

ON THE EXISTENCE OF SMOOTH VARIETIES OF ANY DIMENSION CARRYING NO ULRICH BUNDLES FOR SOME VERY AMPLE LINE BUNDLE

ANGELO FELICE LOPEZ

ABSTRACT. Using some recent examples of [A] in dimension 2, we prove that for any $n \geq 3$ there exist many smooth n -dimensional varieties X with a very ample line bundle H , such that there is no Ulrich bundle for (X, H) . Moreover, we find many new examples in dimension 2.

1. INTRODUCTION

In their seminal 2003 paper [ES], Eisenbud, Schreyer and Weyman introduced Ulrich vector bundles in the realm of algebraic geometry. Let X be a smooth irreducible variety of dimension $n \geq 1$ and let H be a very ample line bundle on X . A vector bundle $\mathcal{E}$ on X is Ulrich if $H^i(\mathcal{E}(-pH)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$. The authors of [ES] showed what important consequences on the geometry of X could be deduced from an Ulrich bundle and posed the main question of their existence (in fact, for an Ulrich sheaf on any projective scheme). In the case of complete intersections, existence followed by previous results in algebra [HUB]. Since then, many papers have been written to show that Ulrich bundles exist on several classes of pairs (X, H) . Some of these existence results include curves (and any polarization), many surfaces (see for example [B2, F, C1, C2, CoHu, L1]), some threefolds, including Fano threefolds [B2, CFK1, CFK2], and other scattered examples of higher dimensional varieties (see [CMR, ES, B2, CMRPL] and references therein). Moreover, several results (see for example [LM, LMS, L2, LS]) confirmed that the existence of an Ulrich bundle gives a lot of insights on the geometry of X .

Despite of all this work, the main existence problem remained open until the 2026 breakthrough paper [A], where several examples of surface pairs (S, H) not carrying Ulrich bundles were produced. Anghel used the following obstruction to the existence of an Ulrich bundle on a surface pair (S, H) (see [A, Prop. 2.1], improving on a previous idea of Beauville [B1]):

$$H^2 < K_S^2 - 8\chi(\mathcal{O}_S).$$

Our starting point was to notice that the above obstruction has a natural generalization in any dimension (see Lemma 3.1):

$$(n+1)H^n < (K_X^2 - 2c_2(X))H^{n-2}.$$

Using the above and some examples present in [A], we produce natural examples of many n -dimensional smooth pairs (X, H) that do not carry any Ulrich bundle. Our main result is the following

Theorem 1. *Let (S_k, L_k) be the pair consisting of a smooth irreducible surface and a very ample line bundle as in [A, Thm. 1.2]. Let $n \geq 3$, let Y be any smooth irreducible $(n-2)$ -dimensional variety and let M be any very ample line bundle on Y . Let*

$$X = S_k \times Y, \quad H = L_k \boxtimes M$$

and let $X' \in |aH|$, $a \geq 1$ be smooth, $H' = H|_{X'}$. Then for $k \gg 0$ the pairs $(X, H), (X', H')$, and all their successive hyperplane sections until dimension 2, do not carry an Ulrich vector bundle. Moreover, X' is not isomorphic to a (non-trivial) product.

Note that the surface sections of X and X' gives new examples of surface pairs, that is, different from the ones in [A], not carrying Ulrich bundles.

We observe that an explicit bound for k , depending on (n, Y, M) (and a in the second case), can be easily deduced from the proof of the theorem.

Author is partially supported by the GNSAGA group of INdAM..

Mathematics Subject Classification: Primary 14J60. Secondary 14F06.

Given the above results, perhaps at this point the main question that remains open is the existence of Ulrich bundles for sufficiently ample polarizations. The latter is known for surfaces by [CoHu].

2. A NON-PRODUCT CRITERION

We give a simple result that shows that a hypersurface section of a product is not a product.

