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Graduation Thesis in Mathematics

by

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On the properties of the Integral Part of Ample and Big Divisors

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A chi dall'alto scruta i nostri passi,
dandoci la forza ed il coraggio
di non fermarci...
mai.

To whom from overhead observes our steps,
giving us the force and the courage
not to stop to us...
never.

Contents

Introduction

For centuries, men have been intrigued by the interplay between algebra and geometry. The ancient Greeks established such a link when they found straightedge-and-compass constructions for the sum, difference, product, quotient and square root of lengths. The next step has been the invention of conics, the first curves that they thoroughly studied after straight lines and circles to solve algebraic problems (intersection of curves for solving equations). Besides planes and spheres, the Greeks also studied some surfaces of revolution, such as cones, cylinders, a few types of quadrics and even tori. Possibly the single greatest step in connecting up algebra and geometry was Descartes introduction in 1637 of *Cartesian geometry* (or *Analytic geometry*). It laid the mathematical foundation for the calculus and the Newtonian physics a half century later.
In the first years of the 18th century a new era begins with the simultaneous introduction of points at infinity and of imaginary points: "geometry" will now, for almost 100 years, for many mathematicians mean geometry in the complex projective plane $\mathbb{P}_2(\mathbb{C})$ or the complex projective 3-dimensional space $\mathbb{P}_3(\mathbb{C})$.

From the projective geometry of Möbius, Plücker and Cayley to the Riemann's birational geometry there has been many different approaches to algebraic geometry. Riemann began a revolution with his introduction of Riemann's surfaces, Hilbert's nonconstructive proof in 1888 of the Nullstellensatz was wildly new. More recently the school including Weil, Chevalley and Zariski revolutionized algebraic geometry by breaking away from the constraints of working over $\mathbb{C}$, replacing $\mathbb{C}$ by an arbitrary ground-field. The

1950's found a new storm brewing, one that ended up absorbing into algebraic geometry nearly all of commutative algebra, and topological notions such as fibre bundles, sheaves and various cohomological theories. Following a suggestion of Cartier, A. Grothendieck undertook around 1957 a gigantic program aiming at a vast generalization of algebraic geometry, absorbing all previous developments and starting from the category of all commutative rings (with unit) instead of reduced finitely generated algebras over an algebraically closed field.

Part of the modern studies are concerned around linear series, that have long stood at the centre of algebraic geometry. Around 1890, the Italian school of algebraic geometry, under the leadership of a trio of great geometers: Castelnuovo, Enriques and (slightly later) Severi, embarked upon a program of study of algebraic surfaces (and later higher dimensional varieties) generalizing the Brill-Noether approach via linear systems: they chiefly worked with purely geometric methods, such as projections or intersections of curves and surfaces in projective space, with as little use as possible of methods belonging either to analysis and topology, or to "abstract" algebra.

Systems of divisors were employed classically to study and define invariants of projective varieties, and it was recognized that varieties share many properties with their hyperplane sections. The classical picture was greatly clarified by the revolutionary new ideas that entered the field starting in the 1950s. To begin with, Serre's great paper (*"Faisceaux algébriques cohérents"*), along with the work of Kodaira (e.g. *"On a differential-geometric method in the theory of analytic stacks"*), brought into focus the importance of amplitude for line bundles.

By the mid 1960s a very beautiful theory was in place, showing that one could recognize positivity geometrically, cohomologically, or numerically. During the same years, Zariski and others began to investigate the more complicated behaviour of linear series defined by line bundles that may not be ample and the theory of $\mathbb{Q}$ and $\mathbb{R}$-divisors (e.g. *"The theorem of Riemann-*

Roch for high multiples of an effective divisor on an algebraic surface").

In this work we will concentrate on algebraic varieties.

Definition 0.0.1. An *abstract variety* is an integral separated scheme of finite type over an algebraically closed field k. If it is proper over k, we will also say it is *complete*.

One of the usual ways for studying properties of algebraic varieties is that of using divisors,

Definition 0.0.2 (Weil divisors). Let X be a noetherian integral separated scheme such that every local ring $\mathcal{O}_{X,x}$ of X of dimension 1 is regular ([Har77]). A *prime divisor* on X is a closed integral subscheme Y of codimension one. A *Weil divisor* is an element of the free abelian group $\mathrm{WDiv}(X)$ generated by the prime divisors. We write a divisor as a finite sum $D = \sum n_i Y_i$ where the Y_i are prime divisors and the n_i are integers. If all the $n_i \geq 0$, we say that D is effective. [Def.1.2.2]

Similarly to Weil divisors we will consider *Cartier divisors* and we will connect those definitions to that of *line bundle*.

Let us denote $\mathrm{Div}(X)$ the group of all Cartier divisors. [Def.1.2.1]

A powerful method to study algebraic varieties is to embed them, if possible, into some projective space $\mathbb{P}^n$.

As it is well known ([Har77]) morphisms in projective spaces are given by line bundles and chosen sections.

Definition 0.0.3. We define a line bundle $\mathscr{L}$ on X to be *very ample* if there is an immersion $i : X \to \mathbb{P}^n$ for some n such that $\mathscr{L} \cong i^*(\mathcal{O}(1))$. $\mathscr{L}$ is said to be *ample* if there exists an integer $m > 0$ such that $\mathscr{L}^{\otimes m}$ is very ample. [Def.2.1.2]

Definition 0.0.4. X is a *projective variety* if it is proper and has an ample line bundle.

The central part of this work is the study of the properties of ampleness for $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$-divisors.
Following well known facts [Laz04a], we will first give a global proof of equivalence of all the already known characterizations of ampleness.
Before stating it we introduce the following notation:

- $\int_V D_1 \cdot \ldots \cdot D_k \overset{\text{not}}{=} (D_1 \cdot \ldots \cdot D_k \cdot V)$ is the *intersection number.* [Sec.1.3]

- $N^1(X) = \text{Div}X/\text{Num}X$ is the *Néron-Severi group* of X, group of numerical equivalence classes of divisors on X, where two divisors $D_1,\ D_2$ are numerically equivalent if $(D_1.C) = (D_2.C)$ for every irreducible curve $C \subseteq X$. [Sec.1.3]

- $\overline{\text{NE}}(X) = \overline{\{\sum a_i[C_i] | C_i \subset X \text{an irreducible curve}, a_i \in \mathbb{R}, a_i \geq 0\}}$. [Def.2.3.5]

Proposition 0.0.5 (Ampleness for $\mathbb{Z}$-divisors). Let $D \in \text{Div}(X)$ be an integral Cartier divisor on a normal projective variety X, and let $\mathcal{O}_X(D)$ be the associated line bundle (sometimes we will think D as a Weil divisor by the canonical correspondence (1.2.5)). The following statements are equivalent: [Pro.2.4.1]

1. There exists a positive integer m such that $\mathcal{O}_X(mD)$ is very ample;

2. Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_1 = m_1(\mathscr{F})$ having the property that

$$H^i(X, \mathscr{F} \otimes \mathcal{O}_X(mD)) = 0 \quad \forall i > 0,\ m \geq m_1;$$

3. Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_2 = m_2(\mathscr{F})$ such that $\mathscr{F} \otimes \mathcal{O}_X(mD)$ is globally generated $\forall m \geq m_2$;

4. There is a positive integer m_3 such that $\mathcal{O}_X(mD)$ is very ample $\forall m \geq m_3$;

5. For every subvariety $V \subseteq X$ of positive dimension, there is a positive integer $m = m(V)$, together with a non-zero section $0 \neq s = s_V \in H^0(V, \mathcal{O}_V(mD))$, such that s vanishes at some point of V;

6. For every subvariety $V \subseteq X$ of positive dimension, $\chi(V, \mathcal{O}_V(mD)) \to +\infty$ as $m \to +\infty$;

7. **(Nakai-Moishezon-Kleiman criterion)**
$$\int_V c_1(\mathcal{O}_X(D))^{\dim(V)} > 0$$
for every positive-dimensional subvariety $V \subseteq X$;

8. **(Seshadri's criterion)** There exists a real number $\varepsilon > 0$ such that
$$\frac{(D.C)}{\text{mult}_x C} \geq \varepsilon$$
for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x;

9. Let H be an ample divisor. There exists a positive number $\varepsilon > 0$ such that
$$\frac{(D.C)}{(H.C)} \geq \varepsilon$$
for every irreducible curve $C \subseteq X$;

10. **(Via cones)** $\overline{\text{NE}}(X) - \{0\} \subseteq D_{>0} = \{\gamma \in N_1(X)_{\mathbb{R}} | (D \cdot \gamma) > 0\}$.

11. There exists a neighborhood U of $[D]_{num} \in \text{N}^1(X)_{\mathbb{R}}$ such that $U \backslash \{[D]_{num}\} \subseteq \text{Amp}(X)$.

For positivity questions, it is very useful to discuss small perturbation of given divisors. The natural way to do so is through the formalism of $\mathbb{Q}$ and $\mathbb{R}$-divisors:

Definition 0.0.6. Let X be an algebraic variety. A Cartier $\mathbb{R}$-*divisor* on X is an element of the $\mathbb{R}$-vector space
$$\text{Div}_{\mathbb{R}}(X) \stackrel{\text{def}}{=} \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Equivalently $D \in \text{Div}_{\mathbb{R}}(X) \Leftrightarrow D = \sum c_i D_i |\ c_i \in \mathbb{R},\ D_i \in \text{Div}(X)$.

The study of those new classes of divisors began in the first part of the 80's. They are fundamental in the birational study of algebraic varieties, in particular for some vanishing theorems as the vanishing theorem of Kawamata and Viehweg ([Laz04b], Theorems 9.1.18 - 9.1.20 - 9.1.21) (also note that there exist singular varieties in which the canonical divisor is a $\mathbb{Q}$-divisor).

Our aim will be to give a characterization of ampleness for those two new classes of divisors similar to that one for $\mathbb{Z}$-divisors, again following well-known results.
Some properties of Proposition 0.0.5 (6-11) dipend only upon the numerical class of D and it is well-known that they characterize ampleness [Laz04a].
On the other hand, some properties of integral divisors that characterize ampleness are connected with the concept of line bundle associated to a divisor (1-5 of Proposition 0.0.5), that is defined only for integral divisors. To overcome this problem there are different possible ways. In this work we have chosen to substitute any real divisor by its integral part any time we had to consider the associated line bundle.

Given an $\mathbb{R}$-divisor $D = \sum_i a_i D_i$ $a_i \in \mathbb{R}$, $D_i \in \mathrm{Div}(X)$ prime divisors, we define its integral part as

$$[D] = \sum_i [a_i] D_i \in \mathrm{Div}(X).$$

Definition 0.0.7 (Amplitude for $\mathbb{Q}$ and $\mathbb{R}$-divisors). A $\mathbb{Q}$-divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ (resp. $\mathbb{R}$-divisor $D \in \mathrm{Div}_{\mathbb{R}}(X)$) is said to be *ample* if it can be written as a finite sum [Def.2.1.9]

$$D = \sum c_i A_i$$

where $c_i > 0$ is a positive rational (resp. real) number and A_i is an ample Cartier divisor.

One of the original contributions of this work has been to obtain a characterization of ampleness whenever the $\mathbb{Q}$ or $\mathbb{R}$-divisor is replaced by its integral *part. This is summarized in the following results:*

Proposition 0.0.8 (Ampleness for $\mathbb{Q}$-divisors). Let $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ be a Cartier divisor on a normal projective variety X, and let $\mathcal{O}_X([D])$ be the associated line bundle (sometimes we will think $[D]$ as a Weil divisor by the canonical correspondence). The following statements are equivalent to the definition of ampleness for $\mathbb{Q}$-divisors: [Pro.2.5.3]

I) Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_1 = m_1(\mathscr{F})$ having the property that

$$H^i(X, \mathscr{F} \otimes \mathcal{O}_X([mD])) = 0 \quad \forall i > 0,\ m \geq m_1;$$

II) Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_2 = m_2(\mathscr{F})$ such that $\mathscr{F} \otimes \mathcal{O}_X([mD])$ is globally generated $\forall m \geq m_2$;

III) There is a positive integer m_3 such that $\mathcal{O}_X([mD])$ is very ample $\forall m \geq m_3$;

IV) For every subvariety $V \subseteq X$ of positive dimension, there is a positive integer $m_4 = m_4(V)$, such that for every $m \geq m_4$ there exists a non-zero section $0 \neq s = s_{V,m} \in H^0(V, \mathcal{O}_V([mD]))$, such that s vanishes at some point of V;

V) For every subvariety $V \subseteq X$ of positive dimension, $\chi(V, \mathcal{O}_V([mD])) \to +\infty$ as $m \to +\infty$;

VI) **(Nakai-Moishezon-Kleiman criterion)**

$$\int_V c_1(\mathcal{O}_X(D))^{\dim(V)} > 0$$

for every positive-dimensional subvariety $V \subseteq X$;

VII) **(Seshadri's criterion)** There exists a real number $\varepsilon > 0$ such that

$$\frac{(D.C)}{\mathrm{mult}_x C} \geq \varepsilon$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x;

VIII) Let H be an ample divisor. There exists a positive number $\varepsilon > 0$ such that
$$\frac{(D.C)}{(H.C)} \geq \varepsilon$$
for every irreducible curve $C \subseteq X$;

IX) **(Via cones)** $\overline{\mathrm{NE}}(X) - \{0\} \subseteq D_{>0}$.

X) There exists a neighborhood U of $[D]_{num} \in \mathrm{N}^1(X)_{\mathbb{R}}$ such that $U \backslash \{[D]_{num}\} \subseteq \mathrm{Amp}(X)$.

Remarks 0.0.9.

1. The equivalences VI-X with the concept of ampleness where already known, the equivalences I-V with the concept of ampleness are original.

2. It is easy to find examples where (1) and (5) of Proposition 0.0.5 don't hold if we use the integral part.

3. To extend property (5) of Proposition 0.0.5 we have chosen to substitute the existence of "$m(V)$" by "$\forall m \geq m_4(V)$".

For $\mathbb{Q}$-divisors it has been quite easy to extend the properties because we have been helped by the existence, for a $\mathbb{Q}$-divisor D, of an integer k such that $kD \in \mathrm{Div}(X)$. Over $\mathbb{R}$ there are serious difficulties, however we have been able to prove the following statements:

Proposition 0.0.10. Let $D \in \mathrm{Div}_{\mathbb{R}}(X)$ be a Cartier divisor on a normal projective variety X, and let $\mathcal{O}_X([D])$ be the associated line bundle (sometimes we will think $[D]$ as a Weil divisor by the canonical correspondence). Consider the following properties: [Pro.2.6.4]

i) Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_2 = m_2(\mathscr{F})$ such that $\mathscr{F} \otimes \mathcal{O}_X([mD])$ is globally generated $\forall m \geq m_2$;

ii) **(Nakai-Moishezon-Kleiman criterion)**

$$\int_V c_1(\mathcal{O}_X(D))^{\dim(V)} > 0$$

for every positive-dimensional subvariety $V \subseteq X$;

iii) **(Seshadri's criterion)** There exists a real number $\varepsilon > 0$ such that

$$\frac{(D.C)}{\mathrm{mult}_x C} \geq \varepsilon$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x;

iv) Let H be an ample divisor. There exists a positive number $\varepsilon > 0$ such that

$$\frac{(D.C)}{(H.C)} \geq \varepsilon$$

for every irreducible curve $C \subseteq X$;

v) **(Via cones)** $\overline{\mathrm{NE}}(X) - \{0\} \subseteq D_{>0}$.

vi) There exists a neighborhood U of $[D]_{num} \in \mathrm{N}^1(X)_{\mathbb{R}}$ such that $U \backslash \{[D]_{num}\} \subseteq \mathrm{Amp}(X)$.

vii) Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_1 = m_1(\mathscr{F})$ having the property that

$$H^i(X, \mathscr{F} \otimes \mathcal{O}_X([mD])) = 0 \quad \forall i > 0,\ m \geq m_1;$$

viii) There is a positive integer m_3 such that $\mathcal{O}_X([mD])$ is very ample $\forall m \geq m_3$;

ix) For every subvariety $V \subseteq X$ of positive dimension, there is a positive integer $m_4 = m_4(V)$, such that for every $m \geq m_4$ there exists a non-zero section $0 \neq s = s_{V,m} \in H^0(V, \mathcal{O}_V([mD]))$, such that s vanishes at some point of V;

x) For every subvariety $V \subseteq X$ of positive dimension, $\chi(V, \mathcal{O}_V([mD])) \to +\infty$ as $m \to +\infty$;

Then:

- (i)-(vi) are equivalent to the concept of ampleness for $\mathbb{R}$-divisors.
- Ampleness implies either one of (vii)-(x).

Remarks 0.0.11.

- It remains an open problem the equivalence of (iii) with ample, however we have been able to prove the equivalence over a surface (Remark 2.6.3).
- The equivalences (ii)-(vi) with the concept of ampleness where already known, the equivalence (i) is original.
- The fact that ampleness implies (vii)-(x) is original.

In the third chapter we introduced the *big* divisors, that have played an important role in the last twenty years. Sometimes it is very difficult to distinguish an ample from a big divisor and for this we have searched a characterization of bigness similar to that one of ampleness, again replacing divisors by integral part when needed.

Definition 0.0.12 (Big). A line bundle $\mathscr{L}$ on a projective variety X is *big* if $\kappa(X, \mathscr{L}) = \dim X$. A Cartier divisor D on X is *big* if $\mathcal{O}_X(D)$ is so. [Def.3.3.1]

Definition 0.0.13 (Big $\mathbb{R}$-divisors). An $\mathbb{R}$-divisor $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is big if it can written in the form [Def.3.3.7]

$$D = \sum a_i \cdot D_i$$

where each D_i is a big integral divisor and a_i is a positive real number.

