NON-BIG ULRICH BUNDLES: THE CLASSIFICATION ON QUADRICS AND THE CASE OF SMALL NUMERICAL DIMENSION

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Abstract. On any smooth n-dimensional variety we give a pretty precise picture of rank r Ulrich vector bundles with numerical dimension at most $\frac{n}{2}+r-1$. Also, we classify non-big Ulrich vector bundles on quadrics and on the Del Pezzo fourfold of degree 6.

1. Introduction

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible complex closed variety of dimension $n \ge 1$. In this paper we consider the investigation of positivity properties of Ulrich vector bundles on X. Recall that a vector bundle $\mathcal{E}$ on X is Ulrich if $H^i(\mathcal{E}(-p)) = 0$ for all $i \ge 0$ and $1 \le p \le n$. The importance of Ulrich vector bundles and the consequences on the geometry of X are well described for example in [ES, B, CMRPL] and references therein.

It was highlighted in our previous work [Lo, LM, LS] that most Ulrich vector bundles should be at least big, unless X is covered by linear spaces of positive dimension. In particular any Ulrich vector bundle is very ample if X does not contain lines by [LS, Thm. 1]. If $n \le 3$, the classification of non-big Ulrich vector bundles was achieved in [LM]. On the other hand, when studying higher dimensional varieties, several new difficulties appear. For example, before this paper, as far as we know, no class of varieties of arbitrary dimension $n \ge 1$ for which non-big Ulrich vector bundles were classified was known except for $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ (in which case either $d = 1$ and the only Ulrich vector bundles are $\mathcal{O}_{\mathbb{P}^n}^{\oplus r}, r \ge 1$ or $d \ge 2$ and they are all very ample by what we said above).

A nice class of n-dimensional varieties, just coming after projective spaces, and for which Ulrich vector bundles are known, is the one of quadrics $Q_n \subset \mathbb{P}^{n+1}$. In fact, it follows by [BGS, Rmk. 2.5(4)] (see also [B, Prop. 2.5], [AHMPL, Exa. 3.2]) that the only indecomposable Ulrich vector bundles on Q_n are the spinor bundles $\mathcal{S}, \mathcal{S}'$ and $\mathcal{S}''$ (see Definition 3.1). Note that they are never ample as their restriction to lines is not ample by [O, Cor. 1.6]. On the other hand, it was not known which of these are big, unless $n \le 8$, this case following easily from [O, Rmk. 2.9].

Our first result is a classification of non-big Ulrich vector bundles on quadrics.

Theorem 1.
An Ulrich vector bundle $\mathcal{E}$ on Q_n is not big if and only if $\mathcal{E}$ is one of the following

Table 1

n	$\mathcal{E}$
2	$(\mathcal{S}')^{\oplus r}, (\mathcal{S}'')^{\oplus r}, r \ge 1$
3	$\mathcal{S}$
4	$\mathcal{S}', \mathcal{S}'', \mathcal{S}' \oplus \mathcal{S}''$
5	$\mathcal{S}$
6	$\mathcal{S}', \mathcal{S}'', (\mathcal{S}')^{\oplus 2}, (\mathcal{S}'')^{\oplus 2}$
10	$\mathcal{S}', \mathcal{S}''$

The proof of the above theorem introduces, in fact on any variety X, a new geometrical estimate on the relation between Ulrich vector bundles and the Fano variety of some linear subspaces contained in X, see Proposition 4.1. This estimate and the one given in Proposition 4.6 are of independent interest and might have several applications. We show this by classifying Ulrich vector bundles of small numerical dimension $\nu(\mathcal{E}) := \nu(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$, a measure of positivity of $\mathcal{E}$. In general, in the presence

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of a rank r Ulrich vector bundle $\mathcal{E}$, [LS, Thm. 2] shows that we can find two kinds of linear spaces covering X, namely the fibers of $\Phi_{\mathcal{E}} : X \to \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$ and the images on X of the fibers of $\varphi_{\mathcal{E}} : \mathbb{P}(\mathcal{E}) \to \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. Whenever the dimension of these spaces is at least $\frac{n}{2}$, Sato's classification [S] can be applied. In [LS, Cor. 4] we dealt with the case of small numerical dimension of $\det \mathcal{E}$, corresponding to fibers of $\Phi_{\mathcal{E}}$. Instead, dealing with $\nu(\mathcal{E})$, we have that X is covered by linear spaces of dimension $n+r-1-\nu(\mathcal{E})$, thus giving rise to the natural bound $\nu(\mathcal{E}) \le \frac{n}{2}+r-1$, appearing in the theorem below. We have:

Theorem 2.
Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$ and let $\mathcal{E}$ be a rank r Ulrich vector bundle on X. Then $\nu(\mathcal{E}) > \frac{n}{2}+r-1$ unless $(X, \mathcal{O}_X(1))$ is either:
- (i) *$(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$, where $\mathcal{F}$ is a rank $n-b+1$ very ample vector bundle over a smooth irreducible variety B of dimension b with $0 \le b \le \frac{n}{2}$.*
- (ii) *$(Q_{2m}, \mathcal{O}_{Q_{2m}}(1))$ with $1 \le m \le 3$, $\nu(\mathcal{E}) = m+r-1$ and $\mathcal{E} \cong (\mathcal{S}')^{\oplus r}$ or $(\mathcal{S}'')^{\oplus r}$ when $m=1$, $\mathcal{S}'$ or $\mathcal{S}''$ when $2 \le m \le 3$.*

Moreover, in case (i) with $\nu(\mathcal{E}) \le \frac{n}{2}+r-1$, denoting by $p : X \cong \mathbb{P}(\mathcal{F}) \to B$ the projection map, we have two cases:
- (i1) *If $c_1(\mathcal{E})^n = 0$, then $\nu(\mathcal{E}) = b+r-1$ and $\mathcal{E} \cong p^*(\mathcal{G}(\det \mathcal{F}))$, where $\mathcal{G}$ is a rank r vector bundle on B such that $H^q(\mathcal{G} \otimes S^k\mathcal{F}^*) = 0$ for $q \ge 0, 0 \le k \le b-1$ and $c_1(\mathcal{G}(\det \mathcal{F}))^b \ne 0$.*
- (i2) *If $c_1(\mathcal{E})^n > 0$, then $b \le \frac{n}{2}-1, \nu(\mathcal{E}) \ge b+r$ and if $\nu(\mathcal{E}) = b+r$ then $\mathcal{E}_{|f} \cong T_{\mathbb{P}^{n-b}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-b}}^{\oplus(r-n+b)}$ for any fiber $f = \mathbb{P}^{n-b}$ of p.*

Note that the cases (i1) and (i2) actually occur at least in some special instances (for (i2) see Example 4.5, for (i1) pick any $\mathcal{E} \cong p^*(\mathcal{G}(\det \mathcal{F}))$ with $\mathcal{G}$ as in (i1), see [Lo, Lemma 4.1(ii), Prop.'s 6.1 and 6.2], [LMS, Thm. 1] for some specific examples), while the cases in (ii) actually occur by Proposition 3.3(ii).

It is usually difficult to compute the numerical dimension of Ulrich vector bundles, even for simple classes of varieties. Aside from projective spaces, perhaps the simplest ones are quadrics and linear $\mathbb{P}^k$-bundles. While for quadrics this can be now done, see Remark 4.2, the same cannot be said for linear $\mathbb{P}^k$-bundles. In the latter case we have useful information when $k \le 2$ by Lemma 4.4: there are only two cases: $\nu(\mathcal{E}) = n-k+r-1$ (and this is well known) or $n-k+r$. A special but classical example with $k=2$ is the Del Pezzo fourfold of degree 6. Another nice application of the methods in this paper is the classification of non-big Ulrich vector bundles on it, obtained using Lemma 4.4 and the resolution of Ulrich bundles given in [M].