Lemma 2.1. *Let X_1, X_2 be two smooth irreducible varieties of dimension at least 2 and let H be a very ample line bundle on $X := X_1 \times X_2$. Let $a \geq 1$ and let $D \in |aH|$ be smooth. Then D is not isomorphic to a (nontrivial) product.*

Proof. Consider, for a given smooth variety V , the k -th Newton class

$$p_k(V) := p_k(T_V) = k!ch_k(T_V).$$

For $i \in \{1, 2\}$, let $n_i = \dim X_i \geq 2$, $n = n_1 + n_2$, $n_i < n - 1$ and let $\pi_i : X = X_1 \times X_2 \rightarrow X_i$ be the projections. Then $T_X = \pi_1^*T_{X_1} \oplus \pi_2^*T_{X_2}$, hence

$$(2.1) \quad p_{n-1}(X) = \pi_1^*p_{n-1}(X_1) + \pi_2^*p_{n-1}(X_2) = 0.$$

Suppose that

$$D \cong Y_1 \times Y_2$$

for two smooth projective positive dimensional varieties $Y_i, i \in \{1, 2\}$. Let $p_i : D \cong Y_1 \times Y_2 \rightarrow Y_i$ be the projections, so that $T_D \cong p_1^*T_{Y_1} \oplus p_2^*T_{Y_2}$. Since $\dim D = n - 1 = \dim Y_1 + \dim Y_2$, we have that $\dim Y_i < n - 1$, hence

$$(2.2) \quad p_{n-1}(D) = p_1^*p_{n-1}(Y_1) + p_2^*p_{n-1}(Y_2) = 0.$$

On the other hand, the exact sequence

$$0 \rightarrow T_D \rightarrow T_{X|D} \rightarrow \mathcal{O}_D(aH) \rightarrow 0$$

and the additivity of Newton classes give, using (2.1) and (2.2), the contradiction

$$0 = p_{n-1}(D) = p_{n-1}(X)|_D - p_{n-1}(\mathcal{O}_D(aH)) = a^n H^n.$$

□

3. AN n -DIMENSIONAL OBSTRUCTION TO THE EXISTENCE OF AN ULRICH BUNDLE

The following is a simple generalization of Anghel's [A, Prop. 2.1] (and Beauville's [B1]) obstruction.

Lemma 3.1. *Let X be a smooth irreducible variety of dimension $n \geq 2$, let H be a very ample line bundle on X . If*

$$(3.1) \quad (n+1)H^n < (K_X^2 - 2c_2(X))H^{n-2}$$

then there is no Ulrich bundle for (X, H) and for all successive smooth hyperplane sections $(X_i, H|_{X_i}), 2 \leq i \leq n-1$, where $X_i = X \cap H_1 \cap \dots \cap H_{n-i}$ with $H_j \in |H|$.

Proof. Assume that there exists a rank r Ulrich bundle $\mathcal{E}$ for (X, H) . Set

$$\alpha = c_1(\mathcal{E}) - \frac{r}{2}(K_X + (n+1)H).$$

As is well known (see for example [LR, Lemma 3.1(iii)]), we have that $\alpha H^{n-1} = 0$. Let S be a smooth surface section of $X \subset \mathbb{P}H^0(H)$. Then $\alpha|_S H|_S = \alpha H^{n-1} = 0$ and therefore $\alpha|_S^2 \leq 0$ by the Hodge index theorem. Therefore

$$(3.2) \quad \alpha^2 H^{n-2} \leq 0.$$

By [LR, Lemma 3.1(viii)] we have that

$$c_2(\mathcal{E})H^{n-2} = \frac{1}{2}(c_1(\mathcal{E})^2 - c_1(\mathcal{E})K_X)H^{n-2} + \frac{r}{12}(K_X^2 + c_2(X) - \frac{3n^2 + 5n + 2}{2}H^2)H^{n-2}$$

and therefore an easy calculation gives that

$$\Delta(\mathcal{E}) = (2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2)H^{n-2} = \alpha^2 H^{n-2} + \frac{r^2}{12}(2c_2(X) - K_X^2 + (n+1)H^2)H^{n-2}.$$

Since $\mathcal{E}$ is μ_H -semistable by [CH, Thm. 2.9(a)], it follows by Bogomolov's inequality that $\Delta(\mathcal{E}) \geq 0$. Using (3.2) we therefore get that

$$(2c_2(X) - K_X^2 + (n+1)H^2)H^{n-2} \geq 0$$

that is

$$(n+1)H^n \geq (K_X^2 - 2c_2(X))H^{n-2}$$

a contradiction. This proves that if (3.1) holds, then there is no Ulrich bundle for (X, H) .

