UNIVERSITÀ DEGLI STUDI ROMA TRE
FACOLTÀ DI SCIENZE MM. FF. NN.

Graduation Thesis in Mathematics
by

Gabriele Nocco

Zariski Decomposition of Pseudo-Effective Divisors

Supervisor

Prof. Angelo Felice Lopez

The Candidate   The Supervisor

ACADEMIC YEAR 2008 - 2009
9 JULY 2009

AMS Subject Class: 14C20, 14J99
Key Words: Divisors, pseudo-effective, Zariski decomposition

A chi ha sempre
vegliato su di me

Introduction

The main concept in the algebraic geometry is the concept of *algebraic variety.* We can consider an algebraic variety as the set of such points where a family of polynomials vanishes (if we are in a projective space this family must be homogeneous).

One of the most interesting problems in the algebraic geometry is to study algebraic or geometric invariants of algebraic varieties. A usefull geometric tool is the group of divisors on a variety X.

For our introduction we need to define some concepts:

For simplicity suppose that X is a non-singular algebraic variety.

A divisor is an element of the free abelian group generated by the subvarieties of codimension one, then we can think a divisor as a formal finite sum $D = \sum a_i C_i$, where C_i is a subvariety of codimension one and the coefficients a_i are in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ (then we have the group of divisors, $\mathbb{Q}$-divisors or $\mathbb{R}$-divisors).

We define an *effective* divisor, E, if the coefficient a_i is not negative for all i.

At this point it is natural to define an intersection theory. The *intersection number* is a multilinear form associated at divisors that can be defined in various ways (see [3, Appendix A], or [9, chapter IV]). For example if we are on a surface, C and D are two distinct subvarieties of dimension one, we can think $C \cdot D$ as the number of the point of the intersection between C and D.

We say that two divisors D' and D'' are *numerically equivalents* if $D' \cdot C = D'' \cdot C$ for every irreducible curve C. Obviously this is an equivalence relation, then we can take the quotient of the group of divisors by this relation. The resulting space is called the *Neron-Severi* space of X. It is a vector space of finite dimension and we can take it with the Euclidean topology.

This is a very important intrinsic invariant of X.

By the intersection theory we have a simple way to describe some central definitions for this thesis.

An *ample* divisor is a divisor A such that $A^{dimV} \cdot V > 0$ for every subvariety V of positive dimension (Nakai-Moishezon Criterion).

A *nef* divisor is a divisor D such that $D^{dimV} \cdot V \geqslant 0$ for every subvariety V of positive dimension (Kleiman's Theorem).

A *pseudo-effective* class of the Neron-Severi space is a limit class of effective divisors.

The linear series of a divisor D, we write $|D|$, is the set of all effective divisors linearly equivalents to D.

In the first chapter we recall the definition and the basic properties of this notions.

The following problem is fundamental in algebraic geometry:

To study the linear system $|nD|$ for $n \geqslant 1$.

One of the basic costruction in this direction is the Riemann-Roch theorem. It links geometric, as the number of intersection between divisors, and algebraic invariants, as the Euler characteristic, in an unique equation. In this sense it is fundamental.

The Riemann-Roch theorem gives us an important way to study the linear series of a multiple of a divisor, $|mD|$.

To this problem, there is a rather well developed theory in the case of $dimX = 1$.

In the case of $dimX = 2$, in early 60-th, O. Zariski reduced this problem to the case that D is nef.

In his article [1], Zariski solved the Riemann-Roch problem for high multiples of an effective divisor on a smooth projective surface.

Zariski needed a decomposition of an *effective* divisor D into a nef $\mathbb{Q}$-divisor P, and an effective $\mathbb{Q}$-divisor N such that $h^0(nP) = h^0(nD)$ whenever n is a positive integer for which nP is an integral divisor. Such a decomposition is called a *Zariski decomposition.* P and N are called the positive and the negative part of D.

More specifically, Zariski proved the following result for $\mathbb{K} = \mathbb{Q}$:

Theorem 0.0.1 (**Zariski**, Theorem 2.1.3)**.** *Let D be an effective $\mathbb{K}$-divisor on a smooth projective surface X. Then there are uniquely determined effective (possibly zero) $\mathbb{K}$-divisors P and N with $D = P + N$ such that*

1. *P is nef,*
2. *N is zero or has negative definite intersection matrix,*
3. *$P \cdot C = 0$ for every irreducible component C of N.*

A simple proof of the existence and the uniqueness of the Zariski decomposition for effective $\mathbb{Q}$-divisors, is due to Bauer [4] (our section 2.1).

While Zariski built his decomposition working with the negative part, in his article Bauer gives a way to build the positive part of the decomposition using the important property that P_D is the maximal nef subdivisor of D. The theorem is the same as Zariski's, but the proof does not require hard work.

Fujita [5] extends the existence and the uniqueness of the decomposition to *pseudo-effective* $\mathbb{Q}$-divisors (our section 2.2) using hardly the finite dimension of the Neron-Severi space. He proved this result if $\mathbb{K} = \mathbb{Q}$:

Theorem 0.0.2 (**Zariski-Fujita**, Theorem 2.2.8)**.** *Let D be a pseudo-effective $\mathbb{K}$-divisor on a smooth projective surface X. Then there are uniquely determined $\mathbb{K}$-divisors P and N such that $D = P + N$, N is effective (possibly zero) and:*

1. *P is nef,*
2. *N is zero or has negative definite intersection matrix,*
3. *$P \cdot C = 0$ for every irreducible component C of N.*

In this case the proof is based on the construction of the negative part, removing effective divisors as long as it remains a nef divisor.

A simple original result obtained in this thesis is the extention of the costruction of Bauer to an effective $\mathbb{R}$-divisor and of the Fujita's costruction to a pseudo-effective $\mathbb{R}$-divisor, that is we extend theorems 0.0.1 and 0.0.2 in the case $\mathbb{K} = \mathbb{R}$.

Examples show that Zariski decomposition cannot be generalized in higher dimensions without significant modifications.

A lot of higher-dimensional generalizations have been based on two properties of the Zariski decomposition: P_D is nef and $H^0(X, \lfloor kP_D \rfloor) = H^0(X, \lfloor kD \rfloor)$ for all k, sometimes passing through a birational map.

Another possible costruction is the σ-decomposition, due to Nakayama [8]. This costruction is linked to the function σ_Γ defined as follow:

Definition 0.0.3 (Definition 3.1.2)**.** *For a prime divisor Γ and for a big $\mathbb{R}$-divisor B, we define:*

$$\sigma_\Gamma(B) := \inf\{mult_\Gamma E \mid E \in |B|_{num}\}$$

where $|B|_{num}$ is the set of effective $\mathbb{R}$-divisors E satisfying $E \equiv_{num} B$.

For pseudo-effective $\mathbb{R}$-divisors we can extend this function in a simple way:

Definition 0.0.4 (Definition 3.1.11)**.** *For a pseudo-effective $\mathbb{R}$-divisor D and a prime divisor Γ, we define*

$$\sigma_\Gamma(D) := \lim_{\epsilon \longrightarrow 0} \sigma_\Gamma(D + \epsilon A)$$

for all A ample divisor.

Now we define the σ-decomposition in this way:

Definition 0.0.5 (Definition 3.2.1)**.** *Let D be a pseudo-effective $\mathbb{R}$-divisor of a non–singular projective variety X. We define*

$$N_\sigma(D) := \sum \sigma_\Gamma(D)\Gamma,$$

and $P_\sigma(D) := D - N_\sigma(D)$.

The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is called the σ-decomposition of D. Here, $P_\sigma(D)$ and $N_\sigma(D)$ are called the positive and the negative parts of the σ-decomposition of D, respectively.

We note that $N_\sigma(D)$ is well defined by the following fact:

Corollary 0.0.6 (Corollary 3.1.18). *For any pseudo-effective $\mathbb{R}$-divisor D, the number of prime divisors Γ satisfying $\sigma_\Gamma(D) > 0$ is less than the Picard number $\rho(X)$.*

We have that

Definition 0.0.7 (Definition 3.2.6). *The σ-decomposition $D = P_\sigma(D) + N_\sigma(D)$ for a pseudo-effective $\mathbb{R}$-divisor is called a Zariski decomposition if $P_\sigma(D)$ is nef.*

In general the positive part of a σ-decomposition is not a nef divisor, it is a class in the movable cone, but on a surface we have:

Remark 0.0.8 (Remark 3.2.7). *If X is a surface, then the movable cone $\overline{Mv(X)}$ coincides with the nef cone $Nef(X)$. Therefore the σ-decomposition of a pseudo-effective $\mathbb{R}$-divisor D is nothing but the usual Zariski decomposition.*

Then we prove the continuity of this function:

Proposition 0.0.9 (Proposition 3.2.11). *The function $\sigma_\Gamma : \overline{Eff(X)} \to \mathbb{R}_{\geqslant 0}$ for a prime divisor Γ on a non-singular projective surface X is continuous.*

Ringraziamenti

Ci sono proprio tante persone che mi sento di ringraziare alla fine di questo percorso formativo. Come non ringraziare il mio relatore, il Prof. Angelo Felice Lopez, non solo per l'infinita e continua pazienza avuta durante questi ultimi mesi ma soprattutto perché è stato proprio per Lui che mi sono convinto a scegliere gli studi matematici.

Volevo ringraziare in particolar modo il Prof. Ugo Bessi, per avermi convinto a non mollare in un momento molto difficile per me. Se sono arrivato fino a qui, di sicuro, lo devo anche alle Sue parole.

Volevo inoltre ringraziare tutti i professori con i quali, grazie anche al mio impegno da rappresentante degli studenti, ho avuto un rapporto buono e sincero, come il Prof. Bruno, il quale mi ha sempre sostenuto con la Sua fiducia nelle mie capacità, il Prof. Pappalardi, il Prof. Caputo, il Prof. Martinelli, il Prof. Esposito, il Prof. Biasco e l'elenco potrebbe continuare quasi esaurendo il corpo docente del nostro Dipartimento. Volevo ringraziare anche quei pochi professori con i quali il rapporto non è stato semplice, per avermi spinto a migliorare sempre.

Volevo ringraziare ancora tutti i dottorandi che mi sono stati vicini. In particolare penso a Salvatore Cacciola, Lorenzo Di Biagio, Stefano Urbinati, Michele Nesci e alla loro incredibile pazienza nei miei confronti.

Poi come non ringraziare gli studenti. La lista qui potrebbe essere davvero smisurata, in questi anni coloro che mi sono stati vicini sono stati innumerevoli, come Daniele Esposito, con il quale abbiamo ideato il sito autogestito degli studenti; Carlo Ciliberto, per il suo supporto costante; Mario Marchetti, per tutti gli anni in cui la nostra amicizia è stata vissuta giornalmente; Elisa Di Gloria, per il suo appoggio e gli anni condivisi, poi Enzo, Valerio, Fra, Padu, Bizio, Fabione, Filippo, Sandrino, Stefano, Maria Chiara e così via. Mi scuso per tutti quelli che non ho nominato ma in tutti questi anni ho conosciuto, in un modo o nell'altro, tutti gli studenti che mi sono passati vicino, tutti mi sono rimasti dentro e quindi l'elenco potrebbe proprio non finire mai. In particolare volevo ringraziare tutti gli studenti con i quali ho diviso il terreno di gioco e che mi hanno sempre considerato il loro capitano.

Un ringraziamento va anche ai miei amici, Marco, Valerio, Antonio e Francesca per la loro pazienza a sopportare l'amicizia di un matematico.

Ringraziamenti infiniti vanno alla mia famiglia, per il sostegno e la pazienza dimostratemi, so che è stato molto difficile in questi anni sopportare me, le mie lunghe assenze e la mia testa sempre rivolta alla Matematica.

Per chiudere in bellezza, un ringraziamento particolare va a Fabiana Marchetti, per questo ultimo anno in cui ci siamo dati forza a vicenda.

Un grazie veramente a tutti.

Notation and Conventions

Some notation and conventions:

- A *scheme* is an algebraic scheme of finite type over $\mathbb{C}$. A *variety* is a reduced and irreducible scheme. We deal exclusively with closed points of schemes.
- Given a real-valued function $f : \mathbb{N} \to \mathbb{R}$ defined on the natural numbers, we say that $f(m) = O(m^k)$ if
$$\limsup_{n\to\infty} \frac{|f(m)|}{m^k} < \infty$$
- A *divisor* is a Cartier divisor. If D is a $\mathbb{R}$-divisor, we denote by $\lfloor D \rfloor$ the integer part of D and by $\langle D \rangle = D - \lfloor D \rfloor$ the fractional part of D.

Contents

Chapter 1

Ample and Nef Line Bundles

1.1 Divisors, Line Bundles and Classical Theory

We start with a quick review of the definitions and the facts concerning Cartier divisors.

Consider then a complex variety X, and denote by $\mathcal{K}_X = \mathbf{C}(X)$ the (costant) sheaf of rational functions on X. It contains the structure sheaf $\mathcal{O}_X$ as a subsheaf, and hence there is an inclusion $\mathcal{O}_X^* \subseteq \mathcal{K}_X^*$ of sheaves of multiplicative abelian groups.

Definition 1.1.1 (**Cartier divisors**). *A Cartier divisor on X is a global section of the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. We denote by $Div(X)$ the group of all such, hence that*

$$Div(X) = \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

Concretely, then the divisor $D \in \mathrm{Div}(X)$ is represented by data $\{(U_i, f_i)\}$ consisting of an open covering $\{U_i\}$ of X together with elements $f_i \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. f_i is called a *local equation* for D at any point $x \in U_i$.

The group operation on $\mathrm{Div}(X)$ is always written additively: if D, $D' \in \mathrm{Div}(X)$ are represented respectively by data $\{(U_i, f_i)\}$ and $\{(U_i, f_i')\}$, then $D + D'$ is given by $\{(U_i, f_i f_i')\}$

Definition 1.1.2 (**Cycles and Weil divisors**). *Let X be a variety or scheme of pure dimension n. A k-cycle on X is a $\mathbb{Z}$-linear combination of subvarieties of dimension k. The group of all such is written $Z_k(X)$. A*

Weil divisor on X is an $(n-1)$-cycle, that is a formal sum of codimension one subvarieties with integer coefficients. We often use $WDiv(X)$ in place of $Z_{n-1}(X)$ to denote the group of Weil divisors.

Remark 1.1.3. *There is a cycle map*

$$Div(X) \to WDiv(X), \;\; D \longmapsto [D] = \sum ord_V(D)[V]$$

where $ord_V(D)$ is the order of D along a codimension one subvariety. In general this homomorphism is neither injective nor suriective, although it is one-to-one when X is a normal variety and an isomorphism when X is non-singular.

A global section $f \in \Gamma(X, \mathcal{K}_X^*)$ determines in the evident manner a divisor

$$D = div(f) \in Div(X).$$

As usual, a divisor of this form is called *principal* and the subgroup of all such is $Princ(X) \subseteq Div(X)$. Two divisors D_1, D_2 are *linearly equivalent*, written $D_1 \equiv_{lin} D_2$, if $D_1 - D_2$ is principal.

Let D be a divisor on X. Given a morphism $f : Y \to X$, one would like to define a divisor f^*D on Y by pulling back the local equations for D. The following condition is sufficient to guarantee that this is meaningful:

Let $V \subseteq Y$ be any *associated subvariety* of Y, that is the subvariety defined by an associated prime of $\mathcal{O}_Y$ in the sense of primary decomposition. Then $f(V)$ should not be contained in the support of D.

If Y is reduced the requirement is just that no component of Y map into the support of D.

A Cartier divisor $D \in Div(X)$ determines a line bundle $\mathcal{O}_X(D)$ on X, leading to a canonical homomorphism:

$$Div(X) \to Pic(X), \;\; D \longmapsto \mathcal{O}_X(D)$$

of abelian groups, where $Pic(X)$ denotes as usual the Picard group of isomorphism classes of line bundles on X.

Definition 1.1.4. *A divisor D represented by data $\{(U_i, f_i)\}$ is effective if $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$, for all i.*

Definition 1.1.5 (Canonical bundle and divisor). *Let X be a non-singular complete variety of dimension n. We denote by $\omega_X = \Omega_X^n$ the canonical line bundle on X, and by K_X any canonical divisor on X. Thus $\mathcal{O}_X(K_X) = \omega_X$.*

Definition 1.1.6 (Intersection numbers). *Let X be a complete complex variety. Given Cartier divisors $D_1, \ldots, D_k \in Div(X)$ together with a sub-variety $V \subset X$ of dimension k, the intersection number*

$$(D_1 \cdot \ \ldots \ \cdot D_k \cdot V) \in \mathbb{Z}$$

can be defined in various ways (see [2, Appendix A], or [9, chapter IV]).

Definition 1.1.7 (Numerical equivalence). *Two Cartier divisors $D_1, D_2 \in Div(X)$ are numerically equivalent, written $D_1 \equiv_{num} D_2$, if $(D_1 \cdot C) = (D_2 \cdot C)$ for every irreducible curve $C \subset X$, or equivalently if $(D_1 \cdot \gamma) = (D_2 \cdot \gamma)$ for all one-cycles γ on X. Numerical equivalence of line bundles is defined in the analogous manner. A divisor or line bundle is numerically trivial if it is numerically equivalent to zero, and $Num(X) \subseteq Div(X)$ is the subgroup consisting of all numerically trivial divisors.*

Definition 1.1.8 (Neron-Severi group). *The Neron-Severi group of X is the group*

$$\mathcal{N}^1(X) = Div(X)/Num(X)$$

of numerical equivalence classes of divisors on X. The first basic fact is that this group is finitely generated:

Proposition 1.1.9 (Theorem of the base). *The Neron-Severi group $\mathcal{N}^1(X)$ is a free abelian group of finite rank.*

See [3, 1.1.16.].

Definition 1.1.10 (Picard number). *The rank of $\mathcal{N}^1(X)$ is called the Picard number of X, written $\rho(X)$.*

Theorem 1.1.11 (Riemann-Roch for surfaces). *If D is any divisor on the smooth surface X, then*

$$h^0(X, mD) - h^1(X, mD) + h^2(X, mD) = \frac{1}{2} mD \cdot (mD - K_X) + 1 - p_a.$$

Theorem 1.1.12 (Asymptotic Riemann-Roch I). *Let X be a projective variety of dimension n, and let D be a divisor on X. Then the Euler characteristic $\chi(X, \mathcal{O}_X(mD))$ is a polynomial of degree $\leqslant n$ in m, with*

$$\chi(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!}m^n + O(m^{n-1}).$$

Definition 1.1.13 (Base locus and base ideal). *The base ideal of $|V|$, written*

$$b(|V|) = b(X, |V|) \subseteq \mathcal{O}_X$$

is the image of the map $V \otimes_{\mathbb{C}} L^* \to \mathcal{O}_X$ determined by $eval_V$ (see [3, 1.1.B]). The base locus

$$Bs(|V|) \subseteq X$$

of $|V|$ is the closed subset of X cut out by the base ideal $b(|V|)$. When we wish to emphasize the scheme structure on $Bs(|V|)$ determined by $b(|V|)$ we will refer to $Bs(|V|)$ as the base scheme of $|V|$.