Proposition 0.0.14 (Bigness for $\mathbb{R}$-divisors). Let D be an $\mathbb{R}$-divisor on a projective variety X. The following are equivalent: [Pro. 3.3.14]

(i) D is big;

(ii) there exists an integer $a \in \mathbb{N}$ such that $\varphi_{|[mD]|}$ is birational for all $m \in \mathbf{N}(X, D)_{\geq a}$;

(iii) $\varphi_{|[mD]|}$ is generically finite for some $m \in \mathbf{N}(X, D)$;

(iv) for any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m = m(\mathscr{F})$ such that $\mathscr{F} \otimes \mathcal{O}_X([mD])$ is generically globally generated, that is such that the natural map

$$H^0(X, \mathscr{F} \otimes \mathcal{O}_X([mD])) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathscr{F} \otimes \mathcal{O}_X([mD])$$

is generically surjective;

(v) for any ample $\mathbb{R}$-divisor A on X, there exists an effective $\mathbb{R}$-divisor N such that $D \equiv_{num} A + N$;

(vi) same as in (v) for some ample $\mathbb{R}$-divisor A;

Section 3.4 is an overview of the theory of cones of big divisors. In particular these are used in the proof of *Nakai-Moishezon criterion for* $\mathbb{R}$-*divisors.*

The last section is a very rapid introduction to the volume's theory. It is the natural development of this work: a new birational invariant for the study of algebraic varieties.

Definition 0.0.15. Let X be an irreducible projective variety of dimension n, and let $\mathscr{L}$ be a line bundle on X. The *volume* of $\mathscr{L}$ is defined to be the non-negative real number [Def.3.5.1]

$$\mathrm{vol}(\mathscr{L}) = \mathrm{vol}_X(\mathscr{L}) = \limsup_{m \to \infty} \frac{h^0(X, \mathscr{L}^{\otimes m})}{n^n/n!}.$$

The volume $\mathrm{vol}(D) = \mathrm{vol}_X(D)$ of a Cartier divisor D is defined passing to $\mathcal{O}_X(D)$.

Chapter 1

Divisors and Intersection Theory

1.1 Notation and Conventions

- We work throughout over the complex numbers $\mathbb{C}$.
- A *scheme* is a separated complete algebraic scheme of finite type over $\mathbb{C}$.
- A *variety* is a reduced and irreducible scheme. We deal exclusively with closed points of schemes.
- Throughout all this work we will consider a complex variety X.

1.2 Integral Divisors

We will denote $\mathcal{M}_X = \mathbb{C}(X)$ the constant sheaf of rational functions on X. It contains the structure sheaf $\mathcal{O}_X$ as a subsheaf, and so there is an inclusion $\mathcal{O}_X^* \subseteq \mathcal{M}_X^*$ of sheaves of multiplicative abelian groups.

Definition 1.2.1 (Cartier divisors). A *Cartier divisor* on X is a global section of the quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$. We denote by $\mathrm{Div}(X)$ the group of all such, so that

$$\mathrm{Div}(X) = \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*).$$

A Cartier divisor $D \in \mathrm{Div}(X)$ can be described by giving an open cover $\{U_i\}$ of X, and for each i an element $f_i \in \Gamma(U_i, \mathcal{M}_X^*)$, such that for each i, j,

$$f_i = g_{ij} f_j \text{ for some } g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*).$$

The function f_i is called a *local equation* for D at any point $x \in U_i$. Two such collections determine the same Cartier divisor if there is a common refinement $\{V_k\}$ of the open coverings on which they are defined so that they are given by data $\{(V_k, f_k)\}$ and $\{(V_k, f'_k)\}$ with

$$f_k = h_k f'_k \text{ on } V_k \text{ for some } h_k \in \Gamma(V_k, \mathcal{O}_X^*).$$

If $D, D' \in \mathrm{Div}(X)$ are represented respectively by data $\{(U_i, f_i\}$ and $\{U_i, f'_i\}$, then $D + D'$ is given by $\{U_i, f_i f'_i\}$. The *support* of a divisor $D = \{U_i, f_i\}$ is the set of points $x \in X$ at which a local equation of D at x is not a unit in $\mathcal{O}_{X,x}$. D is effective if $f_i \in \Gamma(U_i, \mathcal{O}_X)$ is regular on U_i.

Definition 1.2.2 (Weil divisors). Let X be a noetherian integral separated scheme such that every local ring $\mathcal{O}_{X,x}$ of X of dimension 1 is regular ([Har77]). A *prime divisor* on X is a closed integral subscheme Y of codimension one. A *Weil divisor* is an element of the free abelian group $\mathrm{WDiv}(X)$ generated by the prime divisors. We write a divisor as a finite sum $D = \sum n_i Y_i$ where the Y_i are prime divisors, the n_i are integers. If all the $n_i \geq 0$, we say that D is effective.

If D is a prime divisor on X, let $\eta \in D$ be its generic point. Then the local ring $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field K, the function field of X. We call the corresponding valuation v_D the *valuation of D*. Now let $f \in K^*$ be a non-zero rational function on X. Then $v_D(f)$ is an integer.

Definition 1.2.3. We define the *divisor* of f, denoted $\operatorname{div}(f)$, by

$$\operatorname{div}(f) = \sum v_D(f) \cdot D,$$

where the sum is taken over all prime divisors on X. Any divisor which is equal to the divisor of a function is called a *principal* divisor.

Definition 1.2.4. Two Weil divisors D and D' are said to be *linearly equivalent*, written $D \equiv_{lin} D'$, if $D - D'$ is a principal divisor.

Proposition 1.2.5 (Weil & Cartier). ([Har77] II.6.11) Let X be an integral, separated, noetherian scheme, all of whose local rings are unique factorization domains. The group $\operatorname{WDiv}(X)$ of Weil divisors on X is isomorphic to the group of Cartier divisors $\operatorname{Div}(X)$, and furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.

Another important aspect of divisors' theory is the relation with the concept of line bundles.

A Cartier divisor $D \in \operatorname{Div}(X)$ determines a line bundle $\mathcal{O}_X(D)$ on X leading to a canonical homomorphism

$$\operatorname{Div}(X) \to \operatorname{Pic}(X) \quad , \quad D \mapsto \mathcal{O}_X(D)$$

of abelian groups, where $\operatorname{Pic}(X)$ denotes the Picard group of isomorphism classes of line bundles on X.

If D is given by the data $\{U_i, f_i\}$, then one can build $\mathcal{O}_X(D)$ by using the g_{ij} of Definition 1.2.1 as transition functions.

One can also view $\mathcal{O}_X(D)$ as the image of D under the connecting homomorphism

$$\operatorname{Div}(X) = \Gamma(\mathcal{M}_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) = \operatorname{Pic}(X)$$

determined by the exact sequence $0 \to \mathcal{O}_X^* \to \mathcal{M}_X^* \to \mathcal{M}_X^*/\mathcal{O}_X^* \to 0$ of sheaves on X, where

$$\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \Leftrightarrow D_1 \equiv_{lin} D_2.$$

If D is effective then $\mathcal{O}_X(D)$ carries a non-zero global section $s = s_D \in \Gamma(X, \mathcal{O}_X(D))$ with $\mathrm{div}(s) = D$. In general $\mathcal{O}_X(D)$ has a rational section with the analogous property.

Note 1.2.6. There are natural hypotheses to guarantee that every line bundle arises from a divisor:

- If X is reduced and irreducible, then the homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is surjective.
- If X is projective then the same statement holds even if it is non-reduced.

Let $\mathscr{L}$ be a line bundle on X, and $V \subseteq H^0(X, \mathscr{L})$ a non-zero subspace of finite dimension. We denote by $|V| = \mathbb{P}_{sub}(V)$ the projective space of one-dimensional subspaces of V. When X is a complete variety there is a correspondence between this set and the complete linear system of D (where $\mathscr{L} = \mathcal{O}_X(D)$), that is the set of all effective divisors linearly equivalent to the divisor D and is denoted $|D|$.

Evaluation of sections in V gives rise to a morphism:

$$\mathrm{eval}_V : V \otimes_{\mathbb{C}} \mathscr{L}^* \to \mathcal{O}_X$$

of vector bundles on X.

Definition 1.2.7. The *base ideal* of $|V|$, written

$$\mathbf{b}(|V|) = \mathbf{b}(X, |V|) \subseteq \mathcal{O}_X,$$

is the image of the map $V \otimes_{\mathbb{C}} \mathscr{L}^* \to \mathcal{O}_X$ determined by eval_V. The *base locus*

$$\mathbf{Bs}(|V|) \subseteq X$$

of $|V|$ is the closed subset of X cut out by the base ideal $\mathbf{b}(|V|)$ (set of points at which all the section in V vanishes). When $V = H^0(X, \mathscr{L})$ or

$V = H^0(X, \mathcal{O}_X(D))$ are finite-dimensional, we write respectively $\mathbf{b}(|\mathscr{L}|)$ and $\mathbf{b}(|D|)$ for the base ideals of the indicated complete linear series.

Definition 1.2.8 (Free linear series). We say that $|V|$ is *free*, or *base-point free*, if its base locus is empty (that is $\mathbf{b}(|V|) = \mathcal{O}_X$). A divisor D or line bundle $\mathscr{L}$ is *free* if the corresponding complete linear series is so. In the case of line bundles we say that $\mathscr{L}$ is *generated by its global sections* or *globally generated* (for each point $x \in X$ we can find a section $s = s_x \in V$ such that $s(x) \neq 0$).

Assume now that $\dim V \geq 2$, and set $\mathbf{B} = \mathbf{Bs}(|V|)$. Then $|V|$ determines a morphism

$$\varphi : \varphi_{|V|} : X - B \to \mathbb{P}(V)$$

from the complement of the base locus in X to the projective space of one-dimentional quotients of V. Given $x \in X - B$, $\varphi(x)$ is the hyperplane in V consisting of those sections vanishing at x. If we choose a basis $s_0, \ldots, s_r \in V$, this amounts to say that φ is given in homogeneous coordinates by the expression

$$\varphi(x) = [s_0(x), \ldots, s_r(x)] \in \mathbb{P}^r.$$

When X is a variety it is useful to view $\varphi_{|V|}$ as a rational mapping $\varphi : X \dashrightarrow \mathbb{P}(V)$. If $|V|$ is free then the morphism is globally defined.

When $\mathbf{B} = \emptyset$ a morphism to projective space gives rise to a linear series. Suppose given a morphism

$$\varphi : X \to \mathbb{P} = \mathbb{P}(V),$$

then the pullback of sections via φ realizes $V = H^0(\mathbb{P}, \mathcal{O}_P(1))$ as a subspace of $H^0(X, \varphi^*(\mathcal{O}_{\mathbb{P}}(1))$, and $|V|$ is a free linear series on X. Moreover, φ is identified with the corresponding morphism $\varphi_{|V|}$.

1.3 Intersection theory

Given Cartier divisors $D_1, \dots, D_k \in \mathrm{Div}(X)$ together with an irreducible subvariety $V \subseteq X$ of dimension k, we want to define the *intersection number*

$$(D_1 \cdot \ldots \cdot D_k \cdot V) \overset{\text{not}}{=} \int_V D_1 \cdot \ldots \cdot D_k.$$

We know that each of the line bundles $\mathcal{O}_X(D_i)$ has a Chern class $c_1(\mathcal{O}_X(D_i)) \in H^2(X, \mathbb{Z})$, the cohomology group being ordinary singular cohomology of X with the classical topology. The cup product of these classes is then an element

$$c_1(\mathcal{O}_X(D_1)) \wedge \ldots \wedge c_1(\mathcal{O}_X(D_k)) \in H^{2k}(X, \mathbb{Z}).$$

Denoting by $[V] \in H_{2k}(X, \mathbb{Z})$ the fundamental class of V, cap product leads finally to an integer

$$(D_1 \cdot \ldots \cdot D_k \cdot V) \overset{\text{def}}{=} (c_1(\mathcal{O}_X(D_1)) \wedge \ldots \wedge c_1(\mathcal{O}_X(D_k))) \cap [V] \in \mathbb{Z}.$$

that is the intersection number.

Note 1.3.1. Let $n = \dim X$, then

$$(D_1 \cdot ... \cdot D_n) = \int_X D_1 \cdot \ldots \cdot D_n$$

$$(D^n) = \int_X \underbrace{D \cdot \ldots \cdot D}_{n-times}$$

The most important features of this product are:

- the integer $(D_1 \cdot \ldots \cdot D_n)$ is symmetric and multilinear as a function of its arguments;
- $(D_1 \cdot \ldots \cdot D_n)$ depends only on the linear equivalence classes of the D_i;

- if $D_1, \ldots, D_n$ are effective divisors that meet transversely at smooth points of X, then $(D_1 \cdot \ldots \cdot D_n) = \#\{D_1 \cap \ldots \cap D_n\}$.

Note 1.3.2. Given an irreducible subvariety $V \subseteq X$ of dimension k, $(D_1 \cdot ... \cdot D_k \cdot V)$ is then defined by replacing each divisor D_i with a linearly equivalent divisor D'_i whose support does not contain V, and intersecting the restrictions of the D'_i on V.

Furthermore, the intersection product satisfies the *projection formula*: if $f : Y \to X$ is a generically finite surjective proper map, then

$$\int_Y f^*D_1 \cdot ... \cdot f^*D_n = (\deg f) \cdot \int_X D_1 \cdot ... \cdot D_n.$$

Definition 1.3.3. Two Cartier divisors D_1, D_2 are *numerically equivalent*, $D_1 \equiv_{\text{num}} D_2$, if $(D_1.C) = (D_2.C)$ for every irreducible curve $C \subseteq X$. Equivalently if $(D_1.\gamma) = (D_2.\gamma)$ for all one-cycles γ in X.

Definition 1.3.4. A divisor or line bundle is *numerically trivial* if it is numerically equivalent to zero, and $\text{Num}(X) \subseteq \text{Div}(X)$ is the subgroup consisting of all numerically trivial divisors.

The *Néron-Severi group* of X is the free abelian group

$$N^1(X) = \text{Div}X/\text{Num}X$$

of numerical equivalence classes of divisors on X.

Proposition 1.3.5. The Néron-Severi group $N^1(X)$ is a free abelian group of finite rank.

The rank of $N^1(X)$ is called the *Picard number* of X and denoted $\rho(X)$.

Lemma 1.3.6. Let X be a variety, and let $D_1, \ldots, D_k, D'_1, \ldots, D'_k \in \text{Div}X$ be Cartier divisors on X. If $D_i \equiv_{\text{num}} D'_i$ for each i, then

$$(D_1 \cdot \ldots \cdot D_k \cdot [V]) = (D'_1 \cdot ... \cdot D'_k \cdot [V])$$

for every subscheme $V \subseteq X$ of pure dimension k.

The Lemma allows the following:

Definition 1.3.7. Given classes $\delta_1, \ldots, \delta_k \in N^1(X)$, we denote by $(\delta_1 \cdot \ldots \cdot \delta_k \cdot [V])$ the intersection number of any representatives of the classes in question.

Definition 1.3.8. Let X be a variety of dimension n, and let $\mathscr{F}$ be a coherent sheaf on X. Then the rank $rank(\mathscr{F})$ of $\mathscr{F}$ is the length of the stalk of $\mathscr{F}$ at the generic point of X.

Theorem 1.3.9 (Asymptotic Riemann-Roch, I). Let X be a projective variety of dimension n and let D be a divisor on X. Then the Euler characteristic $\chi(X, \mathcal{O}_X(mD))$ is a polynomial of degree $\leq n$ in m, with

$$\chi(X, \mathcal{O}_X(mD)) = \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

More generally, for any coherent sheaf $\mathscr{F}$ on X,

$$\chi(X, \mathscr{F} \otimes \mathcal{O}_X(mD)) = rank(\mathscr{F}) \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

Proof:(by [Kol96])
Let Y/S be a Noetherian scheme. Let $\mathscr{G}$ be a coherent sheaf on Y whose support is proper over a 0-dimensional subscheme of S. We define the *Grothendieck group* of Y, $K(Y)$, as the abelian group generated by the symbols $\overline{\mathscr{G}}$ where for every short exact sequence

$$0 \to \mathscr{G}_1 \to \mathscr{G}_2 \to \mathscr{G}_3 \to 0$$

we have

$$\overline{\mathscr{G}}_2 = \overline{\mathscr{G}}_1 + \overline{\mathscr{G}}_3.$$

We denote by $K_r(Y) \subseteq K(Y)$ the subgroup generated by those $\mathscr{G}$ whose support has dimension at most r.
Let $\mathscr{L}$ be an invertible sheaf on X. We define an endomorphism of $K(X)$

$$c_1(\mathscr{L}) \cdot \overline{\mathscr{F}} = \overline{\mathscr{F}} - \overline{\mathscr{L}^{-1} \otimes \mathscr{F}}.$$

Let us assume that $m \geq r = \dim \operatorname{Supp}(\mathscr{F})$. The intersection number of $\mathcal{O}_X(D_1), \dots, \mathcal{O}_X(D_m)$ with $\mathscr{F}$ is defined by

$$(\mathcal{O}_X(D_1) \cdot \ldots \cdot \mathcal{O}_X(D_m) \cdot \mathscr{F}) = \chi(X, c_1(\mathcal{O}_X(D_1)) \cdot \ldots \cdot c_1(\mathcal{O}_X(D_m)) \cdot \mathscr{F}).$$

Claim 1.3.10. In $K_r(Y)$ we have the following equivalence:

$$\mathscr{F}(mD) = \sum_{i=0}^{r} \binom{m+i-1}{i} c_1(\mathcal{O}_X(D))^i.\mathscr{F}.$$

<u>*Proof of the Claim:*</u>
Setting $n = -m$ we want to calculate $\mathscr{F}(mD)$ considering $\mathscr{F} \in K_r(X)$.
We have the formal identity

$$(1+x)^n = \sum_{i \geq 0} \binom{n}{i} x^i.$$

If we substitute $x = y^{-1} - 1$ and use that $\binom{-m}{i} = (-1)^i \binom{m+i-1}{i}$ we obtain that

$$y^m = \sum_{i \geq 0} \binom{m+i-1}{i} (1 - y^{-1})^i.$$

If we consider y as the operator $\mathscr{F} \mapsto \mathscr{F}(D)$ we obtain that $1 - y^{-1} = c_1(\mathcal{O}_X(D))$. Also $c_1(\mathcal{O}_X(D))^i.\mathscr{F} = 0$ for $i > r$ by the properties of intersection theory, we have

$$\mathscr{F}(mD) = \sum_{i=0}^{r} \binom{m+i-1}{i} c_1(\mathcal{O}_X(D))^i.\mathscr{F},$$

and the Calim is proved.

