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On the existence of Ulrich vector bundles on blown-up surfaces

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Introduction

The question whether homogeneous polynomials can be expressed as a determinant of linear forms is a well-known classical subject, which has been studied since the middle of 1800 (see Beauville [B1] for more references). Many cases have already been analysed: Dixon [Di], at the beginning of the 20^{th} century, treated plane curves of any degree; Dickson [D] presented a way to determine which general homogeneous forms can be expressed as linear determinants, also producing such a representation for plane curves. A more contemporary treatment of plane curves appears in Cook and Thomas [CT]; smooth irreducible curves are issued in Vinnikov [V]; plane curves and surfaces in $\mathbb{P}^3$ are presented in Beauville [B1]; smooth plane quartics are treated in Plaumann, Sturmfels, Vinzant [PSV].

Before proceeding, let us consider some direct examples. Let K be an algebraically closed field, and let $F(x_0, \dots, x_N)$ be a homogeneous polynomial of degree d over K: under which conditions can F be expressed as the determinant of a $d \times d$ matrix $L(x_0, \dots, x_N)$, whose entries are linear in $x_0, \dots, x_N$? As Vinnikov [V] shows, we can consider at first two basic cases, that is $N = 2$ and, respectively, $d = 2, 3$. As we know, every smooth plane quadric can be written, by a homogeneous change of coordinates, as

$$F(x_0, x_1, x_2) = x_1^2 - x_0 x_2,$$

and one observes that

$$F(x_0, x_1, x_2) = \det \begin{pmatrix} x_1 & x_2 \\ x_0 & x_1 \end{pmatrix}.$$

Therefore any smooth plane quadric possesses a unique representation as a determinant of linear forms. With regard to smooth plane cubics, instead, they can be written, by a homogeneous change of coordinates, as

$$F(x_0, x_1, x_2) = x_2^2 x_0 - x_1(x_1 + \theta_1 x_0)(x_1 + \theta_2 x_2),$$

where $\theta_1, \theta_2 \in K^*$ are distinct. One sees that

$$F(x_0, x_1, x_2) = \det \begin{pmatrix} tx_0 + x_1 & dx_0 + x_2 & qx_0 \\ 0 & lx_0 + x_1 & -dx_0 + x_2 \\ -x_0 & 0 & tx_0 + x_1 \end{pmatrix},$$

where $l \in K$ and t, q, d are determined by l, θ_1, θ_2. Therefore any smooth plane cubic possesses infinite representations as a determinant of linear forms. Another basic case is, for instance, quadrics in four variables ($N = 3$ and $d = 2$). The general quadratic forms in four variables can be transformed, by a homogeneous change of coordinates, into

$$F(x_0, x_1, x_2, x_3) = x_0x_1 - x_2x_3.$$

Clearly, we have

$$F(x_0, x_1, x_2, x_3) = \det \begin{pmatrix} x_0 & x_2 \\ x_3 & x_1 \end{pmatrix},$$

and therefore also every quadric surface in $\mathbb{P}^3$ can be expresses as a linear determinant.

A question arises: how does the above classical problem relate to the existence of Ulrich vector bundles, that is the main subject of this text?

We start by giving one of the known characterizations of such sheaves:

Definition 0.1. *Let X be a smooth variety of dimension n, and let $\mathcal{L}$ be a very ample line bundle on X. Let us consider a vector bundle E on X such that, for all $1 \leq p \leq n$, $E \otimes \mathcal{L}^{\otimes(-p)}$ has vanishing cohomology. Such E is said to be an Ulrich vector bundle for $(X, \mathcal{L})$.*

Although this definition appears straightforward at first, determining the existence of an Ulrich vector bundle on a given smooth variety is not an easy task. In fact (see Proposition 2.4) we will observe that verifying whether a homogeneous polynomial F can be written as a determinant of linear forms is equivalent to finding an Ulrich line bundle on the hypersurface X given by the equation $F = 0$. However it is not possible to find a rank one Ulrich bundle for all varieties, as we will see in the following example. Let us consider a smooth surface $S \subset \mathbb{P}^3$, and a rank r Ulrich vector bundle E for (S, H), where H is the hyperplane divisor: by [C, Prop.2.1] we have that

$$c_1(E) \cdot H = \frac{r}{2} H \cdot (3H + K_S).$$

Let us now suppose that the canonical divisor K_S is equal to zero, and that $\mathrm{Pic}(S) = \mathbb{Z}H$, hence $c_1(E) = aH$ for $a \in \mathbb{Z}$. The above equation becomes

$$aH^2 = \frac{3r}{2}H^2,$$

therefore $\frac{3r}{2} \in \mathbb{Z}$, and so r is even. This shows that we cannot always write homogeneous polynomials as a linear determinant, but we can settle for a weaker property: we ask whether X can be defined by a linear determinant. Again (see Proposition 2.4), we will observe that a hypersurface X, given by the equation $F = 0$, admits a rank r Ulrich vector bundle if and only if F^r is proportional to a determinant of linear forms.

Ulrich bundles first appeared during the $80's$ in commutative algebra in Ulrich [U], but they entered algebraic geometry in 2003 with Eisenbud, Schreyer [ES], even if merely in the latest years they started to receive attention. Nowadays, despite of many results, we are not yet able to find such sheaves for all varieties.

In the case of curves, as we will see in Chapter 2, the characterization above simplifies to the following: E is Ulrich if and only if $E(-1)$ is a sheaf with vanishing cohomology. Therefore, finding a rank one Ulrich bundle E on a smooth curve C of genus g is equivalent to finding a divisor D of degree $g-1$ with no global sections, and indeed they are in a one-to-one correspondence. As a matter of fact, there are divisors satisfying such properties: if we consider the map $\varphi : C^{g-1} \to \mathrm{Pic}^{g-1}(C)$, which sends $g-1$ points $(p_1, \dots, p_{g-1})$ to the line bundle $\mathcal{O}_C(p_1 + \cdots + p_{g-1})$, then it is known that φ is not surjective, and therefore we find the requested divisors in $\mathrm{Pic}^{g-1}(C) \setminus \mathrm{Im}(\varphi)$.

More intricate is, instead, the case of surfaces.

The first part of this text provides the reader with all the preliminary notions, highlighting the process of blowing-up a scheme along a closed subscheme, with a focus on blow-ups of nonsingular projective surfaces at a closed point. This becomes quite relevant in section 1.7 of Chapter 1, where an important result is recalled: every nonsingular projective surface S has a smooth minimal model S_0, and S is obtained from S_0 by a finite number of successive blow-ups at closed points. Furthermore, we recall a classification of minimal surfaces in the case of Kodaira dimension $-\infty$ and 0. Then we move on to describe Ulrich vector bundles and their properties, giving a characterization and producing some examples. Eventually, in Chapter 3, we prove an original result:

Theorem 0.2. *Let $\mathcal{L}$ be a very ample line bundle on a nonsingular projective variety X. Let $p \in X$ be a closed point corresponding to a sheaf of ideals $\mathscr{I}$ on X, and let $\pi : \widetilde{X} \to X$ be the blowing-up of X with respect to $\mathscr{I}$. If there exists an Ulrich vector bundle for $(X, \mathcal{L})$, then there exists an Ulrich vector bundle for $(\widetilde{X}, \pi^*\mathcal{L}^{\otimes 2} \otimes \mathcal{O}_{\widetilde{X}}(-E))$, where $E = \pi^{-1}(\{p\})$ is the exceptional divisor.*

This result yields indeed interesting applications for surfaces. Since every smooth surface is obtained by a finite number of blow-ups of its minimal surface at closed points, then we have the following:

Corollary 0.3. *If every minimal surface carries an Ulrich vector bundle, then there exists an Ulrich vector bundle on every nonsingular projective surface.*

At the end, we show some cases of minimal surfaces which admit an Ulrich vector bundle, based on their Kodaira dimension:

Corollary 0.4. *Let S be a nonsingular projective surface with Kodaira dimension 0. Then S admits an Ulrich vector bundle.*

Corollary 0.5. *Let S be a nonsingular projective surface with Kodaira dimension $-\infty$, and let us assume that the minimal model of S is within one of the following:*

1. $\mathbb{P}^2$,
2. $\mathbb{F}_n$ *for* $n \geq 0$ *and* $n \neq 1$,
3. $\mathbf{P}_C(E)$ *for a rank 2 vector bundle E over a nonsingular projective curve C, with invariant $e > 0$.*

Then S admits an Ulrich vector bundle.

Notations and Conventions

For this reading, all the basic knowledge and definitions will be taken from [H]. If not specified otherwise, a *scheme* is a separated scheme of finite type over an algebraically closed field K. A *variety* is an integral scheme.

Chapter 1

Preliminaries

1.1 Divisors and Line Bundles

1.1.1 Weil Divisors

Let X be a noetherian integral separated scheme which is nonsingular in codimension 1.

Definition 1.1. *We call prime divisor on X a closed integral subscheme Y of codimention 1. Then WDiv(X) is the free abelian group generated by the prime divisors, whose elements $D = \sum n_i Y_i$, where n_i are integers and finitely many are different form zero, are called Weil divisors. We say that D is effective, written $D \geq 0$, if $n_i \geq 0$ for all i and we write $D \geq D'$, for any two divisors D and D', if $D - D'$ is effective. Furthermore we define the degree of D by* $\deg D = \sum n_i \deg Y_i$.

If Y is a prime divisor, we know that the local ring $\mathcal{O}_{X,Y}$ is a discrete valuation ring with quotient field $K(X)$, the function field on X. Therefore we can define the valuation of Y, v_Y, by the discrete valuation of $\mathcal{O}_{X,Y}$. So, for all $f \in K(X)^*$, $v_Y(f)$ is an integer. By the properties of the valuation of Y we know that $\mathcal{O}_{X,Y} = \{f \in K(X)^* : v_Y(f) \geq 0\}$, whose maximal ideal is $M = \{f \in K(X)^* : v_Y(f) > 0\}$. We observe that if U is an open subset of X such that $U \cap Y \neq \emptyset$ and $f \in \mathcal{O}_X(U)$, then, $v_Y(f) \geq 0$. If $v_Y(f)$ is positive, we say that f has a zero along Y; if it is negative, then we say that f has a pole along Y.

By [H, Cor.II.6.1] we know that $v_Y(f) = 0$ for all except finitely many prime divisors Y. Therefore we can make the following definition.

Definition 1.2. *Let $f \in K(X)^*$. We define the principal divisor of f by*

$$(f) = \sum v_Y(f)Y.$$

Because of the properties of the discrete valuation v_Y, we know that, if $f, g \in K(X)^*$, then $(f/g) = (f) - (g)$.

Definition 1.3. *Two Weil divisors D and D' are said to be linearly equivalent, written $D \sim D'$, if $D - D'$ is a principal divisor.*

Definition 1.4. *The support of $D = \sum n_i Y_i$ is defined by $supp(D) = \cup Y_i$.*

1.1.2 Cartier Divisors

Let X be an integral scheme. We denote by $\mathcal{K}_X$ the constant sheaf of rational functions on X, that is the function field $K(X)$. It contains $\mathcal{O}_X$ as a subsheaf. If we consider the sheaves $\mathcal{K}_X^*$ and $\mathcal{O}_X^*$ of invertible elements in, respectively, the sheaves $\mathcal{K}_X$ and $\mathcal{O}_X$, then there is an inclusion $\mathcal{O}_X^* \subseteq \mathcal{K}_X^*$ of sheaves of multiplicative abelian groups.

Definition 1.5. *A Cartier divisor on X is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. By the properties of quotient sheaves, any Cartier divisor can be described as a set $\{(U_i, f_i)\}$ consisting of an open covering $\{U_i\}$ of X together with elements $f_i \in \Gamma(U_i, \mathcal{K}_X^*) = K(X)^*$, such that for each $i \neq j$ $f_i/f_j = g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$. The function f_i is called a local equation of $\{(U_i, f_i)\}$ at any point $x \in U_i$. We set $Div(X) = \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.*

The set $Div(X)$ is an abelian group with respect to the operation

$$\{(U_i, f_i)\} + \{(U'_{i'}, f'_{i'})\} = \{(U_i \cap U'_{i'}, f_i f'_{i'})\}.$$

We say that $D \in Div(X)$ is effective, written $D \geq 0$, if $f_i \in \Gamma(U_i, \mathcal{O}_X)$ for all i.

