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Divisors, linear systems and applications to canonical curves

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Introduction

Algebraic geometry is a subject that somehow connects and unifies several mathematical disciplines, first of all algebra and geometry, but also others (such as number theory, string theory,...), and it has as many applications.
Because of this interdisciplinarity, studying it requires an appropriate background. I will therefore try to make this text as self-contained as possible: only a good knowledge of general topology and commutative algebra will be required, as well as a minimal familiarity with category theory and cohomology. All remaining prerequisites (about sheaves, schemes, varieties,...) will be exposed and summarized in chapter 1-Prerequisites. However it would be useful to have a good smattering of classical algebraic geometry (about quasi-projective varieties, also called prevarieties). Instead we will follow the approach developed by Grothendieck and his many coworkers in the 1960's in Paris, concerning the theory of schemes.

<u>What is algebraic geometry?</u> Roughly, it is the kind of geometry you can describe with polynomials. In particular, the closed subsets of a space are loci of points described by a system of polynomial equations.
What are the benefits? What may seems like a limitation, working only with polynomials, however, becomes a powerful tool for studying singular objects (nonsmooth varieties). Moreover we do not need to work only on $\mathbb{R}$ or $\mathbb{C}$, and we can generalize by taking any field k. Note that starting from chapter 2 (so excluding just the chapter 1-Prerequisites) we will assume that k is algebraically closed.

<u>What is the idea behind the theory of schemes?</u> As described in [5], just as topological manifolds are made by gluing together open balls from Euclidean space, schemes are made by gluing together open sets of a simple kind, called *affine schemes*. There is already some subtlety here: when you glue things together, you have to specify what kind of gluing is allowed. For example, about topological manifolds, if the transition functions are required to be differentiable, then you get the notion of a differentiable manifold.
Note that a differentiable manifold M is obviously a topological space, but it is a little bit more: specifying its structure as differentiable manifold is equivalent to specifying which of the continuous functions on any open subset of M are differentiable, and these functions form a sheaf $\mathcal{C}^\infty(M)$ such that the pair $(M, \mathcal{C}^\infty(M))$ is locally isomorphic to an open subset of $\mathbb{R}^n$ with its sheaf of differentiable functions; hence the idea of associating a sheaf of rings $\mathcal{O}_X$ to a topological space X, and to follow that of *scheme*.

What is the purpose of this text? I want to use the acquired notions to explicitly describe something: curves. The first step towards a greater knowledge of the varieties is clearly to start from the one-dimensional case.
The question this text wants to answer is very simple: what are the curves and how are they made? With *curve*, we mean a one-dimensional smooth variety over an algebraically closed field k (see section 2.1 for more information).
A first more explicit question could be: are all curves isomorphic? Obviously not, indeed some are affine and some (such as $\mathbb{P}^1$) are not. The question could become: are all projective curves isomorphic? In this regard, we will assume that all curves are projective (i.e. complete). Once again the answer to this question is no, indeed we will define an invariant, the genus, and we will show that $\mathbb{P}^1$ has genus 0, while there exist curves of positive genus. One more question: are all curves of genus 0 isomorphic to $\mathbb{P}^1$? This is true.
Regarding curves of genus 1 (called *elliptic*) we will see that they are plane cubics. For genus greater than 1, we will distinguish 2 different types: *hyperelliptic* and non; and in particular we will study non-hyperelliptic curves of low genus.

In order to discuss interesting issues like these we will introduce some useful tools, such as divisors, line bundles and linear systems (in chapter 3 and 4), after briefly summarizing the necessary prerequisites (in chapter 1 and 2), to then deepen the study of curves (in chapter 5).

Contents

Chapter 1

Prerequisites

In this chapter we will quickly review all notions and results of algebraic geometry that we need. Consider it a handbook to keep on the tip of the tongue during the whole reading of the text.
Here the main goal is defining schemes and showing their properties.
Note that with *ring* we will mean a unitary commutative ring.

1.1 Sheaves

First, we recall notions about sheaves on a topological space X.

1. A *presheaf* $\mathcal{F}$ on X is a covariant functor from the open subsets of X to a category C, that is it consists of
   - An object $\mathcal{F}(U) \in C$ for each open subset U of X.
     (An element $s \in \mathcal{F}(U)$ is called *section*).
   - A morphism, called *restriction*, $\rho_V^U \colon \mathcal{F}(U) \to \mathcal{F}(V)$ for each couple of open subsets $V \subseteq U$ of X, such that:
     1) $\rho_U^U = id$ for every open subset U.
     2) $\rho_W^V \circ \rho_V^U = \rho_W^U$ for every triple of open subsets $W \subseteq V \subseteq U$.
     (We will usually indicate with a section s, also its restrictions, as abuse of notation).

   In particular if C is the category of rings/modules/groups/sets, we will call $\mathcal{F}$ presheaf *of rings/modules/groups/sets*.
   From now on, we will assume that C is the category of abelian groups.

2. A *sheaf* $\mathcal{F}$ on X is a presheaf with the following property:
   Let U be an open subset of X, with $\{U_h\}$ an open cover of U
   and $s_h \in \mathcal{F}(U_h)$ sections s.t. $\rho_{U_h \cap U_k}^{U_h}(s_h) = \rho_{U_h \cap U_k}^{U_k}(s_k)$ $\forall h, k$
   then $\exists! s \in \mathcal{F}(U) : \rho_{U_h}^U(s) = s_h$ $\forall h$.

3. We can define a sheaf $\mathcal{F}$ on a topological basis β of X and then extend to all open subsets U in the following way (see [17, Proposizione 2.1]):
   $\mathcal{F}(U) = \varprojlim \mathcal{F}(W)$, where the inverse limit is on $\{W \in \beta | W \subseteq U\}$.

In other words $\mathcal{F}(U) = \{(s_W) \in \prod \mathcal{F}(W) | \rho^W_{W'}(s_W) = s_{W'}\}$.
(Restrictions are projections).

4. A *stalk* of a presheaf $\mathcal{F}$ on X is $\mathcal{F}_p = \varinjlim \mathcal{F}(U)$ where $p \in X$ and the direct limit is on its open neighborhoods U.
In other words $\mathcal{F}_p = \coprod \mathcal{F}(U)/\sim$, where taken $s \in \mathcal{F}(U), t \in \mathcal{F}(V)$ we define the equivalence relation:
$s \sim t \iff$ there is an open neighborhood $W \subseteq U \cap V$ of $p : \rho^U_W(s) = \rho^V_W(t)$.
(An element $s_p := [s] \in \mathcal{F}_p$ is called *germ*).

5. Let $s, t \in \mathcal{F}(U)$ be sections of a sheaf.
$s_p = t_p \ \forall p \in U \iff s = t$ (see [17, Lemma 3.2]).

6. We can extend a presheaf $\mathcal{F}$ to a sheaf $\mathcal{F}^+$ so defined (see [17, Teor. 3.7]):
$\mathcal{F}^+(U) = \{s \colon U \to \coprod_{p \in X} \mathcal{F}_p |$ For any $p \in X$, $s(p) \in \mathcal{F}_p$ and there is an open neighborhood $V \subseteq U$ of p and a section $\sigma \in \mathcal{F}(V)$ s.t. $s(x) = \sigma_x \ \forall x \in V\}$.
(Restrictions are natural restrictions of maps).

7. A *morphism of sheaves* on X, $\phi \colon \mathcal{F} \to \mathcal{G}$, is a natural transformation of functors, that is a collection of maps
$\{\phi(U) \colon \mathcal{F}(U) \to \mathcal{G}(U) | U \subseteq X$ open subset$\}$
compatible with the restrictions.
(We will usually indicate $\phi(U)(s)$ simply with $\phi(s)$, as abuse of notation).

8. Let $\phi \colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves, we can define for each point $p \in X$:
$$\begin{aligned} \phi_p \colon \mathcal{F}_p &\longrightarrow \mathcal{G}_p \\ s_p &\mapsto (\phi(s))_p \end{aligned} \quad \text{(morphism on the stalks).}$$

9. We say that ϕ is *injective/surjective/bijective* if ϕ_p is such $\forall p \in X$.
Note that (see [17, Proposizione 3.3 e 3.5]):
ϕ_p is injective/isomorphism $\iff$ $\phi(U)$ is such for any open subset U of X.

1.2 Cohomology of sheaves

Cohomology is a very useful tool. In particular it is good to note that the zero-index case corresponds to the global sections of a sheaf.
Let X be a topological space and let $\mathcal{F}$ be a sheaf of rings or modules on X.

1. A sequence of sheaves is *exact* if it is exact on the stalks.

2. We define $\Gamma(X, \mathcal{F}) := \mathcal{F}(X)$.
In particular $\Gamma(X, -)$ is a functor from $\{$sheaves of rings on $X\}$ to $\{$rings$\}$.
Note that it is covariant, additive and left-exact (see [17, Proposizione 5.2]).

3. A sheaf is *flasque* if its restrictions are surjective.

4. $H^i(X, \mathcal{F}) := H^i(\Gamma(X, \mathcal{K}^\bullet))$, where $\mathcal{K}^\bullet$ is a flasque resolution of $\mathcal{F}$.
(There exists always a flasque resolution of $\mathcal{F}$, see [17, Definizione 5.5]).

5. $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ (see [17, Proposizione 5.8]).

1.3 Locally ringed spaces

The notion of locally ringed spaces is the starting point to define schemes.

1. A *ringed space* is a couple $(X, \mathcal{O}_X)$ where
   - X is a topological space
   - $\mathcal{O}_X$ is a sheaf of rings on X.

2. Let $f\colon X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a sheaf on X, the *push-forward* of $\mathcal{F}$ is the sheaf $f_*\mathcal{F}$ on Y so defined:
   $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ for each open subset U of X.
   (Restrictions are induced by restrictions of $\mathcal{F}$).

3. A *morphism of ringed spaces* is a pair $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, where:
   - $f\colon X \to Y$ is a continuous map of topological spaces.
   - $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves on Y.

4. Let $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.
   For each $p \in X$ we have a ring homomorphism $f_p^\sharp\colon \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$, defined in the following way:

   $$\mathcal{O}_{Y,f(p)} = \varinjlim_{V \ni f(p)} \mathcal{O}_Y(V) \overset{f^\sharp}{\to} \varinjlim_{V \ni f(p)} \mathcal{O}_X(f^{-1}(V)) \to \varinjlim_{U \ni p} \mathcal{O}_X(U) = \mathcal{O}_{X,p}$$

   Note that $f_p^\sharp$ (with $p \in X$) is different from $(f^\sharp)_q$ (with $q \in Y$).

5. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ s.t the stalk $\mathcal{O}_{X,x}$ is a local ring for each $x \in X$ (i.e. $\mathcal{O}_{X,x}$ has a unique maximal ideal $\mathfrak{m}_x$).

6. A *morphism of locally ringed spaces* $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces s.t $(f_x^\sharp)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ $\forall x \in X$ (i.e. $f_x^\sharp$ is a morphism of local rings).

Now we introduce some conventions:

- We denote a ringed space $(X, \mathcal{O}_X)$ only with X, and a morphism of ringed spaces $(f, f^\sharp)$ only with f.

- Let X be a locally ringed space.
  For each $p \in X$, we define its *residue field* $K(p) := \mathcal{O}_{X,p}/\mathfrak{m}_p$.

- Let $s \in \mathcal{O}_X(U)$ be a section, we can consider it as a function
  $s\colon U \to \coprod_{x \in U} K(x)$, $p \mapsto \overline{s_p}$
  (In particular $s(p) = 0 \iff s_p \in \mathfrak{m}_p$).

1.4 $\mathcal{O}_X$-modules

Dealing with schemes, we will work on particular sheaves, treated below.
Let $(X, \mathcal{O}_X)$ be a ringed space.

1. A *sheaf of $\mathcal{O}_X$-modules* (or simply an *$\mathcal{O}_X$-module*) is a sheaf $\mathcal{F}$ on X such that for each open subset U, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module
(and the restrictions of $\mathcal{F}$ are compatible with the module structure via the restrictions of $\mathcal{O}_X$).

2. A *morphism of $\mathcal{O}_X$-modules* is a morphism of sheaves ϕ consisting in homomorphisms of modules.
Note that $\mathcal{K}er(\phi), \mathcal{I}m(\phi), \mathcal{C}oker(\phi)$ are $\mathcal{O}_X$-modules
(as defined in [9, Chapter II.1]).

3. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules, then $\mathcal{F}/\mathcal{G}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}), \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ are $\mathcal{O}_X$-modules
(as defined at the beginning of [9, Chapter II.5]).

1.5 Schemes

Let A be a ring. We consider the set $X = \mathrm{Spec}(A) =$\{prime ideals of A\}, and we denote with P_x the corresponding ideal of A to an element $x \in X$, and with $[P] \in X$ the corresponding element to an ideal P of A.
First, we recall notions about the *Zarisky* topology on X:

1. We equip X with the topology whose closed subsets are
$V(S) := \{x \in X | S \subseteq P_x\}$ where $S \subseteq A$.

2. Every closed subset is of the form $V(J)$ where J is a radical ideal of A (see [17, Esercizio 6.1]).

3. Given a closed subset Z of X, we can associate to it an ideal $I(Z) = \bigcap_{z \in Z} P_z$ of A. In particular we have $V(I(Z)) = \overline{Z}$ (see [17, Esercizio 6.1]).

4. A closed subset Z is irreducible $\iff$ $I(Z)$ is prime.
Moreover, a point $\{x\}$ is closed $\iff$ P_x is maximal.
(see [17, Proposizione 6.5]).
In particular, for every $x \in X$ we will say that x is a *generic point* of the closed subset $\overline{\{x\}} = V(P_x)$.
(If A is an ID, then $X = \overline{\{[0]\}}$).

5. A *principal open subset* of X is $U_f := X \setminus V(f)$, where $f \in A$.
The principal open subsets give a topological basis (by [17, Esercizio 6.2]).

6. X is compact (see [17, Esercizio 6.3]).

7. We define a sheaf $\mathcal{O}_X$ on X, called *structure sheaf*, in the following way:

   - Let U_f be a principal open subset, we have
   $\mathcal{O}_X(U_f) = A_f := \{\frac{a}{f^n} | a \in A, n \in \mathbb{N}\}$.

- Let $U_g \subseteq U_f$ be principal open subsets
(i.e. $\exists m \in \mathbb{N}, b \in A$ such that $g^m = bf$, see [17, Esercizio 6.1]).
We have $\rho^{U_f}_{U_g} \colon A_f \to A_g$, $\frac{a}{f^n} \mapsto \frac{ab^n}{g^{nm}}$

Note that it is defined on a topological basis, then we can extend it to all open subsets (by section 1.1(3)).
(If A is an ID, then $\mathcal{O}_X(U) = \bigcap_{x \in U} A_{P_x}$, see [17, Lemma 7.4]).

Finally, we can define a scheme in the following way:

a. A *standard affine scheme* is a ringed space $(X, \mathcal{O}_X)$ with:

- $X = \mathrm{Spec}(A)$ with Zarisky topology (where A is a ring).
- $\mathcal{O}_X$ is its structure sheaf.

Note that $\mathcal{O}_{X,x} = A_{P_x} \forall x \in X$, hence X is locally ringed.
(see [17, Proposizione 7.2]).

b. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X) \cong (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ for some ring A (called *coordinate ring* of X).

c. A *scheme* is a ringed space $(X, \mathcal{O}_X)$ s.t. there exists an open cover $\mathcal{U}$ of X: $(U, \mathcal{O}_{X|U})$ is an affine scheme $\forall U \in \mathcal{U}$.

d. A *morphism of schemes* is a morphism of locally ringed spaces which are schemes.

Some remarks about schemes:

i. Let $X = \mathrm{Spec}(A)$ be an affine scheme.
A principal open subset $U_f \cong \mathrm{Spec}(A_f)$ is an affine scheme.

ii. Every irreducible closed subset Z of a scheme has a unique generic point $z \in Z : Z = \overline{\{z\}}$ (by [17, Lemma 10.2]).

iii. There is a category equivalence: $\{\text{affine schemes}\} \leftrightarrow \{\text{rings}\}^{op}$
(see [17, Teorema 8.2]).
Given a ring homomorphism $\phi \colon A \to B$, we can define a morphism of schemes in the following way:

- $\bar{\phi} \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A), [P] \mapsto [\phi^{-1}(P)]$ is a continuous map.
- $\bar{\phi}^\sharp \colon \mathcal{O}_{\mathrm{Spec}(A)} \to \bar{\phi}_* \mathcal{O}_{\mathrm{Spec}(B)}$ is a morphism of sheaves so defined (on principal open subsets):
$\bar{\phi}^\sharp(U_g) \colon A_g \to B_{\phi(g)}$, $\frac{a}{g^n} \mapsto \frac{\phi(a)}{\phi(g)^n}$.

Conversely, given a morphism of ringed spaces $f \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, we can define a ring homomorphism $\phi_f := f^\sharp(\mathrm{Spec}(A)) \colon A \to B$.
More generally (see [17, Es. 11.4]): let X be a scheme, let $Y = \mathrm{Spec}(A)$ be an affine scheme, then there is a bijection: $\mathrm{Mor}(X, Y) \leftrightarrow \mathrm{Mor}(A, \mathcal{O}_X(X))$.

1.6 $\mathbb{A}^n$ and $\mathbb{P}^n$

Let k be a field.

1. Let $R = k[X_1, ..., X_n]$ be the ring of polynomials.
   The scheme $\mathbb{A}^n := \operatorname{Spec}(R)$ is called *affine n-space.*

2. Let $S = \oplus_{d\geq 0} S_d$ be a graded ring.
   $S_+ := \oplus_{d>0} S_d$ is maximal ideal.
   $\operatorname{Proj}(S) := \{P \subseteq S | P \text{ is a homogenous prime ideal, } P \neq S_+\}$
   is the topological space whose closed subsets are of the form
   $V(Q) = \{P \in \operatorname{Proj}(S) | P \supseteq Q\}$ for some homogeneus ideal Q of S.

   $X = \operatorname{Proj}(S)$ is a scheme, where the structure sheaf (analogously to affine schemes) is so defined (as presheaf on principal open subsets):
   Let $f \in S$ be homogeneous of degree d,
   $\mathcal{O}_{\operatorname{Proj}(S)}(U_f) = S_{(f)} := \{\frac{m}{f^n} | m \in S \text{ homogeneous of degree } dn\}$.
   In particular for each $x \in X$, we have
   $\mathcal{O}_{X,x} = S_{(P_x)} := \{\dfrac{m}{a} | m, a$ homogeneus polynomials in S of the same degree, $a \in S \setminus P_x\}$.
   Moreover principal open subsets $U_f \cong \operatorname{Spec}(S_{(f)})$ are affine.

3. Let $P = k[X_0, ..., X_n]$ be a (graded) ring of polynomials.
   The scheme $\mathbb{P}^n := \operatorname{Proj}(P)$ is called *projective n-space.*
   Its *standard open cover* is $\{U_0, ..., U_n\}$ with $U_i := U_{X_i}$.

4. An *hypersurface* of $\mathbb{P}^n$ is a closed subset $V = V(F)$ for some homogeneus polynomial F, and its *degree* is $\deg(V) := \deg(F)$.
   An *hyperplane* of $\mathbb{P}^n$ is a hypersurface of degree 1.

5. Let $P = k[X_0, ..., X_n]$. We consider the graded ring $P(l) := \oplus_{d\geq 0} P_{l+d}$.
   The *twist sheaf* $\mathcal{O}(l)$ is the sheaf on $\mathbb{P}^n$ associated to the graded ring $P(l)$ (in analogous way of above). Moreover, by [17, Teorema 15.1]:

   - $\Gamma(\mathbb{P}^n, \mathcal{O}(l)) \cong \begin{cases} P_l & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases}$
   - $\dim_k \Gamma(\mathbb{P}^n, \mathcal{O}(l)) = \binom{n+l}{l}\ \forall l \geq 0$.
   - $\mathcal{O}(l) \otimes \mathcal{O}(m) \cong \mathcal{O}(l + m)$.

1.7 Subschemes

Let X be a scheme.

1. An *open subscheme* of X is a scheme $(U, \mathcal{O}_{X|U})$ with U open subset of X.

2. A *closed subscheme* of X is a scheme $Z \subseteq X$ such that there exists a *closed embedding* $i \colon Z \hookrightarrow X$, that is a morphism of schemes s.t.

- $i\colon Z \to i(Z)$ is homeomorphism.
- $i^\sharp\colon \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.

3. Given a closed subscheme Z, we can associate to it a sheaf of ideals $\mathcal{I}_{Z/X} := \mathcal{K}er(i^\sharp)$ such that $\mathcal{O}_Z \cong \mathcal{O}_X/\mathcal{I}_{Z/X}$.

4. Note that a closed subscheme of an affine scheme is affine.
   In detail if $X = \mathrm{Spec}(A)$ and Z is closed subscheme, we can associate to Z an ideal I of A s.t $Z \cong \mathrm{Spec}(A/I)$.

5. Let Z, Z' be closed subschemes of X, the *intersection scheme* $Z \cap Z'$ is the closed subscheme associated to the ideal sheaf $\mathcal{I}_{Z/X} + \mathcal{I}_{Z'/X}$.

1.8 Properties of schemes

We give some useful definitions. Let X be a scheme.

1. X is *reduced* if $\mathcal{O}_X(U)$ is a reduced ring for every open subset U.

2. X is *irreducible* if it is irreducible as topological space.

3. X is *integral* if $\mathcal{O}_X(U)$ is an ID for every open subset U.
   Note that: integral $\iff$ reduced and irreducible (see [9, Prop. II.3.1]).

4. Let k be a field. X is a *k-scheme* if it is equipped with a morphism of schemes $X \to \mathrm{Spec}(k)$, called *structure morphism*.
   A *morphism of k-schemes* is a morphism of schemes compatible with the structure morphisms.

5. A k-scheme X is *projective* if it is a closed k-subscheme of $\mathbb{P}^n$.

Now, let $f\colon X \to Y$ be a morphism of k-schemes (for some field k).

a. f is *of finite type* if there is an affine open cover $\{V_i = \mathrm{Spec}(B_i)\}$ of Y such that $\forall i$, $f^{-1}(V_i)$ can be covered by a finite affine open cover $\{U_j = \mathrm{Spec}(A_j)\}$, where A_j are finitely generated B_i-algebras.

   X is *of finite type* if $X \to \mathrm{Spec}(k)$ is of finite type, i.e. there exists a finite affine open cover $\{U_i = \mathrm{Spec}(A_i)\}$ of X, where A_i are finitely generated k-algebras (equivalently it holds on every affine open subset, see [17, Proposizione 10.4]).

b. f is *affine* if there is an open cover $\mathcal{V}$ of Y s.t. $f^{-1}(V)$ is affine $\forall V \in \mathcal{V}$.
   (E.g. closed embeddings are affine).

c. f is *finite* if for every affine open subset $V = \mathrm{Spec}(B)$ of Y, we have that $f^{-1}(V) = \mathrm{Spec}(A)$ is affine and A is a B-algebra, finitely generated as module.

d. f is *separated* if the diagonal morphism $X \to X \times_Y X$ is a closed embedding.
   X is *separated* if $X \to \mathrm{Spec}(k)$ is separated.

e. f is *proper* if it is separated and universally closed (i.e. it is closed and for each morphisms $g\colon Z \to Y$, we have that $Z \times_Y X \to Z$ is closed).
Note that: finite $\iff$ proper with finite fibers (see [6, Appendix B2]).
X is *proper* (or *complete*) if $X \to \mathrm{Spec}(k)$ is proper.

1.9 Varieties

A *variety* X is an integral (reduced and irreducible) k-scheme of finite type over an algebraically closed field k.
In this case we can associate X to a prevariety (see [17, Proposizione 10.6]).
Prevarieties (or *quasi-projective varieties*) are topic of classical algebraic geometry. See [9, Chapter I] for more information, in particular for the meaning of rational and regular functions, and that of dimension.
In detail, there exists a scheme $(X', \mathcal{O}_{X'})$ where

- X' is a prevariety.
- $\mathcal{O}_{X'}$ is the sheaf of regular functions, so defined:
  $\mathcal{O}_{X'}(U) = \{f \in k(X) | f \text{ is regular on } U\}$.

such that
1) $X = X' \cup \{[W] | W \subseteq X' \text{ is irreducible sub-prevariety of positive dimension}\}$.
2) The open subsets of X are of the form $U := U' \cup \{[W] \in X | W \cap U' \neq \emptyset\}$ where U' is an open subset of X'.
3) $\mathcal{O}_X(U) = \mathcal{O}_{X'}(U')$ for each open subset U of X.