Let us now consider the Euler characteristic, then, if $n = \dim X$,

$$\chi(\mathscr{F}(mD)) = \sum_{i=0}^{n} \binom{m+i-1}{i} \chi(c_1(\mathcal{O}_X(D))^i.\mathscr{F}),$$

where the right hand side is a polynomial in m of degree at most n and the degree n term is

$$\binom{m+n-1}{n} \chi(\mathscr{F}.c_1(\mathcal{O}_X(D))^n) = \frac{(D^n.\mathscr{F})}{n!} m^n + O(m^{n-1}).$$

□

Corollary 1.3.11. In the setting of the theorem, if $H^i(X, \mathscr{F} \otimes \mathscr{L}(mD)) = 0$ for $i > 0$ and $m \gg 0$, or more generally, if for $i > 0$, $h^i(X, \mathscr{F} \otimes \mathscr{L}(mD)) = O(m^{n-1})$, then

$$h^0(X, \mathscr{F} \otimes \mathscr{L}(mD)) = rank(\mathscr{F}) \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

1.4 $\mathbb{Q}$ and $\mathbb{R}$-divisors

Definition 1.4.1. Let X be an algebraic variety. A Cartier $\mathbb{Q}$-*divisor* on X is an element of the $\mathbb{Q}$-vector space

$$\mathrm{Div}_{\mathbb{Q}}(X) \stackrel{\text{def}}{=} \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Equivalently $D \in \mathrm{Div}_{\mathbb{Q}}(X) \Leftrightarrow D = \sum c_i D_i |\ c_i \in \mathbb{Q},\ D_i \in \mathrm{Div}(X)$.

Definition 1.4.2 (Equivalence and operations on $\mathbb{Q}$-divisors). Assume henceforth that X is complete.

- Given a subscheme $V \subseteq X$ of pure dimension k, a $\mathbb{Q}$-valued intersection product
$$\mathrm{Div}_{\mathbb{Q}}(X) \times \ldots \times \mathrm{Div}_{\mathbb{Q}}(X) \to \mathbb{Q},$$
$$(D_1, \ldots, D_k) \mapsto \int_{[V]} D_1 \cdot \ldots \cdot D_k = (D_1 \cdot \ldots \cdot D_k \cdot [V])$$
is defined via extension of scalars from the analogous product on $\mathrm{Div}(X)$.

- Two $\mathbb{Q}$-divisors $D_1, D_2 \in \mathrm{Div}_{\mathbb{Q}}(X)$ are *numerically equivalent*, written $D_1 \equiv_{num} D_2$ if $(D_1 \cdot C) = (D_2 \cdot C)$ for every curve $C \subseteq X$. We denote by $N^1(X)_{\mathbb{Q}}$ the resulting finite-dimensional $\mathbb{Q}$-vector space of numerical equivalence classes of $\mathbb{Q}$-divisors.
We will denote $[D]_{num}$ the numerical equivalence class of D.

- Two $\mathbb{Q}$-divisors $D_1, D_2 \in \text{Div}_{\mathbb{Q}}(X)$ are *linearly equivalent*, written $D_1 \equiv_{lin} D_2$ if there is an integer r such that rD_1 and rD_2 are integral and linearly equivalent in the usual sense.

Definition 1.4.3. Let X be an algebraic variety. A Cartier $\mathbb{R}$-*divisor* on X is an element of the $\mathbb{R}$-vector space

$$\text{Div}_{\mathbb{R}}(X) \stackrel{\text{def}}{=} \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Equivalently $D \in \text{Div}_{\mathbb{R}}(X) \Leftrightarrow D = \sum c_i D_i |\ c_i \in \mathbb{R},\ D_i \in \text{Div}(X)$.

Definition 1.4.4. Let $D \in \text{Div}_{\mathbb{R}}(X)$, we say that D is *effective* if

$$D = \sum c_i A_i$$

with $c_i \in \mathbb{R},\ c_i \geq 0$ and A_i is an effective integral divisor.

Definition 1.4.5 (Numerical euivalence for $\mathbb{R}$-divisors). Two $\mathbb{R}$-divisors $D_1, D_2 \in \text{Div}_{\mathbb{R}}(X)$ are *numerically equivalent*, written $D_1 \equiv_{num} D_2$ if $(D_1 \cdot C) = (D_2 \cdot C)$ for every curve $C \subseteq X$. We denote by $N^1(X)_{\mathbb{R}}$ the resulting finite-dimensional $\mathbb{R}$-vector space of numerical equivalence classes of $\mathbb{R}$-divisors.
We will denote $[D]_{num}$ the numerical equivalence class of D.

Chapter 2

Ample and Nef divisors

2.1 Ample Divisors

Definition 2.1.1. If X is any scheme over Y, a line bundle $\mathscr{L}$ on X is said to be *very ample* relative to Y, if there is an immersion $i : X \to \mathbb{P}^r_Y$ for some r, such that $i^*(\mathcal{O}(1)) \cong \mathscr{L}$. We say that a morphism $i : X \to Z$ is an *immersion* if it gives an isomorphism of X with an open subscheme of a closed subscheme of Z.

Definition 2.1.2. Let X be a finite type scheme over a noetherian ring A, and let $\mathscr{L}$ be a line bundle on X. Then $\mathscr{L}$ is said to be *ample* if $\mathscr{L}^m$ is very ample over Spec$A$ for some $m > 0$.

Proposition 2.1.3. Let $f : Y \to X$ a finite mapping of complete schemes, and $\mathscr{L}$ an ample line bundle on X. Then $f^*\mathscr{L}$ is an ample line bundle on Y.

Note 2.1.4. In particular, if $Y \subseteq X$ is a subscheme of X, then the restriction $\mathscr{L}|_Y$ is ample.

Corollary 2.1.5. Let $\mathscr{L}$ be a globally generated line bundle on a complete scheme X, and let $\varphi = \varphi_{|\mathscr{L}|} : X \to \mathbb{P} = \mathbb{P}H^0(X, \mathscr{L})$ be the resulting map to projective space defined by the complete linear system $|\mathscr{L}|$. Then $\mathscr{L}$

is ample if and only if φ is a finite mapping, or equivalently if and only if $(C \cdot c_1(\mathscr{L})) > 0$ for every irreducible curve $C \subseteq X$.

Corollary 2.1.6 (Asymptotic Riemann-Roch, II). Let D be an ample Cartier divisor on a projective variety X of dimension n. Then

$$h^0(X, \mathscr{L}(mD)) = \frac{(D^n)}{n!} m^n + O(m^{n-1})$$

Example 2.1.7 (Upper bounds on h^0). If E is any divisor on a variety X of dimension n, there exists a constant $C > 0$ such that:

$$h^0(X, \mathcal{O}_X(mE)) \leq Cm^n \text{ for all } m.$$

Corollary 2.1.8. Let $f : Y \to X$ be a finite and surjective mapping of projective schemes, and $\mathscr{L}$ be a line bundle on X. If $f^*\mathscr{L}$ is ample on Y, then $\mathscr{L}$ is ample on X.

Definition 2.1.9 (Amplitude for $\mathbb{Q}$ and $\mathbb{R}$-divisors). A $\mathbb{Q}$-divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ (resp. $\mathbb{R}$-divisor $D \in \mathrm{Div}_{\mathbb{R}}(X)$) is said to be *ample* if it can be written as a finite sum

$$D = \sum c_i A_i$$

where $c_i > 0$ is a positive rational (resp. real) number and A_i is an ample Cartier divisor.

Note 2.1.10 (A useful way to write divisors). Let D be an $\mathbb{R}$-divisor, and suppose that $D = \sum a_i D_i$ where $a_i \in \mathbb{R}$ and $D_i \in \mathrm{Div}(X)$, *not necessarily prime.* For every integer $m \geq 1$ we can write

$$mD = m \sum a_i D_i = \sum ([ma_i]D_i + \{ma_i\}D_i)$$

so that we obtain:

$$[mD] = \left[\sum ([ma_i]D_i + \{ma_i\}D_i)\right] = \sum [ma_i]D_i + \left[\sum \{ma_i\}D_i\right].$$

Now $\{[\sum\{ma_i\}D_i]\} = \{T_m\}$ is a finite set of integral divisors, $\{T_m\} = \{T_{k_1}, \dots, T_{k_s}\}$.

Remark 2.1.11. If D is an integral divisor, D is ample in the sense of $\mathbb{Z}$-divisors if and only if it is ample in the sense of $\mathbb{R}$-divisors.

Proof:

If D is ample in the sense of $\mathbb{Z}$-divisors, obviously D can be written as $1 \cdot D$ where D is an ample divisor, so that it is an ample real divisor.
If $D = \sum c_i A_i$ in an ample $\mathbb{R}$-divisor, by Note 2.1.10 we can write $[mD] = \sum [ma_i] A_i + T_k$ for finitely many divisors T_k. As A_1 is ample, by Proposition 2.4.1, there exists an integer $r > 0$ such that $rA_1 + T_k$ is globally generated for every k and there exists an iteger $s > 0$ such that tA_i is very ample for all i and for all $t \geq s$. Then, if $m \geq \frac{r+s}{a_i}$ $\forall i$, we have

$$[mD] = \sum_{i \geq 2} [ma_i] A_i + ([ma_1] - r) A_1 + (rA_1 + T_k)$$

that is a sum of a very ample and a globally generated integral divisor, that is very ample. But in this case $[mD] = mD$ and we get the statement.

□

Proposition 2.1.12 (Nakai-Moishezon). D is an ample $\mathbb{R}$-divisor if and only if

$$(D^{\dim V} \cdot V) > 0$$

for every irreducible $V \subseteq X$ of positive dimension.

We will give the proof for this Proposition in 3.4.3; the difficult part is to prove that if the inequalities hold then the divisor D is ample.

Remark 2.1.13. If $D = \sum c_i A_i$ with $c_i > 0$ and A_i integral and ample, then

$$(D^{\dim V} \cdot V) \geq \left(\sum c_i \right)^{\dim V}.$$

Corollary 2.1.14. The amplitude of an $\mathbb{R}$-divisor depends only upon its numerical equivalence class.

Proof:

We will show that if D and B are $\mathbb{R}$-divisors, with D ample and $B \equiv_{num} 0$, then $D + B$ is ample.

First we want to prove that B is an $\mathbb{R}$-linear combination of numerically trivial integral divisors. Now B is given as a finite sum

$$B = \sum r_i B_i, \ r_i \in \mathbb{R}, \ B_i \in \mathrm{Div}(X).$$

The condition of being numerically trivial is given by finitely many linear equations on the r_i, determined by integrating over a set of generators of the subgroup of $H_2(X, \mathbb{Z})$ spanned by algebraic 1-cycles on X. The assertion then follows from the fact that any real solution of these equations is an $\mathbb{R}$-linear combination of integral ones.

We are now reduced to showing that if A and B are integral divisors, with A ample and $B \equiv_{num} 0$, then $A + rB$ is ample for any $r \in \mathbb{R}$. If r is rational we already know this. In general, we can fix rational numbers $r_1 < r < r_2$, together with a real number $t \in [0, 1]$, such that $r = tr_1 + (1 - t)r_2$. Then

$$A + rB = t(A + r_1 B) + (1 - t)(A + r_2 B),$$

exhibiting $A + rB$ as a positive $\mathbb{R}$-linear combination of ample $\mathbb{Q}$-divisors.

$\square$

Definition 2.1.15. A numerical equivalence class $\delta \in N^1(X)$ is ample if it is the class of an ample divisor.

Proposition 2.1.16 (Openness of amplitude for $\mathbb{Q}$ and $\mathbb{R}$-divisors). Let X be a projective variety and let H be an ample $\mathbb{Q}$-divisor (respectively $\mathbb{R}$-divisor) on X. Given finitely many $\mathbb{Q}$-divisors (resp. $\mathbb{R}$-divisors) $E_1, ..., E_r$, the $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor)

$$H + \varepsilon_1 E_1 + ... + \varepsilon_r E_r$$

is ample for all sufficiently small real numbers $0 \leq |\varepsilon_i| \ll 1$.

Proof:

When H and each E_i are rational, clearing denominators we can assume that H and each E_i are integral. By taking $m \gg 0$ we can arrange for any of the $2r$ divisors $mH \pm E_1, \ldots, mH \pm E_r$ to be ample. Now, provided that $|\varepsilon_i| \ll 1$ we can write any divisor of the form $H + \varepsilon_1 E_1 + ... + \varepsilon_r E_r$ as a positive $\mathbb{Q}$-linear combination of H and some of the $\mathbb{Q}$-divisors $H + \frac{1}{m}E_i$. But a positive linear combination of ample $\mathbb{Q}$-divisors is ample.

Since each E_j is a finite $\mathbb{R}$-linear combination of integral divisors, there is no loss of generality in assuming at the outset that all the E_j are integral. Now write $H = \sum c_i A_i$ with $c_i > 0$ and A_i ample and integral, and fix a rational number $0 < c \leq c_1$. Then

$$H + \sum \varepsilon_j E_j = (cA_1 + \sum \varepsilon_j E_j) + (c_1 - cA_1) + \sum_{i \geq 2} c_i A_i.$$

Here the first term on the right is ample by the above proof and the remaining summands are ample.

□

2.2 Nef Divisors

Definition 2.2.1 (Nef line bundles and divisors). Let X be a complete variety. A line bundle $\mathscr{L}$ on X is *numerically effective*, or *nef*, if

$$\int_C c_1(\mathscr{L}) \geq 0$$

for every irreducible curve $C \subseteq X$.

A Cartier $\mathbb{R}$-divisor D on X is *nef* if

$$(D \cdot C) \geq 0$$

for all irreducible curves $C \subset X$.

A Cartier $\mathbb{R}$-divisor D on X is *strictly nef* if

$$(D \cdot C) > 0$$

for all irreducible curves $C \subset X$.

Theorem 2.2.2 (Kleiman). Let X be a complete variety. If D is a nef $\mathbb{R}$-divisor on X, then

$$(D^k \cdot V) \geq 0$$

for every irreducible subvariety $V \subseteq X$ of dimension $k > 0$.

Theorem 2.2.3 (Higher cohomology of nef divisors). Let X be a projective variety of dimension n, and D an integral Cartier divisor on X. If D is nef, then for every coherent sheaf $\mathscr{F}$ on X

$$h^i(X, \mathscr{F}(mD)) = O(m^{n-i}).$$

Corollary 2.2.4. Let X be a projective variety, and D a nef $\mathbb{R}$-divisor on X. If H is an ample $\mathbb{R}$-divisor on X, then

$$D + \varepsilon H$$

is ample for every $\varepsilon > 0$. Conversely, if D and H are any two $\mathbb{R}$-divisors such that $D + \varepsilon H$ is ample for all sufficiently small $\varepsilon > 0$, then D is nef.

Proof:

If $D + \varepsilon H$ is ample for $\varepsilon > 0$, then

$$(D.C) + \varepsilon(H.C) = ((D + \varepsilon H).C) > 0$$

for every irreducible curve C. Letting $\varepsilon \to 0$ it follows that $(D.C) \geq 0$, and hence D is nef.

Assume conversely that D is nef and H is ample. Replacing εH by H, it suffices to show that $D + H$ is ample. To this end, as we will see, we only need to prove that

$$((D + H)^{\dim V}.V) > 0$$

for every subvariety $V \subseteq X$ of positive dimension (this is called Nakai's criterion of ampleness).

First suppose that $D + H$ is a rational divisor (then the general case will follow by an approximation argument).
Fix a variety $V \subseteq X$ of dimension $k > 0$. Then

$$\big((D+H)^k.V\big) = \sum_{s=0}^{k} \binom{k}{s} (H^s.D^{k-s}.V). \tag{2.1}$$

Since H is a positive $\mathbb{R}$-linear combination of integral ample divisors, the intersection $(H^s.V)$ is represented by an effective $(k-s)$-cycle. Applying Kleiman's theorem to each of the components of this cycle, it follows that $(H^s.D^{k-s}.V) \geq 0$. Thus each of the terms in (2.1) is non-negative for $s \neq k$, and the last intersection number $(H^k.V)$ is strictly positive. Therefore $((D+H)^k.V) > 0$ for every V, and in particular if $D+H$ is rational then it is ample (by Proposition 2.1.12).

It remains to prove that $D + H$ is ample even when it is irrational. To this end, choose ample divisors $H_1, \dots, H_r$ whose classes span $N_1(X)_{\mathbb{R}}$. By the open nature of amplitude (2.1.16), the $\mathbb{R}$-divisor $H(\varepsilon_1, \dots, \varepsilon_r) = H - \varepsilon_1 H_1 - \dots - \varepsilon_r H_r$ remains ample for all $0 < \varepsilon_i \ll 1$. Obviously there exist $0 < \varepsilon_i \ll 1$ such that $D' = D + H(\varepsilon_1, \dots, \varepsilon_r)$ represents a rational class in $N^1(X)_{\mathbb{R}}$. The case of the corollary already treated shows that D' is ample. Consequently so too is

$$D + H = D' + \varepsilon_1 H_1 + \dots + \varepsilon_r H_r.$$

□

Example 2.2.5 (Strictly nef but not ample). [Har70](Appendix 10) Now we will give an example by Mumford of a divisor over a surface, that is strictly nef but not ample, to give sense to the definition of nefness.
Let us consider a non-singular complete curve C of genus $g \geq 2$ over $\mathbb{C}$; we know that there exists a stable bundle E of rank two and degree zero such that all its symmetric powers $S^m(E)$ are stable. Let $X = \mathbb{P}(E)$ be the ruled surface over C, let $\pi : X \to C$ be the canonical projection and let D be the

divisor corresponding to $\mathcal{O}_X(1)$. Then, for every irreducible curve $Y \subseteq X$, we have:

- If Y is a fibre of π, then $(D.Y) = 1$;
- If Y is an irreducible curve of degree m over C, then Y corresponds to a sub-line bundle $M \subseteq S^m(E)$. But $S^m(E)$ is stable of degree zero, so $\deg M < 0$. Therefore $(D.Y) = -\deg M > 0$.