Definition 1.6. *A Cartier divisor D is principal if it is in the image of the natural map $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, that is D can be represented by $\{(X, f)\}$, with $f \in \Gamma(X, \mathcal{K}_X^*) = K(X)^*$. We denote $D = div(f)$.*

Definition 1.7. *Two Cartier divisors are said to be linearly equivalent if their difference is principal.*

Definition 1.8. *The support of a divisor* $D = \{(U_i, f_i)\}$*, written* $supp(D)$*, is the set of points* $x \in X$ *such that a local equation of* D *at* x *is not a unit in* $\mathcal{O}_{X,x}$.

Proposition 1.9. *Let* X *be an integral, separated noetherian scheme which is locally factorial. Then the group* $WDiv(X)$ *is isomorphic to the group* $Div(X)$*, and furthermore the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.*

Proof. See [H, Prop.II.6.11] □

1.1.3 Line Bundles

From now on we will consider only Cartier divisors. We recall at first the definition of vector bundle.

Definition 1.10. *A coherent sheaf* $\mathcal{F}$ *on a scheme* X *is said to be locally free of rank* r *if* X *can be covered by open sets* $\{U_i\}_{i\in I}$ *such that* $\forall\ i \in I$ *there exists an isomorphism*

$$\varphi_{U_i} : \mathcal{F}_{|U_i} \to \mathcal{O}_{U_i}^{\oplus r}.$$

Note that if X *is connected, then the rank is the same on every open covering of* X*. Such a sheaf is also called vector bundle of rank* r*. In particular a locally free sheaf of rank 1 (or line bundle) is called an invertible sheaf.*

For any divisor D, described by $\{(U_i, f_i)\}$, we can define a line bundle $\mathcal{O}_X(D)$ on X by the transition functions $g_{ij} = f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$. Let us note that $\mathcal{O}_X(D)$ is therefore a subsheaf of $\mathcal{K}_X$. Conversely, given a line bundle $\mathcal{L}$ on X, we can recover a divisor D by taking f_i for all i, where $f_i^{-1} : \mathcal{O}_{U_i} \to \mathcal{L}_{|U_i}$ is an isomorphism. More precisely, we show the following isomorphism.

Proposition 1.11. *Let* X *be an integral scheme. There is an isomorphism of abelian groups*

$$Div(X)/Pr(X) \cong Pic(X),$$

where $Pr(X)$ *denotes the subgroup of principal divisors of* $Div(X)$ *and* $Pic(X)$ *denotes the Picard group of isomorphism classes of line bundles on* X.

Proof. Let us consider the exact sequence of sheaves on X

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0.$$

Since $\mathcal{K}_X^*$ is flasque, we obtain the following exact sequence of abelian groups

$$0 \to \Gamma(X, \mathcal{O}_X^*) \to \Gamma(X, \mathcal{K}_X^*) \xrightarrow{\varphi} \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \to 0.$$

By [H, Ex.III.4.5] we know that $H^1(X, \mathcal{O}_X^*) = Pic(X)$ and by definition $\mathrm{Im}(\varphi) = Pr(X)$. Therefore $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)/\mathrm{Im}(\varphi) \cong H^1(X, \mathcal{O}_X^*)$, and this proves the statement. □

1.2 Linear Systems

Let $X \subseteq \mathbb{P}^r$ be nonsingular. In this case, by Proposition 1.9, the notions of Weil divisor and Cartier divisor coincide. Furthermore by Proposition 1.11 we have a one-to-one correspondence between linear equivalence classes of divisors and isomorphism classes of line bundles. Given a line bundle $\mathcal{L}$ on X, we first take a look at the structure of $\Gamma(X, \mathcal{L})$.

Definition 1.12. *We define the divisor of zeros of $s \in \Gamma(X, \mathcal{L})$, written $(s)_0$, as the effective divisor $\{(U, \varphi_U(s))\}$, where U is any open subset of X on which $\mathcal{L}$ is trivial and $\varphi_U : \mathcal{L}_{|U} \to \mathcal{O}_U$ is the associated isomorphism.*

Proposition 1.13. *Let $X \subseteq \mathbb{P}^r$ be nonsingular. Let D_0 be a divisor on X and let $\mathcal{L} = \mathcal{O}_X(D_0)$ the corresponding line bundle. Then:*

1. *for each nonzero $s \in \Gamma(X, \mathcal{L})$, $(s)_0$ is an effective divisor, which is linearly equivalent to D_0.*

2. *Every effective divisor linearly equivalent to D_0 is $(s)_0$ for some $s \in \Gamma(X, \mathcal{L})$.*

3. *Two sections $s, s' \in \Gamma(X, \mathcal{L})$ are such that $(s)_0 = (s')_0$ if and only if there exists $\lambda \in K^*$ such that $s = \lambda s'$.*

Proof. See [H, Prop.II.7.7] □

Remark 1.14. *Statements 1. and 2. of the proposition give us a way to describe the global sections of a line bundle $\mathcal{L} = \mathcal{O}_X(D_0)$, that is $\Gamma(X, \mathcal{L}) = \{f \in K(X) : D_0 + div(f) \geq 0\}$.*

Definition 1.15. *A complete linear system on a nonsingular projective variety $X \subseteq \mathbb{P}^r$ is the set of all effective divisors linearly equivalent to some given divisor D_0, denoted by $|D_0|$. By the previous proposition notice that there is a one-to-one correspondence between $|D_0|$ and $\mathbb{P}(\Gamma(X, \mathcal{O}_X(D_0)))$. By notation, we will at times write $|\mathcal{L}|$, where $\mathcal{L} = \mathcal{O}_X(D_0)$, instead of $|D_0|$.*

Definition 1.16. *A linear system Q on X is a subset of a complete linear system $|D_0|$, which corresponds to a sub-vector space V of $\Gamma(X, \mathcal{O}_X(D_0))$, that is $Q = \{(s)_0 : \forall s \in V\}$. By notation, we will write at times $|V|$, instead of Q.*

Definition 1.17. *A point $x \in X$ is a base point of a linear system $|V|$ if $x \in supp(D)$ for all $D \in |V|$. The base locus $Bs(|V|)$ is the set of all base points. One can see the base locus as the set of points in X at which all the sections in V vanish. Furthermore we call base ideal $\mathfrak{b}(|V|)$ the ideal sheaf associated to $Bs(|V|)$.*

Remark 1.18. *If $\{s_0, \dots s_r\}$ is a base for V, we can determine a rational map $\phi_{|V|} : X \dashrightarrow \mathbb{P}^r$. If $|V|$ is base point free, then $\phi_{|V|}$ is a globally defined morphism.*

1.3 Amplitude

Definition 1.19. *Let $\mathcal{L}$ be an invertible sheaf on a complete scheme X.*

1. *$\mathcal{L}$ is very ample if there exists a closed immersion $\phi_{|\mathcal{L}|} : X \hookrightarrow \mathbb{P}^r$ such that $\mathcal{L} = \mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^r}(1)_{|X}$.*
2. *$\mathcal{L}$ is ample if $\mathcal{L}^{\otimes m}$ is very ample for some $m > 0$.*

Theorem 1.20. *Let $\mathcal{L}$ be an invertible sheaf on a complete scheme X. Then the following conditions are equivalent:*

1. *$\mathcal{L}$ is ample.*
2. *$\exists\ m_0 \in \mathbb{N}$ such that $\mathcal{L}^{\otimes m}$ is very ample $\forall\ m \geq m_0$.*
3. *For each coherent sheaf $\mathcal{F}$ on X, $\exists\ m_0(\mathcal{F}) \in \mathbb{N}$ such that $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0\ \forall\ i > 0$ and $\forall\ m \geq m_0(\mathcal{F})$.*
4. *For each coherent sheaf $\mathcal{F}$ on X, $\exists\ m_0(\mathcal{F}) \in \mathbb{N}$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated $\forall\ m \geq m_0(\mathcal{F})$.*

Proof. See [L, Thm.1.2.6]. □

1.4 Some Useful Results about Cohomology

Theorem 1.21. *Let $P = K[x_0, \dots, x_r]$ and let $\mathbb{P}^r$ be the r-projective space over K. Then:*

1. $P \cong \Gamma_*(\mathcal{O}_{\mathbb{P}^r}) := \bigoplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))$ *as graded K-modules. Therefore*

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) = \begin{cases} P_n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

and in particular

$$h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) = \begin{cases} \binom{n+r}{r} & n \geq 0 \\ 0 & n < 0 \end{cases}.$$

2. $H^i(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) = 0$ *for $i \neq 0, r$ and for all $n \in \mathbb{Z}$.*

3. $H^r(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) = 0$ *for $n \geq -r$, and*
$H^r(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) = \langle x_0^{l_0} \cdots x_r^{l_r} : l_j < 0, \sum_{j=0}^{r} l_j = n \rangle$ *for $n \leq -r-1$.*
In particular

$$h^r(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) = \binom{-n-1}{r}, \quad n \leq -r-1$$

.

4. *The natural map*

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \times H^r(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(-n-r-1)) \to H^r(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(-r-1)) \cong K$$

is a perfect pairing of finitely generated free K-modules, $\forall\ n \in \mathbb{Z}$.

Proof. See [H, Thm.III.5.1]. □

Definition 1.22. *Let $X \subseteq \mathbb{P}^r$ be a projective scheme and $\mathcal{F}$ a coherent sheaf on X. Then the Euler Characteristic of $\mathcal{F}$ is defined by*

$$\chi(X, \mathcal{F}) = \sum_{i\geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Proposition 1.23. *Given a short exact sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of coherent sheaves on $X \subseteq \mathbb{P}^r$, we have $\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}') + \chi(X, \mathcal{F}'')$.

Proof. Let us apply the rank-nullity theorem to the long exact sequence

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to \dots$$

and we deduce that

$$0 = h^0(X, \mathcal{F}') - h^0(X, \mathcal{F}) + h^0(X, \mathcal{F}'') - h^1(X, \mathcal{F}') + \dots$$

The statement is then clear. □

Theorem 1.24. *If $f : X \to Y$ is an affine morphism and if $\mathcal{F}$ is a quasi-coherent sheaf on X, then*

$$H^\bullet(Y, f_*\mathcal{F}) = H^\bullet(X, \mathcal{F})$$

Proof. See [S, Thm.13.5] □

1.5 Riemann-Roch Theorem

Theorem 1.25. *Let $C \subseteq \mathbb{P}^r$ be a nonsingular curve of genus g, and let D be a divisor on C of degree d. Then:*

$$\chi(C, \mathcal{O}_C(D)) = d + 1 - g$$

Proof. See [H, Thm.IV.1.3] □

Theorem 1.26. *Let $S \subseteq \mathbb{P}^r$ be a nonsingular surface, and let D be a divisor on S. Then:*

$$\chi(S, \mathcal{O}_S(D)) = \chi(S, \mathcal{O}_S) + \frac{1}{2}D.(D - K_S),$$

where K_S is the canonical divisor.

Proof. See [H, Thm.V.1.6] □

1.6 Blowing-Up

In this section we will consider the following conditions:

(*)X is a noetherian scheme, $\mathscr{S}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules with a structure of a sheaf of graded $\mathcal{O}_X$-algebras. Thus $\mathscr{S} \cong \bigoplus_{d\geq 0} \mathscr{S}_d$, where $\mathscr{S}_d$ is the homogeneous part of degree d. We assume that $\mathscr{S}_0 = \mathcal{O}_X$, that $\mathscr{S}_1$ is a coherent $\mathcal{O}_X$-module and that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as a $\mathscr{S}_0$-algebra.