1.10 Cohomology of projective varieties

There are many important results about cohomology of schemes, in particular involving the notions of coherent and quasi-coherent sheaves, which an interested reader can deepen in [9, Chapter III].
We recall just two of these about a projective variety X.

1. *Global sections* [9, Theorem I.3.4].
   $H^0(X, \mathcal{O}_X(X)) \cong k$.

2. *Serre duality* [9, Remark III.7.12.1].
   Let X be nonsingular of dimension n (see definitions in section 2.2).
   Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module (see definition in section 4.1).
   Let ω_X be the canonical sheaf (see definition in section 5.4).
   There are isomorphisms $H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee$.

1.11 The Gluing Lemma

Finally we recall a very useful tool, used in particular to characterize the notion of line bundle (in section 4.1).

Gluing Lemma. [17, Lemma 17.1]
Let X be a topological space.
Let $\{U_i\}$ be an open cover of X.
Given a sheaf $\mathcal{F}_i$ on U_i, for each i,
and ("transition") isomorphisms $\phi_{ij} \colon \mathcal{F}_{j|U_{ij}} \xrightarrow{\cong} \mathcal{F}_{i|U_{ij}}$ on $U_{ij} := U_i \cap U_j$ s.t.

- $\phi_{ii} = id_{\mathcal{F}_i}$
- $\phi_{ik} = \phi_{ij} \circ \phi_{jk}$ on U_{ijk}.

Then there exists a unique sheaf $\mathcal{F}$ (up to isomorphism) s.t.
1) $\mathcal{F}_{|U_i} \cong \mathcal{F}_i \ \forall i$ (via isomorphisms ϕ_i)
2) $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ on each U_{ij}.

Chapter 2

First notions

After having seen notions about schemes in the previous chapter, we will redefine them in a more congenial way. Then using the new definitions, we will review other notions and results of algebraic geometry about the local case, which will be essential for understanding what we are going to see in this text.

2.1 Notations and conventions

Let k be an algebraically closed field.

- A *scheme* will be a separated k-scheme of finite type.
- A *variety* will be an integral (reduced and irreducible) scheme.
- A *subvariety* of a scheme will be a closed subscheme which is a variety.
- A *point* will be a closed point (and we will write just "$p \in X$").

In the next section we will define the dimension of a variety and the notion of smoothness. In particular

- A *curve* will be a one-dimensional projective (i.e. complete) smooth variety.

(Note that for a one-dimensional smooth variety: projective $\iff$ complete, see [9, Proposition II.6.7])

2.2 Local geometry of schemes

First, we give some definitions.

1. The *field of rational functions* on a variety X is $R(X) := \mathcal{O}_{X,\eta}$, where η is the generic point of X.
   Note that $R(X) \cong R(U) = Q(A)$, where $U = \mathrm{Spec}(A)$ is an affine open subscheme of X (and $Q(A)$ is the quotient field of A).
   The non-zero elements of this field form the multiplicative group $R(X)^*$.

2. A *regular function* is an element $f \in \mathcal{O}_X(X)$.
   We can see f as a function $f \colon X \to \amalg_{p \in X} K(p)$.
   The *zero-set* of f is $V(f) := \{p \in X | f(p) = 0\} = \{p \in X | f_p \in \mathfrak{m}_p\}$.

3. A *rational function* is an element $f \in R(X)$, i.e. $f = \dfrac{g}{h}$, with $g, h \in \mathcal{O}_X(U)$ for some affine open subscheme U of X.
   Moreover, we say that f is *regular* at $p \in X$ if $f = \dfrac{g}{h}$, with $g, h \in \mathcal{O}_X(U)$ for some affine open neighborhood U of p such that $h(p) \neq 0$.

4. Remark. Let $p \in X$ be a point where $f \in R(X)^*$ is regular, then $f(p) = 0$ (i.e. p is a *zero*) $\iff$ f^{-1} is not regular at p (i.e. p is a *pole*). (See [3, Corollario 3.6.7])

5. The *local ring* of a scheme X along a subvariety V is $\mathcal{O}_{X,V} := \mathcal{O}_{X,\mu}$, where μ is the generic point of V.
   Note that $\mathcal{O}_{X,V} \cong A_P$ (where $U = \operatorname{Spec}(A)$ is an affine open subscheme such that $U \cap V \neq \emptyset$ and P is the ideal associated to $U \cap V$). In particular $\mathcal{O}_{X,V}$ is a local ring (with maximal ideal $\mathfrak{m}_{X,V}$).
   If X is a variety, $\mathcal{O}_{X,V} = \{f \in R(X) | f \in \mathcal{O}_X(U)$ for some affine open subscheme U s.t. $U \cap V \neq \emptyset\}$.

6. If V is a point $x \in X$, the notion of local ring coincides with that of stalk, moreover $\mathcal{O}_{X,x} \cong \mathcal{O}_{U,x}$ for every open subset U of X containing x.
   In particular we can locally assume $X = \operatorname{Spec}(A)$ affine, and $\mathcal{O}_{X,x} = A_{P_x}$.

7. Let X be a scheme, its *dimension* is
   $\dim(X) :=$ max.lenght$\{\emptyset \neq V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n \subseteq X | V_i$ are irreducible closed subsets$\}$.
   If X is variety, $\dim(X) = \operatorname{Trdeg}_k R(X)$.

Now some notions of local geometry.
Let X be a variety of dimension n. Let $x \in X$ be a point.

a. The *Zarisky cotangent space* of X at x is $(T_x X)^* := \mathfrak{m}_x / \mathfrak{m}_x^2$ regarded as a k-vector space (of dimension at most n).
   In particular if we assume X affine, $(T_x X)^* \cong P_x / P_x^2$.

b. The *Zarisky tangent space* $T_x X$ is the dual space of $(T_x X)^*$.
   In particular if $X \subset \mathbb{A}^n = \operatorname{Spec} k[X_1, ..., X_n]$ and $x = (a_1, ..., a_n)$.
   $T_x \mathbb{A}^n = V(d_x F | F \in P_x)$ where $d_x F := \sum \partial_{X_i} F(x)(X_i - a_i)$.

c. Note that $\dim_k(T_x X) \geq \dim(X)$ (see [3, Teorema 5.4.3]).
   x is a *non-singular* point if $\dim_k(T_x X) = \dim(X)$, otherwise it is *singular*.
   (If $X = V(F) \subset \mathbb{A}^n$ is a hypersurface, x is singular $\iff$ $\partial_{X_i} F(x) = 0\ \forall i$).

d. X is a *non-singular* (or *smooth*) variety if every point is non-singular.

e. Let $Z \subseteq X$ be a closed subscheme.
   Z is *locally principal* if $(\mathcal{I}_{Z/X})_p$ is a principal ideal $\forall p \in X$.

f. X is *locally factorial* if every local ring $\mathcal{O}_{X,p}$ is an UFD.

g. X is *normal* if every local ring $\mathcal{O}_{X,p}$ is integrally closed.

h. Remark. X is smooth $\Rightarrow$ X is locally factorial (see [12, Th.48, p.142]) $\Rightarrow$ X is normal (by [18, Lemma 10.119.11]).

Chapter 3

Divisors

We take a smooth variety X (e.g. a curve).
We could define divisors in a more general case, but we are interested to this case. Divisors are the first important tool that we introduce. Here the main goal is showing that (in our case) the notions of Weil divisors and Cartier divisors coincide (as well as showing their properties).
Note that on curves, the divisors could be simply seen as finite sums of points.

3.1 Weil divisors

We report below the definition of Weil divisors (which we call just divisors), followed by a very useful notion: the degree of a divisor on a projective space.

Definition 3.1.1.

1. A *prime divisor* on X is a subvariety V of codimension 1.
2. $\mathrm{Div}(X)$ is the free abelian group generated by the prime divisors on X.
3. A *Weil divisor* (or simply a *divisor*) on X is an element $D = \sum n_i V_i$ of $\mathrm{Div}(X)$. (Note that $n_i \neq 0$ for at most a finite number of indexes).
4. A divisor $D = \sum n_i V_i$ is *effective* if $n_i \geq 0$ $\forall i$. (We will write "$D \geq 0$").
5. The *support* of a divisor D is the closed subscheme $\mathrm{Supp}(D) := \bigcup_{i:n_i \neq 0} V_i$.

Definition 3.1.2. Let $X = \mathbb{P}^n$.

1. Let $D = \sum n_i V_i \in \mathrm{Div}(\mathbb{P}^n)$, the *degree* of D is $\deg(D) := \sum n_i \deg(V_i)$ where $\deg(V_i)$ is the degree as hypersurface.
2. $\mathrm{Div}^d(\mathbb{P}^n) := \{$divisors on $\mathbb{P}^n$ of degree d$\}$.

Note that $\mathrm{Div}^0(\mathbb{P}^n)$ is a subgroup of $\mathrm{Div}(\mathbb{P}^n)$.

3.2 Orders of Zeros and Poles

In the next section we will define a kind of divisors called principal, but first we need another notion: the *order* of $f \in R(X)^*$ along a prime divisor V.
Note that $\mathcal{O}_{X,V} = O_{X,\eta}$ where η is the generic point of V, hence $\mathcal{O}_{X,V}$ is integrally closed (as seen in section 2.2(h)).
Since $\mathcal{O}_{X,V}$ has dimension 1, applying [2, Proposition 9.2], we have that:

- $\mathcal{O}_{X,V}$ is a DVR.
- $\mathfrak{m}_{X,V}$ is a principal ideal.
- Each ideal of $\mathcal{O}_{X,V}$ is of the form (t^d) where t is a generator of $\mathfrak{m}_{X,V}$.

Definition 3.2.1. ord_V is the discrete valuation associated to $\mathcal{O}_{X,V}$. Explicitly:

- If $f \in \mathcal{O}_{X,V}$, we have $\mathrm{ord}_V(f) = \max\{d \in \mathbb{N} | f \in (t^d)\}$.
- If $f \in R(X)^*$, that is $f = \frac{a}{b}$ where $a, b \in \mathcal{O}_{X,V}$, we have $\mathrm{ord}_V(f) = \mathrm{ord}_V(a) - \mathrm{ord}_V(b)$.

Definition 3.2.2.

1. If $\mathrm{ord}_V(f) > 0$, we say that f has a *zero* along V.
2. If $\mathrm{ord}_V(f) < 0$, we say that f has a *pole* along V.

3.3 Principal divisors

Principal divisors are essential to studying divisors, in particular to defining the linear equivalence and the Class group (section 3.5).
In section 3.7, we will see that Cartier divisors are exactly the locally principal divisors; we will also see that in our case Cartier divisors and Weil divisors coincide, hence every divisor is locally principal.

Definition 3.3.1. Let $f \in R(X)^*$.
The divisor associated to f is $\mathrm{div}(f) := \sum_{\text{prime divisor } V} \mathrm{ord}_V(f)V$.

Remark 3.3.2. Note that $\mathrm{div}(f) \in \mathrm{Div}(X)$, that is $\mathrm{ord}_V(f) \neq 0$ for at most a finite number of prime divisors V.

Proof. Let $U \subseteq X$ be an open affine subset on which f is regular.
Let V be a prime divisor on X such that $\mathrm{ord}_V(f) \neq 0$. We consider two cases:

1. If $V \cap U = \emptyset$, then $V \subseteq X \smallsetminus U$ and it is an irreducible component. Since $X \smallsetminus U$ has a finite number of irreducible components, we have a finite number of possible choices for V.
2. If $V \cap U \neq \emptyset$, then f is regular on an open subset meeting V, hence $f \in \mathcal{O}_{X,V}$, or better $\mathrm{ord}_V(f) > 0$, that is $f \in \mathfrak{m}_{X,V} = I_U(V \cap U)\mathcal{O}_{U,V\cap U}$.
It follows that $f \in I_U(V \cap U)$, hence $V \cap U \subseteq V_U(f)$ and it is an irreducible component. If we take the closures, we get that $V \subseteq V_X(f)$ and it is an irreducible component, hence we have a finite number of possible choices for V.

$\square$

Remark 3.3.3. $\mathrm{Supp}(\mathrm{div}(f)) = \{\text{zeros of } f\} \cup \{\text{poles of } f\}$
(by definition of zeros and poles).

Proposition 3.3.4. Let $f, g \in R(X)^*$.

1. $\mathrm{div}(\frac{f}{g}) = \mathrm{div}(f) - \mathrm{div}(g)$.
2. f is regular (i.e. $f \in \mathcal{O}_X(X)$) $\iff$ $\mathrm{div}(f) \geq 0$.
3. If $f \in k^*$, then $\mathrm{div}(f) = 0$.
4. If X is projective, then
   $f \in k^* \iff \mathrm{div}(f) = 0$.

Proof.

1. By properties of the valuations ord_V.
2. [$\Rightarrow$] Let V be a prime divisor on X. Since f is regular, we have $f \in \mathcal{O}_{X,V}$, hence $\mathrm{ord}_V(f) \geq 0$. In conclusion $\mathrm{div}(f)$ is effective.
   [$\Leftarrow$] Let U be the open subset where f is regular.
   Now if we assume that f is not regular (on X), then $X \setminus U \neq \emptyset$.
   By [3, Corollario 4.6.3], $X \setminus U$ has pure codimension 1.
   Let V be an irreducible component of $X \setminus U$, then V is a prime divisor on X. For any $p \in V$, f is not regular at p (i.e. $f^{-1}(p) = 0$), then $f^{-1} \in I_X(V)$, in particular $f^{-1} \in \mathfrak{m}_{X,V}$, that is $\mathrm{ord}_V(f^{-1}) > 0$, hence $\mathrm{ord}_V(f) = -\mathrm{ord}_V(f^{-1}) < 0$. It follows that $\mathrm{div}(f)$ is not effective.
3. Let $f = c$ be a non-zero constant, then $c \in \mathcal{O}_{X,V}$ is invertible for each prime divisor V, hence $\mathrm{ord}_V(c) = 0$. It follows that $\mathrm{div}(c) = 0$.
4. By point 2, $\mathrm{div}(f) = 0$ implies that f is regular. Since X is projective, $\Gamma(X, \mathcal{O}_X) = k$, that is every regular function is constant, hence $f \in k^*$.

$\square$

Definition 3.3.5.

1. $D \in \mathrm{Div}(X)$ is *principal* if there exists $f \in R(X)^*$ such that $D = \mathrm{div}(f)$.
2. $\mathrm{Princ}(X) := \{\text{principal divisors on } X\}$

Note that $\mathrm{Princ}(X)$ is a subgroup of $\mathrm{Div}(X)$ by Proposition 3.3.4(1).

3.4 Principal divisors on $\mathbb{A}^n$ and $\mathbb{P}^n$

On $\mathbb{A}^n$ and $\mathbb{P}^n$ there is an easy way to see principal divisors.

Proposition 3.4.1. Let $X = \mathbb{A}^n$.

1. Let $f \in R(\mathbb{A}^n)^*$, that is $f = \frac{F}{G}$ with $F, G \in k[X_1, ..., X_n]$.
   Let $F = F_1^{d_1}...F_t^{d_t}$, $G = G_1^{r_1}...G_s^{r_s}$ be their factorizations into irreducible polynomials. Then
   $\mathrm{div}(f) = \sum d_i V(F_i) - \sum r_i V(G_i)$.
2. $\mathrm{div}\colon R(\mathbb{A}^n)^* \longrightarrow \mathrm{Div}(\mathbb{A}^n)$ is surjective.
   In particular $\mathrm{Div}(\mathbb{A}^n) = \mathrm{Princ}(\mathbb{A}^n)$.

Proof.

1. By Proposition 3.3.4(1).
2. Let $D = \sum_{i=0}^{l} n_i V_i$ be an effective divisor. We have $V_i = V(F_i)$, where F_i are irreducible polynomials. Taken $f_D := \prod_{i=0}^{l} F_i^{n_i}$, then $\mathrm{div}(f_D) = D$.
   Let $D \in \mathrm{Div}(\mathbb{A}^n)$, then $D = E - E'$, with $E, E' \geq 0$, and $\mathrm{div}(\dfrac{f_E}{f_{E'}}) = D$.

$\square$

Proposition 3.4.2. Let $X = \mathbb{P}^n$.

1. Let $f \in R(\mathbb{P}^n)^*$, that is $f = \frac{F}{G}$ with $F, G \in k[X_0, ..., X_n]$ homogenous polynomials of the same degree.
   Let $F = F_1^{d_1}...F_t^{d_t}$, $G = G_1^{r_1}...G_s^{r_s}$ be their factorizations into irreducible polynomials, then
   $\mathrm{div}(f) = \sum d_i V(F_i) - \sum r_i V(G_i)$.
2. Note that $\mathrm{div}(f) \in \mathrm{Div}^0(\mathbb{P}^n)$.
3. $\mathrm{div}\colon R(\mathbb{P}^n)^* \longrightarrow \mathrm{Div}(\mathbb{P}^n)$ is not surjective, but it has image $\mathrm{Div}^0(\mathbb{P}^n)$.
   In particular $\mathrm{Div}^0(\mathbb{P}^n) = \mathrm{Princ}(\mathbb{P}^n)$.

Proof.
Note that $\deg(\mathrm{div}(f)) = \deg(\mathrm{div}(F)) - \deg(\mathrm{div}(G)) = \deg(F) - \deg(G) = 0$.
The remaining proof is analogous to the case $X = \mathbb{A}^n$. $\square$

3.5 Linear equivalence and Class Group

Finally we can define the Class Group and see some particular cases.

Definition 3.5.1.

1. Let $D, D' \in \mathrm{Div}(X)$,
   D, D' are *linearly equivalent* ($D \sim D'$) if $D - D' \in \mathrm{Princ}(X)$.
2. The *Class Group* of X is $\mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{Princ}(X)$.

Examples 3.5.2.

1. $\mathrm{Cl}(\mathbb{A}^n) = 0$ (by Proposition 3.4.1).
   Equivalently, every divisor on $\mathbb{A}^n$ is principal.
   More generally, let $X = \mathrm{Spec}(A)$ (with X still smooth variety), we have A is an UFD $\iff$ $\mathrm{Cl}(X) = 0$ (see [9, Proposition II.6.2]).

2. $\deg\colon \mathrm{Cl}(\mathbb{P}^n) \longrightarrow \mathbb{Z}$ is a group isomorphism (by Proposition 3.4.2).
   In particular $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$ and it is generated by a hyperplane H.
   Note that each equivalence class is of the form dH ($d \in Z$).

3.6 Cartier divisors

We have assumed X to be a smooth variety, but the definition of a Cartier divisor applies to any scheme.
We consider the *sheaf of total quotient rings* $\mathcal{K}$, defined on the open affine subsets $U = \mathrm{Spec}(A)$ in the following way: $\mathcal{K}(U) := Q(A)$.
We denote with $\mathcal{K}^*$ the sheaf (of multiplicative groups) of invertible elements in the sheaf of rings $\mathcal{K}^*$.
In our case, since X is variety, we have that:

- A is a domain.
- $Q(A)$ is the quotient field of A (equal to $R(X)$).
- $\mathcal{K}$ is a constant sheaf, constantly $R(X)$.

Definition 3.6.1.
A *Cartier divisor* D on X is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$.
In other words D is represented by $\{(U_i, f_i)\}_{i\in I}$, where

- $\{U_i\}_{i\in I}$ is an open cover of X.
- $f_i \in \mathcal{K}^*(U_i)$ such that $\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)$ $\forall i, j$.

As abuse of notation, we will write $D = \{(U_i, f_i)\}$.
Moreover let $D' = \{(V_j, g_j)\}$ be a Cartier divisor, we have
$D = D' \iff \frac{f_i}{g_j}, \frac{g_i}{f_j} \in \mathcal{O}^*(U_i \cap V_j)$ $\forall i, j$.

Remark 3.6.2. The following conditions are equivalent:

1. $\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)$ $\forall i, j$.
2. $\frac{f_i}{f_j}$ is a unit on $U_i \cap U_j$ (i.e. regular and nowhere vanishing function) $\forall i, j$.
3. $\mathrm{div}(f_i) = \mathrm{div}(f_j)$ on $U_i \cap U_j$ $\forall i, j$.

Proof.

- [1 $\iff$ 2] $\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)$ (i.e. regular in $U_i \cap U_j$) $\forall i, j$
  $\iff \frac{f_i}{f_j}, \frac{f_j}{f_i}$ are regular in $U_i \cap U_j$ $\forall i, j$
  $\iff \frac{f_i}{f_j}, \frac{f_j}{f_i}$ are nowhere vanishing in $U_i \cap U_j$ $\forall i, j$ (by section 2.2(4))
  $\iff \frac{f_i}{f_j}$ is unit in $U_i \cap U_j$ $\forall i, j$.

- [3 $\iff$ 1] $\mathrm{div}(f_i) = \mathrm{div}(f_j)$ on $U_i \cap U_j$ $\forall i, j$
  $\iff \mathrm{div}(\frac{f_i}{f_j}) = 0$ on $U_i \cap U_j$ $\forall i, j$
  $\iff \frac{f_i}{f_j}$ is regular on $U_i \cap U_j$ $\forall i, j$ (by Propositon 3.3.4(2)).

□

Definition 3.6.3. Let $D = \{(U_i, f_i)\}$ be a Cartier divisor on X.
The support of D is $\mathrm{Supp}(D) := \bigcup_i \{\text{zeros and poles of } f_i \text{ in } U_i\}$

Definition 3.6.4.

1. $\mathrm{CaDiv}(X)$ is the group of Cartier divisor on X, with the following operation: $\{(U_i, f_i)\} + \{(V_j, g_j)\} = \{(U_i \cap V_j, f_i g_j)\}$.
   Note that $(X, 1)$ is the zero and $\{(U_i, \frac{1}{f_i})\}$ is the inverse of $\{(U_i, f_i)\}$.

2. A Cartier divisor D is *principal* if $D \in \mathrm{Im}\{\mathcal{K}^*(X) \longrightarrow \mathcal{K}^*/\mathcal{O}^*(X)\}$, in other words, if $D = (X, f)$ with $f \in R(X)^*$. We will write $D = (f)$.

3. $\mathrm{CaPrinc}(X)$ is the subgroup of $\mathrm{CaDiv}(X)$ of principal divisors on X.

4. Two Cartier divisors D, D' are *linearly equivalent* if $D - D' \in \mathrm{CaPrinc}(X)$.

5. $\mathrm{CaCl}(X) := \mathrm{CaDiv}(X)/\mathrm{CaPrinc}(X)$.

3.7 CaDiv(X)=Div(X)

Now we see that, in our case, Cartier divisors and Weil divisors are the same (from smoothness, or better from locally factoriality), but this is not true for any scheme.

Definition 3.7.1.

1. Let $D = \{(U_i, f_i)\} \in \mathrm{CaDiv}(X)$, we can define the Weil divisor associated to it as $D := \sum_{\text{prime-divisor } V} \mathrm{ord}_V(f_i)V$, where i is such that $U_i \cap V \neq \emptyset$.

2. Let $D = \sum n_V V \in \mathrm{Div}(X)$, we can define the Cartier divisor associated to it in the following way: let $p \in X$, then there is an its open neighborhood $U_p = \mathrm{Spec}(A)$ such that A is an UFD (because X is locally factorial).
   Now by Example 3.5.2(1), $\mathrm{Cl}(U_p) = 0$, that is every divisor is principal, in particular $D_{|U_p} = \mathrm{div}_{U_p}(f_p)$ for some rational function $f_p \in R(U_p)^*$.
   The Cartier divisor associated is $D := \{(U_p, f_p)\}_{p \in X}$.
   (Note that $n_V = \mathrm{ord}_V(f_p)$ for every prime divisor V such that $V \cap U_p \neq \emptyset$, in fact this construction is inverse to the previous one).

Remark 3.7.2. The previous definitions are well-defined, in particular:

1. It does not depend on the choice of f_i.
2. It does not depend on the choice of f_p.

Proof.