To prove that the same property holds on $(X_i, H|_{X_i})$, it is enough to show that (3.1) is equivalent to the corresponding one on a smooth hyperplane section of X . Set $X' = X_{n-1}$, so that $X' \in |H|$, and $H' = H|_{X'}$. We have

$$(3.3) \quad (H')^{n-1} = H^n$$

and

$$(3.4) \quad K_{X'}^2(H')^{n-3} = (K_X + H)^2 H^{n-2} = K_X^2 H^{n-2} + 2K_X H^{n-1} + H^n.$$

From the exact sequence

$$0 \rightarrow T_{X'} \rightarrow T_{X|X'} \rightarrow \mathcal{O}_{X'}(H') \rightarrow 0$$

we obtain $c_2(X') = (c_2(X) + K_X H + H^2)|_{X'}$, and consequently,

$$(3.5) \quad c_2(X')(H')^{n-3} = c_2(X)H^{n-2} + K_X H^{n-1} + H^n.$$

Now, using (3.3), (3.4) and (3.5), we see that (3.1) for (X', H') , that is

$$n(H')^{n-1} < (K_{X'}^2 - 2c_2(X'))(H')^{n-3}$$

is equivalent to

$$nH^n < K_X^2 H^{n-2} - 2c_2(X)H^{n-2} - H^n$$

hence to

$$(n+1)H^n < (K_X^2 - 2c_2(X))H^{n-2}$$

that is to (3.1) for (X, H) . □

4. PROOF OF THE MAIN THEOREM

Proof of Theorem 1. We set $S = S_k$ and $L = L_k$ and let $p : X = S \times Y \rightarrow S$ and $q : X = S \times Y \rightarrow Y$ be the two projections.

We compute the quantities in (3.1).

We have

$$H^n = (p^*L + q^*M)^n = \sum_{j=0}^n \binom{n}{j} p^*L^j q^*M^{n-j}$$

and therefore

$$(4.1) \quad H^n = \binom{n}{2} L^2 M^{n-2}.$$

Similarly, if $n \geq 4$ we get

$$(4.2) \quad \begin{aligned} H^{n-2} &= (p^*L + q^*M)^{n-2} = \sum_{j=0}^{n-2} \binom{n-2}{j} p^*L^j q^*M^{n-2-j} \\ &= q^*M^{n-2} + (n-2)p^*Lq^*M^{n-3} + \binom{n-2}{2} p^*L^2q^*M^{n-4}. \end{aligned}$$

Now, $K_X = p^*K_S + q^*K_Y$. If $n \geq 4$, using (4.2), we get

$$(4.3) \quad \begin{aligned} K_X^2 H^{n-2} &= (p^*K_S^2 + q^*K_Y^2 + 2p^*K_Sq^*K_Y)(q^*M^{n-2} + (n-2)p^*Lq^*M^{n-3} + \binom{n-2}{2} p^*L^2q^*M^{n-4}) \\ &= K_S^2 M^{n-2} + \binom{n-2}{2} L^2 (K_Y^2 M^{n-4}) + 2(n-2)(LK_S)(K_Y M^{n-3}). \end{aligned}$$