Thus $(D.Y) > 0$ for every effective curve $Y \subseteq X$, but D is not ample, because

$$(D^2) = 0.$$

We will now give an example of Ramanujan of a divisor strictly nef but not ample on a threefold that is based on the Example 2.2.5 of Mumford.

Example 2.2.6 (Strictly nef and big but not ample). (Definition 3.3.1) Let X be a non-singular surface, and D a divisor with $(D.Y) > 0$ for all effective curves, and $(D^2) = 0$ as in the Example 2.2.5 by Mumford. Let H be an effective ample divisor on X, then we define $\overline{X} = \mathbb{P}(\mathcal{O}_X(D-H) \otimes \mathcal{O}_X)$, and let $\pi : \overline{X} \to X$ be the projection.
Let X_0 be the zero-section of the associated vector bundle, so that $(X_0^2) = (D-H)_X$. We define $\overline{D} = X_0 + \pi^*H$ that is effective by construction.

$\overline{D}$ is positive over all effective curves Y:

- If Y is a fibre of π, then
$$(\overline{D}.Y) = (X_0.Y) + (\pi^*H.Y) = 1 + 0 = 1.$$
- If $Y \subset X_0$, then
$$(\overline{D}.Y) = (\overline{D}|_{X_0}.Y)|_{X_0} = ((D-H+H).Y)_X = (D.Y)_X > 0.$$
- If $Y \not\subseteq X_0$, and $\pi(Y)$ is a curve Y' in X, then
$$(\overline{D}.Y) = (X_0.Y) + (\pi^*H.Y)$$
where $(X_0.Y) \geq 0$ and $(\pi^*H.Y) = (H.Y')_X > 0$.

On the other hand, $\overline{D}$ is not ample. In fact

$$(\overline{D}^2) = (\overline{D}|^2_{X_0})_{X_0} = (D^2)_X = 0,$$

and therefore, by Nakai-Moishezon (Proposition 2.1.12) $\overline{D}$ is not ample.
On the other hand $\overline{D}$ is big (as we will see in the next chapter (Theorem 3.3.16)) because $(\overline{D}^3) > 0$, in fact:

$$\begin{aligned}(\overline{D}^3) &= (\overline{D}^2.(X_0 + \pi^* H)) = \\ &= ((X_0 + \pi^* H)^2.\pi^* H) = \\ &= ((D - H)_X + 2H_X + \pi^* H^2).\pi^* H) = \\ &= ((D + H).H)_X + (\pi^* H^3) > 0\end{aligned}$$

because $(\pi^* H^3) = 0$ and $(D + H).H > 0$ by Nakai-Moishezon (2.1.12).

Theorem 2.2.7 (Fujita's vanishing theorem). Let X be a variety and let H be an ample integral divisor on X. Given any coherent sheaf $\mathscr{F}$ on X, there exists an integer $m(\mathscr{F}, H)$ such that

$$H^i(X, \mathscr{F} \otimes \mathcal{O}_X(mH + D)) = 0 \text{ for all } i > 0,\ m \geq m(\mathscr{F}, H),$$

and any nef divisor D on X.

2.3 Ample and Nef Cones

Definition 2.3.1 (CONES). Let V be a finite-dimensional real vector space. A *cone* in V is a set $K \subseteq V$ stable under multiplication by positive scalars.

Definition 2.3.2 (Ample and nef cones).

- The *ample cone* $\mathrm{Amp}(X) \subset N^1(X)_{\mathbb{R}}$ of X is the convex cone of all ample $\mathbb{R}$-divisor classes on X.

- The *nef cone* $\mathrm{Nef}(X) \subset N^1(X)_{\mathbb{R}}$ is the convex cone of all nef $\mathbb{R}$-divisor classes.

Theorem 2.3.3 (Kleiman). Let X be any projective variety or scheme.

- The nef cone is the closure of the ample cone: $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$
- The ample cone is the interior of the nef cone: $\mathrm{Amp}(X) = int(\mathrm{Nef}(X))$

Definition 2.3.4 (Numerical equivalence classes of curves). Let X be a variety. We denote by $Z_1(X)_{\mathbb{R}}$ the $\mathbb{R}$-vector space of *real one cycles* of X, consisting of all finite $\mathbb{R}$-linear combinations of irreducible curves on X. An element $\gamma \in Z_1(X)_{\mathbb{R}}$ is thus a formal finite sum

$$\gamma = \sum a_i \cdot C_i$$

where $a_i \in \mathbb{R}$ and $C_i \subset X$ is an irreducible curve.
Two one-cycles $\gamma_1, \gamma_2 \in Z_1(X)_{\mathbb{R}}$ are *numerically equivalent* if $(D \cdot \gamma_1) = (D \cdot \gamma_2)$ for every $D \in \mathrm{Div}_{\mathbb{R}}(X)$
The corresponding vector space of *numerical equivalence classes* of one-cycles is written $N_1(X)_{\mathbb{R}}$. Thus one has a perfect pairing

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}, \quad (\delta, \gamma) \mapsto (\delta \cdot \gamma) \in \mathbb{R}$$

Definition 2.3.5 (Cone of curves). Let X be a complete variety. The *cone of curves* $\mathrm{NE}(X) \subseteq N_1(X)_{\mathbb{R}}$ is the cone spanned by the classes of all effective one-cycles on X.

$$\mathrm{NE}(X) = \left\{ \sum a_i [C_i] | C_i \subset X \text{an irreducible curve}, a_i \in \mathbb{R}, a_i \geq 0 \right\}$$

Proposition 2.3.6. $\overline{\mathrm{NE}}(X)$ is the closed cone dual to $\mathrm{Nef}(X)$:

$$\overline{\mathrm{NE}}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} | (\gamma \cdot \delta) \geq 0 \quad \forall \delta \in \mathrm{Nef}(X)\}$$

Definition 2.3.7. We denote by

$$D^{\perp} = \{\gamma \in N_1(X)_{\mathbb{R}} | (D \cdot \gamma) = 0\};$$

$$D_{>0} = \{\gamma \in N_1(X)_{\mathbb{R}} | (D \cdot \gamma) > 0\}.$$

2.4 Ampleness for $\mathbb{Z}$-divisors

Now is the time to give a global characterization of ampleness for $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$-divisors. We will try to understand the differences among those three classes and study the common properties. We start with the $\mathbb{Z}$-divisors.

Proposition 2.4.1 (Ampleness for $\mathbb{Z}$-divisors). Let $D \in \mathrm{Div}(X)$ be an integral Cartier divisor on a normal projective variety X, and let $\mathcal{O}_X(D)$ be the associated line bundle (sometimes we will think D as a Weil divisor by the canonical correspondence (1.2.5)). The following statements are equivalent:

1. There exists a positive integer m such that $\mathcal{O}_X(mD)$ is very ample;
2. Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_1 = m_1(\mathscr{F})$ having the property that
$$H^i(X, \mathscr{F} \otimes \mathcal{O}_X(mD)) = 0 \quad \forall i > 0,\ m \geq m_1;$$
3. Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_2 = m_2(\mathscr{F})$ such that $\mathscr{F} \otimes \mathcal{O}_X(mD)$ is globally generated $\forall m \geq m_2$;
4. There is a positive integer m_3 such that $\mathcal{O}_X(mD)$ is very ample $\forall m \geq m_3$;
5. For every subvariety $V \subseteq X$ of positive dimension, there is a positive integer $m = m(V)$, together with a non-zero section $0 \neq s = s_V \in H^0(V, \mathcal{O}_V(mD))$, such that s vanishes at some point of V;
6. For every subvariety $V \subseteq X$ of positive dimension, $\chi(V, \mathcal{O}_V(mD)) \to +\infty$ as $m \to +\infty$;
7. **(Nakai-Moishezon-Kleiman criterion)**
$$\int_V c_1(\mathcal{O}_X(D))^{\dim(V)} > 0$$
for every positive-dimensional subvariety $V \subseteq X$;

8. **(Seshadri's criterion)** There exists a real number $\varepsilon > 0$ such that
$$\frac{(D.C)}{\mathrm{mult}_x C} \geq \varepsilon$$
for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x;

9. Let H be an ample divisor. There exists a positive number $\varepsilon > 0$ such that
$$\frac{(D.C)}{(H.C)} \geq \varepsilon$$
for every irreducible curve $C \subseteq X$;

10. **(Via cones)** $\overline{\mathrm{NE}}(X) - \{0\} \subseteq D_{>0}$.

11. There exists a neighborhood U of $[D]_{num} \in \mathrm{N}^1(X)_{\mathbb{R}}$ such that $U \backslash \{[D]_{num}\} \subseteq \mathrm{Amp}(X)$.

Proof:

1 ⇒ 2) By (1), there exists $m_0 > 0$ such that $\mathcal{O}_X(m_0 D)$ is very ample.

Now we want to consider the embedding of X into some projective space $\mathbb{P}$ induced by $\mathcal{O}_X(m_0 D)$ and the sheaf induced by $\mathscr{F}$ extending it by zero to a coherent sheaf in $\mathbb{P}$. The image of $\mathcal{O}_X(m_0 D)$ is the very ample invertible sheaf $\mathcal{O}_X(1)$ and we know ([Har77] III, 5.2 - Serre) that there exists an integer n_0 depending on $\mathscr{F}$ such that for each $i > 0$ and each $m > n_0$, $H^i(X, \mathscr{F} \otimes (\mathcal{O}_X(m_0 D))^{\otimes m}) = H^i(X, \mathscr{F} \otimes (\mathcal{O}_X(1))^{\otimes m}) = H^i(X, \mathscr{F}(m)) = 0$.

Now we only need to replace $\mathscr{F}$ with the other coherent sheaves $\mathscr{F}_1 = \mathscr{F} \otimes \mathcal{O}_X(D), ..., \mathscr{F}_{m_0 - 1} = \mathscr{F} \otimes \mathcal{O}_X((m_0 - 1)D)$ and consider the integers $n_j = n_j(\mathscr{F}_j)$. So we are done with
$$m_1 = (n \cdot m_0) \quad \text{where} \quad n = \max_j n_j.$$

2 ⇒ 3) We begin fixing a closed point $x \in X$ and denoting $\mathbf{m}_x \subset \mathcal{O}_{X,x}$ the maximal ideal sheaf of x. By the hypothesis there is an integer $m_2 = m_2(x)$ depending on $\mathscr{F}$ and x such that $H^1(X, \mathbf{m}_x \cdot \mathscr{F} \otimes \mathcal{O}_X(mD)) = 0$ for $m \geq m_2$.

Now consider the exact sequence

$$0 \to \mathbf{m}_x \cdot \mathscr{F} \to \mathscr{F} \to \mathscr{F}/(\mathbf{m}_x \cdot \mathscr{F}) \to 0;$$

twisting by $\mathcal{O}_X(mD)$ and taking cohomology we get:

$$0 \to H^0(X, \mathbf{m}_x \cdot \mathscr{F} \otimes \mathcal{O}_X(mD)) \to H^0(X, \mathscr{F} \otimes \mathcal{O}_X(mD)) \to$$
$$\to H^0(X, \mathscr{F}/(\mathbf{m}_x \cdot \mathscr{F}) \otimes \mathcal{O}_X(mD)) \to 0$$

and $H^0(X, \mathscr{F} \otimes \mathcal{O}_X(mD)) \neq 0$ so that $\mathscr{F} \otimes \mathcal{O}_X(mD)$ is globally generated in an open neighborhood U of x depending on m. If we consider $\mathscr{F} = \mathcal{O}_X$, there will exist an integer $n > 0$ and a neighborhood V of x such that $\mathcal{O}_X(nD)$ is globally generated over V. Also, for each $r = 1, \dots, n-1$, like above there exists a neighborhood U_r of x such that $\mathscr{F}((m_2 + r)D)$ is globally generated over U_r. Now, considering

$$U_x = V \cap U_1 \cap \dots \cap U_{n-1}$$

we have that $\mathscr{F}(m)$ is globally generated for every $m \geq m_2$, writing

$$\mathscr{F}((m_2 + r)D) \otimes \mathcal{O}_X(nD)^t$$

for suitable $0 \leq r < n$ and $t \geq 0$.

By quasi-compactness, we can cover X by a finite number of the open sets U_x of x, so that now we can choose a single natural number $m_3 = \max m_2(x)$ depending only on $\mathscr{F}$. Then $\mathscr{F} \otimes (mD)$ is generated by global sections over all X, for all $m \geq m_3$.

3 ⇒ 4) We want to show that for $m \gg 0$, $\mathcal{O}_X(mD)$ induces a morphism in some projective space that is an embedding.

By the hypothesis, we know that there exists a positive integer $n > 0$ such that for every $m \geq n$, $\mathcal{O}_X(mD)$ is globally generated. We can now consider the map:

$$\varphi_m : X \to \mathbb{P}H^0(X, \mathcal{O}_X(mD))$$

to the corresponding projective space. We want to find $m \gg 0$ such that φ_m is an embedding (one-to-one and unramified) [Har77](II,7.3). We now consider the open sets

$$U_m = \{y \in X | \mathcal{O}_X(mD) \otimes \mathbf{m}_y \quad \text{is globally generated}\}.$$

Given a point $x \in X$, thanks to the hypothesis, we can find an integer $m_2(x)$ such that $x \in U_m$ for every $m \geq m_2(x)$. Therefore we can write $X = \cup U_m$ and by quasi-compactness we can find a single integer $m_3 \geq n$ such that for every $m \geq m_3$ and every $x \in X$, $\mathcal{O}_X(mD) \otimes \mathbf{m}_x$ is globally generated. But this implies that $\varphi_m(x) \neq \varphi_m(x')$ for all $x \neq x'$ and that φ_m is unramified at x. Thus φ_m is an embedding for all $m \geq m_3$ and $\mathcal{O}_X(mD)$ is very ample.

4 ⇒ 5) By the hypothesis we consider $m \gg 0$ such that $\mathcal{O}_X(mD)$ is very ample. There is a morphism in some projective space $\mathbb{P}$ such that the image of $\mathcal{O}_X(mD)$ is the very ample invertible sheaf $\mathcal{O}_X(1)$. Let V be an irreducible subvariety $V \subseteq X$ of positive dimension; then for every point P of X there exists a divisor $H \equiv_{lin} mD$, such that $P \in H$ and $H \not\supseteq V$. In particular the section associated to H vanishes at P and does not vanish on V.

5 ⇒ 6) By the hypothesis we know that for every subvariety V of positive dimension, $H^0(V, \mathcal{O}_V(mD)) \neq 0$ some $m \gg 0$. We first prove that D is ample by induction on the dimension of X.
Take in a first time $V = X$ and replacing D by a multiple we can assume that D is effective. Now, by induction $\mathcal{O}_D(D)$ is ample, and so

by [Laz04a] (Example 1.2.30) $\mathcal{O}_X(mD)$ is free for $m \gg 0$. Now, by the hypothesis, $\mathcal{O}_X(D)$ is not trivial on any curve and by [Laz04a] (Cor. 1.2.15) we have that D is ample.
Then the assertion follows by Asymptotic Riemann-Roch, I (1.3.9).

6 $\Rightarrow$ 7) We want to do it by induction passing through the definition of ampleness. If the dimension of V is 1, D is ample for $H^0(V, \mathcal{O}_V(mD)) \neq 0$, $\exists\, m \gg 0$. Let us assume inductively that $\mathcal{O}_E(D)$ is ample for every effective divisor E on X. We assert first that

$$H^0(X, \mathcal{O}_X(mD)) \neq 0 \text{ for } m \gg 0.$$

In fact, asymptotic Riemann-Roch (1.3.9) gives that

$$\chi(X, \mathcal{O}_X(mD)) = m^n \frac{(D^n)}{n!} + O(m^{n-1}),$$

and we know by hypothesis that $\chi(X, \mathcal{O}_X(mD)) \to +\infty$. Now write $D \equiv_{lin} A - B$ as a difference of very ample effective divisors. We have the two exact sequences:

$$0 \to \mathcal{O}_X(mD - B) \xrightarrow{.A} \mathcal{O}_X((m+1)D) \to \mathcal{O}_A((m+1)D) \to 0,$$

$$0 \to \mathcal{O}_X(mD - B) \xrightarrow{.B} \mathcal{O}_X(mD) \to \mathcal{O}_B(mD) \to 0.$$

By induction, $\mathcal{O}_A(D)$ and $\mathcal{O}_B(D)$ are ample so that the higher cohomology of each of the two sheaves on the right vanishes when $m \gg 0$, then

$$H^i(X, \mathcal{O}_X(mD)) = H^i(X, \mathcal{O}_X(mD - B)) = H^i(X, \mathcal{O}_X((m+1)D))$$

for $i \geq 2$. Therefore we can write $\chi(X, \mathcal{O}_X(mD)) = h^0(X, \mathcal{O}_X(mD)) - h^1(X, \mathcal{O}_X(mD)) + C$ for some constant C and $m \gg 0$.
We now want to show that $\mathcal{O}_X(mD)$ is globally generated, that is no

point of D is a base point of the linear series $|\mathcal{O}_X(mD)|$. Consider to this end the exact sequence

$$0 \to \mathcal{O}_X((m-1)D) \xrightarrow{.D} \mathcal{O}_X(mD) \to \mathcal{O}_D(mD) \to 0.$$

As before $\mathcal{O}_D(D)$ is ample by induction. Consequently $\mathcal{O}_D(mD)$ is globally generated and $H^1(X, \mathcal{O}_D(mD)) = 0$ for $m \gg 0$. It then follows that the natural homomorphism

$$H^1(X, \mathcal{O}_X((m-1)D)) \to H^1(X, \mathcal{O}_X(mD))$$

is surjective for every $m \gg 0$. The spaces in question being finite-dimensional, the map must be an isomorphism for sufficiently large m. Therefore the restriction mappings

$$H^0(X, \mathcal{O}_X(mD)) \to H^0(X, \mathcal{O}_D(mD))$$

are surjective for $m \gg 0$. But since $\mathcal{O}_D(mD)$ is globally generated, it follows that no point of $\mathrm{Supp}(D)$ is a basepoint of $|mD|$, as required. Finally, the amplitude of $\mathcal{O}_X(mD)$ follows from Corollary 2.1.5, in fact by the hypothesis $\mathcal{O}_X(D)$ restricts to an ample bundle on every irreducible curve in X.