Definition 1.27. *Let X be a noetherian scheme, and let $\mathscr{I}$ be a coherent sheaf of ideals on X. If $\mathscr{I}^d$ is the d-th power of $\mathscr{I}$ and $\mathscr{I}^0 = \mathcal{O}_X$, then we consider $\mathscr{S} = \bigoplus_{d\geq 0} \mathscr{I}^d$. Clearly X and $\mathscr{S}$ satisfy (*), so we consider $\widetilde{X} = \mathbf{Proj}\ \mathscr{S}$. We define $\widetilde{X}$ to be the blowing-up of X with respect to the coherent sheaf of ideals $\mathscr{I}$.*

Proposition 1.28. *Let X be a noetherian scheme, $\mathscr{I}$ a coherent sheaf of ideals on X, and let $\pi : \widetilde{X} \to X$ be the blowing-up of X with respect to $\mathscr{I}$. Then:*

1. *the inverse image ideal sheaf $\widetilde{\mathscr{I}} = \pi^{-1}\mathscr{I} \cdot \mathcal{O}_{\widetilde{X}}$ is an invertible sheaf on $\widetilde{X}$, which is equal to $\mathcal{O}_{\widetilde{X}}(1)$.*

2. *If Y is the closed subscheme of X corresponding to $\mathscr{I}$ and $U = X - Y$, then $\pi : \pi^{-1}(U) \to U$ is an isomorphism.*

Proof. See [H, Prop.II.7.13]. □

The subscheme $\widetilde{Y} \subseteq \widetilde{X}$, corresponding to the inverse image ideal sheaf $\widetilde{\mathscr{I}}$, will be called the exceptional divisor.

Now we need to recall the definition of projective space bundle associated to a locally free coherent sheaf.

Definition 1.29. *Let X be a noetherian scheme, and let $\mathscr{E}$ be a locally free coherent sheaf on X. We define the projective space bundle associated to $\mathscr{E}$ $\mathbf{P}(\mathscr{E})$ as follows. Let $S(\mathscr{E})$ be the symmetric algebra of $\mathscr{E}$. Then X and $S(\mathscr{E})$ satisfy (*) and we define $\mathbf{P}(\mathscr{E}) = \mathbf{Proj}\ S(\mathscr{E})$,*

Theorem 1.30. *Let X be a nonsingular projective variety over K, and let $Y \subseteq X$ be a nonsingular closed subvariety, with ideal sheaf $\mathscr{I}$.*
Let $\pi : \widetilde{X} \to X$ be the blowing-up of X with respect to $\mathscr{I}$, and let $Y' \subseteq \widetilde{X}$ be the subscheme defined by the inverse image sheaf $\mathscr{I}' = \pi^{-1}\mathscr{I} \cdot \mathcal{O}_{\widetilde{X}}$. Then:

1. *$\widetilde{X}$ is also nonsingular.*

2. *Y', together with the induced projective map $\pi : Y' \to Y$, is isomorphic to $\mathbf{P}(\mathscr{I}/\mathscr{I}^2)$, the projective space bundle associated to the locally free sheaf $\mathscr{I}/\mathscr{I}^2$ on Y.*

3. *Under this isomorphism, the normal sheaf $\mathcal{N}_{Y'/\widetilde{X}} = \mathscr{H}om(\mathscr{I}'/\mathscr{I}'^2, \mathcal{O}_{Y'})$ corresponds to $\mathcal{O}_{\mathbf{P}(\mathscr{I}/\mathscr{I}^2)}(-1)$.*

Proof. See [H, Thm.II.8.24]. □

Proposition 1.31. *Let X be a nonsingular projective variety of dimension n, $\mathscr{I}$ a coherent sheaf of ideals on X corresponding to a closed point $P \in X$, and let $\pi : \widetilde{X} \to X$ be the blowing-up of X with respect to $\mathscr{I}$. Then we have $\pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$, and $R^i \pi_* \mathcal{O}_{\widetilde{X}} = 0$ for $i > 0$.*

Proof. Let E be the exceptional divisor, and let $\widetilde{\mathscr{I}}$ be the corresponding ideal sheaf. Since π is an isomorphism of $\widetilde{X} - E$ onto $X - \{P\}$, it is clear that the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_{\widetilde{X}}$ is an isomorphism, except possibly at P, and that the sheaves $\mathcal{F}^i = R^i \pi_* \mathcal{O}_{\widetilde{X}}$ for $i > 0$ have support at P. We use [H, Thm.III.11.1] to compute these $\mathcal{F}^i$. That is, taking completions of the stalks at P, we have

$$\hat{\mathcal{F}}^i \cong \varprojlim H^i(E_m, \mathcal{O}_{E_m})$$

where E_m is the closed subscheme of $\widetilde{X}$ corresponding to $\widetilde{\mathscr{I}^m}$. There are natural exact sequences

$$0 \to \widetilde{\mathscr{I}^m}/\widetilde{\mathscr{I}^{m+1}} \to \mathcal{O}_{E_{m+1}} \to \mathcal{O}_{E_m} \to 0$$

for each m. Furthermore, by Theorem 1.30 we have $\widetilde{\mathscr{I}}/\widetilde{\mathscr{I}^2} = \mathcal{O}_E(1)$, and by [H, Thm.II.8.21A(e)] we have that $\widetilde{\mathscr{I}^m}/\widetilde{\mathscr{I}^{m+1}} \cong S^m(\widetilde{\mathscr{I}}/\widetilde{\mathscr{I}^2}) \cong \mathcal{O}_E(m)$. Now $E \cong \mathbb{P}^{n-1}$, so $H^i(E, \mathcal{O}_E(m)) = 0$ for $i > 0$ and all $m \geq 0$ by Theorem 1.21. Since $E_1 = E$, we will prove by induction on m that $H^i(E_m, \mathcal{O}_{E_m}) = 0$ for all $i > 0$ and $m > 0$. Let us consider the exact sequences

$$0 = H^i(E, \mathcal{O}_E(m)) \to H^i(E_{m+1}, \mathcal{O}_{E_{m+1}}) \to H^i(E_m, \mathcal{O}_{E_m})$$

for $m > 0$.

If $m = 1$, $H^i(E_1, \mathcal{O}_{E_1}) = H^i(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}) = 0$ for all $i > 0$. If $m \geq 2$, suppose that $H^i(E_m, \mathcal{O}_{E_m}) = 0$ for all $i > 0$, then, from the previous exact sequence, we obtain that $H^i(E_{m+1}, \mathcal{O}_{E_{m+1}}) = 0$ for all $i > 0$. It follows that $\hat{\mathcal{F}}^i = 0$ for $i > 0$. Since $\mathcal{F}^i$ is a coherent sheaf with support at P, $\hat{\mathcal{F}}^i = \mathcal{F}^i$, thus $\mathcal{F}^i = 0$ for all $i > 0$.

The equality $\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ follows from the fact that X is normal and π is birational. (See proof of [H, Cor.III.11.4]). □

The following result is taken from [BEL, Lemma 1.4].

Proposition 1.32. *Let X be a nonsingular projective variety of dimension n, $\mathscr{I}$ a coherent sheaf of ideals on X corresponding to a closed point $P \in X$, and let $\pi : \widetilde{X} \to X$ be the blowing-up of X with respect to $\mathscr{I}$. Then, if E is the exceptional divisor, for any locally free sheaf $\mathcal{F}$ on X we have*

$$H^i(\widetilde{X}, \pi^*\mathcal{F} \otimes \mathcal{O}_{\widetilde{X}}(tE)) = H^i(X, \mathcal{F})$$

for all $i \geq 0$ and for $0 \leq t \leq n-1$.

Proof. By the projection formula [H, Ex.II.5.1(d)] and [H, Ex.III.8.1], it is sufficient to show that $\pi_*\mathcal{O}_{\widetilde{X}}(tE) = \mathcal{O}_X$ and $R^i\pi_*\mathcal{O}_{\widetilde{X}}(tE) = 0$ for $i > 0$ and $0 \leq t \leq n-1$. If $t = 0$, we can apply Proposition 1.31. For $t \geq 1$, let us consider the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}((t-1)E) \to \mathcal{O}_{\widetilde{X}}(tE) \to \mathcal{O}_E(tE) \to 0.$$

Then we can prove the statement by induction on t, using the facts that $E = \mathbf{P}(\mathscr{I}/\mathscr{I}^2)$, that $\mathcal{O}_E(E) = \mathcal{O}_{\mathbf{P}(\mathscr{I}/\mathscr{I}^2)}(-1)$, and using [H, Ex.III.8.4(a)] □

The following result is taken from [BS, Thm.1.3].

Theorem 1.33. *Let X be a nonsingual projective variety. Let $\mathcal{L}$ be a line bundle on X and let $Q \subseteq |\mathcal{L}|$ be a nonempty linear system, with base locus B. Then, if $\varphi : X \dashrightarrow \mathbb{P}^N$ is the rational map associated to Q, we have:*

1. *the graph, $\Gamma_\varphi \subseteq X \times \mathbb{P}^N$, of φ is isomorphic to the blowing up, $\pi : \widetilde{X} \to X$, of X along B.*

2. *Let E be the exceptional divisor. Then the pullback of $\mathcal{O}_{\mathbb{P}^N}(1)$ to Γ_φ under the map induced by the projection to $\mathbb{P}^N$ is the sheaf $\pi^*\mathcal{L} \otimes \mathcal{O}_{\widetilde{X}}(-E)$.*

Proof. It follows from [H, Example II.7.17.3], [Fu, Chap.4.4] and [Ha, Example 7.18]. □

1.7 Minimal Surfaces

Results appearing in this section are taken from [B].

Definition 1.34. *For a nonsingular projective surface S we denote by $B(S)$ the set of isomorphism classes of surfaces birationally equivalent to S.*
If S_1, $S_2 \in B(S)$, then we say that S_1 dominates S_2 if there is a birational morphism $S_1 \to S_2$. Therefore we can define an order on $B(S)$.
A surface S is minimal if its class in $B(S)$ is minimal.

Proposition 1.35. *Every nonsingular projective surface dominates a minimal surface.*

Proof. See [B, Thm.II.16] □

Theorem 1.36. *Let $f : S \to S_0$ be a birational morphism of nonsingular projective surfaces. Then there is a sequence of blow-ups $\pi_k : S_k \to S_{k-1}$ at a closed point $P_{k-1} \in S_{k-1}$, for $k = 1, \dots, n$, and an isomorphism $u : S \to S_n$ such that $f = \pi_1 \circ \cdots \circ \pi_n \circ u$.*

Proof. See [B, Thm.II.11]. □

Corollary 1.37. *Every nonsingular projective surface is obtained from a minimal surface by a finite number of successive blow-ups.*

Proof. It follows by Proposition 1.35 and Theorem 1.36. □

Our intent is to give a classification of minimal surfaces in the case of Kodaira dimension $-\infty$, 0. In order to do so, we define an important birational invariant of nonsingular projective varieties, the Kodaira dimension. Let us now fix some notation. For a nonsingular projective variety X we define:

1. the m^{th} plurigenus $P_m = h^0(X, \mathcal{O}_X(mK_X))$.

   If X is a surface:

2. $p_g = h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X(K_X))$,

3. $q = h^1(X, \mathcal{O}_X)$.

There are many ways to define the Kodaira dimension, and we will give the following:

Definition 1.38. *Let X be a nonsingular projective variety of dimension n, K_X be a canonical divisor of X, and let $\phi_{|mK_X|} : X \dashrightarrow \mathbb{P}^{P_m - 1}$ be the rational map associated with the linear system $|mK_X|$. Then, the Kodaira dimension of X, written $k(X)$, is the maximum dimension of the images $\phi_{|mK_X|}(X)$, for $m \geq 1$.*

Let us notice that, if $|mK_X| = \emptyset$, then $\phi_{|mK_X|}(X) = \emptyset$, and we say $\dim(\emptyset) = -\infty$. Furthermore one sees that $k(X) \in \{-\infty, 0, 1, \dots, n\}$.

Remark 1.39. *An equivalent definition states that the Kodaira dimension of X can be given by the minimal $k \in \mathbb{N} \cup \{-\infty\}$ such that $h^0(X, \mathcal{O}_X(mK_X))/m^k$ is bounded.*

Now we can give a classification of minimal surfaces with Kodaira dimension equal to $-\infty$, 0.