Very concretely, then, $Bs(|V|)$ is the set of points at which all the sections in V vanish, and $b(|V|)$ is the ideal sheaf spanned by these sections.

Definition 1.1.14. *We say that $|V|$ is free, or basepoint-free, if its base-locus is empty. A divisor D or line bundle L is free if the corresponding complete linear series is free. In the case of line bundles, we say that L is generated by its global sections or globally generated.*

Definition 1.1.15. *Let $D \in Div(X)$, we can consider D as a Weil divisor on X, then we define $Supp(D)$ in this way:*

$$Supp(D) := \bigcup_i Y_i$$

where the union is over all subvarieties of codimension one Y_i, such that $ord_{Y_i}(D) > 0$.

Definition 1.1.16 (Ample and very ample line bundles and divisors on complete scheme). *Let X be a complete scheme, and L a line bundle on X.*

1. *L is very ample if there exists a closed embedding $X \subseteq \mathbb{P}$ of X into some projective space $\mathbb{P} = \mathbb{P}^N$ such that*

$$L = \mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1)|_X.$$

2. *L is ample if $L^{\otimes m}$ is very ample for some $m > 0$.*

A Cartier divisor D on X is ample or very ample if the corresponding line bundles $\mathcal{O}_X(D)$ is ample or very ample.

1.2 $\mathbb{Q}$-Divisors and $\mathbb{R}$-Divisors

For questions of positivity, it is very useful to be able to discuss small perturbations of a given divisor class. The natural way to do that is through the formalism of $\mathbb{Q}$- and $\mathbb{R}$-divisors, which we develop in this section. As an application, we establish that amplitude is an open condition on numerical equivalence classes.

As one would expect, a $\mathbb{Q}$-divisor is simply a $\mathbb{Q}$-linear combination of integral Cartier divisors:

Definition 1.2.1 (**$\mathbb{Q}$-divisors**)**.** *Let X be an algebraic variety or scheme. A $\mathbb{Q}$-divisor on X is an element of the $\mathbb{Q}$-vector space*

$$Div_{\mathbb{Q}}(X) := Div(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

We represent a $\mathbb{Q}$-divisor $D \in Div_{\mathbb{Q}}(X)$ as a finite sum

$$D = \sum c_i A_i,$$

where $c_i \in \mathbb{Q}$ and $A_i \in Div(X)$. The $\mathbb{Q}$-divisor D is effective if it is of the form $D = \sum c_i A_i$ with $c_i \geqslant 0$ and A_i effective.

Definition 1.2.2 (**Supports**)**.** *Let $D \in Div_{\mathbb{Q}}(X)$ be a $\mathbb{Q}$-divisor, $D = \sum_i c_i A_i$. The support of D is*

$$Supp(D) = \bigcup_i Supp(A_i).$$

All the usual operations and properties of Cartier divisors extend naturally to this setting simply by tensoring with $\mathbb{Q}$.

Definition 1.2.3. *Assume henceforth that X is complete. Given a subvariety or subscheme $V \subset X$ of dimension k, a $\mathbb{Q}$-valued intersection product*

$$Div_{\mathbb{Q}}(X) \times \ldots \times Div_{\mathbb{Q}}(X) \to \mathbb{Q}, \ (D_1, \ldots, D_k) \longmapsto (D_1 \cdot \ \ldots \ \cdot D_k \cdot V)$$

is defined via a extension of scalars from the analogous product on $Div(X)$.

Definition 1.2.4. *Two $\mathbb{Q}$-divisors $D_1, D_2 \in Div_{\mathbb{Q}}(X)$ are numerically equivalent, written $D_1 \equiv_{num} D_2$ (or $D_1 \equiv_{num,\mathbb{Q}} D_2$ when confusion seems possible) if $(D_1 \cdot C) = (D_2 \cdot C)$ for every curve $C \subset X$. We denote by $\mathcal{N}^1(X)_{\mathbb{Q}}$ the resulting finite dimensional $\mathbb{Q}$-vector space of numerical equivalence classes of $\mathbb{Q}$-divisors. Two $\mathbb{Q}$-divisors $D_1, D_2 \in Div_{\mathbb{Q}}(X)$ are $\mathbb{Q}$-linearly equivalent, written $D_1 \equiv_{lin,\mathbb{Q}} D_2$ if there is an integer r such that rD_1 and rD_2 are integral and linearly equivalent in the usual sense, that is if $r(D_1 - D_2)$ is the image of a principal divisor in $Div(X)$.*

Remark 1.2.5. *More concretely, these operations and equivalences are determined from those on integral divisors by writing $D = \sum c_i A_i$ and expanding by linearity. Then for instance $D \equiv_{num,\mathbb{Q}} 0$ if and only if $\sum c_i(A_i \cdot C) = 0$ for every curve $C \subset X$. Note also that there is an isomorphism*

$$\mathcal{N}^1(X)_{\mathbb{Q}} = \mathcal{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Continue to assume that X is complete. The definition of ampleness for $\mathbb{Q}$-divisors likewise presents no problems:

Definition 1.2.6 (Amplitude for Q-divisors). *A $\mathbb{Q}$-divisor $D \in Div_{\mathbb{Q}}(X)$ is ample if any one of the following three equivalent conditions is satisfied:*

1. *D is of the form $D = \sum c_i A_i$ where $c_i > 0$ is a positive rational number and A_i is an ample Cartier divisor.*

2. *There is a positive integer $r > 0$ such that rD is integral and ample.*

3. *D satisfies the statement of Nakai's Criterion, that is*

$$(D^{dimV} \cdot V) > 0$$

for every subvariety $V \subset X$ of positive dimension.

We prove that amplitude is an open condition under small perturbations of a divisor:

Proposition 1.2.7 (**Openness of amplitude for $\mathbb{Q}$-divisors**). *Let X be a projective variety, H an ample $\mathbb{Q}$-divisor on X, and E an arbitrary $\mathbb{Q}$-divisor. Then $H + \epsilon E$ is ample for all sufficiently small rational numbers $0 < |\epsilon| \ll 1$. More generally, given finitely many $\mathbb{Q}$-divisors $E_1, \ldots, E_r$ on X,*

$$H + \epsilon_1 E_1 + \ldots + \epsilon_r E_r$$

is ample for all sufficiently small rational numbers $0 \leqslant |\epsilon_i| \ll 1$.

Proof. Clearing denominators, we may assume that H and each E_i are integral. By taking $m \gg 0$ we can arrange for each of the $2r$ divisors $mH \pm E_1, \ldots, mH \pm E_r$ to be ample. Now provided that $|\epsilon_i| \ll 1$ we can write any divisor of the form

$$H + \epsilon_1 E_1 + \ldots + \epsilon_r E_r$$

as a positive $\mathbb{Q}$-linear combination of H and some of the $\mathbb{Q}$-divisors $H \pm \frac{1}{m} E_i$. But a positive linear combination of ample $\mathbb{Q}$-divisors is ample.

□

The definition of $\mathbb{R}$-divisors proceeds in an exactly analogous fashion.

Definition 1.2.8. *Let X be an algebraic variety or scheme. A $\mathbb{R}$-divisor on X is an element of the $\mathbb{R}$-vector space:*

$$Div_{\mathbb{R}}(X) := Div(X) \otimes \mathbb{R}$$

of $\mathbb{R}$-divisors on X. An $\mathbb{R}$-divisor is represented by a finite sum $D = \sum c_i A_i$ where $c_i \in \mathbb{R}$ and $A_i \in Div(X)$.

Definition 1.2.9. *We say that D is effective if $D = \sum c_i A_i$ with $c_i \geqslant 0$ and A_i effective.*

Definition 1.2.10. *Let X be a complete variety, there is an associated $\mathbb{R}$-valued intersection theory, giving rise in particular to the notion of numerical equivalence.*

The resulting vector space of equivalence classes is denoted by $\mathcal{N}^1(X)_{\mathbb{R}}$ *(there is a natural isomorphism* $\mathcal{N}^1(X)_{\mathbb{R}} = \mathcal{N}^1(X) \otimes \mathbb{R}$*, see [3, 1.3.10]).*

Let $D \in Div_{\mathbb{R}}(X)$ *be a* $\mathbb{R}$*-divisor, we define* $[D] := \{E \in Div_{\mathbb{R}}(X) | \ E \equiv_{num} D\} \in \mathcal{N}^1(X)_{\mathbb{R}}$.

D *is numerically trivial if and only if* $[D] = [0]$.

Definition 1.2.11 (Amplitude for $\mathbb{R}$-divisors). *Assume that* X *is complete. An* $\mathbb{R}$*-divisor* D *on* X *is ample if it can be expressed as a finite sum* $D = \sum c_i A_i$ *where* $c_i > 0$ *is a positive real number and* A_i *is an ample Cartier divisor. Observe that a finite positive* $\mathbb{R}$*-linear combination of ample* $\mathbb{R}$*-divisors is therefore ample*

Proposition 1.2.12 (Ample classes for $\mathbb{R}$-divisors). *The amplitude of an* $\mathbb{R}$*-divisor depends only upon its numerical equivalence class.*

Proof. It is sufficient to show that if D and B are $\mathbb{R}$-divisors, with D ample and $B \equiv_{num} 0$, then $D+B$ is again ample. To this end, observe first that B is an $\mathbb{R}$-linear combination of numerically trivial integral divisors. Indeed, the condition that an $\mathbb{R}$-divisor

$$B = \sum r_i B_i \ \ (r_i \in \mathbb{R}, \ B_i \in Div(X))$$

be numerically trivial is given by finitely many integer linear equations on the r_i, determined by integrating over a set of generators of the subgroup of $H_2(X, \mathbb{Z})$ spanned by algebraic 1-cycles on X. The assertion then follows from the fact that any real solution to these equations is an $\mathbb{R}$-linear combination of integral ones.

We are now reduced to showing that if A and B are integral divisors, with A ample and $B \equiv_{num} 0$, then $A + rB$ is ample for any $r \in \mathbb{R}$. If r is rational we already know this. In general, we can fix rational numbers $r_1 < r < r_2$, plus a real number $t \in [0,1]$ such that $r = tr_1 + (1-t)r_2$. Then

$$A + rB = t(A + r_1 B) + (1-t)(A + r_2 B)$$

which exhibits $A+rB$ as a positive $\mathbb{R}$-linear combination of ample $\mathbb{Q}$-divisors.

□

Remark 1.2.13 (Openness of amplitude for $\mathbb{R}$-divisors). *The statement of Proposition 1.2.7 remains valid for $\mathbb{R}$-divisors. In other words:*

Let X be a projective variety and H an ample $\mathbb{R}$-divisor on X. Given finitely many $\mathbb{R}$-divisors $E_1, \ldots, E_r$, the $\mathbb{R}$-divisor

$$H + \epsilon_1 E_1 + \ldots + \epsilon_r E_r$$

is ample for all sufficiently small real numbers $0 \leqslant |\epsilon_i| \ll 1$.

Definition 1.2.14 (Nef divisors). *Let X be a complete variety or scheme. A Cartier divisor D on X (with $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ coefficients) is nef if for every irreducible curve $C \subset X$ we have $(D \cdot C) \geqslant 0$*

The definition evidently only depends on the numerical equivalence class of D, and hence one has a notion of nef classes in $\mathcal{N}^1(X)$, $\mathcal{N}^1(X)_{\mathbb{Q}}$ and $\mathcal{N}^1(X)_{\mathbb{R}}$. Note that any ample class is nef, as is the sum of two nef classes.

Theorem 1.2.15 (Kleiman's Theorem). *Let X be a complete variety (or scheme). If D is a nef $\mathbb{R}$-divisor on X, then*

$$(D^k \cdot V) \geqslant 0$$

for every subvariety $V \subset X$ of dimension k.

Proof. One can assume that X is reduced (and irreducible if X is a scheme), and replacing it by a Chow cover one can assume in addition that X is projective. We proceed by induction on $n = dimX$, the assertion being evident if X is a curve. We therefore suppose that $(D^k \cdot V) \geqslant 0$ for all $V \subset X$ of dimension $\leqslant n-1$, and the issue is to show that $(D^n) \geqslant 0$. Until further notice we suppose that D is a $\mathbb{Q}$-divisor: the argument reducing the general case to this one appears at the end of the proof.

Fix an ample divisor H on X, and consider for $t \in \mathbb{R}$ the self-intersection number

$$P(t) := (D + tH)^n \in \mathbb{R}.$$

Expanding out the right hand side, we can view $P(t)$ as a polynomial in t, and we are required to verify that $P(0) \geqslant 0$. Aiming for a contradiction, we assume to the contrary that $P(0) < 0$.

Note first that if $k < n$, then

$$(D^k \cdot H^{n-k}) \geqslant 0. \tag{1.1}$$

In fact, H being ample, H^{n-k} is represented by an effective rational $k-$cycle. Therefore (1.1) follows by applying the induction hypothesis to the components of this cycle. In particular, for $k < n$ the coefficient of t^{n-k} in $P(t)$ is non-negative. Since by assumption $P(0) < 0$, it follows that $P(t)$ has a single real root $t_0 > 0$.

We claim next that for any rational number $t > t_0$, the $\mathbb{Q}$-divisor $D+tH$ is ample. To verify this, it is equivalent to check that

$$((D + tH)^k \cdot V) > 0$$

for every $V \subsetneq X$ of dimension k. When $V = X$ this follows from the fact that $P(t) > P(t_0) = 0$. If $V \subsetneq X$ one expands out the intersection number in question as a polynomial in t. As above all the coefficients are non-negative, while the leading term $(H^k \cdot V)$ is strictly positive. The claim is established.

Now write $P(t) = Q(t) + R(t)$, where

$$Q(t) = (D \cdot (D + tH)^{n-1}) \text{ and } R(t) = (tH \cdot (D + tH)^{n-1}).$$

It follows as in (1.1) from the nefness of D and the amplitude of $(D+tH)$ that $Q(t) \geqslant 0$ for all rational $t > t_0$. Consequently $Q(t_0) \geqslant 0$ by continuity. On the other hand, thanks to (1.1) all the coefficients of $R(t)$ are non-negative, and the highest one (H^n) is strictly positive. Therefore $R(t_0) > 0$. But then $P(t_0) > 0$, a contradiction. Thus we proved the Theorem in case D is rational.

It remains only to check that the Theorem holds when D is an arbitrary nef $\mathbb{R}$-divisor. To this end, choose ample divisors $H_1, \ldots, H_r$ whose classes span $\mathcal{N}^1(X)_{\mathbb{R}}$. Then $\epsilon_1 H_1 + \ldots + \epsilon_r H_r$ is ample for all $\epsilon_i > 0$. In particular,

$$D(\epsilon_1, \ldots, \epsilon_r) = D + \epsilon_1 H_1 + \ldots + \epsilon_r H_r$$

being the sum of a nef and an ample $\mathbb{R}$-divisor, is (evidently) nef. But the classes of these divisors fill up an open subset in $\mathcal{N}^1(X)_{\mathbb{R}}$, and therefore we can find arbitrarily small $0 < \epsilon_i \ll 1$ such that $D(\epsilon_1, \ldots, \epsilon_r)$ is (numerically equivalent to) a rational divisor. For such divisors, the case of the Theorem already treated shows that

$$(D(\epsilon_1, \ldots, \epsilon_r)^k \cdot V) \geqslant 0$$

for all V of dimension k. Letting the $\epsilon_i \to 0$, it follows that $(D^k \cdot V) \geqslant 0$. $\square$

Corollary 1.2.16. *Let X be a projective variety or scheme, and D a nef $\mathbb{R}$-divisor on X. If H is any ample $\mathbb{R}$-divisor on X, then*

$$D + \epsilon H$$

is ample for for every $\epsilon > 0$. Conversely, if D and H are any two divisors such that $D + \epsilon H$ is ample for all sufficiently small $\epsilon > 0$, then D is nef.

1.3 Cones

The meaning of Kleiman's Theorem 1.2.15 is clarified by introducing some natural and important cones in the Neron-Severi space $\mathcal{N}^1(X)_{\mathbb{R}}$ and its dual.

Let X be a complete complex variety or scheme.

Definition 1.3.1 (Ample and Nef cones). *The ample cone*

$$Amp(X) \subset \mathcal{N}^1(X)_{\mathbb{R}}$$

of X is the convex cone of all ample $\mathbb{R}$-divisor classes on X. The nef cone

$$Nef(X) \subset \mathcal{N}^1(X)_{\mathbb{R}}$$

is the convex cone all nef $\mathbb{R}$-divisor classes.

Theorem 1.3.2. *Let X be any projective variety or scheme.*

1. *The nef cone is the closure of the ample cone:*

$$Nef(X) = \overline{Amp(X)},$$

2. *The ample cone is the interior of the nef cone:*

$$Amp(X) = int(Nef(X)).$$

Proof. It is evident that the nef cone is closed, and it follows from Remark 1.2.13 that $Amp(X)$ is open. This and the Kleiman's Theorem 1.2.15 gives the inclusions

$$\overline{Amp(X)} \subseteq Nef(X) \text{ and } Amp(X) \subseteq int(Nef(X)).$$

The remaining two inclusions

$$Nef(X) \subseteq \overline{Amp(X)} \text{ and } int(Nef(X)) \subseteq Amp(X) \tag{1.2}$$

are consequences of Corollary 1.2.16. In fact let H be an ample divisor on X. If D is any nef $\mathbb{R}$-divisor then 1.2.16 shows that $D + \epsilon H$ is ample for all $\epsilon > 0$. Therefore D is a limit of ample divisors, establishing the first inclusion in (1.2). For the second, observe that if the class of D lies in the interior of $Nef(X)$, then $D - \epsilon H$ remains nef for $0 < \epsilon \ll 1$. Consequently

$$D = (D - \epsilon H) + \epsilon H$$

is ample thanks again to Corollary 1.2.16. □

Definition 1.3.3. *Let D a divisor on X. Assume now $V \subseteq H^0(X, \mathcal{O}_X(D))$ such that $dim(V) > 0$ and $B = Bs(|V|)$, then $|V|$ determines a morphism*

$$\psi_{|V|} : X \backslash B \to \mathbb{P}(V) = \mathbb{P}^r.$$

Informally, if $\{s_0, \ldots, s_r\}$ is a basis of V, we can describe this map, using homogeneous coordinates, with

$$\psi_{|V|}(x) = [s_0(x) : \ldots : s_r(x)] \quad \forall x \in X \backslash B.$$

Definition 1.3.4 (Iitaka dimension). *Assume that X is normal. Then the Iitaka dimension of L is defined to be*

$$\kappa(L) = \kappa(X, L) = \max_{m \in N(L)} \{dim \psi_{|mL|}(X)\}$$

provided that $\{m > 0 | H^0(X, \mathcal{O}_X(mL) \neq 0\} = N(L) \neq \emptyset$. If $H^0(X, L^{\otimes m}) = 0$ for all $m > 0$, one puts $\kappa(X, L) = -\infty$. If X is non-normal, pass to its normalization $\nu : X' \to X$ and set $\kappa(X, L) = \kappa(X', \nu^* L)$. Finally, for a Cartier divisor D one takes $\kappa(X, D) = \kappa(X, \mathcal{O}_X(D))$.