Now if D is ample, mD is very ample for some $m \gg 0$, and

$$m^{\dim V} \int_V D^{\dim V} = \int_V (mD)^{\dim V}$$

is the degree of V in the corresponding projective embedding of X; consequently this integral is strictly positive.

7 ⇒ 8) We first prove that D is ample. If $\dim X = 1$ is clear. By induction, we assume that the theorem is true for all the varieties of dimension $\leq n-1$. Then we proceed like above ((6) ⇒ (7)) using the fact that $(D^n) > 0$ when we use Riemann-Roch.

Now, if D is ample, there exists $m \gg 0$ such that mD is very ample.

So we can embed X in a suitable projective space $\mathbb{P}$ in such a way that $mD = X \cap H$ for some hyperplane $H \in \mathbb{P}$. Now for every irreducible curve C, we have

$$(mD.C)_X = (H.C)_{\mathbb{P}} = \deg C \geq \mathrm{mult}_x(C)$$

for every $x \in C$. Then

$$(mD.C) \geq \mathrm{mult}_x(C) \Rightarrow \frac{(D.C)}{\mathrm{mult}_x(C)} \geq \frac{1}{m},$$

so the result follows with $\varepsilon = \frac{1}{m}$.

8 $\Rightarrow$ 9) By (8) we know that $\frac{(D.C)}{\mathrm{mult}_x C} \geq \varepsilon$; in particular, considering the divisor $(D - \varepsilon' H)$ (for $0 < \varepsilon' \ll 1$),

$$(D - \varepsilon' H).C \geq 0 \Rightarrow \frac{(D.C)}{(H.C)} \geq \varepsilon'.$$

9 $\Rightarrow$ 10) Let $\gamma \in \overline{\mathrm{NE}}(X)$; we want to prove that $(D.\gamma) \geq 0$ and $(D.\gamma) = 0$ if and only if $\gamma \equiv_{num} 0$:
if $\gamma \in \overline{\mathrm{NE}}(X)$, $\gamma = \lim_{n \to +\infty} \gamma_n$ where $\gamma_n \in \mathrm{NE}(X), \gamma_n = \sum a_{i,n} C_{i,n}$ where $C_{i,n}$ is an irreducible curve and $a_{i,n} \in \mathbb{R}$, $a_{i,n} \geq 0$. Now by (9), for every $C \subseteq X$, $((D - \varepsilon H).C) \geq 0 \Rightarrow (D.C) > 0$ that is $(D.\gamma_n) \geq 0$. Also by (9) $(D.\gamma_n) = \sum a_{i,n}(D.C_{i,n}) \geq \sum a_{i,n}\varepsilon$. If $(D.\gamma) = 0$, then

$$0 = (D.\gamma) = \lim_{n \to +\infty} \sum a_{i,n}(D.C_{i,n}) \geq \lim_{n \to +\infty} \sum a_{i,n}\varepsilon \geq 0$$

so that $\lim_{n \to +\infty} a_{i,n} = 0$ for all i and so $\gamma = \lim_{n \to +\infty} \gamma_n = 0$.

10 $\Rightarrow$ 11) Suppose that (11) does not hold. If we consider the discs $\mathcal{D}_n = \mathcal{D}([D]_{num}, \frac{1}{n})$ of centre $[D]_{num}$ and radius $\frac{1}{n}$. For every n we know that there exists an element $[D_n]_{num} \in \mathcal{D}_n - [D]_{num}$ such that $[D_n]_{num} \notin \mathrm{Amp}(X)$. We even know that $\mathrm{Amp}(X)$ is open, so that $N^1(X) \backslash \mathrm{Amp}(X)$ is closed. Since

$$\lim_{n \to +\infty} [D_n]_{num} = [D]_{num}$$

it follows that $D \in N^1(X) \backslash \mathrm{Amp}(X)$.
Now

$$\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)^* = \{\delta \in \mathrm{Nef}(X) | (\delta.\gamma) \geq 0 \ \forall \gamma \in \overline{\mathrm{NE}}(X)\}$$

by (2.3.6); and so

$$\mathrm{Amp}(X) = \mathrm{int}(\mathrm{Nef}(X)) = \{\delta \in \mathrm{Nef}(X) | (\delta.\gamma) > 0 \ \forall \gamma \in \overline{\mathrm{NE}}(X) - \{0\}\}$$

so that if $D \in N^1(X) \backslash \mathrm{Amp}(X)$ there exists a one-cycle $\gamma \neq 0$ such that $(D.\gamma) \leq 0$, that is absurd for the hypothesis.

11 $\Rightarrow$ 1) By hypothesis, there exists a neighborhood of $[D]_{num}$ that, except $[D]_{num}$, is all contained in $\mathrm{Amp}(X)$; so that there exists an open punctured disc all contained in that neighborhood and so there exists ε such that $(D - \varepsilon H) = N$ is ample for some ample divisor H. Then, as we have seen 2.2.4, $D = N + \varepsilon H$ is ample.

□

Proposition 2.4.2. ***By Definition 2.1.2 every divisor that verifies at least one of the properties of the Proposition 2.4.1 is an ample $\mathbb{Z}$-divisor.***

2.5 Ampleness for $\mathbb{Q}$-divisors

We have just given a definition of ampleness for $\mathbb{Q}$ and $\mathbb{R}$-divisors (2.1.9, 2.1.9); we want to try to understand when one of those two definitions is equivalent to some of the properties of Proposition 2.4.1. Some of them are automatically transferable (and for those we will directly prove that they are equivalent to the concept of ampleness for $\mathbb{Q}$ and $\mathbb{R}$-divisors). Other properties depend directly on the line bundle and not on the divisor (as 2.4.1 (1) or (2)); in this case a possibility is to substitute the divisor mD by its integral part $[mD]$ and see if the equivalence still holds. This is what we will try to do in this thesis.

We will now discuss the case of $\mathbb{Q}$-divisors:

Example 2.5.1. The affirmation (1) in Proposition 2.4.1, when we replace mD with its integral part, is not equivalent to the concept of ampleness for $\mathbb{Q}$-divisors.

To prove that it is not possible to extend (1), or rather that the property of existence of an integer m such that $[mD]$ is very ample is not sufficient to characterize the amplitude for a $\mathbb{Q}$-divisor, it is enough to find an example. We consider a ruled rational surface X_e, $e \geq 2$ defined as $\mathbb{P}(\mathscr{E}) = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e))$ over $\mathbb{P}^1$. We now consider a divisor in the form

$$D = \frac{3}{2}C_0 + (e+1)f$$

where C_0 is a section and f is a fibre of the canonical morphism over $\mathbb{P}^1$. Now

$$[D] = C_0 + (e+1)f$$

is a very ample divisor by [Har77] (Theorem V.2.17) **but**

$$D.C_0 = \left(\frac{3}{2}C_0 + (e+1)f\right).C_0 = 1 - \frac{e}{2} < 0$$

so that D is not ample!

□

Remark 2.5.2. For the same reason the affirmation (5) in Proposition 2.4.1, when we replace mD with its integral part, is not equivalent to the concept of ampleness for $\mathbb{Q}$-divisors. In this case we will replace "$\exists m$" by "$\forall\, m \geq m_4$".

□

We can now enunciate the following proposition:

Proposition 2.5.3 (Ampleness for $\mathbb{Q}$-divisors). Let $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ be a Cartier divisor on a normal projective variety X, and let $\mathcal{O}_X([D])$ be the associated line bundle (sometimes we will think $[D]$ as a Weil divisor by the canonical correspondence). The following statements are equivalent to the definition of ampleness for $\mathbb{Q}$-divisors (Definition 2.1.9):

(I) Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_1 = m_1(\mathscr{F})$ having the property that

$$H^i(X, \mathscr{F} \otimes \mathcal{O}_X([mD])) = 0 \quad \forall i > 0,\ m \geq m_1;$$

(II) Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_2 = m_2(\mathscr{F})$ such that $\mathscr{F} \otimes \mathcal{O}_X([mD])$ is globally generated $\forall m \geq m_2$;

(III) There is a positive integer m_3 such that $\mathcal{O}_X([mD])$ is very ample $\forall m \geq m_3$;

(IV) For every subvariety $V \subseteq X$ of positive dimension, there is a positive integer $m_4 = m_4(V)$, such that for every $m \geq m_4$ there exists a non-zero section $0 \neq s = s_{V,m} \in H^0(V, \mathcal{O}_V([mD]))$, such that s vanishes at some point of V;

(V) For every subvariety $V \subseteq X$ of positive dimension, $\chi(V, \mathcal{O}_V([mD])) \to +\infty$ as $m \to +\infty$;

(VI) **(Nakai-Moishezon-Kleiman criterion)**

$$\int_V c_1(\mathcal{O}_X(D))^{\dim(V)} > 0$$

for every positive-dimensional subvariety $V \subseteq X$;

(VII) **(Seshadri's criterion)** There exists a real number $\varepsilon > 0$ such that

$$\frac{(D.C)}{\mathrm{mult}_x C} \geq \varepsilon$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x;

(VIII) Let H be an ample divisor. There exists a positive number $\varepsilon > 0$ such that

$$\frac{(D.C)}{(H.C)} \geq \varepsilon$$

for every irreducible curve $C \subseteq X$;

(IX) **(Via cones)** $\overline{\mathrm{NE}}(X) - \{0\} \subseteq D_{>0}$.

(X) There exists a neighborhood U of $[D]_{num} \in \mathrm{N}^1(X)_{\mathbb{R}}$ such that $U \backslash \{[D]_{num}\} \subseteq \mathrm{Amp}(X)$.

Proof:

Claim 2.5.4. Either one of (I), (II), (III), (IV) and (V) implies ampleness for $\mathbb{Q}$-divisors

Proof:

(I) $\Rightarrow$ Ample: let $a \in \mathbb{N}$ such that $aD \in \mathrm{Div}(X)$ and let $m_0 = \lceil \frac{m_1}{a} \rceil \geq 1$; if $m \geq m_0 \Rightarrow am \geq am_0 = a\lceil \frac{m_1}{a} \rceil \geq a \cdot \frac{m_1}{a} = m_1$.
Then by hypothesis

$$H^i(\mathscr{F}([amD]) = 0 \qquad \forall i > 0,$$

but $H^i(\mathscr{F}([amD]) = H^i(\mathscr{F}(m(aD)) = 0$ so that by Proposition 2.4.2 aD is an ample integral divisor and so $D = \frac{1}{a}(aD)$ is an ample $\mathbb{Q}$-divisor.

The implications $(*) \Rightarrow$ Ample, where $(*)$ is one of $(II), (III), (IV), (V)$, can be proved in a same way.

□

Now we want to show that if a property holds for every multiple of mD than it is also valid for D; we will use this simple fact:

Lemma 2.5.5. Let $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ and let $k \in \mathbb{N}$ such that $kD \in \mathrm{Div}(X)$. Then for every $m \in \mathbb{N}$ there exist $i, t \in \mathbb{N}$ such that

$$[mD] = tkD + [iD], \quad 0 \leq i \leq k-1.$$

<u>*Proof:*</u>
D is a finite sum $D = \sum a_j D_j$ where $D_j \in \mathrm{Div}(X)$ are prime divisors and $a_j \in \mathbb{Q}$. Now

$$[mD] = \sum [ma_j] D_j;$$

and we can always write $m = tk+i$ with $0 \leq i \leq k-1$ so that $ma_j = tka_j + ia_j$ where ka_j is an integer and therefore $[ma_j] = tka_j + [ia_j]$. Hence:

$$\begin{aligned}[mD] = \sum_j [ma_j] D_j &= \sum_j (tka_j + [ia_j]) D_j = \\ &= tk \sum_j a_j D_j + \sum_j [ia_j] D_j = tkD + [iD].\end{aligned}$$

□

Claim 2.5.6. Ample implies either (I), (II), (III) and (IV)

<u>*Proof:*</u>
Let us consider $k \in \mathbb{N}$ such that $kD = H$ is an ample integral divisor and let us use the notation of Lemma 2.5.5.

Ample $\Rightarrow$ (I) Consider $\mathscr{F}_i = \mathscr{F}([iD])$ for $0 \leq i \leq k-1$ and $n_i = n_i(\mathscr{F}_i)$ such that $H^j(\mathscr{F}([iD]) \otimes \mathcal{O}_X(nkD)) = 0$ for every $j > 0$ and every $n \geq n_i$.
Then the assertion holds with $m_1 = k(\max_i n_i)$.

Ample $\Rightarrow$ (II) If H is ample, for every coherent sheaf $\mathscr{F}$ there exists an integer $m_0 = m_0(\mathscr{F})$ such that $\mathscr{F}(mH)$ is globally generated for every $m \geq m_0$. Consider $\mathscr{F}_i = \mathscr{F}([iD])$ for $0 \leq i \leq k-1$ and $m_i = m_i(\mathscr{F}_i)$ such that $\mathscr{F}_i(mkD)$ is globally generated for every $m \geq m_i$.
Then the assertion holds with $m_2 = k(\max_i m_i)$.

Ample $\Rightarrow$ (III) If H is ample, by Proposition 2.4.2, for every i with $0 \leq i \leq k-1$, there exists an integer t_i such that $tH + [iD]$ is globally generated for every $t \geq t_i$. Also there exists $s \in \mathbb{N}$ such that tH is very ample for every $t \geq s$.
Let $r = \max t_i$ and $t \geq s + r$. We get

$$[mD] = tH + [iD] = (t-r)H + (rH + [iD])$$

that is a very ample divisor because it is a sum of an ample and a globally generated divisor. To conclude we get the statement for $m_3 = k(s+r)$.

Ample $\Rightarrow$ (IV) For what we said above *Ample $\Rightarrow$ (III)* and obviously *(III)$\Rightarrow$ (IV)* with $m_4 = m_3$.

Ample $\Rightarrow$ (V) By Theorem 1.3.9 and by Lemma 2.5.5 as $\mathcal{O}_X([iD])$ has rank 1 we have that:

$$\chi(\mathcal{O}_X([mD]) = \chi(\mathcal{O}_X([iD])(tH)) = \frac{(H^n)}{n!}t^n + O(t^{n-1}) \overset{t\to+\infty}{\longrightarrow} +\infty$$

by Proposition 2.4.2 because of the ampleness of H.

$\square$

Claim 2.5.7. For any $D \in \mathbf{Div}_{\mathbb{R}}(X)$ we have that (VI), (VII), (VIII), (IX) and (X) are equivalent to the definition of ampleness for $\mathbb{R}$-divisors.

(VI) The implication *Ample* $\Rightarrow$ *(VI)* is obvious both for $\mathbb{Q}$ and $\mathbb{R}$-divisors. In fact if D is an ample $\mathbb{R}$-divisor, it is a finite sum in the form $\sum c_i A_i$ where $c_i > 0$ and A_i is an ample and integral divisor so that by Remark 2.1.13, $(D^{(\dim V)}.V) \geq (\sum c_i)^{(\dim V)} > 0$.

The other implication is natural for $\mathbb{Q}$-divisors: in fact if we consider $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ such that (VI) holds, we also know that there exists an integer $m > 0$ such that mD is an integral divisor and (VI) is also valid for mD, so mD is ample and so is D.

For $\mathbb{R}$-divisors it has been proved by Campana-Peternell as we will see in the next chapter (3.4.3).

(VII) The implication *Ample* $\Rightarrow$ *(VII)* is obvious. In fact by Definition 2.1.9 if D is an ample divisor, $D = \sum c_i A_i$ where A_i is an ample integral divisor and $\mathbb{R} \ni c_i \geq 0$. Then by Seshadri's criterion (Proposition 2.4.1) over $\mathbb{Z}$ for all i, there exists $\varepsilon_i > 0$ such that

$$\frac{(A_i.C)}{\mathrm{mult}_x C} \geq \varepsilon_i$$

then

$$\frac{(D.C)}{\mathrm{mult}_x C} \geq \sum_i (c_i \varepsilon_i) = \varepsilon > 0.$$

For the other implication we know that there exists $\varepsilon > 0$ such that $\frac{(D.C)}{\mathrm{mult}_x C} \geq \varepsilon$ for every irreducible curve $C \subseteq X$ passing through x. We will proceed by induction over $n = \dim X$. In $n = 1$ there is nothing to prove. For every subvariety $V \subseteq X$ such that $0 < \dim V < \dim X$, $D|_V$ is ample by induction so that by 2.1.12 we only need to prove that $(D^n) > 0$.