Theorem 1.40. *Let S_0 be a minimal surface such that $k(S_0) = 0$. Then S_0 belongs to one of the following 4 cases:*

1. *$p_g = 0$, $q = 0$; we say that S_0 is an "Enriques surface".*
2. *$p_g = 0$, $q = 1$; then S_0 is a bielliptic surface.*
3. *$p_g = 1$, $q = 0$; we say that S_0 is a "K3 surface".*
4. *$p_g = 1$, $q = 2$; then S_0 is an Abelian surface.*

Proof. See [B, Thm.VIII.2] □

Remark 1.41. *Let S be a nonsingular projective surface. By [B, Prop.III.21] and [B, Thm.VI.17] we deduce that:*

$$k(S) = -\infty \iff S \text{ is a ruled surface.}$$

Proposition 1.42. *Let S_0 be a minimal surface such that $k(S_0) = -\infty$. Then S_0 belongs to one of the following 2 cases:*

1. *$S_0 = \mathbb{P}^2$ or $S_0 = \mathbb{F}_n := \mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$, $n \neq 1$.*
2. *S_0 is a geometrically ruled surfaces over a nonsingular projective curve C, that is the projective bundle $\mathbf{P}_C(E)$, where E is a rank 2 vector bundle over C.*

Proof. For case 1. apply [B, Thm.V.10]. Case 2. follows by [B, Thm.III.10 and Prop.III.7]. □

1.8 Finite Morphisms

Theorem 1.43. *Let $f : X \to Y$ be a finite surjective morphism of nonsingular varieties, and let d be its degree. Let $\mathcal{F}$ be a locally free sheaf of rank r on X. Then $f_*\mathcal{F}$ is a locally free sheaf of rank rd on Y.*

Proof. It follows from [H, Cor.III.12.9] for $i = 0$. Furthermore it is useful to notice that, by [H, Ex.III.9.3(a)], it follows that if f is a finite surjective morphism of nonsingular varieties over K, then f is flat. □

1.9 Castelnuovo-Mumford Regularity

Definition 1.44. *Let $\mathcal{F}$ be a coherent sheaf on the projective space $\mathbb{P}^r$, and let m be an integer. $\mathcal{F}$ is said to be m-regular (in the sense of Castelnuovo-Mumford) if*

$$H^i(\mathbb{P}^r, \mathcal{F}(m-i)) = 0 \quad \forall i > 0.$$

Theorem 1.45. *Let $\mathcal{F}$ be an m-regular sheaf on $\mathbb{P}^r$. Then $\forall\ k \geq 0$:*

1. *$\mathcal{F}(m+k)$ is globally generated.*
2. *The natural maps*

$$H^0(\mathbb{P}^r, \mathcal{F}(m)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(\mathbb{P}^r, \mathcal{F}(m+k))$$

   are surjective.

3. *$\mathcal{F}$ is (m+k)-regular.*

Proof. See [L, Thm.1.8.3]. □

Chapter 2

Ulrich Bundles

2.1 General Properties

Proposition 2.1. *Let E be a rank r vector bundle on a smooth variety $X \subseteq \mathbb{P}^N$ of dimension n and degree d, such that $H^i(X, E(-p)) = 0 \ \forall \ i \geq 0$ and for $1 \leq p \leq n$. Then*

1. *E is 0-regular and globally generated.*
2. *$H^p(X, E) = 0 \ \forall \ p > 0$.*
3. *$h^0(X, E) = rd$.*

Proof. (1.) By hypothesis we observe that $H^p(X, E(-p)) = 0$ for $1 \leq p \leq n$, therefore $\forall \ p > 0$ by Grothendieck's Theorem [H, Thm.III.2.7]. This means that E is 0-regular, and by Theorem 1.45 we have that E is globally generated.

(2.) By condition 1. and Theorem 1.45 we have that E is k-regular $\forall \ k \geq 0$, which means that $H^p(X, E(k-p)) = 0 \ \forall \ p > 0$. If we take $k = p$, then we have $H^p(X, E) = 0 \ \forall \ p > 0$.

(3.) By [L, Thm.1.1.24] and [H, Thm.4.1, Appendix A], $\chi(X, E(m))$ is a polynomial $P(m)$ in $m \in \mathbb{Z}$ of degree n and leading coefficient $r\frac{(H)^n}{n!}$, where H is a hyperplane section of X. We also gather by [L, Thm.1.1.24] that $\chi(X, \mathcal{O}_X(m))$ is a polynomial on degree n and leading coefficient $\frac{(H)^n}{n!}$. Now $\deg X = d := n!\frac{(H)^n}{n!}$, therefore $(H)^n = d$, obtaining that the leading coefficient of $P(m)$ is $\frac{rd}{n!}$. Since $\chi(X, E(t)) = 0$ for $-n \leq t \leq -1$, $P(m)$ vanishes for $m = -n, \ldots, -1$, hence $P(m) = \frac{rd}{n!}(m+1)\ldots(m+n)$. Therefore $rd = P(0) = \chi(X, E) = h^0(X, E)$, because $h^i(X, E) = 0 \ \forall \ i > 0$ by (2.). $\square$

The following result is taken from [B4, Thm.1].

Theorem 2.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n, and let E be a rank r vector bundle on X. The following conditions are equivalent:*

1. *There exists a linear resolution*

$$0 \to L_c \to L_{c-1} \to \cdots \to L_0 \to E \to 0 \tag{2.1}$$

with $c = \mathrm{codim}(X, \mathbb{P}^N)$ and $L_i = \mathcal{O}_{\mathbb{P}^N}(-i)^{\oplus b_i}$, for $0 \leq i \leq c$ and some $b_i \geq 1$.

2. *$H^i(X, E(-p)) = 0 \ \forall \ i \geq 0$ and for $1 \leq p \leq n$.*

3. *If $\pi : X \to \mathbb{P}^n$ is a finite linear projection, then the vector bundle $\pi_* E$ is trivial.*

Proof. **Case** $X = \mathbb{P}^N$

If 3. holds, then $\pi : \mathbb{P}^N \to \mathbb{P}^N$ is the identity map and $\pi_* E = E$ is trivial.

If 1. holds, we have $c = 0$. Therefore the sequence $0 \to L_0 \to E \to 0$ is exact $\Leftrightarrow E \cong L_0 = \mathcal{O}_{\mathbb{P}^N}^{\oplus b_0}$ is trivial.

Now we want to prove that E is trivial $\Leftrightarrow H^i(\mathbb{P}^N, E(-p)) = 0 \ \forall \ i \geq 0$ and for $1 \leq p \leq N$.

($\Rightarrow$) It is sufficient to notice that $H^i(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-p)) = 0$ for $i \neq N$ and for $p > 0$, and for $i = N$ and for $p \leq N$.

($\Leftarrow$) Applying Proposition 2.1 to E we have that $h^0(\mathbb{P}^N, E) = r \cdot \deg \mathbb{P}^N = r$ and that E is globally generated. This implies that

$$H^0(\mathbb{P}^N, E) = K^{\oplus r}.$$

Since E is globally generated, we have a surjective morphism

$$H^0(\mathbb{P}^N, E) \otimes \mathcal{O}_{\mathbb{P}^N} \cong \mathcal{O}_{\mathbb{P}^N}^{\oplus r} \xrightarrow{\varphi} E.$$

This means that, $\forall \ x \in \mathbb{P}^N$,

$$0 \to \ker \varphi_x \to \mathcal{O}_{\mathbb{P}^N, x}^{\oplus r} \xrightarrow{\varphi_x} E_x \to 0$$

is an exact sequence of $\mathcal{O}_{\mathbb{P}^N, x}$-modules. Let us notice that $\ker \varphi_x$ is finitely generated by [AM, Ex.2.12]. Now, since K is a $\mathcal{O}_{\mathbb{P}^N, x}$ flat module (it follows by [AM, Ex.2.20]), tensoring with K we obtain the following exact sequence of K-vector spaces

$$0 \to \ker \varphi_x \otimes K \to \mathcal{O}_{\mathbb{P}^N,x}^{\oplus r} \otimes K \xrightarrow{\varphi_x \otimes id} E_x \otimes K \to 0,$$

where $\varphi_x \otimes id$ is surjective. Since E is locally free of rank r, then clearly $\varphi_x \otimes id$ is an isomorphism, hence $\ker \varphi_x \otimes K = 0$. Since $K \neq 0$, by [AM, Ex.2.3], we have that $\ker \varphi_x = 0$. Thus we have $\mathcal{O}_{\mathbb{P}^N}^{\oplus r} \cong E$.

General Case

If $\pi : X \to \mathbb{P}^n$ is a finite morphism, thus an affine morphism, then $H^i(X, E(-p)) = H^i(\mathbb{P}^n, \pi_* E(-p))$ $\forall\ i \geq 0$ by Theorem 1.24.

(3. $\Rightarrow$ 2.) By hypothesis we have that $\pi_* E$ is trivial, hence $H^i(X, E(-p)) = H^i(\mathbb{P}^n, \pi_* E(-p)) = 0$ $\forall\ i \geq 0$ and for $1 \leq p \leq n$.

(2. $\Rightarrow$ 3.) The vector bundle $F = \pi_* E$ on $\mathbb{P}^n$ is such that $H^i(\mathbb{P}^n, F(-p)) = H^i(X, E(-p)) = 0$ $\forall i \geq 0$ and for $1 \leq p \leq n$. Therefore, by using the case $X = \mathbb{P}^n$, we have that F is trivial.

(1. $\Rightarrow$ 2.) Let $1 \leq p \leq n$ and $i \geq 0$. Tensoring (2.1) by $\mathcal{O}_{\mathbb{P}^N}(-p)$, we obtain the the exact sequence

$$0 \to L_c(-p) \to L_{c-1}(-p) \to \cdots \to L_0(-p) \to E(-p) \to 0 \qquad (2.2)$$

where $\varphi_k : L_{k+1}(-p) \to L_k(-p)$ for $k = 0, \ldots, c-1$ are the morphisms appearing in (2.2). Let us clarify that the sheaf E on $\mathbb{P}^N$ appearing in (2.1), by abuse of notation, is in fact the sheaf $j_* E$, where $j : X \subseteq \mathbb{P}^N$ is the inclusion. Therefore the cohomology of E, considered as a sheaf on $\mathbb{P}^N$, coincides with the cohomology of E as a sheaf on X, by Theorem 1.24. Defining $G_k = \operatorname{Im}(\varphi_k) = \ker(\varphi_{k-1})$, we can extract from (2.2) the following exact sequences:

$$0 \to G_0 \to L_0(-p) \to E(-p) \to 0$$
$$0 \to G_k \to L_k(-p) \to G_{k-1} \to 0, \quad k = 1, \ldots, c-1.$$

Let us prove that $H^{i+k}(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-k-p)) = 0$ for $k = 0, \ldots, c$. If $i + k \neq N$ the statement is trivial; if $i + k = N$ we have $H^{i+k}(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(-k-p)) = H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k+p-N-1))$, which vanishes for $k + p - N - 1 \leq -1$. Recalling that $c - N = -n$, we have $k + p - N - 1 \leq -1 \Leftrightarrow k - N \leq -p$. Now $k - N \leq c - N = -n \leq -p$, hence $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k+p-N-1)) = 0$. This implies that $H^{i+k}(\mathbb{P}^N, L_k(-p)) = 0$ $\forall\ i \geq 0$, for $1 \leq p \leq n$ and for $k = 0, \ldots, c$. Let us now prove that $H^{i+k}(\mathbb{P}^N, G_{k-1}) = 0$ for $k = 1, \ldots, c$:

$$0 \to G_{c-1} \cong L_c(-p) \to L_{c-1}(-p) \to G_{c-2} \to 0$$

is exact, therefore

$$0 = H^{i+c-1}(\mathbb{P}^N, L_{c-1}(-p)) \to H^{i+c-1}(\mathbb{P}^N, G_{c-2}) \to H^{i+c}(\mathbb{P}^N, L_c(-p)) = 0$$

is exact, thus $H^{i+c-1}(\mathbb{P}^N, G_{c-2}) = 0$. Repeating the process from

$$0 \to G_k \to L_k(-p) \to G_{k-1} \to 0$$

we conclude that $H^{i+k}(\mathbb{P}^N, G_{k-1}) = 0$ for $k = c-2, \dots, 1$. In particular $H^{i+1}(\mathbb{P}^N, G_0) = 0$ and, having the exact sequence

$$0 \to G_0 \to L_0(-p) \to E(-p) \to 0$$

we observe that

$$0 = H^i(\mathbb{P}^N, L_0(-p)) \to H^i(\mathbb{P}^N, E(-p)) \to H^{i+1}(\mathbb{P}^N, G_0) = 0$$

is exact, achieving $H^i(\mathbb{P}^N, E(-p)) = 0 \ \forall\ i \geq 0$ and for $1 \leq p \leq n$. Therefore $H^i(X, E(-p)) = 0 \ \forall\ i \geq 0$ and for $1 \leq p \leq n$.