Thus either $\kappa(X,L) = -\infty$, or else

$$0 \leqslant \kappa(X,L) \leqslant dimX.$$

Example 1.3.5 (**Kodaira dimension**). *Let X be a smooth projective variety, and K_X a canonical divisor on X. Then $\kappa(X) = \kappa(X, K_X)$ is the Kodaira dimension of X: it is the most basic birational invariant of a variety. The Kodaira dimension of a singular variety is defined to be the Kodaira dimension of any smooth model.*

Theorem 1.3.6 (**Iitaka**). *Let $D \in Div(X)$ be a divisor on the scheme X such that $\kappa(X,D) \geqslant 0$. Then $\kappa(X,D)$ coincides with the number κ of either the following conditions:*

1. *There is a unique non-negative integer κ and integers $m_0, a, b > 0$ such that for all $m \gg 0$ one has the inequalities*

$$am^\kappa \leqslant dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mm_0D)) \leqslant bm^\kappa.$$

2. *$\kappa(X,D) = -\infty$ if $N(X,D) = \emptyset$, else $\kappa(X,D) = trdeg_{\mathbb{C}}(R(X,D)) - 1$*

For the proof see [7, Theorem 1].

Definition 1.3.7 (**Big**). *A line bundle L on an projective variety X is big if $\kappa(X,L) = dimX$. A Cartier divisor D is big if $\mathcal{O}_X(D)$ is such.*

Example 1.3.8 (**Varieties of general type**). *A smooth projective variety X is of general type if and only if its canonical bundle $\mathcal{O}_X(K_X)$ is big.*

Lemma 1.3.9. *Assume that X is a projective variety of dimension n. A divisor D on X is big if and only if there is a constant $C > 0$ such that*

$$h^0(X, \mathcal{O}_X(mD)) \geqslant C \cdot m^n$$

for all sufficiently large $m \in N(X,D)$.

For the proof see [3, 2.2.3.].

Lemma 1.3.10 (**Kodaira's Lemma**). *Let D be a big Cartier divisor and F an arbitrary effective Cartier divisor on X. Then*

$$H^0(X, \mathcal{O}_X(mD - F)) \neq 0$$

for all sufficiently large $m \in N(X,D)$.

Proof. Suppose that dim $X = n$, and consider the exact sequence

$$0 \to \mathcal{O}_X(mD - F) \to \mathcal{O}_X(mD) \to \mathcal{O}_F(mD) \to 0.$$

Since D is big, there is a constant $c > 0$ such that $h^0(X, \mathcal{O}_X(mD)) \geqslant cm^n$ for sufficiently large $m \in N(X, D)$. On the other hand, F being a scheme of dimension $n-1$, $h^0(F, \mathcal{O}_F(mD))$ grows at most like m^{n-1} by [3, 1.2.33.]. Therefore

$$h^0(X, \mathcal{O}_X(mD)) > h^0(F, \mathcal{O}_F(mD))$$

for large $m \in N(X, D)$, and the assertion follows from the displayed sequence.

□

Kodaira's Lemma has several important consequences. First, it leads to a useful characterization of big divisors:

Theorem 1.3.11 (**Characterization of big divisors**). *Let D be a divisor on a projective variety X. Then the following are equivalent:*

1. *D is big,*
2. *For any ample integer divisor A on X, there exists a positive integer $m > 0$ and an effective divisor N on X such that $mD \equiv_{lin} A + N$,*
3. *Same as in 2) for some ample divisor A,*
4. *There exists an ample divisor A, a positive integer $m > 0$ and an effective divisor N such that*

$$mD \equiv_{num} A + N.$$

Proof. Assuming that D is big, take $r \gg 0$, hence $rA \equiv_{lin} H_r$ and $(r+1)A \equiv_{lin} H_{r+1}$ are both effective. Apply the Kodaira's Lemma 1.3.10 with $F = H_{r+1}$ to find an effective divisor N' with:

$$mD \equiv_{lin} H_{r+1} + N' \equiv_{lin} A + (H_r + N').$$

Taking $N = H_r + N'$ gives 2). The implications 2) $\Rightarrow$ 3) $\Rightarrow$ 4) being trivial, we assume 4) and deduce 1). If $mD \equiv_{num} A + N$, then $mD - N$ is numerically equivalent to an ample divisor, and hence ample. Then after possibly passing to an even larger multiple of D we can assume that $mD \equiv_{lin} H + N'$, where H is very ample and N' is effective. But then $\kappa(X, D) \geqslant \kappa(X, H) = dim X$, then D is big. □

Corollary 1.3.12 (**Numerical nature of bigness**). *The bigness of a divisor D depends only on its numerical equivalence class.*

Proof. This is a consequence of characterization 4) in the previous Theorem. □

Corollary 1.3.13 (**Restrictions of big divisors**). *Let L be a big line bundle on a projective variety X. There is a proper Zariski closed subset $V \subset X$ having the property that if $Y \subset X$ is any subvariety of X not contained in V then the restriction $L|_Y$ of L to Y is a big line bundle on Y. In particular, if H is a general member of a very ample linear series, then $L|_H$ is big.*

Proof. Say $L = \mathcal{O}_X(D)$, and using Theorem 1.3.11 write $mD \equiv_{lin} H + N$ where N is effective and H is very ample. Take V to be the support of N. If $Y \not\subset V$, then the restriction $mD|_Y$ of mD to Y is again the sum of a very ample and an effective divisor, and hence is big. □

Note that an integral divisor A is big if and only if every (or equivalently some) positive multiple of A is big. This leads to a natural notion of bigness for a $\mathbb{Q}$-divisor:

Definition 1.3.14 (**Big $\mathbb{Q}$-divisors**). *A $\mathbb{Q}$-divisor D is big if there is a positive integer $m > 0$ such that mD is integral and big.*

As in Corollary 1.3.12, bigness is a numerical property of $\mathbb{Q}$-divisors. Now we extend the definition to $\mathbb{R}$-divisors and discuss the corresponding cone in $\mathcal{N}^1(X)_{\mathbb{R}}$.

Theorem 1.3.15 (**A numerical condition for bigness**). *Let X be a projective variety of dimension n, and let D and E be nef $\mathbb{Q}$-divisors on X. Assume that*

$$(D^n) > n \cdot (D^{n-1} \cdot E).$$

Then $D - E$ is big.

For the proof see [3, 2.2.15.].

As a first consequence, one obtains an important characterization of bigness for nef divisors.

Theorem 1.3.16 (Bigness of nef divisors). *Let D be a nef divisor on a projective variety X of dimension n. Then D is big if and only if its top self-intersection is strictly positive, that is $(D^n) > 0$.*

Proof. Suppose to begin with that $(D^n) > 0$. Then the hypothesis of Theorem 1.3.15 is satisfied with $E = 0$, and hence D is big. Conversely, suppose that D is nef and big. Then $mD \equiv_{lin} H + N$ for some very ample H and effective N, and suitable $m > 0$. But $(D^{n-1} \cdot N) \geqslant 0$ by Kleiman's Theorem 1.2.15, and therefore

$$m \cdot (D^n) = ((H + N) \cdot D^{n-1}) \geqslant (H \cdot D^{n-1}).$$

In light of Corollary 1.3.13 we may assume moreover that $D|_H$ is a big and nef divisor on H. Hence by induction $(D^{n-1} \cdot H) > 0$, and the required inequality $(D^n) > 0$ follows. □

Now we rephrase the Theorem 1.3.11 to characterize the cone of all big divisors. As before X is a projective variety of dimension n.

We start by extending the definitions to $\mathbb{R}$-divisors.

Definition 1.3.17 (Big $\mathbb{R}$-divisors). *An $\mathbb{R}$-divisor $D \in Div_{\mathbb{R}}(X)$ is big if it can be written in the form*

$$D = \sum a_i \cdot D_i$$

where each D_i is a big integral divisor and a_i is a positive real number.

This is justified by the observation that if D_1 and D_2 are big $\mathbb{Q}$-divisors, then $a_1 D_1 + a_2 D_2$ is big for any positive rational numbers a_1, a_2.

Proposition 1.3.18 (Formal properties of big $\mathbb{R}$-divisors). *Let D and D' be $\mathbb{R}$-divisors on X.*

1. *If $D \equiv_{num} D'$, then D is big if and only if D' is big.*

2. *D is big if and only if $D \equiv_{num} A + N$ where A is an ample and N is an effective $\mathbb{R}$-divisor.*

Proof. For 1) argue as in the proof of Proposition 1.2.12.

Turning to 2), it follows immediately from Theorem 1.3.11 (4) that a big $\mathbb{R}$-divisor has an expression of the indicated sort. For the converse one reduces to showing that if B and N are integral divisors, with B big and N effective, then $B + sN$ is big for any real number $s > 0$. If $s \in \mathbb{Q}$ this follows again from Theorem 1.3.11. In general, choose rational numbers $s_1 < s < s_2$ and $t \in [0, 1]$ such that $s = ts_1 + (1 - t)s_2$. Then

$$B + sN = t(B + s_1N) + (1 - t)(B + s_2N)$$

exhibits $B + sN$ as a positive linear combination of big $\mathbb{Q}$-divisors.

□

Corollary 1.3.19. *Let $D \in Div_{\mathbb{R}}(X)$ be a big $\mathbb{R}$-divisor, and let $E_1, \dots, E_t \in Div_{\mathbb{R}}(X)$ be arbitrary $\mathbb{R}$-divisors. Then*

$$D + \epsilon_1 E_1 + \ldots + \epsilon_t E_t$$

remains big for all sufficiently small real numbers $0 \leqslant |\epsilon_i| \ll 1$.

Proof. This follows from statement 2) of the previous Proposition thanks to the open nature of amplitude. □

Definition 1.3.20. *A class $e \in \mathcal{N}^1(X)_{\mathbb{Q}}$ (respectively $e \in \mathcal{N}^1(X)_{\mathbb{R}}$) is effective if there exists an effective $\mathbb{Q}$-divisor (respectively $\mathbb{R}$-divisor) such that $[E] = e$.*

Remark 1.3.21. *Let B a big divisor on X, by Lemma 1.3.9, we know that the linear series $|mB| \neq \emptyset$ for all sufficiently large m. Hence $\frac{1}{m}E \equiv_{num} B$, then every big class is an effective class in $\mathcal{N}^1(X)_{\mathbb{R}}$.*

Note that in view of Proposition 1.3.18 (1) it makes sense to talk about a big $\mathbb{R}$-divisor class. We can then define some additional cones in $\mathcal{N}^1(X)_{\mathbb{R}}$.

Definition 1.3.22 (**Big and pseudo-effective cones**)**.** *The big cone*

$$Big(X) \subset \mathcal{N}^1(X)_{\mathbb{R}}$$

is the convex cone of all big $\mathbb{R}$*-divisor classes on* X*. The pseudo-effective cone*

$$\overline{Eff(X)} \subset \mathcal{N}^1(X)_{\mathbb{R}}$$

is the closure of the convex cone spanned by the classes of all effective $\mathbb{R}$*-divisors. A divisor* $D \in Div_{\mathbb{R}}(X)$ *is pseudo-effective if its class lies in the pseudo-effective cone.*

Remark 1.3.23. *A* $\mathbb{R}$*-divisor* D *on a surface* X *is pseudo-effective if* $(D \cdot A) \geqslant 0$ *for all* $A \in Amp(X)$*, because a pseudo-effective* $\mathbb{R}$*-divisor is limit of effective* $\mathbb{R}$*-divisors. Moreover we have that* D *is pseudo-effective if and only if* $(D \cdot H) \geqslant 0$ *for all* $H \in Nef(X)$ *(see [3, 2.2.25.]).*

One then has:

Theorem 1.3.24. *The big cone is the interior of the pseudo-effective cone and the pseudo-effective cone is the closure of the big cone:*

$$Big(X) = int(\overline{Eff}(X)) \quad , \quad \overline{Eff}(X) = \overline{Big}(X).$$

Proof. The pseudo-effective cone is closed by definition, the big cone is open by Corollary 1.3.19, and $Big(X) \subseteq \overline{Eff}(X)$ thanks to Proposition 1.3.18. It remains to establish the inclusions

$$\overline{Eff(X)} \subseteq \overline{Big(X)} \ , \ int(\overline{Eff(X)}) \subseteq Big(X).$$

We focus on the first of these. Given $\zeta \in \overline{Eff(X)}$, one can write ζ as the limit $\zeta = \lim_k \zeta_k$ of the classes of effective divisors. Fixing an ample class $\alpha \in \mathcal{N}^1(X)_{\mathbb{R}}$ one has

$$\zeta = \lim_k (\zeta_k + \frac{1}{k}\alpha).$$

Each of the classes $\zeta_k + \frac{1}{k}\alpha$ is big thanks to Proposition 1.3.18, therefore ζ is a limit of big classes. Hence $\overline{Eff(X)} \subseteq \overline{Big(X)}$.

Now we take an element $[D] \in Int(\overline{Eff(X)})$, then there is $r > 0$ such that $U_r([D]) = \{[E] \in \mathcal{N}^1(X)_{\mathbb{R}} | \ \ \|[E] - [D]\| < r\} \subset \overline{Eff(X)}$. By density there exists a big class $[B]$ such that $[B] \in U_r([D])$, but $Big(X)$ is an open cone by Corollary 1.3.19, therefore there exists $\delta > 0$ such that $U_\delta([B]) \subset U_r([D]) \cap Big(X)$. If $D \in U_\delta([B])$ we are done, else we can take the symmetrical with respect to $[D]$ of $U_\delta([B])$, let be $\mathcal{A}$ this open (it is an open disk). Then there exists a big class $[\overline{B}]$ such that $[\overline{B}] \in \mathcal{A}$, then by the convexity of $Big(X)$ we know that $[D]$ is big. $\square$

Remark 1.3.25. *Let E be a pseudo-effective $\mathbb{R}$-divisor and B a big $\mathbb{R}$-divisor, then $E + B$ is big.*

Proof. As above, if we take the open ball $U_r = \{l \in \mathcal{N}^1(X)_{\mathbb{R}} | \ \|l\| < r\}$, then there exists $\epsilon > 0$ such that $B + U_\epsilon \subset Big(X) \subset \overline{Eff(X)}$. Then $D+B+U_\epsilon \subset \overline{Eff(X)}$, hence $D+B \in Int(\overline{Eff(X)}) = Big(X)$ by Theorem 1.3.24. $\square$

Lemma 1.3.26. *If X is a smooth surface, D an effective $\mathbb{Q}$-divisor such that $D^2 > 0$, then D is big.*

Proof. By the Riemann-Roch Theorem for surfaces 1.1.11 we know that

$$h^0(X, mD) - h^1(X, mD) + h^2(X, mD) = \frac{1}{2} mD \cdot (mD - K_X) + 1 - p_a$$

and by Serre duality (see [2, III.7]) we have $h^2(X, mD) = h^0(X, K_X - mD)$.

Then $h^0(X, K_X - mD) = 0$ for $m \gg 0$, because, by the way of contradiction, if E is an effective divisor such that $E \equiv_{lin} K_X - mD$, then for all A ample, we have

$$0 \leqslant E \cdot A = (K_X - mD) \cdot A = K_X \cdot A - mD \cdot A.$$

But this is a contradiction because D is effective, $D \neq 0$, hence we have $D \cdot A > 0$, then $K_X \cdot A < mD \cdot A$, for $m \gg 0$. At this point we have

$$h^0(X, mD) \geqslant \frac{m^2}{2} D^2 - \frac{m}{2} K_X \cdot D + 1 - p_a,$$

then by Lemma 1.3.9 D is big. $\square$

Moreover we have:

Lemma 1.3.27. *Let D be a divisor on the surface X. Assume that D is not pseudo-effective. Then for every $Z \in Div(X)$ we have $H^0(X, \mathcal{O}_X(nD + Z)) = 0$ for every $n \gg 0$.*

Proof. Since D is not pseudo-effective, there exists an ample divisor H such that $(D \cdot H) < 0$. Then for every Z we have $(nD \cdot H + Z \cdot H) < 0$ for $n \gg 0$, and therefore $|nD + Z| = \emptyset$. $\square$

Proposition 1.3.28. *If D is big then there exists m_0 such that*

$$H^0(X, \mathcal{O}_X(mD)) \neq \emptyset,$$

for all $m \geqslant m_0$.

For the proof, see [3, 2.2.10.].

1.4 Volume of a Big divisor

We turn now to an interesting invariant of a big divisor D that measures the asymptotic growth of the linear series $|mD|$ for $m \gg 0$:

Definition 1.4.1 (Volume of a line bundle). *Let X be a projective variety of dimension n, and let L be a line bundle on X. The volume of L is defined to be the non-negative real number*

$$vol(L) = vol(X, L) = \limsup_{m \to \infty} \frac{h^0(X, L^{\otimes m})}{m^n/n!}. \tag{1.3}$$

The volume $vol(D) = vol(X, D)$ of a Cartier divisor D is defined similarly, or by passing to $\mathcal{O}_X(D)$.

Note that $vol(L) > 0$ if and only if L is big.

Remark 1.4.2 (Irrational volumes). *The volume of a big line bundle can be an irrational number. See [3, 2.2.34.].*

The first formal properties of this invariant are summarized in the following

Proposition 1.4.3 (Properties of the volume). *Let D be a big divisor on a projective variety X of dimension n.*

1. *For a fixed natural number $a > 0$,*

$$vol(aD) = a^n vol(D).$$

2. *Fix any divisor N on X, and any $\epsilon > 0$. Then there exists an integer $p_0 = p_0(N, \epsilon)$ such that*

$$\frac{1}{p^n}|vol(pD - N) - vol(pD)| < \epsilon$$

for every $p > p_0$.

For the proof see [3, 2.2.35.].

Remark 1.4.4 (**Volume of a $\mathbb{Q}$-divisor**). *When D is a $\mathbb{Q}$-divisor, one can define the volume $vol(D)$ of D, taking the limsup over those m for which mD is integral. However it is perhaps quicker to choose some $a \in \mathbb{N}$ for which aD is integral, and then set $vol(D) = \frac{1}{a^n}vol(aD)$. It follows from Proposition 1.4.3 (1) that this is independent of the choice of a.*

Remark 1.4.5 (**Numerical nature of volume**). *If D, D' are numerically equivalent divisors on X , then*

$$vol(D) = vol(D').$$

In particular, it makes sense to talk of the volume $vol(\zeta)$ of a class $\zeta \in \mathcal{N}^1(X)_{\mathbb{Q}}$ (see [3, 2.2.41.]).

Theorem 1.4.6 (**Continuity of volume**). *Let X be a projective variety of dimension n, and fix any norm $\|\cdot\|$ on $\mathcal{N}^1(X)_{\mathbb{R}}$ inducing the usual topology on that finite dimensional vector space. Then there is a positive constant $C > 0$ such that*

$$|vol(\zeta) - vol(\zeta')| \leqslant C(max(\|\zeta\|, \|\zeta'\|))^{n-1}\|\zeta - \zeta'\| \tag{1.4}$$

for any two classes $\zeta, \zeta' \in \mathcal{N}^1(X)_{\mathbb{Q}}$.

