c} \{h_i \geq 0\}$.*

Let us assume Claim 2.2.6. With notation as above, we have that

$$I \subseteq \bigcap_{i=1}^{c} \{h_i \geq 0\}.$$

Indeed, let $y \in F(v_1, \dots, v_s)$, with $y = \sum_{j=1}^{s} \gamma_j v_j$. Since $\{\sigma_1, \dots \sigma_s, h_1, \dots, h_c\}$ is the dual basis of the basis defined in the claim above, we have $h_i(v_j) = 0\ \forall i, j$ and then

$$h_i(y) = \sum_{j=1}^{s} \gamma_j h_i(v_j) = 0 \quad \forall i. \tag{2.4}$$

This in particular implies that $y \in \cap_{i=1}^{c} \{h_i \geq 0\}$. Now let $y \in I \setminus F(v_1, \dots, v_s)$. Then

$$y = -\sum_{j=1}^{s} \alpha_j v_j + \sum_{i=1}^{c} a_i x_i$$

for some uniquely determined coefficients α_j, a_i. Moreover

$$\sigma_j(y) = -\alpha_j \sigma_j(v_j) \geq 0 \qquad \forall j$$

by assumption (b). Since $\sigma_j(v_j) < 0$, we must have $\alpha_j \geq 0\ \forall j$. Then $\sum_{j=1}^s \alpha_j v_j + y = \sum_{i=1}^c a_i x_i \in K(I) \cap T$, so that $a_i \geq 0$ by choice of the basis x_i in Claim 2.2.6 (ii). This shows that

$$K(I) \subseteq \bigcap_{i=1}^{c} \{h_i \geq 0\}.$$

To prove that $F(v_1, \ldots, v_s)$ is a perfect face we still need to see

$$\bigcap_{i=1}^{c} \{h_i = 0\} = \langle v_1, \ldots, v_s \rangle. \tag{2.5}$$

$(\supseteq)$: Consider $y \in \langle v_1, \ldots, v_s \rangle$. As seen in (2.4) (the sign of the coefficients does not matter) we have $h_i(y) = 0\ \forall i$.
$(\subseteq)$: Consider $y \in \bigcap_{i=1}^c \{h_i = 0\}$. Since $\{\sigma_1, \ldots \sigma_s, h_1, \ldots, h_c\}$ is the dual basis of the basis $\{v_1, \ldots, v_s, x_1, \ldots, x_c\}$ we have that, supposing $y = \sum_{j=1}^s b_j v_j + \sum_{i=1}^c a_i x_i$,

$$a_i = h_i(y) = 0 \quad \forall i.$$

In other words, $a_i = 0\ \forall i$, and then

$$y = \sum_{j=1}^{s} b_j v_j \in \langle v_1, \ldots, v_s \rangle$$

as required.
Now that equality (2.5) is proven, we can conclude that $F(v_1, \ldots, v_s)$ is a perfect face for $K(I)$ as required.
To prove Claim 2.2.6, let $B_0 := B \cap T$. Since $B = K(I) \cap H$, we have that

$$B_0 \subseteq H \cap T = \{x \in T \mid \Phi(x) = 1\},$$

where as before Φ is a linear function in $K(I)^*$ so that $H = \{x \in V \mid \Phi(x) = 1\}$. Now B_0 is nonempty. In fact since $s < N$ we can always find $x \neq 0,\ x \in K(I \backslash F(v_1, \ldots, v_s))$, and then by (a) we have $\sigma_j(x) \geq 0\ \forall j$, so putting $\beta_j = \sigma_j(x)$, $\gamma_j = \sigma_j(v_j)$ we can define

$$x' = x - \sum_{j=1}^{s} \frac{\beta_j}{\gamma_j} v_j \qquad \text{with} \quad \frac{\beta_j}{\gamma_j} \leq 0 \quad \forall j.$$