(2. $\Rightarrow$ 1.) Assuming 2. holds, our objective is to define by induction a sequence of 0-regular sheaves K_i on $\mathbb{P}^N$, for $0 \leq i \leq c$, such that:

a) $K_0 = E$;

b) $K_{i+1}(-1)$ is the kernel of the evaluation map $H^0(\mathbb{P}^N, K_i) \otimes \mathcal{O}_{\mathbb{P}^N} \to K_i$;

c) $H^q(\mathbb{P}^N, K_i(-j)) = 0$ for $1 \leq j \leq n+i$ and $\forall\ q \geq 0$.

For $i = 0$, it follows from 2. that E satisfies c) and that E is 0-regular by Proposition 2.1. Let us suppose now that the K_t are defined as requested for $0 \leq t \leq i$; we define K_{i+1} by b). Since K_i is 0-regular by induction hypothesis, applying Theorem 1.45 we get the exact sequence

$$0 \to K_{i+1}(-1) \to H^0(\mathbb{P}^N, K_i) \otimes \mathcal{O}_{\mathbb{P}^N} \to K_i \to 0 \tag{2.3}$$

Tensoring (2.3) with $\mathcal{O}_{\mathbb{P}^N}(1-j)$, we get the following exact sequence

$$0 \to K_{i+1}(-j) \to H^0(\mathbb{P}^N, K_i) \otimes \mathcal{O}_{\mathbb{P}^N}(1-j) \to K_i(1-j) \to 0$$

Taking $q \geq 1$, if we consider the exact sequences

$$H^{q-1}(\mathbb{P}^N, K_i(1-j)) \to H^q(\mathbb{P}^N, K_{i+1}(-j)) \to H^0(\mathbb{P}^N, K_i) \otimes H^q(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1-j))$$
$$0 \to H^0(\mathbb{P}^N, K_{i+1}(-j)) \to H^0(\mathbb{P}^N, K_i) \otimes H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1-j))$$

we have that: by induction hypothesis $H^{q-1}(\mathbb{P}^N, K_i(1-j)) = 0$ for $1 \leq j-1 \leq n+i$, $H^q(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1-j)) = 0$ for $j \leq N+1$ and $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1-j)) = 0$ for $j \geq 2$. Since $i \leq c$ and therefore $n+i+1 \leq n+c+1 = N+1$, we have that $H^q(\mathbb{P}^N, K_{i+1}(-j)) = 0$ for $2 \leq j \leq n+i+1$ and $\forall\, q \geq 0$.

Let us now consider the case $j = 1$. Since $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}) \cong K$, then $H^0(\mathbb{P}^N, K_i) \otimes H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}) \cong H^0(\mathbb{P}^N, K_i)$ and

$$H^0(\mathbb{P}^N, K_i) \otimes H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}) \to H^0(\mathbb{P}^N, K_i)$$

is an isomorphism. Hence $H^q(\mathbb{P}^N, K_{i+1}(-1)) = 0$ for $q = 0, 1$. Furthermore, since $H^{q-1}(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}) = 0$ for $q \geq 2$, from the exact sequence

$$0 \to H^{q-1}(\mathbb{P}^N, K_i) \to H^q(\mathbb{P}^N, K_{i+1}(-1)) \to 0$$

we get that $H^q(\mathbb{P}^N, K_{i+1}(-1)) = H^{q-1}(\mathbb{P}^N, K_i) = 0$ for $q \geq 2$, because K_i is 0-regular and $H^l(\mathbb{P}^N, K_i) = 0\ \forall\, l \geq 1$ by Theorem 1.45.

From (2.3) we also get $H^q(\mathbb{P}^N, K_{i+1}(-q)) = H^{q-1}(\mathbb{P}^N, K_i(-q+1)) = 0$ $\forall\, q > 1$, and, as shown above, $H^1(\mathbb{P}^N, K_{i+1}(-1)) = 0$. Thus K_{i+1} is 0-regular and satisfies c). Put $L_i := H^0(\mathbb{P}^N, K_i) \otimes \mathcal{O}_{\mathbb{P}^N}(-i)$ for $0 \leq i \leq c-1$; the exact sequences

$$0 \to K_{i+1}(-i-1) \to L_i \to K_i(-i) \to 0, \quad 0 \leq i \leq c-1$$

give a long exact sequence

$$0 \to K_c(-c) \to L_{c-1} \to \cdots \to L_0 \to E \to 0$$

Now c) means that $H^q(\mathbb{P}^N, K_c(-j)) = 0$ for $1 \leq j \leq n+c = N$ and $\forall\, q \geq 0$, and, by the case $X = \mathbb{P}^N$, we have that K_c is trivial. Now define $L_c = K_c(-c)$: since $K_c = \mathcal{O}_{\mathbb{P}^N}^{\oplus b_c}$, for some $b_c \geq 1$, we have that $L_c = \mathcal{O}_{\mathbb{P}^N}^{\oplus b_c}(-c)$. □

Definition 2.3. *A vector bundle E is an Ulrich vector bundle if it satisfies one of the equivalent conditions of Theorem 2.2.*

Let us now consider a particular case, that is $X \subseteq \mathbb{P}^N$ is a hypersurface of degree d, given by the equation $F = 0$. We will see that determining whether X can be defined by a determinant of linear forms is equivalent to finding a rank r Ulrich vector bundle on X, and we will consider condition 1. of Theorem 2.2. By Proposition 2.1, we have that $h^0(X, E) = rd$, therefore one sees easily that $b_0 = rd$ (this follows by the linear resolution appearing in condition 1. of Theorem 2.2, which is a short exact sequence in this case).

The following result is taken from [B4, Prop.1].

Proposition 2.4. *Let $X \subseteq \mathbb{P}^N$ be a smooth hypersurface of degree d, given by the equation $F = 0$, and let $r \geq 1$ be an integer. The following conditions are equivalent:*

1. *$F^r = \det L$, where L is a $(rd) \times (rd)$ matrix of linear forms on $\mathbb{P}^N$;*

2. *there exists a rank r vector bundle E on X and an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^N}(-1)^{\oplus rd} \xrightarrow{L} \mathcal{O}_{\mathbb{P}^N}^{\oplus rd} \to E \to 0$.*

Proof. Let us assume that 2. holds. L can be seen as a $(rd) \times (rd)$ matrix (L_{ij}), where $L_{ij} \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$, and we observe that, for all $x \in \mathbb{P}^N$, L_x is the matrix $(\frac{L_{ij}}{1})$. Therefore $\det L = \det L_x$ for all $x \in \mathbb{P}^N$. Let $\mathfrak{p}_x$ be the homogeneous prime ideal associated to x: we denote by $D_{rd-1}(L)$ the set $\{x \in \mathbb{P}^N | \mathrm{rk}(L_x) \leq rd - 1\} = \{x \in \mathbb{P}^N | \det(L_x) \in \mathfrak{p}_x\}$. Now, $x \notin X \iff E_x = 0 \iff L_x$ is an isomorphism $\iff \mathrm{rk}(L_x) = rd$. Therefore X coincides with $D_{rd-1}(L)$. We observe that $\det L \in \cap_{x \in X} \mathfrak{p}_x = \cap_{F \in \mathfrak{p}_x} \mathfrak{p}_x = \sqrt{F} = (F)$, hence $\det L = F \cdot G_1$, for some homogeneous polynomial G_1. Then G_1 must be in (F), otherwise X would be strictly contained in $D_{rd-1}(L)$, thus $\det L = F^2 \cdot G_2$. If we iterate the process, we obtain that $\det L$ is proportional to some power of F, hence F^r for degree reasons.

Let us now assume that 1. holds, and let E be the cokernel of the injective morphism

$$L : \mathcal{O}_{\mathbb{P}^N}(-1)^{\oplus rd} \to \mathcal{O}_{\mathbb{P}^N}^{\oplus rd}.$$

We observe at first that the support of E coincides with X: $x \in \mathrm{Supp}(E) \iff E_x \neq 0 \iff L_x$ is not surjective $\iff \det L \in \mathfrak{p}_x$. Therefore, by hypothesis, $\mathrm{Supp}(E) = \{x \in \mathbb{P}^N | F \in \mathfrak{p}_x\} = X$.

Since $\mathcal{O}_{\mathbb{P}^N,x}$ and $\mathcal{O}_{X,x}$ have the same residue field, that is K, then for all $x \in X$ the depth of the stalk E_x as a $\mathcal{O}_{\mathbb{P}^N,x}$-module or as a $\mathcal{O}_{X,x}$-module is the same. Furthermore we have, for all $x \in X$, a projective resolution

$$0 \to \mathcal{O}_{\mathbb{P}^N,x}(-1)^{\oplus rd} \to \mathcal{O}_{\mathbb{P}^N,x}^{\oplus rd} \to E_x \to 0 \tag{2.4}$$

of E_x, therefore $0 \leq \mathrm{pd}_{\mathcal{O}_{\mathbb{P}^N,x}}(E_x) \leq 1$. Notice that, for $x \in X$, E_x is not a free $\mathcal{O}_{\mathbb{P}^N,x}$-module, because, otherwise, we would have $\mathrm{rk}(E_x) = 0$, hence $E_x = 0$, which is a contradiction. Then, since $\mathcal{O}_{\mathbb{P}^N,x}$ is a local ring, we have that E_x is not a projective $\mathcal{O}_{\mathbb{P}^N,x}$-module, so we have $\mathrm{pd}_{\mathcal{O}_{\mathbb{P}^N,x}}(E_x) = 1$. Now, for all $x \in X$, E_x is a finitely generated $\mathcal{O}_{\mathbb{P}^N,x}$-module, because we have that

E_x is the quotient of a free $\mathcal{O}_{\mathbb{P}^N,x}$-module of finite rank by the exact sequence (2.4). So for all $x \in X$, by [H, Prop.III.6.12A], we have that

$$\operatorname{depth}_{\mathcal{O}_{X,x}}(E_x) = \operatorname{depth}_{\mathcal{O}_{\mathbb{P}^N,x}}(E_x) = \dim \mathcal{O}_{\mathbb{P}^N,x} - \operatorname{pd}_{\mathcal{O}_{\mathbb{P}^N,x}}(E_x) =$$
$$= \dim \mathcal{O}_{\mathbb{P}^N,x} - 1 = \dim \mathcal{O}_{X,x},$$

hence $\operatorname{pd}_{\mathcal{O}_{X,x}}(E_x) = 0$ and E_x is a free $\mathcal{O}_{X,x}$-module, so E is locally free on X. Now we observe that, if $\mathscr{I}_{X/\mathbb{P}^N}$ is the ideal sheaf corresponding to X, then $\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{P}^N,x}/(\mathscr{I}_{X/\mathbb{P}^N})_x$ as rings, and so, since E_x is both a $\mathcal{O}_{X,x}$-module and a finitely generated $\mathcal{O}_{\mathbb{P}^N,x}$-module, we have that E_x is also a finitely generated $\mathcal{O}_{X,x}$-module. This follows by the fact that if $\{f_1, \ldots, f_h\}$ is a set of generators for E_x as a $\mathcal{O}_{\mathbb{P}^N,x}$-module, then $\{(f_1 + (\mathscr{I}_{X/\mathbb{P}^N})_x), \ldots, (f_h + (\mathscr{I}_{X/\mathbb{P}^N})_x)\}$ is a set of generators for E_x as a $\mathcal{O}_{X,x}$-module. Finally we have that E_x is a free $\mathcal{O}_{X,x}$-module of finite rank. Let us say $\operatorname{rk}(E_x) = s$. Therefore E satisfies condition 1. of Theorem 2.2 , which is equivalent to condition 2. of the same Theorem. Hence E satisfies the hypothesis of Proposition 2.1, and so we have $h^0(X, E) = sd$. Now, considering the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-1)^{\oplus rd} \to \mathcal{O}_{\mathbb{P}^N}^{\oplus rd} \to E \to 0$$

we achieve, by the corresponding long exact sequence, that $K^{\oplus rd} \cong K^{\oplus sd}$, so $r = s$. □

We can now look at some consequences of Theorem 2.2 ([B4, Section 2]).