Theorem 3.
Let $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ be the Segre embedding and let $\mathcal{E}$ be a rank r non-big Ulrich vector bundle on $\mathbb{P}^2 \times \mathbb{P}^2$. Then $\mathcal{E} \cong p^*(\mathcal{O}_{\mathbb{P}^2}(2))^{\oplus r}$, where $p : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ is one of the two projections.

Finally, we emphasize that Theorems 1, 2 and 3 will be important in our classification of non-big Ulrich vector bundles on fourfolds given in [LMS].

2. Notation and standard facts about (Ulrich) vector bundles

Throughout this section we will let $X \subseteq \mathbb{P}^N$ be a smooth irreducible closed complex variety of dimension $n \ge 1$, degree d and H a hyperplane divisor on X.

Definition 2.1. For $k \in \mathbb{Z} : 1 \le k \le n$ we denote by $F_k(X)$ the Fano variety of k-dimensional linear subspaces of $\mathbb{P}^N$ that are contained in X. For $x \in X$, we denote by $F_k(X,x) \subset F_k(X)$ the subvariety of k-dimensional linear subspaces passing through x.

Definition 2.2. Given a nef line bundle $\mathcal{L}$ on X we denote by

$$\nu(\mathcal{L}) = \max\{k \ge 0 : c_1(\mathcal{L})^k \ne 0\}$$

the *numerical dimension* of $\mathcal{L}$.

As is well known (see for example [F, (3.8)]), when $\mathcal{L}$ is globally generated, $\nu(\mathcal{L})$ is the dimension of the image of the morphism induced by $\mathcal{L}$.

Definition 2.3. Let $\mathcal{E}$ be a rank r vector bundle on X. We denote by $c(\mathcal{E})$ its Chern polynomial and by $s(\mathcal{E})$ its Segre polynomial. We set $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}(\mathcal{E}))$ with projection map $\pi : \mathbb{P}(\mathcal{E}) \to X$ and tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. We say that $\mathcal{E}$ is *nef (big, ample, very ample)* if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef (big, ample, very ample). If $\mathcal{E}$ is nef, we define the numerical dimension of $\mathcal{E}$ by $\nu(\mathcal{E}) := \nu(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. When $\mathcal{E}$ is globally generated we define the map determined by $\mathcal{E}$ as

$$\Phi = \Phi_{\mathcal{E}} : X \to \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$$

and we set $\phi(\mathcal{E})$ for the dimension of the general fiber of $\Phi_{\mathcal{E}}$. Moreover, we set

$$\varphi = \varphi_{\mathcal{E}} = \varphi_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)} : \mathbb{P}(\mathcal{E}) \to \mathbb{P}H^0(\mathcal{E})$$

$$\Pi_y = \pi(\varphi^{-1}(y)), y \in \varphi(\mathbb{P}(\mathcal{E}))$$

and

$$P_x = \varphi(\mathbb{P}(\mathcal{E}_x)).$$

Note that $\Phi(x) = [P_x]$ is the point in $\mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$ corresponding to P_x.

Lemma 2.4. *Let $\mathcal{E}$ be a rank r globally generated vector bundle on X. Let $x \in X$, so that $\mathbb{P}^{r-1} \cong P_x \subseteq \mathbb{P}H^0(\mathcal{E})$. For any $y \in \varphi(\mathbb{P}(\mathcal{E}))$ we have that*

$$y \in P_x \iff x \in \Pi_y.$$

Proof. Since $P_x = \varphi(\mathbb{P}(\mathcal{E}_x))$ and $\Pi_y = \pi(\varphi^{-1}(y))$, we have

$$y \in P_x \iff \exists z \in \mathbb{P}(\mathcal{E}_x) \cap \varphi^{-1}(y) \iff \exists z \in \varphi^{-1}(y) : \pi(z) = x \iff x \in \Pi_y.$$

□

We recall the following well-known fact (see for example [EH, Prop. 10.2]).

Lemma 2.5. *Let $\mathcal{E}$ be a rank r globally generated vector bundle on X. Then*

$$\nu(\mathcal{E}) = r - 1 + \max\{k \geq 0 : s_k(\mathcal{E}) \neq 0\}. \tag{2.1}$$

In order to check bigness of direct sums we will use the ensuing

Lemma 2.6. *Let $\mathcal{E}, \mathcal{F}$ be two globally generated vector bundles on X. Then $s_i(\mathcal{E}^*)s_{n-i}(\mathcal{F}^*) \geq 0$ for all $0 \leq i \leq n$. Moreover $\mathcal{E} \oplus \mathcal{F}$ is big if and only if $s_i(\mathcal{E}^*)s_{n-i}(\mathcal{F}^*) > 0$ for some $i \in \{0, \dots, n\}$. In particular, if $\mathcal{E}$ is big then $\mathcal{E} \oplus \mathcal{F}$ is big.*

Proof. Set $\xi := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. First, note that the Segre classes $s_i(\mathcal{E}^*)$ and $s_i(\mathcal{F}^*)$ are effective and nef. In fact, $s_i(\mathcal{E}^*) = \pi_*\xi^{r-1+i}$ is effective because ξ is globally generated. Also $s_i(\mathcal{E}^*)$ is nef because for every subvariety $Z \subseteq X$ of dimension i we have that

$$s_i(\mathcal{E}^*) \cdot Z = \xi^{r-1+i} \cdot \pi^*Z \geq 0.$$

Therefore $s_i(\mathcal{E}^*)s_{n-i}(\mathcal{F}^*) \geq 0$ for all $0 \leq i \leq n$. Since $c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E})c(\mathcal{F})$ we get that

$$s(\mathcal{E})s(\mathcal{F})c(\mathcal{E} \oplus \mathcal{F}) = s(\mathcal{E})s(\mathcal{F})c(\mathcal{E})c(\mathcal{F}) = 1$$

hence $s(\mathcal{E} \oplus \mathcal{F}) = s(\mathcal{E})s(\mathcal{F})$. It follows that

$$s_n((\mathcal{E} \oplus \mathcal{F})^*) = \sum_{i=0}^{n} s_i(\mathcal{E}^*)s_{n-i}(\mathcal{F}^*).$$

Hence $\mathcal{E} \oplus \mathcal{F}$ is big if and only if $s_n((\mathcal{E} \oplus \mathcal{F})^*) > 0$, if and only if $s_i(\mathcal{E}^*)s_{n-i}(\mathcal{F}^*) > 0$ for some $i \in \{0, \dots, n\}$. Also, if $\mathcal{E}$ is big then $s_n(\mathcal{E}^*) > 0$, hence $\mathcal{E} \oplus \mathcal{F}$ is big. □

Definition 2.7. Let $\mathcal{E}$ be a vector bundle on $X \subseteq \mathbb{P}^N$. We say that $\mathcal{E}$ is an *Ulrich vector bundle* if $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$.

The following properties will be often used without mentioning.

Remark 2.8. Let $\mathcal{E}$ be a rank r Ulrich vector bundle on $X \subseteq \mathbb{P}^N$ and let $d = \deg X$. Then

(i) $\mathcal{E}$ is 0-regular in the sense of Castelnuovo-Mumford, hence $\mathcal{E}$ is globally generated (by [La, Thm. 1.8.5]).
(ii) $\mathcal{E}_{|Y}$ is Ulrich on a smooth hyperplane section Y of X (by [B, (3.4)]).

Remark 2.9. On $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ the only rank r Ulrich vector bundle is $\mathcal{O}_{\mathbb{P}^n}^{\oplus r}$ by [ES, Prop. 2.1], [B, Thm. 2.3].

3. Ulrich bundles on quadrics

For $n \ge 2$ we let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric. We let S (n odd), and S', S'' (n even), be the vector bundles on Q_n, as defined in [O, Def. 1.3].