Clearly $x' \neq 0$, otherwise x would be 0 since $K(I)$ is salient, $x' \in K(I)$ and $\sigma_j(x') = 0$ $\forall j$ so $x' \in T$. As argued previously, some positive multiple of x' lies in H, therefore in B_0. Take $T_0 = T \cap \{w \in V \mid \Phi(w) = 0\} \subseteq T$. This subspace is $(N - s - 1)$-dimensional because if we take the linear function $\sum_{i=1}^s b_i \sigma_i + \gamma \Phi = 0$ and $x \in B_0$ we have $\Phi(x) > 0$ and

$$\Big(\sum_{i=1}^{s} b_i \sigma_i + \gamma \Phi\Big)(x) = \gamma \Phi(x) = 0.$$

This implies that $\gamma = 0$ independently from the choice of the b_i, and in particular that $\sigma_1, \ldots, \sigma_s, \Phi$ are linearly independent, since the linearly independence of the σ_i has already been shown in the first part. Observe that $T_0 \cap B_0 = \emptyset$. Let x_1' be a normal vector to T_0 in T with $\Phi(x_1') > 0$. Let $x_2', \ldots, x_c'$ be a basis for T_0. This means that $x_1', x_2', \ldots, x_c'$ is a basis for T. Now let h_i' denote the coordinate functions associated to this basis, and by boundedness of B_0 we have that for each i, $h'_i(B_0) \subseteq [a_i, b_i]$ for some finite a_i, b_i. Note that by assumption $a_1 > 0$. In fact, taking a generic $u \in B_0 \subseteq T$, we have

$$u = \sum_{i=1}^{c} c_i x'_i$$

and so $h'_1(u) = c_1$. On the other hand

$$1 = \Phi(u) = c_1 \Phi(x'_1),$$

so $c_1\Phi(x'_1) = 1$, and if $c_1 < 0$ this identity would be false, since $\Phi(x'_1) > 0$. Now define new coordinates by $x_i = x'_i$ for $i \geq 2$, and

$$x_1 = x'_1 + \sum_{\substack{i=2 \\ a_i<0}}^{c} \frac{a_i}{a_1} x'_i.$$

If $y \in B_0$, then $y = \lambda_1 x'_1 + \sum_{i=2}^{c} \lambda_i x'_i$ for $\lambda_i \geq a_i$. Substituting to express y with respect to the basis $\{x_i\}$. we obtain

$$y = \lambda_1 x_1 + \sum_{\substack{i=2 \\ a_i<0}}^{c} \Big(-\frac{\lambda_1}{a_1} a_i + \lambda_i \Big) x_i + \sum_{\substack{i=2 \\ a_i \geq 0}}^{c} \lambda_i x_i.$$

All coordinates of vectors in B_0 are non negative with respect to the new basis, since $-\frac{\lambda_1}{a_1} a_i + \lambda_i \geq -\frac{\lambda_1}{a_1} a_i + a_i \geq 0$ for $a_i < 0$. In conclusion, note that $\langle x_1, \dots, x_c \rangle = T$ and by (a) $T \cap \langle v_1, \dots, v_s \rangle = 0$. It is then clear that $x_1, \dots, x_c, v_1, \dots, v_s$ is a basis for V.

□

2.3 Criteria of extremality for cones of divisors

In this last section we will go back to the algebraic geometry point of view. Indeed, our final goal is to apply the results studied in this second chapter to the pseudoeffective cone of a smooth projective variety. Once these applications are shown, we will proceed with some practical examples of projective varieties in which the hypotheses of our theorems hold.

Corollary 2.3.1. *Let X be a smooth projective surface. The class of an irreducible curve on X satisfying $C^2 < 0$ spans an extremal ray of* $\mathrm{Pseff}_1(X)$.

Proof. We find ourselves in the hypotheses of Corollary 2.1.6. We assume that our finite dimensional $\mathbb{R}$-vector space is $N_1(X)$, consider then the cone $\mathrm{Pseff}_1(X)$ in it. It is a well-known fact (cfr. [BFJ, Proposition 1.3]) that the pseudoeffective cone of a smooth projective variety is salient. We can then take an irreducible curve $C \in \mathrm{Pseff}_1(X)$ in place of the $v_1, \dots, v_s$ seen in 2.1.6. What is left to do is to find a linear function $\sigma : N_1(X) \to \mathbb{R}$ with suitable properties. This function can be easily chosen by putting $\sigma(C') = C' \cdot C,\ \forall C' \in N_1(X)$. □

We will now expose a more general result that holds for a generic smooth projective variety X. Begin with a set of irreducible divisors $D_1, \ldots, D_s \in \mathrm{Pseff}^1(X)$ with $D_i \cap D_j = \emptyset$ while $i \neq j$. If there exist irreducible curves $C_1, \ldots, C_s$ so that for each i, $C_i \cdot D_i < 0$ and D_i is covered by irreducible curves numerically equivalent to C_i, then $D_1, \ldots, D_s$ generate a perfect face of the pseudoeffective cone.