Example 2.5. *Let E be a rank r Ulrich vector bundle on $X \subseteq \mathbb{P}^N$ of degree d and dimension n. Then $H^i(X, E(j)) = 0 \ \forall\ j \in \mathbb{Z}$ and for $0 < i < n$. Furthermore $h^0(X, E) = rd$.*

Proof. By condition 3. of Theorem 2.2 we have that $H^i(X, E(j)) =$ $H^i(\mathbb{P}^n, \pi_* E(j))$, which vanishes for $1 \leq i \leq n-1$ and for all $j \in \mathbb{Z}$. Furthermore it follows by Proposition 2.1 that $h^0(X, E) = rd$. □

Example 2.6. *The Ulrich vector bundles on a curve $C \subseteq \mathbb{P}^N$ are the bundles $E(1)$, where E is a vector bundle with vanishing cohomology.*

Proof. If F is an Ulrich bundle on a curve $C \subseteq \mathbb{P}^N$, by condition 2. of Theorem 2.2 we have that $H^i(C, F(-1)) = 0 \ \forall\ i \geq 0$. Setting $E = F(-1)$, then E has vanishing cohomology and $F = E(1)$. On the other hand, if E is a vector bundle on a curve C with vanishing cohomology, then $H^i(C, (E(1))(-1)) = 0$ $\forall\ i \geq 0$. Therefore $E(1)$ satisfies condition 2. of Theorem 2.2, hence it is an Ulrich bundle. □

Example 2.7. *If E is an Ulrich bundle on $X \subseteq \mathbb{P}^N$ of dimension n and Y is a hyperplane section of X, then $E_{|Y}$ is an Ulrich bundle on Y.*

Proof. Let us take the exact sequence

$$0 \to E(-1) \to E \to E_{|Y} \to 0$$

Tensoring with $\mathcal{O}_X(-j)$ we get that

$$0 \to E(-1-j) \to E(-j) \to E_{|Y}(-j) \to 0$$

and therefore also

$$H^i(X, E(-j)) \to H^i(Y, E_{|Y}(-j)) \to H^{i+1}(X, E(-1-j)) \quad i \geq 0$$

are exact sequences. By condition 2. of Theorem 2.2 it follows that for all $i \geq 0$ we have $H^i(X, E(-j)) = 0$ for $1 \leq j \leq n$ and $H^{i+1}(X, E(-1-j)) = 0$ for $0 \leq j \leq n-1$. Hence $H^i(Y, E_{|Y}(-j)) = 0$ for $1 \leq j \leq n-1 = \dim Y$, proving that $E_{|Y}$ is an Ulrich vector bundle on Y. □

Example 2.8. *Let $\pi : X \to Y$ be a finite surjective morphism, L a very ample line bundle on Y and E a vector bundle on X. Then E is an Ulrich bundle for (X, π^*L) if and only if π_*E is an Ulrich bundle for (Y, L).*

Proof. Since L is very ample, there exists a closed embedding $Y \subseteq \mathbb{P}^N$ such that $L \cong \mathcal{O}_Y(1)$ and $\pi^*L \cong \mathcal{O}_X(1)$. Furthermore $L^{\otimes -k} \cong \mathcal{O}_Y(-k)$. Therefore $E \otimes \pi^*L^{\otimes -k} \cong E(-k) = E \otimes \mathcal{O}_X(-k)$ and $\pi_*E \otimes L^{\otimes -k} \cong (\pi_*E)(-k) = (\pi_*E) \otimes \mathcal{O}_Y(-k)$.
By Theorem 1.24 and by the projection formula [H, Ex.II.5.1(d)], we have $H^\bullet(X, E \otimes \pi^*L^{\otimes -k}) = H^\bullet(Y, \pi_*E \otimes L^{\otimes -k})$. Since π is finite, we have $\dim X = \dim Y$, and then, by condition 2. of Theorem 2.2, we have that E is an Ulrich bundle for (X, π^*L) if and only if π_*E is an Ulrich bundle for (Y, L). □

Example 2.9. *Let E and F be Ulrich vector bundles for $(X, \mathcal{O}_X(1))$ and $(Y, \mathcal{O}_Y(1))$; put $n = \dim X$. Then $E \boxtimes F(n)$ is an Ulrich vector bundle for $(X \times Y, \mathcal{O}_X(1) \boxtimes \mathcal{O}_Y(1))$.*

Proof. By the Künneth formula [SW], we have that

$$H^\bullet(X \times Y, E(-p) \boxtimes F(n-p)) = H^\bullet(X, E(-p)) \otimes H^\bullet(Y, F(n-p))$$

By condition 2. of Theorem 2.2, the first factor vanishes for $1 \leq p \leq n$ and the second one for $1 \leq p - n \leq \dim Y$.
Therefore $H^\bullet(X \times Y, E(-p) \boxtimes F(n-p)) = 0$ for $1 \leq p \leq n + \dim Y = \dim X \times Y$. □

It is useful to notice that Example 2.9 can be generalized for a finite number of vector bundles:

Remark 2.10. *Let E_i be an Ulrich vector bundle for $(X_i, \mathcal{O}_{X_i}(1))$, where $n_i = \dim X_i$, for $i = 0, \dots, m$. Then $E_0 \boxtimes E_1(n_0) \boxtimes \cdots \boxtimes E_m(n_0 + \cdots + n_{m-1})$ is an Ulrich vector bundle for $(X_0 \times \cdots \times X_m, \mathcal{O}_{X_0}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{X_m}(1))$.*

Proof. Let us proceed by induction on m.
For $m = 1$, it is sufficient to apply Example 2.9.
For $m > 1$, by induction hypothesis, we have that $F := E_0 \boxtimes E_1(n_0) \boxtimes \cdots \boxtimes E_{m-1}(n_0 + \cdots + n_{m-2})$ is an Ulrich vector bundle for $(Y := X_0 \times \cdots \times X_{m-1}, \mathcal{O}_Y(1) = \mathcal{O}_{X_0}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{X_{m-1}}(1))$, with $n_0 + \cdots + n_{m-1} = \dim Y$. Applying Example 2.9 to F and E_m we have that $F \boxtimes E_m(\dim Y) = E_0 \boxtimes E_1(n_0) \boxtimes \cdots \boxtimes E_m(n_0 + \cdots + n_{m-1})$ is an Ulrich vector bundle for $(Y \times X_m = X_0 \times \cdots \times X_m, \mathcal{O}_Y(1) \boxtimes \mathcal{O}_{X_m}(1) = \mathcal{O}_{X_0}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{X_m}(1))$. □

Example 2.11. *There exists an Ulrich vector bundle of rank $n!$ for $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ for all $d \geq 1$.*

Proof. Let X be $v_d(\mathbb{P}^1)$, where $v_d : \mathbb{P}^1 \to \mathbb{P}^d$ is the d-uple embedding. If we consider the quotient map $\pi : (\mathbb{P}^1)^n \to \mathrm{Sym}^n\mathbb{P}^1 = \mathbb{P}^n$, whose degree is $n!$, then $\pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ is the sheaf $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$, hence $\pi^*\mathcal{O}_{\mathbb{P}^n}(d) = \mathcal{O}_{\mathbb{P}^1}(d) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^1}(d)$, which is the sheaf $\mathcal{O}_X(1) \boxtimes \cdots \boxtimes \mathcal{O}_X(1)$.
Let us take $E = \mathcal{O}_{\mathbb{P}^1}(d-1)$: we observe that E is Ulrich for $(X, \mathcal{O}_X(1))$, because the sheaf $E(-1) = \mathcal{O}_{\mathbb{P}^1}(d-1) \otimes \mathcal{O}_X(-1) = \mathcal{O}_{\mathbb{P}^1}(d-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-d) = \mathcal{O}_{\mathbb{P}^1}(-1)$ has vanishing cohomology by Theorem 1.21. Therefore, by Remark 2.10, we have that $\mathcal{L} := E \boxtimes E(1) \boxtimes \cdots \boxtimes E(n-1) = \mathcal{O}_{\mathbb{P}^1}(d-1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^1}(nd-1)$ is Ulrich for $((\mathbb{P}^1)^n, \pi^*\mathcal{O}_{\mathbb{P}^n}(d))$, hence, by Example 2.8, we have that $\pi_*\mathcal{L}$ is Ulrich for $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. Lastly, $\mathcal{L}$ is a line bundle, therefore, by Theorem 1.43, we have that $\pi_*\mathcal{L}$ has rank $n!$. □

Example 2.12. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n. If there exists a rank r Ulrich vector bundle for $(X, \mathcal{O}_X(1))$, then there exists a rank $rn!$ Ulrich vector bundle for $(X, \mathcal{O}_X(d))$ for all $d \geq 1$.*

Proof. Let $\pi : X \to \mathbb{P}^n$ be a finite linear projection, and let F be an Ulrich vector bundle of rank $n!$ for $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, which exists by Example 2.11. Let assume E is a rank r Ulrich bundle for $(X, \mathcal{O}_X(1))$: we want to show that the sheaf $G := E \otimes \pi^*F$ is an Ulrich vector bundle for $(X, \mathcal{O}_X(d))$ of rank $rn!$. Let us fix the rank of G equal to s. Clearly the degree of π is equal to $\deg X$,

therefore, by Theorem 1.43, we have that π_*G and π_*E are vector bundles of rank $s \cdot \deg X$ and $r \cdot \deg X$ respectively. Furthermore, by the projection formula [H, Ex.II.5.1(d)], we have that $\pi_*G \cong \pi_*E \otimes F$, whose rank is equal to $r \cdot \deg X \cdot n!$. Hence $s = rn!$. Lastly, in order to satisfy condition 2. of Theorem 2.2, we need to show that $G \otimes \mathcal{O}_X(-dp)$ has vanishing cohomology for $1 \leq p \leq n$.
By condition 3. of Theorem 2.2 we have that π_*E is trivial, that is $\pi_*E = \mathcal{O}_{\mathbb{P}^n}^{\oplus r \cdot \deg X}$, therefore, by the projection formula [H, Ex.II.5.1(d)] and by Theorem 1.24, we have that $H^\bullet(X, G \otimes \mathcal{O}_X(-dp)) = H^\bullet(X, G \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(-dp)) = H^\bullet(\mathbb{P}^n, \pi_*[G \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(-dp)]) = H^\bullet(\mathbb{P}^n, \pi_*G \otimes \mathcal{O}_{\mathbb{P}^n}(-dp)) = H^\bullet(\mathbb{P}^n, \pi_*E \otimes F \otimes \mathcal{O}_{\mathbb{P}^n}(-dp)) = \bigoplus_{r \cdot \deg X} H^\bullet(\mathbb{P}^n, F \otimes \mathcal{O}_{\mathbb{P}^n}(-dp))$, which vanishes for $1 \leq p \leq n$ by hypothesis. □

2.2 Ulrich bundles on curves

Proposition 2.13. *Let E be a rank r vector bundle on a curve $C \subseteq \mathbb{P}^N$ of degree d and genus g. The following conditions are equivalent:*

1. *E is an Ulrich vector bundle.*
2. *$h^0(C, E(-1)) = 0$ and $\deg E = r(d + g - 1)$.*

Proof. Applying the Riemann-Roch theorem to $E(-1)$ we have

$$\chi(C, E(-1)) = \deg E(-1) + r(1 - g) = \deg E - rd + r(1 - g); \tag{2.5}$$

on the other hand,

$$\chi(C, E(-1)) = h^0(C, E(-1)) - h^1(C, E(-1)). \tag{2.6}$$

Now, if 1. holds, by condition 2. of Theorem 2.2 we have $h^0(C, E(-1)) = h^1(C, E(-1)) = 0$. Therefore (2.5) vanishes and this proves 2. If 2. holds, then (2.5) vanishes and by (2.6) we have that $h^1(C, E(-1)) = 0$. Therefore E satisfies condition 2. of Theorem 2.2, then E is Ulrich. □

Definition 2.14. *Let E be a vector bundle on $X \subseteq \mathbb{P}^N$, $\dim X = n$. Then E is arithmetically Cohen-Macaulay (ACM) if $H^i(X, E(t)) = 0\ \forall\ t \in \mathbb{Z}$ and for $0 < i < n$.*

Let d be the degree of X in the embedding defined by $\mathcal{O}_X(1)$. If E is an ACM vector bundle of rank r, then the number $m(E)$ of generators of the graded module $H^0_*(X,E) := \bigoplus_{t\in\mathbb{Z}} H^0(X,E(t))$ is $\leq rd$, by [CS1, Thm.3.1]. Furthermore we notice that every vector bundle on a curve is ACM, since the condition in Definition 2.14 is empty for n=1.