Definition 3.1. The *spinor bundles* on Q_n are $\mathcal{S} = \mathcal{S}_n = S(1)$ if n is odd and $\mathcal{S}' = \mathcal{S}'_n = S'(1)$, $\mathcal{S}'' = \mathcal{S}''_n = S''(1)$, if n is even. They all have rank $2^{\lfloor \frac{n-1}{2} \rfloor}$.

Lemma 3.2. *With the above notation we have:*

(i) $s(\mathcal{S}) = c(S), s(\mathcal{S}') = c(S''), s(\mathcal{S}'') = c(S')$.
(ii) $\nu(\mathcal{S}) = 2^{\frac{n-1}{2}} - 1 + \max\{k \ge 0 : c_k(S) \ne 0\}$,
$\nu(\mathcal{S}') = \nu(\mathcal{S}'') = 2^{\frac{n-2}{2}} - 1 + \max\{k \ge 0 : c_k(S') \ne 0\}$.
(iii) *The spinor bundles on Q_n are Ulrich.*
(iv) *The only indecomposable Ulrich vector bundles on Q_n are the spinor bundles.*
(v) *Spinor bundles are not ample.*

Proof. By [O, Thm. 2.8] there are exact sequences

$$0 \to S \to \mathcal{O}_Q^{\oplus 2^{\lfloor \frac{n+1}{2} \rfloor}} \to \mathcal{S} \to 0 \tag{3.1}$$

$$0 \to S' \to \mathcal{O}_Q^{\oplus 2^{\lfloor \frac{n+1}{2} \rfloor}} \to \mathcal{S}'' \to 0, \quad 0 \to S'' \to \mathcal{O}_Q^{\oplus 2^{\lfloor \frac{n+1}{2} \rfloor}} \to \mathcal{S}' \to 0. \tag{3.2}$$

From (3.1) we get $c(S)c(\mathcal{S}) = 1$, hence $s(\mathcal{S}) = c(S)$. Similarly, from (3.2) we get that $s(\mathcal{S}'') = c(S')$ and $s(\mathcal{S}') = c(S'')$. This gives (i). Now (ii) follows by (i), (2.1) and the fact that if n is even and f is an automorphism of Q_n exchanging the $\frac{n}{2}$-planes in Q_n, then $f^*S' \cong S''$ and $f^*S'' \cong S'$ (see [O, page 304]). (iii) follows by [O, Thms. 2.3 and 2.8]. (iv) follows by [K] (see also [BGS, Rmk. 2.5(4)], [B, Prop. 2.5], [AHMPL, Exa. 3.2]). As for (v), we know that $\mathcal{S}'_2 \cong \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{S}''_2 \cong \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$, while for $n \ge 3$, [O, Cor. 1.6] gives that $\mathcal{S}_{|L}, \mathcal{S}'_{|L}, \mathcal{S}''_{|L}$ all decompose as $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2^{\lfloor \frac{n-3}{2} \rfloor}} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2^{\lfloor \frac{n-3}{2} \rfloor}}$ for any line L contained in Q_n, hence $\mathcal{S}, \mathcal{S}'$ and $\mathcal{S}''$ are not ample. □

We collect a preliminary result about the numerical dimension of spinor bundles.

Proposition 3.3. *With the above notation we have:*

(i) *$\mathcal{S}_n$ is big if and only if $c_n(S) \ne 0$, $\mathcal{S}'_n$ or $\mathcal{S}''_n$ is big if and only if $c_n(S') \ne 0$.*
(ii) *For $2 \le n \le 10$ we have:*

n	$\nu(\mathcal{S})$
3	3
5	6
7	14
9	24

n	$\nu(\mathcal{S}') = \nu(\mathcal{S}'')$
2	1
4	3
6	6
8	15
10	24

(iii) *For $2 \le n \le 10$ the vector bundles in Table 1 of Theorem 1 are the only non-big Ulrich vector bundles $\mathcal{E}$ on Q_n.*

Proof. Using Lemma 3.2(i) we have that $\mathcal{S}$ is big if and only if

$$0 < s_n(\mathcal{S}^*) = (-1)^n s_n(\mathcal{S}) = (-1)^n c_n(S)$$

and we get (i) for $\mathcal{S}$. Similarly, (i) holds for $\mathcal{S}'$ and $\mathcal{S}''$.

To see (ii), for $n \ne 7, 9, 10$, just use Lemma 3.2(ii) and [O, Rmk. 2.9]. If $n = 7$ we have by [O, Thm. 1.4] that $S = (S'_8)_{|Q_7}$ and picking $H \in |\mathcal{O}_{Q_8}(1)|$ we deduce by [O, Rmk. 2.9] that

$$c_7(S) = c_7(S'_8) \cdot H = -H^8 = -2.$$

Thus we get (ii) by Lemma 3.2(ii).

If $n = 10$, by [O, Thm. 2.6(ii)], on any $\mathbb{P}^5$ in one of the families of 5-planes in Q_{10}, we have that

$$(S')_{|\mathbb{P}^5} = \bigoplus_{i=0}^{2} \Omega^{2i}_{\mathbb{P}^5}(2i), \qquad (S'')_{|\mathbb{P}^5} = \bigoplus_{i=0}^{2} \Omega^{2i+1}_{\mathbb{P}^5}(2i+1)$$

and, in the other family

$$(S'')_{|\mathbb{P}^5} = \bigoplus_{i=0}^{2} \Omega^{2i}_{\mathbb{P}^5}(2i), \qquad (S')_{|\mathbb{P}^5} = \bigoplus_{i=0}^{2} \Omega^{2i+1}_{\mathbb{P}^5}(2i+1).$$

This allows to compute the first five Chern classes of S' and S''. In the standard basis $\{e_0, \dots, e_4, e_5, e_5', e_6, \dots, e_{10}\}$ of the cohomology ring of Q_{10} one gets that for S' (respectively for S''):

$$c_1 = -8e_1, c_2 = 32e_2, c_3 = -84e_3, c_4 = 160e_4, c_5 = -244e_5 - 220e_5' (\text{resp. } c_5 = -220e_5 - 244e_5').$$

To get c_i, $i > 5$, recall that any automorphism of Q_{10} interchanging its two families of 5-planes provides an isomorphism between S' and S'', hence $c_i(S') = c_i(S'')$ for $i \neq 5$. From the exact sequence

$$0 \to S' \to \mathcal{O}_{Q_{10}}^{\oplus 2^5} \to S'(1) \to 0$$

and the isomorphism $(S')^* \cong S'(1)$ (see [O, Thm. 2.8]), we get the following relations in the Chern polynomials of S' and S'':

$$c(S')c((S')^*) = 1 \tag{3.3}$$

and

$$c((S')^*) = c(S''(1)). \tag{3.4}$$

By (3.3) one gets inductive formulae for the even Chern classes:

$$c_{2k} = (-1)^{k+1}\frac{c_k^2}{2} + \sum_{i=1}^{k-1}(-1)^{k+1} c_i c_{2k-i}.$$

Respectively, (3.4) leads to inductive formulae for the odd ones:

$$c_{2k+1} = -\frac{1}{2}\sum_{i=0}^{2k}\binom{16-i}{2k+1-i} c_i e_1^{2k+1-i}.$$

Hence, in coordinates in the standard basis quoted above, and recalling the relations $e_1^i = 2e_i$ for $6 \le i \le 10$, $e_5^2 = (e_5')^2 = 0$ and $e_5 e_5' = e_{10}$, the Chern classes of S' (resp. S'') can be computed to give:

$$\begin{aligned} &c_1 = -8, c_2 = 32, c_3 = -84, c_4 = 160, c_5 = (-244, -220) \text{ (resp. } c_5 = (-220, -244)), \\ &c_6 = 528, c_7 = -484, c_8 = 352, c_9 = -176, c_{10} = 0. \end{aligned} \tag{3.5}$$

If $n = 9$, since $(S'_{10})_{|Q_9} \cong S$ by [O, Thm. 1.4], we get that $c_9(S) = -176$, thus $\mathcal{S}$ is big and $\nu(\mathcal{S}) = 24$.