Definition 2.3.2 (Collection of curves). A set of curves $\{C_i\}_{i \in I}$ of a scheme X is a *collection* if $C_i \equiv_{\mathrm{num}} C_{i'}$ for all $i \neq i \in I$.

Corollary 2.3.3. *Let X be a smooth projective variety. Given irreducible effective divisors $D_1, \ldots, D_s \in \mathrm{Pseff}^1(X)$ with $D_i \cap D_j = \emptyset$ while $i \neq j$ and collections of curves $\{C_{i,t}\}_{t \in T_i}$ so that for each i we have $\cup_{t \in T_i} C_{i,t} = D_i$ and that $C_{i,t} \cdot D_i < 0$ for some $t \in T_i$, then the divisors $D_1, \ldots, D_s$ generate a perfect face of the pseudoeffective cone of X.*

Proof. Our purpose is to apply Lemma 2.2.5 to $\mathrm{Pseff}^1(X)$. Choose $i \in \{1, \ldots, s\}$, fix $t_{i,1} \in T$ and consider the function

$$\sigma_i : N^1(X) \to \mathbb{R}$$
$$B \mapsto B \cdot C_{t_{i,1}}.$$

By hypothesis $\sigma_i(D_i) < 0\ \forall i$, moreover $\sigma_i(D') \geq 0$ for any irreducible divisor $D' \neq D_i$. In fact, if there existed D' such that $\sigma_i(D') < 0$, this would mean that $C_{i,t} \cdot D' < 0$ $\forall t \in T_i$ since $\{C_{i,t}\}_t$ is a collection. But this would also imply that $C_{i,t} \subseteq D'$ for all $t \in T_i$, and then $D_i = \cup_t C_{i,t} \subseteq D'$. Since D' is irreducible, it can not contain another irreducible divisor, so in this case we would have $D_i = D'$. This argument can evidently be repeated for each i. Moreover, since $D_i \cap D_j = \emptyset$ if $i \neq j$ we can also conclude that $\sigma_i(D_j) = 0$ if $i \neq j$. In other words the requirement (a) of Lemma 2.2.5 is satisfied by the σ_i. On the other hand, it is a well-known fact that the pseudoeffective cone is salient (cfr. [BFJ, Proposition 1.3]), so (b) is also satisfied. We can then apply Lemma 2.2.5 and state that $D_1, \ldots, D_s$ generate a perfect face of $\mathrm{Pseff}^1(X)$.

□

We conclude our work with a practical example of a smooth algebraic variety of arbitrary dimension in which the hypotheses of Corollary 2.3.3 are satisfied. Before introducing this example, we will recall for completion the standard definitions about blowing-ups. We will first of all impose a condition which is required from now on.

Condition 2.3.4. Let X be a noetherian scheme, and let $\mathscr{S}$ be a quasi-coherent sheaf of $\mathscr{O}_X$-modules, which has a structure of a sheaf of graded $\mathscr{O}_X$-algebras. Thus $\mathscr{S} \cong \bigoplus_{d \geq 0} \mathscr{S}_d$, where $\mathscr{S}_d$ is the homogeneous part of degree d. We assume furthermore that $\mathscr{S}_0 = \mathscr{O}_X$, that $\mathscr{S}_1$ is a coherent $\mathscr{O}_X$-module, and that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathscr{O}_X$-algebra.