1. Let V be a prime divisor. Let $f_i \in R(U_i)^*, f_j \in R(U_j)^*$ be such that $U_i \cap V \neq \emptyset$ and $U_j \cap V \neq \emptyset$. By definition of Cartier divisor, $\frac{f_i}{f_j}$ is a unit on $U_i \cap U_j$, hence $0 = \mathrm{ord}_V(\frac{f_i}{f_j}) = \mathrm{ord}_V(f_i) - \mathrm{ord}_V(f_j)$. In conclusion $\mathrm{ord}_V(f_i) = \mathrm{ord}_V(f_j)$.
   Moreover note that since X is of finite type (hence there is a finite affine cover), the definition gives a finite sum (hence a Weil divisor).
2. Given $f_p, g_p \in R(U_p)^*$ for any $p \in X$.
   Let $p, q \in X$, we have $\mathrm{div}_{U_p \cap U_q}(f_p) = D_{|U_p \cap U_q} = \mathrm{div}_{U_p \cap U_q}(g_q)$.
   By Remark 3.6.2, we have $\{(U_p, f_p)\} = \{(U_p, g_p)\}$.
   Moreover, for the same reason, $\{(U_p, f_p)\}$ is well-defined as a Cartier divisor.

□

These two constructions are inverse to each other, in particular they give an isomorphism $\mathrm{CaDiv}(X) \cong \mathrm{Div}(X)$ (see [9, Proposition II.6.11]).
It is clear that this isomorphism carries principal divisors to principal divisors, hence we get an isomorphism $\mathrm{CaCl}(X) \cong \mathrm{Cl}(X)$.
From now, we talk just of *divisors*.

From these constructions, we can see that every divisor is locally principal.
Moreover by this (and by Remark 3.3.3), the two definitions of support agree.

3.8 Effective Divisors

In the next chapter we will introduce the notion of linear system, and effective divisors play a key role for it. In this last section of this chapter we will go onto them before moving on the next chapter.

Definition 3.8.1. A Cartier divisor D is *effective* (we will write "$D \geq 0$") if $D = \{(U_i, f_i)\}$ with $f_i \in \mathcal{O}_X(U_i)\ \forall i$.

Remark 3.8.2. Note that the definition of effective for a Cartier divisor agrees with the definition for a Weil divisor (see Definition 3.1.1(4)).
Again, we will talk just of *effective divisors.*

Proof. Let D be an effective Cartier divisor. Given a prime divisor V, we take an index $i : U_i \cap V \neq \emptyset$, then $f_i \in \mathcal{O}_{X,V}$, hence $\mathrm{ord}_V(f_i) \geq 0$.
Conversely, let D be an effective Weil divisor. Let $f \in R(U)^*$ and let U be an open subset such that $D_{|U} = \mathrm{div}_U(f)$. Now we have that $\mathrm{div}_U(f) \geq 0$, hence by Proposition 3.3.4(2), $f \in \mathcal{O}_X(U)$. □

Remark 3.8.3. Let D be an effective divisor. We can define the closed subscheme defined by the ideal sheaf $\mathcal{O}_X(-D)$ (see Definiton 4.2.1).
We can identify an effective divisor D with this subscheme, and write $D \subseteq X$.

Chapter 4

Line bundles and linear systems

Now we introduce other two essential tools: line bundles and linear systems. Main goals of this chapter are: proving that in our case line bundles and divisors coincide (up to isomorphism or to linear equivalence), showing the relation between linear systems and global sections of line bundles, and finally their relation with projective morphisms (in particular with closed embeddings).

Let X be a scheme.
Note that (as specified in the following sections) we will assume X to be smooth when we will work with divisors, and also projective when we will work with linear systems. In particular everything in this chapter holds for curves.

4.1 Line Bundles

Line bundles are particular sheaves. We introduce this notion because when X is smooth, it gives a new way to see divisors (as we will see in the next section).

Definition 4.1.1. Let $\mathcal{L}$ be an $\mathcal{O}_X$-module.
$\mathcal{L}$ is *locally free of rank r* if there is an open cover $\mathcal{U}$ of X such that $\mathcal{L}_{|U} \cong \mathcal{O}_U^{\oplus r}$ for each $U \in \mathcal{U}$.
$\mathcal{L}$ is a *line bundle* (or an *invertible sheaf*) if it is locally free of rank 1. In particular there exist isomorphisms $f_i \colon \mathcal{L}_{|U_i} \to \mathcal{O}_{U_i}$ for an open cover $\{U_i\}$ of X.
In other words $\mathcal{L}$ is represented by $\{(U_i, f_{ij})\}$ where

- $\{U_i\}$ is an open cover of X,
- $f_{ij} \colon \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$ are transition isomorphisms (as in the Gluing Lemma, see section 1.11), and we can identify f_{ij} with a section $f_{ij}(1)$ of $\mathcal{O}^*_{U_{ij}}$.

Example 4.1.2. Let $X = \mathbb{P}^n$, then $\mathcal{O}(l)$ is a line bundle $\forall l \in \mathbb{Z}$.
In particular its transition maps are $f_{ij} = \frac{X_j^l}{X_i^l}$ (seen as sections).
(See definition of $\mathcal{O}(l)$ in section 1.6(5)).

Proof. Let $X = \mathbb{P}^n = \text{Proj}(P)$, where $P = k[X_0, ..., X_n]$.
Let $\mathcal{U} = \{U_0, ..., U_n\}$ be its standard open cover.
We consider the multiplications by X_i^{-l}:

$$f_i \colon \Gamma(U_i, \mathcal{O}_{\mathbb{P}^n}(l)) = (P_{X_i})_l \to \Gamma(U_i, \mathcal{O}_{\mathbb{P}^n}) = (P_{X_i})_0$$
$$\frac{r}{X_i^h} \mapsto \frac{r}{X_i^{h+l}}$$
These are isomorphisms, hence $\mathcal{O}(l)$ is invertible.
In particular its transition maps $f_{ij} = f_i \circ f_j^{-1}$ are the isomorphisms given by the multiplications by $X_i^{-l} X_j^l = \frac{X_j^l}{X_i^l}$. □

Definition 4.1.3. The *Picard Group* of X, $\text{Pic}(X)$, is the group of the line bundles on X, up to isomorphism, with the operation $\otimes$ (tensor product between $\mathcal{O}_X$-modules).

Note that $\text{Pic}(X)$ is a group by the following remark:

Remark 4.1.4. Let $\mathcal{L}, \mathcal{L}' \in \text{Pic}(X)$, with transition maps f_{ij}, g_{ij} respectively on an open cover $\{U_i\}$ of X.

1. $\mathcal{L} \otimes \mathcal{L}'$ is the line bundle with transition maps $f_{ij} g_{ij}$.
2. The neutral element is $\mathcal{O}_X$.
3. $\mathcal{L}^\vee := \mathcal{H}om_{O_X}(\mathcal{L}, \mathcal{O}_X)$ is the inverse of $\mathcal{L}$.
   In particular it is the line bundle with transition maps f_{ij}^{-1}.

Proof.

1. Note that$(\mathcal{L} \otimes \mathcal{L}')_{|U_i} = \mathcal{L}_{|U_i} \otimes \mathcal{L}'_{|U_i} \cong \mathcal{O}_{U_i} \otimes \mathcal{O}_{U_i} \cong \mathcal{O}_{U_i}$.
   The transition maps are
   $$\begin{array}{ccc} \mathcal{O}_{U_{ij}} \cong \mathcal{O}_{U_{ij}} \otimes \mathcal{O}_{U_{ij}} & \xrightarrow{f_{ij} \otimes g_{ij}} & \mathcal{O}_{U_{ij}} \otimes \mathcal{O}_{U_{ij}} \cong \mathcal{O}_{U_{ij}} \\ 1 \mapsto 1 \otimes 1 & & \mapsto f_{ij} \otimes g_{ij} \mapsto f_{ij} g_{ij} \end{array}.$$
2. $\mathcal{O}_X$ is the line bundle with transition map 1 on whole X.
   By 1, it is neutral element.
3. $\mathcal{L}^\vee$ is a line bundle, indeed $\mathcal{L}^\vee{}_{|U_i} = \mathcal{H}om_{\mathcal{O}_{U_i}}(\mathcal{O}_{U_i}, \mathcal{O}_{U_i}) \cong \mathcal{O}_{U_i}$.
   By [9, Ex.II.5.1(b)], $\mathcal{L}^\vee \otimes \mathcal{L} \cong \mathcal{H}om_{O_X}(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}_X$, hence $\mathcal{L}^\vee$ is the inverse of $\mathcal{L}$. In particular, by 1, its transition maps are f_{ij}^{-1}.

□

4.2 Pic(X)=CaCl(X)

Assume in the remainder of this chapter, that X is a smooth variety (so divisors are well defined). We will show that line bundles and divisors coincide (up to isomorphism or to linear equivalence).

Definition 4.2.1. Let $D = \{(U_i, f_i)\} \in \text{CaDiv}(X)$.
The line bundle associated to D is the line bundle $\mathcal{O}_X(D) \subseteq \mathcal{K}$ generated by f_i^{-1} on U_i, that is $\mathcal{O}_X(D)_{|U_i} = \mathcal{O}_{U_i} f_i^{-1}$ (see definition of $\mathcal{K}$ in section 3.6),
i.e. it is the line bundle with transition maps $f_{ij} = \dfrac{f_i}{f_j} \in \mathcal{O}_X^*(U_{ij})$).

By this construction, we have a one-to-one correspondence:
$\mathrm{CaDiv}(X) \leftrightarrow \{\text{invertible subsheaves of } \mathcal{K}\}$.

Lemma 4.2.2. Let $D, D' \in \mathrm{CaDiv}(X)$.

1. $\mathcal{O}_X(D - D') \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{-1}$.
2. $\mathcal{O}_X(D)^{-1} \cong \mathcal{O}_X(-D)$.
3. $D \sim 0 \iff \mathcal{O}_X(D) \cong \mathcal{O}_X$,
in particular $D \sim D' \iff \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$.

Proof.

1. Let $D = \{(U_i, f_i)\}, D' = \{(U_i, g_i)\}$, then $\mathcal{O}_X(D - D') = \mathcal{O}_X(\{(U_i, f_i g_i^{-1})\})$ is the line bundle generated by $f_i^{-1} g_i$ on U_i, that is $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{-1}$.
2. By definition.
3. [$\Rightarrow$] Assume $D \sim 0$, that is $D = (X, f)$ with $f \in R(X)^*$.
We have that $\mathcal{O}_X(D)$ is generated by f^{-1} on X, that is there exists an isomorphism $\mathcal{O}_X \to \mathcal{O}_X(D)$, $1 \mapsto f^{-1}$; hence $\mathcal{O}_X(D) \cong \mathcal{O}_X$.
[$\Leftarrow$] Let $D = \{(U_i, f_i)\}$. We have isomorphisms $\mathcal{O}_{U_i} \to \mathcal{O}_X(D)_{|U_i}$, $1 \mapsto f_i^{-1}$.
Taken an isomorphism $\mathcal{O}_X \to \mathcal{O}_X(D)$, $1 \mapsto g$, then we have $g_{|U_i} = a f_i^{-1}$, where $a \in \mathcal{O}_X(U_i)^*$; hence $g f_i \in \mathcal{O}_X(U_i)^*$, that is $D = (X, g^{-1})$.

□

By this Proposition, we have that $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$, $D \mapsto \mathcal{O}_X(D)$ is an injective homomorphism of groups, and it should be also surjective (since the one-to-one correspondence above) if every line bundle on X is isomorphic to a subsheaf of $\mathcal{K}$. Now we will show that in our case this happens (because X is integral, in particular $\mathcal{K}$ is a constant sheaf), hence the map is a group isomorphism.

Theorem 4.2.3. $\mathrm{Pic}(X) \cong \mathrm{CaCl}(X)$

Proof.
As we said above we should show that, given $\mathcal{L} \in \mathrm{Pic}(X)$, there exists a line bundle $\mathcal{L}' \subseteq \mathcal{K}$ such that $\mathcal{L} \cong \mathcal{L}'$.
In our case, by the smoothness, $\mathcal{K}$ is constantly $R(X)$.
Let $\mathcal{U}$ be an open cover of X, where $\mathcal{L}_{|U} \cong \mathcal{O}_U$ $\forall U \in \mathcal{U}$, then $\mathcal{L} \otimes \mathcal{K}_{|U} \cong \mathcal{K}_{|U}$ constantly $R(X)$ $\forall U \in \mathcal{U}$, hence $\mathcal{L} \otimes \mathcal{K}$ is isomorphic to the constant sheaf constantly $R(X)$, that is $\mathcal{K}$.
Let $i\colon \mathcal{O}_X \to \mathcal{K}$ be the injective morphism given by the structure of $\mathcal{O}_X$-module.
Tensoring with $\mathcal{L}$, we get $i\colon \mathcal{L} \to \mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$, then $\mathcal{L} \cong i(\mathcal{L}) \subseteq \mathcal{K}$. □

Example 4.2.4. Let $X = \mathbb{P}^n$.
$\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$ and it is generated by $\mathcal{O}(1)$.
In particular every line bundle is isomorphic to $\mathcal{O}(l)$, $\exists l \in \mathbb{Z}$.

Proof. $\mathbb{P}^n = \text{Proj } k[X_0, ..., X_n]$ and $\text{Pic}(\mathbb{P}^n) \cong \text{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$.
Let $H = \{X_0 = 0\}$ be the hyperplane class of $\text{Cl}(\mathbb{P}^n)$, that is a generator.
Let $\{U_0, .., U_n\}$ be the standard open affine cover of $\mathbb{P}^n$, then $H_{|U_i} = \text{div}_{U_i}(\frac{X_0}{X_i})$, hence the Cartier divisor associated to H is $H = \{(U_i, \frac{X_0}{X_i})\}$, and the line bundle associated has transition maps $f_{ij} = \frac{X_j}{X_i}$.
Now by Example 4.1.2, $\mathcal{O}_X(H) = \mathcal{O}(1)$. □

4.3 Pullback of line bundles

Let $\phi\colon X \to Y$ be a morphism of schemes.
Let $\mathcal{L}$ be a line bundle on Y with transition maps f_{ij} on an open cover $\{U_i\}$.
First, we define the pullback of a line bundle, then the pullback of a section and finally we will see how this notion becomes on divisors.

Definition 4.3.1. The *pullback* of $\mathcal{L}$ is the line bundle $\phi^*\mathcal{L}$ with transition maps $\phi^\sharp f_{ij}$ on the open cover $\{\phi^{-1}(U_i)\}$ of X.
It defines a group homomorphism $\phi^*\colon \text{Pic}(Y) \to \text{Pic}(X)$.

Remark 4.3.2. The pullback is well-defined.

Proof. Note that $\phi^\sharp(f_{ij}) \in \Gamma(U_{ij}, \phi_*\mathcal{O}_X) = \Gamma(\phi^{-1}(U_{ij}), \mathcal{O}_X)$.
Moreover since $f_{ij} \in \mathcal{O}^*_{U_{ij}}$ (i.e. f_{ij} is a unit, that is nowhere vanishing), we have $(f_{ij})_{\phi(x)} \notin \mathfrak{m}_{\phi(x)} = (\phi^\sharp_x)^{-1}(\mathfrak{m}_x)\ \forall x \in \phi^{-1}(U_{ij})$, that is $\phi^\sharp_x((f_{ij})_{\phi(x)}) = \phi^\sharp(f_{ij})_x \notin \mathfrak{m}_x$ $\forall x \in \phi^{-1}(U_{ij})$; hence $\phi^\sharp f_{ij} \in \mathcal{O}^*_{\phi^{-1}(U_{ij})}$. □

Definition 4.3.3. Let $s \in \Gamma(Y, \mathcal{L})$ be a global section,
we can write $s = \{s_i\}$, where $s_i \in \Gamma(U_i, \mathcal{O}_{U_i})$.
The *pullback* of s is the global section $\phi^*s = \{\phi^\sharp s_i\}$ of $\phi^*\mathcal{L}$.

Remark 4.3.4. Let $\mathcal{L}$ be the line bundle generated by $f_i \in \mathcal{L}(U_i)$ on U_i, then $\phi^*(f_i^{-1}) = (\phi^*f_i)^{-1}$ and $\phi^*(\mathcal{L}^{-1}) = (\phi^*\mathcal{L})^{-1}$ (by definition).
Hence on the divisors the definition of pullback becomes $\phi^*\{(U_i, f_i)\} := \{(\phi^{-1}(U_i), \phi^*f_i)\}$.

Note that the pullback sends effective divisors to effective divisors,
indeed if $f_i \in \mathcal{O}_Y(U_i)$ then $\phi^*f_i \in \mathcal{O}_X(f^{-1}(U_i))$.

4.4 Line bundles generated by global sections

After having introduced line bundles and a useful tool that is the pullback, we want to define linear systems. A notion related to them is that of line bundle generated by global sections. This is particularly relevant for the correspondence with projective morphisms (at the end of this chapter).

Let $\mathcal{L}$ be a line bundle on X.

Remark 4.4.1. Let $s \in \Gamma(U, \mathcal{L})$ be a section.

1. Note that we can see s as a function
$s\colon U \to \coprod_{p \in U} K(p)$, where $s(p) := \bar{s_p} \in \mathcal{L}_p/\mathfrak{m}_p\mathcal{L}_p \cong \mathcal{O}_{X,p}/\mathfrak{m}_p = K(p)$.
In particular for any $p \in U, s(p) = 0 \iff s_p \in \mathfrak{m}_p\mathcal{L}_p$.

2. Let $p \in U$. Since $\mathcal{L}_p \cong \mathcal{O}_{X,p}$ (where $l_p \leftrightarrow 1$), there is $f_s \in \mathcal{O}_{X,p} : s_p = f_s l_p$. (In particular $s(p) = 0 \iff f_s \in \mathfrak{m}_p$).

3. Let $\phi \colon X \to Y$ be a morphism of schemes, then for any $p \in X$ we have $s(\phi(p)) = 0 \iff \phi^* s(p) = 0$
(as in the proof of Remark 4.3.2).

Definition 4.4.2.

1. $\mathcal{L}$ is *generated by global sections* at $p \in X$ if there is $s \in \Gamma(X, \mathcal{L}) : s(p) \neq 0$.

2. $\mathcal{L}$ is *generated by global sections* if it is such at every point $p \in X$.
In other words there are global sections $\{s_i\}$ s.t. $\forall p \in X$, $\exists i : s_i(p) \neq 0$,
i.e. $\forall p \in X$, $\{(s_i)_p\}$ generate $\mathcal{L}_p$ as $\mathcal{O}_{X,p}$-module.

Example 4.4.3. $\mathcal{O}_{\mathbb{P}^r}(1)$ is generated by global sections.

Proof. Let $\mathbb{P}^r = \text{Proj}(P)$ with $P = k[X_0, ..., X_r]$. Let $x \in \mathbb{P}^r$ be a point corresponding to the homogeneus prime ideal $P_x \subseteq k[X_0, ..., X_r]$.
Since $P_x \neq P_+ = (X_0, .., X_r)$, we have that $\exists i : X_i \notin P_x$. Now note that:

- $X_i \in P_1 = \Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$.
- $(X_i)_x = \dfrac{X_i}{1} \in (\mathcal{O}_{\mathbb{P}^r}(1))_x = P(1)_{(P_x)} = \{\dfrac{m}{a} | m \in P(1), a \in P \smallsetminus P_x$ homogeneus polynomials of the same degree$\}$.
- $(X_i)_x = \dfrac{X_i}{1} \notin \mathfrak{m}_x = \{\dfrac{m}{a} | m \in P_x, a \in P \smallsetminus P_x$ homogeneus polynomials of the same degree$\}$, that is $X_i(x) \neq 0$.

In conclusion $\mathcal{O}_{\mathbb{P}^r}(1)$ is generated by the global sections $X_0, ..., X_r$. □

4.5 (Effective) divisors of zeros

Before defining linear systems, we see as global sections of a line bundle are related with effective divisors. In this way we can describe a linear system through them.

Definition 4.5.1. Let $\mathcal{L}$ be a line bundle of X.
There are an open cover $\mathcal{U}$ of X and isomorphisms $\phi_U \colon \mathcal{L}_{|U} \xrightarrow{\cong} \mathcal{O}_U$ $(U \in \mathcal{U})$, which define $\mathcal{L}$.
Let $s \in \Gamma(X, \mathcal{L})$ be a non-zero global section.
The *divisor of zeros* of s is the effective divisor $(s)_0 := \{(U, \phi_U(s))\}$.

Proposition 4.5.2. Let $D \in \text{Div}(X)$ and let $\mathcal{L} = \mathcal{O}_X(D)$.

1. For any $s \in \Gamma(X, \mathcal{L})$ we have that $D \sim (s)_0$.

2. Let E be an effective divisor on X such that $D \sim E$, then
there exists $s \in \Gamma(X, \mathcal{L}) : E = (s)_0$ (or better $E = D + \text{div}(s)$).

3. Let X be projective. Let $s, s' \in \Gamma(X, \mathcal{L})$.
   $(s)_0 = (s')_0 \iff s' = \lambda s \; \exists \lambda \in k^*$.

Proof. Let $D = \{(U_i, f_i)\}$.

1. $\mathcal{L}$ is generated by f_i^{-1} on U_i, hence $s = f_s f_i^{-1}$ with $f_s \in \mathcal{O}_{U_i}(U_i)$, or better $f_s = \phi_{U_i}(s)$ (note that $f_s = sf_i$).
   We have $(s)_0 = \{(U_i, sf_i)\} = \operatorname{div}(s) + D$. (Note that $s \in \mathcal{L} \subseteq \mathcal{K}$, hence it is a rational function and $\operatorname{div}(s)$ is well-defined). In conclusion $(s)_0 \sim D$.

2. Let $s \in R(X)^*$ be such that $D - E = \operatorname{div}(s)$, then $E = \{(U_i, f_i s)\}$. Since E is effective, we have $f_i s \in \mathcal{O}_{U_i}(U_i)$, in particular $s = f_s f_i^{-1}$ with $f_s \in \mathcal{O}_{U_i}(U_i)$, hence $s \in \Gamma(X, \mathcal{L})$ and $(s)_0 = \{(U_i, sf_i)\} = E$.

3. By definition of Cartier divisors, $(s)_0 = (s')_0 \iff \frac{\phi_{U_i}(s)}{\phi_{U_i}(s')} \in \mathcal{O}^*_{U_i}(U_i) \; \forall i$.
   Since X is projective, $\mathcal{O}^*_{U_i}(U_i) \cong k^*$, hence $\phi_{U_i}(s) = \lambda_i \phi_{U_i}(s') = \phi_{U_i}(\lambda_i s')$ for some scalar $\lambda_i \in k^*$. Now since ϕ_{U_i} is an isomorphism, $s = \lambda_i s'$ on U_i.
   Note that $\lambda_i = \lambda_j =: \lambda \; \forall i, j$ (because $\lambda_i s'_{|U_{ij}} = s_{|U_{ij}} = \lambda_j s'_{|U_{ij}}$).
   In conclusion we have $s = \lambda s'$ on each U_i, hence $s = \lambda s'$.

□

Remark 4.5.3. Let $D \in \operatorname{Div}(X)$ and let $\mathcal{L} = \mathcal{O}_X(D)$.

1. $\Gamma(X, \mathcal{L}) = \{s \in R(X)^* | D + \operatorname{div}(s) \geq 0\} \cup \{0\}$.

2. $\Gamma(X, \mathcal{L})$ is a k-vector space.

Proof.

1. [$\subseteq$] Let $s \in \Gamma(X, \mathcal{L})$ be a non-zero section, then $D + \operatorname{div}(s) = (s)_0 \geq 0$ by Proposition 4.5.2(2).
   [$\supseteq$] Let $s \in R(X)^*$. If $D + \operatorname{div}(s) \geq 0$ then there is $s' \in \Gamma(X, \mathcal{L})$ such that $D + \operatorname{div}(s) = (s')_0$ by Proposition 4.5.2(2), and $\operatorname{div}(s) = (s')_0 - D = \operatorname{div}(s')$, hence $\frac{s}{s'} = f \in \Gamma(X, \mathcal{O}_X)$. In conclusion $s = fs' \in \Gamma(X, \mathcal{L})$.

2. Since X is of finite type, we have that $R(X)$ is k-vector space.
   We want to show that $\Gamma(X, \mathcal{L}) \subseteq R(X)$ is sub-vector space.
   Let $D = \sum n_V V$. Let $f, g \in \Gamma(X, \mathcal{L})$ and $c \in k^*$.