To see (iii) note that Lemma 3.2(iv) gives that

$$\mathcal{E} \cong (\mathcal{S}')^{\oplus a} \oplus (\mathcal{S}'')^{\oplus b} \text{ when } n \text{ is even, with } a \ge 0, b \ge 0, a+b \ge 1$$

and

$$\mathcal{E} \cong \mathcal{S}^{\oplus a} \text{ when } n \text{ is odd, with } a \ge 1.$$

If $n = 2$ then $\mathcal{S}' = \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1), \mathcal{S}'' = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}$ and $\det \mathcal{E} = a\mathcal{S}' + b\mathcal{S}'', c_2(\mathcal{E}) = ab$. Hence

$$s_2(\mathcal{E}^*) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = (a\mathcal{S}' + b\mathcal{S}'')^2 - ab = ab.$$

Thus $\mathcal{E}$ is not big if and only if $s_2(\mathcal{E}^*) = 0$, that is if and only if $a = 0$ or $b = 0$.

If $n = 3$ it is proved in [LM, Rmk. 2.7] that $\mathcal{E}$ is not big if and only if $a = 1$.

If $n = 4$ we need to show that $\mathcal{E}$ is not big if and only if $(a, b) \in \{(0,1), (1,0), (1,1)\}$. We know by (ii) that $\mathcal{S}'$ and $\mathcal{S}''$ are not big since their numerical dimension is less than $n + r - 1 = 5$. Consider now $\mathcal{E} = \mathcal{S}' \oplus \mathcal{S}''$. By Lemma 2.6 and Lemma 3.2(i) we need to prove that $c_i(\mathcal{S}')c_{4-i}(\mathcal{S}'') = 0$ for all $0 \le i \le 4$. Since they have rank 2 we just need the case $i = 2$. By [O, Rmk. 2.9] we know that $c_2(\mathcal{S}') = \ell, c_2(\mathcal{S}'') = h^2 - \ell$ where h is the class of a hyperplane section and ℓ of a plane in Q_4, with $h^2\ell = \ell^2 = 1$. Hence $c_2(\mathcal{S}')c_2(\mathcal{S}'') = \ell(h^2 - \ell) = 0$ and $\mathcal{E}$ is not big. Vice versa, to conclude the case $n = 4$, using Lemma 2.6, it remains to show that $(\mathcal{S}')^{\oplus 2}$ and $(\mathcal{S}'')^{\oplus 2}$ are big. This follows as above since $c_2(\mathcal{S}')^2 = \ell^2 = 1$ and $c_2(\mathcal{S}'')^2 = (h^2 - \ell)^2 = 1$.

If $n = 5$ we know by (ii) that $\mathcal{S}$ is not big. Proceeding as above, we need to show that $\mathcal{S}^{\oplus 2}$ is big. Now [O, Rmk. 2.9] gives that $c_2(S)c_3(S) = (2h^2)(-h^3) = -4 \neq 0$, hence $\mathcal{S}^{\oplus 2}$ is big.

If $n = 6$ we need to show that $\mathcal{E}$ is not big if and only if $(a, b) \in \{(0, 1), (1, 0), (0, 2), (2, 0)\}$. To see that these are not big note that the case $(2, 0)$ implies the case $(1, 0)$ and the case $(0, 2)$ implies the case $(0, 1)$ by Lemma 2.6. Consider $\mathcal{E} = (\mathcal{S}')^{\oplus 2}$ or $(\mathcal{S}'')^{\oplus 2}$. By Lemma 2.6 and Lemma 3.2(i) we need to prove that $c_i(S'')c_{6-i}(S'') = 0$ or $c_i(S')c_{6-i}(S') = 0$, for all $0 \le i \le 6$. On the other hand [O, Rmk. 2.9] gives that $c_i(S') = c_i(S'') = 0$ for all $4 \le i \le 6$ and $c_3(S') = -2\ell, c_3(S'') = -2(h^3 - \ell)$, where h is the class of a hyperplane section and ℓ of a 3-plane in Q_6, with $h^3\ell = 1, \ell^2 = 0$. Hence $c_3(S'')^2 = 4\ell^2 = 0, c_3(S'')^2 = 4(h^3 - \ell)^2 = 0$ and $\mathcal{E}$ is not big. Vice versa, to conclude the case $n = 6$, using Lemma 2.6, it remains to show that $\mathcal{S}' \oplus \mathcal{S}'', (\mathcal{S}')^{\oplus 3}$ and $(\mathcal{S}'')^{\oplus 3}$ are big. Since $c_3(S')c_3(S'') = 4\ell(h^3 - \ell) = 4$ we get, as above, that $\mathcal{S}' \oplus \mathcal{S}''$ is big. Now Lemma 3.2(i) gives

$$s_3((\mathcal{S}')^{\oplus 2})s_3(\mathcal{S}') = 2(s_3(\mathcal{S}') + s_1(\mathcal{S}')s_2(\mathcal{S}'))c_3(\mathcal{S}'') = -8(\ell - 3h^3)(h^3 - \ell) = 16$$

and Lemma 2.6 implies that $(\mathcal{S}')^{\oplus 3}$ is big. Similarly, $(\mathcal{S}'')^{\oplus 3}$ is big.

If $n = 7, 8, 9$ the spinor bundles are big by (ii), hence so is $\mathcal{E}$ by Lemma 2.6.

Finally, if $n = 10$ (see (3.5)) we have the following values of the Chern classes of S' and S'':

$$c_1 = -8, c_9 = -176 \text{ and } c_{10} = 0,$$

thus $\mathcal{S}'$ and $\mathcal{S}''$ are not big by (i). On the other hand, $c_1(S')c_9(S') = c_1(S'')c_9(S'') = c_1(S')c_9(S'') = c_1(S'')c_9(S') = 1408 \neq 0$ so that $\mathcal{E}$ is big if $(a, b) \notin \{(1, 0), (0, 1)\}$ by Lemma 2.6 and Lemma 3.2(i). □

4. Behaviour of Ulrich bundles on linear subspaces

We start by analyzing the behaviour on lines.

Proposition 4.1. *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 2$. Let $\mathcal{E}$ be a non-big rank r Ulrich vector bundle on X. Let $x \in X$ and let*

$$h(\mathcal{E}, x) = \max\{h \ge 1 : \exists L \in F_1(X, x) \text{ such that } \mathcal{O}_{\mathbb{P}^1}^{\oplus h} \text{ is a direct summand of } \mathcal{E}_{|L}\}.$$

Then

$$\dim F_1(X, x) + h(\mathcal{E}, x) \ge r. \tag{4.1}$$

Moreover, if $h(\mathcal{E}) = \max\{h(\mathcal{E}, x), x \in X\}$, then

$$\nu(\mathcal{E}) \le \dim F_1(X) + h(\mathcal{E}) - 1. \tag{4.2}$$

Proof. Set $\varphi = \varphi_{\mathcal{E}}$. We will use the fact, following by [LS, Lemma 3.3], that for any line $L \subset X$, we have that $H^0(\mathcal{E}) \to H^0(\mathcal{E}_{|L})$ is surjective, hence the restriction of φ to $\mathbb{P}(\mathcal{E}_{|L})$ is the tautological morphism $\varphi_{(\mathcal{E}_{|L})}$ associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E}_{|L})}(1)$.