If $n = 3$, the same calculation shows that

$$(4.4) \quad K_X^2 H = K_S^2 \deg(M) + 2(LK_S) \deg(K_Y).$$

Next, $T_X \cong p^*T_S \oplus q^*T_Y$ and therefore

$$c_2(X) = p^*c_2(S) + p^*K_S q^*K_Y + q^*c_2(Y).$$

Using again (4.2) when $n \geq 4$, we find

$$(4.5) \quad \begin{aligned} c_2(X)H^{n-2} &= (p^*c_2(S) + p^*K_S q^*K_Y + q^*c_2(Y))(q^*M^{n-2} + (n-2)p^*Lq^*M^{n-3} + \binom{n-2}{2}p^*L^2q^*M^{n-4}) \\ &= c_2(S)M^{n-2} + (n-2)(LK_S)(K_Y M^{n-3}) + \binom{n-2}{2}L^2(c_2(Y)M^{n-4}). \end{aligned}$$

When $n = 3$ we get

$$(4.6) \quad \begin{aligned} c_2(X)H &= (p^*c_2(S) + p^*K_S q^*K_Y)(p^*L + q^*M) \\ &= c_2(S) \deg(M) + (LK_S) \deg(K_Y). \end{aligned}$$

We now consider (3.1).

First, assume that $n \geq 4$. Using (4.1), (4.3) and (4.5), we see that (3.1) holds if and only if

$$\begin{aligned} (n+1)\binom{n}{2}L^2M^{n-2} &< K_S^2M^{n-2} + \binom{n-2}{2}L^2(K_Y^2M^{n-4}) + 2(n-2)(LK_S)(K_Y M^{n-3}) \\ &\quad - 2c_2(S)M^{n-2} - 2(n-2)(LK_S)(K_Y M^{n-3}) - 2\binom{n-2}{2}L^2(c_2(Y)M^{n-4}) \end{aligned}$$

if and only if

$$(4.7) \quad ((n+1)\binom{n}{2}M^{n-2} - \binom{n-2}{2}K_Y^2M^{n-4} + 2\binom{n-2}{2}c_2(Y)M^{n-4})L^2 < M^{n-2}(K_S^2 - 2c_2(S)).$$

Set

$$f(n, Y, M) = (n+1)\binom{n}{2}M^{n-2} - \binom{n-2}{2}K_Y^2M^{n-4} + 2\binom{n-2}{2}c_2(Y)M^{n-4}$$

so that (4.7) becomes

$$(4.8) \quad f(n, Y, M)L^2 < M^{n-2}(K_S^2 - 2c_2(S)).$$

By [A, Thm. 1.2] we have that $L^2 = 7k^9$ and $K_S^2 - 2c_2(S) = 3\sigma(S) = 3(3k^2 - 11)k^9 > 0$. Therefore, if $f(n, Y, M) \leq 0$, then certainly (4.8) holds. On the other hand, if $f(n, Y, M) > 0$, (4.8) is equivalent to

$$\frac{L^2}{\sigma(S)} = \frac{7}{3k^2 - 11} < \frac{3M^{n-2}}{f(n, Y, M)}$$

that certainly holds for $k \gg 0$.

Finally, assume that $n = 3$. Using (4.1), (4.4) and (4.6), we see that (3.1) holds if and only if

$$\begin{aligned} 12L^2 \deg(M) &< K_S^2 \deg(M) + 2(LK_S) \deg(K_Y) \\ &\quad - 2c_2(S) \deg(M) - 2(LK_S) \deg(K_Y) \end{aligned}$$

if and only if

$$L^2 < \frac{K_S^2 - 2c_2(S)}{12} = \frac{\sigma(S)}{4}$$

that is

$$(4.9) \quad \frac{L^2}{\sigma(S)} = \frac{7}{3k^2 - 11} < \frac{1}{4}$$

that certainly holds for $k \geq 4$.

Therefore, in all cases, (3.1) is satisfied and we conclude by Lemma 3.1 that (X, H) and all its successive hyperplane sections until dimension 2, do not carry an Ulrich vector bundle.

As for (X', H') , the plan is to check again (3.1).