   - By Prop.3.3.4(1), $D + \operatorname{div}(cf) = D + \operatorname{div}(c) + \operatorname{div}(f) = D + \operatorname{div}(f) \geq 0$, hence $cf \in \Gamma(X, \mathcal{L})$.
   - Assume $f + g \neq 0$.
     Note that $\operatorname{div}(f) = \sum \operatorname{ord}_V(f) V$ and $\operatorname{ord}_V(f)$ is a valuation. We have $n_V + \operatorname{ord}_V(f+g) \geq n_V + \min\{\operatorname{ord}_V(f), \operatorname{ord}_V(g)\} \geq 0$ for each prime divisor V; hence $D + \operatorname{div}(f+g) \geq 0$, that is $f + g \in \Gamma(X, \mathcal{L})$).

□

4.6 Linear systems

Assume now (and in the following sections) that X is also projective.
We can finally define linear systems.

Definition 4.6.1. A *complete linear system* on X is
$|D| := \{E \in \mathrm{Div}(X) \text{ effective } | E \sim D\}$, where $D \in \mathrm{Div}(X)$.

Remark 4.6.2.
By Proposition 4.5.2, taken $\mathcal{L} = \mathcal{O}_X(D)$, we have a one-to-one correspondence $|D| \leftrightarrow (\Gamma(X, \mathcal{L}) \smallsetminus \{0\})/k^*$.
By [9, Theorem II.5.19], $\Gamma(X, \mathcal{L})$ is a vector space over k of finite dimension.
We define $l(D) := \dim_k \Gamma(X, \mathcal{L})$ and $\dim |D| := l(D) - 1$.

Definition 4.6.3.
A *linear system* on X is $\Lambda \subseteq |D|$, where $D \in \mathrm{Div}(X)$, corresponding to a k-vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, where $\mathcal{L} = \mathcal{O}_X(D)$; that is $\Lambda = \{(s)_0 | s \in V \smallsetminus \{0\}\}$.
We will also denote $\Lambda = (V, \mathcal{L})$.
Moreover $\dim(\Lambda) := \dim(V) - 1$.

Definition 4.6.4. Let Λ be a linear system on X.
A *basepoint* of Λ is a point $p \in X$ such that $p \in \mathrm{Supp}(E)$, $\forall E \in \Lambda$.

4.7 Linear systems and projective morphisms

We introduce one of the theorems that we will use most, which relates linear systems and projective morphisms. First we will see the version for line bundles and then we will rephrase it for linear systems.

Lemma 4.7.1. Let $\mathcal{L}$ be a line bundle on X and let s be a global section of $\mathcal{L}$.

1. $\mathrm{Supp}(s)_0 = V(s) := \{\text{zeros of } s\}$.
   Equivalently, for any $p \in X$, $p \in \mathrm{Supp}((s)_0) \iff s(p) = 0$.
2. Let $\Lambda = (V, \mathcal{L})$ be a linear system on X.
   Λ is basepoint-free $\iff$ $\mathcal{L}$ is generated by the global sections in V.
3. $X_s := V(s)^c = \{x \in X | s(x) \neq 0\}$ is open in X.

Proof.

1. Let $(s)_0 = \{(U, \phi_U(s))\}$. Since it is effective, we have that $\phi_U(s) \in \mathcal{O}_U(U)$ is regular in U, i.e. it has no poles.
   Moreover since ϕ_U is an isomorphism, we have
   p is a zero of $s \iff p$ is a zero of $\phi_U(s)$.
   Hence $\mathrm{Supp}(s)_0 = \bigcup \{\text{zeros and poles of } \phi_U(s) \text{ in } U\} = \{\text{zeros of } s\}$.
2. Λ is basepoint-free $\iff \forall p \in X, \exists s \in V : p \notin \mathrm{Supp}((s)_0)$
   $\iff \forall p \in X, \exists s \in V : s(p) \neq 0$ (by 1)
   $\iff \mathcal{L}$ is generated by the global sections in V.

3. Let $\mathcal{U}$ be an open affine cover of X such that $\mathcal{L}_{|U} \cong \mathcal{O}_U\ \forall U \in \mathcal{U}$.
   For every $U = \mathrm{Spec}(A) \in \mathcal{U}$ we have
   $\rho^X_U \colon \Gamma(X, \mathcal{L}) \to \Gamma(U, \mathcal{L}_{|U}) = \Gamma(U, \mathcal{O}_U) = A$. Let $\tilde{s} := \rho^X_U(s) \in A$.
   Moreover $\forall x \in U$, $\mathcal{L}_x \cong \mathcal{O}_{U,x} \cong A_{P_x}$.
   Now $U \cap X_s = U_{\tilde{s}}$ (principal open subset),
   indeed $\forall x \in U$, we have $x \in X_s \iff s_x \notin \mathfrak{m}_x\mathcal{L}_x \iff \frac{\tilde{s}}{1} \notin P_xA_{P_x}$
   $\iff \tilde{s} \notin P_x \iff x \notin V(\tilde{s}) \iff x \in U_{\tilde{s}}$.
   In conclusion $U \cap X_s$ is open in the open U, hence in X. It follows that X_s is union of open subsets, hence it is open in X.

□

Theorem 4.7.2 (Projective morphisms and line bundles)**.**

1. To give a morphism of schemes $f \colon X \to \mathbb{P}^n$ is equivalent to give a line bundle $\mathcal{L}$ and $n+1$ global sections $s_0, ..., s_n$ which generate $\mathcal{L}$.
   In detail, f is the unique morphism s.t. $\mathcal{L} \cong f^*(\mathcal{O}(1))$ and $s_i = f^*X_i$.

2. Let f be the morphism corresponding to the line bundle $\mathcal{L}$ and global sections $s_0, ..., s_n$ which generate a sub-vector space $V \subseteq \Gamma(X, \mathcal{L})$.
   Then f is a closed embedding if and only if

   - *V separates points*,
     i.e. for any two distinct points $p, q \in X$, there exists $s \in V$ such that $s(p) = 0$ and $s(q) \neq 0$.
   - *V separates tangent vectors*,
     i.e. for any point $p \in X$, $\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p$ is spanned by $\{s \in V | s(p) = 0\}$.

Proof.

1. [⇒] $f^*\mathcal{O}(1)$ is a line bundle (by definition of pullback) and it is generated by global sections $f^*X_1, ..., f^*X_n$ (by Remark 4.4.1(3)).
   [⇐] Since $\mathcal{L}$ is generated by global sections $\{s_i\}$, we have that $\{X_{s_i}\}$ is an open cover of X.
   Let $\{U_i\}$ be the standard open cover of $\mathbb{P}^n$, where $U_i \cong \mathrm{Spec}\ k[\frac{X_0}{X_i}, ..., \frac{X_n}{X_i}]$.
   Consider the ring homomorphisms
   $$\phi_i \colon k[\frac{X_0}{X_i}, ..., \frac{X_n}{X_i}] \to \Gamma(X_{s_i}, \mathcal{O}_{X_{s_i}})$$
   $$\frac{X_j}{X_i} \to \frac{s_j}{s_i}$$
   By section 1.5(iii), they correspond to morphisms of schemes $f_i \colon X_{s_i} \to U_i$, that we can glue to a morphism $f \colon X \to \mathbb{P}^n$.
   Moreover $f^*X_i = f^*\{\frac{X_i}{X_j}\}_j = \{\phi_j\frac{X_i}{X_j}\}_j = \{\frac{s_i}{s_j}\}_j = s_i$ and $f^*\mathcal{O}(1) = \mathcal{L}$.
   Finally this morphism is unique by construction, indeed let $f' \colon X \to \mathbb{P}^n$ be a such morphism. It induces morphisms $f'_i \colon X_{s_i} \to U_i$
   (because $t \in X_{s_i} \iff s_i(t) \neq 0 \iff X_i(f(t)) \neq 0 \iff f(t) \in U_i$), and we call ϕ'_i the corresponding ring homomorphisms.

Now $f^*X_i = s_i = (f')^*X_i$, where $X_i = \{\frac{X_i}{X_j}\}_j$,
hence $f_i^*(\frac{X_i}{X_j}) = (f_i')^*(\frac{X_i}{X_j})$, that is $\phi_i(\frac{X_i}{X_j}) = (\phi_i')(\frac{X_i}{X_j})$, hence $f = f'$.

2. [$\Rightarrow$] Since f is a closed embedding, we can consider $X \subset \mathbb{P}^n$.
In this case $\mathcal{L} = \mathcal{O}_X(1)$, and $V \subseteq \Gamma(\mathbb{P}^n, \mathcal{O}_X(1)) = \{$hyperplanes in $\mathbb{P}^n$ meeting $X\}$ is just spanned by the images of $X_0, ..., X_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

   - Let $p, q \in X$ be two distinct points. Since $p \neq q$, there is a hyperplane $V(s)$ in $\mathbb{P}^n$ containing p but not q, that is there is $s = \sum a_i X_i$ such that $s(p) = 0$ and $s(q) \neq 0$, hence $f^*s \in V$ separates p and q.
   - Let $p = (a_0 : ... : a_n) \in X$. We can consider $p = (1 : 0 : ... : 0)$ (without loss of generality).
   In $U_0 \cong \text{Spec } k[\frac{X_1}{X_0}, ..., \frac{X_n}{X_0}] = \text{Spec } k[y_1, ..., y_n]$, we have $p = (0, ..., 0)$.
   The space $\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p \cong \mathfrak{m}_p/\mathfrak{m}_p^2$ is spanned by $y_1, ..., y_n$.

   [$\Leftarrow$] Note that f is injective (by Remark 4.7.3 below).
   By [9, Theorem II.4.9], f is proper, hence closed. Being a morphism, it is also continuous. It follows that f is homeomorphism on $f(X)$.
   Moreover we have

   - $\mathcal{O}_{\mathbb{P}^n,p}/\mathfrak{m}_{\mathbb{P}^n,p} \cong k \cong \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$ (because they are projective).
   - $\mathfrak{m}_{\mathbb{P}^n,p} \to \mathfrak{m}_{X.p}/\mathfrak{m}_{X,p}^2, t_p \mapsto s_p = f^*t_p$ is surjective
   (indeed let s_p be a generator of $\mathfrak{m}_{X.p}/\mathfrak{m}_{X,p}^2$, with $s \in V$, then $\exists t \in \mathcal{O}(1)$ such that $s = f^*t$. Since $s(p) = 0$, i.e. $s_p \in \mathfrak{m}_{X,p}$, we have $t_p \in \mathfrak{m}_{\mathbb{P}^n,p}$).
   - $\mathcal{O}_{X,p}$ is finitely generated as $\mathcal{O}_{\mathbb{P}^n,p}$-module (by Corollary [9, II.5.20]).

   It follows that $f^\sharp$ is surjective (by [9, Lemma II.7.4]).
   In conclusion f is a closed embedding.

$\square$

Given global sections $s_0, ..., s_n$ which generate a line bundle $\mathcal{L}$. The morphism associated to them is set-theoretically given by

$$f\colon X \longrightarrow \mathbb{P}^n, p \mapsto (s_0(p) : ... : s_n(p))$$

Note that $s_i(p)$ are not all zero because $\mathcal{L}$ is generated by the global sections s_i.

Remark 4.7.3. *V separates two points p and q $\iff$ $f(p) \neq f(q)$*

Proof.

$f(p) \neq f(q) \iff \exists H = V(F)$ hyperplane in $\mathbb{P}^n$ *s.t.* $\begin{cases} f(p) \in H \\ f(q) \notin H \end{cases}$

$\iff \exists F = \sum a_i X_i : \begin{cases} F(f(p)) = 0 \\ F(f(q)) \neq 0 \end{cases}$

$\iff \exists s = f^*F = \sum a_i f^*X_i \in V : \begin{cases} s(p) = f^*F(p) = 0 \\ s(q) = f^*F(q) \neq 0 \end{cases}$

$\iff V$ separates p and q. □

Remark 4.7.4. By Theorem 4.7.2(1), a morphism $f\colon X \to \mathbb{P}^n$ corresponds to a basepoint-free linear system $\Lambda = (V, \mathcal{L})$.
Indeed given a morphism f we can take $\mathcal{L} \cong f^*(\mathcal{O}(1))$ and V generated by $s_i = f^*X_i$. The linear system $(V, \mathcal{L})$ is basepoint-free (by Lemma 4.7.1(2)).
Conversely, given a basepoint-free linear system $(V, \mathcal{L})$ we can take a basis $\{s_0, ..., s_n\}$ of V as global sections. They generate $\mathcal{L}$ (by Lemma 4.7.1(2)), hence we can apply Theorem 4.7.2(1) and get the morphism f.

Lemma 4.7.5. A basepoint-free linear system $(V, \mathcal{L})$ on X of dimension n corresponds (taking a basis $s_0, ..., s_n$ of V) to a morphism $f\colon X \to \mathbb{P}^n$. Moreover

1. f depends by the chosen of the basis of V,
   but it is unique up to automorphism of $\mathbb{P}^n$.
2. Assume that $s_0, ..., s_n$ are generators of V.
   $\{s_0, ..., s_n\}$ is basis of $V \iff f(X)$ is not contained in any hyperplane.
3. $\mathrm{Supp}((s)_0)$ is preimage of a hyperplane $\forall s \in V$.

Proof. Taken a basis $s_0, ..., s_n$ of V, by the theorem for line bundles it corresponds to a unique morphism $f\colon X \to \mathbb{P}^n$.

1. Let $s'_0, ..., s'_n$ be another basis of V associated to the morphism f', and let $A = (a_{ij}) \in \mathrm{GL}_{n+1}(k)$ be the change-of-basis matrix. We have that $s'_i = \sum_j a_{ij} s_j$. Taken the isomorphism
   $\phi_A\colon k[X_0, ..., X_n] \to k[X_0, ..., X_n]$
   $$X_i \mapsto \sum_j a_{ij} X_j$$
   it induces an automorphism $\phi_A\colon \mathbb{P}^n \to \mathbb{P}^n$ such that $f' = \phi_A \circ f$.
2. $s_0, ..., s_n$ are linear independent (i.e. basis)
   $\iff \sum a_i s_i \neq 0$ ($\forall a_i \in k$ not-all zero)
   $\iff \sum a_i s_i(p) \neq 0 \ \exists p \in X$ ($\forall a_i \in k$ not-all zero)
   $\iff \sum a_i X_i(f(p)) \neq 0 \ \exists p \in X$ ($\forall a_i \in k$ not-all zero)
   $\iff \sum a_i X_{i|f(X)} \neq 0$ ($\forall a_i \in k$ not-all zero)
   $\iff f(X) \nsubseteq V(\sum a_i X_i)$ for each hyperplane $V(\sum a_i X_i)$.
3. $s = \sum a_i f^* X_i = f^*F$ with $F \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.
   $V(s) = V(f^*F) = \{p \in X | f^*F(p) = 0\} = \{p \in X | F(f(p)) = 0\} =$
   $= \{p \in X | f(p) \in V(F)\} = f^{-1}V(F)$ and $V(F)$ is a hyperplane.

□

Theorem 4.7.6 (Projective morphisms and linear systems).

1. To give a non-degenerate morphism $f\colon X \to \mathbb{P}^n$ is equivalent to give a basepoint-free linear system $\Lambda = (V, \mathcal{L})$ on X of dimension n.
   (*Non-degenerate* means that $f(X)$ is not contained in any hyperplane).
   Moreover $\mathcal{L} \cong f^*(\mathcal{O}(1))$ and it is generated by global sections $s_i = f^*X_i$.

2. Let f be the morphism corresponding to a linear system Λ.
   f is a closed embedding if and only if

   - Λ *separates points*,
     i.e. for any distinct points $p, q \in X$, $\exists E \in \Lambda : \begin{cases} p \in \mathrm{Supp}(E) \\ q \notin \mathrm{Supp}(E) \end{cases}$
   - Λ *separates tangent vectors*,
     i.e. for any point $p \in X$ and any tangent vector $t \in T_p(X)$,
     $\exists E \in \Lambda : \begin{cases} p \in \mathrm{Supp}(E) \\ t \notin T_p(E) \end{cases}$
     (Note that E is effective, so we can see it as a closed subscheme of X).

Proof. Rephrasing Theorem 4.7.2 in terms of linear systems. □

We can see an application of the theorem:

Example 4.7.7.
Every automorphism of $\mathbb{P}^n$ is of the form ϕ_A (as defined in the proof of Lemma 4.7.5(1)). In particular $\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{GL}_{n+1}(k)/k^*$.

Proof. Let $A \in \mathrm{GL}_{n+1}(k)$. Clearly $\phi_A = \phi_{\lambda A}\ \forall \lambda \in k^*$.
Let $\phi\colon \mathbb{P}^n \to \mathbb{P}^n$ be an automorphism, it induces a group isomorphism $\phi^*\colon \mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(\mathbb{P}^n)$.
We know that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$ and $\mathcal{O}(1)$ is a generator of $\mathrm{Pic}(\mathbb{P}^n)$, hence also $\phi^*\mathcal{O}(1)$ is a generator, that is $\mathcal{O}(1)$ or $\mathcal{O}(-1)$. Note that $\Gamma(\mathbb{P}^n, \mathcal{O}(-1)) = 0$, i.e. it has no non-zero global sections. It follows that $\phi^*\mathcal{O}(1) \cong \mathcal{O}(1)$.
Now the sections $s_i = \phi^* X_i$ give a basis of the vector space $\Gamma(\mathbb{P}^n, \mathcal{O}(1))$, hence $\exists A = (a_{ij}) \in \mathrm{GL}_{n+1}(k)$ such that $s_i = \sum_j a_{ij} X_j$.
Note that ϕ is the morphism associated to these global sections but also ϕ_A is such. By the uniqueness in Theorem 4.7.2(1), $\phi = \phi_A$. □

Chapter 5

Curves

Finally we can apply our knowledge so far acquired to study the one-dimensional case, hence to better describe curves.
As already established in section 2.1, with *curve* we mean a one-dimensional projective (i.e. complete) smooth variety over an algebraically closed field k.
Our main interest is to understand in which projective spaces a curve can be embedded and in what way. Studying closed embeddings, we will use the relation between morphisms and linear systems. In this light, divisors and line bundles assume fundamental importance.
In the first part of this chapter, we will review what a divisor on a curve is and we will introduce the notion of degree of a curve, in particular we will see the Bézout's theorem. After that we will define a particular divisor called *canonical* and we will introduce the notion of genus of a curve. Finally we will arrive at the Riemann-Roch Theorem, that is a formula involving notions of genus, degree and dimension of a complete linear system.
In the second part, our first goal is showing that every curve can be embedded in $\mathbb{P}^3$; after that we will study curves of low genus, in particular we will distinguish two different kinds of curves of genus at least 2: hyperelliptic and non-hyperelliptic curves, and we will focus on the latter, which correspond to canonical curves. Finally, we will give a brief exhibition about higher genus.

5.1 Divisors on curves

On a curve X, we have $\operatorname{Div}(X) \cong \operatorname{CaDiv}(X)$ and $\operatorname{Cl}(X) \cong \operatorname{Pic}(X)$.
The prime divisors are the points of X, hence a divisor is of the form $D = \sum n_i P_i$ (with P_i points of X), and its degree is $\deg(D) := \sum n_i$.
In particular $n_i =: m_D(P_i)$ is called *multiplicity* of P_i in D.

To every divisor D we can associate a line bundle $\mathcal{L} = \mathcal{O}_X(D)$ and a complete linear system $|D|$. We have defined $l(D) = \dim_k H^0(X, \mathcal{L})$ and $\dim |D| = l(D) - 1$.
Now, we want to define the degree of a linear system.

Remark 5.1.1. Let X be a curve. By [9, Corollary II.6.10], we have that the map $\deg \colon \operatorname{Cl}(X) \to \mathbb{Z}$, $D \mapsto \deg(D)$ is a surjective homomorphism.
In particular, let $D, D' \in \operatorname{Div}(X) : D \sim D'$, then $\deg(D) = \deg(D')$.

Definition 5.1.2. Let X be a curve.
The *degree* of a linear system Λ on X is the degree of any divisor in Λ.

Note that it is well-defined by the previous remark, because in Λ the divisors are linearly equivalent.

5.2 Degree of curves and degree of morphisms

The first important notion for curves is that of degree of a curve in $\mathbb{P}^n$.
Note that this is not an invariant, but depends by the closed embedding. For example in the next sections we will see that plane curves of degree 1 and plane curves of degree 2 have genus 0 (by the Genus-degree Formula), hence they are both isomorphic to $\mathbb{P}^1$.

Definition 5.2.1. Let X be a curve, with a closed embedding $i\colon X \hookrightarrow \mathbb{P}^n$.
Taken the divisor D corresponding to the line bundle $i^*(\mathcal{O}(1))$,
the *degree* of X is $\deg(X) := \deg(D)$.

Lemma 5.2.2.
Let $f\colon X \to Y$ be a morphism of curves, then f is constant or surjective.
Moreover if f is non-constant, then
$R(Y) \subseteq R(X)$ is a finite extension of fields and f is finite.

Proof.

Since X is complete, then $f(X)$ is closed (and complete) in Y (by [9, Ex. II.4.4]).
Since $\dim(Y) = 1$ and $f(X)$ is an irreducible closed subset, we either have that $f(X)$ is a point or $f(X) = Y$, that is f is either constant or surjective.

Now if f is non-constant, hence surjective, $R(Y) \hookrightarrow R(X)$ is an extension of fields. We want to show that it is finite.
We know that $R(X) \cong A_{(0)}$, where $U = \mathrm{Spec}(A)$ is an affine open subscheme of X, hence $R(X)$ is a finitely generated k-algebra. Since $\mathrm{Trdeg}_k R(X) = \dim(X) = 1$, we have that $k \subset R(X)$ is a finitely generated extension of fields of transcendence degree 1.
Analogously, $k \subset R(Y)$ is such.
Now $\mathrm{Trdeg}_{R(Y)} R(X) = \mathrm{Trdeg}_k R(X) - \mathrm{Trdeg}_k R(Y) = 1 - 1 = 0$. It follows that $R(Y) \subseteq R(X)$ is an algebraic extension, hence finite.

Finally, we want to show that f is finite. Let $V = \mathrm{Spec}(B)$ be an affine open subscheme of Y. Note that $B \subseteq Q(B) \cong R(Y) \subseteq R(X)$, and we can take $A := \overline{B}^{R(X)}$, that is the integral closure of B in $R(X)$.
We know that A is a B-algebra, finitely generated as B-module (by [21, Ch.V, Th.9, p.267]) and there is an open subset $U \cong \mathrm{Spec}(A)$ of X (by [9, I.6.7]), or better $U = f^{-1}(V)$, hence f is finite. □

Definition 5.2.3. Let $f\colon X \to Y$ be a non-constant (finite) morphism of curves.
The *degree* of f is $\deg(f) := [R(X) : R(Y)]$.

Note that $\deg(f)$ is finite and well-defined by the previous lemma.

Remark 5.2.4. Let $D \in \mathrm{Div}(Y)$, then $\deg(f^*D) = \deg(f)\deg(D)$.
(See [9, Proposition II.6.9]).

Examples 5.2.5. Let Λ be the linear system associated to a non-constant morphism $f\colon X \to \mathbb{P}^1$, then $\deg(\Lambda) = \deg(f)$.

Proof. Let $\Lambda \subseteq |D|$, then $D = f^*H$ with H hyperplane (by Theorem 4.7.6(1)). By Remark 5.2.4, we have $\deg(\Lambda) = \deg(D) = \deg(f)\deg(H) = \deg(f)$. $\square$

5.3 Bézout's Theorem

Let $X \subset \mathbb{P}^n$ be a curve.
We saw that we can see the divisors of a basepoint-free linear system as pullbacks of hyperplanes (so as intersections, if the corresponding morphism is a closed embedding). For this reason, it is useful to study the intersection of X and a hypersurface $V = V(F) \subset \mathbb{P}^n$ s.t. $X \not\subseteq V$.

Definition 5.3.1. Let $j\colon X \hookrightarrow \mathbb{P}^n$ be a closed embedding.
We define the intersection $X \cdot V := j^*V$ in $\mathrm{Pic}(X)$.

Note that it is well-defined, indeed
since $X \not\subseteq V$, we have $F\big|_X \neq 0$, hence $j^*V = j^*\mathrm{div}(F) = j^*(\mathbb{P}^n, F) = (X, F\big|_X)$.
Moreover, let $V' \subset \mathbb{P}^n$ be another hypersurface s.t. $X \not\subseteq V'$,
if $\deg(V) = \deg(V')$ (i.e. $V = V'$ in $\mathrm{Pic}(\mathbb{P}^n)$), then $X \cdot V = X \cdot V'$ in $\mathrm{Pic}(X)$.