The following result is taken from [CS2, Prop.2.3].

Theorem 2.15. *Let E be a rank r line bundle on a curve $C \subseteq \mathbb{P}^N$ of degree d and genus g. Assume $h^0(C,E(-1)) = 0$. Then:*

1. $h^0(C,E) \leq rd$.
2. $\deg E \leq r(d+g-1)$.
3. $\chi(C,E(n)) \leq rd(n+1)\ \forall\ n \in \mathbb{Z}$.

Furthermore, equality in any of the three conditions implies equality on the other two, and is equivalent to E being an Ulrich bundle.

Proof. From the following exact sequence

$$0 \to E(-1) \to E$$

and the vanishing of $h^0(C,E(-1))$, it is clear that $h^0(C,E(-t)) = 0\ \forall\ t \geq 1$. Therefore $H^0_*(C,E) := \bigoplus_{t\in\mathbb{Z}} H^0(C,E(t)) = \bigoplus_{t\geq 0} H^0(C,E(t))$ and $h^0(C,E) \leq m(E)$. By the previous observation, E is ACM and $m(E) \leq rd$, hence $h^0(C,E) \leq rd$, which gives 1. To prove 2. we apply the Riemann-Roch theorem to $E(-1)$ and we find

$$\chi(C,E(-1)) = \deg E - rd + r(1-g) = -h^1(C,E(-1)) \leq 0.$$

Hence $\deg E \leq r(d+g-1)$. For statement 3., applying the Riemann-Roch theorem to $E(n)$ we have that

$$\chi(C,E(n)) = \deg E + nrd + r(1-g),$$

Then, substituting 2., we have $\chi(C,E(n)) = \deg E + nrd + r(1-g) \leq r(d+g-1) + nrd + r(1-g) = rd(n+1)$. Now, it is clear that equality in 2. is equivalent to equality in 3. Equality in 2. implies that $\chi(C,E(-1)) = 0$, so $H^1(C, E(-1)) = 0$, since $H^0(C, E(-1)) = 0$ by hypothesis. Hence from the exact sequence

$$0 \to E(-1) \to E \to E_{|H} \to 0,$$

where H is a hyperplane section consisting of d points, that is $H = p_1 \sqcup \cdots \sqcup p_d$, we have that

$$0 \to H^0(C, E) \to H^0(C, E_{|H}) \to 0$$

is exact, therefore $h^0(C, E) = h^0(C, E_{|H})$. Since $E_{|H} = E_{|p_1} \oplus \cdots \oplus E_{|p_d} = \mathcal{O}_{C|p_1}^{\oplus r} \oplus \cdots \oplus \mathcal{O}_{C|p_d}^{\oplus r}$, we have $h^0(C, E_{|H}) = \sum_{i=1}^{d} h^0(C, \mathcal{O}_{C|p_i}^{\oplus r}) = \sum_{i=1}^{d} r \cdot h^0(C, \mathcal{O}_{C|p_i}) = rd$, implying that $h^0(C, E) = rd$, which gives equality in 1. Conversely, equality in 1. gives us that $H^0(C, E) = K^{rd}$ and, as previously shown, we also have $H^0(C, E_{|H}) = K^{rd}$. Therefore, using the same exact sequence, we have that

$$0 \to H^0(C, E) \xrightarrow{\varphi} H^0(C, E_{|H}) \xrightarrow{\psi} H^1(C, E(-1)) \xrightarrow{\alpha} H^1(C, E)$$

is exact and that φ is an isomorphism. Hence $\mathrm{Im}(\varphi) = H^0(C, E_{|H}) = \ker(\psi)$, so $\ker(\alpha) = \mathrm{Im}(\psi) = 0$, implying that α is injective. Furthermore, we observe that, since H consists of d points, we have $E_{|H} \cong E_{|H}(n)$, hence $H^0(C, E_{|H}) = H^0(C, E_{|H}(n))\ \forall\ n \geq 0$. Let us consider a dividor D in the linear system $|\mathcal{O}_X(n)|$, which is nonempty for $n \geq 0$, and the following exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

Tensoring with $\mathcal{O}_X(D)$, we get the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D \otimes \mathcal{O}_X(D) \to 0,$$

where $\mathcal{O}_X(D) = \mathcal{O}_X(n)$. Now, if we consider the injective morphism

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(n),$$

tensoring with E, we obtain another injective morphism

$$0 \to E \to E(n),$$

which gives us the inclusion $H^0(C, E) \subseteq H^0(C, E(n))\ \forall\ n \geq 0$. Furthermore we have a commutative diagram of sheaves

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, hook] \arrow[d] & E(n) \arrow[d] \\
0 \arrow[r] & E_{|H} \arrow[r, "\cong"'] & E_{|H}(n)
\end{tikzcd}$$

from which we obtain a commutative diagram of K-vector spaces

$$\begin{tikzcd}
0 \arrow[r] & H^0(C,E) \arrow[r, hook] \arrow[d, "\varphi"] \arrow[dr, "\beta"] & H^0(C,E(n)) \arrow[d, "\varphi(n)"] \\
0 \arrow[r] & H^0(C,E_{|H}) \arrow[r, "\cong"'] & H^0(C,E_{|H}(n))
\end{tikzcd}$$

where $\varphi(n)$ is the map appearing in the following exact sequence:

$$H^0(C,E(n)) \xrightarrow{\varphi(n)} H^0(C,E_{|H}(n)) \xrightarrow{\psi(n)} H^1(C,E(n-1)) \xrightarrow{\alpha(n)} H^1(C,E(n)).$$

Since φ is an isomorphism, we have that β is an isomorphism, hence $\varphi(n)$ is surjective. Eventually we have that $\alpha(n)$ is injective $\forall\ n \geq 0$. We can conclude that $0 \leq h^1(C,E(-1)) \leq \cdots \leq h^1(C,E(n-1)) \leq h^1(C,E(n))$, which vanishes for some $n \gg 0$ by Serre's vanishing theorem (condition 3. of Thm. 1.20). Therefore $h^1(C,E(-1)) = 0$, which implies $\chi(C,E(-1)) = 0$, which gives equality in 2., equivalent to equality in 3.

Moreover, by Proposition 2.13, we have that equality in 2. is equivalent to E being an Ulrich bundle. $\square$

Chapter 3

Ulrich Vector Bundles on Blow-ups

Our objective is to show that there exists an Ulrich vector bundle on the blowing-up $\pi : \widetilde{X} \to X$ of a nonsingular projective variety X along a closed point $P \in X$, given an Ulrich vector bundle $\mathcal{E}$ on $(X, \mathcal{L})$, where $\mathcal{L}$ is a very ample line bundle. In order to achieve our goal, some useful results are needed.

Lemma 3.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on a scheme X. Assume that there exists a surjective morphism $f : \mathcal{F} \to \mathcal{G}$ and that $\mathcal{F}$ is globally generated. Then $\mathcal{G}$ is also globally generated.*

Proof. By hypothesis we can write the following commutative diagram:

$$\begin{tikzcd}
H^0(X,\mathcal{F}) \otimes \mathcal{O}_X \arrow[r, "\alpha"] \arrow[d, "\psi"'] & \mathcal{F} \arrow[d, "f"] \\
H^0(X,\mathcal{G}) \otimes \mathcal{O}_X \arrow[r, "\beta"] & \mathcal{G}
\end{tikzcd},$$

where $\psi := f(X) \otimes id : H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \to H^0(X, \mathcal{G}) \otimes \mathcal{O}_X$. Since α and f are surjective, and since the diagramm is commutative, then it follows that β is also surjective. $\square$

Lemma 3.2. *Let X be a scheme, and let $\mathcal{L}$ be a very ample line bundle on X corresponding to a closed immersion $i : X \to \mathbb{P}^N$. Let $P \in X$ be a closed point. Then the natural restriction map $\varphi : \mathscr{I}_{\{P\}/\mathbb{P}^N} \to i_* \mathscr{I}_{\{P\}/X}$ is a surjective morphism.*

Proof. In order to avoid confusion, let us consider $X \subseteq \mathbb{P}^N$.
If $Q \notin X$, then $(i_*\mathscr{I}_{\{P\}/X})_Q = 0$, so φ_Q is surjective.
For $Q \in X$, let us take into consideration the natural morphism

$$\psi : \mathcal{O}_{\mathbb{P}^N} \to i_*\mathcal{O}_X,$$

which is surjective by hypothesis. Furthermore, since $Q \in X$, we have that $(i_*\mathscr{I}_{\{P\}/X})_Q = (\mathscr{I}_{\{P\}/X})_Q$ and that $(i_*\mathcal{O}_X)_Q = \mathcal{O}_{X,Q}$.
If $Q \neq P$, we have that $(\mathscr{I}_{\{P\}/\mathbb{P}^N})_Q = \mathcal{O}_{\mathbb{P}^N,Q}$ and $(\mathscr{I}_{\{P\}/X})_Q = \mathcal{O}_{X,Q}$, therefore $\varphi_Q = \psi_Q$, which is surjective.
If $Q = P$, we have that $(\mathscr{I}_{\{P\}/\mathbb{P}^N})_P$ and $(\mathscr{I}_{\{P\}/X})_P$ are, respectively, the maximal ideals of $\mathcal{O}_{\mathbb{P}^N,P}$ and $\mathcal{O}_{X,P}$, and that $\varphi_P = (\psi_P)_{|(\mathscr{I}_{\{P\}/\mathbb{P}^N})_P}$. Since ψ_P is a local ring surjective homomorphism, we have that $\psi_P((\mathscr{I}_{\{P\}/\mathbb{P}^N})_P) = (\mathscr{I}_{\{P\}/X})_P$, therefore φ_P is surjective. □

Proposition 3.3. *Let $P \in \mathbb{P}^N$ be a closed point, and let $\mathscr{I}$ be the corresponding sheaf of ideals. Then $\mathscr{I}(1)$ is globally generated for all $N \geq 1$.*

Proof. In order to achieve the expected result, we show that $\mathscr{I}(1)$ is 0-regular, and we will be done by applying Theorem 1.45(1).