To this end, fix any smooth point $x \in X$, and consider the blowing up in this point with exceptional divisor E:

$$\begin{array}{cccc} \mu : & X' & \to & X \\ & \cup & & \cup \\ & E & \to & x \end{array}$$

we claim that the divisor $\mu^*D - \varepsilon E$ is nef on X'. Then by Theorem 2.2.2

$$(D^n)_X - \varepsilon^n = ((\mu^*D - \varepsilon E)^n)_{X'} \geq 0;$$

therefore $(D^n) > 0$ as required.
For the nefness of $(\mu^*D - \varepsilon E)$, fix an irreducible curve $C' \subset X'$ not contained in E and set $C = \mu(C')$, so that C' is the proper transform of C. Then by Lemma 2.5.8 below

$$(C'.E) = \text{mult}_x C.$$

On the other hand,

$$(C'.\mu^*D)_{X'} = (C.D)_X$$

by the projection formula. So the hypothesis of the criterion implies that $((\mu^*D - \varepsilon E).C') \geq 0$. Since $\mathcal{O}(E)$ is a negative line bundle on the projective space E the same inequality certainly holds if $C' \subset E$. Therefore $\mu^* - \varepsilon E$ is nef and the proof is complete.

Lemma 2.5.8. ([Laz04a] Lemma 5.1.10) Let V be a variety and $x \in V$ a fixed point. Denote by $\mu : V' \to V$ the blowing-up of V at x, with exceptional divisor $E \subseteq V'$. Then

$$(-1)^{(1+\dim V)} \cdot (E^{\dim V}) = \text{mult}_x V.$$

(VIII) The inequality is equivalent to the condition that $D - \varepsilon H$ be nef. If we consider that the inequality holds, then by Corollary 2.2.4 we have that $D = (D - \varepsilon H) + \varepsilon H$ is ample.
Conversely, by the openness of the ample cone (2.1.16), if D is ample then $D - \varepsilon H$ is even ample for $0 \leq \varepsilon \ll 1$.

(IX) & (X) The proof of this equivalence is the same as the one we used in Proposition 2.4.1 (*Ample* $\Rightarrow$ *(VIII)* $\Rightarrow$ *(IX)* $\Rightarrow$ *(X)* $\Rightarrow$ *Ample*).

□

2.6 Ampleness for $\mathbb{R}$-divisors

In this section we will try to formulate a proposition as complete as possible for $\mathbb{R}$-divisors, always referring to the properties of Proposition 2.4.1.

We are now beginning a discussion similar to that one made for $\mathbb{Q}$-divisors:

Claim 2.6.1. The affirmations (1) and (4) in Proposition 2.4.1, when we replace mD with its integral part, are not equivalent to the concept of ampleness for $\mathbb{R}$-divisors.

Proof:

Obviously, the example in the previous section (2.5.1) is still valid for $\mathbb{R}$-divisors.

□

Claim 2.6.2. If (III) of Proposition 2.5.3 holds for an $\mathbb{R}$-divisor D, then D is nef.

Proof:

Suppose that there exists an irreducible curve C such that $D.C < 0$. Since in $N^1(X)_{\mathbb{R}}$ we have that $\lim_{n\to\infty} \left[\frac{[mD]}{m}\right]_{num} = [D]_{num}$ we get

$$0 > D.C = \left(\lim_{m\to\infty} \frac{[mD]}{m}\right).C = \lim_{m\to\infty} \frac{[mD].C}{m} \geq 0$$

contradiction.

□

Remark 2.6.3. If (III) of Proposition 2.5.3 holds for an $\mathbb{R}$-divisor D over a surface, then D is ample.

Proof:

We first prove that D is strictly nef.

By Claim 2.6.2 D is nef. Suppose that there exists an irreducible curve $C \subset X$ such that $D.C = 0$.

We even know that for every $m > m_0$, $[mD].C \geq 1$.
Also

$$0 = mD.C = [mD].C + \{mD\}.C \Rightarrow \{mD\}.C = -[mD].C \leq -1.$$

If we write $D = \sum a_i D_i,\ a_i \in \mathbb{R}$, D_i prime divisors, then there exists j such that $(D_j.C) < 0$ and so $D_j = C$, $(C^2) < 0$ and $(D_i.C) \geq 0$ for all $i \neq j$.
By Weyl's principle ([KN74]) for every $a \in \mathbb{R}$ and for every $0 < \varepsilon < 1$ there exists an integer $k \gg 0$ such that $\{ka\} < \varepsilon$. Choose $\varepsilon = \frac{-1}{C^2}$ and $k > m_0$ to obtain that $\{ka_j\}(D_j.C) = \{ka_j\}(C^2) > -1$ and we get an absurd.

For the ampleness, by Theorem 3.3.15, $D^2 > 0$ because, for $m \geq m_0$, $mD = [mD] + \{mD\}$ is a sum of an ample and an effective divisor that is a big divisor (Proposition 3.3.14).

□

Proposition 2.6.4 (Ampleness for $\mathbb{R}$-divisors). Let $D \in \mathrm{Div}_{\mathbb{R}}(X)$ be a Cartier divisor on a normal projective variety X, and let $\mathcal{O}_X([D])$ be the associated line bundle (sometimes we will think $[D]$ as a Weil divisor by the canonical correspondence). The following statements are equivalent to the definition of ampleness for $\mathbb{R}$-divisors:

i) Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_2 = m_2(\mathscr{F})$ such that $\mathscr{F} \otimes \mathcal{O}_X([mD])$ it is globally generated $\forall m \geq m_2$;

ii) **(Nakai-Moishezon-Kleiman criterion)**

$$\int_V c_1(\mathcal{O}_X(D))^{\dim(V)} > 0$$

for every positive-dimensional subvariety $V \subseteq X$;

iii) **(Seshadri's criterion)** There exists a real number $\varepsilon > 0$ such that

$$\frac{(D.C)}{\mathrm{mult}_x C} \geq \varepsilon$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x;

iv) Let H be an ample divisor. There exists a positive number $\varepsilon > 0$ such that

$$\frac{(D.C)}{(H.C)} \geq \varepsilon$$

for every irreducible curve $C \subseteq X$;

v) **(Via cones)** $\overline{\mathrm{NE}}(X) - \{0\} \subseteq D_{>0}$.

vi) There exists a neighborhood U of $[D]_{num} \in \mathrm{N}^1(X)_{\mathbb{R}}$ such that $U \backslash \{[D]_{num}\} \subseteq \mathrm{Amp}(X)$.

Proof:

If D is ample then $D = \sum a_i A_i$ where A_i is an ample integral divisor and $a_i > 0$, $a_i \in \mathbb{R}$. We can now consider $n \gg 0$ such that $nA_i = H_i$ is a very ample integral divisor so that, if $c_i = \frac{a_i}{n}$, $D = \sum c_i H_i$.

Ample $\Rightarrow$ (i) With the notation of Note 2.1.10, for every $j \in \{1, \ldots, s\}$ there exists n_j such that $\mathscr{F}(T_{k_j})(nH_1)$ is globally generated for every $n \geq n_j$. Let $n'_i \geq \frac{n_i}{c_1}$ and consider

$$m_1 = \max_i(n'_i);$$

then for every $m \geq m_1$,

$$\mathscr{F}([mD]) = \mathscr{F}(T_{k_j})([mc_1]H_1)\left(\sum_{i \geq 2}[mc_i]H_i\right)$$

for some j. But this is a tensor product of a globally generated coherent sheaf and a very ample divisor, whence globally generated.

(i) $\Rightarrow$ Ample We first prove that $[mD]$ is very ample for all $m \geq m_0$. Let H be a very ample divisor and consider $\mathscr{F} = \mathcal{O}_X(-H)$. Then, by (i), there exists m_0 such that for all $m \geq m_0$, $[mD] - H$ is globally generated, so that

$$[mD] = ([mD] - H) + H$$

is a sum of a very ample and a globally generated integral divisor, that is very ample and we are done. Also by Claim 2.6.2 D is nef.

In particular, if D it is not ample, by Proposition 2.5.3, there exists $0 \neq \gamma \in \overline{\mathrm{NE}}(X)$ such that $(D.\gamma) = 0$. If $D = \sum a_i D_i$ $a_i \in \mathbb{R}$, D_i prime divisor, then

$$|(\{mD\}.\gamma)| \leq \sum \{ma_i\}|(D_i.\gamma)| < +\infty.$$

For $m \geq m_0$, $([mD].\gamma) > 0$, so $\{mD\}.\gamma < 0$ so that there exists D_i such that $(D_i.\gamma) < 0$. We want to show that in this case we obtain

$$\lim_{m \to \infty} |(\{mD\}.\gamma)| = +\infty$$

that is absurd. In fact, for $m \geq m_0$,

$$|(\{mD\}.\gamma)| = -(\{mD\}.\gamma) = ([mD].\gamma).$$

Also, if we fix any real number M, there exists $k \in \mathbb{N}$ such that $-k(D_i.\gamma) > M$.
By (i) there exists m_1 such that $kD_i + [mD]$ is globally generated for all $m \geq m_1$.

Let us now consider $m \geq m_0,\ m_1$:

$$|(\{mD\}.\gamma)| = ([mD].\gamma) = (([mD] + kD_i - kD_i).\gamma) > M$$

since $[mD] + kD_i$ is nef, and we are done.

To conclude we get the statement by Claim 2.5.7.

□

Remark 2.6.5. The equivalences (ii)-(vi) with the concept of ampleness where already known, the equivalence of (i) with the concept of ampleness is original.

For $\mathbb{Q}$-divisors it has been quite easy to extend the properties because we have been helped by the existence, for a $\mathbb{Q}$-divisor D, of an integer k such that $kD \in \mathrm{Div}(X)$. For $\mathbb{R}$-divisors we found serious difficulties in working with the integral part so that we haven't been able to make a complete discussion on the equivalence of all the properties. However we have been able to prove the following statement:

Proposition 2.6.6 (Properties of ampleness for $\mathbb{R}$-divisors). Let $D \in \mathrm{Div}_{\mathbb{R}}(X)$ be an ample Cartier divisor on a normal projective variety X, and let $\mathcal{O}_X([D])$ be the associated line bundle (sometimes we will think $[D]$ as a Weil divisor by the canonical correspondence). Then:

a) Given any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m_1 = m_1(\mathscr{F})$ having the property that

$$H^i(X, \mathscr{F} \otimes \mathcal{O}_X([mD])) = 0 \quad \forall i > 0,\ m \geq m_1;$$

b) There is a positive integer m_2 such that $\mathcal{O}_X([mD])$ is very ample $\forall m \geq m_2$;

c) For every subvariety $V \subseteq X$ of positive dimension, there is a positive integer $m_3 = m_3(V)$, such that for every $m \geq m_3$ there exists a non-zero section $0 \neq s = s_{V,m} \in H^0(V, \mathcal{O}_V([mD]))$, such that s vanishes at some point of V;

d) For every subvariety $V \subseteq X$ of positive dimension, $\chi(V, \mathcal{O}_V([mD])) \to +\infty$ as $m \to +\infty$;

Proof:

If D is ample then $D = \sum a_i A_i$ where A_i is an ample integral divisor and $a_i > 0$, $a_i \in \mathbb{R}$. We can now consider $n \gg 0$ such that $nA_i = H_i$ is a very ample integral divisor so that, if $c_i = \frac{a_i}{n}$, $D = \sum c_i H_i$.

Ample ⇒ (a) In analogous way to Ample ⇒ (i) of Proposition 2.6.4 and by Theorem 2.2.7, there exists an integer m_2 such that

$$H^i(X, \mathscr{F}([mD])) = H^i\left(X, \mathscr{F}(T_{k_j})([mc_1]H_1)\left(\sum_{i\geq 2}[mc_i]H_i\right)\right) = 0$$

for all $m \geq m_2$ and all $i \geq 1$.

Ample ⇒ (b) In analogous way to Ample ⇒ (i) of Proposition 2.6.4 with $\mathscr{F} = \mathcal{O}_X$.

Ample ⇒ (c) Passing through the property (b) we get the statement. In fact (c) obviously holds for any very ample divisor.

Ample ⇒ (d) Let us consider two ample integral effective divisors $A, B \subseteq V$ and the canonical exact sequence of A:

$$0 \to \mathcal{O}_V(B) \to \mathcal{O}_V(A+B) \to \mathcal{O}_A(A+B) \to 0$$

so that we obtain the cohomological long exact sequence:

$$0 \to H^0(\mathcal{O}_V(B)) \xrightarrow{f} H^0(\mathcal{O}_V(A+B)) \xrightarrow{h} H^0(\mathcal{O}_A(A+B)) \to \cdots$$

By the ampleness of A and B, we get that f cannot be surjective, so that h is not the zero map, whence

$$h^0(\mathcal{O}_V(A+B)) - h^0(\mathcal{O}_V(B)) \geq 1.$$

By *ample ⇒ (b)* we know that there exists a positive integer m_0, such that $[mD]$ is very ample for all $m \geq m_0$. Also, by Note 2.1.10, $[mD] = \sum [mc_i]H_i + T_k$.
Let $n > m > m_0$, such that $T_k(m) = T_k(n)$; then we obtain a new divisor in the form $[nD] = \sum[nc_i]H_i + T_k$, that is $[nD] = [mD] + \sum e_iH_i$, $e_i \in \mathbb{N}$ where, if $e_i = 0\ \forall i$, we take $n' > n$. Let us consider an increasing sequence of those n.
Since $[mD]$ and $\sum e_iH_i$ are very ample, we get that

$$h^0(\mathcal{O}_V([nD]|_V)) \xrightarrow{n\to\infty} +\infty.$$

To conclude, since *ample* ⇒ *(a)* we get $h^i(V, \mathcal{O}_V([mD]|_V)) = 0$ for all $i \geq 1$ and for all $m \gg 0$, whence:

$$\chi(\mathcal{O}_V([nD])) \stackrel{n\to\infty}{\longrightarrow} +\infty.$$

□

Chapter 3

Big and Pseudoeffective Divisors

3.1 Fields of Rational Functions

We will now introduce a bit of a theory that is not properly pertinent but it will be useful subsequently for a more complete definition of a new class of divisors, the *big divisors*.

Let k be a field, and let $R = \bigoplus_\nu R_\nu$ be a graded k-domain such that $R_\nu = 0$ for all $\nu < 0$. Let $Q(R)$ be the quotient field of R and R^* the multiplicative subset of all nonzero homogeneous elements. Then the quotient ring $(R^*)^{-1}R$ is a graded k-domain, and its degree 0 part $((R^*)^{-1}R)_0$ is a field which we denote by $Q((R))$.
Let $\mathbf{N}(R) = \{\nu \in \mathbb{N} | R_\nu \neq 0\}$.

Lemma 3.1.1. Let R be as above with $R \neq R_0$, and $S = \bigoplus_\nu S_\nu$ a graded k-subalgebra of R. For $\nu \geq 0$, let $S_0[S_\nu]$ denote the graded subdomain of S generated by S_0 and S_ν. Then the integral closure of S in R is graded, and

(i) if $Q(R)$ is finitely generated over k, then there exists $n \gg 0$ such that $Q((S_0[S_\nu])) = Q((S))$ for all $\nu \in \mathbf{N}(S)_{\geq n}$;

(ii) if S is integrally closed in R, then $Q((S))$ is algebraically closed in $(S^*)^{-1}R$.

<u>*Proof:*</u>

(i) If we consider $\mu,\ \nu \in \mathbf{N}(S)$ with $\mu|\nu$, since $S_\mu^{\nu/\mu} \subseteq S_\nu$, for every $\xi \in Q((S_0[S_\mu]))$, $\xi = \frac{a}{b}$ where $a \in S_0[S_\mu]$, $b \in S_0[S_\mu]^*$ we have that $a^{\nu/\mu}$, $b \cdot a^{(\nu/\mu)-1} \in S_0[S_\nu]$ and:

$$Q((S_0[S_\nu])) \ni \frac{a^{\nu/\mu}}{b \cdot a^{(\nu/\mu)-1}} = \frac{a}{b} \Rightarrow Q((S_0[S_\mu])) \subseteq Q((S_0[S_\nu])).$$

Now $Q(R)$ is finitely generated over k so that even $Q((S)) = \bigcup_{\mu \in \mathbf{N}(S)} Q((S_0[S_\mu]))$ is, and we can choose $\mu \in \mathbf{N}(S)$ such that $Q((S)) = Q((S_0[S_\mu]))$. Consider $d = gcd(\mathbf{N}(S))$, let $n \gg 0$ such that $\mathbf{N}(S)_{\geq n-\mu} = (d\mathbb{N})_{\geq n-\mu}$, then if $\nu \in \mathbf{N}(S)_{\geq n}$ we have that $S_{\nu-\mu} \neq 0$ so there exists an element η so that for every $\xi = \frac{a}{b} \in Q((S_0[S_\mu]))$:

$$Q(S_0[S_\nu]) \ni \frac{\eta \cdot a}{\eta \cdot b} = \frac{a}{b} \Rightarrow Q((S_0[S_\mu])) \subseteq Q((S_0[S_\nu])).$$

(ii) Since S is integrally closed in R, so is $(S^*)^{-1}S$ in $(S^*)^{-1}R$, and hence $Q((S))$ is algebraically closed in $(S^*)^{-1}R$.

□

Let X be a variety, and let D be a Cartier divisor on it. Let

$$R(X,D) = \bigoplus_{\nu \geq 0} H^0(X, \mathcal{O}_X(\nu D))$$

be the graded $\mathbb{C}$-domain, which may be viewed as the coordinate ring of the line bundle $\mathscr{L}(D)$, and let $\mathbf{N}(X,D) = \mathbf{N}(R(X,D))$ and $Q((X,D)) = Q((R(X,D)))$.

When X is normal and projective, the $\mathcal{D}$-*dimension* $\kappa(X,D)$ is defined as

$$\kappa(X,D) = \begin{cases} -\infty & \text{if } \mathbf{N}(X,D) = \emptyset \\ tr.deg_{\mathbb{C}} Q((X,D) & \text{if } \mathbf{N}(X,D) \neq \emptyset \end{cases}$$

If $\nu \in \mathbf{N}(X,D)$, then $R(X,D)_\nu = H^0(X, \mathcal{O}_X(\nu D))$ induces a rational map

$$\varphi_{|\nu D|} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(\nu D))).$$

Corollary 3.1.2. Assume that X is normal and projective. If $\mathbf{N}(X,D) \neq \emptyset$, then there exists $n \in \mathbb{N}$ such that $Q(\varphi_{|\nu D|}(X)) = Q((X,D))$ for all $\nu \in \mathbf{N}(X,D)_{\geq n}$.