Corollary 1.4.7 (**Volume of real classes**). *The function $\zeta \to vol(\zeta)$ on $\mathcal{N}^1(X)_{\mathbb{Q}}$ extends uniquely to a continuous function*

$$vol : \mathcal{N}^1(X)_{\mathbb{R}} \to \mathbb{R}.$$

Proof. Equation (1.4) guarantees that if we choose a sequence $\zeta_i \in \mathcal{N}^1(X)_{\mathbb{Q}}$ converging to a given real class $\zeta \in \mathcal{N}^1(X)_{\mathbb{R}}$, then $\lim_{i\to\infty} vol(\zeta_i)$ exists and is independent of the choice of $\{\zeta_i\}$. □

Chapter 2

Zariski Decomposition on Surfaces

Now we have the notions to study the Zariski decomposition. This is a very interesting idea:

Let D a pseudo-effective divisor (or $\mathbb{Q}$- or $\mathbb{R}$-divisor) we want to decompose D in two $\mathbb{Q}$-divisors (or $\mathbb{R}$-), as a sum of a nef and an effective divisor (respectively the positive and the negative part), with the property that positive and negative part are orthogonal and the negative part is a linear combination of prime components with negative self-intersection.

Now we give a proof of the existence and uniqueness of this decomposition for $\mathbb{K}$-divisors, where $\mathbb{K}$ will be $\mathbb{Q}$ or $\mathbb{R}$.

2.1 Zariski Decomposition for Effective Divisors

For all prime divisor Γ, we have that for all big divisor B,

Now we report a simple proof of the existence and the uniqueness of Zariski decompositions for effective $\mathbb{K}$-divisors on a smooth surface X. This proof is due to Bauer, [4].

Remark 2.1.1. *We will use the partial ordering $\leqslant$ in $\mathbb{K}^r$ that is defined by $(x_1, \ldots, x_r) \leqslant (y_1, \ldots, y_r)$ if $x_i \leqslant y_i$ for all i.*

Lemma 2.1.2. *Let X is a smooth surface, and let $N \neq 0$ is a $\mathbb{K}$-divisor, whose intersection matrix S is not negative definite, then there is an effective* *non-zero nef $\mathbb{K}$-divisor E, whose components are among those of N.*

Proof. We distinguish between two cases:

Case 1: S is not negative semi-definite. In this case there is a $\mathbb{K}$-divisor B whose components are among those of N such that $B^2 > 0$. Multiplying for $m \gg 0$ and taking the integer part, $\lfloor mB \rfloor$, we may assume that B is integer and $B^2 > 0$:

In fact, if $B = \sum_i c_i C_i$, then $0 \leqslant \langle mc_i \rangle \leqslant 1$ and $mc_i - 1 \leqslant \lfloor mc_i \rfloor \leqslant mc_i$, for all i and for all m, therefore we have for $m \gg 0$ that, if $c_i > 0$, then $\lfloor mc_i \rfloor \langle mc_j \rangle \leqslant \lfloor mc_i \rfloor \leqslant mc_i \leqslant mc_i + 1$. If $c_i < 0$, then we have $\lfloor mc_i \rfloor \langle mc_j \rangle \geqslant \lfloor mc_i \rfloor \geqslant mc_i - 1$. Therefore we have that $|\lfloor mc_i \rfloor \langle mc_j \rangle| \leqslant m|c_i| + 1$, then

$$\sum_{i,j} |\lfloor mc_i \rfloor \langle mC_j \rangle||C_i \cdot C_j| \leqslant \sum (m|c_i| + 1)|C_i \cdot C_j| = O(m)$$

and

$$\lim_{m\to\infty} \frac{|\langle mc_i \rangle \langle mc_j \rangle|}{m} = 0,$$

hence, by the finite number of $\{C_i\}$, we have

$$0 < (mB)^2 = (\sum_i \lfloor mc_i \rfloor C_i + \sum_i \langle mc_i \rangle C_i)^2 \leqslant$$

$$\leqslant \sum_{i,j,i\neq j} \lfloor mc_i \rfloor \lfloor mc_j \rfloor (C_i \cdot C_j) + \sum_i \lfloor mc_i \rfloor^2 (C_i)^2 + O(m) =$$

$$= (\lfloor mB \rfloor)^2 + O(m).$$

Then, for $m \gg 0$ we have that $(\lfloor mB \rfloor)^2 > 0$.

Then, writing $B = B' - B''$ as a difference of effective divisors having no common components, we have $0 < B^2 = (B' - B'')^2 = (B')^2 - 2B' \cdot B'' + (B'')^2$, and hence $(B')^2 > 0$ or $(B'')^2 > 0$. Therefore, replacing B by B' or B'' respectively, we may assume that B is effective. But then it follows from the Theorem 1.3.26 that B is big, then there is a constant $C > 0$ such that $h^0(X, \mathcal{O}_X(lB)) \geqslant Cl^2$ for $l \gg 0$ (by Theorem 1.3.9). Therefore $dim|lB| \geqslant Cl^2 - 1$ for $l \gg 0$. Hence we can write $lB = E_l + F_l$, where $|E_l|$ is the moving part of $|lB|$, then E_l is nef. By Lemma 1.3.26 we note that E_l is non-zero. Then $E = E_l$ is a nef divisor as required, then the proof is complete in this case.

Case 2: S is negative semi-definite. Let $C_1, \dots, C_k$ be the components of N. We argue by induction on k. If $k = 1$, then $N^2 = {C_1}^2 = 0$, hence C_1 is nef and we are done taking $E = C_1$. Suppose $k > 1$. The hypotheses on S imply that S does not have full rank. Therefore there is a non-zero $\mathbb{K}$-divisor R, whose components are among $C_1, \dots, C_k$, having the property that $R \cdot C_i = 0$ for $i = 1, \dots, k$:

In fact if $R = \sum a_i C_i$ then

$$\begin{cases} a_i C_1 \cdot C_1 & + & \dots & + & a_k C_k \cdot C_1 & = & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_i C_1 \cdot C_k & + & \dots & + & a_k C_k \cdot C_k & = & 0 \end{cases}$$

This is a linear system in $\{a_1, \dots, a_k\}$, but $rank(S) < k$, then there is $\infty^{k-rank(S)}$ solutions, then the set of all $\mathbb{K}$-divisors orthogonal at every C_i for all i, is non-empty.

If one of the $\mathbb{K}$-divisors R or $-R$ is effective, then is nef, and we are done, taking $E = R$ or $E = -R$ respectively. In alternative case we write $R = R' - R''$, where R' and R'' are effective non-zero $\mathbb{K}$-divisors without common components. We have

$$0 = R^2 = (R')^2 - 2R' \cdot R'' + (R'')^2.$$

As by hypothesis $(R')^2 \leqslant 0$ and $(R'')^2 \leqslant 0$, we must have $(R')^2 = 0$. The divisor R' has fewer components than R, and its intersection matrix is still negative semi-definite, but not negative definite. It now follows by induction that there is a divisor as claimed, consisting entirely of components of R'.

□

Theorem 2.1.3 (**Zariski decomposition**). *Let D be an effective $\mathbb{K}$-divisor on a smooth projective surface X. Then there are uniquely determined effective (possibly zero) $\mathbb{K}$-divisors P and N with $D = P + N$ such that*

1. *P is nef,*

2. *N is zero or has negative definite intersection matrix,*

3. *$P \cdot C = 0$ for every irreducible component C of N.*

The decomposition $D = P + N$ is called *Zariski decomposition* of D, the $\mathbb{K}$-divisors P and N are respectively the *positive* and *negative* parts of D. Given an effective $\mathbb{K}$-divisor D, Zariski's original proof employs a rather sophisticated procedure to construct the negative part N out of those components C of D satisfying $D \cdot C \leqslant 0$. Our purpose here is to provide a quick and simple proof based on the idea that the positive part P can be constructed as a maximal nef $\mathbb{K}$-subdivisor of D.

Proof of existence. Write $D = \sum_{i=1}^{r} a_i C_i$ with distinct irreducible curves C_i and positive (rational or real) numbers a_i. Consider now all effective $\mathbb{K}$-subdivisors P of D, that is all $\mathbb{K}$-divisors of the form $P = \sum_{i=1}^{r} x_i C_i$ with coefficients x_i satisfying $0 \leqslant x_i \leqslant a_i$. A divisor P of this kind is nef if and only if

$$\sum_{i=1}^{r} x_i C_i \cdot C_j \geqslant 0 \quad \forall j = 1, \ldots, r. \tag{2.1}$$

We claim that this system of linear inequalities for the numbers x_i has a maximal solution (with respect to $\leqslant$) in the $\mathbb{K}$-cuboid

$$[0, a_1] \times \ldots \times [0, a_r] \subset \mathbb{K}^r.$$

To see this, note first that the subset K of the cuboid that is described by (2.1) is a convex polytope defined by finitely many halfspaces with coordinates in $\mathbb{K}$. We are done if $(a_1, \ldots, a_r) \in K$. In the alternative case consider for $t \in \mathbb{K} \cap [0, 1]$ the family of hyperplanes $H_t = \{(x_1, \ldots, x_r) \in \mathbb{K}^r | \sum_i x_i = t \sum_i a_i\}$.

Let us see that there is a maximal t such that $H_t \cap K \neq \emptyset$ and such that $H_t \cap K$ is a vertex of K.

Let us take the completion of K in $\mathbb{R}^r$, that is

$$\overline{K} = \{(x_1, \ldots, x_r) \in \mathbb{R}^r | x_i \in [0, a_i] \text{ and } \sum_{i=1}^{r} x_i C_i \cdot C_j \geqslant 0, \, \forall j \in [1, r]\}.$$

This is closed and bounded, then is compact in $\mathbb{R}^r$. Now we take the application $\pi : \overline{K} \to \mathbb{R}$, $(x_1, \ldots, x_r) \to \sum_i x_i$. We note that π is obviously continue and then it has maximum in $\overline{K}$, let it be M. If $\sum_i a_i = 0$, by $a_i \geqslant 0$, we have $a_i = 0$ for all i, then $D = \sum_i a_i C_i = 0$, and we are done because D is nef. Therefore there exists m such that $M = m \sum_i a_i$. Let $(b_1, \ldots, b_r) = b \in \overline{K}$ such that $\pi(b) = M$. We claim:

1. b is maximal,

2. b is a vertex of K.

b is maximal because if we have $b \leqslant x \in \overline{K}$, that is $b_i \leqslant x_i$ for all i then we have $\sum b_i \geqslant \sum x_i$ because $\pi(b)$ is a maximum, but $\sum x_i \geqslant \sum b_i$, then $x_i = b_i$, for all i. Now we have to prove that b is a vertex (for the rational case this implies that b is in $\mathbb{Q}^r$ and then in K because the coordinate of a vertex of K are combinations of rational numbers).

Let H_m^- the subspace below the hyperplane H_m. It is obvious that $\overline{K} \subset H_m^-$ by definition of m. Since $\overline{H_m} \cap \overline{K} \neq \emptyset$, then $\overline{H_m} \cap \overline{K} \subset Fr(\overline{K})$, that is $\overline{H_m} \cap \overline{K}$ is a vertex, an edge or a face. Now $\overline{H_m} \cap \overline{K}$ can be only a vertex, because the edge of $\overline{K}$ either lies on the border of the cuboid or joins the origin with the border of $\overline{K}$. Now $\overline{H_m}$ does not contain any edge of the cuboid (π is not constant on an edge of the cuboid) and $\overline{H_m}$ does not contain the origin. Then $\overline{H_m}$ cannot contain any edge of $\overline{K}$, then it cannot contain any face generated by this edge. Then $\overline{H_m} \cap \overline{K}$ can be only a vertex.

Therefore there exists m such that $\overline{H_m} \cap \overline{K}$ is a vertex and this m is maximal.

Let now $P = \sum_{i=1}^r b_i C_i$ be a divisor that is determined by a maximal solution, and put $N = D - P$. Then both P and N are effective, and P is a maximal nef $\mathbb{K}$-subdivisor of D. We will now show that 2) and 3) are satisfied as well.

As for 3): suppose $P \cdot C_i > 0$ for some component C_i of N. As $C_i \leqslant N$, we have $b_i < a_i$, hence that for sufficiently small $\epsilon > 0$ in $\mathbb{K}$, the divisor $P + \epsilon C_i$ is a subdivisor of D. For curves C different from C_i we clearly have $(P + \epsilon C_i) \cdot C \geqslant 0$. Moreover, $(P + \epsilon C_i) \cdot C_i = P \cdot C_i + \epsilon {C_i}^2$ for small ϵ. Then $P + \epsilon C_i$ is nef, contradicting the maximality of P.

As for 2): supposing that the divisor N is non-zero, we need to show that its intersection matrix is negative definite. Assume by way of contradiction that the intersection matrix of N is not negative definite, and take E as in Lemma 2.1.2. Consider then for $\epsilon > 0$ in $\mathbb{K}$, the $\mathbb{K}$-divisor $P' := P + \epsilon E$. Certainly P' is effective and nef. As all components of E are among the components of N, it is clear that P' is a subdivisor of D when ϵ is small enough. But this is a contradiction, because P' is strictly bigger than P.

We now give the:

Proof of uniqueness. We claim first that in any decomposition $D = P + N$ satisfying the conditions of the Theorem, the $\mathbb{K}$-divisor P is necessarily a *maximal* nef $\mathbb{K}$-subdivisor of D. To see this we suppose that P' is any nef $\mathbb{K}$-divisor with $P \leqslant P' \leqslant D$. Then $P' = P + \sum_{i=1}^{r} q_i C_i$, where $C_1, \dots, C_r$ are the components of N and $q_1, \dots, q_r$ are elements of $\mathbb{K}$ with $q_i \geqslant 0$. We have

$$0 \leqslant P' \cdot C_j = \sum_{i=1}^{r} q_i C_i \cdot C_j \quad \text{for } j = 1, \dots, r\,,$$

and hence

$$(\sum_{i=1}^{r} q_i C_i)^2 = \sum_{j=1}^{r} q_i \sum_{i=1}^{r} q_i C_i \cdot C_j \geqslant 0.$$

As the intersection matrix of $C_1, \dots, C_r$ is negative definite, we get $q_i = 0$ for all i. So $P' = P$.

To complete the proof it is now enough to show that a maximal effective nef $\mathbb{K}$-subdivisor of D is in fact *unique*. This in turn follows from:

(3.2) If $P' = \sum_{i=1}^{r} x_i' C_i$ and $P'' = \sum_{i=1}^{r} x_i'' C_i$ are effective nef $\mathbb{K}-$divisors of D, then $P = \sum_{i=1}^{r} x_i C_i$ is effective, where $x_i = \max(x_i', x_i'')$.

As for (3.2): The divisor P is of course an effective $\mathbb{K}$-subdivisor of D, hence it remains to show that it is nef, that is the tuple $(x_1, \dots, x_r)$ satisfies the inequalities (2.1). This, finally, is a consequence of the following elementary fact: let $H \subset \mathbb{K}^r$ be a halfspace, given by a linear inequality $\sum_{i=1}^{r} \alpha_i x_i \geqslant 0$, where the coefficients α_i are numbers with at most one of them negative. If two points $(x_1', \dots, x_r')$ and $(x_1'', \dots, x_r'')$ with $x_i' \geqslant 0$ and $x_i'' \geqslant 0$ lie in H, then $(x_1, \dots, x_r)$, where $x_i = \max(x_i', x_i'')$, lies in H. This is simple, because if for all i we have that α_i is not negative, then $(x_1, \dots, x_r)$ lies obviously in H. If there exists i such that $\alpha_i < 0$ then let $x_i' = \max\{x_i', x_i''\}$ for simplicity, then we know that

$$\sum_{j \neq i} \alpha_j \max\{x_j', x_j''\} \geqslant \sum_{j \neq i} \alpha_j x_j' \geqslant -\alpha_i x_i'.$$

□

2.2 Pseudo-Effective Case

Now we extend the Zariski Theorem to pseudo-effective divisors. This costruction is due to Fujita, [5],[6].

To do it, we need some technical results.

Remark 2.2.1. *Let H a nef $\mathbb{K}$-divisor and E an effective $\mathbb{K}$-divisor on the surface X. If $((H+E)\cdot C) \geqslant 0$ for every component C of E then $H+E$ is nef.*

Lemma 2.2.2. *Let $C_1,\dots,C_q$ be integral curves on the surface X such that the intersection matrix $||(C_i \cdot C_j)||_{i,j}$ is negative definite. Let $D = \sum_{i=1}^q a_iC_i$ be a $\mathbb{K}$-divisor such that $(D\cdot C_i) \leqslant 0$, for all $i = 1,\dots,q$. Then $D \geqslant 0$.*

Proof. Let $D = A-B$ be the decomposition of D with $A \geqslant 0$, $B \geqslant 0$, and A and B having no common components. Since $B \geqslant 0$ we have $0 \geqslant (D\cdot B) = (A\cdot B) - (B^2)$. Since A and B have no common components, $(A\cdot B) \geqslant 0$, whence $(B^2) \geqslant 0$. On the other hand, because the matrix $||(C_i \cdot C_j)||_{i,j}$ is negative definite, $(B^2) \leqslant 0$. It follows that $(B^2) = 0$, whence $B = 0$ (again because the intersection matrix is negative definite). □

Lemma 2.2.3. *Let $C_1,\dots,C_q$ be integral curves on the surface X such that the intersection matrix $||(C_i \cdot C_j)||_{i,j}$ is negative definite. Let $D = \sum_{i=1}^q a_iC_i$ be a $\mathbb{K}$-divisor and let D' be a pseudo-effective $\mathbb{K}$-divisor such that $((D'-D)\cdot C_i) \leqslant 0$, for all $i = 1,\dots,q$. Then the divisor $D'-D$ is pseudo-effective.*

Proof. Let H be an arbitrary nef $\mathbb{K}$-divisor on X. Since $||(C_i \cdot C_j)||_{i,j}$ is negative definite, there exists a $\mathbb{K}$-divisor of the form $Y = \sum_{i=1}^q a_iC_i$ such that $(Y\cdot C_i) = -(H\cdot C_i)$, for all $i = 1,\dots,q$. Since $(Y\cdot C_i) = -(H\cdot C_i) \leqslant 0$, for all $i = 1,\dots,q$, we have $Y \geqslant 0$ by Lemma 2.2.2. Then from the hypotheses we get $((D'-D)\cdot Y) \leqslant 0$. On the other hand, $((H+Y)\cdot C_i) = 0$, for all $i = 1,\dots,q$ implies that $H+Y$ is nef by Remark 2.2.1, and since D' is pseudo-effective, $((H+Y)\cdot D') \geqslant 0$. Combining these inequalities we get $((D'-D)\cdot H) = (D'\cdot H) - (D\cdot H) = (D'\cdot H) + (D\cdot Y) \geqslant -(D'\cdot Y) + (D\cdot Y) = -((D'-D)\cdot Y) \geqslant 0$. □

Lemma 2.2.4. *Let $C_1, \ldots, C_q$ be integral curves on the surface X such that the intersection matrix $I(C_1, \ldots, C_q) = ||(C_i \cdot C_j)||_{i,j=1,\ldots,q}$ is semidefinite of signature $(0, r)$, with $r < q$. Assume also that the matrix $I(C_1, \ldots, C_r) = ||(C_i \cdot C_j)||_{i,j=1,\ldots,r}$ is negative definite. Then for every $j > r$ there exists an effective $\mathbb{K}$-divisor $D_j = \sum_{i=1}^{r} a_{ji} C_i$ such that $((D_j + C_j) \cdot C_i) = 0$, for all $i = 1, \ldots, q$.*