We need to show that $H^i(\mathbb{P}^N, \mathscr{I}(1-i)) = 0$ for all $1 \leq i \leq N$. For $i > N$, it follows by Grothendieck's Theorem [H, Thm.III.2.7].
Tensoring

$$0 \to \mathscr{I} \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\{P\}} \to 0,$$

with $\mathcal{O}_{\mathbb{P}^N}(1-i)$, and considering the corresponding long exact sequence, we obtain the following exact sequence:

$$H^{i-1}(\mathbb{P}^N, \mathscr{I}(1-i)) \to H^{i-1}(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1-i)) \xrightarrow{\alpha_{i-1}}$$
$$H^{i-1}(\mathbb{P}^N, \mathcal{O}_{\{P\}}(1-i)) \xrightarrow{\varphi_{i-1}} H^i(\mathbb{P}^N, \mathscr{I}(1-i)) \xrightarrow{\psi_i} H^i(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1-i)).$$

For $i = 1$, we observe that $H^0(\mathbb{P}^N, \mathscr{I}) = H^1(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}) = 0$ and that $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}) = H^0(\mathbb{P}^N, \mathcal{O}_{\{P\}}) = K$. It follows that α_0 is an isomorphism and that

1. $\ker(\varphi_0) = \mathrm{Im}(\alpha_0) = H^0(\mathbb{P}^N, \mathcal{O}_{\{P\}}) \Rightarrow \mathrm{Im}(\varphi_0) = 0$,
2. $\mathrm{Im}(\varphi_0) = \ker(\psi_1) = H^1(\mathbb{P}^N, \mathscr{I})$,

hence $0 = \mathrm{Im}(\varphi_0) = H^1(\mathbb{P}^N, \mathscr{I})$.
For $i \geq 2$, since $\dim \{P\} = 0 < i-1$, we have that $H^{i-1}(\mathbb{P}^N, \mathcal{O}_{\{P\}}(1-i)) = 0$. On the other hand, by Theorem 1.21, we have $H^i(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1-i)) = 0$ for $1 \leq i \leq N$. Therefore $H^i(\mathbb{P}^N, \mathscr{I}(1-i)) = 0$ for $2 \leq i \leq N$. □

Corollary 3.4. *Let X be a scheme, and let $\mathcal{L}$ be a very ample line bundle on X corresponding to a closed immersion $i : X \to \mathbb{P}^N$. Let $P \in X$ be a closed point. Then $\mathscr{I}_{\{P\}/X} \otimes \mathcal{L}$ is globally generated.*

Proof. By Lemma 3.2 and by tensoring with $\mathcal{O}_{\mathbb{P}^N}(1)$, we have a surjective morphism $\mathscr{I}_{\{P\}/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(1) \to i_*\mathscr{I}_{\{P\}/X} \otimes \mathcal{O}_{\mathbb{P}^N}(1)$. Therefore, by Lemma 3.1 and Proposition 3.3, we have that $i_*\mathscr{I}_{\{P\}/X} \otimes \mathcal{O}_{\mathbb{P}^N}(1)$ is globally generated. Furthermore we observe that, by the projection formula [H, Ex.II.5.1(d)], $i_*\mathscr{I}_{\{P\}/X} \otimes \mathcal{O}_{\mathbb{P}^N}(1) \cong i_*(\mathscr{I}_{\{P\}/X} \otimes i^*\mathcal{O}_{\mathbb{P}^N}(1))$.
This implies that $\mathscr{I}_{\{P\}/X} \otimes i^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathscr{I}_{\{P\}/X} \otimes \mathcal{L}$ is globally generated. □

The following result is taken from [BS, Thm.2.1].

Theorem 3.5. *Let $\mathcal{L}$ be a very ample line bundle on a nonsingular projective variety X, and let $Y \subseteq X$ be a closed subscheme corresponding to a sheaf of ideals $\mathscr{I}$ on X. Let $\pi : \widetilde{X} \to X$ be the blowing-up of X with respect to $\mathscr{I}$, and let $E = \pi^{-1}(Y)$ be the exceptional divisor.*
Assume that $\mathscr{I}(t) = \mathcal{L}^{\otimes t} \otimes \mathscr{I}$ is generated by global sections for some positive integer t. Then $(\pi^*\mathcal{L})^{\otimes t'} \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is very ample for $t' \geq t+1$.

Proof. We notice at first that it is sufficient to prove the theorem for $t' = t+1$. Indeed, since $\mathcal{L}$ is globally generated, we have that, for $t' \geq t+1$, $(\pi^*\mathcal{L})^{\otimes t'-t-1}$ is globally generated. Therefore, if $(\pi^*\mathcal{L})^{\otimes t+1} \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is very ample, we have that $(\pi^*\mathcal{L})^{\otimes t'} \otimes \mathcal{O}_{\widetilde{X}}(-E) = [(\pi^*\mathcal{L})^{\otimes t'-t-1}] \otimes [(\pi^*\mathcal{L})^{\otimes t+1} \otimes \mathcal{O}_{\widetilde{X}}(-E)]$ is very ample for $t' \geq t+1$ by [H, Ex.II.7.5(d)]. Let $Q \subseteq |\mathcal{L}^{\otimes t}|$ be the linear system corresponding to the image V of the canonical injecive map

$$H^0(X, \mathscr{I}(t)) \xrightarrow{j} H^0(X, \mathcal{L}^{\otimes t}).$$

We want to show that $Bs(|V|)$ is equal to Y.
For all $s \in V$, $\exists\ \sigma \in H^0(X, \mathscr{I}(t))$ such that $j(\sigma) = s$. Since, by definition of ideal sheaf corresponding to Y, the following sequence is exact

$$0 \to H^0(X, \mathscr{I}(t)) \xrightarrow{j} H^0(X, \mathcal{L}^{\otimes t}) \to H^0(X, \mathcal{L}^{\otimes t} \otimes \mathcal{O}_Y) \to 0,$$

we have that the sections in V vanishes at the all the points in Y.
On the other hand, for all $x \notin Y$, $\mathscr{I}(t)_x \cong (\mathcal{L}^{\otimes t})_x$ and $\mathscr{I}(t)$ is globally generated by hypothesis, therefore $\exists\ \sigma \in H^0(X, \mathscr{I}(t))$ such that $\sigma(x) \neq 0$, hence $s(x) = j(\sigma)(x) \neq 0$ for some $s \in V$.
Let $\varphi : X \dashrightarrow \mathbb{P}^N$ be the rational map associated to $|V|$. Then by Theorem 1.33(1) we have

$$\Gamma_\varphi \cong \widetilde{X} \subseteq X \times \mathbb{P}^N,$$

where Γ_φ is the graph of φ. Note that the line bundle $\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^N}(1)$ is very ample, then, restricting to $\widetilde{X}$ and using Theorem 1.33(2), we have that

$$\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^N}(1)_{|\widetilde{X}} \cong \pi^*\mathcal{L} \otimes (\varphi \circ \pi)^*\mathcal{O}_{\mathbb{P}^N}(1) \cong (\pi^*\mathcal{L})^{\otimes t+1} \otimes \mathcal{O}_{\widetilde{X}}(-E)$$

is very ample. □

Corollary 3.6. *Let $\mathcal{L}$ be a very ample line bundle on a nonsingular projective variety X, and let $P \in X$ be a closed point corresponding to a sheaf of ideals $\mathscr{I}$ on X. Let $\pi : \widetilde{X} \to X$ be the blowing-up of X with respect to $\mathscr{I}$, and let $E = \pi^{-1}(\{P\})$ be the exceptional divisor. Then $(\pi^*\mathcal{L})^{\otimes t} \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is a very ample line bundle on $\widetilde{X}$ for all $t \geq 2$.*

Proof. By Corollary 3.4, $\mathcal{L} \otimes \mathscr{I}$ is globally generated. Then, by applying Theorem 3.5, we achieve our goal. □

Theorem 3.7. *Let $\mathcal{L}$ be a very ample line bundle on a nonsingular projective variety X. Let $P \in X$ be a closed point corresponding to a sheaf of ideals $\mathscr{I}$ on X, and let $\pi : \widetilde{X} \to X$ be the blowing-up of X with respect to $\mathscr{I}$. If there exists an Ulrich vector bundle for $(X, \mathcal{L})$, then there exists an Ulrich vector bundle for $(\widetilde{X}, \pi^*\mathcal{L}^{\otimes 2} \otimes \mathcal{O}_{\widetilde{X}}(-E))$, where $E = \pi^{-1}(\{P\})$ is the exceptional divisor.*

Proof. Let us fix $n = \dim X$. By Example 2.12, there exists an Ulrich vector bundle $\mathcal{F}$ for $(X, \mathcal{L}^{\otimes 2})$. We aim to prove that $\pi^*\mathcal{F} \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is an Ulrich vector bundle for $(\widetilde{X}, \pi^*\mathcal{L}^{\otimes 2} \otimes \mathcal{O}_{\widetilde{X}}(-E))$. Indeed:

$$(\pi^*\mathcal{F} \otimes \mathcal{O}_{\widetilde{X}}(-E)) \otimes (\pi^*\mathcal{L}^{\otimes 2} \otimes \mathcal{O}_{\widetilde{X}}(-E))^{\otimes(-p)} = \pi^*(\mathcal{F} \otimes \mathcal{L}^{\otimes(-2p)}) \otimes \mathcal{O}_{\widetilde{X}}((p-1)E),$$

and by Proposition 1.32 we have that

$$H^i(\widetilde{X}, \pi^*(\mathcal{F} \otimes \mathcal{L}^{\otimes(-2p)}) \otimes \mathcal{O}_{\widetilde{X}}((p-1)E)) = H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes(-2p)})$$

for all $i \geq 0$ and for $0 \leq p-1 \leq n-1$. Therefore, since $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes(-2p)}) = 0$ for all $i \geq 0$ and for $1 \leq p \leq n$ by applying to $\mathcal{F}$ condition 2. of Theorem 2.2, and since $\dim X = \dim \widetilde{X}$, we achieve the expected result. □

Corollary 3.8. *If every minimal surface carries an Ulrich vector bundle, then there exists an Ulrich vector bundle on every nonsingular projective surface.*

Proof. It is sufficient to apply both Corollary 1.37 and Theorem 3.7. □

Let us remark that a similar result to Corollary 3.8 is proved in [ACM, Cor.3.8], while Theorem 3.7 provides an enhancement of [K, Thm.0.1].

We now take into account some applications of Corollary 3.8. By Theorem 1.40, we know that every minimal surface with Kodaira dimension equal to 0 can be classified by looking at the pair (p_g, q) in $\{(0,0),(0,1),(1,0),(1,2)\}$. An Ulrich vector bundle has been found for each of the four cases above, so we achieve the following:

Corollary 3.9. *Let S be a nonsingular projective surface such that $k(S) = 0$. Then S admits an Ulrich vector bundle.*

Proof. By [B2], [B3] (see also [C]), [F] and [B4], we obtain, respectively, that Abelian, Enriques, K3 and bielliptic surfaces admit an Ulrich vector bundle. Those surfaces are the minimal models for all nonsingular projective surfaces with Kodaira dimension 0, by Theorem 1.40. Therefore it is sufficient to apply both Corollary 1.37 and Theorem 3.7. □

With regard to the minimal surfaces with Kodaira dimension $-\infty$ (Proposition 1.42), we can find some cases in which an Ulrich vector bundle is admitted. If S_0 is equal to $\mathbb{P}^2$ or $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, then S_0 admits an Ulrich vector bundle (for instance, respectively, $\mathcal{O}_{\mathbb{P}^2}$ and $\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$). Let us suppose that $S_0 = \mathbb{F}_n$, $n > 1$ (recall that $\mathbb{F}_n = \mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ and note that in this case the invariant e is equal to n), or $S_0 = \mathbf{P}_C(E)$, where E is a rank 2 vector bundle over a nonsingular projective curve C of genus g, with invariant $e > 0$. Let $\pi : S_0 \to C$ be the associated morphism, F be the fiber of π over a fixed point $p \in C$, and let us denote by C_0 the section with self-intersection $-e$. Take $A = C_0 + bF$ with

$$b > \max\{e, (g-1+e)/2\}. \tag{3.1}$$

Since $b > e$, we deduce that A is ample by [H, Prop.V.2.20], hence there exists $m_0 \geq 0$ such that $H := mA = mC_0 + mbF$ is very ample for all $m \geq m_0$. Let us also assume that $m > 3$. In order to prove the existence of an Ulrich vector bundle on S_0, it is sufficient to verify that the very ample line bundle

H on S_0 satisfies condition (3) of [ACM, Thm.3.4], which translates into the following:

$$2mb > \max\{(m-3)(g-1)+me, g-1+me\}. \tag{3.2}$$

If $g \geq 1$, then (3.2) is equivalent to

$$2b > (1-3/m)(g-1)+e,$$

therefore $(1-3/m)(g-1)+e \leq g-1+e < 2b$ by the hypothesis (3.1) on b. If $g=0$, then condition (3.2) is equivalent to $2b > -1/m+e$, and it is satisfied since $b > e$ by the hypothesis (3.1) on b.
So H satisfies condition (3.2), and therefore there exists an Ulrich vector bundle for (S_0, H) by [ACM, Thm.3.4].

Corollary 3.10. *Let S be a nonsingular projective surface with Kodaira dimension $k(S) = -\infty$, and let us assume that the minimal model of S is within one of the following:*

1. *$\mathbb{P}^2$,*
2. *$\mathbb{F}_n$ for $n \geq 0$ and $n \neq 1$,*
3. *$\mathbf{P}_C(E)$ for a rank 2 vector bundle E over a nonsingular projective curve C, with invariant $e > 0$.*

Then S admits an Ulrich vector bundle.

Proof. By the previous observations, it is sufficient to apply both Corollary 1.37 and Theorem 3.7. □

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