Definition 5.3.2. Let $P \in X$ be a point.
The *intersection multiplicity* at P is $(X \cdot V)_P := \mathrm{ord}_P(\frac{F}{G}\big|_X)$ where G is a homogeneus polynomial of the same degree of F s.t. $G(P) \neq 0$.

Proposition 5.3.3.

1. $X \cdot V = \sum (X \cdot V)_P P$ in $\mathrm{Pic}(X)$.
2. $(X \cdot V)_P$ does not depend by the choice of G.
   We could redefine $X \cdot V := \sum (X \cdot V)_P P$ (well-defined in $\mathrm{Div}(X)$).
3. Let $P \in X$.
   If $P \in V$, then $(X \cdot V)_P \geq 1$.
   If $P \notin V$, then $(X \cdot V)_P = 0$,.
   In particular $\sharp(X \cap V) \leq \deg(X \cdot V)$.
4. $(X \cdot V)_P \geq 2 \iff T_pX \subseteq T_PV$ (i.e. V is tangent to X at P).

Proof.

1. Note that $V = V(F) = \{(U_G, \frac{F}{G})\}$ as divisor.
   Indeed we assume that F is an irreducible polynomial of degree d, then $\mathrm{div}_{U_i}(\frac{F}{X_i^d}) = \mathrm{div}_{U_i}(F) - d \cdot \mathrm{div}_{U_i}(X_i) = V_{U_i}(F)$. Hence $\mathrm{div}_{U_i}(\frac{F}{X_i^d}) = V \cap U_i$, that is $V = \{(U_i, \frac{F}{X_i^d})\}$, or better $V = \{(U_G, \frac{F}{G})\}$.
   Now $X \cdot V = j^*V = \{(U_G \cap X, \frac{F}{G}\big|_X)\} = \sum \mathrm{ord}_P(\frac{F}{G})P$ where $P \in U_G \cap X$.

2. By the proof of previous point.

3. If $P \in V(F)$, then $\dfrac{F}{G}$ is regular at P; hence $(X \cdot V)_P := \mathrm{ord}_P(\dfrac{F}{G}|_X) \geq 1$.
   If $P \notin V(F)$, then $P \notin \mathrm{Supp}(\mathrm{div}_{U_G}(\dfrac{F}{G}))$; hence $\mathrm{ord}_P(\dfrac{F}{G}) = 0$.

4. Let $P \in U_G$. $\dfrac{F}{G}$ is regular in U_G and $T_P V = T_P(V \cap U_G) = V(d_P \dfrac{F}{G})$.
   $T_P X \subseteq T_P V \iff d_P \dfrac{F}{G}|_X = 0 \iff \dfrac{F}{G}|_X \in \mathfrak{m}_P^2$
   $\iff (X \cdot V)_P = \mathrm{ord}_P(\dfrac{F}{G}|_X) \geq 2.$

□

Theorem 5.3.4 (Bézout)**.**

1. $\deg(X) = \deg(X \cdot H)$, with H hyperplane of $\mathbb{P}^n$ s.t. $X \not\subseteq H$.
   In particular we have that
   $\deg(X) = \max\{\sharp(X \cap H) | H \subset \mathbb{P}^n$ hyperplane s.t. $X \not\subseteq H\}$.

2. $\deg(X \cdot V) = \deg(X) \cdot \deg(V)$.
   In particular $\sharp(X \cap V) \leq \deg(X) \cdot \deg(V)$.

Proof.

1. By the definition of degree, $\deg(X) = \deg(j^* H) = \deg(X \cdot H)$.
   Let H' be a hyperplane not tangent to X, then $(X \cdot H')_P = 1 \; \forall P \in X \cap H'$; hence $\deg(X) = \deg(X \cdot H') = \sharp(X \cap H')$.

2. Let $d = deg(V)$. We have $V \sim dH$ with H hyperplane, $d \in \mathbb{Z}$.
   $\deg(X \cdot V) = \deg(X \cdot dH) = d \cdot \deg(X \cdot H) = d \cdot \deg(X) = \deg(X) \cdot \deg(V)$.

□

5.4 The canonical divisor

The most important divisor is the canonical divisor. We will find it in the definition of genus, in the Riemann-Roch Theorem and in the notion of canonical embedding (closely related to that of non-hyperelliptic curves). Therefore the notion of canonical divisor will be strongly present throughout this chapter.

Definition 5.4.1. Let X be a scheme.
Since X is separated, the diagonal map $\Delta \colon X \to X \times_{\mathrm{Spec}(k)} X$ is a closed embedding.
Let $\mathcal{I}$ be the sheaf of ideals of $\Delta(X)$.
The *sheaf of relative differentials* is the sheaf $\Omega_{X/k} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$.

Remark 5.4.2. If X is variety of dimension n, then
X is smooth $\iff \Omega_{X/k}$ is locally free of rank n
(see [9, Theorem II.8.15]).
If X is a curve, then $\Omega_{X/k}$ is a line bundle.

Definition 5.4.3.

1. Let X be a smooth variety of dimension n.
   The *canonical sheaf* of X is $\omega_X := \bigwedge^n \Omega_{X/k}$.
   In particular if X is a curve, then it is the line bundle $\omega_X = \Omega_{X/k}$.

2. Let X be a curve.
   A *canonical divisor* on X is a divisor $K \in \mathrm{Div}(X)$ such that $\mathcal{O}_X(K) \cong \omega_X$.
   Note that K is unique in $\mathrm{Cl}(X)$.

5.5 Genus of a curve

The most important invariant for curves is the genus. It allows us to make a first distinction between curves, hence (as seen in the introduction) to answer some first questions.

Definition 5.5.1. Let X be a smooth projective variety of dimension n.

1. The *Euler characteristic* of a coherent sheaf $\mathcal{F}$ on X is
   $\chi(\mathcal{F}) := \sum_{i=0}^n (-1)^i \dim_k H^i(X, \mathcal{F})$.

2. The *arithmetic genus* of X is $p_a(X) := (-1)^n(\chi(\mathcal{O}_X) - 1)$.

3. The *geometric genus* of X is $p_g(X) := \dim_k \Gamma(X, \omega_X)$.

Remark 5.5.2. Let X be a curve, then $p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X)$ and we call this number the *genus* of X. (See [9, Remark III.7.12.2]).

Note that the genus of a curve is invariant under isomorphism.
Moreover it is always non-negative, and conversely for any $g \geq 0$, there exist curves of genus g (see [9, Remark IV.1.1.1]).

Remark 5.5.3 (Genus-degree formula)**.**
If X is a plane curve of degree d, then $p_g(X) = \frac{1}{2}(d-1)(d-2)$.

Proof. By [9, Ex. II.8.4(e)], we have that $\omega_X \cong \mathcal{O}_X(d-3)$.
It follows that $p_g(X) = \binom{2+d-3}{d-3} = \frac{1}{2}(d-1)(d-2)$. □

5.6 The Riemann-Roch Theorem for curves

A first important formula involving the genus is given by the Riemann-Roch Theorem.

Theorem 5.6.1. (Riemann-Roch)
Let X be a curve of genus g and let $D \in \mathrm{Div}(X)$. Then
$l(D) - l(K - D) = \deg(D) + 1 - g$.

Proof.

- We will show first that $\chi(\mathcal{O}_X(D)) = l(D) - l(K-D)$.
  By Serre duality (see section 1.10(2)) we have that
  $H^0(X, \mathcal{O}_X(K-D)) = H^0(X, \omega_X \otimes \mathcal{O}_X(D)^\vee) = H^1(X, \mathcal{O}_X(D))^\vee$.
  Hence $l(D)-l(K-D) = \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(K-D)) =$
  $= \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(X, \mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D))$.

- We will show now that $\chi(\mathcal{O}_X(D)) = \deg(D) + 1 - g$.
  If $D = 0$, i.e. $\mathcal{O}_X(D) = \mathcal{O}_X$, then $H^0(X, \mathcal{O}_X) \cong k$ (because X is projective) and $\dim_k H^1(X, \mathcal{O}_X) = g$ (by definition), hence we have
  $\chi(\mathcal{O}_X) = \dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = 1 - g$.
  Now we want to show that the formula is true for D if and only if it is true for $D + P$ (for every divisor D and every point $P \in X$). In this way, starting from the case $D = 0$, we can get any case.
  Let $D' = D + P$, then it has degree $\deg(D') = \deg(D) + 1$.
  We know that $\mathcal{I}_{P/X} = \mathcal{O}_X(-P)$ (by Remark 3.8.3).
  We can define $k(P) := \mathcal{O}_X/\mathcal{I}_{P/X}$ the sheaf of residue fields of P, in detail this is constantly k on each open neighborhood of P, and 0 everywhere else.
  Therefore we have the exact sequence

  $$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to k(P) \to 0$$

  Note that $k(P) \otimes \mathcal{O}_X(D') \cong k(P)$ (analogously to the proof of Theorem 4.2.3), hence tensoring with $\mathcal{O}_X(D') = \mathcal{O}_X(D + P)$, we get

  $$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D') \to k(P) \to 0$$

  The Euler characteristic is additive on short exact sequences (by [9, Ex III.5.1]) and $\chi(k(P)) = 1$, hence we have
  $\chi(\mathcal{O}_X(D')) = \chi(\mathcal{O}_X(D)) + \chi(k(P)) = \chi(\mathcal{O}_X(D)) + 1$.

□

5.7 Applications of Riemann-Roch Theorem

Let X be a curve of genus g. An immediate application is computing $\deg(K)$.

Lemma 5.7.1. $l(K) = g$ and $\deg(K) = 2g - 2$.

Proof. By definition, $l(K) = \dim H^0(X, \omega_X) = p_g = g$.
Now note that $l(K - K) = l(0) = 1$.
By Riemann-Roch, $g - 1 = \deg(K) + 1 - g$, hence $\deg(K) = 2g - 2$. □

Before seeing other applications, it is advisable to see the following lemma.

Lemma 5.7.2.

1. If $l(D) \neq 0$, then $\deg(D) \geq 0$.

2. If $l(D) \neq 0$ and $\deg(D) = 0$, then $D \sim 0$ (i.e. $\mathcal{O}_X(D) \cong \mathcal{O}_X$).

Proof.

1. If $l(D) \neq 0$, then $|D| \neq \emptyset$, that is there exists a divisor $E \geq 0$ such that $D \sim E$, hence $\deg(D) = \deg(E) \geq 0$.

2. Let E be as above. In this case $\deg(E) = \deg(D) = 0$.
   Since E is effective, we have $E = 0$, hence $D \sim 0$.

□

A very useful application of Riemann-Roch is the following:

Proposition 5.7.3. If $\deg(D) > 2g - 2$, then $\dim |D| = \deg(D) - g$.

Proof. Since $\deg(K - D) = \deg(K) - \deg(D) < (2g - 2) - (2g - 2) = 0$, then $l(K - D) = 0$ by Lemma 5.7.2. By Riemann-Roch, $\dim |D| = \deg(D) - g$. □

Using Riemann-Roch, we can finally see that there exists only one curve of genus 0 (up to isomorphism).

Proposition 5.7.4. $g = 0 \iff X \cong \mathbb{P}^1$ (i.e. X is rational).

Proof. Note that X is rational $\iff X \cong \mathbb{P}^1$ (by [9, Example II.6.10.1]).
Now we want to show the proposition.
[$\Leftarrow$] We can see $\mathbb{P}^1$ as hyperplane of $\mathbb{P}^2$. By Remark 5.5.3 we have $g = 0$.
[$\Rightarrow$] We assume $g = 0$. Let P, Q be two distinct points of X. Since $\deg(K-P+Q) = \deg(K)-\deg(P-Q) = 2g-2-0 = -2$, we have $l(K-P+Q) = 0$ (by Lemma 5.7.2). Applying Riemann-Roch, we have
$l(P - Q) = \deg(P - Q) + 1 - g = 1$, hence $l(P - Q) \neq 0$ and $\deg(P - Q) = 0$, then $P - Q \sim 0$ (by Lemma 5.7.2), that is $P \sim Q$.
In conclusion X is rational (by [9, Example II.6.10.1]), i.e. $X \cong .\mathbb{P}^1$. □

5.8 Linear systems on curves

In this section we will introduce the notion of very ample divisors. They are particularly important in the study of linear systems as they correspond to closed embeddings in projective spaces.
Let $\mathcal{L}$ be a line bundle on a curve X corresponding to a divisor D.

Definition 5.8.1.

1. $\mathcal{L}$ (resp.D) is *very ample* if there exists a closed embedding $j\colon X \to \mathbb{P}^n$ such that $\mathcal{L} \cong j^*\mathcal{O}(1) \cong \mathcal{O}_X(1)$.

2. $\mathcal{L}$ (resp.D) is *ample* if there is $n > 0$ such that $\mathcal{L}^n$ (resp.nD) is very ample.

Clearly, very ample implies ample.

Remark 5.8.2.

- D is very ample $\Rightarrow \mathcal{L}$ is generated by global sections (because $\mathcal{O}(1)$ is such) $\Rightarrow |D|$ is basepoint-free (by Lemma 4.7.1(2)).
- Let $f\colon X \to \mathbb{P}^n$ be the morphism corresponding to $|D|$, then: D is very ample $\iff$ f is a closed embedding (by definition).

Lemma 5.8.3. Let $P \in X$ be a point.

1. $\dim |D - P| \in \{\dim |D|, \dim |D| - 1\}$.
2. P is a basepoint of $|D| \iff \dim |D - P| = \dim |D|$.

Proof.

1. As in the proof of Riemann-Roch, we have an exact sequence
$$0 \to \mathcal{I}_P = \mathcal{O}_X(-P) \to \mathcal{O}_X \to k(P) \to 0$$
Tensoring by $\mathcal{O}_X(D)$, we get
$$0 \to \mathcal{O}_X(D - P) \to \mathcal{O}_X(D) \to k(P) \to 0$$
Taken the global sections, we get the left-exact sequence
$$0 \to \Gamma(X, \mathcal{O}_X(D - P)) \to \Gamma(X, \mathcal{O}_X(D)) \to \Gamma(X, k(P)) = k$$
Hence
$\dim_k \Gamma(X, \mathcal{O}_X(D - P)) \leq \dim_k \Gamma(X, \mathcal{O}_X(D)) \leq \dim_k \Gamma(X, \mathcal{O}_X(D - P)) + 1$,
that is $l(D) \in \{l(D-P), l(D-P)+1\}$, hence $l(D-P) \in \{l(D), l(D)-1\}$.

2. By Remark 4.5.3, $\Gamma(X, \mathcal{O}_X(D)) = \{0\} \cup \{s \in R(X)^* | D + \operatorname{div}(s) \geq 0\}$.
In particular $\Gamma(X, \mathcal{O}_X(D - P)) \subseteq \Gamma(X, \mathcal{O}_X(D))$. Now we have that:
P is a basepoint of $|D|$
$\iff \forall s \in \Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}$, $P \in \operatorname{Supp}(D + \operatorname{div}(s))$
$\iff \forall s \in R(X)^*$, we have $D + \operatorname{div}(s) \geq 0$ iff $D - P + \operatorname{div}(s) \geq 0$
$\iff \Gamma(X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{O}_X(D - P))$
$\iff l(D) = l(D - P)$.

$\square$

Remark 5.8.4. Let D be effective.

1. $\dim |D| \leq \deg(D)$.
2. $\dim |D| = \deg(D) \iff D = 0$ or $X \cong \mathbb{P}^1$.

Proof.

1. By Lemma 5.8.3, $\dim |D + P| \leq \dim |D| + 1\ \forall P \in X$.
Now we can iterate, starting by $\dim |P| \leq \dim |0| + 1 = 1 = \deg(P)$.

2. [$\Rightarrow$] Assume $D \neq 0$. If $\dim |D| = \deg(D)$, then by the iteration in the proof of 1, we have $\dim |P| = 1$ for any $P \in \mathrm{Supp}(D)$, in particular $|P|$ corresponds to a morphism $f\colon X \to \mathbb{P}^1$ of degree 1, hence f is an isomorphism and $X \cong \mathbb{P}^1$.
[$\Leftarrow$] If $D = 0$, then $\dim |0| = l(0) - 1 = 0 = \deg(0)$.
If $X \cong \mathbb{P}^1$, then $D \sim nP$, and we have
$\dim |nP| = l(nP) - 1 = \dim_k \Gamma(\mathbb{P}^1, \mathcal{O}(n)) - 1 = \binom{n+1}{1} - 1 = n = \deg(nP)$.

$\square$

Proposition 5.8.5.

1. $|D|$ is basepoint-free $\iff \forall P \in X$, $\dim |D - P| = \dim |D| - 1$.

2. D is very ample $\iff \forall P, Q \in X$, $\dim |D - P - Q| = \dim |D| - 2$.
(Note that we include the case $P = Q$).

Proof.

1. By Lemma 5.8.3(2).

2. We can consider $|D|$ to be basepoint-free, indeed:
If D is very ample, then D is basepoint-free (by Remark 5.8.2(1)).
If D satisfies the condition on the right, then we have that for any $P \in X$, $\dim |D - 2P| = \dim |D| - 2$, hence P is not a basepoint of $|D|$ (otherwise $\dim |D - 2P| \geq \dim |D - P| - 1 = \dim |D| - 1$, by Lemma 5.8.3)

Now, by 1, $\forall P \in X$, $\dim |D - P| = \dim |D| - 1$.

By Theorem 4.7.6(1), $|D|$ corresponds to a unique morphism
$f\colon X \to \mathbb{P}^n$ (with $n = \dim |D|$) such that $\mathcal{L} \cong f^*\mathcal{O}(1)$.
By Theorem 4.7.6(2), D is very ample $\iff$ f is a closed embedding
$\iff \begin{cases} |D| \text{ separates points, and} \\ |D| \text{ separates tangent vectors.} \end{cases}$

We will study the two conditions:

- $|D|$ separates points $\iff \forall P \neq Q$ in $X, \exists E \in |D| : \begin{cases} P \in \mathrm{Supp}(E) \\ Q \notin \mathrm{Supp}(E) \end{cases}$
$\iff \forall P \neq Q$ in $X, \exists E \in |D| : \begin{cases} E - P \geq 0 \text{ (i.e. } E - P \in |D - P|) \\ Q \notin \mathrm{Supp}(E - P) \end{cases}$
$\iff \forall P \neq Q$ in X, Q is not a basepoint of $|D - P|$
$\iff$ (by Lemma 5.8.3(2)) $\forall P \neq Q$ in X,
$\dim |D - P - Q| = \dim |D - P| - 1 = (\dim |D| - 1) - 1 = \dim |D| - 2$.

- $|D|$ separates tangent vectors
$\iff \forall P \in X, \forall t \in T_P X, \exists E \in |D| : \begin{cases} P \in \mathrm{Supp}(E) \\ t \notin T_P E \end{cases}$

$$\iff \forall P \in X, \exists E \in |D| : \begin{cases} E - P \geq 0 \\ T_P E = 0 \end{cases}$$
($\dim T_P X = \dim X = 1$, because X is smooth, and $T_P E \subsetneq T_P X$)
$$\iff \forall P \in X, \exists E \in |D| : \begin{cases} m_E(P) \geq 1 \\ m_E(P) \leq 1 \end{cases}$$
$$\iff \forall P \in X, \exists E \in |D| : m_E(P) = 1$$
$$\iff \forall P \in X, \exists E \in |D| : \begin{cases} E - P \geq 0 \text{ (i.e. } E - P \in |D - P|) \\ P \notin \operatorname{Supp}(E - P) \end{cases}$$
$\iff \forall P \in X, P$ is not basepoint of $|D - P|$
$\iff$ (by Lemma 5.8.3(2)) $\forall P \in X$,
$\dim |D - P - P| = \dim |D - P| - 1 = \dim |D| - 2$.

□

Corollary 5.8.6.

1. If $\deg(D) \geq 2g$, then $|D|$ is basepoint-free.
2. If $\deg(D) \geq 2g + 1$, then D is very ample.
3. $\deg(D) > 0 \iff D$ is ample.

Proof.

1. By Proposition 5.7.3 if $\deg(D) \geq 2g > 2g - 2$, then $\dim |D| = \deg(D) - g$. For any $P \in X$, $\deg(D-P) \geq 2g-1 > 2g-2$, then by the same proposition $\dim |D - P| = \deg(D - P) - g = \deg(D) - 1 - g$; hence we have that $\dim |D - P| = \dim |D| - 1$. By Proposition 5.8.5(1) $|D|$ is basepoint-free.
2. As in the previous point, $\dim |D| = \deg(D) - g$.
   In analogous way, since for any $P, Q \in X$, $\deg |D-P-Q| \geq 2g-1 > 2g-2$, we have $\dim |D-P-Q| = \deg(D-P-Q)-g = \deg(D)-2-g = \dim |D|-2$. By Proposition 5.8.5(2), D is very ample.
3. [$\Leftarrow$] If D is ample, i.e. $\exists n > 0 : nD$ is very ample, then there is a closed embedding $j\colon X \hookrightarrow \mathbb{P}^m$ s.t. $nD \sim j^*H$ (where H is a hyperplane of $\mathbb{P}^m$). Since the pullback sends effective divisors to effective divisors, we have $\deg(nD) = \deg(j^*H) > 0$, hence $\deg(D) > 0$.
   [$\Rightarrow$] If $\deg(D) > 0$, then $\exists n > 0 : \deg(nD) = n \cdot \deg(D) \geq 2g + 1$, and then nD is very ample (by 2); hence D is ample.

□

5.9 Curves of low genus

Now we exhibit some first results about curves of low genus.
Applying Corollary 5.8.6 we can give an alternative proof that the only curve of genus 0 is $\mathbb{P}^1$. We can also show that the curves of genus 1 (called *elliptic* curves)

can be seen as plane cubic curves. In the same way we can show that every curve of genus 2 can be embedded in $\mathbb{P}^3$; in fact we will see in the next section that every curve can be embedded in $\mathbb{P}^3$.
Let X be a curve of genus g.

Examples 5.9.1.

1. If $g = 0$, then for any divisor D we have
   D is very ample $\iff$ D is ample (i.e. $\deg(D) > 0$).
   Moreover, $X \cong \mathbb{P}^1$.

2. $g = 1 \iff X$ can be embedded in $\mathbb{P}^2$ as cubic curve.

3. If $g = 2$, then X can be embedded in $\mathbb{P}^3$ (as curve of degree 5).

Proof.

1. We know by definition that very ample implies ample.
   Let $D \in \mathrm{Div}(X)$. Applying Corollary 5.8.6, we have that if D is ample, then $\deg(D) \geq 1 = 2g + 1$, and then D is very ample.
   Moreover, taken $P \in X$, since $\deg(P) > 0$ we have that P is very ample, hence X is embedded in $\mathbb{P}^1$, or better $X \cong \mathbb{P}^1$.

2. [$\Rightarrow$] Let D be a divisor of degree 3. Since $\deg(D) \geq 2g+1$, we have that D is very ample (by Corollary 5.8.6) and $\dim |D| = 2$ (by Proposition 5.7.3).
   Hence $|D|$ corresponds to a closed embedding $j \colon X \to \mathbb{P}^2$ and
   $\deg(X) = \deg(D) = 3$.
   [$\Leftarrow$] By Remark 5.5.3, $g = \frac{1}{2}(3-1)(3-2) = 1$.

3. Analogous to 2.

$\square$

5.10 Embedding in $\mathbb{P}^3$

Now our goal is to prove that every curve can be embedded in $\mathbb{P}^3$.
Let $X \subset \mathbb{P}^n$ be a curve.

Definition 5.10.1. Let $O \in \mathbb{P}^n \setminus X$ be a point.
The *projection* from O in $\mathbb{P}^{n-1}$ is $\psi \colon \mathbb{P}^n \setminus \{O\} \to \mathbb{P}^{n-1}, P \mapsto \overline{OP} \cap \mathbb{P}^{n-1}$
where $\mathbb{P}^{n-1}$ is a hyperplane of $\mathbb{P}^n$ not containing O, and $\overline{OP}$ is the line in $\mathbb{P}^n$ passing through P and O.

We consider the restriction $\phi := \psi_{|X} \colon X \to \mathbb{P}^{n-1}$.
By [9, Ex. I.3.14], ϕ is a morphism.
We will investigate when it is a closed embedding.