For any $L \in F_1(X, x)$ we can write

$$\mathcal{E}_{|L} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus h_L} \oplus \mathcal{O}_{\mathbb{P}^1}(a_{1,L}) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{r-h_L,L})$$

for some integers $0 \le h_L \le r$ and $a_{i,L} \ge 1$ for $1 \le i \le r - h_L$. Suppose that $h_L \ge 1$. Then $\varphi(\mathbb{P}(\mathcal{E}_{|L})) \subseteq \mathbb{P}H^0(\mathcal{E}_{|L})$ is either a rational normal scroll with vertex $V_L \cong \mathbb{P}^{h_L - 1}$ when $h_L \le r - 1$ (including the case $h_L = r - 1, a_{1,L} = 1$, in which $\varphi(\mathbb{P}(\mathcal{E}_{|L})) = \mathbb{P}H^0(\mathcal{E}_{|L})$) or $\varphi(\mathbb{P}(\mathcal{E}_{|L})) = \mathbb{P}^{r-1}$ when $h_L = r$. In the latter case we define $V_L = \varphi(\mathbb{P}(\mathcal{E}_{|L}))$. Moreover $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus h_L}) \subseteq \mathbb{P}(\mathcal{E}_{|L})$ and if we call $\psi_L : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus h_L}) \to V_L$ the map associated to the tautological line bundle on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus h_L})$, the fibers of ψ_L are the curves $\psi_L^{-1}(y), y \in V_L$. Note that they all intersect $\mathbb{P}(\mathcal{E}_x)$ in one point.

Consider the incidence correspondence

$$\mathcal{I} = \{(z, L) \in \mathbb{P}(\mathcal{E}_x) \times F_1(X, x) : h_L \ge 1 \text{ and } \exists y \in V_L \text{ with } z \in \psi_L^{-1}(y) \cap \mathbb{P}(\mathcal{E}_x)\}$$

together with its projections

$$\begin{array}{ccccc} & & \mathcal{I} & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \mathbb{P}(\mathcal{E}_x) & & & & F_1(X, x) \end{array}.$$

We claim that p_1 is surjective.

Let $z \in \mathbb{P}(\mathcal{E}_x)$, so that $x = \pi(z)$. Let $y = \varphi(z)$. Since $\mathcal{E}$ is not big, we have that $\dim \varphi(\mathbb{P}(\mathcal{E})) < \dim \mathbb{P}(\mathcal{E})$, hence all fibers of φ are positive dimensional. Thus $\dim \varphi^{-1}(y) > 0$, hence we can find $z' \in \varphi^{-1}(y)$ with $z' \neq z$. Let $x' = \pi(z')$. Note that $x \neq x'$, because otherwise $z, z' \in \mathbb{P}(\mathcal{E}_x)$, contradicting the fact that $\varphi_{|\mathbb{P}(\mathcal{E}_x)}$ is an embedding. Consider the line $L = \langle x, x' \rangle$. If $L \not\subset X$, then [LS, Lemma 3.2] implies that $P_x \cap P_{x'} = \emptyset$. On the other hand, since $z, z' \in \varphi^{-1}(y)$, we have that $x, x' \in \Pi_y$. Therefore Lemma 2.4 implies that $y \in P_x \cap P_{x'}$, a contradiction. Thus we have proved that $L \subset X$, that is $L \in F_1(X, x)$.

Since z and z' are not separated by $\varphi_{(\mathcal{E}_{|L})}$, we have that $h_L \geq 1, z \in \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus h_L})$ and $y = \varphi(z) = \psi_L(z) \in V_L$. Therefore $z \in \psi_L^{-1}(y) \cap \mathbb{P}(\mathcal{E}_x)$ and then $(z, L) \in \mathcal{I}$, giving that $z \in \operatorname{Im} p_1$. Hence p_1 is surjective, $h(\mathcal{E}, x)$ is well defined and let us see that

$$\varphi(\mathbb{P}(\mathcal{E})) = \bigcup_{L \in F_1(X): h_L \geq 1} V_L. \tag{4.3}$$

In fact, on one side, the inclusion $V_L \subseteq \varphi(\mathbb{P}(\mathcal{E}))$ follows by definition. Now let $y \in \varphi(\mathbb{P}(\mathcal{E}))$. Then there is $z \in \mathbb{P}(\mathcal{E})$ such that $y = \varphi(z)$ and set $x = \pi(z)$. Then $z \in \mathbb{P}(\mathcal{E}_x)$ and the surjectivity of p_1 implies that there is a line $L \in F_1(X, x)$ such that $(z, L) \in \mathcal{I}$, hence $h_L \geq 1$ and $y = \varphi(z) = \psi_L(z) \in V_L$. This proves (4.3).

Now observe that the nonempty fibers of p_2 are all isomorphic to V_L for some $L \in F_1(X, x)$. In fact, if $L \in F_1(X, x)$ is such that $p_2^{-1}(L) \neq \emptyset$, then there is $z_0 \in \mathbb{P}(\mathcal{E}_x)$ such that $(z_0, L) \in \mathcal{I}$, hence $h_L \geq 1$. Then, for any $(z, L) \in p_2^{-1}(L)$, there exists $y \in V_L$ with $z \in \psi_L^{-1}(y) \cap \mathbb{P}(\mathcal{E}_x)$, that is $\psi_L(z) = y \in V_L$. This defines a morphism $f : p_2^{-1}(L) \to V_L$ by $f((z, L)) = \psi_L(z)$. We have that f is injective since all the fibers $\psi_L^{-1}(y), y \in V_L$ intersect $\mathbb{P}(\mathcal{E}_x)$ in one point. Also, if $y \in V_L$, let $\{z\} = \psi_L^{-1}(y) \cap \mathbb{P}(\mathcal{E}_x)$. Then $(z, L) \in p_2^{-1}(L)$ and $y = \psi_L(z)$. Thus f is also surjective.

Let W be an irreducible component of $\mathcal{I}$ such that W dominates $\mathbb{P}(\mathcal{E}_x)$. We have, for a general $L \in p_2(W)$,

$$\dim W = \dim p_2(W) + h_L - 1 \leq \dim F_1(X, x) + h(\mathcal{E}, x) - 1.$$

Therefore we deduce that

$$r - 1 = \dim \mathbb{P}(\mathcal{E}_x) \leq \dim W \leq \dim F_1(X, x) + h(\mathcal{E}, x) - 1$$

and (4.1) holds. Now consider the incidence correspondence

$$\mathcal{I}' = \{(y, L) \in \mathbb{P}H^0(\mathcal{E}) \times F_1(X) : h_L \geq 1 \text{ and } y \in V_L\}$$

together with its projections

$$\begin{array}{ccc} & \mathcal{I}' & \\ {\scriptstyle p'_1} \swarrow & & \searrow {\scriptstyle p'_2} \\ \mathbb{P}H^0(\mathcal{E}) & & F_1(X) \end{array} .$$

Then $\operatorname{Im} p'_1 = \varphi(\mathbb{P}(\mathcal{E}))$ by (4.3). Also, the nonempty fibers of p'_2 are isomorphic to $V_L \cong \mathbb{P}^{h_L - 1}$, since if $(p'_2)^{-1}(L) \neq \emptyset$ then $h_L \geq 1$ and $(p'_2)^{-1}(L) = \{(y, L) : y \in V_L\} \cong V_L$. Therefore, choosing an irreducible component W' of $\mathcal{I}'$ such that W' dominates $\varphi(\mathbb{P}(\mathcal{E}))$, we get, for a general $L \in p_2(W')$, that

$$\nu(\mathcal{E}) = \dim \varphi(\mathbb{P}(\mathcal{E})) \leq \dim W' \leq \dim F_1(X) + h_L - 1 \leq \dim F_1(X) + h(\mathcal{E}) - 1$$

and we get (4.2). □

We can now prove our first theorem.