Proof. For each $1 \leqslant p \leqslant q$ denote by $V(C_1, \ldots, C_p)$ the $\mathbb{K}$-vector space spanned by $C_1, \ldots, C_p$ (that is the set of $\mathbb{K}$-divisors of the form $\sum_{i=1}^{p} a_i C_i$, with $a_i \in \mathbb{K}$). Let $V_1 = V(C_1, \ldots, C_r)$, and let V_2 be the singular $\mathbb{K}$-subspace of the matrix $I(C_1, \ldots, C_q)$ in $V = V(C_1, \ldots, C_q)$, that is,
$V_2 = \{D \in V | (D \cdot D') = 0, \forall D' \in V\}$. From our hypotheses it follows that $V = V_1 \oplus V_2$. In particular, for every $j > r$ there exist $D_j \in V_1$ and $E_j \in V_2$ such that $C_j = -D_j + E_j$. Since $E_j = C_j + D_j \in V_2$ we get $((D_j + C_j) \cdot C_i) = 0$, for all $i = 1, \ldots, q$. Moreover, $(D_j \cdot C_i) = -(C_j \cdot C_i) \leqslant 0$, for all $i = 1, \ldots, r$, hence by Lemma 2.2.2, $D_j \geqslant 0$, for all $j > r$. □

Lemma 2.2.5. *Let $C_1, \ldots, C_q$ be integral curves on the surface X, and let D be a pseudo-effective $\mathbb{K}$-divisor on X such that $(D \cdot C_i) \leqslant 0$, for all $i = 1, \ldots, q$. Assume that there is an r such that $1 \leqslant r < q$ and $(D \cdot C_j) < 0$, for all $j = r+1, \ldots, q$. If the intersection matrix $I(C_1, \ldots, C_r) = ||(C_i \cdot C_j)||_{i,j=1,\ldots,r}$ is negative definite or $(D \cdot C_j) < 0$ for all $j = 1, \ldots, q$, then the intersection matrix $I(C_1, \ldots, C_q) = ||(C_i \cdot C_j)||_{i,j=1,\ldots,q}$ is also negative definite.*

Proof. Assume first that $I(C_1, \ldots, C_q)$ is not negative semidefinite. Then there exists a $\mathbb{K}$-divisor $Z \in V(C_1, \ldots, C_q)$ (as above) such that $(Z^2) > 0$. As in the proof of *Case* 1 of Lemma 2.1.2 we may assume that Z is integer, effective with $Z^2 > 0$. Then Z is big by Lemma 1.3.26. Now it follows that $\kappa(X, Z) = 2$. By the Iitaka Theorem 1.3.6 the image of X under the rational map $\varphi_{|nZ|}$ is an surface if $n \gg 0$. Replacing Z by a suitable multiple of it, we may therefore assume that $\varphi_{|Z|}(X)$ is a surface. Denote the fixed part of $|Z|$ by F, that is, the linear system $|Z'| := |Z| - F$ has no fixed components, $Z' \geqslant 0$ and $Z' \in V(C_1, \ldots, C_q)$. Clearly $\kappa(X, Z') = \kappa(X, Z) = 2$. Moreover, since Z' is effective and $|Z'|$ has no fixed components, Z' is nef. Therefore by Theorem 1.3.16 we get $(Z')^2 > 0$.

Since $Z' \in V(C_1, \ldots, C_q)$ and $Z' \geqslant 0$, from the hypotheses on D it follows that $(D \cdot Z') \leqslant 0$ and $(D \cdot Z') < 0$ if $(D \cdot C_j) < 0$, for all j. Recalling that Z' is nef and D is pseudo-effective, by Remark 1.3.23 we get $(D \cdot Z') \geqslant 0$. Thus $(D \cdot Z') = 0$ and we already have a contradiction if $(D \cdot C_j) < 0$, for all j. From the hypotheses on D it again follows that $Z' \in V(C_1, \ldots, C_r)$. Since $(Z')^2 > 0$, this last fact contradicts the hypothesis that the matrix $I(C_1, \ldots, C_r)$ is negative definite.

Therefore we have proved that the matrix $I(C_1, \ldots, C_q)$ is negative semidefinite. Assume that $I(C_1, \ldots, C_q)$ is not negative definite. By Lemma 2.2.4, for every $j = r+1, \ldots, q$, there is an effective $\mathbb{K}$-divisor $D_j \in V(C_1, \ldots, C_r)$ such that $((D_j + C_j) \cdot C_i) = 0$, for all $i = 1, \ldots, q$. By Remark 2.2.1, $D_j + C_j$ is nef (take $H = 0$ and $E = D_j + C_j$). But from the hypotheses on D it follows that $(D \cdot (D_j + C_j)) \leqslant (D \cdot C_j) < 0$, which contradicts the fact that D is pseudo-effective. □

Corollary 2.2.6. *Led D be a pseudo-effective $\mathbb{K}$-divisor on the surface X. Then there are only finitely many integral curves C on X such that*

$$(D \cdot C) < 0.$$

Proof. Let $C_1, \ldots, C_q$ be integral curves on X such that $(D \cdot C) < 0$, for all $i = 1, \ldots, q$. By Lemma 2.2.5, the intersection matrix $I(C_1, \ldots, C_q)$ is negative definite. This implies that the classes of the curves $C_1, \ldots, C_q$ define linearly independent elements of $\mathcal{N}^1(X)_{\mathbb{K}}$. Then the corollary is a consequence of the fact that the $\mathcal{N}^1(X)_{\mathbb{K}}$ is a finite dimensional vector space.

□

Lemma 2.2.7. *Let $C_1, \ldots, C_q$ be integral curves on the surface X such that the intersection matrix $I(C_1, \ldots, C_q)$ is negative definite. Let D' be an effective $\mathbb{K}$-divisor on X, and let $D \in V(C_1, \ldots, C_q)$ be a $\mathbb{K}$-divisor such that $((D' - D) \cdot C_i) \leqslant 0$, for all $i = 1, \ldots, q$. Then $D' - D \geqslant 0$.*

Proof. Write $D' = Z + D'_1$, with $Z \in V(C_1, \ldots, C_q)$ and $C_i \not\subset \operatorname{Supp}(D'_1)$, for all $i = 1, \ldots, q$. Then $(D'_1 \cdot C_i) \geqslant 0$, and hence $(Z \cdot C_i) = (D' \cdot C_i) - (D'_1 \cdot C_i) \leqslant (D' \cdot C_i) \leqslant (D \cdot C_i)$, for all $i = 1, \ldots, q$. Then by Lemma 2.2.2, $Z - D \geqslant 0$, and therefore $D' - D \geqslant 0$. □

Theorem 2.2.8 (**Zariski-Fujita**). *Let D be a pseudo-effective $\mathbb{K}$-divisor on a smooth projective surface X. Then there are uniquely determined $\mathbb{K}$-divisors P and N such that $D = P + N$, N is effective (possibly zero) and:*

1. *P is nef,*
2. *N is zero or has negative definite intersection matrix,*
3. *$P \cdot C = 0$ for every irreducible component C of N.*

Proof of the existence. If D is nef, put $P = D$ and $N = 0$. Otherwise, let $C_1, \ldots, C_{q_1}$ be all the integral curves C_i on X such that $(D \cdot C_i) < 0$ (by Corollary 2.2.6, there are only finitely many such curves). By Lemma 2.2.5, the intersection matrix $I(C_1, \ldots, C_{q_1})$ is negative definite. Therefore there exists a unique $\mathbb{K}$-divisor $N_1 \in V(C_1, \ldots, C_{q_1})$ such that $(N_1 \cdot C_i) = (D \cdot C_i)$, for all $i = 1, \ldots, q_1$. Since $(D \cdot C_i) < 0$, for all $i = 1, \ldots, q_1$ we infer that $N_1 \geqslant 0$ by Lemma 2.2.2. If $D_1 := D - N_1$ is nef, we take $P = D_1$ and $N = N_1$. If not, from the definition of N_1 and Lemma 2.2.3 it follows that D_1 is pseudo-effective. Let then $C_{q_1+1}, \ldots, C_{q_2}$ be all the integral curves C_j on X such that $(D_1 \cdot C_j) < 0$ (there are finitely many by Corollary 2.2.6). From Lemma 2.2.5 we deduce that the intersection matrix $I(C_1, \ldots, C_{q_2})$ is negative definite, and consequently there exists a unique $\mathbb{K}$-divisor $N_2 \in V(C_1, \ldots, C_{q_2})$ such that $(N_2 \cdot C_j) = (D_1 \cdot C_j)$, for all $j = 1, \ldots, q_2$. As earlier, $N_2 \geqslant 0$, and $D_2 := D_1 - N_2$ is pseudo-effective. If D_2 is nef we conclude by taking $P = D_2$ and $N = N_1 + N_2$. If not, we continue the above procedure, which has to stop after finitely many steps because the dimension of $\mathcal{N}^1(X)_{\mathbb{K}}$ is finite. This proves the existence part of our Theorem.

Proof of uniqueness. Let $D = P + N = P' + N'$ be two decompositions as in the Theorem. If $\mathrm{Supp}(N) = C_1 \cup \ldots \cup C_q$, we have $(P \cdot C_i) = 0$ and $(P' \cdot C_i) \geqslant 0$, for all $i = 1, \ldots, q$ by the nefness of P'. It follows that $(N \cdot C_i) = (D \cdot C_i) = (P' \cdot C_i) + (N' \cdot C_i) \geqslant (N' \cdot C_i)$, for all $i = 1, \ldots, q$. Then by Lemma 2.2.7 we get $N' \geqslant N$. Similarly $N \geqslant N'$, whence $N' = N$.

□

Remark 2.2.9. *Every D pseudo-effective $\mathbb{K}$-divisor admits a Zariski decomposition. Conversely, every divisor D admitting a Zariski decomposition, is necessarily pseudo-effective, because is sum of two pseudo-effective.*

Proposition 2.2.10. *Let D a divisor, D pseudo-effective, and let $D = P + N$ the Zariski decomposition of D, then the natural map*

$$H^0(X, \mathcal{O}_X(mD - [mN])) \to H^0(X, \mathcal{O}_X(mD))$$

is bijective for every $m \geqslant 1$.

In other words at least after passing to a multiple to clear denominators (hence that $mD - [mN] = mP$) this means that all the sections of a line bundle on a surface come from a nef divisor.

Proof. The issue is to show that if $D' \equiv_{lin} mD$ is an effective divisor, then $mN \leqslant D'$. It is enough to prove this after replacing each of the divisors in question by a multiple, hence without loss of generality we may assume to begin with P and N are integral and that $m = 1$. Given $D' \equiv_{lin} D$ effective, write $D' = N_1 + M_1$, where N_1 is an effective linear combination of the E_i and M_1 does not contain any of these components. We are required to prove that $N \leqslant N_1$. Since $D' - N \equiv_{num} P$ is perpendicular to each of the E_i and since M_1 is an effective divisor meeting each E_i properly, we see that

$$((N_1 - N) \cdot E_i) \leqslant 0 \tag{2.2}$$

for all i. Now write $N_1 - N = N' - N''$ where N' and N'' are non-negative linear combinations of the E_i with no common components, and assume for a contradiction that $N'' \neq 0$. Then $N'' \cdot N'' < 0$, and hence

$$((N_1 - N) \cdot N'') > 0$$

But this contradicts (2.2). $\square$

Corollary 2.2.11 (**Volume of divisors on a surface**)**.** *In the situation of Proposition 2.2.10,*

$$vol(D) = (P^2).$$

In particular, the volume of a integral pseudo-effective divisor on a surface is always a rational number.

Proof. In fact, the proposition implies that $vol(D) = vol(P) = (P^2)$, because P is nef. For a integral divisor we can take a $\mathbb{Q}$-Zariski decomposition, then we can take P a $\mathbb{Q}$-divisor. $\square$

Chapter 3

σ-decomposition

Now we try to face the Zariski decomposition from a different point of view. We will study the function σ_Γ, its continuity, and its link with the negative part of the Zariski decomposition. This contruction is due to Nakayama, [8].

3.1 Definitions of σ and τ functions

Definition 3.1.1. *Let X be a non-singular projective variety of dimension n and let B a big $\mathbb{R}$-divisor of X. The linear system $|B|$ is the set of effective $\mathbb{R}$-divisors linearly equivalent to B. Similarly, we define $|B|_{\mathbb{Q}}$ and $|B|_{num}$ to be the sets of effective $\mathbb{R}$-divisors E satisfying $E \equiv_{lin,\mathbb{Q}} B$ and $E \equiv_{num} B$, respectively.*

By definition, we write

$$|B|_{\mathbb{Q}} = \bigcup_{m \in \mathbb{N}} \frac{1}{m}|mB|.$$

There is a positive integer m_0 such that $|mB| \neq \emptyset$ for $m \geqslant m_0$, by the bigness, by Proposition 1.3.28.

Definition 3.1.2. *For a prime divisor Γ, we define:*

$$\sigma_\Gamma(B)_{\mathbb{Z}} := \inf\{mult_\Gamma E \mid E \in |B|\},$$

if $|B| \neq \emptyset$ ($= \infty$ else), and similarly we define

$$\begin{aligned} \sigma_\Gamma(B)_{\mathbb{Q}} &:= \inf\{mult_\Gamma E \mid E \in |B|_{\mathbb{Q}}\}; \\ \sigma_\Gamma(B) &:= \inf\{mult_\Gamma E \mid E \in |B|_{num}\}. \end{aligned}$$

Remark 3.1.3. *These three functions $\sigma_\Gamma(\cdot)_*$ ($* = \mathbb{Z}, \mathbb{Q}$, and $\emptyset$) satisfy the triangle inequality:*

$$\sigma_\Gamma(B_1 + B_2)_* \leqslant \sigma_\Gamma(B_1)_* + \sigma_\Gamma(B_2)_*. \tag{3.1}$$

Proof. $\sigma_\Gamma(B) \geqslant 0$ for all B big, and if $\Delta_1 \in |B_1|_*$ and $\Delta_2 \in |B_2|_*$ then $\Delta_1 + \Delta_2 \in |B_1 + B_2|_*$, hence the set $\{|B_1|_* + |B_1|_*\} \subset |B_1 + B_2|_*$, therefore applying σ_Γ at the left hand side and at the right hand side of the inclusion we are done because σ_Γ is an inf. □

Definition 3.1.4. *Similarly to the above, we define:*

$$\tau_\Gamma(B)_\mathbb{Z} := \sup\{mult_\Gamma E \mid E \in |B|\},$$

if $|B| \neq \emptyset$ ($= -\infty$ else), and

$$\begin{aligned} \tau_\Gamma(B)_\mathbb{Q} &:= \sup\{mult_\Gamma E \mid E \in |B|_\mathbb{Q}\}; \\ \tau_\Gamma(B) &:= \sup\{mult_\Gamma E \mid E \in |B|_{num}\}. \end{aligned}$$

Then these three functions $\tau_\Gamma(\cdot)_*$ satisfy the triangle inequality:

$$\tau_\Gamma(B_1 + B_2)_* \geqslant \tau_\Gamma(B_1)_* + \tau_\Gamma(B_2)_*$$

as above.

Remark 3.1.5. *The function $\tau_\Gamma(\cdot)$ is expressed also by*

$$\tau_\Gamma(B) = \max\{t \in \mathbb{R}_{\geqslant 0} \mid [B - t\Gamma] \in \overline{Eff(X)}\}.$$

Proof. First we note that on the right hand side of the equality we have that the $\sup\{t \in \mathbb{R}_{\geqslant 0} \mid [B - t\Gamma] \in \overline{Eff(X)}\}$ is a max because $\overline{Eff(X)}$ is a closed cone.

First we have to show that

$$\sup\{mult_\Gamma E \mid E \in |B|_{num}\} \leqslant \max\{t \geqslant 0 \mid [\ B - t\Gamma] \in \overline{Eff(X)}\}.$$

This is simple because, for all $E \in |B|_{num}$, $B - (mult_\Gamma E)\Gamma \equiv_{num} E - (mult_\Gamma E)\Gamma \geqslant 0$, then

$$[B - (mult_\Gamma E)\Gamma] \in \overline{Eff(X)}.$$

Hence

$$[B - \sup_E\{mult_\Gamma E\}\Gamma] \in \overline{Eff(X)}$$

because $\overline{Eff(X)}$ is closed.

Now it is sufficient to show the other inequality.

If $B - t\Gamma$ is pseudo-effective then for all $\epsilon > 0$ we have that $B - t\Gamma + \epsilon B = B_\epsilon$ is big by Remark 1.3.25. Then $(1+\epsilon)B = t\Gamma + B_\epsilon$. Now B_ϵ is big, then there exists $\Delta_\epsilon \geqslant 0$ such that $\Delta_\epsilon \equiv_{num} B_\epsilon$. Then $(1+\epsilon)B \equiv_{num} t\Gamma + \Delta_\epsilon$, hence $B \equiv_{num} \frac{t}{(1+\epsilon)}\Gamma + \frac{1}{(1+\epsilon)}\Delta_\epsilon \in Eff(X)$. Hence $\frac{t}{1+\epsilon} \leqslant t \leqslant \sup\{mult_\Gamma E \mid E \in |B|_{num}\}$ for all ϵ.

□

In particular, $B - \tau_\Gamma(B)\Gamma$ is pseudo-effective but not big, in fact $B - \tau_\Gamma(B)\Gamma - \epsilon\Gamma$ is not pseudo-effective (hence it is not big) for all $0 < \epsilon \ll 1$.

For $t < \tau_\Gamma(B)$, we have $\tau_\Gamma(B - t\Gamma) = \tau_\Gamma(B) - t$, obviously by the alternative definition. The inequality $(B - \tau_\Gamma(B)\Gamma) \cdot A^{n-1} \geqslant 0$ holds for any ample divisor A. In particular

$$\tau_\Gamma(B) \leqslant \frac{B \cdot A^{n-1}}{\Gamma \cdot A^{n-1}} < \infty. \tag{3.2}$$

Remark 3.1.6. *For all prime divisor Γ and for all big divisor B, we have that $B - \sigma_\Gamma(B)\Gamma$ is pseudo-effective.*

Proof. Since $\sigma_\Gamma(B) \leqslant \tau_\Gamma(B)$, then $B - \sigma_\Gamma(B)\Gamma = B - \tau_\Gamma(B)\Gamma + (\tau_\Gamma(B) - \sigma_\Gamma(B))\Gamma$, but $B - \tau_\Gamma(B)\Gamma$ is pseudo-effective by Remark 3.1.5, $(\tau_\Gamma(B) - \sigma_\Gamma(B))\Gamma$ is an effective $\mathbb{R}$-divisor, hence by the convexity of the pseudo-effective cone we have that $B - \sigma_\Gamma(B)\Gamma$ is pseudo-effective. □

Remark 3.1.7. *The following equalities and inequalities hold for the functions $\sigma_\Gamma(\cdot)_*$ and $\tau_\Gamma(\cdot)_*$:*

$$\begin{aligned} \sigma_\Gamma(B) \leqslant \sigma_\Gamma(B)_\mathbb{Q} \leqslant \tfrac{1}{m}\sigma_\Gamma(mB)_\mathbb{Z}, &\qquad \tau_\Gamma(B) \geqslant \tau_\Gamma(B)_\mathbb{Q} \geqslant \tfrac{1}{m}\tau_\Gamma,(mB)_\mathbb{Z}, \\ \sigma_\Gamma(qB)_\mathbb{Q} = q\sigma_\Gamma(B)_\mathbb{Q} &\qquad \tau_\Gamma(qB)_\mathbb{Q} = q\tau_\Gamma(B)_\mathbb{Q} \\ \sigma_\Gamma(tB) = t\sigma_\Gamma(B) &\qquad \tau_\Gamma(tB) = t\tau_\Gamma(B) \end{aligned}$$

for $m \in \mathbb{N}$, $q \in \mathbb{Q}_{>0}$, and $t \in \mathbb{R}_{>0}$.