Definition 5.10.2. Let $P, Q \in X$ be two distinct points.

1. A *secant line* of X is a line in $\mathbb{P}^n$ joining two distinct points of X.
   We call $\mathrm{Sec}(X)$ the union of secant lines of X.

2. A *tangent line* of X at a point P is the line L_P passing through P such that $T_P L_P = T_P X$ as subspaces of $T_P\mathbb{P}^n$.
   We call $\mathrm{Tan}(X)$ the union of tangents lines of X.

Lemma 5.10.3. Let $O \in \mathbb{P}^n \smallsetminus X$ be a point.
Let $\phi\colon X \to \mathbb{P}^{n-1}$ be the projection from O.
ϕ is a closed embedding $\iff O \notin \mathrm{Sec}(X) \cup \mathrm{Tan}(X)$.

Proof. By Theorem 4.7.6(1), ϕ corresponds to a basepoint-free linear system $\Lambda = (V, \mathcal{L})$ on X. By Lemma 4.7.5(3), for any section $s \in V$, $\mathrm{Supp}(s)_0 = X \cap H$ (for some hyperplane H in $\mathbb{P}^n$ passing through O).
By Theorem 4.7.6(2), we have

$$\phi \text{ is a closed embedding } \iff \begin{cases} \Lambda \text{ separates points, and} \\ \Lambda \text{ separates tangent vectors.} \end{cases}$$

We will study the two conditions:

- Λ separates points $\iff \forall P \neq Q$ in X, $\exists s \in V : \begin{cases} P \in \mathrm{Supp}(s)_0 \\ Q \notin \mathrm{Supp}(s)_0 \end{cases}$

  $\iff \forall P \neq Q$ in X, $\exists H$ hyperplane in $\mathbb{P}^n$ passing through $O : \begin{cases} P \in H \\ Q \notin H \end{cases}$

  $\iff \forall P \neq Q$ in X, $O \notin \overline{PQ} \iff O \notin \mathrm{Sec}(X)$.

- Λ separates tangent vectors $\iff \forall P \in X, t \in T_P X, \exists s \in V : \begin{cases} P \in \mathrm{Supp}(s)_0 \\ t \notin T_P(s)_0 \end{cases}$

  $\iff \forall P \in X$, $\exists H$ hyperplane in $\mathbb{P}^n$ passing through $O : \begin{cases} P \in H \\ m_P(X \cap H) = 1 \end{cases}$

  ($\dim T_P X = \dim X = 1$, because X is smooth, and $T_P(X \cap H) \subsetneq T_P X$, then $T_P(X \cap H) = 0$, hence $m_P(X \cap H) \leq 1$)
  $\iff \forall P \in X$, $O \notin L_P \iff O \notin \mathrm{Tan}(X)$.

□

Proposition 5.10.4. Let $n \geq 4$.
There exists $O \in \mathbb{P}^n \smallsetminus X$ s.t. the projection ϕ from O is a closed embedding.

Proof.

$\mathrm{Sec}(X)$ is a locally closed subvariety of $\mathbb{P}^n$ of dimension at most 3, indeed:
we consider the morphism $f\colon (X \times X \smallsetminus \Delta) \times \mathbb{P}^1 \to \mathbb{P}^n$, who carries (P, Q, t) to the point t of the secant line $\overline{PQ}$ (suitably parameterized).
Then $\mathrm{Sec}(X) = \mathrm{Im}(f)$ and it has dimension at most $\dim(X \times X \times \mathbb{P}^1) = 3$.

$\mathrm{Tan}(X)$ is a locally closed subvariety of $\mathbb{P}^n$ of dimension at most 2, indeed:
we consider the morphism $g\colon X \times \mathbb{P}^1 \to \mathbb{P}^n$, who carries (P, t) to the point t of the tangent line L_P (suitably parameterized).
Then $\mathrm{Tan}(X) = \mathrm{Im}(g)$ and it has dimension at most $\dim(X \times \mathbb{P}^1) = 2$.

Now $\dim(\mathrm{Sec}(X) \cup \mathrm{Tan}(X)) \leq 3$. Since $n \geq 4$, $\mathrm{Sec}(X) \cup \mathrm{Tan}(X) \subsetneq \mathbb{P}^n$, hence $\exists O \notin \mathrm{Sec}(X) \cup \mathrm{Tan}(X)$. Finally, by Lemma 5.10.3, ϕ is a closed embedding. $\square$

Corollary 5.10.5. Any curve can be embedded in $\mathbb{P}^3$.

Proof. Let X be a curve, and let $j\colon X \hookrightarrow \mathbb{P}^n$ be a closed embedding.
If $n \leq 3$, we can consider $\mathbb{P}^n$ a subspace of $\mathbb{P}^3$, and X is embedded in $\mathbb{P}^3$.
If $n \geq 4$, there exists a closed embedding $X \hookrightarrow \mathbb{P}^{n-1}$ (by Proposition 5.10.4), and repeating we get a closed embedding $X \hookrightarrow \mathbb{P}^3$. $\square$

5.11 Canonical embedding and non-hyperelliptic curves

We make a further distinction between two types of curves: hyperelliptic and non. The latter correspond to the notion of canonical embedded curves.
Let X be a curve of genus g.
We will denote with g_d^r a linear system on X of degree d and dimension r.
Note that when it is basepoint-free it corresponds to a morphism $f\colon X \longrightarrow \mathbb{P}^r$.

Definition 5.11.1. Let $g \geq 2$.
X is *hyperelliptic* if there exists a finite morphism $f\colon X \to \mathbb{P}^1$ of degree 2.
In other words if there exists a g_2^1. (Otherwise X is *non-hyperelliptic*).

Note that in this case a g_2^1 is a basepoint-free linear system.
Indeed, as we will see in the proof of the following proposition, $g_2^1 = |P+Q|$.
Now since $g > 0$, i.e. $X \not\cong \mathbb{P}^1$, we have that $\dim |P+Q-R| \leq 1-1 = 0$ for any $R \in X$ (by Remark 5.8.4). Finally by Proposition 5.8.5, g_2^1 is basepoint-free.

Proposition 5.11.2. Let $g \geq 2$.
X is hyperelliptic $\iff \dim |P+Q| = 1 \ \exists P, Q \in X$.
(Equivalently, X is non-hyperelliptic $\iff \dim |P+Q| = 0 \ \forall P, Q \in X$).

Proof. Since $g \neq 0$, we have that $X \not\cong \mathbb{P}^1$.
By Remark 5.8.4, $\dim |P+Q| \leq \deg(P+Q) - 1 = 1$; hence $\dim |P+Q| \in \{0,1\}$.
[$\Leftarrow$] We can take $g_2^1 = |P+Q|$.
[$\Rightarrow$] Since X is hyperelliptic, there exists a g_2^1. Let $P+Q \in g_2^1$, we have that $g_2^1 \subseteq |P+Q|$, hence $\dim |P+Q| \geq 1$, or better $\dim |P+Q| = 1$. $\square$

Now we want to study the morphism corresponding to the canonical divisor. To do this we first study the properties of $|K|$.

Proposition 5.11.3.

1. If $g = 0$ then $|K| = \emptyset$.

2. If $g = 1$, then $|K| = 0$, or better $K \sim 0$.

3. If $g \geq 2$, then $|K|$ is basepoint-free.
   (Hence $|K|$ corresponds to a morphism $\phi\colon X \to \mathbb{P}^{g-1}$).

4. Let $g \geq 2$.
   X is non-hyperelliptic $\iff$ K is very ample (i.e. ϕ is a closed embedding).

Proof.

1. We know that $\deg(K) = 2g - 2 = -2$.
   Now any divisor $D \sim K$ has degree $\deg(D) = \deg(K) < 0$, hence $|K| = \emptyset$.
2. Since $l(K) = g = 1$, we have $\dim |K| = 0$.
   Since $\deg(K) = 2g - 2 = 0$, we have $K \sim 0$ (by Lemma 5.7.2(2)).
3. Let $P \in X$ be a point. Since $g \neq 0$ we have that $X \ncong \mathbb{P}^1$. It follows that $\dim |P| = 0$ (by Remark 5.8.4). Applying Riemann-Roch we have that
   $\dim |K - P| = \dim |P| - \deg(P) - 1 + g = g - 2 = \dim |K| - 1$.
   In conclusion, by Proposition 5.8.5(1), $|K|$ is basepoint-free.
4. $|K|$ is very ample
   $\iff$ (by Proposition 5.8.5(2))
   $\dim |K - P - Q| = \dim |K| - 2 = g - 3\ \forall P, Q \in X$
   $\iff$ (by Riemann-Roch)
   $\dim |P + Q| = \dim |K - P - Q| + \deg(P + Q) + 1 - g = 0\ \forall P, Q \in X$
   $\iff$ (by Proposition 5.11.2) X is non-hyperelliptic.

□

Definition 5.11.4. Let $g \geq 2$.

- $|K|$ corresponds to $\phi\colon X \to \mathbb{P}^{g-1}$ called *canonical morphism*.
- $\phi(X)$ is called a *canonical curve*.
- X is *canonically embedded* in $\mathbb{P}^{g-1}$ if ϕ is a closed embedding.
  In this case, $\deg(X) = \deg(\phi^* H) = \deg(K) = 2g - 2$.

Note that, by Proposition 5.11.3(4), we have
X is *canonically embedded* in $\mathbb{P}^{g-1}$ $\iff$ X is non-hyperelliptic.

Remark 5.11.5. Every curve of genus 2 is hyperelliptic.

Proof.
Since $\dim |K| = g - 1 = 1$ and $\deg(K) = 2g - 2 = 2$, we have that the canonical morphism is $\phi\colon X \to \mathbb{P}^1$ of degree 2; hence X is (canonically) hyperelliptic. □

5.12 Non-hyperelliptic curves of low genus

Now we focus on non-hyperelliptic curves. We saw that every curve of genus 2 is hyperelliptic, so we want to study the case $g \geq 3$, in particular we want to show that there exist non-hyperelliptic curves (and what they are) for $g = 3, 4, 5$. Let X be a curve of genus g.

Definition 5.12.1. Let $X \subset \mathbb{P}^n$.
X is a *complete intersection* if $X = F_1 \cap ... \cap F_{n-1}$ such that

- F_i are hypersurfaces of $\mathbb{P}^n$.

- $\mathcal{I}_{X/\mathbb{P}^n} = \mathcal{I}_{F_1/\mathbb{P}^n} + ... + \mathcal{I}_{F_{n-1}/\mathbb{P}^n}$
(i.e. $F_1 \cap ... \cap F_{n-1}$ is an intersection scheme).

Note that, by [9, Theorem I.7.7.], $\deg(X) = \prod_i \deg(F_i)$.

Proposition 5.12.2. Working up to isomorphism, we have that

1. $g = 3$ and X is non-hyperelliptic $\iff$ $X \subset \mathbb{P}^2$ is a plane curve of degree 4

2. $g = 4$ and X is non-hyperelliptic $\iff$ $X = Q \cap F \subset \mathbb{P}^3$ (complete intersection) where Q, F are irreducible hypersurfaces of degree (resp.) $2, 3$.

3. If $g = 5$ and X is non-hyperelliptic, then $X \subseteq Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^4$ where Q_1, Q_2, Q_3 are irreducible hypersurfaces of degree 2.
On the other hand, if X is a complete intersection of three quadrics in $\mathbb{P}^4$, then X is a non-hyperelliptic curve of genus 5.

Proof.

1. [$\Rightarrow$] The canonical embedding is $X \hookrightarrow \mathbb{P}^2$ and X has degree $2g - 2 = 4$.
[$\Leftarrow$] By [9, Ex.II.8.4(e)], $\omega_X \cong \mathcal{O}_X(4 - 2 - 1) = \mathcal{O}_X(1)$. Let $i \colon X \hookrightarrow \mathbb{P}^2$ be the inclusion, then $i^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_X(1) = \omega_X$, hence i is the morphism associated to $|K|$, that is the canonical embedding and $g = 2 + 1 = 3$.

2. [$\Rightarrow$] The canonical embedding is $\phi \colon X \hookrightarrow \mathbb{P}^3$ and X has degree 6.
Let $\mathcal{I} := \mathcal{I}_{X/\mathbb{P}^3}$. We consider the short exact sequence (*):
$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^3}/\mathcal{I} \to 0$$
   - (*) induces a left-exact sequence
$$0 \to \Gamma(\mathbb{P}^3, \mathcal{I}(2)) \to \Gamma(\mathbb{P}^3, \mathcal{O}(2)) \to \Gamma(X, \mathcal{O}_X(2))$$
   Now we know that:
   1) $\dim_k \Gamma(\mathbb{P}^3, \mathcal{O}(2)) = \binom{3+2}{2} = 10$.
   2) $\dim_k \Gamma(X, \mathcal{O}_X(2)) = 9$.
   (Because $\omega_X = \mathcal{O}_X(K) = \phi^*\mathcal{O}(1) = \mathcal{O}_X(1)$, hence $\mathcal{O}_X(2K) = \omega_X^2 =$
   $= \mathcal{O}_X(1)^2 = \mathcal{O}_X(2)$, and by Riemann-Roch we have $l(2K) =$
   $= l(2K) - l(K - 2K) = \deg(2K) + 1 - g = 2(2g - 2) + 1 - g = 9$).
   By left-exactness, $\dim_k \Gamma(\mathbb{P}^3, \mathcal{I}(2)) \geq 10 - 9 = 1$.
   Hence there exists $s \in \Gamma(\mathbb{P}^3, \mathcal{I}(2)) = \{\sigma \in \Gamma(\mathbb{P}^3, \mathcal{O}(2)) | \sigma_{|X} = 0\}$.
   Set $Q := V(s)$, it is a surface of degree 2 containing X.
   Moreover Q is irreducible (because X is not contained in any plane, and if we assume $s = l \cdot l'$ with l, l' homogeneous polynomials of degree 1, then since X is irreducible we have that X is contained in $V(l)$ or $V(l')$, that are planes, and this is a contradiction).
   Note that Q is unique (because if X is contained in a surface $Q' \neq Q$ of degree 2, then Q' is irreducible, in particular it does not have common

components with Q, hence $Q \cap Q'$ is a complete intersection and it is a curve of degree $2 \cdot 2 = 4$ containing X, but X has degree 6 and this is a contradiction)

- (*) induces a left-exact sequence

$$0 \to \Gamma(\mathbb{P}^3, \mathcal{I}(3)) \to \Gamma(\mathbb{P}^3, \mathcal{O}(3)) \to \Gamma(X, \mathcal{O}_X(3))$$

  Now we know that
  1) $\dim_k \Gamma(\mathbb{P}^3, \mathcal{O}(3)) = \binom{3+3}{3} = 20$.
  2) $\dim_k \Gamma(X, \mathcal{O}_X(3)) = 15$.
  (Indeed, in analogous way of above, we have that $\mathcal{O}_X(3) = \mathcal{O}_X(3K)$ and $l(3K) = \deg(3K) + 1 - g = 3(2g-2) + 1 - g = 15$).
  By left-exactness, $\dim_k \Gamma(\mathbb{P}^3, \mathcal{I}(3)) \geq 20 - 15 = 5$.
  Hence there exists $t \in \Gamma(\mathbb{P}^3, \mathcal{I}(3)) = \{\sigma \in \Gamma(\mathbb{P}^3, \mathcal{O}(3)) | \sigma_{|X} = 0\}$.
  Set $F := V(t)$, it is a surface of degree 3 containing X.
  Note that $\{t \in \Gamma(\mathbb{P}^3, \mathcal{I}(3)) | t = l \cdot s$, with $l \in \Gamma(\mathbb{P}^3, \mathcal{O}(1))\}$ has dimension equal to $\dim_k \Gamma(\mathbb{P}^3, \mathcal{O}(1)) = 4$, hence we can choose t such that it has not s as factor, hence F has no common components with Q, in particular $Q \cap F$ is a complete intersection.
  Moreover F is irreducible (because if we assume $t = l \cdot s'$ with l, s' homogeneous polynomials of degree respectively 1 and 2, then we have that $X \not\subseteq V(s')$ by the uniqueness of Q, hence X is contained in the plane $V(l)$, but X is not contained in any plane; this is a contradiction).

In conclusion $X \subseteq Q \cap F$ is a curve of degree $2 \cdot 3 = 6 = \deg(X)$, hence $X = Q \cap F$.
[$\Leftarrow$] By [9, Ex.II.8.4.(e)] $\omega_X \cong \mathcal{O}_X(2 + 3 - 3 - 1) = \mathcal{O}_X(1)$. Let $i \colon X \hookrightarrow \mathbb{P}^3$ be the inclusion, then $i^*\mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}_X(1) = \omega_X$, hence i is the morphism associated to $|K|$, that is the canonical embedding and $g = 3 + 1 = 4$.

3. [$\Rightarrow$] The canonical embedding is $\phi \colon X \hookrightarrow \mathbb{P}^4$ and X has degree 8.
Analogously to the previous point we have a left-exact sequence

$$0 \to \Gamma(\mathbb{P}^4, \mathcal{I}(2)) \to \Gamma(\mathbb{P}^4, \mathcal{O}(2)) \to \Gamma(X, \mathcal{O}_X(2))$$

and we know that
1) $\dim_k \Gamma(\mathbb{P}^4, \mathcal{O}(2)) = \binom{4+2}{2} = 15$.
2) $\dim_k \Gamma(X, \mathcal{O}_X(3)) = 12$.
(Indeed, in analogous way of the previous point, we have $\mathcal{O}_X(2) = \mathcal{O}_X(2K)$ and $l(2K) = \deg(2K) + 1 - g = 2(2g-2) + 1 - g = 12$).
By left-exactness, $\dim_k \Gamma(\mathbb{P}^4, \mathcal{I}(2)) \geq 15 - 12 = 3$.
Hence there exist linearly independent sections $s_1, s_2, s_3 \in \Gamma(\mathbb{P}^4, \mathcal{I}(2))$.
Set $Q_i := V(s_i)$ $(i = 1, 2, 3)$, then Q_i are hypersurfaces containing X and (as in the previous point) they are irreducible.
[$\Leftarrow$] By [9, Ex.II.8.4.(e)] $\omega_X \cong \mathcal{O}_X(2+2+2-4-1) = \mathcal{O}_X(1)$. Let $i \colon X \hookrightarrow \mathbb{P}^4$ be the inclusion, then $i^*\mathcal{O}_{\mathbb{P}^4}(1) = \mathcal{O}_X(1) = \omega_X$, hence i is the morphism associated to $|K|$, that is the canonical embedding and $g = 4 + 1 = 5$.

$\square$

Note that in the case $g = 5$ if the intersection $Q_1 \cap Q_2 \cap Q_3$ is a curve, it has degree $2 \cdot 2 \cdot 2 = 8 = \deg(X)$ and it contains X, hence X is a complete intersection of three quadrics. We will better study curves of genus 5 in the next sections. Finally, note that if $g \geq 6$ we are not so lucky, as we can see in this last remark.

Remark 5.12.3.
If $g \geq 6$ and X is non-hyperelliptic, then X is not a complete intersection in $\mathbb{P}^{g-1}$.

Proof. Assume that X is a complete intersection in $\mathbb{P}^{g-1}$, then
$X = F_1 \cap ... \cap F_{g-2}$ where F_i are hypersurfaces of degree d_i.
Let $i \colon X \hookrightarrow \mathbb{P}^{g-1}$ be the canonical embedding, then $\mathcal{O}_X(1) = i^*\mathcal{O}(1) = \omega_X$.
By [9, Ex.II.8.4.(e)] $\omega_X \cong \mathcal{O}_X(d_1 + ... + d_{g-2} - (g-1) - 1)$, hence we have $1 = d_1 + ... + d_{g-2} - g$.
Note that $d_i \geq 2 \ \forall i$, because X is not contained in any hyperplane.
Now $1 = d_1 + ... + d_{g-2} - g \geq 2(g-2) - g = g - 4$, hence $g \leq 5$. $\square$

5.13 Trigonal curves

We will show that not all curves of genus 5 are canonically complete intersections. In order to show it we will introduce the notion of trigonal curves.

Definition 5.13.1. A curve is *trigonal* if it is non-hyperelliptic and it has a g^1_3.

Remark 5.13.2. Let $X \subset \mathbb{P}^4$ be a canonical curve of genus 5.
X is trigonal $\iff$ X has a trisecant line.

Proof. Let P, Q, R be points of X.
By Riemannn-Roch, $\dim |P+Q+R| - \dim |K-P-Q-R| = -1$ (note that X has degree $2g-2 = 8$). Now we have that:
X is trigonal $\iff \exists g^1_3 = |P+Q+R|$
$\iff \exists P, Q, R \in X, \dim |P+Q+R| = 1$
$\iff \exists P, Q, R \in X, \dim |K-P-Q-R| = 2$
$\iff \exists P, Q, R \in X$ s.t. the linear system of hyperplanes in $\mathbb{P}^4$ containing P, Q, R has dimension 2
$\iff \exists P, Q, R \in X$ collinear points (otherwise they span a plane, and the linear system of hyperplanes containing this plane has dimension 1)
$\iff$ X has a trisecant line. $\square$

Now we want to show that trigonal curves X of genus 5, in their canonical embedding, are not complete intersection, or even better they are not intersection of quadrics:
if X be an intersection of quadrics Q_i (for example, as in Proposition 5.12.2(3)), then X has no trisecant lines L (otherwise L meets each Q_i in three points, hence L is contained in every Q_i, or better $L \subseteq X$. Contradiction!). In conclusion X is not trigonal.
Basically if X is trigonal, in the proof of Remark 5.13.2 we can see that the canonical embedding carries every divisor of the g^1_3 to a triad of collinear points, hence X has infinitely many trisecant lines. It follows that the intersection of

quadrics $Y := \bigcap_{X\subseteq Q} Q$ contains a surface, that is the union of the trisecant lines to X, in particular Y contains X properly (in this case, in the Proposition 5.12.2(3) $Q_1 \cap Q_2 \cap Q_3$ is a surface, not a curve).

5.14 Canonical curves of genus 5

We saw that non-hyperelliptic curves of genus 3 or 4 are canonically complete intersections. We will see that this holds for curves of genus 5 if and only if they are not *trigonal.* This is a particular case of the Enriques-Petri's Theorem.
In order to show it, we will introduce the Steiner Construction and the Castelnuovo Lemma. First, we give some definitions:

Definition 5.14.1.

1. A *Veronese map* is the morphism associated to a linear system $|dH|$ on $\mathbb{P}^n$, that is $v_d^n\colon \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}, (x_0 : ... : x_n) \mapsto (x_0^d : x_0^{d-1}x_1 : ... : x_n^d)$ given by all monomials of degree d.
   The image of a Veronese map is said a *Veronese variety.*
2. A *rational normal curve* is a Veronese variety corresponding to $v_d^1\colon \mathbb{P}^1 \to \mathbb{P}^d, (x : y) \mapsto (x^d : x^{d-1}y : ... : xy^{d-1} : y^d)$.
3. A *Veronese surface* is a Veronese variety corresponding to $v_2^2\colon \mathbb{P}^2 \to \mathbb{P}^5$.

Now, we see two useful remarks:

Remark 5.14.2.

- We recall that a curve in $\mathbb{P}^n$ is *non-degenerate* if it is not contained in a hyperplane. A such curve has degree at least n.
- Clearly a rational normal curve in $\mathbb{P}^n$ is non-degenerate of degree n.
  Conversely, a non-degenerate curve of degree n in $\mathbb{P}^n$ is a rational normal curve.

Proof.

1. Any n points of a non-degenerate curve $X \subset \mathbb{P}^n$ are contained in a hyperplane H, hence $X \cap H$ consists of at least n points; since $X \not\subseteq H$ then $\deg(X) \geq n$.
2. Let X be a non-degenerate curve of degree n in $\mathbb{P}^n$.
   First, we want to show that $X \cong \mathbb{P}^1$.
   We take $n-1$ points $P_1, ..., P_{n-1} \in X$ which span a $(n-2)$-plane V. Let $\{H_\lambda\}$ be the family of hyperplanes containing V, parameterized by $\lambda \in \mathbb{P}^1$. Each hyperplane H_λ meets X in $P_1, .., P_{n-1}$ and in another point which we call $q(\lambda)$ (If H_λ is tangent to X at P_i, we set $q(\lambda) = P_i$). We get an isomorphism $\mathbb{P}^1 \to X, \lambda \mapsto q(\lambda)$.
   In conclusion, let $X \cong \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ be a closed embedding, then it is associated to the unique divisor of $\mathbb{P}^1$ of degree n, that is nH; hence it is a Veronese map.