Proof of Theorem 1. The vector bundles in Table 1 are Ulrich and non-big by Proposition 3.3(iii). Vice versa, by the same proposition, we can assume that $n \geq 11$. Let $\mathcal{E}$ be a spinor bundle on Q_n. Note that $r = 2^{\lfloor \frac{n-1}{2} \rfloor}$ and $h(\mathcal{E}, x) = 2^{\lfloor \frac{n-3}{2} \rfloor}$ by [O, Cor. 1.6] for any $x \in Q_n$. We claim that $\mathcal{E}$ is big. In fact, if not, we get by Proposition 4.1 that

$$n - 2 + 2^{\lfloor \frac{n-3}{2} \rfloor} \geq 2^{\lfloor \frac{n-1}{2} \rfloor}$$

contradicting $n \geq 11$. Now just apply Lemma 2.6 together with Lemma 3.2(iv) to get that any Ulrich vector bundle on $Q_n, n \geq 11$ is big. $\square$

Remark 4.2. It follows by Theorem 1 that the numerical dimension of any rank r Ulrich vector bundle $\mathcal{E}$ on Q_n is known. In fact, if $\mathcal{E}$ does not belong to Table 1, then $\nu(\mathcal{E}) = n + r - 1$. Now suppose that $\mathcal{E}$ is as in Table 1. Then $\nu(\mathcal{E})$ is given in Proposition 3.3(ii) if $\mathcal{E}$ is a spinor, $\nu(\mathcal{E}) = n + r - 2$ if $\mathcal{E}$ is not a spinor.

Remark 4.3. The example $\mathcal{E} = \mathcal{S}' \oplus \mathcal{S}''$ on Q_4 shows that, even when $\operatorname{Pic}(X) \cong \mathbb{Z}$, the fibers of φ can have different dimensions. In fact, $\nu(\mathcal{E}) = 6$ so that a general fiber is 1-dimensional, while $\nu(\mathcal{S}') = 3$ hence the fibers over points in $\mathbb{P}(\mathcal{S}')$ are 2-dimensional.

Under a suitable hypothesis, we will now study Ulrich vector bundles with minimal numerical dimension on linear $\mathbb{P}^k$-bundles and determine their restriction to fibers in the next case. We will use the notation in Definition 2.3.

The following lemma allows us to identify $\mathcal{E}_{|f}$ for all f, rather than on a general f, thus giving a better description of $\mathcal{E}$. This is needed, in the present paper, in Theorem 2(i2) and in the proof of [LMS, Thm. 1] (for example in case (xii)).

Lemma 4.4. *Let $(X, \mathcal{O}_X(1)) = (\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$, where $\mathcal{F}$ is a rank $n-b+1$ very ample vector bundle over a smooth irreducible variety B of dimension b with $1 \leq b \leq n-1$. Let $\mathcal{E}$ be a rank r Ulrich vector bundle on X, let $p : X \to B$ be the projection morphism and suppose that*

$$p_{|\Pi_y} : \Pi_y \to B \text{ is constant for every } y \in \varphi(\mathbb{P}(\mathcal{E})). \tag{4.4}$$

Then, for every fiber f of p we have $\nu(\mathcal{E}) = b + \dim \varphi(\pi^{-1}(f)) \geq b + r - 1$. Moreover we have the following two extremal cases:

(i) *If $\nu(\mathcal{E}) = b + r - 1$ there is a rank r vector bundle $\mathcal{G}$ on B such that $\mathcal{E} \cong p^*(\mathcal{G}(\det \mathcal{F}))$ and $H^j(\mathcal{G} \otimes S^k\mathcal{F}^*) = 0$ for all $j \geq 0, 0 \leq k \leq b-1$.*
(ii) *If $\nu(\mathcal{E}) = b + r$ then either $b = n-1$ and $\mathcal{E}$ is big or $b \leq n-2$ and $\mathcal{E}_{|f} \cong T_{\mathbb{P}^{n-b}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-b}}^{\oplus(r-n+b)}$ for any fiber $f = \mathbb{P}^{n-b}$ of p.*

Proof. Set $f_v = p^{-1}(v), v \in B$. Let $x, x' \in X$ be such that $p(x) \neq p(x')$. If there exists an $y \in P_x \cap P_{x'}$, then $x, x' \in \Pi_y$ by Lemma 2.4, contradicting (4.4). Therefore $P_x \cap P_{x'} = \emptyset$. It follows that

$$\varphi(\mathbb{P}(\mathcal{E})) = \bigsqcup_{v \in B} \varphi(\pi^{-1}(f_v)). \tag{4.5}$$

Consider the incidence correspondence

$$\mathcal{I} = \{(y, v) \in \varphi(\mathbb{P}(\mathcal{E})) \times B : y \in \varphi(\pi^{-1}(f_v))\}$$

together with its two projections $p_1 : \mathcal{I} \to \varphi(\mathbb{P}(\mathcal{E}))$ and $p_2 : \mathcal{I} \to B$. Then (4.5) implies that p_1 is bijective, hence $\mathcal{I}$ is irreducible and $\dim \mathcal{I} = \dim \varphi(\mathbb{P}(\mathcal{E})) = \nu(\mathcal{E})$. Since p_2 is surjective, it follows that for any $v \in B$ we have that

$$\dim \varphi(\pi^{-1}(f_v)) \geq \nu(\mathcal{E}) - b.$$

On the other hand, for every $y \in \varphi(\pi^{-1}(f_v))$ there is a $z \in \pi^{-1}(f_v)$ such that $y = \varphi(z)$, hence $\pi(z) \in \Pi_y \cap f_v$ and (4.4) gives that $p(\Pi_y) = \{v\}$. Hence $\Pi_y \subseteq f_v$ and therefore $\varphi^{-1}(y) \subseteq \pi^{-1}(f_v)$. Picking a general $y \in \varphi(\pi^{-1}(f_v))$ we deduce that

$$\dim \varphi(\pi^{-1}(f_v)) = \dim \pi^{-1}(f_v) - \dim \varphi^{-1}(y) \leq n - b + r - 1 - (n + r - 1 - \nu(\mathcal{E})) = \nu(\mathcal{E}) - b.$$

Therefore $\dim \varphi(\pi^{-1}(f)) = \nu(\mathcal{E}) - b$ for every fiber f of $p : X \to B$. In particular, being $\varphi(\pi^{-1}(f_v))$ union of linear spaces $P_x = \mathbb{P}^{r-1}$ for $x \in f_v$, we see that $\nu(\mathcal{E}) \geq b + r - 1$.

Now consider the morphism $\Phi_{|f} : f = \mathbb{P}^{n-b} \to \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$. Observe that $\Phi_{|f}$ is constant if and only if $P_x = P_{x'}$ for any $x, x' \in f$, that is if and only if $\varphi(\pi^{-1}(f)) = P_x = \mathbb{P}^{r-1}$, or, equivalently, if and only if $\nu(\mathcal{E}) = b + r - 1$. Hence, when $\nu(\mathcal{E}) = b + r - 1$, we get that $\det(\mathcal{E})_{|f}$ is trivial. This implies that there is a globally generated line bundle M on B such that $\det \mathcal{E} = p^*M$ and therefore $\mathcal{E}$ is as in (i) by [Lo, Lemmas 5.1 and 4.1].