Proof:

Follows directly from Lemma 3.1.1

□

3.2 Asymptotic theory

Definition 3.2.1 (Semigroup and exponent of a line bundle). Let $\mathscr{L}$ be a line bundle on the projective variety X:

- the *semigroup* of $\mathscr{L}$ is the set

$$\mathbf{N}(\mathscr{L}) = \mathbf{N}(X, \mathscr{L}) = (m \geq 0 | H^0(X, \mathscr{L}^{\otimes m}) \neq 0);$$

- assuming $\mathbf{N}(\mathscr{L}) \neq (0)$, the *exponent* $e = e(\mathscr{L})$ of $\mathscr{L}$ is a natural number such that all sufficiently large elements of $\mathbf{N}(\mathscr{L})$ are multiples of e and all sufficiently large multiples of e appear in $\mathbf{N}(\mathscr{L})$ and it is the largest number with those properties.

Given $m \in \mathbf{N}(\mathscr{L})$ we consider the rational mapping

$$\varphi_m = \varphi_{|\mathscr{L}^{\otimes m}|} : X \dashrightarrow \mathbb{P}H^0(X, \mathscr{L}^{\otimes m})$$

and we denote $Y_m = \overline{\varphi_m(X)} \subseteq \mathbb{P}H^0(X, \mathscr{L}^{\otimes m})$ the closure of its image.

Definition 3.2.2 (Iitaka dimension).

- *if X is normal* the *Iitaka dimension* of $\mathscr{L}$ is defined to be

$$\kappa(\mathscr{L}) = \kappa(X, \mathscr{L}) = \max_{m \in \mathbf{N}(\mathscr{L})} \{\dim Y_m\},$$

  if $H^0(X, \mathscr{L}^{\otimes m}) = 0 \; \forall m > 0$, we put $\kappa(\mathscr{L}) = -\infty$;

- *if X is not normal*, consider the normalization $\nu : X' \to X$ and set $\kappa(X, \mathscr{L}) = \kappa(X', \nu^* \mathscr{L})$.

Note 3.2.3. By Corollary 3.1.2 we have that the definitions of $\mathcal{D}$-dimension and Iitaka dimension are equivalent.

Corollary 3.2.4. Let $\mathscr{L}$ be a line bundle on a normal projective variety X, and set $\kappa = \kappa(X, \mathscr{L})$. Then there are constants $a, A > 0$ such that

$$a \cdot m^\kappa \leq h^0(X, \mathscr{L}^{\otimes m}) \leq A \cdot m^\kappa$$

for all sufficiently large $m \in \mathbf{N}(X, \mathscr{L})$.

3.3 Big line bundles and divisors

Definition 3.3.1 (Big). A line bundle $\mathscr{L}$ on a projective variety X is *big* if $\kappa(X, \mathscr{L}) = \dim X$. A Cartier divisor D on X is *big* if $\mathcal{O}_X(D)$ is so.

Lemma 3.3.2. Assume that X is a projective variety of dimension n. A divisor D on X is big if and only if there is a constant $C > 0$ such that

$$h^0(X, \mathcal{O}_X(mD)) \geq C \cdot m^n$$

for all sufficiently large $m \in \mathbf{N}(X, D)$.

Proposition 3.3.3 (Kodaira's lemma). Let D be a big Cartier divisor and F an arbitrary effective Cartier divisor on X. Then

$$H^0(X, \mathcal{O}_X(mD - F)) \neq 0$$

for all sufficiently large $m \in \mathbf{N}(X, D)$.

Proof:

Suppose that $\dim X = n$ and consider the exact sequence of F

$$0 \to \mathcal{O}_X(-F) \to \mathcal{O}_X \to \mathcal{O}_F \to 0$$

tensored by $\mathcal{O}_X(mD)$:

$$0 \to \mathcal{O}_X(mD - F) \to \mathcal{O}_X(mD) \to \mathcal{O}_F(mD) \to 0.$$

Since D is big, by the Lemma 3.3.2 there is a constant $C > 0$ such that $h^0(X, \mathcal{O}_X(mD)) \geq c \cdot m^n$ for sufficiently large $m \in \mathbf{N}(X, D)$. On the other hand $\dim F = n-1$ so that $h^0(F, \mathcal{O}_F(mD))$ grows at most like m^{n-1} (Example 2.1.7). Therefore

$$h^0(X, \mathcal{O}_X(mD)) > h^0(F, \mathcal{O}_F(mD)$$

for large $m \in \mathbf{N}(X, D)$ and the assertion follows by the exact sequence.

□

Corollary 3.3.4 (Characterization of big integral divisors). Let D be a Cartier divisor on a normal variety X. Then the following are equivalent:

1. D is big;
2. there exists an integer $a \in \mathbb{N}$ such that $\varphi_{|mD|}$ is birational for all $m \in \mathbf{N}(X, D)_{\geq a}$;
3. $\varphi_{|mD|}$ is generically finite for some $m \in \mathbf{N}(X, D)$;
4. for any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m = m(\mathscr{F})$ such that $\mathscr{F} \otimes \mathcal{O}_X(mD)$ is generically globally generated, that is such that the natural map

$$H^0(X, \mathscr{F} \otimes \mathcal{O}_X(mD)) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathscr{F} \otimes \mathcal{O}_X(mD)$$

is generically surjective;

5. for any ample integral divisor A on X, there exists a positive integer $m > 0$, and an effective divisor N such that $mD \equiv_{lin} A + N$;

6. same as in (5) for some integral ample divisor A;

7. there exists an ample integral divisor A, a positive integer $m > 0$ and an effective divisor N such that $mD \equiv_{num} A + N$.

Proof:

(1) $\Rightarrow$ (2) Follows directly from Corollary 3.1.2.

(2) $\Rightarrow$ (3) Obvious by the birationality.

(3) $\Rightarrow$ (4) Let A be a very ample divisor such that $\mathscr{F} \otimes \mathcal{O}_X(A)$ is generated by the global sections. By the hypothesis $\dim \varphi_{|mD|}(X) = \dim X$ and $\dim \varphi_{|mD|}(A) \neq \dim \varphi_{|mD|}(X)$ so that $\varphi_{|mD|}(A)$ is contained in a hypersurface section of $\varphi_{|mD|}(X)$, say of degree c. Thus $|mcD - A| \neq \emptyset$, and let $E \in |mcD - A|$.
Consider the natural application by the exact sequence of E:

$$H^0(\mathscr{F}(A)) \to H^0(\mathscr{F}(E + A)),$$

now let ξ be the generic point of X and tensoring by $\mathcal{O}_{X,\xi} = k(\xi)$ we obtain another natural sequence (of $\mathcal{O}_{X,\xi}$-modules):

$H^0(\mathscr{F}(A)) \otimes k(\xi) \to$

$$\begin{aligned} \to H^0(\mathscr{F}(E + A)) \otimes k(\xi) &\xrightarrow{\varphi} \mathscr{F}(E+A) \otimes k(\xi) \cong \mathscr{F}(A) \otimes k(\xi) \\ s \otimes f &\mapsto f \cdot s|_\xi \end{aligned}$$

Since $\mathscr{F}(A)$ is generated by the global sections, we get that $H^0(\mathscr{F}(A)) \to \mathscr{F}(A)$ is surjective, whence φ is surjective and the implication is proved for $E + A \equiv_{lin} mcD$.

(4) ⇒ (5) For every ample integral divisor A, let us consider $\mathscr{F} = \mathcal{O}_X(-A)$. Then, by (4) there exists m such that $\mathcal{O}_X(-A) \otimes \mathcal{O}_X(mD)$ is generically globally generated. This implies that $H^0(X, \mathcal{O}_X(mD - A)) \neq 0$ that is what we needed.

(5) ⇒ (6) Trivial.

(6) ⇒ (7) Trivial.

(7) ⇒ (1) If $mD \equiv_{num} A + N$, then $mD - N \equiv_{num} A$ that is $mD - N$ is ample; for an opportune big multiple r, $rA = H$ is very ample and $rN = N'$ is effective, so that (considering $mr = n$) we have that

$$nD \equiv_{lin} H + N'.$$

But now

$$\kappa(X, D) \geq \kappa(X, H) = \dim X$$

so D is big.

□

Corollary 3.3.5 (Exponent of a big divisor). If D is big then $e(D) = 1$, that is every sufficiently large multiple of D is effective.

Definition 3.3.6 (Big $\mathbb{Q}$-divisors). A $\mathbb{Q}$-divisor D is big if there is a positive integer $m > 0$ such that mD is integral and big.

Definition 3.3.7 (Big $\mathbb{R}$-divisors). An $\mathbb{R}$-divisor $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is big if it can written in the form

$$D = \sum a_i \cdot D_i$$

where each D_i is a big integral divisor and a_i is a positive real number.

Proposition 3.3.8. Let D and D' be $\mathbb{R}$-divisors on X. If $D \equiv_{num} D'$, then D is big if and only if D' is big.

Proof: Like in Corollary 2.1.14.

□

Remark 3.3.9. If D is an integral divisor, D is big in the sense of $\mathbb{Z}$-divisors if and only if it is big in the sense of $\mathbb{R}$-divisors.

Proof: If D is big in the sense of $\mathbb{Z}$-divisors, obviously D can be written as $1 \cdot D$, so that it is a big $\mathbb{R}$-divisor.
If $D = \sum c_i B_i$, where $c_i \in \mathbb{R}$, $c_i > 0$ and B_i is a big integral divisor, as we will see in the Claim 3.3.13, there exists m_0 such that $[m_0 D]$ is an integral big divisor. But in this case we have that $[m_0 D] = m_0 D$ and we get the statement by Corollary 3.3.4.

□

As we have just done for the ampleness, we would extend the various properties of bigness, when it is possible, to $\mathbb{Q}$ and $\mathbb{R}$-divisors referring us to Corollary 3.3.4;
The first step will be to redefine the notion of semigroup:

Definition 3.3.10. Let D be a $\mathbb{R}$-divisor; the *semigroup* of D is the set

$$\mathbf{N}(D) = \mathbf{N}(X, D) = \{m \geq 0 | H^0(X, \mathcal{O}_X([mD])) \neq 0\};$$

Proposition 3.3.11 (Bigness for $\mathbb{Q}$-divisors). Let D be a $\mathbb{Q}$-divisor on a projective variety X. Then the following are equivalent:

I) D is big;

II) there exists an integer $a \in \mathbb{N}$ such that $\varphi_{|[mD]|}$ is birational for all $m \in \mathbf{N}(X, D)_{\geq a}$;

III) $\varphi_{|[mD]|}$ is generically finite for some $m \in \mathbf{N}(X, D)$;

IV) for any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m = m(\mathscr{F})$ such that $\mathscr{F} \otimes \mathcal{O}_X([mD])$ is generically globally generated, that is such that the natural map

$$H^0(X, \mathscr{F} \otimes \mathcal{O}_X([mD])) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathscr{F} \otimes \mathcal{O}_X([mD])$$

is generically surjective;

V) for any ample $\mathbb{Q}$-divisor A on X, there exists an effective $\mathbb{Q}$-divisor N such that $D \equiv_{lin} A + N$;

VI) same as in (V) for some ample $\mathbb{Q}$-divisor A;

VII) there exists an ample $\mathbb{Q}$-divisor A on X and an effective $\mathbb{Q}$-divisor N such that $D \equiv_{num} A + N$.

Proof:

(I)⇒ (II): If D is a big $\mathbb{Q}$-divisor, we know that there exists $k \gg 0$ such that kD is a multiple of an integral big divisor, so that, by Corollary 3.3.4, $kD \equiv_{lin} A + E$, A ample and E effective $\mathbb{Z}$-divisors. By Lemma 2.5.5:

$$[mD] \equiv_{lin} tkD + [iD] \equiv_{lin} tA + tE + [iD], \qquad 0 \leq i \leq k-1.$$

Let $r \in \mathbb{N}$ such that rA is very ample and let $s \in \mathbb{N}$ such that $sA+[iD]$ is globally generated for all $i = 1, \ldots, k-1$. Then, for every $m \geq k(r+s)$, we obtain:

$$[mD] \equiv_{lin} (t-s)A + (sA + [iD]) + tE = H + tE,$$

where H is very ample (for very ample + globally generated is very ample) and tE is effective. To conclude

$$\kappa(X, [mD]) \geq \kappa(X, H) = \dim X,$$

hence $\kappa(X, [mD]) = \dim X$ by Definition 3.2.2.

(II)⇒ (I) If (II) holds it is also true that $\varphi_{|m(kD)|}$ is birational for all $m \in \mathbf{N}(X, kD)_{\geq ka}$. Now, by Corollary 3.3.4, kD is a big integral divisor, and so D is a big $\mathbb{Q}$-divisor by definition.

- **(III),...,(VII) are equivalent to (I):**

The implication *(I) ⇒ (III),...,(VII)* is obvious, in fact if D is big we only need to consider an integer m for which $mD = D'$ is an integral big divisor. Now for those properties it is sufficient that there exists an integer satisfying them. Then we only need to consider the product of this integer with m and we obtain the statement.

Now we need to prove that if those properties hold, D is a big divisor.

(III) ⇒ (I) By (III) and by Corollary 3.3.4 we know that $[mD]$ is big; now $mD = [mD] + \{mD\}$ where $\{mD\}$ is an effective $\mathbb{Q}$-divisor. Also, there exists an integer $k > 0$ such that kmD is an integral divisor, where

$$kmD = k[mD] + k\{mD\}$$

that is a sum of a big and an effective integral divisor. Now, by Corollary 3.3.4, kmD is big and accordingly $D = \frac{1}{km}(kmD)$ is.

(IV) ⇒ (V) For every ample $\mathbb{Q}$-divisor A, let us consider an effective integral divisor E such that $E - \{A\}$ is effective. Let us consider $\mathscr{F} = \mathcal{O}_X(-[A]-E)$. Then, by (IV) there exists m such that $\mathcal{O}_X(-[A]-E)\otimes\mathcal{O}_X([mD])$ is generically globally generated. This implies that $H^0(X, \mathcal{O}_X([mD] - [A] - E)) \neq 0$ so that, there exists an effective integral divisor F such that

$$[mD] \equiv_{lin} [A] + E + F \Rightarrow mD \equiv_{lin} [mD] + \{mD\} \equiv_{lin}$$
$$\equiv_{lin} A + (E - \{A\}) + (F + \{mD\}),$$

that is what we needed.

(V) ⇒ (I) if this property holds, there exists an integer $k > 0$ such that kmD, kA and kN are integral divisors, where kA is ample and

kN is effective. Now, by Corollary 3.3.4, kmD is big, and so $D = \frac{1}{km}(kmD)$ is.

(VI) $\Rightarrow$ (I) like in (V).

(VII) $\Rightarrow$ (I) like in (V).

$\square$

Remark 3.3.12. If B is a big rational divisor and N is an effective rational divisor, then $B + sN$ is big for all $s \in \mathbb{R}, s > 0$.

Proof:

If $s \in \mathbb{Q}$ it is obvious by Proposition 3.3.11. If $s \in \mathbb{R} - \mathbb{Q}$ we only need to choose two positive rational numbers s_1, s_2 with $s_1 < s < s_2$ and $t \in [0, 1]$ such that $s = ts_1 + (1 - ts_2)$. Then

$$B + sN = t(B + s_1N) + (1 - t)(B + s_2N)$$

that is a positive linear combination of big $\mathbb{Q}$-divisors.

$\square$

Claim 3.3.13. Let D be an $\mathbb{R}$-divisor on a projective variety X. Then the following are equivalent:

1. D is big;
2. There exists an integer $m_0 > 0$ such that $[mD]$ is a big integral divisor for all $m \geq m_0$;
3. There exists an integer $m_1 > 0$ such that $[m_1D]$ is a big integral divisor.

Proof:

1 ⇒ 2) If D is a big divisor then $D = \sum a_i D_i$, $a_i \in \mathbb{R}$, $a_i > 0$ and $D_i \in \mathrm{Div}(X)$ big divisors. By Note 2.1.10 we can write $[mD] = \sum [ma_i] D_i + T_k$ for finitely many integral divisors T_k. We also point out that $\{mD\} \in \mathrm{Div}_{\mathbb{R}}(X)$ is an effective divisor.

By Corollary 3.3.4

$$[mD] \equiv_{lin} \sum [ma_i](A_i + E_i) + T_k$$

where A_i is an ample integral divisor and E_i is an effective integral divisor. Now we can choose $r \in \mathbb{N}$ such that tA_i is very ample for every i and for every $t \geq r$. Let $s \in \mathbb{N}$ such that $sA_1 + T_k$ is globally generated for every k. If we take m_0 such that $[m_0 a_i] \geq r$ $\forall i$ and $[m_0 a_1] \geq r + s$, then for all $m \geq m_0$

$$[mD] \equiv_{lin} \sum_{i \geq 2} [ma_i](A_i + E_i) + ([ma_1] - s)A_1 + [ma_1]E_1 + (sA_1 + T_k)$$

where $\sum_{i \geq 2} [ma_i] A_i + ([ma_1] - s)A_1 + (sA_1 + T_k) = H$ is a very ample integral divisor and $\sum_i [ma_i] E_i = E$ is an effective integral divisor so that

$$\kappa(X, [mD]) \geq \kappa(X, H) = \dim X,$$

and we get the statement.

2 ⇒ 3) Trivial.