Proof. We have that $\sigma_\Gamma(B) \leqslant \sigma_\Gamma(B)_\mathbb{Q}$ because $|B|_\mathbb{Q} \subset |B|_{num}$, and $\sigma_\Gamma(B)_\mathbb{Q} \leqslant \frac{1}{m}\sigma_\Gamma(mB)_\mathbb{Z}$ because $\frac{1}{m}|mB| \subset \bigcup_{m\in\mathbb{N}} \frac{1}{m}|mB| = |B|_\mathbb{Q}$, by the definition of $\sigma_\Gamma(\cdot)_*$.

If $\Delta \in |B|_\mathbb{Q}$, then we have that $q\Delta \in |qB|_\mathbb{Q}$, for all $q \in \mathbb{Q}$. In the same way, if $\Delta \in |B|_{num}$, then we have that $t\Delta \in |tB|_{num}$, for all $t \in \mathbb{R}$.

For $\tau_\Gamma(\cdot)_*$ we can proceed in the specular way.

□

Lemma 3.1.8. *Let d be a positive integer and let f be a function $\mathbb{N}_{\geqslant d} \to \mathbb{R}$ such that*

$$f(k_1 + k_2) \leqslant f(k_1) + f(k_2)$$

for any k_1, $k_2 \geqslant d$. Furthermore, suppose that the sequence $\{f(k)/k\}$ for $k \geqslant d$ is bounded below. Then the limit $\lim_{k\to\infty} f(k)/k$ exists.

The following simple proof is due to S.Mori:

Proof. Let us fix a sequence of integer $\{l_k\}$ such that for all k, $l_k > d$ and $\lim_{k\to\infty} \frac{f(l_k)}{l_k} = \liminf_m \frac{f(m)}{m}$. For all k, an integer $m > l_k$ has an expression $m = ql_k + r$ for $0 \leqslant q \in \mathbb{Z}$ and $l_k \leqslant r \leqslant 2l_k - 1$. Thus $f(m) \leqslant qf(l_k) + f(r)$. Hence

$$\frac{f(m)}{m} \leqslant \frac{qf(l_k) + f(r)}{ql_k + r} = (\frac{ql_k}{ql_k + r})\frac{f(l_k)}{l_k} + (\frac{r}{ql_k + r})\frac{f(r)}{r}.$$

By taking $m \to \infty$, we have:

$$\limsup_{m\to\infty} \frac{f(m)}{m} \leqslant \frac{f(l_k)}{l_k}.$$

Hence by taking $k \to \infty$ we have that $\limsup_{m\to\infty} \frac{f(m)}{m} \leqslant \liminf_{m\to\infty} \frac{f(m)}{m}$. Thus the limit exists.

□

By Lemma 3.1.8 and Remark 3.1.7 we have the following inequalities

$$\sigma_\Gamma(B)_\mathbb{Q} \leqslant \liminf_{\mathbb{N}\ni m\to\infty} \frac{1}{m}\sigma_\Gamma(mB)_\mathbb{Z} = \lim_{\mathbb{N}\ni m\to\infty} \frac{1}{m}\sigma_\Gamma(mB)_\mathbb{Z}, \tag{3.3}$$

$$\tau_\Gamma(B)_\mathbb{Q} \geqslant \limsup_{\mathbb{N}\ni m\to\infty} \frac{1}{m}\tau_\Gamma(mB)_\mathbb{Z} = \lim_{\mathbb{N}\ni m\to\infty} \frac{1}{m}\tau_\Gamma(mB)_\mathbb{Z}. \tag{3.4}$$

Lemma 3.1.9. *Let B be a big $\mathbb{R}$-divisor and Γ a prime divisor.*

1. $\sigma_\Gamma(A)_\mathbb{Q} = 0$ *for any ample* $\mathbb{R}$*-divisor* A.

2. $\lim_{\epsilon\to 0}\sigma_\Gamma(B+\epsilon A) = \sigma_\Gamma(B)$ *and* $lim_{\epsilon\to 0}\tau_\Gamma(B+\epsilon A) = \tau_\Gamma(B)$ *for any ample* $\mathbb{R}$*-divisor* A.

3. $\sigma_\Gamma(B)_\mathbb{Q} = \sigma_\Gamma(B)$ *and* $\tau_\Gamma(B)_\mathbb{Q} = \tau_\Gamma(B)$.

4. *The* $\mathbb{R}$*-divisor* $B^o := B - \sigma_\Gamma(B)\Gamma$ *satisfies* $\sigma_\Gamma(B^o) = 0$ *and* $\sigma_{\Gamma'}(B^o) = \sigma_{\Gamma'}(B)$ *for any other prime divisor* Γ'*. Furthermore,* B^o *is also big.*

5. *Let* $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ *be mutually distinct prime divisors with* $\sigma_{\Gamma_i}(B) = 0$ *for all* i*. Then, for any* $\epsilon > 0$*, there is an effective* $\mathbb{R}$*-divisor* $\Delta \in |B|_\mathbb{Q}$ *such that* $mult_{\Gamma_i}\Delta < \epsilon$ *for any* i.

Proof. 1) By (3.1) it suffices to show $\sigma_\Gamma(tA)_\mathbb{Q} = 0$ for any $t \in \mathbb{R}_{>0}$, and for every ample effective divisor A. The equality holds for $t \in \mathbb{Q}$, by the ampleness of A, in fact there exists $m > 0$ such that $|mA|$ has no fixed component by the ampleness, then $mA \equiv \sum_{i=1}^{s} a_i\Gamma_i$ with $\Gamma_i \neq \Gamma$, then $\sigma_\Gamma(mA)_\mathbb{Z} = 0$, hence by Remark 3.1.7 we have that $\sigma_\Gamma(A)_\mathbb{Q} = 0$.

Hence even for $t \notin \mathbb{Q}$,we have, for $q \in \mathbb{Q}_{>0}$, that

$$\sigma_\Gamma(tA)_\mathbb{Q} = \sigma_\Gamma((t-q)A + qA)_\mathbb{Q} \leqslant$$

$$\leqslant \sigma_\Gamma((t-q)A) \leqslant (t-q)mult_\Gamma A$$

then

$$\sigma_\Gamma(tA)_\mathbb{Q} \leqslant \lim_{\mathbb{Q}\ni q\to t}(t-q)mult_\Gamma A = 0$$

2) First we see that the limits have sense. Moreover we prove that for all pseudo-effective $\mathbb{R}$-divisor D and for all A ample divisor, the limits $\lim_{\epsilon\to 0}\sigma_\Gamma(D+\epsilon A)$ and $\lim_{\epsilon\to 0}\tau_\Gamma(D+\epsilon A)$ exist.

First we start with τ_Γ-function.

By Remark 3.1.5, $\tau_\Gamma(B) := \max\{t \geqslant 0 \mid [B - t\Gamma] \in \overline{Eff(X)}\}$, then it is obvious that we can extend this definition to pseudo-effective $\mathbb{R}$-divisors in a continuous way. Then the limit exists.

Now we take σ_Γ-function.

If $0 < t' < t$, using Remark 3.1.3 and (1), we have that

$$\sigma_\Gamma(D+tA) - \sigma_\Gamma(D+t'A) =$$

$$= \sigma_\Gamma(D+t'A+(t-t')A) - \sigma_\Gamma(D+t'A) \leqslant$$

$$\leqslant \sigma_\Gamma(D+t'A) + \sigma_\Gamma((t-t')A) - \sigma_\Gamma(D+t'A) =$$

$$= \sigma_\Gamma((t-t')A) = 0,$$

because $(t-t') > 0$ and A is ample. Then we have $\sigma_\Gamma(D+tA) \leqslant \sigma_\Gamma(D+t'A)$, then $\sigma_\Gamma(D+tA)$ is a monotone function. But now we know that for all t, $\sigma_\Gamma(D+tA) \leqslant \tau_\Gamma(D+tA)$, then for all $t \leqslant 1$ we have that

$$\sigma_\Gamma(D+tA) \leqslant \tau_\Gamma(D+tA) \leqslant$$

$$\leqslant \frac{(D+tA)\cdot A^{n-1}}{\Gamma\cdot A^{n-1}} \leqslant \frac{(D+A)\cdot A^{n-1}}{\Gamma\cdot A^{n-1}} < \infty.$$

Therefore this is a bounded monotone function, then it has limit.

Now $\tau_\Gamma(B+\epsilon A) \geqslant \tau_\Gamma(B)$ and $\sigma_\Gamma(B+\epsilon A) \leqslant \sigma_\Gamma(B)$ for any $\epsilon \in \mathbb{R}_{>0}$, since $\sigma_\Gamma(\epsilon A) = 0$. There exists a number $\delta \in \mathbb{R}_{>0}$ and an effective $\mathbb{R}$-divisor Δ satisfying $B \equiv_{lin,\mathbb{Q}} \delta A + \Delta$ by [8, II,3.16]. The inequalities

$$(1+\epsilon)\sigma_\Gamma(B) \leqslant \sigma_\Gamma(B+\epsilon\delta A) + \epsilon mult_\Gamma\Delta,$$

$$(1+\epsilon)\tau_\Gamma(B) \geqslant \tau_\Gamma(B+\epsilon\delta A) + \epsilon mult_\Gamma\Delta,$$

follow from $(1+\epsilon)B \equiv B + \epsilon\delta A + \epsilon\Delta$. Thus we have 2) by taking $\epsilon \to 0$.

3) Let A be a very ample divisor. Then $\tau_\Gamma(B+\epsilon A)_\mathbb{Q} \geqslant \tau_\Gamma(B)_\mathbb{Q}$ and $\sigma_\Gamma(B+\epsilon A)_\mathbb{Q} \leqslant \sigma_\Gamma(B)_\mathbb{Q}$ for any $\epsilon \in \mathbb{Q}_{>0}$. There exists an effective $\mathbb{R}$-divisor Δ such that $B \equiv_{lin,\mathbb{Q}} \delta A + \Delta$ for some $\delta \in \mathbb{Q}_{>0}$. The inequalities

$$(1+\epsilon)\sigma_\Gamma(B)_\mathbb{Q} \leqslant \sigma_\Gamma(B+\epsilon\delta A)_\mathbb{Q} + \epsilon mult_\Gamma\Delta,$$

$$(1+\epsilon)\tau_\Gamma(B)_\mathbb{Q} \geqslant \tau_\Gamma(B+\epsilon\delta A)_\mathbb{Q} + \epsilon mult_\Gamma\Delta,$$

follow from $(1+\epsilon)B \equiv B + \epsilon\delta A + \epsilon\Delta$. Thus we have

$$\sigma_\Gamma(B)_\mathbb{Q} = \lim_{\mathbb{Q}\ni\epsilon\to 0} \sigma_\Gamma(B+\epsilon A)_\mathbb{Q} \quad \text{and} \quad \tau_\Gamma(B)_\mathbb{Q} = \lim_{\mathbb{Q}\ni\epsilon\to 0} \tau_\Gamma(B+\epsilon A)_\mathbb{Q} \qquad (3.5)$$

The inequalities $\sigma_\Gamma(B)_\mathbb{Q} \geqslant \sigma_\Gamma(B)$ and $\tau_\Gamma(B)_\mathbb{Q} \leqslant \tau_\Gamma(B)$ follow from Remark 3.1.7. For an effective $\mathbb{R}$-divisor $\Delta \in |B|_{num}$, $B + \epsilon A - \Delta$ is ample for any

$\epsilon \in \mathbb{Q}_{>0}$. Here $\sigma_\Gamma(B+\epsilon A-\Delta)_\mathbb{Q}=0$ by 1) and $\lim_{\epsilon\to 0}\tau_\Gamma(B+\epsilon A-\Delta)_\mathbb{Q}=0$ by (3.2). Therefore, by (3.5), we have $\sigma_\Gamma(B)_\mathbb{Q}\leqslant mult_\Gamma\Delta\leqslant\tau_\Gamma(B)_\mathbb{Q}$. Thus the equalities in 3) hold.

4) If $\Delta\in|mB|$ for some $n\in\mathbb{N}$, then $mult_\Gamma\Delta\geqslant\sigma_\Gamma(mB)_\mathbb{Z}\geqslant m\sigma_\Gamma(B)$. Hence $\Delta-m\sigma_\Gamma(B)\Gamma\in|mB^o|$. In particular, $|B^o|_\mathbb{Q}+\sigma_\Gamma(B)\Gamma=|B|_\mathbb{Q}$, which implies the first half assertion of 4). To see that B^o is big we will prove the isomorphisms $|\lfloor mB\rfloor|\cong|\lfloor mB^o\rfloor|$ and we will use [8, II 3.7].

In fact if $B^o=\sum_i b_i\Gamma_i+b\Gamma$ with $\Gamma_i\neq\Gamma$ for all i, then $B=\sum_i b_i\Gamma_i+(b+\sigma)\Gamma$ where $\sigma=\sigma_\Gamma(B)$. Hence $mB=\sum_i mb_i\Gamma_i+m(b+\sigma)\Gamma$ and $\lfloor mB\rfloor=\sum_i\lfloor mb_i\rfloor\Gamma_i+\lfloor m(b+\sigma)\rfloor\Gamma$, but $\lfloor mb+m\sigma\rfloor=\lfloor mb\rfloor+\lfloor m\sigma\rfloor+v$, where $v=0$ or $v=1$, then

$$\lfloor mB\rfloor=\sum_i\lfloor mb_i\rfloor\Gamma_i+(\lfloor mb\rfloor+\lfloor m\sigma\rfloor+v)\Gamma=$$

$$=\lfloor mB^o\rfloor+(\lfloor m\sigma\rfloor+v)\Gamma.$$

Similarly

$$\langle mB\rangle=mB-\lfloor mB\rfloor=$$

$$=mB^o+m\sigma\Gamma-\lfloor mB^o\rfloor-(\lfloor m\sigma\rfloor+v)\Gamma=$$

$$=\langle mB^o\rangle+(\langle m\sigma\rangle-v)\Gamma.$$

First we note that the equality $\lfloor mB\rfloor=\lfloor mB^o\rfloor+(\lfloor m\sigma\rfloor+v)\Gamma$ gives the inclusion $|\lfloor mB^o\rfloor|\subset|\lfloor mB\rfloor|$ naturally.

Now let $\Delta\in|\lfloor mB\rfloor|$, we have to show that there exists an effective $\Delta_1\in|\lfloor mB^o\rfloor|$ such that $\Delta=\Delta_1+(\lfloor m\sigma\rfloor+v)\Gamma$. Hence let $\Delta\in|\lfloor mB\rfloor|$, then $\Delta+\langle mB\rangle$ is effective and $\Delta+\langle mB\rangle\equiv_{lin}\lfloor mB\rfloor+\langle mB\rangle=mB$. Hence

$$m\sigma\leqslant mult_\Gamma(\Delta+\langle mB\rangle)=mult_\Gamma\Delta+mult_\Gamma\langle mB\rangle=$$

$$=mult_\Gamma\Delta+mult_\Gamma\langle mB^o\rangle+\langle m\sigma\rangle-v.$$

Then

$$mult_\Gamma\Delta\geqslant m\sigma-mult_\Gamma\langle mB^o\rangle-\langle m\sigma\rangle+v$$

therefore

$$mult_\Gamma\Delta-\lfloor m\sigma\rfloor-v+mult_\Gamma\langle mB^o\rangle\geqslant 0.$$

But now $mult_\Gamma\Delta - \lfloor m\sigma \rfloor - v \in \mathbb{Z}$ and $mult_\Gamma\langle mB^o\rangle \in [0,1)$, then $mult_\Gamma\Delta - \lfloor m\sigma \rfloor - v + mult_\Gamma\langle mB^o\rangle \geqslant 0$ if and only if $mult_\Gamma\Delta - \lfloor m\sigma \rfloor - v \geqslant 0$.

Now $\Delta_1 := \Delta - (\lfloor m\sigma \rfloor + v)\Gamma \geqslant 0$, and

$$\Delta_1 = \Delta - (\lfloor m\sigma \rfloor + v)\Gamma \equiv_{lin}$$

$$\equiv_{lin} \lfloor mB \rfloor - (\lfloor m\sigma \rfloor + v)\Gamma = \lfloor mB^o \rfloor.$$

Hence there is an injective map $|\lfloor mB \rfloor| \hookrightarrow |\lfloor mB^o \rfloor|$.

5) There exist a number $m \in \mathbb{N}$ and effective $\mathbb{R}$-divisors $\Delta_i \in |mB|$ for $1 \leqslant i \leqslant l$ such that $mult_{\Gamma_i}\Delta_i < m\epsilon$. For an $\mathbb{R}$-divisor $\Delta \in |mB|$, the condition: $mult_{\Gamma_i}\Delta < m\epsilon$, is a Zariski-open condition in the projective space $|mB|$. Thus we can find an $\mathbb{R}$-divisor $\Delta \in |mB|$ satisfying $mult_{\Gamma_i}\Delta < m\epsilon$ for any i. $\square$

Lemma 3.1.10. *Let D be a pseudo-effective $\mathbb{R}$-divisor of X.*

1. *For any ample $\mathbb{R}$-divisor A,*

$$\lim_{\epsilon\to 0}\sigma_\Gamma(D+\epsilon A) \leqslant \lim_{\epsilon\to 0}\tau_\Gamma(D+\epsilon A) \leqslant \frac{D\cdot A^{n-1}}{\Gamma\cdot A^{n-1}} < +\infty.$$

2. *The limits $\lim_{\epsilon\to 0}\sigma_\Gamma(D+\epsilon A)$ and $\lim_{\epsilon\to 0}\tau_\Gamma(D+\epsilon A)$ do not depend on the choice of ample divisors A.*

Proof. 1) In the proof of Lemma 3.1.9 (2) we proved that the limits have sense, then this is a consequence of (3.2).