On the other hand, if $\nu(\mathcal{E}) = b + r$, then $\Phi_{|f}$ is finite-to-one onto its image. If $b = n - 1$ then $\nu(\mathcal{E}) = n + r - 1$ and $\mathcal{E}$ is big. If $b \le n - 2$ then $\varphi(\pi^{-1}(f)) \subseteq \mathbb{P}H^0(\mathcal{E})$ is swept out by a family $\{P_x, x \in f\}$ of dimension $n - b \ge 2$ of linear $\mathbb{P}^{r-1}$'s. Therefore

$$\varphi(\pi^{-1}(f)) = \mathbb{P}^r \text{ for every fiber } f \text{ of } p : X \to B. \tag{4.6}$$

This gives that $\mathcal{E}_{|f}$ can be generated by $r + 1$ global sections and we get an exact sequence

$$0 \to \mathcal{O}_f(-a) \to \mathcal{O}_f^{\oplus(r+1)} \to \mathcal{E}_{|f} \to 0 \tag{4.7}$$

where $c_1(\mathcal{E}_{|f}) = \mathcal{O}_f(a)$. Note that $H^1(\mathcal{O}_f(-a-1)) = 0$ since $f = \mathbb{P}^{n-b}, n-b \ge 2$ and then (4.7) implies that

$$H^0(\mathcal{E}_{|f}(-1)) = 0. \tag{4.8}$$

Since $\varphi(\pi^{-1}(f)) = \mathbb{P}^r$, there must be two points $x, x' \in f$ such that $P_x \ne P_{x'}$ and (4.6) gives that $\varphi(\pi^{-1}(L)) = \mathbb{P}^r$, where L is the line in f joining x and x'. Now [LS, Lemma 3.2] gives that $H^0(\mathcal{E}) \to H^0(\mathcal{E}_{|L})$ is surjective, hence $\varphi_{|\pi^{-1}(L)} = \varphi_{\mathcal{E}_{|L}}$. Since $c_1(\mathcal{E}_{|L}) = \mathcal{O}_L(a)$ we get that

$$a = \deg \varphi_{\mathcal{E}_{|L}}(\pi^{-1}(L)) = \deg \varphi(\pi^{-1}(L)) = \deg \mathbb{P}^r = 1$$

where the degrees are meant as subvarieties of $\mathbb{P}H^0(\mathcal{E})$.

Now for any line $L' \subset f$ we have that $c_1(\mathcal{E}_{|L'}) = \mathcal{O}_{L'}(1)$, hence, being $\mathcal{E}$ globally generated, we get that $(\mathcal{E}_{|f})_{|L'} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(r-1)}$. Then [E, Prop. IV.2.2] implies that $\mathcal{E}_{|f} \cong T_{\mathbb{P}^{n-b}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-b}}^{\oplus(r-n+b)}$ or $\mathcal{O}_{\mathbb{P}^{n-b}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n-b}}^{\oplus(r-1)}$ or $\Omega_{\mathbb{P}^{n-b}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-b}}(1)^{\oplus(r-n+b)}$. Now the second case is excluded by (4.8). On the other hand, if the third case occurs, then $\mathcal{O}_f(1) = c_1(\mathcal{E}_{|f}) = \mathcal{O}_f(r-1)$, hence $2 = r \ge n - b \ge 2$, therefore equality holds, and this is also the first case. Thus $\mathcal{E}$ is as in (ii). □

We now give an example showing that the restriction of $\mathcal{E}$ in Lemma 4.4(ii) actually occurs. For an example with $b = 2$ see [LMS, Ex. 5.11].

Example 4.5. Let $n \ge 3$, let $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$ and let $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. It is easily seen that, for every $r \ge n - 1$, the vector bundle

$$\mathcal{E} = [\mathcal{O}_{\mathbb{P}^1}(n-2) \boxtimes T_{\mathbb{P}^{n-1}}(-1)] \oplus [\mathcal{O}_{\mathbb{P}^1}(n-1) \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}]^{\oplus(r-n+1)}$$

is Ulrich, $c_1(\mathcal{E})^n > 0, \nu(\mathcal{E}) = r+1$ and $\mathcal{E}_{|f} = T_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}^{\oplus(r-n+1)}$ on any fiber f of the first projection $p : X \to \mathbb{P}^1$.

A first use of Lemma 4.4 is the following.

Proof of Theorem 3. If $c_1(\mathcal{E})^4 = 0$ we have by [LS, Cor. 4.9] that $\mathcal{E} \cong p^*(\mathcal{O}_{\mathbb{P}^2}(2))^{\oplus r}$, where $p : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ is one of the two projections.

Assume now that $c_1(\mathcal{E})^4 > 0$. First, we claim that we can apply Lemma 4.4(ii) with one of the two projections $p : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ or $q : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$. In fact, [LS, Cor. 2] implies that $r + 3 - \nu(\mathcal{E}) = 1$, so that $\nu(\mathcal{E}) = r + 2$. Also, for any $y \in \varphi(\mathbb{P}(\mathcal{E}))$, we have that Π_y is a linear subspace in $\mathbb{P}^8$ of dimension $k \ge r + 3 - \nu(\mathcal{E}) = 1$, hence, as is well known, Π_y is contained in a fiber of p or in a fiber of q. Let $U \subseteq \varphi(\mathbb{P}(\mathcal{E}))$ be the non-empty open subset on which $\dim \varphi^{-1}(y) = 1$ for every $y \in U$. This gives a morphism $\gamma : U \to F_1(X)$ defined by $\gamma(y) = \Pi_y$. Since $F_1(X)$ has two irreducible disjoint components, namely W_p, the lines contained in a fiber of p and W_q the lines contained in a fiber of q, we get that either $\gamma(U) \subseteq W_p$ or $\gamma(U) \subseteq W_q$. In the first case, by specialization, we deduce that Π_y is contained in a fiber of p for every $y \in \varphi(\mathbb{P}(\mathcal{E}))$. In the second case the same happens for q. Therefore we can assume that (4.4) holds for p. Then Lemma 4.4(ii) applies to p and we get that

$$\mathcal{E}_{|f} \cong T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus(r-2)} \tag{4.9}$$

on any fiber $f \cong \mathbb{P}^2$ of p. Let $A = p^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and $B = q^*(\mathcal{O}_{\mathbb{P}^2}(1))$, so that we can write

$$\det \mathcal{E} = \alpha A + \beta B, c_2(\mathcal{E}) = \gamma A^2 + \delta A \cdot B + \epsilon B^2$$

for some $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{Z}$. Now (4.9) implies that $\alpha = \gamma = 1$. Moreover, if C is the curve section of $X \subset \mathbb{P}^8$, then $C \subset \mathbb{P}^5$ is an elliptic curve of degree 6 and $\mathcal{E}_{|C}$ is a rank r Ulrich vector bundle on C by Remark 2.8(ii). Hence $\chi(\mathcal{E}_{|C}(-1)) = 0$, that is $\deg(\mathcal{E}_{|C}) = 6r$ and we get

$$3 + 3\beta = (A + \beta B)(A + B)^3 = c_1(\mathcal{E}_{|C}) = 6r$$

that is $\beta = 2r - 1$ and this gives

$$\det \mathcal{E} = A + (2r-1)B, c_2(\mathcal{E}) = A^2 + \delta AB + \epsilon B^2. \tag{4.10}$$

Let now Y be the hyperplane section of X, so that $\mathcal{E}_{|Y}$ is a rank r Ulrich vector bundle on Y by Remark 2.8(ii). By [M, Thm. 5.1], setting $p_1 = p_{|Y} : Y \to \mathbb{P}^2, p_2 = q_{|Y} : Y \to \mathbb{P}^2$ and $\mathcal{G}_i = p_i^*(\Omega_{\mathbb{P}^2}(1)), i = 1, 2$, there is a resolution

$$0 \to p_1^*(\mathcal{O}_{\mathbb{P}^2}(1))^{\oplus d} \oplus p_2^*(\mathcal{O}_{\mathbb{P}^2}(1))^{\oplus c} \to \mathcal{G}_1(1)^{\oplus b} \oplus \mathcal{G}_2(1)^{\oplus a} \to \mathcal{E}_{|Y} \to 0. \tag{4.11}$$

Computing rank and the first Chern class in (4.11) and using (4.10) we get the three equations

$$2a + 2b = c + d + r, 1 = b + 2a - d, 2r - 1 = 2b + a - c$$

which imply that $a + c = 1$, thus giving the only possible solutions $(a, b, c, d) = (0, r, 1, r-1)$ or $(1, r-1, 0, r)$. Now computing the second Chern class in (4.11) and using (4.10) we have two possibilities. In the first case we get that $r = 1$, a contradiction since $\mathcal{E}$ is not big. In the second case we get that $r = 2, \det \mathcal{E} = A + 3B$ and $c_2(\mathcal{E}) = A^2 + AB + 4B^2$, but then $s_4(\mathcal{E}^*) = 6 > 0$, a contradiction since $\mathcal{E}$ is not big. □

Next we prove a useful result that will later allow to give an upper bound on the rank. We will use the notation in Definition 2.3.