We have

$$(4.10) \quad (H')^{n-1} = aH^n$$

and

$$(4.11) \quad K_{X'}^2(H')^{n-3} = aK_X^2H^{n-2} + 2a^2K_XH^{n-1} + a^3H^n.$$

From the exact sequence

$$0 \rightarrow T_{X'} \rightarrow T_{X|X'} \rightarrow \mathcal{O}_{X'}(aH') \rightarrow 0$$

we obtain $c_2(X') = (c_2(X) + aK_XH + a^2H^2)|_{X'}$, and consequently,

$$(4.12) \quad c_2(X')(H')^{n-3} = ac_2(X)H^{n-2} + a^2K_XH^{n-1} + a^3H^n.$$

Now, using (4.10), (4.11) and (4.12), we see that (3.1) for (X', H') , that is

$$n(H')^{n-1} < (K_{X'}^2 - 2c_2(X'))(H')^{n-3}$$

is equivalent to

$$naH^n < aK_X^2H^{n-2} - 2ac_2(X)H^{n-2} - a^3H^n$$

hence to

$$(4.13) \quad (n + a^2)H^n < (K_X^2 - 2c_2(X))H^{n-2}.$$

The same calculations as the case (X, H) show that (4.13) is the same, for $n \geq 4$, as (4.8) with the new function

$$f(n, a, Y, M) = (n + a^2) \binom{n}{2} M^{n-2} - \binom{n-2}{2} K_Y^2 M^{n-4} + 2 \binom{n-2}{2} c_2(Y) M^{n-4}.$$

That is, (4.13) is equivalent to

$$f(n, a, Y, M)L^2 < M^{n-2}(K_S^2 - 2c_2(S))$$

and the latter is certainly satisfied for $k \gg 0$ as we showed for (4.8). Similarly, when $n = 3$, the same calculations as the case (X, H) show that (4.13) is similar to (4.9), namely is

$$\frac{L^2}{\sigma(S)} = \frac{7}{3k^2 - 11} < \frac{1}{3 + a^2}$$

that is certainly satisfied for $k \gg 0$. We therefore conclude by again by Lemma 3.1.

It remains to show that X' is not a product. The latter follows from Lemma 2.1 when $n \geq 4$. If $n = 3$, assume that $X' \cong C_1 \times C_2$, for two smooth curves $C_i, i \in \{1, 2\}$. Let $p_i : D = Y_1 \times Y_2 \rightarrow Y_i$ be the projections, so that $T_D \cong p_1^*T_{Y_1} \oplus p_2^*T_{Y_2}$. As in the proof of Lemma 2.1, we have that $p_2(D) = p_2(Y) = 0$, while $p_2(X) = \pi_1^*p_2(S)$. Therefore, the normal bundle sequence of X' gives, using (4.1),

$$\begin{aligned} 3a^3L^2 \deg(M) &= a^3H^3 = p_2(\mathcal{O}_{X'}(aH)) = p_2(X)|_{X'} = a\pi_1^*p_2(S)H = a(\pi_1^*c_1(S)^2 - 2\pi_1^*c_2(S))(\pi_1^*L + \pi_2^*M) \\ &= a(c_1(S)^2 - 2c_2(S)) \deg(M) = 3a\sigma(S) \deg(M) \end{aligned}$$

and dividing by $3a \deg(M)$, the above becomes

$$a^2L^2 = \sigma(S)$$

that is

$$7a^2 = 3k^2 - 11$$

hence $3k^2 \equiv 4 \pmod{7}$, a contradiction. Therefore X' is not a product. $\square$

Acknowledgement

We thank R. Vacca for a couple of useful remarks on a previous version.