$\square$

5.14.1 Steiner Construction

We will say that a collection of at least $n+1$ points in $\mathbb{P}^n$ is *in general linear position* if no $n+1$ of these points are on the same hyperplane.

1. Let $P_1, ..., P_5 \in \mathbb{P}^2$ be points in general linear position (i.e. no 3 points are collinear). We can find a smooth conic passing through them.
First of all we consider P_1 and P_2 and we define the families $\{L_1(\lambda)\}, \{L_2(\lambda)\}$ of lines through P_1 and P_2 respectively, parameterized by $\lambda \in \mathbb{P}^1$ so that the unique common line $\overline{P_1P_2}$ corresponds to different values of λ, that is so that $L_1(\lambda) \neq L_2(\lambda)$ for all λ.
Now we can define $C := \bigcup_\lambda (L_1(\lambda) \cap L_2(\lambda))$. This is a non-degenerate and irreducible curve, containing P_1 and P_2. Moreover C has degree 2:
indeed its intersection with a line $L \subset \mathbb{P}^2$ consists of the fixed points of the automorphism $L \to L, L \cap L_1(\lambda) \mapsto L \cap L_2(\lambda)$, and they can be at most 2.
We can choose our parameterizations of the two families so that

    - $P_3 \in L_1(0) \cap L_2(0)$,
    - $P_4 \in L_1(1) \cap L_2(1)$,
    - $P_5 \in L_1(\infty) \cap L_2(\infty)$.

    Note that in this way $\overline{P_1P_2}$ still corresponds to different values of λ:
indeed if $\overline{P_1P_2} = L_1(\lambda_0) = L_2(\lambda_0)$, then taken $L := \overline{P_3P_4}$ the automorphism $L \to L, L \cap L_1(\lambda) \mapsto L \cap L_2(\lambda)$ would fix the three points P_3, P_4 and $L \cap \overline{P_1P_2}$, hence it would be the identity, that is $L \cap L_1(\lambda) = L \cap L_2(\lambda) = L_1(\lambda) \cap L_2(\lambda)$ for any λ, in particular $P_5 = L_1(\infty) \cap L_2(\infty) = L_1(\infty) \cap L \in L$, hence P_3, P_4, P_5 would be collinear. Contradiction!.
We can define C as above, and it contains $P_1, ..., P_5$.

2. Let $P_1, ..., P_{n+3} \in \mathbb{P}^n$ be points in general linear position. We can generalize the construction in 1 and find a rational normal curve passing through them.
First of all we consider $P_1, ..., P_n$. Let V be the hyperplane spanned by $P_1, ..., P_n$ and let V_i be the $(n-2)$-plane spanned by $P_1, .., \hat{P}_i, ..., P_n$. We define the families $\{H_i(\lambda)\}$ of hyperplanes through V_i, parameterized by $\lambda \in \mathbb{P}^1$ so that the unique common hyperplane V corresponds to different values of λ. Note that for any λ, $H_1(\lambda) \cap ... \cap H_n(\lambda)$ is a point:
indeed if none of $H_i(\lambda)$ is V, then their intersection cannot meet V, hence it is a point; while if $H_i(\lambda) = V$, then $H_i(\lambda) \cap H_j(\lambda) = V_j$ for any $j \neq i$, hence $H_1(\lambda) \cap ... \cap H_n(\lambda) = \bigcap_{j \neq i} V_j = P_i$.
We can define $C := \bigcup_\lambda (H_1(\lambda) \cap ... \cap H_n(\lambda))$ and it is a non-degenerate and irreducible curve containing P_i for $i = 1, .., n$. Moreover C has degree n:
its intersection with a hyperplane $H \subset \mathbb{P}^n$ consists of the fixed points of the automorphism $H \to H, H \cap H_1(\lambda) \cap ... \cap H_{n-1}(\lambda) \mapsto H \cap H_2(\lambda) \cap ... \cap H_n(\lambda)$, and they can be at most n, hence $\deg(C) \leq n$. Since C is non-degenerate, $\deg(C) \geq n$; hence $\deg(C) = n$.
In conclusion C is a rational normal curve.
We can choose our parameterizations of the families so that

- $P_{n+1} \in H_i(0)$, for all i,
- $P_{n+2} \in H_i(1)$, for all i,
- $P_{n+3} \in H_i(\infty)$, for all i.

Note that in this way V still corresponds to different values of λ:
indeed if $V = H_i(\lambda_0) = H_j(\lambda_0)$, then taken $L := \overline{P_{n+1}P_{n+2}}$ the automorphism $L \to L, L \cap H_i(\lambda) \mapsto L \cap H_j(\lambda)$ would fix the three points P_{n+1}, P_{n+2} and $L \cap V$, hence it would be the identity, in particular L would meet $H_i(\infty) \cap H_j(\infty)$, hence the $n+1$ points $P_{n+1}, P_{n+2}, P_1, ..., \hat{P}_i, ..., \hat{P}_j, ..., P_n$ and P_{n+3} would be on the same hyperplane. Contradiction!.
We can define C as above, and it contains $P_1, ..., P_{n+3}$.

Remark 5.14.3.

1. Through any $n+3$ points in general linear position in $\mathbb{P}^n$ there passes a unique rational normal curve.

2. A rational normal curve is intersection of quadric hypersurfaces.

Proof.

1. Taken $P_1, ..., P_{n+3} \in \mathbb{P}^n$ be points in general linear position. We can define a rational normal curve C passing through them as in section 5.14.1(2).
If D is another rational normal curve passing through them, then each hyperplane $H_i(\lambda)$ meets D in $P_1, .., \hat{P}_i, ..., P_n$ and another point which we may denote $q_i(\lambda)$.
Now the automorphism $D \to D, q_i(\lambda) \mapsto q_j(\lambda)$ fixes the three points $P_{n+1}, P_{n+2}, P_{n+3}$; hence it is the identity, that is $q_i(\lambda) = H_1(\lambda) \cap ... \cap H_n(\lambda)$.
It follows that $D = C$.

2. Let V_1, V_2 be $(n-2)$-planes of $\mathbb{P}^n$.
We define the two families $\{H_1(\lambda)\}, \{H_2(\lambda)\}$ of hyperplanes through V_1 and V_2 respectively, parameterized by $\lambda \in \mathbb{P}^1$ so that $H_1(\lambda) \neq H_2(\lambda)$ for all λ.
Analogously to section 5.14.1(1) we have a non-degenerate and irreducible quadric hypersurface $Q := \bigcup_\lambda (H_1(\lambda) \cap H_2(\lambda))$.
Now, let C be a rational normal curve. We can see it to be constructed as in section 5.14.1(2). If we set $Q_{ij} := \bigcup_\lambda (H_i(\lambda) \cap H_j(\lambda))$, then C is the intersection of the quadrics Q_{ij}.

□

5.14.2 Castelnuovo Lemma

Lemma 5.14.4 (Castelnuovo Lemma)**.** A collection $P_1, ..., P_d \in \mathbb{P}^n$ of $d \geq 2n+3$ points in general linear position which impose only $2n+1$ conditions on quadrics lies on a rational normal curve.

Proof. Let $\{H_i(\lambda)\}$ (for $i = 1, ..., n$) and $\{H(\lambda)\}$ be the families of hyperplanes passing through $P_1, ..., \hat{P}_i, ..., P_n$ (for $i = 1, ..., n$) and $P_{n+1}, ..., P_{2n-1}$ respectively, parameterized by $\lambda \in \mathbb{P}^1$ so that

- $P_{2n} \in H(0), H_i(0)$, for all i,
- $P_{2n+1} \in H(1), H_i(1)$, for all i,
- $P_{2n+2} \in H(\infty), H_i(\infty)$, for all i.

Let $Q_i := \bigcup_\lambda (H_i(\lambda) \cap H(\lambda))$, we know that this is a quadric. Since Q_i contains $P_1, ..., \hat{P}_i, ..., P_{2n+2}$, then it contains all the points $P_1, ..., P_d$ (because $2n+1$ points in general linear position impose independent conditions on quadrics), in particular we have $P_{2n+3}, ..., P_d \in H_1(\lambda) \cap ... \cap H_n(\lambda)$ for the same λ.
Now let $C := \bigcup_\lambda (H_1(\lambda) \cap ... \cap H_n(\lambda))$, then it is a rational normal curve (as seen in section 5.14.1(2)) and it contains $P_1, ..., P_n, P_{2n}, ..., P_d$.
Analogously any $d - n + 1 (> n + 3)$ of the points $P_1, ..., P_d$ lie on a rational normal curve. Since a rational normal curve is determined by any $n + 3$ points (see Remark 5.14.3(1)), then all the points $P_1, ..., P_d$ lie on a rational normal curve. □

5.14.3 Rational normal scroll

Before seeing the Enriques-Petri's Theorem, we want to introduce some notions about surfaces.

Definition 5.14.5.

1. As a *surface* we mean a projective variety of dimension 2 over an algebraically closed field k.
   As a *curve on a surface* we mean an effective divisor (not necessarily smooth or irreducible).

2. Let C, D be two curves on a smooth surface S. We say that they meet *transversally* at a point $P \in C \cap D$ if the local equation f, g of C, D at P generate the maximal ideal $\mathfrak{m}_{S,P}$.
   Moreover there is a unique pairing $\cdot : \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ such that

   - if C, D are smooth curves meeting transversally, then $C \cdot D = \sharp(C \cap D)$. Note that if C is smooth and irreducible, then $C \cdot D = \deg(\mathcal{O}_X(D)_{|C})$.
   - $C \cdot D = D \cdot C$.
   - $(C + C') \cdot D = C \cdot D + C' \cdot D$.
   - if $C \sim C'$, then $C \cdot D = C' \cdot D$.

   (see [9, Theorem V.1.1])
   In particular let D be a very ample divisor on S, which gives a closed embedding in $\mathbb{P}^n$, then for any curve C on S, $C \cdot D = \deg(C)$.
   (see [9, Ex.V.1.2]).

3. The *degree* of a surface $S \subset \mathbb{P}^n$ is $\deg(S) = \sharp(S \cap H \cap H')$ where H, H' are general hyperplanes.
   If S is smooth, $\deg(S) = D^2$ where $D \in |\mathcal{O}_S(1)|$ (see [9, Ex.V.1.2]).

At the beginning of this section, we defined a Veronese surface, which is a special case of surface of minimal degree. Now we want to define a rational normal scroll.

Definition 5.14.6.

1. Let C be a curve and let $\mathcal{E}$ be a locally free sheaf on C.
   According to [9, Chapter II.7], we can define $P(\mathcal{E})$ and $\mathcal{O}_{P(\mathcal{E})}(1)$.
   Let $\mathcal{S} = \mathcal{S}(\mathcal{E}) = \oplus_{d\geq 0}\mathcal{S}^d(\mathcal{E})$ be the symmetric algebra of $\mathcal{E}$ (as defined in [9, Ex.II.5.16]). Explicitly (defined as presheaf): for any open subset U of C, we first define $\mathcal{T}^0(\mathcal{E})(U) = \mathcal{O}_C(U)$ and $\mathcal{T}^d(\mathcal{E})(U) = \mathcal{E}(U) \otimes ... \otimes \mathcal{E}(U)$ (d times) for $d \geq 1$, then we define $\mathcal{T}(\mathcal{E})(U) = \oplus_{d\geq 0}\mathcal{T}^d(\mathcal{E})(U)$ and finally we define $\mathcal{S}(\mathcal{E})(U) = \mathcal{T}(\mathcal{E})(U)/\langle x \otimes y - y \otimes x | x, y \in \mathcal{E}(U)\rangle$.
   Note that $\mathcal{S}$ is a sheaf of graded $\mathcal{O}_C$-algebras such that $\mathcal{S}^0 = \mathcal{O}_C$ and $\mathcal{S}$ is generated by $\mathcal{S}^1$ as $\mathcal{O}_C$-algebra.
   For any open affine subset $U = \mathrm{Spec}A$ of C we have an A-algebra $\mathcal{S}(U)$ and a scheme $\mathrm{Proj}\mathcal{S}(U)$ with a projection $\pi_U \colon \mathrm{Proj}\mathcal{S}(U) \to U$. Gluing together these schemes we get the scheme $P(\mathcal{E})$ with the projection $\pi\colon P(\mathcal{E}) \to C$. Moreover gluing together the 1-twist sheaves we get a sheaf called $\mathcal{O}_{P(\mathcal{E})}(1)$.

2. Let C be a curve and let $\mathcal{E}$ be a locally free sheaf on C of rank 2.
   A *ruled surface* is a smooth surface $S = P(\mathcal{E})$.
   In particular there is a projection $\pi\colon P(\mathcal{E}) \to C$ such that
   1) every fiber is isomorphic to $\mathbb{P}^1$
   2) there exists a section $\sigma\colon C \to P(\mathcal{E})$, that is a morphism s.t. $\pi \circ \sigma = id_C$.
   By [9, Proposition V.2.8] there is a section $\sigma\colon C \to P(\mathcal{E})$ with image C_0 such that $\mathcal{O}_S(C_0) \cong \mathcal{O}_{P(\mathcal{E})}(1)$.

3. A *scroll* is a ruled surface embedded in $\mathbb{P}^r$ in such a way that all the fibers f have degree 1.

4. Let $e \geq 0$.
   $X_e = P(\mathcal{E})$ is the rational ruled surface over $\mathbb{P}^1$ with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$.
   In this case, by [9, Proposition V.2.3 and V.2.9] we have

   - $C_0^2 = -e$
   - $f^2 = 0$
   - $C_0.f = 1$

   Moreover since $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$, we have $\mathrm{Pic}(X_e) = \mathbb{Z}C_0 \oplus \mathbb{Z}f$ (see [9, Proposition V.2.3]), hence we can consider every divisor of the form $aC_0 + bf$.
   We can also consider the canonical divisor $K_{X_e} = -2C_0 + (-2-e)f$ (see [9, Corollary V.2.11]).

5. Let $e \geq 0$ and let $n \geq e$.
   By [9, Theorem V.2.17] the linear system $|C_0 + nf|$ on X_e is basepoint-free, hence it corresponds to a morphism $\phi\colon X_e \to \mathbb{P}^r$.
   A *rational normal scroll* is a surface which is image of a such morphism.
   (Note that this is a scroll, by Remark 5.14.7(2a)).

Remark 5.14.7.

1. Let S be a smooth surface and let C be a curve on S.
Let $\mathcal{L}$ be a line bundle on S generated by global sections.
Let $r\colon \Gamma(S,\mathcal{L}) \to \Gamma(C,\mathcal{L}_{|C}), \sigma \mapsto \sigma_{|C}$ and let $V = \operatorname{Im}(r)$.
We have a morphism $\phi_{\mathcal{L}}$ associated to $\Gamma(S,\mathcal{L})$ and a morphism ϕ_V associated to V. Then $\phi_{\mathcal{L}|C} = \phi_V$.

2. Let $S = X_e$ and let $\mathcal{L} = \mathcal{O}_S(C_0 + nf)$, with $n \geq e$. We have that
   (a) $\phi_{\mathcal{L}}(f)$ is a line.
   In particular a rational normal scroll is a scroll.
   (b) Let $D := \phi_{\mathcal{L}}^{-1}(X)$ be the preimage of a smooth irreducible curve X. Then $\deg(X) = (C_0 + nf) \cdot D$.

Proof.

1. First, note that V is basepoint-free: indeed for any $x \in C$, then $x \in S$. Since $\mathcal{L}$ is generated by global sections, there is $\sigma \in \Gamma(S,\mathcal{L})$ such that $\sigma(x) \neq 0$; hence we have $\sigma_{|C} \in V$ and $\sigma_{|C}(x) = \sigma(x) \neq 0$. By Remark 4.7.1(2), V is basepoint-free.
Now, we recall that $\mathcal{O}_S(-C)$ is the ideal sheaf of C. We have a short exact sequence
$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$
Tensoring with $\mathcal{L}$ we get
$$0 \to \mathcal{L}(-C) \to \mathcal{L} \to \mathcal{L}_{|C} \to 0$$
It induces a left-exact sequence
$$0 \to \Gamma(S,\mathcal{L}(-C)) \to \Gamma(S,\mathcal{L}) \xrightarrow{r} \Gamma(C,\mathcal{L}_{|C})$$
Hence there is a basis $\{\sigma_0, ..., \sigma_r\}$ of $\Gamma(S,\mathcal{L})$ such that $\{\sigma_0, ..., \sigma_s\}$ is basis of $\Gamma(S,\mathcal{L}(-C))$ and $\{\sigma_{s+1|C}, ..., \sigma_{r|C}\}$ is basis of V.
Consider the map $\phi_{\mathcal{L}}\colon S \to \mathbb{P}^r$, $x \mapsto (\sigma_0(x) : ... : \sigma_r(x))$.
If $x \in C$, then $\sigma_0(x) = ... = \sigma_s(x) = 0$; hence we get a commutative diagram
$$\begin{tikzcd} C \arrow[r, "\phi_V"] \arrow[d, hook] & \mathbb{P}(V) \arrow[d, hook, "j"] \\ S \arrow[r, "\phi_{\mathcal{L}}"] & \mathbb{P}^r \end{tikzcd}$$
where $j\colon (x_{s+1} : ... : x_r) \mapsto (0 : ... : 0 : x_{s+1} : ... : x_r)$.

2. (a) We have that $\deg(\mathcal{L}_{|f}) = \mathcal{L} \cdot f = (C_0 + nf) \cdot f = 1$. Since $f \cong \mathbb{P}^1$, then $\mathcal{L}_{|f} \cong \mathcal{O}_{\mathbb{P}^1}(1)$.
   Let V be defined as in 1 (with $C = f$). We have that:
   i) $\dim V \geq 1$ (because V is basepoint-free, as seen in 1).
   ii) $\dim V \leq \dim \Gamma(f, \mathcal{L}_{|f}) = \dim \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 2$.

We can conclude that $\dim V = 2$: otherwise if $\dim V = 1$, then $V = k\sigma$ where σ has no zeros; but $\sigma \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, hence it has a zero.
Now we have $V = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. By 1, $\phi_{\mathcal{L}}(f) = \phi_{|\mathcal{O}_{\mathbb{P}^1}(1)|}(\mathbb{P}^1)$ is a line.

(b) If $n > e$, $C_0 + nf$ is very ample (see Remark 5.14.8(1)). By [9, Ex.V.1.2], we have $\deg(X) = (C_0 + nf) \cdot D$.
If $n = e$, $\phi_{\mathcal{L}}(X_e)$ is a cone of vertex P (see again Remark 5.14.8(1)). By [9, Example V.2.11.4], $\phi_{\mathcal{L}}$ is the blowing-up of the cone in P (see [9, Chapter II.7] for the definition of blowing-up) and $\phi_{\mathcal{L}}^{-1}(P) = C_0$; hence $\phi_{\mathcal{L}|X_e \smallsetminus C_0}$ is an isomorphism (by [9, Proposition II.7.13]).
Since $\phi_{\mathcal{L}}(D) = X$ is smooth, D meets C_0 in either 0 or 1 point.
Hence $\phi_{\mathcal{L}|D}$ is a closed embedding. Note that $\phi_{\mathcal{L}|D} = \phi_V$ as in 1.
We have $\deg(X) = \deg(\phi_V^* \mathcal{O}(1)) = \deg(\mathcal{L}_{|D}) = (C_0 + ef) \cdot D$.

□

Remark 5.14.8.

1. A rational normal scroll S is either a cone (if $n = e$) or smooth (if $n > e$):
   - $[n = e]$ Since $(C_0 + ef) \cdot C_0 = 0$, C_0 is contracted to a point P (the vertex of the cone). Moreover any fiber f is mapped to a line passing through P: indeed $f \cdot C_0 = 1$, hence f meets C_0 in a point and the image of f passes through P.
   - $[n > e]$ By [9, Theorem V.2.17] $C_0 + nf$ is very ample, hence the corresponding map ϕ is a closed embedding, in particular S is smooth. Moreover in this case we can see:
     a) $\deg(S) = (C_0 + nf)^2 = 2n - e$,
     b) S is embedded (via ϕ) in $\mathbb{P}^{2n-e+1}$ (by [9, Corollary V.2.19]).
2. *del Pezzo's Theorem* (see [7, Ch. 4, sec. 3, p. 525]).
   A non-degenerate surface in $\mathbb{P}^n$ of degree $n-1$ is either a Veronese surface or a rational normal scroll.

5.14.4 Enriques-Petri's Theorem

Lemma 5.14.9.
Let $X \subset \mathbb{P}^n$ ($n \geq 2$) be a canonical curve (note that it has genus $g = n+1$).
Let $S := \bigcap_{X \subseteq Q} Q$ be the intersection of quadrics in $\mathbb{P}^n$ containing X.

1. The points of a general hyperplane section $X \cap \mathbb{P}^{n-1}$ impose only $2n-1$ conditions on quadrics.
2. Let $P \in \mathbb{P}^n$ be a point not lying on infinitely many secant lines of X. Taken a general hyperplane H passing through P we consider the points $\{P\} \cup (X \cap H)$. We have that
   (a) they are in general linear position (in $H = \mathbb{P}^{n-1}$),
   (b) if $P \in S$, then they impose only $2n-1$ conditions on quadrics.

3. Let $g \geq 4$.
   If $X \subsetneq S$ then there is a point $P \in S \setminus X$ not lying on infinitely many secant lines of X.

Proof.

1. In analogous way as in the proof of Proposition 5.12.2, we can see that:
   a) $\dim_k \Gamma(\mathbb{P}^n, \mathcal{O}(2)) = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$,
   b) $\dim_k \Gamma(X, \mathcal{O}(2)) = 2(2g-2) + 1 - g = 3n$,
   c) $\dim_k \Gamma(\mathbb{P}^n, \mathcal{I}(2)) \geq \frac{(n+2)(n+1)}{2} - 3n = \frac{(n-2)(n-1)}{2}$.
   Now, let W be the linear system of quadrics of $\mathbb{P}^n$ containing X. It has dimension at least $\frac{(n-2)(n-1)}{2} - 1$ (by c).
   Since no quadric containing C can contain a hyperplane, the restriction $W_{|\mathbb{P}^{n-1}}$ of W to a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ is injective. Hence the linear system of quadrics of $\mathbb{P}^{n-1}$ containing $X \cap \mathbb{P}^{n-1}$ (which contains $W_{|\mathbb{P}^{n-1}}$) has dimension at least $\frac{(n-2)(n-1)}{2} - 1$.
   The linear system of quadrics of $\mathbb{P}^{n-1}$ has dimension $\frac{n(n+1)}{2} - 1$ (by a).
   It follows that the points of $X \cap \mathbb{P}^{n-1}$ impose only $\frac{n(n+1)}{2} - \frac{(n-2)(n-1)}{2} = 2n-1$ conditions on quadrics.

2. (a) Note that
   - Since $H \ni P$ is general, then H contains no secant lines of X passing through P (which are finitely many).
   - The projection $\phi\colon X \to M$ from P to a hyperplane $M = \mathbb{P}^{n-1}$ gives a bijection $\phi_{|X \cap H}\colon X \cap H \to \phi(X) \cap H'$, where $H' = H \cap M$: indeed for any $Q \in X \cap H$, the secant line $\overline{PQ}$ is contained in H, hence $\phi(Q) \in \overline{PQ} \subseteq H$, and hence $\phi(Q) \in \phi(X) \cap H'$. On the other hand for any $R \in \phi(X) \cap H'$, there is $Q \in X$ such that $\phi(Q) = R$, in particular $Q \in \overline{RP} \subseteq H$. $\overline{RP}$ cannot contain another point of X (otherwise it would be a secant line of X passing through P contained in H), hence Q is unique.
   - $\phi(X)$ is non-degenerate: indeed assume $\phi(X) \subseteq H''$ where H'' is a hyperplane of M. Let $\langle P, H'' \rangle$ be the hyperplane of $\mathbb{P}^n$ containing P and H''. For any $Q \in X$, we have $Q \in \overline{P\phi(Q)} \subseteq \langle P, H'' \rangle$; hence $X \subseteq \langle P, H'' \rangle$. This is a contradiction, because X is non-degenerate.
   - H' is a general hyperplane of M: indeed there is a surjective map $\{$hyperplanes of $\mathbb{P}^n$ containing $P\} \to \{$hyperplanes of $M\}$, $H \mapsto H' = H \cap M$, which is surjective because for any hyperplane H' of M, $\langle P, H' \rangle \cap M = H'$.