2) Let A' be another ample $\mathbb{R}$-divisor. Then there are an effective $\mathbb{R}$-divisor Δ and a positive number δ such that $A' \equiv_{num} \delta A + \Delta$ by Theorem 1.3.11. Hence we have

$$\sigma_\Gamma(D+\epsilon\delta A) + \epsilon mult_\Gamma\Delta \geqslant \sigma_\Gamma(D+\epsilon A'),$$

$$\tau_\Gamma(D+\epsilon\delta A) + \epsilon mult_\Gamma\Delta \leqslant \tau_\Gamma(D+\epsilon A').$$

They induce inequalities $\lim_{\epsilon\to 0}\sigma_\Gamma(D+\epsilon A) \geqslant \lim_{\epsilon\to 0}\sigma_\Gamma(D+\epsilon A')$ and $\lim_{\epsilon\to 0}\tau_\Gamma(D+\epsilon A) \leqslant \lim_{\epsilon\to 0}\tau_\Gamma(D+\epsilon A')$. Changing A with A', we have the equalities. $\square$

Definition 3.1.11. *For a pseudo-effective $\mathbb{R}$-divisor D and a prime divisor Γ, we define*

$$\sigma_\Gamma(D) := \lim_{\epsilon \to 0} \sigma_\Gamma(D + \epsilon A) \quad \textit{and} \quad \tau_\Gamma(D) := \lim_{\epsilon \to 0} \tau_\Gamma(D + \epsilon A).$$

Note that if $D \equiv_{num} D'$, then $\sigma_\Gamma(D) = \sigma_\Gamma(D')$ and $\tau_\Gamma(D) = \tau_\Gamma(D')$. In particular, σ_Γ and τ_Γ are functions on the closed convex cone $\overline{Eff(X)}$. Here, σ_Γ is lower convex and τ_Γ is upper convex. Obviously, we have another expression of τ_Γ:

$$\tau_\Gamma(D) = \max\{t \in \mathbb{R}_{\geqslant 0} \mid D - t\Gamma \in \overline{Eff(X)}\}.$$

Remark 3.1.12. *Let $D, D' \in \overline{Eff(X)}$, then $\sigma_\Gamma(D+D') \leqslant \sigma_\Gamma(D)+\sigma_\Gamma(D')$.*

Proof. Let $D, D' \in \overline{Eff(X)}$, and let A be an ample divisor. Then

$$\sigma_\Gamma(D + D') = \lim_{\epsilon \to 0} \sigma_\Gamma(D + D' + \epsilon A) \leqslant$$

$$\leqslant \lim_{\epsilon \to 0}(\sigma_\Gamma(D + \frac{\epsilon}{2}A) + \sigma_\Gamma(D' + \frac{\epsilon}{2}A)) =$$

$$= \lim_{\epsilon \to 0} \sigma_\Gamma(D + \frac{\epsilon}{2}A) + \lim_{\epsilon \to 0} \sigma_\Gamma(D + \frac{\epsilon}{2}A) =$$

$$= \sigma_\Gamma(D) + \sigma_\Gamma(D').$$

□

Remark 3.1.13. *For all prime divisor Γ, we have that for all pseudo effective $\mathbb{R}$-divisor D, then we have that $D - \sigma_\Gamma(D)\Gamma$ is pseudo-effective.*

Proof. $\sigma_\Gamma(D) \leqslant \tau_\Gamma(D)$, then we have $D - \sigma_\Gamma(D)\Gamma = D - \tau_\Gamma(D)\Gamma + (\tau_\Gamma(D) - \sigma_\Gamma(D))\Gamma$, but $D - \tau_\Gamma(D)\Gamma$ is pseudo effective, $(\tau_\Gamma(D) - \sigma_\Gamma(D))\Gamma$ is an effective $\mathbb{R}$-divisor, then $D - \sigma_\Gamma(D)\Gamma$ is pseudo-effective. □

Lemma 3.1.14. *1. $\sigma_\Gamma : \overline{Eff(X)} \to \mathbb{R}_{\geqslant 0}$ is lower semi-continuous and $\tau_\Gamma : \overline{Eff(X)} \to \mathbb{R}_{\geqslant 0}$ is upper semi-continuous. Both functions are continuous on $Big(X)$.*

2. $\lim_{\epsilon \to 0} \sigma_\Gamma(D + \epsilon E) = \sigma_\Gamma(D)$ and $\lim_{\epsilon \to 0} \tau_\Gamma(D + \epsilon E) = \tau_\Gamma(D)$ for any pseudo-effective $\mathbb{R}$-divisor E.

3. *Let* $\Gamma_1, \ldots, \Gamma_l$ *be mutually distinct prime divisors such that* $\sigma_{\Gamma_i}(D) = 0$. *Then, for any ample* $\mathbb{R}$*-divisor* A*, there exists an effective* $\mathbb{R}$*-divisor* Δ *such that* $\Delta \equiv_{lin,\mathbb{Q}} D + A$ *and* $\Gamma_i \not\subset Supp(\Delta)$ *for any* i*.*

Proof. 1) Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of pseudo-effective $\mathbb{R}$-divisors such that $[D_n]$ are convergent to $[D]$. Let us take a norm $||\cdot||$ for the finite-dimensional real vector space $\mathcal{N}^1(X)_{\mathbb{R}}$ and let U_r be the open ball $\{e \in \mathcal{N}^1(X)_{\mathbb{R}} \mid ||e|| < r\}$ for $r \in \mathbb{R}_{>0}$. We fix an ample $\mathbb{R}$-divisor A on X. Then, for any $r > 0$, there is a number n_0 such that $[D - D_n] \in U_r$ for $n \geqslant n_0$. For any $\epsilon > 0$, there is an $r > 0$ such that $U_r + \epsilon A$ is contained in the ample cone $Amp(X)$ by Remark 1.2.13. Applying the triangle inequalities to $D + \epsilon A = (D - D_n + \epsilon A) + D_n$, we have

$$\sigma_\Gamma(D) = \lim_{\epsilon\to 0} \sigma_\Gamma(D + \epsilon A) \leqslant \liminf_{n\to\infty} \sigma_\Gamma(D_n),$$
$$\tau_\Gamma(D) = \lim_{\epsilon\to 0} \tau_\Gamma(D + \epsilon A) \geqslant \limsup_{n\to\infty} \tau_\Gamma(D_n).$$

In fact $\sigma_\Gamma(D + \epsilon A) \leqslant \sigma_\Gamma(D - D_n + \epsilon A) + \sigma_\Gamma(D_n) = \sigma_\Gamma(D_n)$, because $(D - D_n + \epsilon A)$ is ample (for τ_Γ it is sufficient the positivity of the function).

Next assume that D is big. Then there is a positive number δ such that $D - \delta A$ is still big by Corollary 1.3.19. We can take $r_1 > 0$ such that $D - \delta A + U_{r_1} \subset Big(X)$ by the openness of the big cone. For any $\epsilon > 0$ there is a real number $r \in (0, r_1)$ such that $U_r + \epsilon A \subset Amp(X)$. Applying the triangle inequalities to $D_n + (\epsilon - \delta)A = (D_n - D + \epsilon A) + D - \delta A$, for $\epsilon < \delta$, we have

$$\limsup_{n\to\infty} \sigma_\Gamma(D_n) \leqslant \sigma_\Gamma(D - \delta A) \quad \text{and} \quad \liminf_{n\to\infty} \tau_\Gamma(D_n) \geqslant \tau_\Gamma(D - \delta A),$$

as above. Hence it is enough to show

$$\lim_{t\to 0} \sigma_\Gamma(D - tA) = \sigma_\Gamma(D) \quad \text{and} \quad \lim_{t\to 0} \tau_\Gamma(D - tA) = \tau_\Gamma(D).$$

Since $D-\delta A$ is big, there exists an effective $\mathbb{R}$-divisor Δ with $D-\delta A \equiv_{num} \Delta$ by Remark 1.3.21. Hence $D - t\delta A \equiv_{num} (1-t)D + t\Delta$ for any $t > 0$, which induce

$$\sigma_\Gamma(D - t\delta A) \leqslant (1-t)\sigma_\Gamma(D) + t mult_\Gamma \Delta,$$

$$\tau_\Gamma(D - t\delta A) \geqslant (1-t)\tau_\Gamma(D) + t mult_\Gamma \Delta.$$

By taking $t \to 0$, we are done.

2) By 1), we have $\liminf_{\epsilon\to 0}\sigma_\Gamma(D+\epsilon E) \geqslant \sigma_\Gamma(D)$ and $\limsup_{\epsilon\to 0}\tau_\Gamma(D+\epsilon E) \leqslant \tau_\Gamma(D)$. On the other hand, $\sigma_\Gamma(D+\epsilon E) \leqslant \sigma_\Gamma(D)+\epsilon\sigma_\Gamma(E)$ and $\tau_\Gamma(D+\epsilon E) \geqslant \tau_\Gamma(D)+\epsilon\tau_\Gamma(E)$ for any $\epsilon > 0$. Thus we have the other inequalities by taking $\epsilon \to 0$.

3) Let us take $m \in \mathbb{N}$ such that $mA+\Gamma_i$ is ample for any i. For any small $\epsilon > 0$, there exist positive rational numbers $\lambda, \{\delta_i\}$, and an effective $\mathbb{R}$-divisor B such that $B+\sum_{i=1}^l \delta_i\Gamma_i \equiv_{lin,\mathbb{Q}} D+\lambda A$, $\Gamma_i \not\subset SuppB$ for any i, and $m(\sum_i \delta_i)+\lambda < \epsilon$. Then

$$B+\sum_{i=1}^{l}\delta_i(mA+\Gamma_i) \equiv_{lin,\mathbb{Q}} D+(m\sum_{i=1}^{l}\delta_i+\lambda)A.$$

Thus we can find an expected effective $\mathbb{R}$-divisor. □

Lemma 3.1.15. *Let D be a pseudo-effective $\mathbb{R}$-divisor, $\Gamma_1,\dots,\Gamma_l$ mutually distinct prime divisors, and let $s_1,\dots,s_l$ be real numbers with $0 \leqslant s_i \leqslant \sigma_{\Gamma_i}(D)$. Then $\sigma_{\Gamma_i}(D-\sum_{j=1}^l s_j\Gamma_j) = \sigma_{\Gamma_i}(D)-s_i$ for all i.*

Proof. If D is big, for simplicity we write $\sigma_i = \sigma_{\Gamma_i}(D)$, and we take Δ an effective $\mathbb{R}$-divisor such that $\Delta \equiv_{num} D$. Then

$$mult_{\Gamma_i}\Delta \geqslant \sigma_i \geqslant s_i$$

for all i. Hence $\Delta-\sum_j s_j\Gamma_j \geqslant 0$ and $\Delta-\sum_j s_j\Gamma_j \equiv_{num} D-\sum_j s_j\Gamma_j$, therefore $mult_{\Gamma_i}(\Delta-\sum_j s_j\Gamma_j) \geqslant \sigma_{\Gamma_i}(D-\sum_j s_j\Gamma_j)$, then $mult_{\Gamma_i}\Delta-s_i \geqslant \sigma_{\Gamma_i}(D-\sum_j s_j\Gamma_j)$ for all $\Delta \equiv_{num} D$, $\Delta \geqslant 0$ and for all i. Then $\sigma_{\Gamma_i}(D)-s_i \geqslant \sigma_{\Gamma_i}(D-\sum_j s_j\Gamma_j)$ for all i.

Now let $\Delta \geqslant 0$ such that $\Delta \equiv_{num} D-\sum_j s_j\Gamma_j$ then $\Delta+\sum_j s_j\Gamma_j \equiv_{num} D$, hence $mult_{\Gamma_i}(\Delta+\sum_j s_j\Gamma_j) \geqslant \sigma_{\Gamma_i}(D)$, therefore $mult_{\Gamma_i}\Delta+s_i \geqslant \sigma_{\Gamma_i}(D)$, for all $\Delta \equiv_{num} D-\sum_j s_j\Gamma_j$, then $\sigma_{\Gamma_i}(D)-s_i \leqslant \sigma_{\Gamma_i}(D-\sum_j s_j\Gamma_j)$.

If D is not big, let $\epsilon > 0$ be a real number satisfying $s_i > \epsilon$ for any i with $s_i > 0$. We define $s(\epsilon)$ to be the following number:

$$s_i(\epsilon) := \begin{cases} s_i-\epsilon & \text{if } s_i > 0 \\ 0 & \text{if } s_i = 0. \end{cases}$$

Let us consider $\mathbb{R}$-divisors $E := D-\sum_{j=1}^l s_j\Gamma_j$ and $E(\epsilon) := D-\sum_{j=1}^l s_j(\epsilon)\Gamma_j$. There exist an ample $\mathbb{R}$-divisor A and a real number $\delta > 0$

satisfying $\sigma_{\Gamma_i}(D+\delta A) \geqslant s_i(\epsilon)$ for all i, in fact if we fix an ample divisor A and we take $\sigma_{\Gamma_i}(D+\frac{1}{n}A)$, by the proof of Lemma 3.1.9 (2) we have that $\sigma_{\Gamma_i}(D+\frac{1}{n}A) \to \sigma_{\Gamma_i}(D)$, then we can take $\overline{n} := \max_i\{\min\{n \text{ such that } |\sigma_{\Gamma_i}(D) - \sigma_{\Gamma_i}(D+\frac{1}{n}A)| < \epsilon\}\}$.

Then $E(\epsilon) + \delta A$ is also big by Remark 1.3.25 and $\sigma_{\Gamma_i}(E(\epsilon) + \delta A) = \sigma_{\Gamma_i}(D+\delta A) - s_i(\epsilon)$. Thus $\sigma_{\Gamma_i}(E(\epsilon)) = \lim_{\delta\to 0}\sigma_{\Gamma_i}(E(\epsilon)+\delta A) = \sigma_{\Gamma_i}(D) - s_i(\epsilon)$ by Lemma 3.1.14 (2). Then $\sigma_{\Gamma_i}(E) \leqslant \sigma_{\Gamma_i}(D) - s_i$ by the semi-continuity shown in Lemma 3.1.14 (1). On the other hand, $\sigma_{\Gamma_i}(D) \leqslant \sigma_{\Gamma_i}(E) + s_i$ follows from $D = E + \sum_{j=1}^{l} s_j\Gamma_j$ by the lower convexity of σ_{Γ_i}.

□

Corollary 3.1.16. *Let D be a pseudo-effective $\mathbb{R}$-divisor and let $\Gamma_1, \ldots, \Gamma_l$ be mutually distinct prime divisors with $\sigma_{\Gamma_i}(D) > 0$ for any i. Then, for $s_i \in \mathbb{R}_{\geqslant 0}$,*

$$\sigma_{\Gamma_i}(D + \sum s_j\Gamma_j) = \sigma_{\Gamma_i}(D) + s_i.$$

Proof. Let E be the $\mathbb{R}$-divisor $D + \sum s_j\Gamma_j$ and let $\sigma_i = \sigma_{\Gamma_i}(D)$. For $0 < c < 1$, we have

$$(1-c)(D - \sum \sigma_j\Gamma_j) + cE = D + \sum(-(1-c)\sigma_j + cs_j)\Gamma_j.$$

Let c be a number with $0 < c < \frac{\sigma_i}{(s_i+\sigma_i)}$ for any i. Then $-\sigma_i < -(1-c)\sigma_i + cs_i < 0$ and $s'_i := (1-c)\sigma_i + cs_i < \sigma_i$. By Lemma 3.1.15, we infer that $\sigma_{\Gamma_i}(E) \geqslant \sigma_i + s_i$:

In fact $\sigma_{\Gamma_i}(D - \sum_j s'_j\Gamma_j) = \sigma_{\Gamma_i}(D) - s'_i = \sigma_i - s'_i$ but

$$\sigma_{\Gamma_i}(D - \sum_j s'_j\Gamma_j) = \sigma_{\Gamma_i}((1-c)(D - \sum \sigma_j\Gamma_j) + cE) \leqslant$$

$$\leqslant (1-c)\sigma_{\Gamma_i}(D - \sum_j \sigma_j\Gamma_j) + c\sigma_{\Gamma_i}(E) \leqslant c\sigma_{\Gamma_i}(E),$$

because $\sigma_{\Gamma_i}(D - \sum_j \sigma_j\Gamma_j) = 0$ by Lemma 3.1.9 (4). Now

$$c\sigma_{\Gamma_i}(E) \geqslant \sigma_i - s'_i = \sigma_i - (1-c)\sigma_i + cs_i =$$

$$= c(\sigma_i + s_i)$$

then $\sigma_{\Gamma_i}(E) \geqslant \sigma_i + s_i$.

The other inequality is derived from the lower convexity of σ_Γ, that is $\sigma_\Gamma(E) \leqslant \sigma_\Gamma(D) + \sigma_\Gamma(\sum_j s_j\Gamma_j)$. □

Proposition 3.1.17. *Let D be a pseudo-effective $\mathbb{R}$-divisor and let $\Gamma_1, \dots, \Gamma_l$ be mutually distinct prime divisors of X with $\sigma_{\Gamma_i}(D) > 0$ for any i. Then*

$$\sigma_{\Gamma_i}(\sum_{j=1}^{l} x_j \Gamma_j) = x_i$$

for any $x_1, \dots, x_l \in \mathbb{R}_{\geqslant 0}$. In particular, $[\Gamma_1], \dots, [\Gamma_l]$ are linearly independent in $\mathcal{N}^1(X)_{\mathbb{R}}$.

Proof. Let us take $\alpha \in \mathbb{R}_{>0}$ with $\sigma_{\Gamma_i}(D) > \alpha x_i$ for any i. Then by Remark 3.1.13 we have that $D - \alpha \sum x_j \Gamma_j$ is pseudo-effective and by Remark 3.1.12

$$\sigma_{\Gamma_i}(D) \leqslant \sigma_{\Gamma_i}(D - \alpha \sum x_j \Gamma_j) + \alpha \sigma_{\Gamma_i}(\sum x_j \Gamma_j).$$

Thus the equality $\sigma_{\Gamma_i}(\sum x_j \Gamma_j) = x_i$ follows from Lemma 3.1.15. Suppose that there is a linear relation

$$\sum_{i=1}^{s} a_i \Gamma_i \equiv \sum_{j=s+1}^{l} b_j \Gamma_j$$

for some $a_i, b_j \in \mathbb{R}_{\geqslant 0}$ and for some $1 \leqslant s < l$. Then

$$a_k = \sigma_{\Gamma_k}(\sum_{i=1}^{s} a_i \Gamma_i) = \sigma_{\Gamma_k}(\sum_{j=s+1}^{l} b_j \Gamma_j) = 0$$

for $k \leqslant s$. Hence $a_i = b_j = 0$ for all i, j. □

Corollary 3.1.18. *For any pseudo-effective $\mathbb{R}$-divisor D, the number of prime divisors Γ satisfying $\sigma_\Gamma(D) > 0$ is less than the Picard number $\rho(X)$.*

3.2 Zariski decomposition problem

Definition 3.2.1. *Let D be a pseudo-effective $\mathbb{R}$-divisor of a non–singular projective variety X. We define*

$$N_\sigma(D) := \sum \sigma_\Gamma(D)\Gamma,$$

and $P_\sigma(D) := D - N_\sigma(D)$.

The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is called the σ-decomposition of D. Here, $P_\sigma(D)$ and $N_\sigma(D)$ are called the positive and the negative parts of the σ-decomposition of D, respectively.