3 ⇒ 1) We have that $[m_1 D]$ is big. Now by Proposition 3.3.4 there exist an ample integral divisor A and an effective integral divisor E such that

$$[m_1 D] \equiv_{num} A + E \Rightarrow m_1 D \equiv_{num} A + (E + \{m_1 D\})$$

that is the sum of an ample and an effective $\mathbb{R}$-divisor. By Proposition 3.3.8 it is enough to prove that $A + E + \{mD\}$ is big. So we get the statement by Remark 3.3.12: if $D = \sum a_i D_i$, $a_i \in \mathbb{R}$, D_i prime divisors, then

$$\{m_1 D\} = \sum_{i=1}^{s} \{m_1 a_i\} D_i$$

and we can write

$$A + (E + \{m_1 D\}) = \sum_{i=1}^{s} \frac{1}{s}(A + E + s\{m_1 a_i\} D_i).$$

$\square$

Now we are discussing the case of $\mathbb{R}$-divisors:

Proposition 3.3.14 (Bigness for $\mathbb{R}$-divisors). Let D be an $\mathbb{R}$-divisor on a projective variety X. The following are equivalent:

(i) D is big;

(ii) there exists an integer $a \in \mathbb{N}$ such that $\varphi_{|[mD]|}$ is birational for all $m \in \mathbf{N}(X, D)_{\geq a}$;

(iii) $\varphi_{|[mD]|}$ is generically finite for some $m \in \mathbf{N}(X, D)$;

(iv) for any coherent sheaf $\mathscr{F}$ on X, there exists a positive integer $m = m(\mathscr{F})$ such that $\mathscr{F} \otimes \mathcal{O}_X([mD])$ is generically globally generated, that is such that the natural map

$$H^0(X, \mathscr{F} \otimes \mathcal{O}_X([mD])) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathscr{F} \otimes \mathcal{O}_X([mD])$$

is generically surjective;

(v) for any ample $\mathbb{R}$-divisor A on X, there exists an effective $\mathbb{R}$-divisor N such that $D \equiv_{num} A + N$;

(vi) same as in (v) for some ample $\mathbb{R}$-divisor A;

Proof:

(i)⇒ (ii) As in the proof of Claim 3.3.13 there exists a positive integer m_0 such that we can write

$$[mD] \equiv_{lin} H + E$$

for all $m \geq m_0$, where H is a very ample integral divisor and E is an effective integral divisor. Then, obviously, $\varphi_{|[mD]|}$ is birational for all $m \geq m_0$.

(ii)⇒ (iii) Trivial.

The implication ***(iii)⇒ (i)*** is obvious, in fact like above we only need to consider the integer m for which $[mD]$ is big and the statement follows by Claim 3.3.13.

(i)⇒ (iv) D is a big $\mathbb{R}$-divisor, then $D = \sum a_i D_i$, $a_i \in \mathbb{R}$, $a_i > 0$ and D_i big integral divisor. Also by Note 2.1.10 we can write $[mD] = \sum [ma_i] D_i + T_k$ for finitely many integral divisors T_k. Let A be an ample integral divisor, then there exists $m_0 = m(\mathscr{F}(T_k))$ such that $\mathscr{F}(T_k)(m_0 A)$ is globally generated for all k. Let us denote $m_0 A = H$
Since D_i is a big integral divisor, by Corollary 3.3.4 there exists $m_i \in \mathbb{N}$ such that $m_i D_i \equiv_{lin} H + E_i$ where E_i is an effective integral divisor. Also, since D_i is big and integral, there exists n_i such that nD_i is effective for all $n \geq n_i$ by Corollay 3.3.5.

Let $m \gg 0$ such that $[ma_i] - m_i \geq n_i$ for all i, then

$$\mathscr{F}(T_k)([mD]) = \mathscr{F}(T_k)(\sum_{i=1}^{s}(([ma_i] - m_i)D_i + m_i D_i))$$

where $([ma_i] - m_i)D_i$ is effective and $m_i D_i \equiv_{lin} H + E_i$.
Then $\mathscr{F}(T_k)([mD]) = \mathscr{F}(T_k)(sH+E)$ where E is effective and $\mathscr{F}(T_k)(sH)$ is globally generated and we are done.

(iv)⇒ (v) Like in Proposition 3.3.11 *(IV)⇒ (V)* where we replace $\equiv_{lin}$ by $\equiv_{num}$.

(v)⇒ (vi) Trivial.

(vi)⇒ (i) By (vi) we have that $D \equiv_{num} A + N$ where A is an ample and N is an effective $\mathbb{R}$-divisor. By the openness of amplitude 2.1.16, with opportune $E_1, \ldots, E_r$ effective $\mathbb{R}$-divisors, the divisor

$$H = A - \varepsilon_1 E_1 - \ldots - \varepsilon_r E_r$$

is an ample $\mathbb{Q}$-divisor, for $0 < \varepsilon_i \ll 1$. Also, $M = N + \varepsilon_1 E_1 + \ldots + \varepsilon_r E_r$ is an effective $\mathbb{R}$-divisor.
Now we can write $M = \sum_{i=1}^{s} c_i D_i$ where $c_i \in \mathbb{R},\ c_i > 0$ and D_i is a prime divisor.
Then the results follows by Remark 3.3.12 and Proposition 3.3.8, since we can write

$$D \equiv_{num} \sum_{i=1}^{s} \left(\frac{1}{s} H + c_i D_i \right).$$

□

Theorem 3.3.15. Let X be a projective variety of dimension n, and let D and E be nef $\mathbb{Q}$-divisors on X. Assume that

$$(D^n) > n \cdot (D^{n-1} \cdot E).$$

Then $D - E$ is big.

Proof:
By continuity we can replace D and E respectively by $D + \varepsilon H$ and $E + \varepsilon H$ for any ample divisor H and sufficiently small $0 < \varepsilon \ll 1$. Therefore we may suppose that both D and E are ample. We can also replace by a multiple both D and E without altering the inequality, so that we can assume D and E to be integral and very ample.
Now choose a sequence $E_1, E_2, E_3 \cdots \in |E|$ of general divisors linearly equivalent to E and fix an integer $m \geq 1$. Then

$$\mathcal{O}_X(m(D - E)) \cong \mathcal{O}_X \left(mD - \sum_{i=1}^{m} E_i \right),$$

so $H^0(X, \mathcal{O}_X(m(D - E))) \cong H^0(\mathcal{O}_X (mD - \sum E_i))$ is identified with the group of sections of $\mathcal{O}_X(mD)$ vanishing on each of the divisors $E_1, \ldots, E_m$.

Now consider the exact sequence of $\sum E_i$ tensored by $\mathcal{O}_X(mD)$:

$$0 \to H^0\left(X, \mathcal{O}_X(mD - \sum E_i)\right) \to H^0(X, \mathcal{O}_X(mD)) \to \bigoplus_{i=1}^{m} H^0(E_i, \mathcal{O}_{E_i}(mD)).$$

By asymptotic Riemann-Roch on X and each E_i:

$$\begin{aligned} h^0(X, \mathcal{O}_X(m(D-E))) &\geq h^0(mD) - \sum_{i=1}^{m} h^0(E_i, \mathcal{O}_{E_i}(mD)) = \\ &= \frac{(D^n)}{n!}m^n - \sum_{i=1}^{m} \frac{(D^{n-1} \cdot E_i)}{(n-1)!}m^{n-1} + O(m^{n-1}) = \\ &= \frac{(D^n)}{n!}m^n - n\frac{(D^{n-1} \cdot E)}{n!}m^n + O(m^{n-1}). \end{aligned}$$

In particular, by the hypothesis, $h^0(X, \mathcal{O}_X(m(D-E)))$ grows like a positive multiple of m^n and the assertion follows.

□

Theorem 3.3.16 (Bigness of nef divisors). Let D be a nef divisor on a projective variety X of dimension n. Then D is big if and only if its top self-intersection is strictly positive, that is $(D^n) > 0$.

3.4 Pseudoeffective and big cones

Definition 3.4.1 (Big and pseudoeffective cones). The *big cone*

$$\mathrm{Big}(X) \subseteq N^1(X)_{\mathbb{R}}$$

is the convex cone of all big $\mathbb{R}$-divisors classes on X. The *pseudoeffective cone*

$$\overline{\mathrm{Eff}}(X) \subseteq N^1_{\mathbb{R}}(X)$$

is the closure of the convex cone spanned by the classes of all effective $\mathbb{R}$-divisors. A divisor $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is *pseudoeffective* if its class lies in the pseudoeffective cone.

Theorem 3.4.2. The big cone is the interior of the pseudoeffective cone and the pseudoeffective cone is the closure of the big cone:

$$\mathrm{Big}(X) = \mathrm{int}(\overline{\mathrm{Eff}}(X)) \quad , \quad \overline{\mathrm{Eff}}(X) = \overline{\mathrm{Big}(X)}.$$

Theorem 3.4.3 (Nakai criterion for $\mathbb{R}$-divisors). Let X be a projective variety, and let $\delta \in N^1(X)_{\mathbb{R}}$ be a class having positive intersection with every irreducible subvariety of X. In other words, assume that

$$(\delta^{\dim V} \cdot V) > 0$$

for every $V \subseteq X$ of positive dimension. Then δ is an ample class.

Proof: **(by Campana and Peternell** ([Laz04a], Theorem 2.3.18)**)**
We want to proceed by induction on the dimension of X: if $\dim X = 1$ it is obvious. Let now $n = \dim X > 1$ and assume that for every proper subvariety $Y \subset X$ the restriction

$$\delta|_Y \in N^1(Y)_{\mathbb{R}}$$

is an ample class on Y.
Let us now choose ample divisors $H_1, \dots, H_r$ whose classes $h_1, \dots, h_r$ span $N^1(X)_{\mathbb{R}}$. Obviously there exists $\varepsilon_1, \dots, \varepsilon_r \in \mathbb{R}$, $\varepsilon_i > 0$ such that

$$\delta' = \delta + \varepsilon_1 h_1 + \cdots + \varepsilon_r h_r$$

is the class of a $\mathbb{Q}$-divisor. Also there exist $\eta_1, \dots, \eta_r \in \mathbb{R}$, $\eta_i > 0$ such that

$$\delta'' = \delta - \eta_1 h_1 \cdots - \eta_r h_r$$

is the class of a $\mathbb{Q}$-divisor.
Let us denote $\alpha' = \sum \varepsilon_i h_i, \alpha = \sum \eta_i h_i$, so that

$$\delta + \alpha' \quad , \quad \alpha' + \alpha \; (= \delta' - \delta'') \quad , \quad \delta - \alpha$$

are rational.
Moreover since $(\delta^n) > 0$, we can suppose by taking α and α' sufficiently small that

$$((\delta + \alpha')^n) > n \cdot \left((\delta + \alpha')^{n-1} \cdot (\alpha + \alpha')\right).$$

Now, by Theorem 3.3.15

$$\delta - \alpha = (\delta + \alpha') - (\alpha + \alpha')$$

is represented by an effective $\mathbb{Q}$-divisor E. Denote by $Y_1, \ldots, Y_t \subset X$ the irreducible components of a support of E.
Now $\delta|_{Y_i}$ is ample by the induction hypothesis, so by taking sufficiently small $0 < \varepsilon \ll 1$ we can arrange that each of the restrictions $(\delta - \varepsilon\alpha)|_{Y_i}$ are nef.
Let now $C \subset X$ be any curve. If $C \subset Y_i$ for some i, then $((\delta - \varepsilon\alpha).C) \geq 0$ for what we have just said. On the other hand, if $C \not\subseteq \operatorname{Supp}(E)$, then

$$E.C = ((\delta - \alpha).C) \geq 0,$$

and so $((\delta - \varepsilon\alpha).C) \geq 0$. Thus $\delta - \varepsilon\alpha$ is nef and then δ is ample.

□

3.5 Volume of a Big Divisor

Definition 3.5.1 (Volume of a line bundle). Let X be a projective variety of dimension n, and let $\mathscr{L}$ be a line bundle on X. The *volume* of $\mathscr{L}$ is defined to be the non-negative real number

$$\operatorname{vol}(\mathscr{L}) = \limsup_{m \to \infty} \frac{h^0(X, \mathscr{L}^{\otimes m})}{m^n/n!}.$$

The volume $\operatorname{vol}(D) = \operatorname{vol}_X(D)$ of a Cartier divisor D is defined passing to $\mathcal{O}_X(D)$.

Note 3.5.2. $\operatorname{vol}(\mathscr{L}) > 0$ if and only if $\mathscr{L}$ is big.

Lemma 3.5.3. Let $\mathscr{L}$ be a big line bundle and let A be a very ample divisor on X. If $E, E' \in |A|$ are very general divisors, then

$$\mathrm{vol}_E(\mathscr{L}|_E) = \mathrm{vol}_{E'}(\mathscr{L}|_{E'}).$$

Lemma 3.5.4. Let D be a divisor on X, and $a \in \mathbb{R}$ a fixed positive integer, then

$$\limsup_m \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!} = \limsup_k \frac{h^0(X, \mathcal{O}_X(akD))}{(ak)^n/n!}.$$

Proposition 3.5.5 (Properties of the volume). Let D be a big divisor on a variety X of dimension n.

(i) For a fixed natural number $a > 0$,

$$\mathrm{vol}(aD) = a^n \cdot \mathrm{vol}(D).$$

(ii) Fix any divisor N on X, and any $\varepsilon > 0$. Then there exists an integer $p_0 = p_0(N, \varepsilon)$ such that

$$\frac{1}{p^n} \cdot |\mathrm{vol}(pD - N) - \mathrm{vol}(pD)| < \varepsilon$$

for every $p \geq p_0$.

Proof:

(i) By Lemma 3.5.4 we get

$$\mathrm{vol}(aD) = \limsup_k \frac{h^0(X, \mathcal{O}_X(akD))}{(k)^n/n!} = $$
$$= a^n \limsup_k \frac{h^0(X, \mathcal{O}_X(akD))}{(ak)^n/n!} = a^n \mathrm{vol}(D)$$

(ii) We can always write $N \equiv_{lin} A - B$ as a difference of effective divisors. Also, by the bigness of D, there exists $r \in \mathbb{N}$ such that $rD - B \equiv_{lin} B_1$ is an effective divisor, so that

$$pD - N = (p + r)D - (A + B_1).$$

Also, by (i)

$$\frac{\mathrm{vol}(pD)}{p^n} = \mathrm{vol}(D)$$

and

$$\lim_{p\to\infty} \frac{\mathrm{vol}((p+r)D)}{p^n} = \lim_{p\to\infty} \frac{(p+r)^n}{p^n}\mathrm{vol}(D) = \mathrm{vol}(D).$$

Thus we can assume that N is effective. If we consider another effective divisor N' we can even say that

$$\mathrm{vol}(pD - (N + N')) \leq \mathrm{vol}(pD - N) \leq \mathrm{vol}(pD).$$

So that we can prove the statement for $N + N'$ instead of N. By the arbitrarity of the divisor N', we can choose N' as a very ample divisor such that $N + N'$ is itself very ample. To conclude we can consider N as a very ample divisor. Let us now consider an effective very general divisor $E \in |N|$, then (like in the proof of Theorem 3.3.15 and using Lemma 3.5.3)

$$h^0(X, \mathcal{O}_X(m(pD - N))) \geq h^0(X, \mathcal{O}_X(mpD)) - mh^0(E, \mathcal{O}_E(mpD)).$$

This implies that

$$\begin{aligned}
\mathrm{vol}_X(pD - N) &= \limsup_m \frac{h^0(X, \mathcal{O}_X(m(pD - N)))}{m^n/n!} \geq \\
&\geq \limsup_m \frac{h^0(X, \mathcal{O}_X(mpD)) - mh^0(E, \mathcal{O}_E(mpD))}{m^n/n!} = \\
&= \mathrm{vol}_X(pD) - n \cdot \mathrm{vol}_E(pD|_E),
\end{aligned}$$

where by (i)

$$\mathrm{vol}_E(pD|_E) = p^{n-1}\mathrm{vol}_E(D|_E)$$

and

$$\frac{1}{p^n}|\mathrm{vol}_X(pD - N) - \mathrm{vol}_X(pD)| \leq \left| \frac{-n \cdot p^{n-1} vol_E(D|_E)}{p^n} \right| < \varepsilon$$

for $p \gg 0$.

$\square$

By Proposition 3.5.5 we can give the following:

Definition 3.5.6 (Volume of a $\mathbb{Q}$-divisor). Let D be a $\mathbb{Q}$-divisor. Let us choose $a \in \mathbb{N}$ such that aD is an integral divisor, then we define the volume of D as

$$\mathrm{vol}(D) = \frac{1}{a^n}\mathrm{vol}(aD).$$

Proposition 3.5.7 (Numerical nature of volume). If D, D' are numerically equivalent divisors on X, then

$$\mathrm{vol}(D) = \mathrm{vol}(D').$$

Theorem 3.5.8 (Continuity of volume). Let X be a variety of dimension n, and fix a norm $\| \ \|$ on $N^1(X)_{\mathbb{R}}$ inducing the usual topology on that finite-dimensional vector space. Then there is a constant $C > 0$ such that

$$|\mathrm{vol}(\xi) - \mathrm{vol}(\xi')| \leq (\max(\| \xi \| \ , \ \| \xi' \|))^{n-1} \cdot \| \xi - \xi' \|$$

for any two classes $\xi, \xi' \in N^1(X)_{\mathbb{Q}}$.

By Theorem 3.5.8 we can give the following:

Corollary 3.5.9 (Volume of real classes). The function $\xi \to \mathrm{vol}(\xi)$ on $N^1(X)_{\mathbb{Q}}$ extends uniquely to a continuous function

$$\mathrm{vol} : N^1(X)_{\mathbb{R}} \to \mathbb{R}.$$

Definition 3.5.10 (Volume of an $\mathbb{R}$-divisor). Let D be an $\mathbb{R}$-divisor. The volume of D is the volume of its class in $N^1(X)_{\mathbb{R}}$

$$\mathrm{vol}(D) = \mathrm{vol}([D]_{num}).$$

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