Definition 2.3.5 (Projective space bundle). Let X be a noetherian scheme, and let $\mathscr{E}$ be a locally free coherent sheaf on X. We define the associated *projective space bundle* $\mathbf{P}(\mathscr{E})$ as follows. Let $\mathscr{S} = \mathscr{S}(\mathscr{E})$ be the symmetric algebra of $\mathscr{E}$, $\mathscr{S} = \bigoplus_{d \geq 0} S^d \mathscr{E}$. Then $\mathscr{S}$ is a sheaf of graded $\mathscr{O}_X$-algebras satisfying Condition 2.3.4, and we define $\mathbf{P}(\mathscr{E}) = \mathbf{Proj}(\mathscr{S})$. As such, it comes with a projection morphism $\pi : \mathbf{P}(\mathscr{E}) \to X$ and a line bundle $\mathscr{O}(1)$. Check [H, p.160] for the details about the construction of $\mathbf{Proj}(\mathscr{S})$.

Definition 2.3.6 (Blowing-ups). Let X be a noetherian scheme, and let $\mathscr{I}$ be a coherent sheaf of ideals on X. Consider the sheaf of graded algebras $\mathscr{S} = \bigoplus_{d\geq 0} \mathscr{I}^d$, where $\mathscr{I}^d$ is the dth power of the ideal $\mathscr{I}$, and we set $\mathscr{I}^0 = \mathscr{O}_X$. Then $X, \mathscr{S}$ clearly satisfy Condition 2.3.4, so we can consider $\mathrm{Bl}_{\mathscr{I}}(X) = \mathbf{Proj}(\mathscr{S})$. We define $\mathrm{Bl}_{\mathscr{I}}(X)$ to be the *blowing-up* of X *with respect to the coherent sheaf of ideals* $\mathscr{I}$. If Y is the closed subscheme of X corresponding to $\mathscr{I}$, then we also write $\mathrm{Bl}_Y(X)$ and we call it the *blowing-up* of X along Y or *with center* Y.

Definition 2.3.7 (Exceptional divisors). Let X be a nonsingular variety over k, and let $Y \subseteq X$ be a nonsingular closed subvariety, with ideal sheaf $\mathscr{I}$. Let $\pi : \mathrm{Bl}_{\mathscr{I}}(X) \to X$ be the blowing-up of $\mathscr{I}$. We call *exceptional divisor* the subscheme $Y' \subseteq \mathrm{Bl}_{\mathscr{I}}(X)$ defined by the inverse image ideal sheaf $\mathscr{I}' = \pi^{-1}\mathscr{I} \cdot \mathscr{O}_{\mathrm{Bl}_{\mathscr{I}}(X)}$.

Example 2.3.8. Let X be a smooth projective variety, and let $Z_1, \dots, Z_s$ be closed smooth subvarieties of X of arbitrary dimensions. Suppose also that $Z_i \cap Z_j = \emptyset$ if $i \neq j$. Now consider the blowing-up of X with center $Z_1 \cup \dots \cup Z_s$, and let $E_1, \dots, E_s$ be the exceptional divisors generated by the Z_i, giving the diagram

$$\begin{array}{ccc} \mathrm{Bl}_{\cup_{i=1}^s Z_i}(X) & \xrightarrow{\varphi} & X \\ \cup\!\shortmid & & \cup\!\shortmid \\ E_i & \longmapsto & Z_i \end{array}$$

In this situation, $E_i \cap E_j = \emptyset$ if $i \neq j$ since $\varphi^{-1}(Z_i) = E_i$ $\forall i$. We should also recall that $E_i \cong \mathbb{P}(N^*_{Z_i}) = \mathbf{P}(\mathscr{E}_i) =: P_i$ and that $E_{i|E_i} \cong \mathscr{O}_{P_i}(-1)$ (cfr. [H, II,8.24]). This property lets us found families of irreducible curves $\{C_{i,j}\}_{j\in J}$, with $\cup_{j\in J} C_{i,j} = E_i$ and $C_{i,j} \cdot E_i < 0$ $\forall j$. The first assertion is clear if we regard the E_i as unions of projective spaces and we pick lines in the inverse images of the points of each Z_i. The second one can be seen in the following manner:

$$C_{i,j} \cdot E_i = C_{i,j} \cdot E_{i|E_i} = C_{i,j} \cdot \mathscr{O}_{P_i}(-1) < 0.$$

We now find ourselves in the hypotheses of Corollary 2.3.3. This means that the E_i generates a perfect face of the pseudoeffective cone of $\mathrm{Bl}_{\cup_{i=1}^s Z_i}(X)$.

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