Definition 3.2.2. *Let $Mv'(X)$ be the convex cone in $\mathcal{N}^1(X)_{\mathbb{R}}$ generated by $[D]$ of all the fixed part free divisors D ($|D|_{fix} = 0$). We denote its closure by $\overline{Mv(X)}$ and the interior of $\overline{Mv(X)}$ by $Mv(X)$. The cones $\overline{Mv(X)}$ and $Mv(X)$ are called the movable cone and the strictly movable cone, respectively. An $\mathbb{R}$-divisor D is called movable if $[D] \in \overline{Mv(X)}$.*

Remark 3.2.3. *There are inclusions $Nef(X) \subset \overline{Mv(X)} \subset \overline{Eff(X)}$ and $Amp(X) \subset Mv(X) \subset Big(X)$.*

Proof. First we note that $Amp(X) \subset Mv(X)$, because if A is an ample integral divisor, there exists an $m \gg 0$ such that mA is very ample, then $|mA|$ has no fixed component, hence $[mA]$ is in $Mv'(X)$, but $Mv'(X)$ is a cone, then $[A]$ is in $Mv'(X)$ (then $Amp(X) \subset Mv'(X)$, but $Amp(X)$ is open by Remark 1.2.13, then $Amp(X) \subset Int(\overline{Mv(X)})$).

For the other inclusion we note that if $[D]$ is in the strictly movable cone then $[D]$ is an effective class, because the linear series $|D| \neq \emptyset$. Then $Mv(X) \subset Eff(X) \subset \overline{Eff(X)}$, then $Mv(X) \subset Int(\overline{Eff(X)}) = Big(X)$, by Theorem 1.3.24.

Now taking the closures of the open cones we have $Nef(X) \subset \overline{Mv(X)} \subset \overline{Eff(X)}$. □

Proposition 3.2.4. *Let D be a pseudo-effective $\mathbb{R}$-divisor.*

1. *$N_\sigma(D) = 0$ if and only if D is movable.*
2. *If $D - \Delta$ is movable for an effective $\mathbb{R}$-divisor Δ, then $\Delta \geqslant N_\sigma(D)$.*

Proof. 1) Assume that $N_\sigma(D) = 0$. Then, by the proof of Lemma 3.1.14 (3), we infer that $[D+A] \in Mv'(X)$ for any ample $\mathbb{R}$-divisor A. In fact, for all Γ, Γ is not a fixed component of $[D+A]$, because by $\sigma_\Gamma(D) = 0$ we know that there exists $\Delta \in [D+A]$ such that $\Gamma \not\subset Supp(\Delta)$. Therefore $[D] \in \overline{Mv(X)}$ because for all A ample divisor, $\lim_{n\to\infty}[D + \frac{1}{n}A] = [D]$. If $[D] \in Mv'(X)$ then we can write $D \equiv_{num} \sum_{i=1}^{s} a_i\Gamma_i$ with $\Gamma_i \neq \Gamma$, then $\sigma_\Gamma(D) = 0$ for all Γ. Now, by Lemma 3.1.14 (1), that is the lower semi-continuity of σ_Γ, we know that $\sigma_\Gamma(D) = 0$, for all $D \in \overline{Mv(X)}$.

2) By (1), $N_\sigma(D-\Delta) = 0$. Thus $\sigma_\Gamma(D) \leqslant \sigma_\Gamma(D-\Delta)+\sigma_\Gamma(\Delta) \leqslant mult_\Gamma\Delta$ for any prime divisor Γ. Therefore $N_\sigma(D) \leqslant \Delta$. □

Lemma 3.2.5. *Let D be a pseudo-effective $\mathbb{R}$-divisor, Γ a prime divisor, and Δ an effective $\mathbb{R}$-divisor with $\Delta \leqslant N_\sigma(D)$. Then*

$$\tau_\Gamma(D) = \tau_\Gamma(D - \Delta) + mult_\Gamma \Delta.$$

In particular, $\tau_\Gamma(D) = \tau_\Gamma(P_\sigma(D)) + \sigma_\Gamma(D)$.

Proof. We know $\tau_\Gamma(D) \geqslant \sigma_\Gamma(D) \geqslant mult_\Gamma \Delta$. If $D - t\Gamma$ is pseudo-effective for some $t \in \mathbb{R}_{\geqslant 0}$, then $\sigma_{\Gamma'}(D - t\Gamma) \geqslant \sigma_{\Gamma'}(D) \geqslant mult_{\Gamma'}\Delta$ for any prime divisor $\Gamma' \neq \Gamma$. We claim that $D - \Delta$ is pseudo-effective. To see this, we set $\Delta = \sum_{i=1}^{s}(mult_{\Gamma_i}\Delta)\Gamma_i$ and we argue by induction on s:

For $s = 1$, $D - (mult_{\Gamma_1}\Delta)\Gamma_1$ is pseudo-effective by Remark 3.1.6.

If $1 \leqslant r < s$ and $D - \sum_{i=1}^{r}(mult_{\Gamma_i}\Delta)\Gamma_i$ is pseudo-effective then

$$\sigma_{\Gamma_{r+1}}(D - \sum_{i=1}^{r}(mult_{\Gamma_i}\Delta)\Gamma_i) \geqslant \sigma_{\Gamma_{r+1}}(D) \geqslant mult_{\Gamma_{r+1}}\Delta,$$

hence $D - \sum_{i=1}^{r}(mult_{\Gamma_i}\Delta)\Gamma_i - (mult_{\Gamma_{r+1}}\Delta)\Gamma_{r+1} = D - \sum_{i=1}^{r+1}(mult_{\Gamma_i}\Delta)\Gamma_i$ is pseudo-effective by Remark 3.1.6. Hence $D - \Delta$ is pseudo-effective.

Now suppose $\Gamma = \Gamma_1$ for simplicity. For all t such that $D - t\Gamma_1$ is pseudo-effective, we have that

$$D - t\Gamma_1 - \sum_{i=2}^{s}(mult_{\Gamma_i}\Delta)\Gamma_i =$$

$$= D - t\Gamma_1 + (mult_{\Gamma_1}\Delta)\Gamma_1 - (mult_{\Gamma_1}\Delta)\Gamma_1 - \sum_{i=2}^{s}(mult_{\Gamma_i}\Delta)\Gamma_i =$$

$$= D - \Delta + (mult_{\Gamma_1}\Delta - t)\Gamma_1$$

is pseudo-effective as above.

If $t \geqslant mult_{\Gamma_1}\Delta$, then we have that $t - mult_\Gamma \Delta \leqslant \tau_{\Gamma_1}(D - \Delta)$, else the inequality is obvious. In particular we have that

$$\tau_{\Gamma_1}(D) - mult_{\Gamma_1}\Delta \leqslant \tau_{\Gamma_1}(D - \Delta).$$

On the other hand,

$$D - \Delta - \tau_{\Gamma_1}(D - \Delta)\Gamma_1 \leqslant D - (\tau_{\Gamma_1}(D - \Delta) + mult_{\Gamma_1}\Delta)\Gamma_1.$$

Then $D-(\tau_{\Gamma_1}(D-\Delta)+mult_{\Gamma_1}\Delta)\Gamma_1$ is pseudo-effective, because

$$D-(\tau_{\Gamma_1}(D-\Delta)+mult_{\Gamma_1}\Delta)\Gamma_1-(D-\Delta-\tau_{\Gamma_1}(D-\Delta)\Gamma_1)\geqslant 0$$

is effective, $D-\Delta-\tau_{\Gamma_1}(D-\Delta)\Gamma_1$ is pseudo-effective by Remark 3.1.5 then we know that the sum of two pseudo-effective is pseudo-effective, then

$$D-(\tau_{\Gamma_1}(D-\Delta)+mult_{\Gamma_1}\Delta)\Gamma_1=$$

$$=D-(\tau_{\Gamma_1}(D-\Delta)+mult_{\Gamma_1}\Delta)\Gamma_1-(D-\Delta-\tau_{\Gamma_1}(D-\Delta)\Gamma_1)+$$

$$+(D-\Delta-\tau_{\Gamma_1}(D-\Delta)\Gamma_1)$$

is pseudo-effective.

Therefore $D-(\tau_{\Gamma_1}(D-\Delta)+mult_{\Gamma_1}\Delta)\Gamma_1$ is pseudo-effective, hence

$$\tau_{\Gamma_1}(D-\Delta)+mult_{\Gamma_1}\Delta\leqslant\tau_{\Gamma_1}(D).$$

Thus, for all Γ, we have the equality

$$\tau_{\Gamma}(D-\Delta)+mult_{\Gamma}\Delta=\tau_{\Gamma}(D).$$

□

Definition 3.2.6. *The σ-decomposition $D=P_\sigma(D)+N_\sigma(D)$ for a pseudo-effective $\mathbb{R}$-divisor is called a Zariski decomposition if $P_\sigma(D)$ is nef.*

Remark 3.2.7. *If X is a surface, then the movable cone $\overline{Mv(X)}$ coincides with the nef cone $Nef(X)$. Therefore Proposition 3.2.4 implies that the σ-decomposition of a pseudo-effective $\mathbb{R}$-divisor D is nothing but the usual Zariski decomposition.*

Proof. If $[D]\in Mv'(X)$ then we have that $D\cdot C\geqslant 0$ for all curve C, because we can write $D=\sum\alpha_i\Gamma_i$ with $\Gamma_i\neq C$, then $Mv'(X)\subset Nef(X)$. Hence we have that $\overline{Mv(X)}\subset Nef(X)$, therefore $\overline{Mv(X)}=Nef(X)$ by Remark 3.2.3.

Let $D=P_D+N_D$ be the Zariski decomposition of D (Theorem 2.2.8), and let $D=P_\sigma(D)+N_\sigma(D)$ be the σ-decomposition. For simplicity we set $P_\sigma:=P_\sigma(D)$ and $N_\sigma:=N_\sigma(D)$. Then N_D is effective, $D-N_D=P_D$ is nef, therefore by Remark 3.2.3, $[P_D]$ is movable, hence by Proposition

3.2.4 (2) $N_D \geqslant N_\sigma$. Now we have that $P_D = D - N_D \leqslant D - N_\sigma = P_\sigma$, hence $P_\sigma \leqslant P_\sigma + N_\sigma = D = P_D + N_D$, then $0 \leqslant P_\sigma - P_D \leqslant N_D$, therefore $Supp(P_\sigma - P_D) \subseteq Supp(N_D)$. Hence by Theorem 2.2.8 automatically the intersection matrix of $P_\sigma - P_D$ is negative definite and for all $\Gamma \in Supp(P_\sigma - P_D)$, then $P_D \cdot \Gamma = 0$.

Thus P_D and $P_\sigma - P_D$ are the positive and the negative part of the Zariski decomposition of P_σ. But P_σ is pseudo-effective by Remark 3.1.13 and $N_\sigma(P_\sigma) = 0$ by Lemma 3.1.15, then by Proposition 3.2.4 (1) we know that P_σ is movable. As we are on a surface, we get that P_σ is nef by what we proved above. Then P_σ has no negative part, therefore $P_\sigma - P_D = 0$, hence $P_\sigma = P_D$ and $N_\sigma = N_D$.

Therefore the σ-decomposition on a surface is nothing but the usual Zariski decomposition.

□

Remark 3.2.8. *If $P_\sigma(D)$ is nef, then the σ-decomposition is a Zariski decomposition in the sense of Fujita (our second chapter). It is not clear that Zariski decomposition in the sense of Fujita is a σ-decomposition.*

Remark 3.2.9. *The following properties are immediate consequences of the definition and Remark 3.1.7. If D' and D'' are pseudo-effective $\mathbb{R}$-divisors, then*

$$N_\sigma(D') + N_\sigma(D'') \geqslant N_\sigma(D' + D''),$$

then

$$P_\sigma(D') + P_\sigma(D'') \leqslant P_\sigma(D' + D'').$$

Let D be a pseudo-effective $\mathbb{R}$-divisor. Then for any $t \in \mathbb{R}_{>0}$ we have

$$N_\sigma(tD) = tN_\sigma(D)$$

and

$$P_\sigma(tD) = tP_\sigma(D).$$

Lemma 3.2.10. *Let D be a nef $\mathbb{R}$-divisor on a non-singular projective surface X with $D^2 = 0$. Then there exist at most finitely many irreducible curves C with $C^2 < 0$ such that $D - \epsilon C$ is pseudo-effective for some $\epsilon > 0$.*

Proof. We may assume that $D \not\equiv 0$. Let $\mathcal{S} = \mathcal{S}_D$ be the set of such curves C. For $C_1 \in \mathcal{S}$, let $\alpha_1 > 0$ be a number with $D - \alpha_1 C_1$ being pseudo-effective. Then $0 = D^2 \geqslant (D - \alpha_1 C_1) \cdot D \geqslant 0$. Hence $D \cdot C_1 = 0$ and $(D - \alpha_1 C_1)^2 < 0$. Hence $D - \alpha_1 C_1$ is not nef. Let N_1 be the negative part of the Zariski-decomposition of $D - \alpha_1 C_1$ and let $F_1 := \alpha_1 C_1 + N_1$. Then $L_1 := D - F_1$ is nef and

$$0 = D^2 = D \cdot F_1 + D \cdot L_1 \geqslant F_1 \cdot L_1 + L_1^2 \geqslant L_1^2 \geqslant 0.$$

Any prime component Γ of F_1 is an element of $\mathcal{S}$. Further, $D \cdot \Gamma = L_1 \cdot \Gamma = F_1 \cdot \Gamma = 0$.

We assume by the way of contradiction that $\sharp \mathcal{S} = \infty$.

Let C_2 be a curve belonging to $\mathcal{S}$ but not contained in $Supp(F_1)$. Similarly let $\alpha_2 > 0$ be a number with $D - \alpha_2 C_2$ being pseudo-effective, N_2 the negative part of the Zariski-decomposition of $D - \alpha_2 C_2$, and let F_2 the $\mathbb{R}$-divisor $\alpha_2 C_2 + N_2$. Then as above we have that $D \cdot C_2 = 0$, hence

$$0 = D \cdot C_2 = L_1 \cdot C_2 + F_1 \cdot C_2.$$

Now L_1 is nef and $C_2 \notin Supp(F_1)$, then $L_1 \cdot C_2 = F_1 \cdot C_2 = 0$, that is $\Gamma \cdot C_2 = 0$ for all $\Gamma \in Supp(F_1)$. Hence we have that C_2 is orthogonal at every Γ contained in $Supp(F_1)$, in particular C_2 is orthogonal at C_1.

Now let C_3 be a curve belonging to $\mathcal{S}$ but not contained in $Supp(F_1) \cup Supp(F_2)$, then as above C_3 is orthogonal at C_1 and at C_2. Hence we have a contradiction, because for all i we have that $Supp(F_i)$ is the union of finitely many irreducible curves, but we can find only a finite number of curves C_i as above, by the finite dimension of $\mathcal{N}^1(X)_{\mathbb{R}}$.

Hence $\mathcal{S}$ is finite.

□

We shall show the following continuity mentioned before:

Proposition 3.2.11. *The function $\sigma_\Gamma : \overline{Eff(X)} \to \mathbb{R}_{\geqslant 0}$ for a prime divisor Γ on a non-singular projective surface X is continuous.*

Proof. We may assume that D is not big by Lemma 3.1.14 (1). Let $\{D_n\}_{n \in \mathbb{N}}$ be a sequence of pseudo-effective $\mathbb{R}$-divisors such that $[D] = \lim_{n \to \infty} [D_n]$. If

Γ is an irreducible curve with $\sigma_\Gamma(D) > 0$, then $\sigma_\Gamma(D) \leqslant \sigma_\Gamma(D_n)$ except for finitely many n as in the proof of Lemma 3.1.14 (1). In fact we know that if we fix an ample A then $\sigma_\Gamma(D + \epsilon A) \leqslant \sigma_\Gamma(D_n)$, then for $n > n_0$

$$\sigma_\Gamma(D) := \lim_{\epsilon \to 0} \sigma_\Gamma(D + \epsilon A) \leqslant \sigma_\Gamma(D_n).$$

In particular $D_n - \sigma_\Gamma(D)\Gamma$ is pseudo-effective for $n > n_0$ by Remark 3.1.6. Hence we may assume that $\sigma_\Gamma(D) = 0$ and moreover that D is nef by Remark 3.2.7. But D is not big, hence $D^2 = 0$ by Theorem 1.3.16. We set $N_n := N_\sigma(D_n)$. Then we claim that $N_\infty := \sum_i \{\limsup_n \sigma_{\Gamma_i}(D_n)\}\Gamma_i$ exists.

In fact if $\limsup \sigma_\Gamma(D_n) = \epsilon > 0$, then there exists $\{j_n\}_{n \in \mathbb{N}}$ such that $\lim_{n\to\infty} \sigma_\Gamma(D_{j_n}) = \epsilon$. Then for all α such that $0 < \alpha < \epsilon$, we have that $\sigma_\Gamma(D_{j_n}) \geqslant \alpha$, hence $\Gamma^2 < 0$, because Γ is a component of the negative part of the Zariski decomposition of D_{j_n} by Remark 3.2.7 and Theorem 2.2.8. Then

$$D_{j_n} - \alpha\Gamma = D_{j_n} - \sigma_\Gamma(D_{j_n})\Gamma + (\sigma_\Gamma(D_{j_n}) - \alpha)\Gamma,$$

but $D_{j_n} - \sigma_\Gamma(D_{j_n})\Gamma$ is pseudo-effective and $(\sigma_\Gamma(D_{j_n}) - \alpha)\Gamma$ is an effective $\mathbb{R}$-divisor. Hence $D_{j_n} - \alpha\Gamma$ is pseudo-effective, then $D - \alpha\Gamma$ is pseudo-effective by the closure of $\overline{Eff(X)}$. Therefore there exists a finite number of such Γ. Here, $D - N_\infty$ is nef. If $N_\infty \neq 0$, then $N_\infty^2 < 0$, since $SuppN_\infty \subset SuppN_n$ for some n. However, $N_\infty^2 = 0$ follows from

$$0 = D^2 \geqslant (D - N_\infty) \cdot D \geqslant (D - N_\infty)^2 \geqslant 0.$$

Therefore, $N_\infty = 0$ and σ_Γ is continuous. □

Bibliography

[1] Oscar Zariski, *The Theorem of Riemann-Roch for High Multiples of an Effective Divisor on an Algebraic Surface*. Ann. of Math, 1962

[2] Robin Hartshorne, *Algebraic Geometry*. Springer, 1977.

[3] Robert Lazarsfeld, *Positivity in Algebraic Geometry*. Springer, 2004.

[4] Thomas Bauer, *A Simple Proof For The Existence Of Zariski Decompositions On Surfaces*. arXiv:0712.1576v1, 2007

[5] Takao Fujita, *On Zariski Problem*. Ann. of Math, 1979

[6] Lucian Bădescu, *Algebraic Surfaces*. Springer, 2001

[7] Shigeru Iitaka, *On D-dimensions of Algebraic Varieties*. Proc. Japan Accad, 1970

[8] Noboru Nakayama, *Zariski-decomposition and Abundance*. MSJ Memoires, 2004

[9] Igor Rostislavovich Shafarevich, *Basic Algebraic Geometry* 1. Springer, edition 1998