REFERENCES

- [A] C. Anghel. *Smooth Counterexamples to the Eisenbud-Schreyer-Weyman Ulrich Existence Problem*. Preprint 2026, arXiv:2609.02718. [1](#), [2](#), [4](#)
- [B1] A. Beauville. *Ulrich bundles: to be or not to be*. Lecture slides, Beijing, October 2017. <https://math.univ-cotedazur.fr/u/beauvill/conf/Ulrich.pdf> [1](#), [2](#)
- [B2] A. Beauville. *An introduction to Ulrich bundles*. Eur. J. Math. **4** (2018), no. 1, 26-36. [1](#)
- [C1] G. Casnati. *Special Ulrich bundles on non-special surfaces with $p_g = q = 0$* . Internat. J. Math. **28** (2017), no. 8, 1750061, 18 pp. [1](#)
- [C2] G. Casnati. *Special Ulrich bundles on regular surfaces with non-negative Kodaira dimension*. Manuscripta Math. **167** (2022), no. 1-2, 151-172. [1](#)
- [CFK1] C. Ciliberto, F. Flamini, A. L. Knutsen. *Ulrich bundles on Del Pezzo threefolds*. J. Algebra **634** (2023), 209-236. [1](#)
- [CFK2] C. Ciliberto, F. Flamini, A. L. Knutsen. *Elliptic curves, ACM bundles and Ulrich bundles on prime Fano threefolds*. Collect. Math. **75** (2024), no. 3, 795-822. [1](#)
- [CH] M. Casanellas, R. Hartshorne. *Stable Ulrich bundles*. With an appendix by F. Geiss, F.-O. Schreyer. Internat. J. Math. **23** (2012), no. 8, 1250083, 50 pp. [3](#)
- [CMR] L. Costa, R. M. Miró-Roig. *$GL(V)$ -invariant Ulrich bundles on Grassmannians*. Math. Ann. **361** (2015), no. 1-2, 443-457. [1](#)
- [CMRPL] L. Costa, R. M. Miró-Roig, J. Pons-Llopis. *Ulrich bundles*. De Gruyter Studies in Mathematics, **77**, De Gruyter 2021. [1](#)
- [CoHu] I. Coskun, J. Huizenga. *Brill-Noether problems, Ulrich bundles and the cohomology of moduli spaces of sheaves*. Mat. Contemp. **47** (2020), 21-72. [1](#), [2](#)
- [ES] D. Eisenbud, F.-O. Schreyer. *Resultants and Chow forms via exterior syzygies*. With an appendix by Jerzy Weyman. J. Amer. Math. Soc. **16** (2003), no. 3, 537-579. [1](#)
- [F] D. Faenzi. *Ulrich bundles on K3 surfaces*. Algebra Number Theory **13** (2019), no. 6, 1443-1454. [1](#)
- [HUB] J. Herzog, B. Ulrich, J. Backelin. *Linear maximal Cohen-Macaulay modules over strict complete intersections*. J. Pure Appl. Algebra **71** (1991), no. 2-3, 187-202. [1](#)
- [L1] A. F. Lopez. *On the existence of Ulrich vector bundles on some irregular surfaces*. Proc. Amer. Math. Soc. **149** (2021), no. 1, 13-26. [1](#)
- [L2] A. F. Lopez. *On the positivity of the first Chern class of an Ulrich vector bundle*. Commun. Contemp. Math. **24** (2022), no. 9, Paper No. 2150071, 22 pp. [1](#)
- [LM] A. F. Lopez, R. Muñoz. *On the classification of non-big Ulrich vector bundles on surfaces and threefolds*. Internat. J. Math. **32** (2021), no. 14, Paper No. 2150111, 18 pp. [1](#)
- [LMS] A. F. Lopez, R. Muñoz, J. C. Sierra. *On the classification of non-big Ulrich vector bundles on fourfolds*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **26** (2025), no. 2, 707-755. [1](#)
- [LR] A. F. Lopez, D. Raychaudhury. *Ulrich subvarieties and the non-existence of low rank Ulrich bundles on complete intersections*. Trans. Amer. Math. Soc. **379** (2026), no. 8, 5809-5846. [2](#)
- [LS] A. F. Lopez, J. C. Sierra. *A geometrical view of Ulrich vector bundles*. Int. Math. Res. Not. IMRN(2023), no. 11, 9754-9776. [1](#)

ANGELO FELICE LOPEZ, DIPARTIMENTO DI MATEMATICA E FISICA, UNIVERSITÀ DI ROMA TRE, LARGO SAN LEONARDO MURIALDO 1, 00146, ROMA, ITALY. E-MAIL angelo.lopez@uniroma3.it