Proposition 4.6. *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 2$ and let $\mathcal{E}$ be a rank r Ulrich vector bundle on X such that $\mathcal{E}$ is not big and $c_1(\mathcal{E})^n > 0$. Assume that for a general point $x \in X$ we have that $F_{n+r-\nu(\mathcal{E})}(X, x) = \emptyset$. Then there is a morphism, finite onto its image,*

$$\psi : \mathbb{P}^{r-1} \cong P_x \to F_{n+r-1-\nu(\mathcal{E})}(X, x)$$

and

$$\dim F_{n+r-1-\nu(\mathcal{E})}(X, x) \geq r - 1.$$

Proof. By [LS, Thm. 2] and Lemma 2.4, for any $y \in P_x$ we have that Π_y is a linear subspace in $\mathbb{P}^N$ of dimension $k \geq n + r - 1 - \nu(\mathcal{E}) \geq 1$ such that $x \in \Pi_y \subseteq X$. Since $F_{n+r-\nu(\mathcal{E})}(X, x) = \emptyset$ we get that $k = n + r - 1 - \nu(\mathcal{E})$ and we can define a morphism

$$\psi : \mathbb{P}^{r-1} \cong P_x \to F_k(X, x)$$

by $\psi(y) = \Pi_y$. If ψ is constant, we claim that for any $y \in P_x$ we have that

$$\Pi_y \subseteq \Phi^{-1}(\Phi(x)).$$

In fact, let $x' \in \Pi_y$. For every $y' \in P_x$, since $\Pi_y = \Pi_{y'}$, we have that $x' \in \Pi_{y'}$, that is $y' \in P_{x'}$ by Lemma 2.4. Hence $P_x \subseteq P_{x'}$ and we deduce that $P_x = P_{x'}$ and therefore

$$\Phi(x) = [P_x] = [P_{x'}] = \Phi(x')$$

and the claim is proved. Now Π_y has dimension $k \geq 1$, hence the fibers of Φ have dimension at least 1. This implies that $\det \mathcal{E}$ is not big, a contradiction. Therefore ψ is finite onto its image and we deduce that $\dim F_k(X, x) \geq r - 1$. □

We can now prove our second theorem.

Proof of Theorem 2. Suppose that $\nu(\mathcal{E}) \leq \frac{n}{2} + r - 1$.

If $n = 1$ then $\nu(\mathcal{E}) \leq r - 1$. But $\varphi(\mathbb{P}(\mathcal{E}_x)) = P_x = \mathbb{P}^{r-1}$, hence $\nu(\mathcal{E}) = r - 1$ and $P_x = P_{x'}$ for every $x \neq x' \in X$, that is $\Phi(x) = \Phi(x')$. Hence [LS, Thm. 2] gives that $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ and we are in case (i). Vice versa, if we are in case (i) then $b = 0$ and $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. It follows by Remark 2.9 that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$, hence $\nu(\mathcal{E}) = r - 1$.

Suppose from now on that $n \geq 2$.

Let $x \in X$. By [LS, Thm. 2], for any $y \in P_x$ we have that Π_y is a linear subspace of dimension k in $\mathbb{P}^N$ such that, using Lemma 2.4, $x \in \Pi_y \subseteq X$ and $k \geq n+r-1-\nu(\mathcal{E}) \geq \frac{n}{2}$.

Assume that we are not in case (i).

Then [S, Main Thm.] implies that (X, H) is one of the following:

(1) $(Q_{2m}, \mathcal{O}_{Q_{2m}}(1))$.
(2) $(\mathbb{G}(1, m+1), \mathcal{O}_{\mathbb{G}(1,m+1)}(1))$ (the Plücker line bundle).

We can assume that $m \geq 2$, for otherwise, when $m = 1$, case (2) is also case (i) and in case (1) we know that $\mathcal{E} \cong (\mathcal{S}')^{\oplus r}$ or $(\mathcal{S}'')^{\oplus r}$ by Proposition 3.3(iii), thus giving case (ii). Note that $\det \mathcal{E}$ is globally generated and big by [Lo, Lemma 3.2], since we are excluding (i). In particular $r \geq 2$. Now $n = 2m$ and $F_{m+1}(X, x) = \emptyset$. Hence $m \geq k \geq 2m+r-1-\nu(\mathcal{E}) \geq m$ and we get that $k = m$ and $\nu(\mathcal{E}) = m+r-1$. Then $\dim F_m(X, x) \geq r-1$ by Proposition 4.6. In case (1) we know by Lemma 3.2(iv) that $r \geq 2^{m-1}$ and $\dim F_m(X, x) = \frac{m(m-1)}{2}$, thus the only possibilities are for $2 \leq m \leq 3$ and $r = 2^{m-1}$. Hence $\mathcal{E}$ is a spinor bundle by Lemma 3.2(iv) and we get case (ii). In case (2) we can assume that $m \geq 3$ for otherwise we are in case (1). We know that $\dim F_m(X, x) = 1$, hence $r = 2$ and $\varphi(\mathbb{P}(\mathcal{E}))$ has dimension $\nu(\mathcal{E}) = m+1$. Moreover $\varphi(\mathbb{P}(\mathcal{E}))$ is covered by a family of dimension $2m$ of lines $\{P_x, x \in X\}$: the family has dimension $2m$ because $c_1(\mathcal{E})^n > 0$, hence Φ is birational onto its image. Therefore $\varphi(\mathbb{P}(\mathcal{E})) = \mathbb{P}^{m+1} = \mathbb{P}H^0(\mathcal{E}) = \mathbb{P}^{2d-1}$ where $d = \deg X$. But $d = \frac{(2m)!}{m!(m+1)!}$ giving the contradiction $m+1 = 2d-1$.

Assume now that we are in case (i).

If $b = 0$ then $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and Remark 2.9 gives that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$, hence $\nu(\mathcal{E}) = r-1$.

Now suppose that $b \geq 1$. By [S, Main Thm.] we have that $\Pi_y \subseteq \mathbb{P}^{n-b}$ for a general y.

If $c_1(\mathcal{E})^n = 0$ we have by [LS, Cor. 3] that Π_y is a general fiber F of Φ and therefore $\Phi_{|\mathbb{P}^{n-b}}$ must be constant. Thus $F = \Pi_y = \mathbb{P}^{n-b}, \nu(\mathcal{E}) = b+r-1$ and $\det \mathcal{E}$ is trivial on the fibers of p. Hence there is a globally generated line bundle M on B such that $\det \mathcal{E} = p^*M$ and $\mathcal{E}$ is as in (i1) by [Lo, Lemmas 5.1 and 4.1].

If $c_1(\mathcal{E})^n > 0$ we know by [LS, Cor. 2] that through a general point $x \in X$ we can find infinitely many linear spaces Π_y, hence $n+r-1-\nu(\mathcal{E}) \leq n-b-1$, that is $\nu(\mathcal{E}) \geq b+r$ and then $b \leq \frac{n}{2}-1$. Also, if $\nu(\mathcal{E}) = b+r$ we have, for every $y \in \varphi(\mathbb{P}(\mathcal{E}))$,

$$\dim \Pi_y \geq n+r-1-\nu(\mathcal{E}) = n-b-1 \geq b+1.$$

Since Π_y is a linear space, we deduce that $p_{|\Pi_y} : \Pi_y \to B$ is constant. Thus we can apply Lemma 4.4(ii) and find that $\mathcal{E}_{|f} \cong T_{\mathbb{P}^{n-b}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-b}}^{\oplus(r-n+b)}$. This gives case (i2). □

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