   First, we want to show that P and any $n-1$ points of $X \cap H$ are in general linear position. In other words we want to show that, taken $P_1, ..., P_{n-1} \in X \cap H$, then $P \notin \langle P_1, ..., P_{n-1} \rangle$
   (where $\langle P_1, ..., P_{n-1} \rangle$ is the linear variety spanned by $P_1, ..., P_{n-1}$):
   set $\mathbb{P}^n = \mathbb{P}(V)$ with $P = [v]$, $P_i = [v_i]$, and set $M = \mathbb{P}^{n-1} = \mathbb{P}(W)$ such that $V = \langle v \rangle \oplus W$. We have $v_i = a_i v + w_i$ ($a_i \in k$) and

$\phi(P_i) = [w_i] \in M$. By [7, Ch.2, s.3, p.249, Lemma] $w_1, ..., w_{n-1}$ are independent (because $\phi(X)$ is non-degenerate and H' is general). If we assume $P \in \langle P_1, ..., P_{n-1}\rangle$, that is $v \in \langle v_1, ..., v_{n-1}\rangle$, then for some $b_i \in k$ $(1 \leq i \leq n-1)$ we have $v = \sum_{i=1}^{n-1} b_i v_i = \sum_{i=1}^{n-1} b_i a_i v + \sum_{i=1}^{n-1} b_i w_i$, hence $\sum_{i=1}^{n-1} b_i w_i = 0$, and hence $b_i = 0$ for all i, that is $v = 0$. Contradiction.

Now, we want to show that for a general hyperplane H containing P, the points $X \cap H$ are in general linear position. In other words we want to show that, taken $P_1, ..., P_n \in X \cap H$, then $H = \langle P_1, ..., P_n\rangle$: note that if $v_1, ..., v_{n-1}$ (defined above) are dependent, then there is $(b_1, ..., b_n) \neq (0, ..., 0)$ s.t. $0 = \sum_{i=1}^{n-1} b_i v_i = \sum_{i=1}^{n-1} b_i a_i v + \sum_{i=1}^{n-1} b_i w_i$, and then $\sum_{i=1}^{n-1} b_i w_i = 0$, but this is not possible because the w_i are independent. We can conclude that $\dim\langle P_1, ..., P_{n-1}\rangle = n-2$.

We assume that there exist n points $P_1, ..., P_n$ s.t. $\langle P_1, ..., P_n\rangle \neq H$ for a general hyperplane H containing P, that is $\dim\langle P_1, ..., P_n\rangle \leq n-2$. Therefore $\langle P_1, ..., P_n\rangle = \langle P_1, ..., P_{n-1}\rangle$. Since $P \notin \langle P_1, ..., P_{n-1}\rangle$, we have $\dim\langle P, P_1, ..., P_n\rangle = n-1$, that is $\langle P, P_1, ..., P_n\rangle = H$.

Let $\mathrm{Hyp}(\mathbb{P}^n)$ be the variety of hyperplanes in $\mathbb{P}^n$.

We define $J = \{(P_1, ..., P_n, H) | H \in \mathrm{Hyp}(\mathbb{P}^n),\ P_1, ..., P_n \in X \cap H$, and $\dim\langle P_1, ..., P_n\rangle \leq n-2\} \subseteq X^n \times \mathrm{Hyp}(\mathbb{P}^n)$.

We consider the projection $\pi_1 \colon J \to X^n$.

There is a rational map $\pi_1(J) \dashrightarrow \{$hyperplanes of $\mathbb{P}^n$ containing $P\}$, $(P_1, ..., P_n) \mapsto \langle P, P_1, ..., P_n\rangle$ that is defined on a dense open subset of $\pi_1(J)$ and whose image contains an open subset of $\{$hyperplanes containing $P\}$, by what we have seen above.

It follows that $\dim \pi_1(J) \geq \dim\{$hyperplanes containing $P\} = n-1$.

Let $(P_1, ..., P_n) \in \pi_1(J)$ be general, then $\dim\langle P_1, ..., P_n\rangle \leq n-2$, and then $\dim\{$hyperplanes of $\mathbb{P}^n$ containing $\langle P_1, ..., P_n\rangle\} \geq 1$, in particular we have $\dim \pi_1^{-1}((P_1, ..., P_n)) \geq 1$.

By [3, Teorema 4.7.1], $\dim J = \dim \pi_1(J) + \dim \pi_1^{-1}((P_1, ..., P_n)) \geq n$.

Consider now the projection $\pi_2 \colon J \to \mathrm{Hyp}(\mathbb{P}^n)$. Since for any hyperplane H, $\sharp(X \cap H)$ is finite, then π_2 has finite fibers; hence by [3, Teorema 4.7.1] $\dim \pi_2(J) = \dim J + 0 \geq n$. Hence $\pi_2(J) = \mathrm{Hyp}(\mathbb{P}^n)$; that is for any hyperplane H, the points $X \cap H$ are not in general linear position in H. By [7, Ch.2, s.3, p.249, Lemma], this is a contradiction.

(b) Consider the linear systems $W_{|H} = \{Q \cap H | Q$ is a quadric in $\mathbb{P}^n$ containing $X\}$ and $U = \{$quadrics in H containing $P \cup (X \cap H)\}$.
We have $W_{|H} \subseteq U$: indeed let Q be a quadric in $\mathbb{P}^n$ containing X, then $X \cap H \subseteq Q \cap H$. Moreover $P \in S \subseteq Q$ and $P \in H$, hence $P \cup (X \cap H) \subseteq Q \cap H$.
As in 1, the points $\{P\} \cup (X \cap H)$ impose only $2n-1$ conditions on quadrics.

3. We define $\mathrm{Cor}(X)$ the variety of secant lines of X.
Let $P' \in S \smallsetminus X$. We can assume that P' lies on infinitely many secant lines of X (otherwise we can take $P = P'$).
Let S' be the surface given by the union of secant lines of X containing P'.

Let $J := \{(Q, L) | Q \in S', L \in \mathrm{Cor}(X), L \ni Q\} \subseteq S' \times \mathrm{Cor}(X)$.
We consider the projection $\pi_1 \colon J \to S'$. This is surjective: indeed for any $Q \in S'$ there is a secant line L of X containing Q, hence $Q = \pi_1(Q, L)$.
Let $Q \in S'$ be a general point. We want to show that $\dim \pi_1^{-1}(Q) = 0$, so that we can take $P = Q$.
We assume that $\dim \pi_1^{-1}(Q) \geq 1$. By [3, Teorema 4.7.1], we have that $\dim J = \dim S' + \dim \pi_1^{-1}(Q) \geq 2 + 1 = 3$; hence there is an irreducible component Z of J such that $\dim Z \geq 3$.
Now we consider the projection $\pi_2 \colon J \to \mathrm{Cor}(X)$.
Let $L \in \pi_2(Z)$ be a general element. By [3, Teorema 4.7.1], we have that $\dim \pi_{2|Z}^{-1}(L) = \dim Z - \dim \pi_{2|Z}(Z) \geq 3 - \dim \mathrm{Cor}(X) = 3 - 2 = 1$. On the other hand $\pi_{2|Z}^{-1}(L) \subseteq \pi_2^{-1}(L) \cong L \cap S' \subseteq L$, hence $\dim \pi_{2|Z}^{-1}(L) = 1$.
Hence $L \cap S' = L$, that is $L \subseteq S'$.
Moreover $\dim \pi_2(Z) = \dim Z - \dim \pi_{2|Z}^{-1}(L) \geq 3 - 1 = 2$. On the other hand $\pi_2(Z) \subseteq \mathrm{Cor}(X)$, hence $\pi_2(Z) = \mathrm{Cor}(X)$.
It follows that $L \subseteq S'$, $\forall L \in \mathrm{Cor}(X)$. Hence $S' = \bigcup_{L \in \mathrm{Cor}(X)} L = \mathrm{Sec}(X)$.
This is a contradiction, because $\dim \mathrm{Sec}(X) = 3$ (since X is non-degenerate).

□

Theorem 5.14.10 (Enriques-Petri's Theorem for genus 5)**.** Let $X \subset \mathbb{P}^4$ be a canonical curve (of genus 5), then either

- X is an intersection of quadrics, or
- X is trigonal (in which case, the intersection of quadrics containing X is a rational normal scroll).

More generally, this theorem holds for every genus, with the exception that for $g = 6$ there is a third possibility, that is X is a plane quintic (in which case, the intersection of quadrics containing X is a Veronese surface). See [7, Chapter 4, section 3, p. 535].
In our case ($g = 5$), the rational normal scroll is X_1 embedded in $\mathbb{P}^4$ via $|C_0 + 2f|$. (Moreover $X \sim 3C_0 + 5f$).

Proof. Let $S := \bigcap_{X \subseteq Q} Q$ be the intersection of quadrics containing X.
We saw in section 5.13 that if X is an intersection of quadrics, then it is not trigonal. Now assume that X is not an intersection of quadrics, that is $X \subsetneq S$.
By Lemma 5.14.9(3) we may choose a point $P \in S \smallsetminus X$ not lying on infinitely many secant lines of X.
We recall that our curve in $\mathbb{P}^4$ has genus $g = 5$ and degree $d = 8$.
Let $M = \mathbb{P}^3$ be a general hyperplane containing P. By Lemma 5.14.9(2), the 9 points $\{P\} \cup (X \cap M)$ are in general linear position and they impose 7 conditions on quadrics. Applying Lemma 5.14.4 in M, we get that these points lie on a rational normal curve C (note that C has degree 3).
Since any quadric Q containing X meets C in the $8(> 6)$ points $X \cap M$, then any such Q contains C, hence $C \subseteq S$.
On the other hand, by Proposition 5.14.3(2), $C = \cap_{C \subseteq Q'} Q'$ is an intersection of quadrics Q' of M. We consider the two linear systems $W_{|M} := \{Q \cap M | Q$ is a

quadric of $\mathbb{P}^4$ containing $X\}$ and $W' := \{Q' | Q'$ is a quadric of M containing $C\}$. Since for any quadric Q containing X, $Q \cap M$ is a quadric of M containing C, then $W_{|M} \subseteq W'$. As seen in the proof of Lemma 5.14.9(1), $\dim W_{|M} \geq \frac{2 \cdot 3}{2} - 1 = 2$. Since C is a rational normal curve in $M = \mathbb{P}^3$, $\dim W' = 2$ (by [3, Osservazione 4.1.2]). Hence $W_{|M} = W'$, and $S \cap M = C$.
It follows that S is a surface of degree 3 in $\mathbb{P}^4$. By Remark 5.14.8(2), S is a rational normal scroll. (Note that S cannot be a Veronese surface, otherwise it would be a non-degenerate surface in $\mathbb{P}^5$).
Now since $n \geq e \geq 0$ and $4 = 2n - e + 1$ we have two cases, that is either:
(a) $n = e = 3$ (S is a cone), or
(b) $n = 2, e = 1$ (S is smooth).
We consider the map $\phi \colon X_e \to S \subset \mathbb{P}^4$ given by $|C_0 + nf|$ and we take the smooth curve $D = \phi^{-1}(X) \cong X$. We have

$$\begin{cases} D \cdot (C_0 + nf) = \deg(X) = 8 \text{ (by Remark 5.14.7(2b))} \\ D \cdot (D + K_{X_e}) = 2g - 2 = 8 \text{ (by the Adjunction Formula, see [9, Prop.V.1.5])} \end{cases}$$

We can set $D = aC_0 + bf$ and $K_{X_e} = -2C_0 + (-2 - e)f$, hence we have

$$\begin{cases} -ea + b + na = 8 \\ -ea(a-2) + b(a-2) + a(b-2-e) = 8 \end{cases}$$

We study the two possible cases:
(a) if $n = e = 3$, then the system above has one integer solution: $a = 3, b = 8$.
(b) if $n = 2, e = 1$, then the system above has one integer solution: $a = 3, b = 5$.
By [9, Corollary V.2.18(b)] the only possible case is $n = 2, e = 1, a = 3, b = 5$.
In particular S is smooth and $D = 3C_0 + 5f$ (in detail S is X_1 embedded in $\mathbb{P}^4$ via $|C_0 + 2f|$). Since $f \cdot D = 3$, a fiber is mapped to a trisecant line of X. By Remark 5.13.2, X is trigonal. $\square$

5.14.5 Trigonal canonical curves of genus 5

We saw that non-hyperelliptic curves of genus 5 are either trigonal or complete intersection of three quadrics. Now, we want to give a better description of the trigonal case.
Let X be a canonical curve of genus 5 which is trigonal, then the g^1_3 gives infinitely many secant lines of X (as seen in section 5.13). By Theorem 5.14.10, the union of such lines is a rational normal scroll S, or better it is X_1 embedded in $\mathbb{P}^4$ via $|C_0 + 2f|$. Since $X \sim 3C_0 + 5f$, we have that $X + f \sim 3C_0 + 6f \in |3(C_0 + 2f)|$. Since $|3(C_0 + 2f)|$ is the linear system cut out by cubics in $\mathbb{P}^4$, then there is a line $L \subset \mathbb{P}^4$ and a cubic hypersurface $F \subset \mathbb{P}^4$ containing X such that $X \cup L = S \cap F$. We say that X is *residue* of a line in the intersection $S \cap F$.

5.15 Exhibition about canonical curves of higher genus

We conclude with a brief exhibition about canonical curves of higher genus. First, we give the definition of extendable variety.

Definition 5.15.1. Let $Y \subset \mathbb{P}^r$ be a variety.
Y is *extendable* if there exists a variety $Z \subset \mathbb{P}^{r+1}$ such that $Z \cap \mathbb{P}^r = Y$ and Z is not a cone over Y.

Remark 5.15.2. Let $X \subset \mathbb{P}^{g-1}$ be a canonical curve of genus g.
If X is extendable to a smooth surface $S \subset \mathbb{P}^g$, then S is a K3-surface (i.e. a smooth proper geometrically connected surface with trivial canonical bundle $K_S \sim 0$ and $\dim_k H^1(S, \mathcal{O}_S) = 0$).

Proof. Let $X = S \cap H$, where H is a hyperplane of $\mathbb{P}^g$.
First, by the Adjunction Formula (see [7, Ch.1, s.1, p.147]) we have that $O_X(H) \cong O_X(K_X) \cong O_X(H + K_S)$; hence $O_X(K_S) \cong O_X$.
Since the map $\mathrm{Pic}(S) \to \mathrm{Pic}(X)$, $\mathcal{L} \mapsto \mathcal{L}_{|X}$ is injective (see [8, Exposé XII, Cor. 3.6]), we have $O_S(K_S) \cong O_S$.
Now, consider the short exact sequence

$$0 \to O_S \to O_S(1) \to O_X(1) \to 0$$

By Kodaira Vanishing Theorem (see [9, Remark III.7.15]), we have that $H^1(O_S(1)) = H^1(K_S + H) = 0$; hence by the long exact sequence of cohomology we get

$$0 \to H^0(O_S) \to H^0(O_S(1)) \to H^0(O_X(1)) \to H^1(O_S) \to 0$$

We know that $h^0(O_S) = 1$ and $h^0(O_X(1)) = g$, hence $h^0(O_S(1)) \leq g + 1$.
But S is embedded in $\mathbb{P}^g$, hence $h^0(O_S(1)) \geq g + 1$, or better $h^0(O_S(1)) = g + 1$.
By exactness, we have $H^1(O_S) = 0$. □

Let $X \subset \mathbb{P}^{g-1}$ be a canonical curve of genus $g \geq 3$.
The question is: when is X extendable? If extendable, when is it a hyperplane section of a smooth surface, i.e. of a K3-surface (in this case we say K3-extendable)?
For example if $g = 3$, then X is a plane curve, hence extendable.
If $g = 4$, we showed that $X = Q \cap F$ is a complete intersection in $\mathbb{P}^3$ of a quadric Q and a cubic F. We can see that there are a quadric Q' and a cubic F' in $\mathbb{P}^4$ such that $Q' \cap \mathbb{P}^3 = Q$, $F' \cap \mathbb{P}^3 = F$ and $Q' \cap F'$ is not a cone. It follows that X is K3-extendable. If $g = 5$ and X is not trigonal, it is analogous.
Mukai showed that this still happens for $g \leq 9$ (see [16, §6]). He showed more:

- If $g = 6$, then X is a linear section of a quadric section of the Grassmannian $G(2, 5)$ embedded in $\mathbb{P}^9$ via Plücker if and only if X has at most finitely many g^2_6. See [15, §6] and [1, Proposition 1.2].
- If $g = 7$, then X is a linear section of the Orthogonal Grassmannian $OG(5, 10)$ embedded in $\mathbb{P}^{15}$ and 9 hyperplanes if and only if X has no g^1_4. See [14, Theorem 2].

- If $g = 8$, then X is a linear section of the Grassmannian $G(2,6)$ embedded in $\mathbb{P}^{14}$ via Plücker if and only if X has no g^2_7. See [14, Theorem 1].
- If $g = 9$, then X is a linear section of the Symplectic Grassmannian $SpG(3,6)$ embedded in $\mathbb{P}^{13}$ if and only if X has no g^1_5. See [14, Theorem 2].

Moreover a general curve of degree 11 is K3-extendable (see [13]), but a general curve of degree 10 is not K3-extendable (see [16, Theorem 0.7]). Finally, as we can see in [16, §0], the space M_g of curves of genus g has dimension $3g - 3$; on the other hand the space F_g of pairs (S, X) of a K3-surface S and a curve $X \subset S$ of genus g has dimension $19 + g$. Since there is a map $F_g \to M_g$, $(S, X) \mapsto X$ which cannot be surjective if $19 + g < 3g - 3$ (i.e. $g > 11$), we can conclude that a general curve of degree at least 12 is not K3-extendable.
Finally, about extendability, we can see:

Theorem 5.15.3 (Zak, see [20] and [11])**.**
Let $X \subset P^r$ be a smooth variety of codimension at least 2.
Let N_X be the normal bundle of X.
If $\dim_k \Gamma(X, N_X(-1)) \leq r + 1$, then X is not extendable.

Theorem 5.15.4 (Wahl, see [19])**.**
Let $X \subset P^{g-1}$ be a canonical curve.
Let $\Phi_{\omega_X} \colon \bigwedge^2 \Gamma(X, \omega_X) \to \Gamma(X, \omega_X^{\otimes 3})$ be the Wahl map.
Then $\dim_k \Gamma(X, N_X(-1)) = g + \operatorname{corank}(\Phi_{\omega_X})$.

Theorem 5.15.5 (Ciliberto-Harris-Miranda, see [4])**.**
Let $X \subset P^{g-1}$ be a general canonical curve.
If either $g = 10$ or $g \geq 12$, then Φ_{ω_X} is surjective.

Corollary 5.15.6.
Let $X \subset P^{g-1}$ be a general canonical curve.
If either $g = 10$ or $g \geq 12$, then X is not extendable.

We can conclude that a general canonical curve of genus g is extendable if and only if $g \leq 9$ or $g = 11$ (and in this case it is K3-extendable).

5.16 Conclusion

Finally, we can summarize what has been shown so far about curves.
After having defined in the previous chapters three important tools (divisors, line bundles and linear systems), we adapted what we have seen about them to the case of curves, and we used it to achieve our goals, including that:
$\boxed{\text{Every curve can be embedded in } \mathbb{P}^3}$.
First of all we defined the main notions associated to a curve, such as the degree, the canonical divisor and the genus. The latter is an important invariant that allowed us to make a first distinction; in particular we focused on the cases of low genus and we showed that (up to isomorphism):

Genus	Curves
$g = 0$	The only curve is $\mathbb{P}^1$
$g = 1$	Plane cubic curve

For $g \geq 2$ we distinguished two types of curves: hyperelliptic and non.
First we saw that: $\boxed{\text{Every curve of genus 2 is hyperelliptic}}$.
After that we studied the non-hyperelliptic case of low genus, and we saw that in this case we have:

Genus	Canonical curves
$g = 3$	Plane curve of degree 4
$g = 4$	Complete intersection in $\mathbb{P}^3$ of a quadric and a cubic
$g = 5$	Either trigonal or complete intersection in $\mathbb{P}^4$ of three quadrics
$g \geq 6$	Not complete intersection in $\mathbb{P}^{g-1}$

In order to better describe a trigonal canonical curve X of genus 5, we showed that X is contained in a rational normal scroll S in $\mathbb{P}^4$ (where S is the intersection of quadrics containing X). Such curve X is residue of a line in the intersection of S with a cubic.

Finally, we briefly saw an exhibition of results about the extendability of curves of higher genus. We have that: $\boxed{\text{A canonical curve of genus at most 9 is K3-extendable}}$.
Moreover, for general curves we have:

Genus	General curves
$g \leq 9$	K3-extendable
$g = 10$	Not extendable
$g = 11$	K3-extendable
$g \geq 12$	Not extendable

Bibliography

[1] M Artebani, S. Kondo *The moduli of curves of genus six and K3 surfaces* American Mathematical Society, Vol. 363, N.3, March 2011, pp. 1445-1462

[2] M.F. Atiyah, I.G. Macdonald *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.

[3] L. Caporaso *Introduzione alla geometria algebrica. Corso di lezioni*

[4] C. Ciliberto, J. Harris, R. Miranda. *On the surjectivity of the Wahl map.* Duke Math. J. 57 (1988), no. 3, 829-858

[5] D. Eisenbud, J. Harris *The geometry of schemes.* Graduate Texts in Mathematics, 197. Springer-Verlag, New York, 2000. x+294 pp.

[6] W. Fulton *Intersection theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 2. Springer-Verlag, Berlin, 1984. xi+470 pp.

[7] P. Griffiths, J. Harris *Principles of algebraic geometry.* Reprint of the 1978 original. John Wiley & Sons, Inc., New York, 1994. xiv+813 pp.

[8] A. Grothendieck *Cohomologie locale des faisceaux cohérents et théor'emes de Lefschetz locaux et globaux (SGA 2).* Documents Mathématiques (Paris), 4, Société Mathématique de France, Paris, 2005, pp. x+208. Séminaire de Géométrie Algébrique du Bois-Marie, 1962.

[9] R. Hartshorne *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.

[10] M. Ide *Every curve of genus not greater than eight lies on a K3 surface.* Nagoja Math J., Vol. 190 (2008) pp. 183-197

[11] S.M. L'vovskii *On the extension of varieties defined by quadratic equations.* Math. USSR Sbornik 63, (1989) 305-317.

[12] H. Matsumura *Commutative algebra.* W. A. Benjamin, Inc., New York 1970 xii+262 pp.

[13] S. Mori, S. Mukai *The uniruledness of the moduli space of curves of genus 11.* Lecture Notes in Math. 1016, Springer-Verlag, 1983, pp. 334-353.

[14] S. Mukai *Curves and symmetric spaces.* Proc. Japan Acad., 68 (1992), 7-10.

[15] S. Mukai *Curves and Grassmannians, Algebraic Geometry and Related Topics.* Inchoen, Korea, 1992, International Press, Boston, 1993, pp. 19-40.

[16] S. Mukai *Curves K3 surfaces and Fano 3-folds of genus* ≤ 10. Algebraic Geometry and Commutative Algebra, Vol. 1, Kinokuniya, Tokyo, 1988, pp. 357-377

[17] E. Sernesi *Appunti del corso di GE510.* `http://www.mat.uniroma3.it/users/sernesi/GE5101617/GE5101617appunti.pdf`

[18] The Stacks Project `https://stacks.math.columbia.edu/`

[19] J. Wahl *Introduction to Gaussian maps on an algebraic curve.* In: Complex Projective Geometry, Trieste-Bergen 1989. London Math. Soc. Lecture Notes Series 179. Cambridge Univ. Press: 1992, 304-323.

[20] F.L. Zak *Some properties of dual varieties and their application in projective geometry.* In: Algebraic Geometry, Proceedings Chicago 1989. Lecture Notes in Math. 1479. Springer, Berlin-New York: 1991, 273-280.

[21] O. Zariski, P. Samuel *Commutative algebra, Volume I.* With the cooperation of I. S. Cohen. The University Series in Higher Mathematics. D. Van Nostrand Company, Inc., Princeton, New Jersey, 1958. xi+329 pp.